Page No. ____
Date ____

Ordinary Differential Equations

① What are ODE?

→ When only 1 independent variable but can have one or more dependent variable, then it is called ODE.

② Order = Highest order derivative

Degree = Degree of highest order derivative

★ Don't forget to resolve fraction or square root before.

③ How to form ODE?

Differentiate twice and then try by arrangements.

④ Linear Differential equation with constant coefficient:

$$f(D)y = Q$$

where $f(D) = (a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_{n-1} D + a_n)$

$$D = \frac{d}{dx} \qquad \frac{1}{D} = \int dx$$

If $Q = 0$; Homogeneous L.D.E

If $Q \neq 0$; Non-Homogeneous L.D.E

⑤ General or Complete solution of LDE →

$$y = C.F + P.I$$

↳ Particular integral

↳ Complementary function

Page No.
Date

⑥ How to colwlate C.F.? $f(D)y = Q$

Step1 First find the Auxillary equation. For that put $Q=0$, $y=1$ & $D=m$. You will get a polynomial eq^n in m. Solve these & then compare

Step 2 (i) If roots are real & distinct:

$$C.F = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x} + \cdots + C_n e^{m_n x}$$

(ii) If roots are real & equal & 1st 2 are equal others are distinct.

$$C.F = (C_1 + C_2 x) e^{m_1 x} + C_3 e^{m_3 x} + C_4 e^{m_4 x} + \cdots$$

(iii) If roots are imaginary in pair; $m = \alpha \pm i\beta$

$$C.F = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x)$$

(iv) If roots are irrational ($m = \alpha \pm \sqrt{\beta}$)

$$C.F = C_1 e^{(\alpha + \sqrt{\beta})x} + C_2 e^{(\alpha + \sqrt{\beta})x}$$

⑦ How to Calculate P.I.? $f(D)y = Q \Rightarrow P.I = \frac{1}{f(D)} Q$

Rule 1 $f(D) = D$, then $P.I = \frac{1}{D} Q = \int Q\,dx$

Rule 2 (General Rule) $f(D) = D - a$, then $P.I = \frac{1}{D-a} Q = e^{ax} \int e^{-ax} Q\,dx$

Rule 3 If $Q = e^{ax}$ where a = constant then

$$P.I = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax} \quad (D \to a)$$

(given $f(a) \neq 0$)

if $f(a) = 0$ then $f(D) = (D-a)^n \phi(D)$ where $\phi(a) \neq 0$

$$P.I = \frac{1}{\phi(a)} \left[\frac{x^n e^{ax}}{n!} \right]$$

Page No. ____
Date

Rule 4 If $Q = \sin ax$ or $\cos ax$, then (given $f(-a^2) \neq 0$)

$$P.I = \frac{1}{f(D^2)} \sin(ax) = \frac{1}{f(-a^2)} \sin ax \qquad D^2 \to -a^2 \qquad D^3 \to D(-a^2)$$

Rule 5 If $Q = x^m$ (m = +ve integer) then

$$P.I = \frac{1}{f(D)} x^m = [f(D)]^{-1} x^m$$ & take expansion till x^m.

Rule 6 If $Q = e^{ax} \cdot v(x)$

$$P.I = \frac{1}{f(D)} e^{ax} \cdot v = e^{ax} \left[\frac{1}{f(D+a)} v \right]$$

Rule 7 If $f(D) = D^2 + a^2$

$$P.I = \frac{1}{D^2+a^2} \sin ax = -\frac{x}{2a} \cos ax$$

$$P.I = \frac{1}{D^2+a^2} \cos ax = \frac{x}{2a} \sin ax$$

Rule 8 If $Q = x^m \cdot v(x)$

$$P.I = x^m \frac{1}{f(D)} v + m x^{m-1} \left[\frac{d}{dD} \frac{1}{f(D)} \right] v + \frac{m(m-1) x^{m-2}}{2!} \left[\frac{d^2}{dD^2} \frac{1}{f(D)} \right] v + \cdots$$

It goes on till x is raised to 0

(8) Variation of Parameter Method

Step 1 Write given D.E into standard forms

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad -(1)$$

Step 2 Considering $R = 0$ in eq^n (1), find its solution using CF PI method. Suppose the sol^n is $y = C_1 u(x) + C_2 v(x)$
C_1 & C_2 are arbitrary constants & $u(x)$ & $v(x)$ are function of x

Page No. ______
Date

Step 3 Consider complete solution of (1) as

$$y = A(x)\,u(x) + B(x)\,v(x) \quad (2)$$ where

$$A(x) = -\int \frac{v(x)\,R}{uv' - u'v}\,dx + C_3 \quad \Big| \quad B(x) = \int \frac{u(x)\,R\,dx}{uv' - vu'} + C_4$$

Put these values back in equation (2) for final answer

(9) Cauchy's Linear Homogeneous Differential Equation:

Standard form: $a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_n y = Q$

$$= (a_0 x^n D^n + a_1 x^{n-1} D^{n-1} + a_2 x^{n-2} D^{n-2} + \dots + a_n) y = Q$$

Step 1 Put $x = e^z$ or $z = \log x$

$$x\frac{d}{dx} = xD = D' \quad \Big| \quad x^2 \frac{d^2}{dx^2} = x^2 D^2 = D'(D'-1)$$

$$x^3 \frac{d^3}{dx^3} = x^3 D^3 = D'(D'-1)(D'-2). \text{ \& so on}$$

Form the equation & then solve using A.E / CF / PI method

At the end change back your equation to x.

(10) Solution of Simultaneous Linear Differential Equation.

⇒ Method 1 (Elimination method)

Multiply both equations with coefficient so as to make same coefficient with any 1 variable to eliminate it & solve the rest by C.F / P.I method after forming the equation after elimination of one variable. Now put value of variable & its differential back in equation to get the other variable

Page No.
Date

→ Method 2 (Substitution Method)
Differentiate any 1 equation & put its value in other equation & then the rest remains same.

(11) Wronskian Method in Variation of Parameter method.
Let $y'' + Py' + Qy = R$ be the second order D.E

Step ① Taking $R=0$, calculate the CF of this equation.
Let $y = c_1 u + c_2 v$

Step ② Find wronskian of u & v $\quad W(u,v) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$

Step ③ $P.I = u f(x) + v g(x)$

$$f(x) = -\int \frac{vR}{W(u,v)}\,dx \quad \Big| \quad g(x) = \int \frac{u \cdot R}{W(u,v)}\,dx$$

Step ④ Get general solution by $y = C.F + P.I.$