Probability and Random Processes

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EE22BTECH11046*

Question:
Let X be a random variable with probability density function

$$p_X(x) = \begin{cases} e^{-x} & if x \geq 0 \\ 0 & otherwise \end{cases} \tag{1}$$

For $a < b$, if $U(a, b)$ denotes the uniform distribution over the interval (a, b), then which of the following statements is/are true?

1) e^{-X} follows $U(-1, 0)$ distribution
2) $1 - e^{-X}$follows $U(0, 2)$ distribution
3) $2e^{-X} - 1$follows $U(-1, 1)$ distribution
4) The probability mass function of $Y = [X]$ is $\Pr(Y = k) = e^{-k}\left(1 - e^{-1}\right)$ for k= 0, 1, 2, ..., where $[X]$denotes the largest integer not exceeding x

(GATE ST 2023)

Solution: Let $Y \sim U(a, b)$, then

$$p_Y(y) = \begin{cases} \frac{1}{b-a} & a < y < b \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

and for $a < y < b$

$$F_Y(y) = \Pr(Y \leq y) \tag{3}$$
$$= \int_a^y \frac{1}{b-a} dy \tag{4}$$
$$= \frac{y-a}{b-a} \tag{5}$$

1) $Y = e^{-X}$

$$F_Y(y) = \Pr\left(e^{-X} \leq y\right) \tag{6}$$
$$= \Pr(X \geq -\ln y) \tag{7}$$
$$= 1 - \int_0^{-\ln y} e^{-y} dy \tag{8}$$
$$= 1 - (1 - y) \tag{9}$$
$$= y \tag{10}$$

Comparing this with CDF of Uniform distribution, we obtain

$$a = 0, b = 1 \tag{11}$$
$$\therefore Y \sim U(0, 1) \tag{12}$$

2) $Y = 1 - e^{-X}$

$$F_Y(y) = \Pr\left(1 - e^{-X} \leq y\right) \tag{13}$$
$$= \Pr\left(e^{-X} \geq 1 - y\right) \tag{14}$$
$$= \Pr(X \leq -\ln(1 - y)) \tag{15}$$
$$= \int_0^{-\ln(1-y)} e^{-y} dy \tag{16}$$
$$= 1 - (1 - y) \tag{17}$$
$$= y \tag{18}$$
$$\implies Y \sim U(0, 1) \tag{19}$$

3) $Y = 2e^{-X} - 1$

$$F_Y(y) = \Pr\left(2e^{-X} - 1 \leq y\right) \tag{20}$$
$$= \Pr\left(X \geq -\ln\left(\frac{y+1}{2}\right)\right) \tag{21}$$
$$= 1 - \int_0^{-\ln\left(\frac{y+1}{2}\right)} e^{-y} dy \tag{22}$$
$$= 1 - \left(1 - \frac{y+1}{2}\right) \tag{23}$$
$$= \frac{y+1}{2} \tag{24}$$

Comparing this with CDF of Uniform distribution, we obtain

$$a = -1, b = 1 \tag{25}$$
$$\therefore Y \sim U(-1, 1) \tag{26}$$

4) $Y = [X]$

$$\Pr(Y = k) = \Pr([X] = k) \tag{27}$$
$$= \Pr(k \leq x < k + 1) \tag{28}$$
$$= \int_k^{k+1} e^{-x} dx \tag{29}$$
$$= e^{-k}\left(1 - e^{-1}\right) \text{for k=0,1,2..} \tag{30}$$