Econometrics - I

Jan 29, 2021

Lecture - 5

Recap:

(yield) (fert) (ability)

$$y_i = \beta_0 + \beta_1 x_i + u_i$$

SLRM

β_0, β_1 : parameters (constant, to be estimated)

$\{y_i, x_i\}_{i=1}^{N}$

u_i : random error $\quad u_i \sim f_u$

Assumptions

A1 : $E(u_i) = 0$ w.l.o.g. assumption whenever β_0 (intercept) is included in the model.

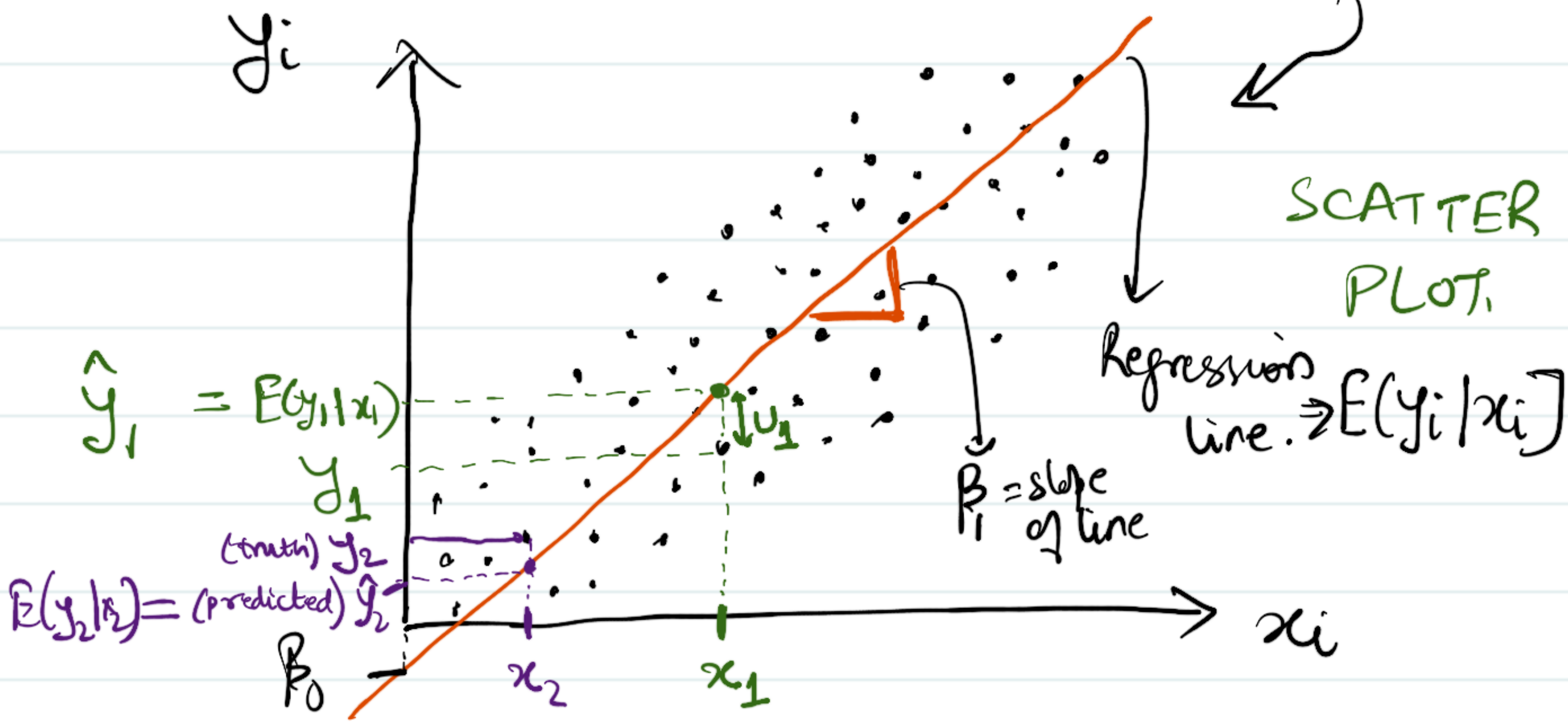
A2 : $E(u_i | x_i) = 0$ $\longrightarrow$ $E(\text{abil.} | x_i = 10) = E(\text{abil} | x = 15) = \ldots = 0$

Implication: u_i and x_i independent

$$\text{Corr}(x_i, u_i) = 0$$
$$E(u_i x_i) = 0$$

Estimate β_0 and β_1 from the given data.

$$\underbrace{y_i = \overbrace{\beta_0 + \beta_1 x_i}^{\text{deterministic}} + \overbrace{u_i}^{\text{random/stochastic component.}}} \qquad \{y_i, x_i\}_{i=1}^{N}$$

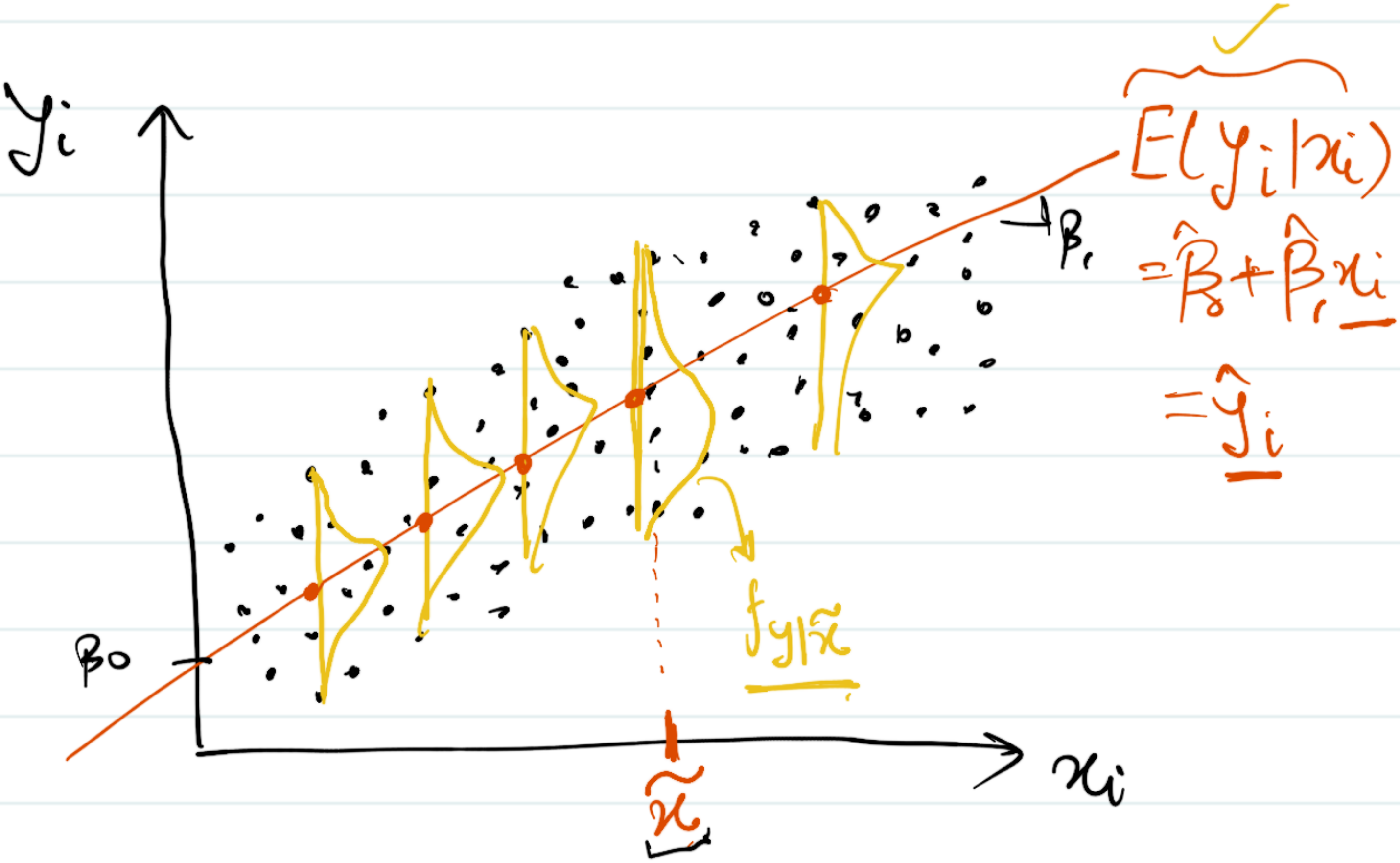


$$E(y_i|x_i) = E_x(\beta_0 + \beta_1 x_i + u_i) = \beta_0 + \beta_1 x_i + 0$$

(By assumption).

$$\| \qquad \hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$$

represent estimated values of β_0, β_1, resp.



① $\boxed{\text{Regression is a model on the MEAN}}$

② Variation in y at different values of x_i is an important driver of estimated β_0 and β_1.

Aside (How did you arrive at $f_{y|\tilde{x}}$ in the graph above?)

Step 1.

y is a r.v. $\Longrightarrow$ $y \sim \underline{f_y}$

Step 2: $y = \beta_0 + \beta_1 x$

fix $\tilde{x}$ $\rightarrow$ $\tilde{y} = \beta_0 + \beta_1 \tilde{x}$

$\tilde{y}$ is a r.v. $\rightarrow$ $\tilde{y} \sim f_{\tilde{y}} = f_{y|\tilde{x}}$

Now, to estimating β_0 and β_1 from the data.

→ Go back to our assumptions for guidance.

① $E(u_i) = 0$ (Mean)

② $E(u_i | x_i) = 0$ (conditional mean)

$\Rightarrow E(u_i x_i) = 0$

Both assumptions are on the first moment of u_i.

①' $E(y_i - \beta_0 - \beta_1 x_i) = 0 \Rightarrow \frac{1}{N}\sum_{i=1}^{N}(y_i - \beta_0 - \beta_1 x_i) = 0$

②' $E((y_i - \beta_0 - \beta_1 x_i) x_i) = 0 \Rightarrow \frac{1}{N}\sum_{i=1}^{N}(y_i x_i - \beta_0 x_i - \beta_1 x_i^2) = 0$

2 eq^ns, 2 unknowns!!

$1' \Rightarrow \bar{y} - \hat{\beta}_0 - \hat{\beta}_1 \bar{x} = 0 \quad \Rightarrow \quad \boxed{\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}}$

$2' \Rightarrow \frac{\sum_{i=1}^{N} y_i x_i}{N} - \hat{\beta}_0 \bar{x} - \hat{\beta}_1 \frac{\sum_{i=1}^{N} x_i^2}{N} = 0$

$\Rightarrow \sum_i y_i x_i - \hat{\beta}_0 N \bar{x} - \hat{\beta}_1 \sum_i x_i^2 = 0$

$$1' \Rightarrow \quad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

$$2' \Rightarrow \quad \frac{\sum_{i=1}^{N} y_i x_i}{N} - \hat{\beta}_0 \bar{x} - \hat{\beta}_1 \frac{\sum_{i=1}^{N} x_i^2}{N} = 0$$

$$2' \Rightarrow \quad \frac{\sum_{i=1}^{N} y_i x_i}{N} - \bar{x}\left(\bar{y} - \hat{\beta}_1 \bar{x}\right) - \hat{\beta}_1 \frac{\sum_{i=1}^{N} x_i^2}{N} = 0$$

$$\Rightarrow \quad \boxed{\begin{array}{l} \hat{\beta}_{1,MM} = \dfrac{\sum_{i=1}^{N} y_i x_i - N\bar{x}\bar{y}}{\sum_{i=1}^{N} x_i^2 - N\bar{x}^2} \\ \hline \hat{\beta}_{0,MM} = \bar{y} - \hat{\beta}_{1,MM}\bar{x} \end{array}}$$

Used the 1st moment prop. of random error. [Strategy 1] for estimation (Method of moments).

→ Strategy 2 for estimation of β_0 and β_1

Minimize ~~of~~ sum of errors

$$\min_{\beta_0, \beta_1} \sum_{i=1}^{N} \underline{u_i}$$

(acceptable?)

issue is that
some u_i's will be +ve
& some u_i's will be -ve
⇒ Nullify the total error.

$$\min_{\beta_0, \beta_1} \sum_{i=1}^{N} u_i^2$$

→ ORDINARY Least squares algorithm.

$$\Rightarrow \left.\min_{\beta_0, \beta_1} \underbrace{\sum_{i=1}^{N} (y_i - \beta_0 - \beta_1 x_i)^2}_{= S(\beta_0, \beta_1)}\right\}$$

First order conditions for minimization (FOC)

$$\min_{\beta_0,\beta_1} \sum_{i=1}^{N} (y_i-\beta_0-\beta_1 x_i)^2 \equiv \min_{\beta_0,\beta_1} S(\beta_0,\beta_1)$$

① $\dfrac{\partial S}{\partial \beta_0} \overset{set}{=} 0 \Rightarrow \sum_{i=1}^{N}\left[-2(y_i-\beta_0-\beta_1 x_i)\right] \overset{set}{=} 0$

$\Rightarrow \sum_{i=1}^{N}(y_i-\beta_0-\beta_1 x_i) = 0 \rightarrow 1''$

② $\dfrac{\partial S}{\partial \beta_1} \overset{set}{=} 0 \Rightarrow \sum_{i=1}^{N}\left[-2x_i(y_i-\beta_0-\beta_1 x_i)\right] \overset{set}{=} 0$

$\Rightarrow \sum_{i=1}^{N}(y_i x_i-\beta_0 x_i-\beta_1 x_i^2) = 0 \rightarrow 2''$

Upon solving $1''$ and $2''$ we get $\underbrace{\hat{\beta}_{0,OLS} \text{ and } \hat{\beta}_{1,OLS}}$.

[ORDINARY LEAST SQUARES ESTIMATES]

$$\hat{\beta}_{0,OLS} = \bar{y} - \hat{\beta}_{1,OLS}\,\bar{x}$$

$$\hat{\beta}_{1,OLS} = \frac{\sum_{i=1}^{N}(y_i-\bar{y})(x_i-\bar{x})}{\sum_{i=1}^{N}(x_i-\bar{x})^2}$$

See that eqn. pair $1', 2'$ is equivalent to the pair $1'', 2''$

$$\Rightarrow \quad \hat{\beta}_{1,MM} = \hat{\beta}_{1,OLS} \quad \Rightarrow \hat{\beta}_{0,OLS} = \hat{\beta}_{0,MM}$$

$$\frac{\sum_{i=1}^{N} y_i x_i - N\bar{x}\bar{y}}{\sum_{i=1}^{N} x_i^2 - N\bar{x}^2} = \frac{\sum_{i=1}^{N} (y_i - \bar{y})(x_i - \bar{x})}{\sum_{i=1}^{N} (x_i - \bar{x})^2}$$

(LHS) (RHS)

should be able to show this

(Quiz 1)

RHS: $\dfrac{\sum_{i=1}^{N} (y_i - \bar{y})(x_i - \bar{x})}{\sum_{i=1}^{N} (x_i - \bar{x})^2}$

N^r: $\sum_{i=1}^{N} (y_i x_i - \bar{y} x_i - \bar{x} y_i + \bar{x}\bar{y})$

$= \sum_{i=1}^{N} y_i x_i - \bar{y} \underbrace{\sum x_i}_{=N\bar{x}} - \bar{x} \underbrace{\sum y_i}_{=N\bar{y}} + N\bar{x}\bar{y}$

$= \sum_{i=1}^{N} y_i x_i - N\bar{x}\bar{y}$

D^r: $\sum_{i=1}^{N} (x_i - \bar{x})^2 = \sum_{i=1}^{N} x_i^2 - N\bar{x}^2.$

✓

Focus on the slope estimate

$$\hat{\beta}_{1,OLS} = \frac{\left[\sum_{i=1}^{N} (x_i - \bar{x})(y_i - \bar{y})\right] / N-1}{\left[\sum_{i=} (x_i - \bar{x})^2\right] / N-1}$$

$$= \frac{Cov(y, x)}{V(x)}$$

$Cov(y, x) \rightarrow$ degree of linear relationship b/w x and y

$$y = \beta_0 + \beta_1 x + u$$

$\downarrow$

Normalized covariance b/w y and x.

Normalization factor is the variance of x_i

Aside (why $N-1$ in the dr. of $\hat{\beta}_1$ ors?)

$$\underbrace{V(x)}_{\downarrow} = \frac{\sum_{i=1}^{N} (x_i - \bar{x})^2}{N-1}$$

- depends on N values of x
- But defn. of $V(x)$ includes $\underline{\bar{x}}$

$\downarrow$ knowledge of

$\rightarrow$ If $\bar{x}$ is known, I can give you any $N-1$ values of x_i's and you can recover the N^{th} value