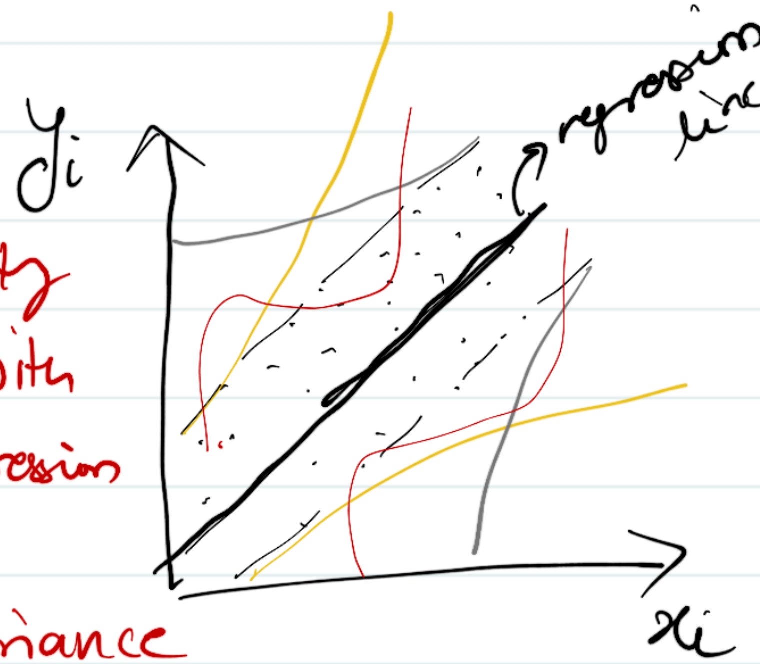
lecture 22

16.04.2021

Econometrics - 1

Heteroskedasticity



- Impact of heteroskedasticity
  - → does not interfere with the unbiasedness of regression coefficient estimates
  - → does impact the variance estimate of regression coefficients, and hence statistical inference.

- OLS fails (no long BLUE)

GLS is apt

$\boxed{\text{Generalized least squares}}$ ✱

$$y_i = \beta_0 + \beta_1 x_i + u_i$$

$$V(u_i | x_i) = \underbrace{\sigma_i^2}_{\text{Heteroskedastic structure.}} \qquad i = 1, 2, \ldots, N$$

Variance-Covariance matrix of regression errors

$$\sigma_i^2 = \begin{bmatrix} \sigma_1^2 & & & 0 \\ & \sigma_2^2 & & \\ & & \ddots & \\ 0 & & & \sigma_N^2 \end{bmatrix}$$

$$= \sigma^2 \begin{bmatrix} \sigma_1^2/\sigma^2 & & & 0 \\ & \sigma_2^2/\sigma^2 & & \\ & & \ddots & \\ 0 & & & \sigma_N^2/\sigma^2 \end{bmatrix}$$

$$\underline{\sigma^2 \Omega_{N \times N}}$$

$$\underset{N\times 1}{\underline{y}} = \underset{N\times K}{X}\,\underset{K\times 1}{\underline{\beta}} + \underset{N\times 1}{\underline{u}}$$

$$V(\underset{N\times 1}{\underline{u}}\,|\,X) = \underset{\substack{\downarrow\\ \text{scalar}\\ (1\times 1)}}{\sigma^2}\,\underset{N\times N}{\underline{\Omega}}$$

Then,

$$\underset{K\times 1}{\underline{\hat{\beta}_{GLS}}} = \underbrace{(\underset{K\times N}{X'}\,\underset{N\times N}{\underline{\Omega}^{-1}}\,\underset{N\times K}{X})^{-1}}_{K\times K}\,\underbrace{\underset{K\times N}{X'}\,\underset{N\times N}{\underline{\Omega}^{-1}}\,\underset{N\times 1}{\underline{y}}}_{K\times 1}$$

$$V(\underset{K\times 1}{\underline{\hat{\beta}_{GLS}}}) = \sigma^2\underbrace{(\underset{K\times N}{X'}\,\underset{N\times N}{\underline{\Omega}^{-1}}\,\underset{N\times K}{X})^{-1}}_{K\times K} < \sigma^2(X'X)^{-1} = V(\hat{\beta}_{OLS})$$

Contrast GLS estimators w/ OLS counterparts

$$\hat{\beta}_{OLS} = (X'X)^{-1}X'\underline{y} \qquad V(\hat{\beta}_{OLS}) = \sigma^2(X'X)^{-1}$$

$$\boxed{E(\hat{\underline{\beta}}_{OLS}) = E(\hat{\underline{\beta}}_{GLS}) = \underline{\beta}}$$

Q. why / how does the GLS work?

* Suppose $\exists$ a $N \times N$ matrix $\underset{N\times N}{T}$ s.t.

$$T \underline{\Omega} T' = \underline{I}_N$$

* Now, for our model $\underline{y} = x \underline{\beta} + \underline{u}$

we transform each component in the following manner:-

Define

$$\underset{N\times 1}{\underline{y}^{NEW}} = \underset{N\times N}{T} \underset{N\times 1}{\underline{y}}$$

$$\underset{N\times K}{X^{NEW}} = \underset{N\times N}{T} \underset{N\times K}{X}$$

$$\underset{N\times 1}{\underline{u}^{NEW}} = \underset{N\times N}{T} \underset{N\times 1}{\underline{u}}$$

* Consider estimating the transformed model

$$\underline{y}^{NEW} = X^{NEW} \underline{\beta} + \underline{u}^{NEW}$$

$$\underline{y}^{NEW} = X^{NEW}\underline{\beta} + \underline{u}^{NEW}$$

$$T\underline{y} = TX\underline{\beta} + T\underline{u}$$ original model premultiplied by $T_{N\times N}$

$$V(T\underline{u}) = E\left[(T\underline{u})(T\underline{u})'\right]$$

$$= E\left[T\underline{u}\,\underline{u}'T'\right]$$

$$= T\,\underbrace{E[\underline{u}\,\underline{u}']}_{=\sigma^2\Omega}\cdot T'$$

$$= T\sigma^2\Omega T'$$

$$= \sigma^2 T\Omega T'$$

$$\boxed{V(T\underline{u}) = \sigma^2 I_N}$$

So, the transformed model is HOMOSKEDASTIC.

$\Rightarrow$ We can efficiently estimate the transformed model using OLS.

$$T\underline{y} = TX\underline{\beta} + T\underline{u}$$

$$V(T\underline{u}) = \sigma^2 I_n$$

Hence The OLS estimators of

$$\underline{y}^{NEW} = X^{NEW}\underline{\beta} + \underline{u}^{NEW}$$

are efficient.

$$\underline{\beta}^{transformed}_{OLS} = \left(X^{NEW'}X^{NEW}\right)^{-1} X^{NEW'}\underline{y}^{NEW}$$

$$= \left[(TX)'(TX)\right]^{-1}(TX)'T\underline{y}$$

$$= [X'T'TX]^{-1}X'T'T\underline{y}$$

$$= (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}\underline{y} \equiv \hat{\beta}_{GLS}.$$

$$V\left(\hat{\beta}^{\text{transformed}}_{OLS}\right) = \sigma^2 \left(X^{NEW'} X^{NEW}\right)^{-1}$$

$$= \sigma^2 \left((TX)' \, TX\right)^{-1}$$

$$= \sigma^2 \left[X' \Omega^{-1} X\right]^{-1}$$

$$\equiv V\left(\hat{\beta}_{GLS}\right)$$

$$\hat{\beta}_{GLS} = \left(X' \Omega^{-1} X\right)^{-1} X' \Omega^{-1} \underline{y}$$

unknown!

$$V\left(\hat{\beta}_{GLS}\right) = \sigma^2 \left(X' \Omega^{-1} X\right)^{-1}$$

$V(u_i | x_i) = \sigma^2 \Omega \rightarrow$ True Ω is not known

Hence, we don't know T! Then how do we estimate GLS?

Feasible Generalised Least Squares

FGLS

Step 1: estimate $\hat{\Omega}$ from the data $\{y_i, \vec{x}_i\}_{i=1}^{N}$

step 2: get $\hat{T}$ s.t. $\hat{\Omega}^{-1} = \hat{T}'\hat{T}$

step 3: given $\hat{T}$, define

$$\underline{y}^{m} = \hat{T}\,\underline{y}$$

$$X^{m} = \hat{T}X$$

Step 4: Run OLS with y^{m} as dependent variable and X^{m} as explanatory variables.

$$\hat{\beta}_{FGLS} = \left(X^{m\prime} X^{m}\right)^{-1} X^{m\prime} \underline{y}^{m}$$

$$= \left[(\hat{T}X)' \hat{T} X\right]^{-1} (\hat{T}X)' \hat{T} \underline{y}$$

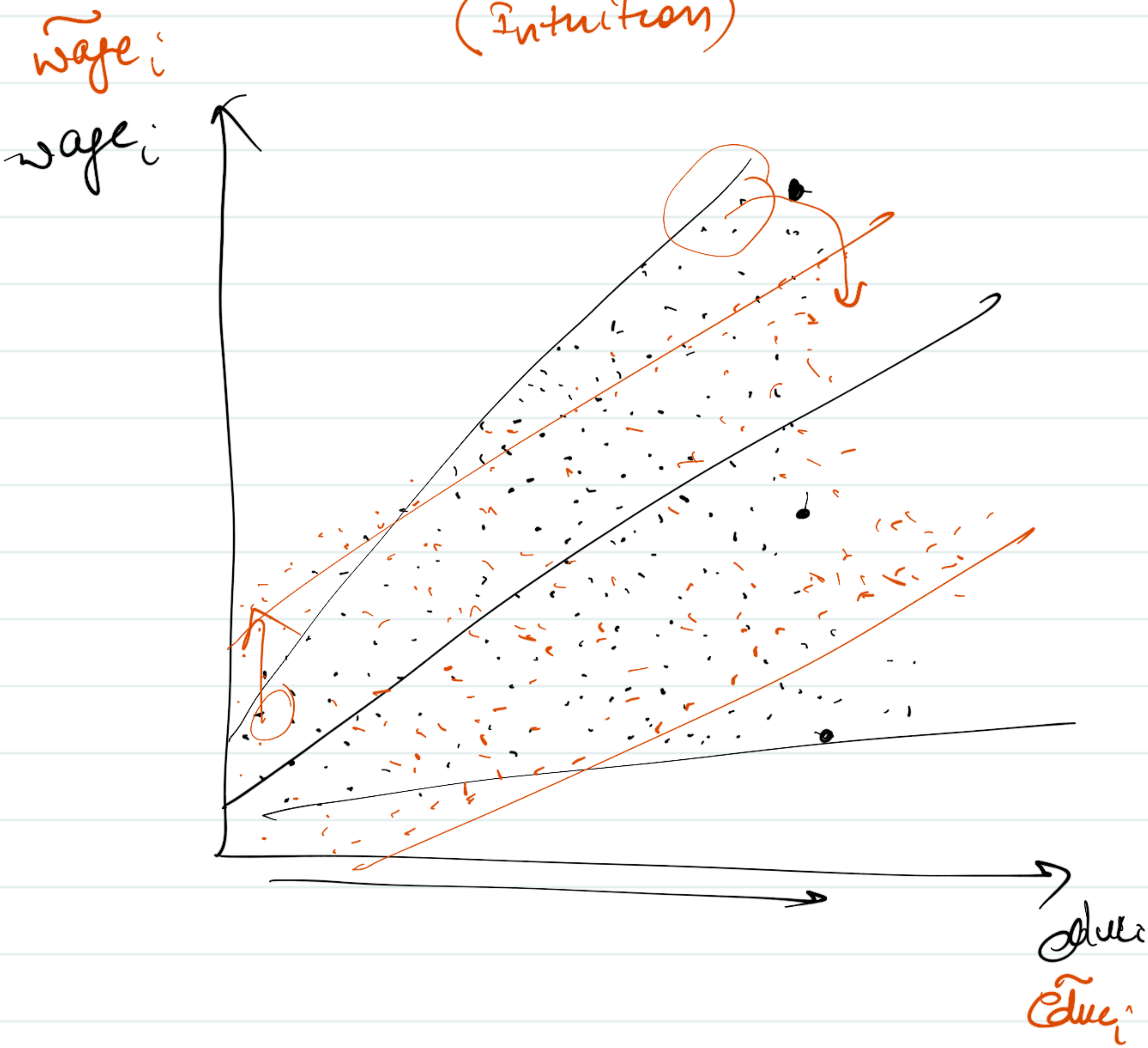
$$\boxed{\hat{\beta}_{FGLS} = \Big(X' \underbrace{\hat{T}'\hat{T}}_{\hat{\Omega}^{-1}} X\Big)^{-1} X' \underbrace{\hat{T}'\hat{T}}_{\hat{\Omega}^{-1}} \underline{y}}$$

$$\hat{\beta}_{GLS} = \Big(X' \underbrace{\Omega^{-1}} X\Big)^{-1} X' \underbrace{\Omega^{-1}} \underline{y}$$

Replace with $\hat{\Omega}^{-1}$

What does the model transformation achieve?

(Intuition)



Example 1:

GROUPWISE HETEROSKEDASTICITY.

$$y = \beta_1 + \beta_2 x_{2i} + \beta_3 x_{3i} + u_i$$

$$i = 1, 2, \dots, N$$

The data can be divided in 3 strata:-

$$i = \underbrace{1, 2, \dots, N_1}_{\text{group } 1},\ \underbrace{N_1+1, N_1+2, \dots, N_1+N_2}_{g_2},\ \underbrace{N_1+N_2+1, \dots, N}_{g_3}$$

$$V(\underset{\sim}{u}) = \begin{array}{c} \begin{bmatrix} \sigma_1^2 & & & & & \\ & \ddots & & & & \\ & & \sigma_1^2 & & & \\ & & & \sigma_2^2 & & \\ & & & & \ddots & \\ & & & & & \sigma_2^2 \\ & & & & & & \sigma_3^2 \\ & & & & & & & \ddots \\ & & & & & & & & \sigma_3^2 \end{bmatrix} \end{array} \begin{array}{l} \left.\right\} N_1 \text{ rows} \\ \left.\right\} N_2 \text{ rows} \\ \left.\right\} N - N_1 - N_2 = N_3 \text{ rows.} \end{array}$$

$H_0: \sigma_1^2 = \sigma_2^2 = \sigma_3^2$; i.e., No heteroskedasticity

we have

Ω

but we need $\hat{\Omega}$

in the next lecture.
(Apply FGLS).