Econometrics-1

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Lecture 11

Multiple Linear Regression Model (MLRM)

$$y_i = \beta_1 x_{i1} + \beta_2 x_{i2} + \dots + \beta_K x_{iK} + u_i$$

$$i = 1, 2, \dots, N$$

$$\underset{N\times 1}{\underline{y}} = \underbrace{\underset{N\times K}{X}\ \underset{K\times 1}{\underline{\beta}}}_{N\times 1} + \underset{N\times 1}{\underline{u}}$$

Assumption to MLRM

A1 : $E(\underline{u} | X) = 0 \Rightarrow E(X'\underline{u}) = 0$ (critical for causality)

A2 : $E(\underline{u}\,\underline{u}' | X) = \sigma^2 I_N$

$\Rightarrow$ Homoskedasticity

$\Rightarrow Cov(u_i, u_j) = 0 \ \forall i \neq j$

$\Rightarrow$ ERRORS ARE SPHERICAL

A3 : X has full rank $\Rightarrow Rank(X) = K$.

A3: $\underset{N\times K}{X}$ has full rank $\Leftrightarrow$ Rank$(X) = K$

- $N \geq K$
- All columns of X are linearly independent
- $X'X$ is non-singular ($\det(X'X) \neq 0$) and $X'X$ is positive definite ($\det(X'X) > 0$)

$\Rightarrow$ Dummy variable trap.

A4: $\underset{N\times K}{X}$ is non-stochastic, i.e., X is fixed.

A5: $\underline{U}$ is normally-distributed.

$\underline{U}$ is a N-dimensional or N-variate random vector that is normally distributed.

$$\underset{N\times 1}{\underline{U}} \sim \mathbb{N}\left(\underset{N\times 1}{\underline{0}}, \underset{N\times N}{\sigma^2 I_N}\right) \rightarrow \text{Matrix notation}$$

independently ($\text{cov}(U_i, U_j) = 0$)

identically (all normally distd.)

$$U_i \overset{iid}{\sim} N(0, \sigma^2) \quad \forall i = 1, 2, \ldots, N \rightarrow \text{Scalar notation}$$

Aside

Q. why do we write $\underbrace{\sigma^2 I_N}_{(N\times N)}$ as the variance of $\underline{u}$ and not just $\underset{N\times 1}{\underline{\sigma^2}}$?

Think about two r.v.s

1st r.v. Z_1, Normally dist^d., $E(Z_1) = \mu_1 = 0$; $V(Z_1) = \sigma_1^2 = \sigma^2$

2nd r.v. Z_2, Normally dist^d, $E(Z_2) = \mu_2 = 0$; $V(Z_2) = \sigma_2^2 = \sigma^2$

$$\underbrace{Z_1 \sim N(\mu_1, \sigma_1^2) \; ; \; Z_2 \sim N(\mu_2, \sigma_2^2)}_{\text{both are univariate distributions}}$$

Bivariate distribution :

$$\underset{2\times 1}{\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}} \sim \mathbb{N}\left(\underset{2\times 1}{\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}}, \underbrace{\begin{bmatrix} \sigma_1^2 & Cov(Z_1, Z_2) \\ Cov(Z_2, Z_1) & \sigma_2^2 \end{bmatrix}}_{\text{Variance–covariance matrix}}_{2\times 2} \right)$$

$$y_i = \beta_0 + \beta_1 x_i + \underset{\text{is a r.v.}}{u_i}$$

u_i can take many values with an attached prob.

each u_i has $E(u_i) = 0$

each u_i has $V(u_i) = \sigma^2$

each u_i : $E(u_i | X) = 0$

$$\underset{N\times 1}{\underline{U}} \sim \underline{\mathbb{N}}\Big(\underset{N\times 1}{\underline{0}}\,,\ \sigma^2 \underbrace{I_N}_{N\times N}\Big)$$

$\mathbb{N}$: By A5

$\underline{0}$: By A1 + A4

$E(U|X) = 0$

If X is non-stochastic

the $E(\underline{U}) = 0$

$\sigma^2 I_N$: By A2

A1, A2, A4, A5 together imply the above.

Next step. $\longrightarrow$ to estimate $\underset{K\times 1}{\underline{\beta}} = \underset{K\times 1}{\begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_K \end{bmatrix}}$

Method of least squares to estimate $\underset{K\times 1}{\underline{\beta}}$

$$\min_{(\beta_1, \beta_2, \ldots, \beta_K)} \underbrace{\sum_{i=1}^{N} \left(y_i - \beta_1 x_{i1} - \beta_2 x_{i2} - \cdots - \beta_K x_{iK}\right)^2}_{= S}$$

Next: we can write down our first-order conditions.

$$\min_{(\beta_1, \beta_2, \ldots, \beta_K)} S\left(\beta_1, \beta_2, \ldots, \beta_K ; \left\{y_i, \{x_{ik}\}_{k=1}^{K}\right\}_{i=1}^{N}\right)$$

FOCs:

(i) $\dfrac{\partial S}{\partial \beta_1} \overset{set}{=} 0$

(ii) $\dfrac{\partial S}{\partial \beta_2} \overset{set}{=} 0$

⋮

(Kth condition) $\dfrac{\partial S}{\partial \beta_K} \overset{set}{=} 0$

K-equations and K unknowns.

$[\hat{\beta}_{1\,OLS}, \hat{\beta}_{2\,OLS} \ldots \hat{\beta}_{K\,OLS}] = \underline{\hat{\beta}}_{OLS}$

$$\min_{\underline{\beta}} \underline{\sum_{i=1}^{N} u_i^2} \equiv \min_{\underline{\beta}} S(\underline{\beta}; \underline{y}, X)$$

$$\underset{1\times 1}{S} = \underset{1\times N}{[u_1\ u_2\ \cdots\ u_N]} \underset{N\times 1}{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}} = \underset{1\times 1}{u_1^2 + u_2^2 + u_3^2 + \cdots + u_N^2}$$

$$= \sum_{i=1}^{N} u_i^2$$

$$\underset{1\times 1}{S} = \underbrace{\underset{1\times N}{\underline{u}'} \underset{N\times 1}{\underline{u}}}_{1\times 1}$$

[Remember $S = \underline{u}'\underline{u}$ is distinct from $E(\underbrace{\underline{u}\,\underline{u}'}_{N\times N} | X) = \sigma^2 I_N$

$$S = \underline{u}'\underline{u} = \overbrace{(\underset{N\times 1}{\underline{y}} - \underset{N\times K}{X}\underset{K\times 1}{\underline{\beta}})'}^{1\times N}\overbrace{(\underset{N\times 1}{\underline{y}} - \underset{N\times K}{X}\underset{K\times 1}{\underline{\beta}})}^{N\times 1}$$

$$= \underbrace{(\underset{1\times N}{y'} - \underbrace{\underset{1\times K}{\underline{\beta}'}\underset{K\times N}{X'}}_{1\times N})}_{1\times N}\underbrace{(y - X\underline{\beta})}_{N\times 1}$$

$$S = \underline{u}'\underline{u} = (\underline{y}' - \underline{\beta}'X')(\underline{y} - X\underline{\beta})$$

$$\underset{1\times 1}{S} = \underbrace{\underset{1\times N}{\underline{y}'}\,\underset{N\times 1}{\underline{y}}}_{1\times 1\ \checkmark} - \underbrace{\underbrace{\underset{1\times N}{\underline{y}'}\,\underset{N\times K}{X}}_{1\times K}\,\underset{K\times 1}{\underline{\beta}}}_{1\times 1\ \checkmark} - \underbrace{\underbrace{\underset{1\times K}{\underline{\beta}'}\,\underset{K\times N}{X'}}_{1\times N}\,\underset{N\times 1}{\underline{y}}}_{1\times 1\ \checkmark} + \underbrace{\underbrace{\underbrace{\underset{1\times K}{\underline{\beta}'}\,\underset{K\times N}{X'}}_{1\times N}\,\underset{N\times K}{X}}_{1\times K}\,\underset{K\times 1}{\underline{\beta}}}_{1\times 1\ \checkmark}$$

both scalar entities that are transpose of each other.

$\because$ Transpose of a scalar quantity is the scalar itself, we can write:

$$S = \underline{y}'\underline{y} - 2\underline{\beta}'X'\underline{y} + \underline{\beta}'X'X\underline{\beta}$$

FOCs:

$$\frac{\partial S_{1\times 1}}{\partial \underline{\beta}_{K\times 1}} \overset{\text{set}}{=} 0$$

Notice that we are differentiating a scalar entity with a vector !!

To be able to evaluate $\partial S / \partial \underline{\beta}$ we will introduce a:-

Jacobian matrix

$$\underset{M\times 1}{\underline{y}} = f(\underset{N\times 1}{\underline{Z}})$$

$\underline{y}$ and $\underline{z}$ are vectors.

$$J = \frac{\partial \underline{y}}{\partial \underline{Z}} = \begin{bmatrix} \frac{\partial y_1}{\partial z_1} & \frac{\partial y_1}{\partial z_2} & \cdots & \frac{\partial y_1}{\partial z_N} \\ \frac{\partial y_2}{\partial z_1} & \frac{\partial y_2}{\partial z_2} & \cdots & \frac{\partial y_2}{\partial z_N} \\ \vdots & & & \\ \frac{\partial y_M}{\partial z_1} & \frac{\partial y_M}{\partial z_2} & \cdots & \frac{\partial y_M}{\partial z_N} \end{bmatrix}_{M\times N} \rightarrow \text{Jacobian } (J)$$

for our case

$M = 1$ and $N = K$.

$$\frac{\partial S_{1\times 1}}{\partial \underline{\beta}_{K\times 1}} = \begin{bmatrix} \frac{\partial S}{\partial \beta_1} & \frac{\partial S}{\partial \beta_2} & \cdots & \frac{\partial S}{\partial \beta_K} \end{bmatrix}_{1\times K}$$

$$S = \underbrace{\underline{y}'\underline{y}} - 2\underline{\beta}'X'\underline{y} + \beta' X' X \underline{\beta}$$

$$\underbrace{\frac{\partial S}{\partial \underline{\beta}}}_{K\times 1} = \underbrace{\underline{0}}_{K\times 1} - \underbrace{2 \underset{K\times N}{X'} \underset{N\times 1}{\underline{y}}}_{K\times 1} + \underbrace{2 \underset{K\times N}{X'} \underset{N\times K}{X} \underset{K\times 1}{\beta}}_{K\times 1} \overset{set}{=} \underset{K\times 1}{\vec{0}}$$

$$\Rightarrow \quad -2X'\underline{y} + 2X'X\underline{\beta} = 0$$

$$X'X\underline{\beta} = X'\underline{y}$$

Pre-multiply both sides with $(X'X)^{-1}$.

$$\boxed{\underset{K\times 1}{\underline{\hat{\beta}}_{OLS}} = \underbrace{\underbrace{\underbrace{(\underset{K\times N}{X'}\underset{N\times K}{X})^{-1}}_{K\times K} \underset{K\times N}{X'}}_{K\times N} \underset{N\times 1}{\underline{y}}}_{K\times 1} \checkmark}$$

for the case of SCRM : $\hat{\beta}_1 = \dfrac{\sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{N}(x_i-\bar{x})^2} = \underbrace{\left[\sum_{i=1}^{N}(X_i - \bar{X})^2\right]^{-1}}_{(X'X)^{-1}} \underbrace{\sum_{i=1}^{N}(X_i-\bar{X})(y_i-\bar{y})}_{X'y}$.

$$\underline{\hat{\beta}_{OLS}} = \overbrace{\underbrace{\underbrace{(X'X)}_{K\times K}^{-1}}_{\text{inverting } (X'X)} \underbrace{X'}_{K\times N}}^{A_{K\times N}} \underline{y}$$

So we need
the $(X'X)$ is non-singular.
(By A3 i.e., $\text{Rank}(X) = K$)

$\underline{\hat{\beta}_{OLS}}$ is linear in $\underline{y}$

$$\hat{\beta}_{OLS} = A \cdot \underline{y} \qquad \text{where} \quad A = (X'X)^{-1}X'$$