

Probability and Random Processes

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Question:

Let X be a random variable with probability density function

$$p_X(x) = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

For $a < b$, if $U(a, b)$ denotes the uniform distribution over the interval (a, b) , then which of the following statements is/are true?

- (A) e^{-X} follows $U(-1, 0)$ distribution
- (B) $1 - e^{-X}$ follows $U(0, 2)$ distribution
- (C) $2e^{-X} - 1$ follows $U(-1, 1)$ distribution
- (D) The probability mass function of $Y = [X]$ is $\Pr(Y = k) = e^{-k}(1 - e^{-1})$ for $k = 0, 1, 2, \dots$, where $[X]$ denotes the largest integer not exceeding x

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Solution: Let $Y \sim U(a, b)$, then

$$p_Y(y) = \begin{cases} \frac{1}{b-a} & a < y < b \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

and for $a < y < b$

$$F_Y(y) = \Pr(Y \leq y) \quad (3)$$

$$= \int_a^y \frac{1}{b-a} dy \quad (4)$$

$$= \frac{y-a}{b-a} \quad (5)$$

Similarly, for $x \geq 0$

$$F_X(x) = \Pr(X \leq x) \quad (6)$$

$$= \int_0^x e^{-x} dx \quad (7)$$

$$= 1 - e^{-x} \quad (8)$$

- (A) $Y = e^{-X} = U(a, b)$
for $a < y < b$

$$F_Y(y) = \Pr(e^{-X} \leq y) \quad (9)$$

$$= \Pr(X \geq -\ln y) \quad (10)$$

$$= 1 - F_X(-\ln y) \quad (11)$$

$$= 1 - (1 - y) \quad (12)$$

$$= y \quad (13)$$

Comparing this with CDF of Uniform distribution,

we obtain

$$a = 0, b = 1 \quad (14)$$

$$\therefore Y \sim U(0, 1) \quad (15)$$

- (B) $Y = 1 - e^{-X} = U(a, b)$
for $a < y < b$

$$F_Y(y) = \Pr(1 - e^{-X} \leq y) \quad (16)$$

$$= \Pr(e^{-X} \geq 1 - y) \quad (17)$$

$$= \Pr(X \leq -\ln(1 - y)) \quad (18)$$

$$= F_X(-\ln(1 - y)) \quad (19)$$

$$= 1 - (1 - y) \quad (20)$$

$$= y \quad (21)$$

$$\implies Y \sim U(0, 1) \quad (22)$$

- (C) $Y = 2e^{-X} - 1 = U(a, b)$
for $a < y < b$

$$F_Y(y) = \Pr(2e^{-X} - 1 \leq y) \quad (23)$$

$$= \Pr\left(X \geq -\ln\left(\frac{y+1}{2}\right)\right) \quad (24)$$

$$= 1 - F_X\left(-\ln\left(\frac{y+1}{2}\right)\right) = 1 - \left(1 - \frac{y+1}{2}\right) \quad (25)$$

$$= \frac{y+1}{2} \quad (26)$$

Comparing this with CDF of Uniform distribution, we obtain

$$a = -1, b = 1 \quad (27)$$

$$\therefore Y \sim U(-1, 1) \quad (28)$$

- (D) $Y = [X]$

$$\Pr(Y = k) = \Pr([X] = k) \quad (29)$$

$$= \Pr(k \leq X < k + 1) \quad (30)$$

$$= \int_k^{k+1} e^{-x} dx \quad (31)$$

$$= e^{-k}(1 - e^{-1}) \text{ for } k=0,1,2,\dots \quad (32)$$

- (E) Generation of Random Variable X in C language

(i) `rand()` / `(double)RAND_MAX` :

This generates a random variable between 0 and RAND_MAX and divides it by RAND_MAX to obtain a uniform distribution between 0 and 1.

- (ii) $-\log(\text{rand}() / (\text{double})\text{RAND_MAX})$:
This transforms the uniform distribution between 0 and 1 into an exponential distribution by making the values vary from 0 to ∞ .

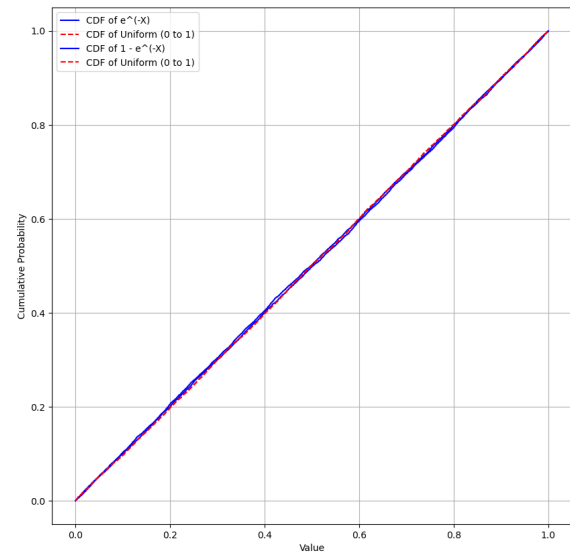


Fig. 5: Comparison of CDF: $1 - e^{-X}$ vs. $U(0, 1)$

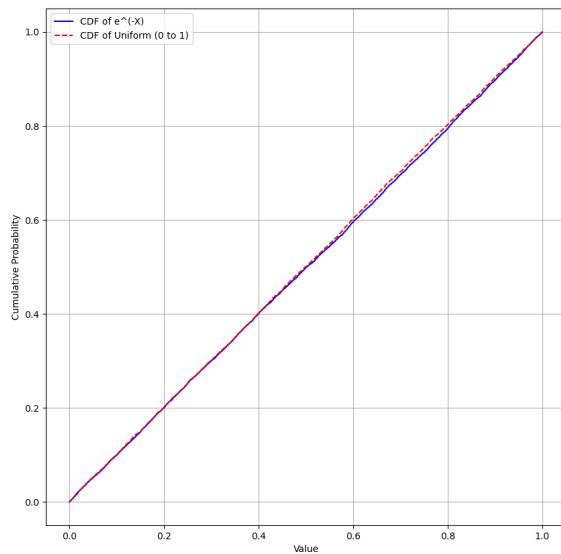


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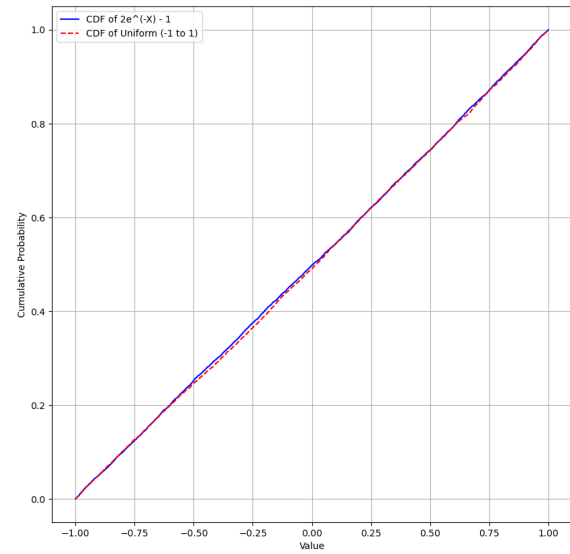


Fig. 5: Comparison of CDF: $2e^{-X} - 1$ vs. $U(-1, 1)$