Lecture 18

# Econometrics - I

02.04.2021

JOINT

Hypothesis testing w/ > 1 linear restriction

[for a set of linear restrictions]

Testing a set of linear restrictions jointly

$$y_i = \beta_1 + \beta_2 x_{2i} + \beta_3 x_{3i} + \cdots + \beta_K x_{Ki} + u_i$$

$i = 1, 2, \ldots, N.$

Test

$$\left.\begin{aligned} H_0: \quad & r_{11}\beta_1 + r_{12}\beta_2 + \cdots + r_{1k}\beta_K = r_1 \\ \text{and} \quad & r_{21}\beta_1 + r_{22}\beta_2 + \cdots + r_{2k}\beta_K = r_2 \\ & \vdots \\ \text{and} \quad & r_{Q1}\beta_1 + r_{Q2}\beta_2 + \cdots + r_{Qk}\beta_K = r_Q \end{aligned}\right\}$$

Q : # of non-redundant linear restriction.

$$\underset{Q\times K}{R} \cdot \underset{K\times 1}{\beta} = \underset{Q\times 1}{r}$$

Assume: $Q \le K$

R is full rank.

$$\underset{Q\times K}{R} = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1K} \\ r_{21} & r_{22} & \cdots & r_{2K} \\ \vdots & & & \\ r_{Q1} & r_{Q2} & \cdots & r_{QK} \end{bmatrix} ; \quad \underline{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_K \end{bmatrix}$$

$$\underline{r} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_Q \end{bmatrix}$$

Simplest example:

$H_0: \beta_1 = 0$ and $\beta_2 = 0$ and $\beta_3 = 0$ and ... and $\beta_K = 0$

$$\underset{Q\times K}{R_{example}} = \underset{K\times K}{\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}}$$

$Q = K$

Now, lets work out how to test >1 linear restriction:

$$H_0: \underset{Q\times K}{R}\ \underset{K\times 1}{\underline{\beta}} = \underset{Q\times 1}{\underline{r}} \qquad Q \le K.$$

$$\underline{\hat{\beta}}_{OLS} \sim \underline{N}\Big(\underset{E(\hat{\beta}_{OLS})}{\overset{\shortparallel}{\underline{\beta}}},\ \underset{V(\hat{\beta}_{OLS})}{\overset{\shortparallel}{\sigma^2(X'X)^{-1}}}\Big)$$

$$R\underline{\hat{\beta}}_{OLS} \sim \underline{N}\left(R\underline{\beta},\ R[\sigma^2(X'X)^{-1}]R'\right)$$

$$R\underline{\hat{\beta}}_{OLS} - \underline{r} \sim \underline{N}\left(R\underline{\beta} - r,\ \sigma^2 R(X'X)^{-1}R'\right)$$

Under $H_0: R\underline{\beta} = \underline{r}$ or $R\underline{\beta} - \underline{r} = \underset{Q\times 1}{\underline{0}}$

$$\underbrace{R\hat{\beta}_{OLS} - \underline{r}}\ \sim \underline{N}\Big(\underset{Q\times 1}{\underline{0}},\ \underbrace{\sigma^2 R(X'X)^{-1}R'}_{Q\times Q}\Big).$$

Under $H_0$, we have

$$\underbrace{R\hat{\beta}_{ols} - \underline{r}}_{Q\times 1} \sim \mathbb{N}\Big(\underset{Q\times 1}{\underline{0}}, \underbrace{\sigma^2 R(X'X)^{-1}R'}_{Q\times Q}\Big)$$

Aside 1: if $\underset{m\times 1}{\underline{w}} \sim \mathbb{N}(0, \Sigma_{m\times m})$

then $\underset{1\times 1}{c} = \underbrace{\underset{1\times m}{w'}\,\underset{m\times m}{\Sigma^{-1}}\,\underset{m\times 1}{w}}_{\text{matrix form of quadratic sum of standard normal r.v.s}} \sim \chi^2_m$

Aside 2: if $c_1 \sim \chi^2_{m_1}$, $c_2 \sim \chi^2_{m_2}$ ; $c_1, c_2$ are independent

then $\dfrac{c_1/m_1}{c_2/m_2} \sim F_{m_1, m_2}$

Aside 3: $\dfrac{\hat{u}'\hat{u}}{\sigma^2} \sim \chi^2_{N-K}$ } prev. lecture had scalar form: $\sum_{i=1}^{N} \frac{\hat{u}_i^2}{\sigma^2} \sim \chi^2_{N-K}$

residual vector for MLRM.

Aside 1 $\Rightarrow$

$$\left(R\underline{\hat{\beta}}_{ols} - \underline{r}\right)'\left[\sigma^2 R\,(X'X)^{-1}R'\right]^{-1}\left(R\underline{\hat{\beta}}_{ols} - \underline{r}\right) \sim \chi^2_Q$$

Aside 3 $\Rightarrow$ $\quad \underline{\hat{u}}'(\sigma^2)^{-1}\,\underline{\hat{u}} \sim \chi^2_{N-K}.$

Overall,

$$\frac{\left[\left(R\underline{\hat{\beta}}_{ols} - \underline{r}\right)'\left(\cancel{\sigma^2} R(X'X)^{-1}R\right)^{-1}\left(R\underline{\hat{\beta}}_{ols} - \underline{r}\right)\right]/Q}{\underbrace{\left[\underline{\hat{u}}'\left(\cancel{\sigma^2}\right)^{-1}\underline{\hat{u}}\right]/N-K}} \sim F_{Q,N-K}$$

$$\hat{\sigma}^2 = \frac{\underline{\hat{u}}'\,\underline{\hat{u}}}{N-K} = \hat{\sigma}^2$$

So,

$$\frac{\left(R\underline{\hat{\beta}}_{ols} - \underline{r}\right)'\left[\hat{\sigma}^2 R\,(X'X)^{-1}R'\right]^{-1}\left(R\underline{\hat{\beta}}_{ols} - \underline{r}\right)}{Q} \sim F_{Q,N-K}.$$

→ Conducting hypothesis tests by estimating a restricted regression model.

Consider a full model:

(1) full model $$y = \beta_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + u$$

Test: $H_0 : \beta_2 + 2\beta_4 = 4$ estimate: $\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3, \hat{\beta}_4$

Recover a restricted model by substituting the restriction in the original model (1)

(2) Restricted model $$y = \beta_1 + (4 - 2\beta_4) x_2 + \beta_3 x_3 + \beta_4 x_4 + u$$

$$\underbrace{y - 4x_2}_{=\tilde{y}} = \beta_1 + \beta_3 x_3 + \beta_4 \underbrace{(x_4 - 2x_2)}_{=z} + u$$

$$\tilde{y} = \beta_1 + \beta_3 x_3 + \beta_4 z + u$$

estimate: $\tilde{\beta}_1, \tilde{\beta}_3, \tilde{\beta}_4$

Test statistic: CHOW TEST

(A) $$\frac{\left(SSE_{Restricted} - SSE_{Full}\right)/Q}{\left(SSE_{Full}\right)/N-K} \sim F_{Q,\,N-K}$$

(B) $$\frac{\left(R^2_{Full} - R^2_{Restricted}\right)/Q}{\left(1-R^2_{Full}\right)/N-K} \sim F_{Q,\,N-K}$$

For our example: $Q = 1$

final step

Inferece :

If F-stat $> F^*_{Q, N-K, \alpha}$ then reject $H_0$.

If F-stat $\leq F^*_{Q, N-K, \alpha}$ then fail to reject $H_0$.

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Next lecture: Maximum Likelihood Estimation