Econometrics-1

9.02.2021.

Lecture 8

$$V(\hat{\beta}_{0,OLS}) = V(\bar{y} - \hat{\beta}_{1,OLS}\bar{x})$$

$$= V(\bar{y}) + \bar{x}^2 V(\hat{\beta}_{1,OLS}) - 2\bar{x}\,\mathrm{Cov}(\hat{\beta}_{1,OLS}, \bar{y})$$

$$\downarrow$$

will figure out next steps by next lecture.
(Tushar)

Unbiasedness of $\hat{\beta}_{1,OLS} \Rightarrow E(\hat{\beta}_{1,OLS}) = \beta_1$

$$\hat{\beta}_{1,OLS} = \frac{\sum_{i=1}^{N}(x_i-\bar{x})y_i}{\sum_{i=1}^{N}(x_i-\bar{x})^2} = \frac{\sum_{i=1}^{N}(x_i-\bar{x})(\beta_0+\beta_1 x_i+u_i)}{\sum_{i=1}^{N}(x_i-\bar{x})^2}$$

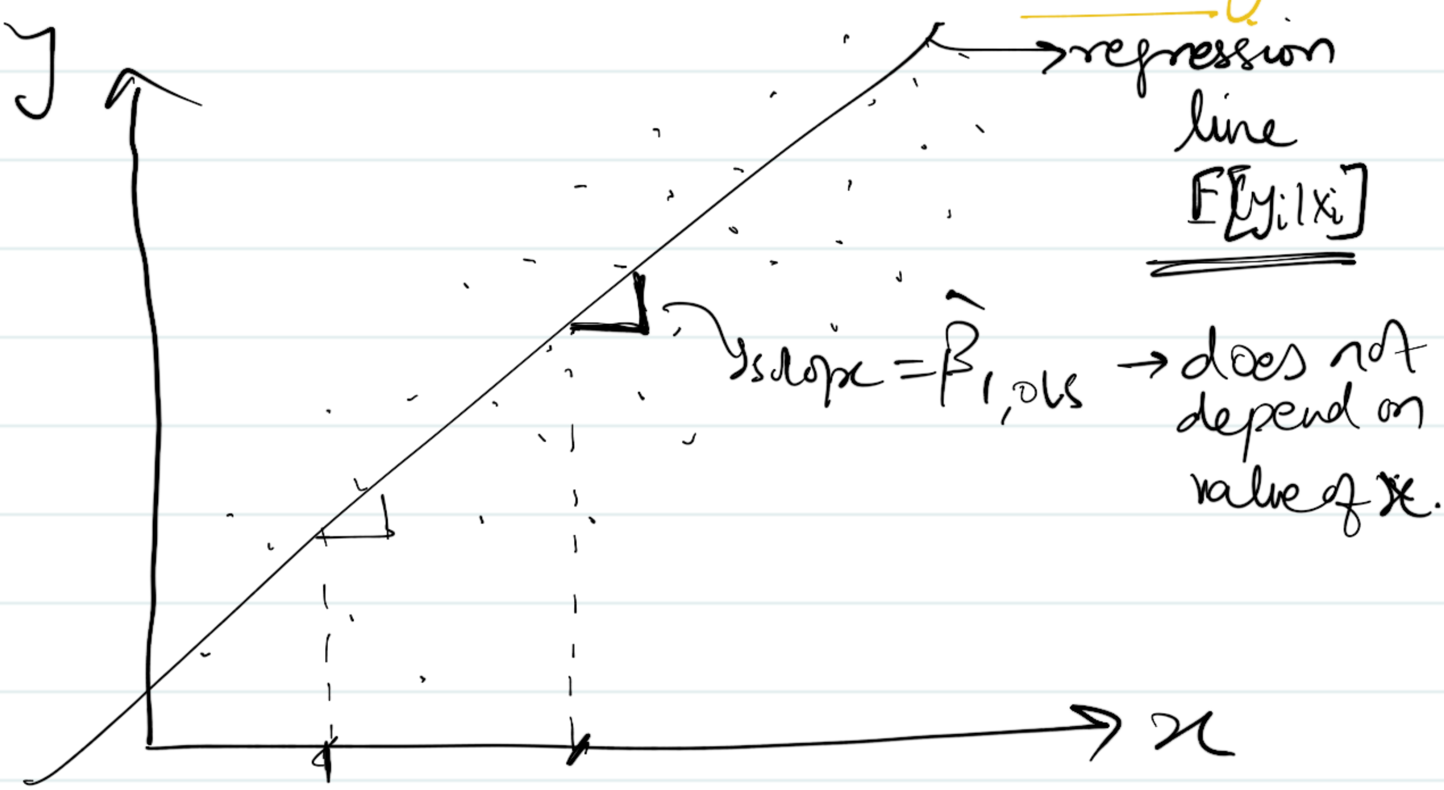
$$E(\hat{\beta}_{1,OLS}) = \frac{1}{\sum(x_i-\bar{x})^2}\left\{E\left[\beta_0 \underbrace{\sum_{i=1}^{N}(x_i-\bar{x})}_{\substack{\text{sum of deviation}\\ \text{from mean}\\ =0}} + \beta_1 \sum_{i=1}^{N}(x_i-\bar{x})\cdot x_i + \sum_{i=1}^{N}(x_i-\bar{x})u_i\right]\right.$$

$$E(\hat{\beta}_{1,OLS}) = \frac{1}{\sum_{i=1}^{N}(x_i-\bar{x})^2} E\left[\underbrace{\beta_1}_{\text{True } \beta_1} \underbrace{\sum_{i=1}^{N}(x_i-\bar{x})x_i}_{=\cancel{\sum_{i=1}^{N}(x_i-\bar{x})^2}} + \sum_{i=1}^{N}(x_i-\bar{x})u_i\right]$$

$$= \beta_1 + \underbrace{E\left[\frac{\sum_{i=1}^{N}(x_i-\bar{x})u_i}{\sum_{i=1}^{N}(x_i-\bar{x})^2}\right]}_{=0 \text{ by A1, A2}}$$

$$\hat{\beta}_{1,OLS} = \beta_1 + \frac{\sum_{i=1}^{N}(x_i-\bar{x})u_i}{\sum_{i=1}^{N}(x_i-\bar{x})^2}$$

Measures the bias in OLS estimators whenever $E(u_i|x_i) \neq 0$ OR we are not able to simulate the ceteris paribus experiment OR we are not able to establish causality.



- $V(\hat{\beta}_{1,OLS}) = \dfrac{\sigma^2}{\sum_{i=1}^{N}(x_i - \bar{x})^2}$

- $\boxed{\text{Population Model}}$

$$y_i = \beta_0 + \beta_1 x_i + u_i$$

$$E(u_i | x_i) = 0$$

$$V(u_i | x_i) = \sigma^2 \rightarrow \text{Homoskedasticity.}$$

3 parameters : $\underbrace{\beta_0, \beta_1}_{}$, σ^2

sample estimates from OLS $\rightarrow \hat{\beta}_{0,OLS}, \hat{\beta}_{1,OLS}$

what about the estimator for σ^2?

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{N} \hat{u}_i^2}{N-2}$$

Claim : $\underline{E}(\hat{\sigma}^2) = \sigma^2$ (To prove).

To prove: $E(\hat{\sigma}^2) = \sigma^2$ where $\hat{\sigma}^2 = \frac{\sum_{i=1}^{N} \hat{u}_i^2}{N-2}$

Proof: $\hat{u}_i = y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i$ $\quad (\because \hat{u}_i = y - \hat{y}_i)$

$$\hat{u}_i = \beta_0 + \beta_1 x_i + u_i - \hat{\beta}_0 - \hat{\beta}_1 x_i$$

$$\hat{u}_i = u_i + (\beta_0 - \hat{\beta}_0) + (\beta_1 - \hat{\beta}_1) x_i \quad -①$$

Take average on both sides $\Rightarrow N^{-1} \sum_{i=1}^{N} (\cdot)$

$$\bar{\hat{u}} = \bar{u} + (\beta_0 - \hat{\beta}_0) + (\beta_1 - \hat{\beta}_1) \bar{x} \quad -②$$

① − ②

$$\hat{u}_i - \bar{\hat{u}} = (u_i - \bar{u}) + (\beta_1 - \hat{\beta}_1)(x_i - \bar{x}) \quad -③$$

Square ③ on both sides

$$(\hat{u}_i - \bar{\hat{u}})^2 = (u_i - \bar{u})^2 + (\beta_1 - \hat{\beta}_1)^2 (x_i - \bar{x})^2 + 2(\beta_1 - \hat{\beta}_1)(x_i - \bar{x})(u_i - \bar{u})$$

$$(\hat{u}_i - \bar{\hat{u}})^2 = (u_i - \bar{u})^2 + (\hat{\beta}_1 - \beta_1)^2 (x_i - \bar{x})^2$$
$$- 2(\hat{\beta}_1 - \beta_1)(x_i - \bar{x})(u_i - \bar{u})$$

Apply $\sum_{i=1}^{N}(\cdot)$ on both sides:

$$\sum_{i=1}^{N} (\hat{u}_i - \bar{\hat{u}})^2 = \overbrace{\sum_{i=1}^{N} (u_i - \bar{u})^2}^{T1} + \overbrace{(\hat{\beta}_1 - \beta_1)^2 \sum_{i=1}^{N} (x_i - \bar{x})^2}^{T2}$$
$$\underbrace{- 2(\hat{\beta}_1 - \beta_1) \sum_{i=1}^{N} (x_i - \bar{x})(u_i - \bar{u})}_{T3}$$

On the R.H.S:

Expected value of T1 given x $= (N-1)\sigma^2$

Expected value of T2 given x $= \sigma^2$

$$V(\hat{\beta}_{1\,ols}) = \frac{\sigma^2}{\sum_{i=1}^{N} (x_i - \bar{x})^2}$$

$$E[\hat{\beta}_{1\,ols} - \underbrace{E(\hat{\beta}_{1\,ols})}_{=\beta_1}]^2$$

$$T3 = -2\left(\hat{\beta}_1 - \beta_1\right)\sum_{i=1}^{N}(x_i-\bar{x})(u_i-\bar{u})$$

$$= -2\left[\left(\hat{\beta}_1-\beta_1\right)\times\sum_{i=1}^{N}(x_i-\bar{x})u_i - \left(\hat{\beta}_1-\beta_1\right)\bar{u}\underbrace{\sum_{i=1}^{N}(x_i-\bar{x})}_{0}\right]$$

$$= -2\left(\hat{\beta}_1-\beta_1\right)\underbrace{\sum_{i=1}^{N}(x_i-\bar{x})u_i}$$

$$= -2\left(\hat{\beta}_1-\beta_1\right)\left(\hat{\beta}_1-\beta_1\right)\cdot\sum_{i=1}^{N}(x_i-\bar{x})^2$$

$$\left[\because \hat{\beta}_1 = \beta_1 + \frac{\sum(x_i-\bar{x})u_i}{\sum_i(x_i-\bar{x})^2}\right]$$

$$= -2\left(\hat{\beta}_1-\beta_1\right)^2\sum_{i=1}^{N}(x_i-\bar{x})^2$$

Expected value of T3 given $x = -2\sigma^2$

$$\left(\because V(\hat{\beta}_1) = \frac{\sigma^2}{\sum_{i=1}^{N}(x_i-\bar{x})^2}\right)$$

So,

$$RHS = T1+T2+T3 = (N-1)\sigma^2 + \sigma^2 - 2\sigma^2$$

$$= (N-2)\sigma^2$$

LHS = RHS :

$$\sum_{i=1}^{N} (\hat{u}_i - \underbrace{\bar{\hat{u}}}_{=0})^2 = (N-2)\sigma^2$$

$$\left(\because \sum_{i=1}^{N} \hat{u}_i = 0\right)$$

$$\Rightarrow \boxed{\frac{\sum_{i=1}^{N} \hat{u}_i^2}{N-2} = \sigma^2} \longrightarrow$$

★ Multiple Linear Regression Model

$$y_i = \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \dots + \beta_K x_{iK} + u_i$$

$$i = 1, 2, \dots, N$$

K regressors.

$\boxed{\text{Alternative notation}}$

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \dots + \beta_K x_{iK} + u_i$$

$$i = 1, 2, \dots, N$$

$K+1$ regressors

→ Matrix Notation

$$\underset{N\times 1}{\underline{y}} = \underset{N\times K}{X}\, \underset{K\times 1}{\underline{\beta}} + \underset{N\times 1}{\underline{u}}$$

$$\underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}_{N\times 1} = \begin{bmatrix} x_{11}^{=1} & x_{12} & x_{13} & \cdots & x_{1K} \\ x_{21}^{=1} & x_{22} & x_{23} & \cdots & x_{2K} \\ \vdots & & & & \\ x_{N1} & x_{N2} & x_{N3} & \cdots & x_{NK} \end{bmatrix}_{N\times K} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_K \end{bmatrix}_{K\times 1} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}_{N\times 1}$$

$$\underset{N\times 1}{\underline{y}} = \underbrace{\underset{N\times K}{X}\ \underset{K\times 1}{\underline{\beta}}}_{N\times 1} + \underset{N\times 1}{\underline{u}}$$

- $\underline{y}$ is specified to be linear in X
- $\underline{\beta}$ is a vector of fixed constants
- $\underline{u}$ carries omitted or unobserved variables