Lecture 14 — Econometrics-I

19.03.2021

* Hypothesis Testing and Statistical Inference

Example: (Just an example) → we will go over all concepts separately.

$$wage_i = \beta_0 + \beta_1 \, educ_i + \beta_2 \, exper_i + u_i$$

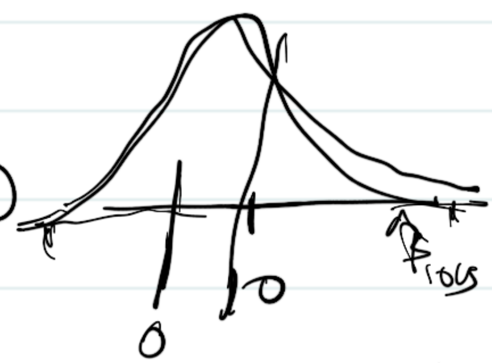
Null Hypothesis

$H_0 : \beta_1 = 0$

$H_a : \beta_1 \neq 0$

$$u_i \sim N(0, \sigma_u^2) \qquad i = 1, 2, \ldots, 5000$$

$$\underbrace{\hat{\beta}_{1,OLS}} = \frac{\partial \, wage_i}{\partial \, educ_i} = 10$$



$V(\hat{\beta}_{1,OLS})$ = 50 eg.

$$spend_i = \beta_0 + \beta_1 \, income_i + \beta_2 \, asset_i + \beta_3 \, decline_i + \varepsilon_i$$

0/1

$i = 1, 2, \ldots, 1000$

$$\hat{\beta}_{3,OLS} \; ; \; V(\hat{\beta}_{3,OLS})$$

Some Useful distributions

1. Standardized normal distribution

(Normal) $X \sim N(\mu, \sigma^2)$ ✓

then

(standardized normal) $Z = \dfrac{X-\mu}{\sigma} \sim N(0,1)$

$$E(Z) = E\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma}[EX - \mu] = 0$$

$$V(Z) = V\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma^2} V(X) = 1$$ ✓

If you are given a sample of X_i distributed normally with mean μ and variance σ^2 then we can write:

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{N}\right)$$

$$X_i \overset{iid}{\sim} N(\mu, \sigma^2)$$

$$\bar{X} = \frac{1}{N} \sum_{i=1}^{N} X_i$$

$$\Rightarrow E(\bar{X}) = \frac{1}{N} \sum_{i=1}^{N} E(X_i) = \frac{N\mu}{N} = \mu$$

$$\Rightarrow V(\bar{X}) = \frac{1}{N^2} V\left(\sum_{i=1}^{N} X_i\right) = \frac{1}{N^2} \sum_{i=1}^{N} \underbrace{V(X_i)}_{=\sigma^2} = \frac{N\sigma^2}{N^2} = \frac{\sigma^2}{N}$$

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{N}\right)$$

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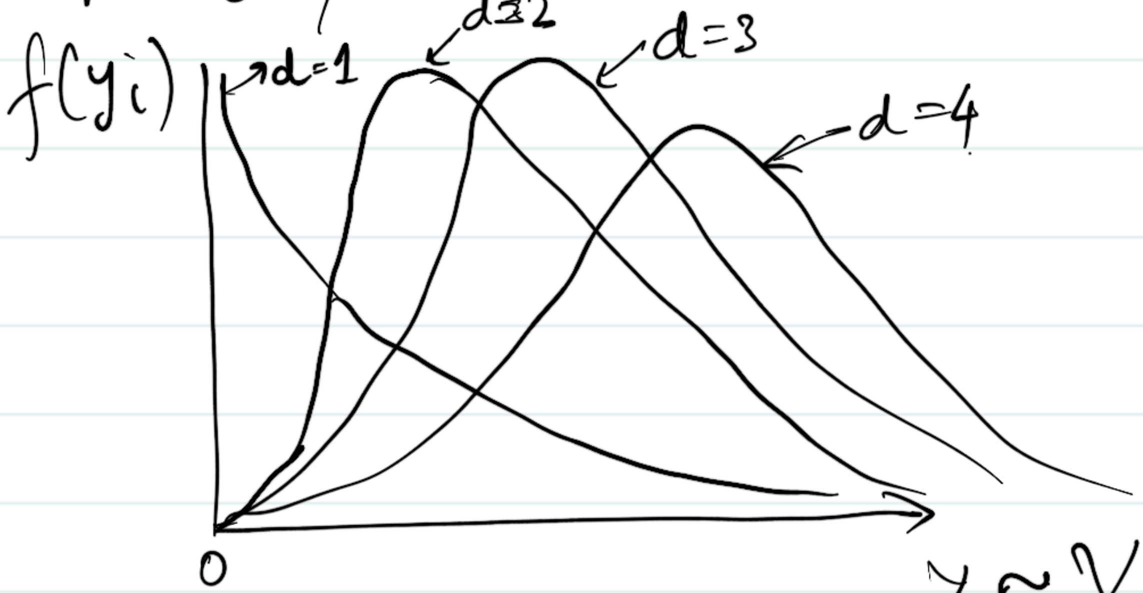
2. Chi-squared distribution

Let $X_i \overset{iid}{\sim} N(0,1)$

then $\boxed{Y_i = \sum_{i=1}^{d} X_i^2 \sim \chi^2_d}$

* a chi-squared distributed random variable cannot be $(-)$ve.

Shape of χ^2 distributed r.v.



$Y_i \sim \chi^2_d$

Claim: if $Y_i \overset{iid}{\sim} N(\mu, \sigma^2)$

then $\sum_{i=1}^{N} \left(\frac{Y_i - \bar{Y}}{\sigma} \right)^2 \sim \chi^2_{N-1}$

(where $\sum_{i=1}^{N} \left(\frac{Y_i - \mu}{\sigma} \right)^2 \sim \chi^2_N$)

Claim: $Y_i \overset{iid}{\sim} N(\mu, \sigma^2) \Rightarrow \sum_{i=1}^{N} \left(\frac{Y_i - \bar{Y}}{\sigma}\right)^2 \sim \chi^2_{N-1}$

Proof:

$$LHS = \sum_{i=1}^{N} \left(\frac{Y_i - \mu + \mu - \bar{Y}}{\sigma}\right)^2$$

$$= \sum_{i=1}^{N} \left[\frac{(Y_i-\mu)^2 + (\mu - \bar{Y})^2 + 2(Y_i - \mu)(\mu - \bar{Y})}{\sigma^2}\right]$$

$$= \sum_{i=1}^{N} \left(\frac{Y_i - \mu}{\sigma}\right)^2 + N\frac{(\mu - \bar{Y})^2}{\sigma^2} + 2\frac{(\mu - \bar{Y})}{\sigma^2} \underbrace{\sum_{i=1}^{N}(Y_i - \mu)}_{\substack{= N\bar{Y} - N\mu \\ = N(\bar{Y} - \mu)}}$$

$$= \underbrace{\sum_{i=1}^{N} \left(\frac{Y_i - \mu}{\sigma}\right)^2}_{Y_i \overset{iid}{\sim} N(\mu, \sigma^2)} - \underbrace{\frac{(\mu - \bar{Y})^2}{\sigma^2/N}}_{\bar{Y} \sim N\left(\mu, \frac{\sigma^2}{N}\right)}$$

$\Rightarrow$ Hence, $\sum_{i=1}^{N} \left(\frac{Y_i - \bar{Y}}{\sigma}\right)^2 \sim \chi^2_{N-1}$

3. t-distribution / Student's t-distribution

If
(a) $X \sim N(0,1)$
(b) $Z \sim \chi^2_d$ → chi-squared w/ d degrees of freedom.
(c) X and Z are independent r.v.s.

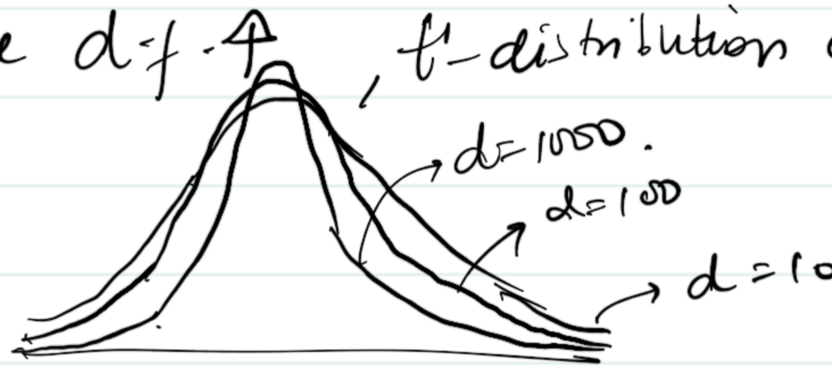
then

$$\to \quad t = \frac{X}{\sqrt{Z/d}} \sim t_d$$

$\boxed{\text{Shape of the t-distribution}}$

→ Typically, the t-distribution has fatter tails relative to a standard normal, but is a symmetric distribution.

→ As the d.f. ↑, t-distribution converges to normal dist.



Consider : $X \overset{iid}{\sim} N(\mu, \sigma^2) \Rightarrow \frac{X-\mu}{\sigma} \sim N(0,1)$

$\Rightarrow$ we can say : $Y = \sum_{i=1}^{N} \left(\frac{X_i - \bar{X}}{\sigma}\right)^2 \sim \chi^2_{N-1}$

Hence ;

$$\frac{\left(\frac{X-\mu}{\sigma}\right)}{\sqrt{\left[\sum_{i=1}^{N}\left(\frac{X_i-\bar{X}}{\sigma}\right)^2 / N-1\right]}} \sim t_{N-1}$$

OR.

$$\frac{\left(\frac{X-\mu}{\sigma}\right)}{\sqrt{Y/N-1}} \sim t_{N-1}$$

Mid-sem exam solution

Q1.

Model 1

$$\underset{n\times1}{\underline{y}} = \underset{n\times k}{x}\,\underset{k\times1}{\underline{\beta}} + \underset{n\times1}{\underline{u}}$$

Model 2.

$$\underset{\frac{n}{2}\times1}{\underline{\tilde{y}}} = \underset{\frac{n}{2}\times k}{\tilde{x}}\,\underset{k\times1}{\underline{\tilde{\beta}}} + \underset{\frac{n}{2}\times1}{\underline{\tilde{u}}}$$

$$\underset{n\times1}{\underline{y}} = \begin{bmatrix} \underline{\tilde{y}}_{\frac{n}{2}\times1} \\ \underline{\tilde{y}}_{n/2\times1} \end{bmatrix} ; \underset{n\times k}{\tilde{x}} = \begin{bmatrix} \tilde{x}_{n/2\times k} \\ \tilde{x}_{n/2\times k} \end{bmatrix} ; \underset{k\times n}{\tilde{x}'} = \begin{bmatrix} \underset{k\times\frac{n}{2}}{\tilde{x}'} & \underset{k\times\frac{n}{2}}{\tilde{x}'} \end{bmatrix}$$

(i) $$\underline{\beta}_{OLS} = (x'x)^{-1}x'\underline{y}$$

$$= \left(\begin{bmatrix} \tilde{x}' & \tilde{x}' \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{x} \end{bmatrix} \right)^{-1} \begin{bmatrix} \tilde{x}' & \tilde{x}' \end{bmatrix} \begin{bmatrix} \tilde{y} \\ \underline{\tilde{y}} \end{bmatrix}$$

$$= \left(\underline{2}\,\tilde{x}'\tilde{x}\right)^{-1}\left(\underline{2}\,\tilde{x}'\tilde{y}\right)$$

$$= (\tilde{x}'\tilde{x})^{-1}(\tilde{x}'\tilde{y}) = \tilde{\beta}_{OLS} \Rightarrow \boxed{\beta_{OLS} = \tilde{\beta}_{OLS}}$$

(iii) $k=2$

$$y_i = \beta_1 + \beta_2 x_{2i} + u_i \qquad x_{1i} = 1 \ \forall i$$

x_{2i} is measured in km. $\xrightarrow{\text{modified}}$ x^*_{2i} in meters

$$x^*_{2i} = 1000\, x_{2i}$$

New model: $y_i = \beta^*_1 + \beta^*_2 x^*_{2i} + u_i$

$$\beta^*_{2,OLS} = \frac{\sum_{i=1}^{N} (\overset{=1000x_{2i}}{x^*_{2i}} - \overset{=1000\bar{x}_2}{\bar{x}^*_2})(y_i - \bar{y})}{\sum_{i=1}^{N} (\underset{1000x_{2i}}{x^*_{2i}} - \underset{1000\bar{x}_2}{\bar{x}^*_2})^2} = \frac{\beta_{2,OLS}}{1000}.$$

Alternatively:

$$\longrightarrow \quad y_i = \beta_1 + \underbrace{\frac{\beta_2}{1000}}_{=\beta^*_2} \underbrace{(x_{2i} \times 1000)}_{=x^*_{2i}} + u_i$$

$$V(\beta^*_{2,OLS}) = V\left(\frac{\beta_{2,OLS}}{1000}\right) = \frac{1}{(1000)^2} V(\beta_{2,OLS})$$

Q2. $S_d = \eta\, I_d^{\alpha} E_d^{\beta} \exp(\varepsilon_d)$

(a) Required transformation :

$$\underbrace{\ln S_d}_{\tilde{S}_d} = \underbrace{\ln \eta}_{\tilde{\eta}} + \alpha \underbrace{\ln I_d}_{\tilde{I}_d} + \beta \underbrace{\ln E_d}_{\tilde{E}_d} + \varepsilon_d.$$

s.t. $\varepsilon_d \overset{iid}{\sim} N(0, \sigma^2)$

For $\tilde{S}_d$, $\tilde{I}_d$ and $\tilde{E}_d$ to be defined we must have

$$(S_d, I_d, E_d) > (0, 0, 0)$$

$$\underset{D\times 1}{\tilde{S}} = \underset{D\times 3}{\tilde{X}}\ \underset{3\times 1}{\underline{\gamma}} + \underset{D\times 1}{\underline{\varepsilon}}$$

where, $\tilde{X}_d = [1, \tilde{I}_d\ \tilde{E}_d]$ and $\underset{D\times 3}{\tilde{X}} = [\underset{D\times 1}{\mathbf{1}}\ \underset{D\times 1}{\tilde{I}_d}\ \underset{D\times 1}{\tilde{E}_d}]$

and $\underline{\gamma} = \begin{bmatrix} \tilde{\eta} \\ \alpha \\ \beta \end{bmatrix}$, $\tilde{\eta} = \ln \eta$

$$\Rightarrow \underline{\gamma}_{OLS} = \begin{bmatrix} \tilde{\eta}_{OLS} \\ \alpha_{OLS} \\ \beta_{OLS} \end{bmatrix} = (\tilde{X}'\tilde{X})^{-1} \tilde{X}'\tilde{S}.$$

(c) $$\hat{\sigma}^2 = \frac{\sum_{i=1}^{D} \hat{\varepsilon}_d^2}{N-K}$$

$$= \frac{\sum_{i=1}^{100} \left(\tilde{S}_d - \hat{\tilde{\eta}}_{ols} - \hat{\alpha}_{ols} \tilde{I}_d - \hat{\beta}_{ols} \tilde{E}_d \right)^2}{97}$$