18-Definite Integrals and Applications of Integrals

EE1030 : Matrix Theory

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9. Let S be the area of the region enclosed by $y = e^{-x^2}$, $y = 0$, $x = 0$, and $x = 1$; then (2012)

(a) $S \geq \frac{1}{e}$
(b) $S \geq 1 - \frac{1}{e}$
(c) $S \leq \frac{1}{4}\left(1 + \frac{1}{\sqrt{e}}\right)$
(d) $S \leq \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}}\left(1 - \frac{1}{\sqrt{2}}\right)$

10. The option(s) with the values of a and L that satisfy the following equation is(are) (JEE Adv. 2015)

$$\frac{\int_0^{4\pi} e^t \left(\sin^6(at) + \cos^4(at)\right) dt}{\int_0^{\pi} e^t \left(\sin^6(at) + \cos^4(at)\right) dt} = L?$$

(a) $a = 2,\ L = \frac{e^{4\pi}-1}{e^{\pi}-1}$
(b) $a = 2,\ L = \frac{e^{4\pi}+1}{e^{\pi}+1}$
(c) $a = 4,\ L = \frac{e^{4\pi}-1}{e^{\pi}-1}$
(d) $a = 4,\ L = \frac{e^{4\pi}+1}{e^{\pi}+1}$

11. Let $f(x) = 7\tan^8 x + 7\tan^6 x - 3\tan^4 x - 3\tan^2 x$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
Then the correct expression(s) is(are) (JEE Adv. 2015)

(a) $\int_0^{\pi/4} x f(x)\, dx = \frac{1}{12}$
(b) $\int_0^{\pi/4} f(x)\, dx = 0$
(c) $\int_0^{\pi/4} x f(x)\, dx = \frac{1}{6}$
(d) $\int_0^{\pi/4} f(x)\, dx = 1$

12. Let $f'(x) = \frac{192x^3}{2+\sin^4(\pi x)}$ for all $x \in \mathbb{R}$ with $f\left(\frac{1}{2}\right) = 0$. If $m \leq \int_{1/2}^{1} f(x)\, dx \leq M$, then the possible values of m and M are (JEE Adv. 2015)

(a) $m = 13, M = 24$
(b) $m = \frac{1}{4}, M = \frac{1}{2}$
(c) $m = -11, M = 0$
(d) $m = 1, M = 12$

13. Let

$$f(x) = \lim_{n\to\infty} \left(\frac{n^n (x+n)\, x + \left(\frac{n}{2}\right) \cdots \left(x + \frac{n}{n}\right)}{n! \left(x^2 + n^2\right)\left(x^2 + \frac{n^2}{4}\right) \cdots \left(x^2 + \frac{n^2}{n^2}\right)} \right)^{\frac{x}{n}},$$

Then for all $x > 0$. (JEE Adv. 2016)

(a) $f\left(\frac{1}{2}\right) \geq f(1)$
(b) $\left(\frac{1}{3}\right) \leq f\left(\frac{2}{3}\right)$
(c) $f'(2) \leq 0$
(d) $\frac{f'(3)}{f(3)} \geq \frac{f'(2)}{f(2)}$

14. Let $f : \mathbb{R} \to (0, 1)$ be a continuous function. Then, which of the following function(s) has (have) the value zero at some point in the interval $(0, 1)$? (JEE Adv 2016)

(a) $x^9 - f(x)$
(b) $x - \int_0^{\frac{\pi}{2}-x} f(t) \cos t\, dt$
(c) $e^x - \int_0^x f(t) \sin t\, dt$
(d) $f(x) + \int_0^{\frac{\pi}{2}} f(t) \sin t\, dt$

15. If $g(x) = \int_{\sin x}^{\sin(2x)} \sin^{-1}(t), dt$, then (JEE Adv 2017)

(a) $g'\left(\frac{\pi}{2}\right) = -2\pi$
(b) $g'\left(-\frac{\pi}{2}\right) = 2\pi$
(c) $g'\left(\frac{\pi}{2}\right) = 2\pi$
(d) $g'\left(-\frac{\pi}{2}\right) = -2\pi$

16. If the line $sx = \alpha$ divides the area of region $R = \left((x, y) \in \mathbb{R}^2 : x^3 \leq y \leq x, 0 \leq x \leq 1\right)$ into two equal parts, then (JEE Adv. 2017)

(a) $0 < \alpha \leq \frac{1}{2}$
(b) $\frac{1}{2} < \alpha < 1$
(c) $2\alpha^4 - 4\alpha^2 + 1 = 0$
(d) $\alpha^4 + 4\alpha^2 - 1 = 0$

17. If $I = \sum_{k=1}^{98} \int_k^{k+1} \frac{k+1}{x(k+1)}\, dx$, then (JEE Adv 2017)

(a) $1 > \ln 99$
(b) $1 < \ln 99$
(c) $1 < \frac{49}{50}$
(d) $1 > \frac{49}{50}$

18. For $a \in \mathbb{R}, |a| \geq 1$, let (JEE Adv 2019)

$$\lim_{n\to\infty} \frac{1 + \sqrt{2} + \cdots + \sqrt{n}}{(an+1)^2 + (an+2)^2 + \cdots + (an+n)^2} = 54$$

Then the possible value(s) of are:

(a) 9
(b) 7
(c) 6
(d) 8

E Subjective Problems

1. Find the area bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$. (1981 - 4 Marks)

2. Show that:

$$\lim_{n\to\infty} 1 + \left(\frac{1}{n}\right)^n = e \quad \text{and} \quad \lim_{n\to\infty} 1 + \frac{1}{n}^{n^2} = e^{1981}$$

(1981 - 4 Marks)

3. Show that: $n\int_0^\pi f(x)\sin x\,dx = \frac{n\pi}{2}\int_0^\pi f(\sin x)\,dx$ (1982 - 2 Marks)

4. Find the value of:

$$p\int_0^p |\sin|\,x\,dx + q\int_p^{2p} \sin x\,dx \quad \text{where} \quad p > q > 0$$

(1982 - 3 Marks)

5. For any real t, show that the point with coordinates given by:

$$x = e^{-t}, \quad y = e^t - e^{-t}$$

lies on the hyperbola defined by: $x^2 - \frac{y^2}{e^2} = 1$ Show that there are two points corresponding to $t = 1$ and $t = -1$; find them.

Show that the area bounded by this hyperbola and these two points corresponding to $t = 1$ and $t = -1$ is

$$\left(\frac{\pi}{4}\right)(e^2 + 1)$$

(1982 - 3 Marks)