Econometrics-1

16.02.2021

Lecture 10

Multiple Linear Regression model

$$\underset{N\times 1}{\underline{y}} = \underbrace{\underset{N\times K}{X}\ \underset{K\times 1}{\underline{\beta}}}_{N\times 1} + \underset{N\times 1}{\underline{u}}$$

K: # of regressors

if K=2 then we have SLRM

A1: $E(\underline{u}|X)=0$

$$\Rightarrow E(\underbrace{\underset{K\times N}{X'}\ \underset{N\times 1}{\underline{u}}}_{K\times 1}) = \underset{K\times 1}{\underline{0}}$$

$$\left\{ \underset{K\times 1}{\underline{0}} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{K\times 1} \right\}$$

we will now expand $E(X'\underline{u})$ to see what it means in the scalar notational context?

$$E[X'\underline{u}] = E\left\{ \begin{bmatrix} x_{11} & x_{21} & \cdots & x_{N1} \\ x_{12} & x_{22} & \cdots & x_{N2} \\ \vdots & & & \\ x_{1K} & x_{2K} & \cdots & x_{NK} \end{bmatrix}_{K\times N} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}_{N\times 1} \right\}$$

$$X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1K} \\ x_{21} & x_{22} & \cdots & x_{2K} \\ \vdots & & & \\ x_{N1} & x_{N2} & & x_{NK} \end{bmatrix}_{N\times K} \qquad X' = ?$$

$$= E\begin{bmatrix} x_{11}u_1 + x_{21}u_2 + \cdots + x_{N1}u_N \\ x_{12}u_1 + x_{22}u_2 + \cdots + x_{N2}u_N \\ \vdots \\ x_{1K}u_1 + x_{2K}u_2 + \cdots + x_{NK}u_N \end{bmatrix}_{K\times 1} = E\begin{bmatrix} \sum_{i=1}^{N} x_{i1}\cdot u_i \\ \sum_{i=1}^{N} x_{i2}\cdot u_i \\ \vdots \\ \sum_{i=1}^{N} x_{iK}\cdot u_i \end{bmatrix}$$

focus on: $\sum_{i=1}^{N} x_{i1}\cdot u_i = \underset{1\times N}{\underline{x}'_{i1}} \; \underset{N\times 1}{\underline{u}}$ $\left[\because \underline{x}'_{i1} = [x_{11}\; x_{21} \cdots x_{N1}] \right]$

$$E[X'\underline{u}] = E\begin{bmatrix} \underline{x}'_{i1}\underline{u} \\ \underline{x}'_{i2}\underline{u} \\ \vdots \\ x'_{iK}\underline{u} \end{bmatrix} \overset{\because \text{ expectation is a linear operator}}{=\!=} \begin{bmatrix} E(x'_{i1}\underline{u}) \\ E(x'_{i2}\underline{u}) \\ \vdots \\ E(x'_{iK}\underline{u}) \end{bmatrix} \overset{A1}{=} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\Rightarrow \boxed{E(x'_{ik}\underline{u}) = 0 \quad \forall k = \{1, 2, \ldots, K\}}$$

Example

(SLRM) $wage_i = \beta_0 + \beta_1 educ_i + u_i$ $\rightarrow$ SLRM.

✓ $E(u_i | educ_i) = 0$

(MLRM) $wage_i = \gamma_0 + \gamma_1 educ_i + \gamma_2 exper_i + \gamma_3 age_i + u_i$

$\hookrightarrow$ MLRM.

$E(u_i | educ_i) = 0$

and $E(u_i | exper_i) = 0$

and $E(u_i | age_i) = 0$

$\boxed{A2}$ * $E(\underline{u}\,\underline{u}' | X) = \sigma^2 I_N$ where $I_N = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}_{N\times N}$

If X is fixed then $\underbrace{E(\underbrace{\underset{N\times 1}{\underline{u}}\,\underset{1\times N}{\underline{u}'}}_{N\times N})}_{} = \underbrace{\underset{\text{scalar}}{\sigma^2}\,\underbrace{I_N}_{N\times N}}_{N\times N}$

$$E(\underline{u}\,\underline{u}') = E\left\{ \underset{N\times 1}{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}} \underset{1\times N}{\begin{bmatrix} u_1 & u_2 & \cdots & u_N \end{bmatrix}} \right\} = E \underset{N\times N}{\begin{bmatrix} u_1^2 & u_1u_2 & \cdots & u_1u_N \\ u_2u_1 & u_2^2 & \cdots & u_2u_N \\ \vdots & & \ddots & \vdots \\ u_Nu_1 & u_Nu_2 & \cdots & u_N^2 \end{bmatrix}}$$

$$= \begin{bmatrix} Eu_1^2 & Eu_1u_2 & \cdots & Eu_1u_N \\ Eu_2u_1 & Eu_2^2 & \cdots & Eu_2u_N \\ \vdots & \vdots & \ddots & \vdots \\ Eu_Nu_1 & Eu_Nu_2 & \cdots & Eu_N^2 \end{bmatrix} \overset{A2}{=} \begin{bmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma^2 \end{bmatrix}$$

$\Rightarrow E u_1^2 = E u_2^2 = E u_3^2 = \cdots = E u_N^2 = \sigma^2$

and $E\, u_i u_j = 0 \quad \forall\, i \neq j,\ i = \{1, 2, \ldots, N\},\ j = \{1, 2, \ldots, N\}$

A2 : $E(\underline{u}\,\underline{u}') = \sigma^2 I_N$

$$\Rightarrow \underbrace{E(u_i^2) = \sigma^2}_{} \quad \forall\, i \in \{1,2,\dots,N\}$$

→ $\boxed{\text{HOMOSKEDASTICITY}}$

$$\underbrace{E(u_i u_j) = 0}_{} \quad \forall\, i,j \in \{1,2,\dots,N\} \text{ and } i \neq j$$

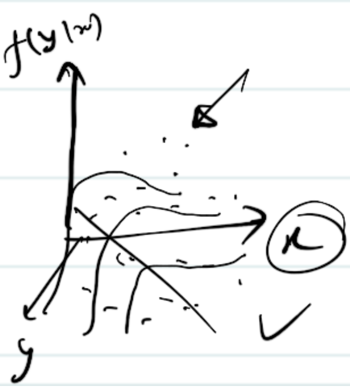
→ a crucial assumption of no correlation in errors across i's.

$$V(u_i) = E\big[u_i - \underbrace{E(u_i)}_{=0}\big]^2 = E(u_i^2) = \sigma^2 \ \forall i$$

$$Cov(u_i u_j) = E\Big[\big[u_i - \underbrace{Eu_i}_{=0}\big]\big[u_j - \underbrace{Eu_j}_{=0}\big]\Big] = E(u_i u_j) = 0 \ \forall\, i \neq j$$

A2 ⇒ (i) Homoskedasticity
(ii) No covariance/correlation across error terms of diff. i's.

ERRORS ARE SPHERICAL



ERRORS ARE SPHERICAL.

$\underline{A1}$: $E(\underline{U}|X) = 0$

$\underline{A2}$: $E(\underline{U}\,\underline{U}'|X) = \sigma^2 I_N$

$\underline{A3}$: $\text{Rank}\underset{N \times K}{(X)} = K \iff X$ has full rank.

Rank of a matrix : # of linearly independent column vectors in a matrix,

$$\text{Rank}(X) \leq \min(N, K) = K$$

Why A3 important?

(MLRM) $wage_i = \beta_0 + \beta_1 educ_i + \beta_2 exper_i + \beta_3 age_i + u_i$

Suppose exper and age are linearly relation.

$$\boxed{exper_i = age_i - 20}$$

then MLRM becomes

$$wage_i = \beta_0 + \beta_1 educ_i + \beta_2 (age_i - 20) + \beta_3 age_i + u_i$$

$$wage_i = (\beta_0 - 20\beta_2) + \beta_1 educ_i + (\beta_2 + \beta_3) \underline{age_i} + u_i$$

Implications of A3 {Rank(X) = K)

(i) $N \geq K$ (we have at least as many observations as we have variables)

(ii) All columns of X are linearly independent.

★ (iii) $\underbrace{X'_{K\times N} X_{N\times K}}_{K\times K}$ is non-singular and positive-definite matrix.

★ $\Rightarrow \det(X'X) \neq 0$ $\det(X'X) > 0$

↓

crucial

we will come back to this.

Dummy variable trap

$$wage_i = \beta_0 + \beta_1 educ_i + \beta_2 exper_i + \beta_3 age_i + \beta_4 \tilde{D}_i + u_i$$

dummy variable $\tilde{D}_i = \begin{cases} 1 & \text{if } i \text{ is a female employee} \\ 0 & \text{otherwise.} \end{cases}$

$\tilde{D}_i = \begin{cases} 1 & \text{if } i \text{ is female} \\ 0 & \text{otherwise} \end{cases}$ | $\hat{D}_i = \begin{cases} 1 & \text{if } i \text{ is male} \\ 0 & \text{otherwise} \end{cases}$

$$wage_i = \underline{\beta_0} + \beta_1 educ_i + \beta_2 exper_i + \beta_3 age_i + \underline{\beta_4} \tilde{D}_i + \beta_5 \hat{D}_i + u_i$$

Sp. 1

vs.

$$wage_i = \underline{\beta_0} + \beta_1 educ_i + \beta_2 exper_i + \beta_3 age_i + \underline{\beta_4} \tilde{D}_i + u_i$$

Sp. 2

Which among the above specification is more appropriate given that the sample comprise of male and female employees?

Sp. 1 :

$$X = \begin{array}{cccccc} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ \end{array}$$
$$X = \begin{bmatrix} 1 & educ_1 & exper_1 & age_1 & 1 & 0 \\ 1 & educ_2 & exper_2 & age_2 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & educ_N & exper_N & age_N & 1 & 0 \end{bmatrix}$$

c_6: excluded

$c_5 = c_1 - c_6$

$c_5 + c_6 = c_1$

DVT is when we include both $\tilde{D}_i$ and $\hat{D}_i$ in the same regression model