Econometrics-1

12.02.2021

Lecture 9

from prev. lecture

$$V(ax + by) = a^2 V(x) + b^2 V(y) + 2ab\,Cov(x,y)$$

$$V(\hat{\beta}_{0,OLS} | X) = V_x\left(\bar{y} - \hat{\beta}_{1,OLS}\,\bar{x}\right)$$

$$= \underbrace{V_x(\bar{y})}_{= \frac{\sigma^2}{N}} + \underbrace{\bar{x}^2\, V_x(\hat{\beta}_{1,OLS})}_{= \bar{x}^2 \frac{\sigma^2}{\sum_{i=1}^{N}(x_i - \bar{x})^2}} - 2\bar{x}\,\underbrace{Cov(\hat{\beta}_1, \bar{y})}_{= ?\,(0)}$$

$$V\left(\sum_{i=1}^{N} \frac{y_i}{N}\right) = \sum_{i=1}^{N} \frac{1}{N^2} V(y_i) = \frac{1}{N^2} \sum_{i=1}^{N} \sigma^2$$

$$= \frac{\sigma^2 \cdot N}{N^2} = \frac{\sigma^2}{N}$$

$$Cov(\hat{\beta}_1, \bar{y}) = E\left\{(\hat{\beta}_{1,OLS} - \beta)(\bar{y} - \bar{y})\right\} = 0$$

$$Cov(X,Y) = E\{[X - E(X)][Y - E(Y)]\}$$

$$\boxed{V(\hat{\beta}_{0,OLS}) = \frac{\sigma^2}{N} + \frac{\bar{x}^2 \sigma^2}{\sum_{i=1}^{N}(x_i-\bar{x})^2}}$$

$\Rightarrow$ MULTIPLE LINEAR REGRESSION MODEL

Scalar notation

$$y_i = \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_k x_{ik} + u_i$$

prob. of getting caught. eg. legal actions eg. ...

crime$_i$ — wages in criminal activities — wages in alternative jobs avenues

$i = 1, 2, \ldots, N$

$$wage_i = \beta_0 + \beta_1 educ_i + \beta_2 educ^2 + u_i \Rightarrow \frac{\partial wage}{\partial edu} = \beta_1 + 2\beta_2 educ_i$$

$$yield_i = \gamma_0 + \gamma_1 fert_i + \gamma_2 (fert_i)^{1/2} + \varepsilon_i$$

$$\Rightarrow wage_i = \beta_0 + \beta_1 educ_i + \beta_2 exper_i + \beta_3 age_i + u_i$$

Multiple Linear Regression Model

SCALAR FORM $\Bigg\{ y_i = \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \cdots + \beta_k x_{ik} + u_i$

$i = 1, 2, \ldots, N.$

Matrix notation:

$$\underset{N\times 1}{\underline{Y}} = \underbrace{\underset{N\times K}{X}\ \underset{K\times 1}{\underline{\beta}}}_{N\times 1} + \underset{N\times 1}{\underline{u}}$$

$$\underset{N\times 1}{\underline{Y}} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} ; \quad \underset{N\times K}{X} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1k} \\ x_{21} & x_{22} & \cdots & x_{2k} \\ x_{31} & x_{32} & \cdots & x_{3k} \\ \vdots & \vdots & & \vdots \\ x_{N1} & x_{N2} & \cdots & x_{Nk} \end{bmatrix}$$

$$\underset{K\times 1}{\underline{\beta}} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_K \end{bmatrix} \qquad \underset{N\times 1}{\underline{u}} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}$$

Alternative matrix notation

$$\underset{N\times 1}{\underline{y}} = \underbrace{\left[\underbrace{\underset{1\times K}{\underline{\beta}'} \underset{K\times N}{X'}}_{1\times N}\right]'}_{N\times 1} + \underset{N\times 1}{\underline{u}} \quad \text{(ok?)}$$

$$(\beta' x')' = X\beta \quad ✓$$

$$X\beta = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1K} \\ x_{21} & x_{22} & \cdots & x_{2K} \\ \vdots & & & \\ x_{N1} & x_{N2} & \cdots & x_{NK} \end{bmatrix}_{N\times K} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_K \end{bmatrix}_{K\times 1} = \begin{bmatrix} x_{11}\beta_1 + x_{12}\beta_2 + \cdots + x_{1K}\beta_K \\ x_{21}\beta_1 + x_{22}\beta_2 + \cdots + x_{2K}\beta_K \\ \vdots \\ x_{N1}\beta_1 + x_{N2}\beta_2 + \cdots + x_{NK}\beta_K \end{bmatrix}_{N\times 1}$$

$$\underline{y} = X\underline{\beta} + \underline{u}$$

MATRIX NOTATION

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}_{N\times 1} = \begin{bmatrix} x_{11}\beta_1 + x_{12}\beta_2 + \cdots + x_{1K}\beta_K \\ x_{21}\beta_1 + x_{22}\beta_2 + \cdots + x_{2K}\beta_K \\ \vdots \\ x_{N1}\beta_1 + x_{N2}\beta_2 + \cdots + x_{NK}\beta_K \end{bmatrix}_{N\times 1} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}_{N\times 1}$$

SCALAR NOTATION $\rightarrow$ $y_i = x_{i1}\beta_1 + x_{i2}\beta_2 + \cdots + x_{iK}\beta_K + u_i \;, i = 1, 2, \ldots, N$

SLRM:

$$y_i = \beta_1 + \beta_2 x_i + u_i$$ (as special case of MLRM)

MLRM:

$$\underline{y} = X\underline{\beta} + \underline{u}$$

$K = 2$

$x_{i1} = 1 \quad \forall i = \{1, 2, \ldots, N\}$

$x_{i2} = x_i \quad \forall i = \{1, 2, \ldots, N\}$

★ { Assumptions on the multiple linear regression model.

A1: $E(\underline{u} \mid X) = 0$

- more general, it applies also when X is a r.v.
- if X_i is fixed, then $E(\underline{u}) = 0$
- Implication of A1: $E(\underbrace{\underset{K\times N}{X'}\ \underset{N\times 1}{\underline{u}}}_{K\times 1}) = 0$ [CETERIS PARIBUS]

PROOF OF $E(X'\underline{u}) = 0$ given $E(u|X)$.

Uses: LAW OF ITERATED Expectations [LIE]

Consider z and y are random variables

then,

$$\text{LIE} \Rightarrow E(Z) = E_y\left\{E_z\left\{(z|y)\right\}\right\}$$

Use LIE for our proof $\Rightarrow$

$$\underbrace{E_x\left[E_{\underline{u}}\left(X'\underline{u}|X\right)\right]}_{\|\text{LIE}} = E_x\left[X' \underbrace{E_u(\underline{u}|x)}_{\|\text{A1} \atop 0}\right]$$

$$E(X'\underline{u}) = 0$$

Hence Proved