Q Conduct the MLE of the parameters of a log normal distribution i.e. (μ & σ^2). (1)

PDF of a log-normal distribution:

$$f(X \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}\cdot X} \exp.\left(\frac{-(\ln(X)-\mu)^2}{2\sigma^2}\right)$$

Probability of observing any particular x_i:

$$f(x = x_i) = \frac{1}{\sqrt{2\pi\sigma^2}\, x_i} \cdot \exp\left(\frac{-(\ln x_i - \mu)^2}{2\sigma^2}\right)$$

Probability of observing the sample of data:

$$L(x; \mu, \sigma^2) = f(x_1) \cdot f(x_2 \mid x_1) \cdots\cdots f(x_n \mid x_{n-1} \cdots x_1)$$

By the i.i.d assumption, we can write this as follows:

$$L(x; \mu, \sigma^2) = f(x_1) \cdot f(x_2) \cdots\cdots f(x_n)$$

$$= \prod_{i=1}^{n} \left(\frac{1}{\sqrt{2\pi\sigma^2}\, x_i} \exp.\left(\frac{-(\ln(x_i)-\mu)^2}{\sigma^2} \right) \right)$$

$$L(x,\theta) = \frac{1}{(2\pi\sigma^2)^{n/2} \cdot \prod_{i=1}^{n} x_i} \exp\left(\frac{-\sum_{i=1}^{n}(\ln(x_i) - \mu)^2}{2\sigma^2}\right)$$

Taking log :
Log - Likelihood fn :

$$\ell(x, \theta) = \ln . L(x,\theta)$$

$$\ell(x,\theta) = -\frac{n}{2} \ln(2\pi) - \frac{n}{2} \ln(\sigma^2) - \ln\left(\prod_{i=1} x_i\right)$$

$$- \frac{1}{2\sigma^2} \cdot \sum_{i=1}^{n} (\ln(x_i) - \mu)^2$$

Maximizing $\ell(x, \theta)$ s.t θ :

FONC :

① w.r.t $\hat{\mu}$: $\frac{\partial \ell}{\partial \mu} = 0$

$$+ \frac{2}{2\hat{\sigma}^2} \cdot \sum_{i=1}^{n} (\ln(x_i) - \hat{\mu}) = 0.$$

$$\sum_{i=1}^{n} \ln x_i - n\hat{\mu} = 0.$$

$$\Rightarrow \boxed{\hat{\mu}_{MLE} = \frac{\sum_i \ln x_i}{n}}$$

(2) w.r.t. σ^2: $\frac{\partial \ell}{\partial \sigma^2} = 0$

$$\frac{-n}{2(\hat{\sigma}^2)} + \frac{1}{2(\hat{\sigma}^2)^2} \sum_{i=1}^{n} (\ln(x_i) - \hat{\mu})^2 = 0.$$

$$\frac{1}{\cancel{2}(\hat{\sigma}^2)^2} \sum_i (\ln(x_i) - \hat{\mu})^2 = \frac{n}{\cancel{2}\sigma^2}.$$

$$\frac{1}{\hat{\sigma}^2} \sum_i (\ln(x_i) - \hat{\mu})^2 = n.$$

$$n \cdot \hat{\sigma}^2 = \sum_i (\ln(x_i) - \hat{\mu})^2.$$

$$\hat{\sigma}^2 = \frac{\sum_i (\ln(x_i) - \hat{\mu})^2}{n}.$$

$$\boxed{\hat{\sigma}^2_{MLE} = \frac{\sum_i (\ln(x_i) - \hat{\mu})^2}{n}.}$$

$$\boxed{\hat{\sigma}^2_{MLE} = \frac{\sum_{i=1}^{n} \left[\left(\ln(x_i) - \left(\frac{\sum_{i=1}^{n} \ln(x_i)}{n} \right) \right) \right]^2}{n}.}$$

↳ Ans.