

# Probability and Random Processes

Sarvesh K  
EE22BTECH11046\*

## Question:

Let  $X$  be a random variable with probability density function

$$p_X(x) = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

For  $a < b$ , if  $U(a, b)$  denotes the uniform distribution over the interval  $(a, b)$ , then which of the following statements is/are true?

- (A)  $e^{-X}$  follows  $U(-1, 0)$  distribution
- (B)  $1 - e^{-X}$  follows  $U(0, 2)$  distribution
- (C)  $2e^{-X} - 1$  follows  $U(-1, 1)$  distribution
- (D) The probability mass function of  $Y = [X]$  is  $\Pr(Y = k) = e^{-k}(1 - e^{-1})$  for  $k = 0, 1, 2, \dots$ , where  $[X]$  denotes the largest integer not exceeding  $x$

(GATE ST 2023)

**Solution:** Let  $Y \sim U(a, b)$ , then

$$p_Y(y) = \begin{cases} \frac{1}{b-a} & a < y < b \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

and for  $a < y < b$

$$F_Y(y) = \Pr(Y \leq y) \quad (3)$$

$$= \int_a^y \frac{1}{b-a} dy \quad (4)$$

$$= \frac{y-a}{b-a} \quad (5)$$

Similarly, for  $x \geq 0$

$$F_X(x) = \Pr(X \leq x) \quad (6)$$

$$= \int_0^x e^{-x} dx \quad (7)$$

$$= 1 - e^{-x} \quad (8)$$

- (A)  $Y = e^{-X} = U(a, b)$   
for  $a < y < b$

$$F_Y(y) = \Pr(e^{-X} \leq y) \quad (9)$$

$$= \Pr(X \geq -\ln y) \quad (10)$$

$$= 1 - F_X(-\ln y) \quad (11)$$

$$= 1 - (1 - y) \quad (12)$$

$$= y \quad (13)$$

Comparing this with CDF of Uniform distribution,

we obtain

$$a = 0, b = 1 \quad (14)$$

$$\therefore Y \sim U(0, 1) \quad (15)$$

- (B)  $Y = 1 - e^{-X} = U(a, b)$   
for  $a < y < b$

$$F_Y(y) = \Pr(1 - e^{-X} \leq y) \quad (16)$$

$$= \Pr(e^{-X} \geq 1 - y) \quad (17)$$

$$= \Pr(X \leq -\ln(1 - y)) \quad (18)$$

$$= F_X(-\ln(1 - y)) \quad (19)$$

$$= 1 - (1 - y) \quad (20)$$

$$= y \quad (21)$$

$$\implies Y \sim U(0, 1) \quad (22)$$

- (C)  $Y = 2e^{-X} - 1 = U(a, b)$   
for  $a < y < b$

$$F_Y(y) = \Pr(2e^{-X} - 1 \leq y) \quad (23)$$

$$= \Pr\left(X \geq -\ln\left(\frac{y+1}{2}\right)\right) \quad (24)$$

$$= 1 - F_X\left(-\ln\left(\frac{y+1}{2}\right)\right) = 1 - \left(1 - \frac{y+1}{2}\right) \quad (25)$$

$$= \frac{y+1}{2} \quad (26)$$

Comparing this with CDF of Uniform distribution, we obtain

$$a = -1, b = 1 \quad (27)$$

$$\therefore Y \sim U(-1, 1) \quad (28)$$

- (D)  $Y = [X]$

$$\Pr(Y = k) = \Pr([X] = k) \quad (29)$$

$$= \Pr(k \leq X < k + 1) \quad (30)$$

$$= \int_k^{k+1} e^{-x} dx \quad (31)$$

$$= e^{-k}(1 - e^{-1}) \text{ for } k=0,1,2,\dots \quad (32)$$

- (E) Generation of Random Variable  $X$  in C language

(i) rand() / (double)RAND\_MAX:

This generates a random variable between 0 and RAND\_MAX and divides it by RAND\_MAX to obtain a uniform distribution between 0 and 1.

- (ii)  $-\log(\text{rand}()) / (\text{double})\text{RAND\_MAX}$  :

This transforms the uniform distribution between 0 and 1 into an exponential distribution by making the values vary from 0 to  $\infty$ .

- (iii) From these values the distributions which are to be plotted are obtained and then their CDF is plotted using python.

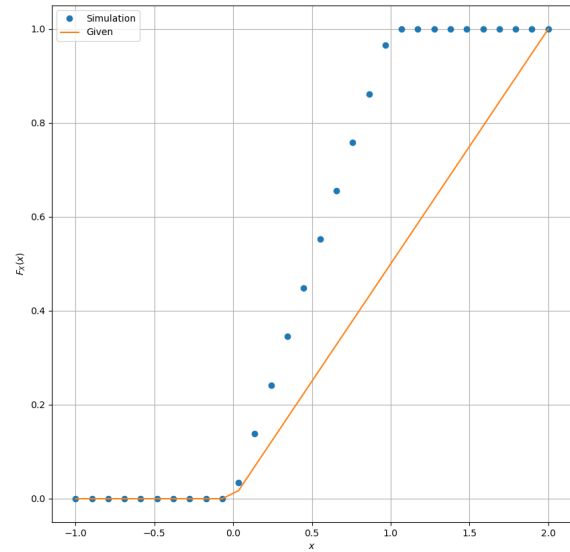


Fig. 5:  $1 - e^{-X}$  vs.  $U(0, 2)$   
Graphs don't match,  $\therefore$  wrong option

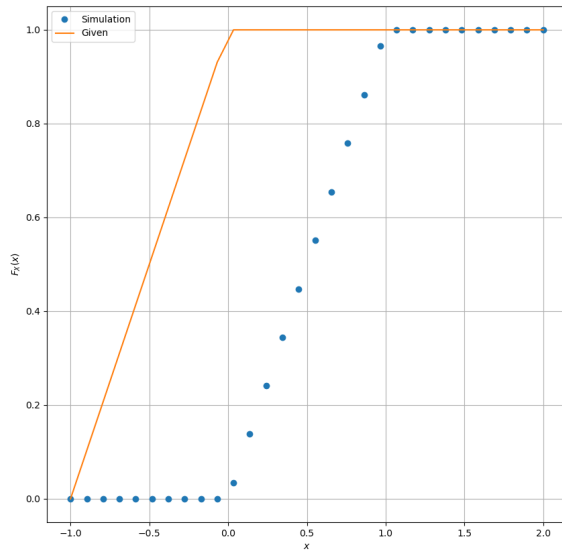


Fig. 5:  $e^{-X}$  vs.  $U(-1, 0)$   
Graphs don't match,  $\therefore$  wrong option

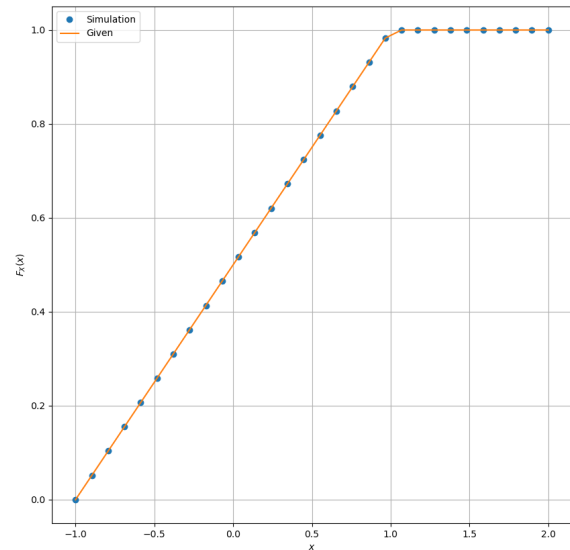


Fig. 5:  $2e^{-X} - 1$  vs.  $U(-1, 1)$   
Graphs match,  $\therefore$  correct option