College of Engineering Pune

Linear Algebra and Univariate Calculus(D.S.Y)

Tutorial 1

Basics of Matrices, System of linear equations and Determinants

A=1 1 1 1 B= 0 1 1 1
0 1 1 1   0 0 1 1
0 0 1 1   0 0 0 1
0 0 0 0   0 0 0 0

1. Let
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$
Compute A^2, A^3, A^4 and B^2, B^3. Generalize A and B to 4x4 matrix.

2. Let A be a diagonal matrix with diagonal elements $a_1, a_2, ..., a_n$. What is A^2, A^3, A^k for any positive integer k.

3. Let A be a square matrix.
   (a) If $A^2 = 0$ show that $I - A$ is invertible.
   (b) If $A^3 = 0$ show that $I - A$ is invertible.
   (c) In general, If $A^n = 0$ for some positive integer n, show that $I - A$ is invertible.

4. If the inverse of A^2 is B, show that the inverse of A is AB. (Thus A is invertible whenever A^2 is invertible)

5. (a) If A is invertible and if $AB = BC$, then prove that $B = C$.
   (b) If A is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, find an example with $AB = BC$ but $B \neq C$.

6. Give examples of A and B such that
   (a) $A + B$ is not invertible although A and B are invertible.
   (b) $A + B$ is invertible although A and B are not invertible.
   (c) All of A, B and $A + B$ are invertible.

a A= 1 0 B = -1 0
0 1   0 -1
b A= 2 2 B = 3 -3
2 2   -3 3
c A = 1 2 B = 1 0
3 4   0 -2

7. (a) Show that for any square matrix, the matrix $A +^t A$ is symmetric.
   (b) Show that for any square matrix, the matrix $A -^t A$ is skew-symmetric.
   (c) If a matrix is skew-symmetric then what can you say about its diagonal entries.
   (d) Show that any square matrix can always be written as sum of symmetric and skew-symmetric matrix.

8. Let A be a skew-symmetric matric with odd order then what can you say about its determinant? Determinant is zero

9. True or false, with reason if true or counterexample if false
   (a) If A and B are symmetric then AB is symmetric. False

(b) If A and B are invertible then BA is invertible. True

10. Let A and B be two matrices of the same size. We say that A is **similar** to B if there exists an invertible matrix T such that $B = TAT^{-1}$. Suppose this is the case. Prove:

    (a) B is similar to A.

    (b) A is invertible iff B is invertible.

    (c) ${}^tA$ is similar to ${}^tB$.

    (d) Suppose $A^n = 0$ and B is an invertible matrix of the same size as A. Show that $(BAB^{-1})^n = 0$.

11. Find solutions to following systems using Gauss Elimination method.

    (a) $3x + y + z = 0$

    (b) $-2x + 3y + z + 4w = 0$
    $x + y + 2z + 3w = 0$
    $2x + y + z - 2w = 0$

    (c) $3x + 4y - 2z = 0$
    $x + y + z = 0$
    $-x - 3y + 5z = 0$

    (d) $-3x + y + z = 0$
    $x - y + z - 2w = 0$
    $-x + y - 3w = 0$

    (e) $x + y + z + w = 0$
    $x + y + z - w = 4$
    $x + y - z + w = -4$
    $x - y + z + w = 2$

    (f) $2x - 2y + 4z + 3w = 9$
    $x - y + 2z + 2w = 6$
    $2x - 2y + z + 2w = 3$
    $x - y + w = 2$

12. Determine the values of a and b for which the system has (i) No solution (ii) Infinite number of solutions (iii) Unique solution.

    (a) $x + 2y + 3z = 6$
    $x + 3y + 5z = 9$
    $2x + 5y + az = b$

    (b) $2x + 3y + 5z = 9$
    $7x + 3y - 2z = 8$
    $2x + 3y + az = b$

a-> let a = 8 and b !=15
System has no soln
let a!=8 and b = 15
System has unique soln
let a=8 and b=15
system has infinitely many soln

b-> let a! = 5 and b =9
System has unique soln
let a=5 and b! = 9
System has no soln
let a!=5 and b!=9
system has unique soln x = 6/5
let a=5 and b=9
system has infinitely many solution

13. Find inverses of the following matrices, if exists.

    (a) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

    (b) $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$

a-> 3 1 3/2
-5/4 -1/4 -3/4
-1/4 -1/4 -1/4

b-> -2 4/5 9/5
-3 -4/5 -14/5
-1 1/5 6/5

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Tutorial 2

Vector Space, Subspace, Linear combination, Linearly dependence and Independence

1. Show that $\mathbb{R}^n$ forms a vector space over R.
2. Show that set of all nxn matrices over $\mathbb{R}$ i.e., $M_{n\times n}(\mathbb{R})$ forms a vector space over $\mathbb{R}$.
3. Show that set of all continuous functions from set of real numbers to set of real numbers i.e., $C(\mathbb{R}, \mathbb{R})$ forms a vector space over $\mathbb{R}$.
4. Which of the following forms subspaces?

Highlighted ones form subspaces

   (a) $S_1 = \{(x, y) \in \mathbb{R}^2 | x = y\}$
   (b) $S_2 = \{(x, y) \in \mathbb{R}^2 | x = 2y\}$
   (c) $S_3 = \{(x, y) \in \mathbb{R}^2 | x = cy, c \in \mathbb{R}\backslash 0\}$
   (d) $S_4 = \{(x, y) \in \mathbb{R}^2 | x = y + 1\}$
   (e) $S_5 = \{(x, y) \in \mathbb{R}^2 | x = y + c, c \in \mathbb{R}\backslash 0\}$
   (f) $S_6 = \{(x, y, z) \in \mathbb{R}^3 | x + y + z = 0\}$
   (g) $S_7 = \{(x, y, z) \in \mathbb{R}^3 | x = y \text{ and } 2y = z\}$
   (h) $S_8 = \{(x, y, z) \in \mathbb{R}^3 | x + y = 3z\}$
   (i) $S_9 = \{(x, y, z) \in \mathbb{R}^3 | x = 0\}$

5. Which of the following forms a subspace for $M_{n\times n}(\mathbb{R})$?

The ones highlighted with yellow and green form subspaces and rest dont

   (a) Set of upper triangular matrices.
   (b) Set of lower triangular matrices.
   (c) Set of diagonal matrices.
   (d) Set of scalar matrices.
   (e) Set of matrices whose determinant is non-zero.
   (f) Set of matrices whose determinant is zero.

(g) Set of matrices whose trace (Sum of diagonal entries) is zero.

(h) Set of matrices whose trace (Sum of diagonal entries) is non-zero.

(i) Set of symmetric matrices.

(j) Set of skew-symmetric matrices.

6. Which of the following forms subspaces for $C(\mathbb{R}, \mathbb{R})$.

   (a) $S_9 = \{f \in C(\mathbb{R}, \mathbb{R}) | f(0) = 0\}$

   (b) $S_9 = \{f \in C(\mathbb{R}, \mathbb{R}) | f(0) = 1\}$

   (c) $S_9 = \{f \in C(\mathbb{R}, \mathbb{R}) | f(x) = f(-x), \forall x \in \mathbb{R}\}$

   (d) $S_9 = \{f \in C(\mathbb{R}, \mathbb{R}) | f(x) = -f(-x), \forall x \in \mathbb{R}\}$

   (e) $S_9 = \{f \in C(\mathbb{R}, \mathbb{R}) | f(x+1) = f(x), \forall x \in \mathbb{R}\}$

7. Which of the following are subspaces of $\mathbb{R}^\infty$.

   (a) All sequence like $(1, 0, 1, 0, 1, 0, ...)$ i.e., zero at even positions.

   (b) All sequences $(x_1, x_2, x_3, ...)$ with $x_j = 0$ from some point onwards.

   (c) All decreasing sequences: $x_{j+1} \leq x_j$ for each j. (c) doesnt form subspace

8. If U and W are subspaces of a vactor space V then show that $U \cap W$ and $U + W$ are also subspaces of V. What can you say about $U \cup W$, does it forms a subspace in general?

9. Construct a subset of the $x - y$ plane in $\mathbb{R}^2$ that is:

   (a) closed under vector addition and subtraction but not under scalar multiplication.

   (b) closed under scalar multiplication but not under vector addition.

10. Express the given vector X as a linear combination of the given vectors A,B, and find the coordinates of X with respect to A,B.

    (a) $X = {}^t(1, 0)$, $A = {}^t(1, 1)$, $B = {}^t(0, 1)$ a1=1, a2=-1

    (b) $X = {}^t(2, 1)$, $A = {}^t(1, -1)$, $B = {}^t(1, 1)$ a1=1/2, a2=3/2

    (c) $X = {}^t(1, 0, 0)$, $A = {}^t(1, 1, 1)$, $B = {}^t(-1, 1, 0)$, $C = {}^t(1, 0, -1)$ a1=a3=1, a2=-1

    (d) $X = {}^t(1, 1, 1)$, $A = {}^t(0, 1, -1)$, $B = {}^t(1, 1, 0)$, $C = {}^t(1, 0, 2)$ a1=a3=1, a2=0

11. Check linear independence and dependence of following vectors.

Highlighted ones are linearly dependent. Rest are linearly independent

(a) ${}^t(1, 2, 3), {}^t(0, 0, 0), {}^t(1, 0, 0)$.

(b) ${}^t(1, 1, 0), {}^t(1, 1, 1), {}^t(0, 1, -1)$.

(c) ${}^t(0, 1, 1), {}^t(0, 2, 1), {}^t(1, 5, 3)$.

(d) ${}^t(1, 1, 2), {}^t(1, 2, 3), {}^t(2, 2, 4)$.

(e) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix}$

12. Show that v_1, v_2, v_3 are independent but v_1, v_2, v_3, v_4 are dependent:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_4 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

13. If w_1, w_2, w_3 are independent vectors, show that the differences $v_1 = w_2 - w_3$, $v_2 = w_1 - w_3$, and $v_3 = w_1 - w_2$ are dependent. (Hint: Find a combination of the $v's$ that gives 0.)

14. If w_1, w_2, w_3 are independent vectors, show that the sum $v_1 = w_2 + w_3$, $v_2 = w_1 + w_3$, and $v_3 = w_1 + w_2$ are linearly independent.

15. Suppose v_1, v_2, v_3, v_4 are vectors in $\mathbb{R}^3$.

(a) These four vectors are dependent because ...

(b) The two vectors v_1 and v_2 will be dependent if ...

(c) The vectors v_1 and $(0, 0, 0)$ are dependent because...

16. True or false. Justify

T (a) Subset of linearly independent set is linearly independent.

F (b) Subset of linearly dependent set is linearly dependent. $\{(2,4,6),(1,2,3)\}$

F (c) Superset of linearly independent set is linearly independent. ← same

T (d) Superset of linearly dependent set is linearly dependent.

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Tutorial 3

Spanning Set, Basis, Dimension, Rank of matrix, Application to system of linear equations

1. Describe the subspace spanned by:

   (a) vector $(1,1) \in \mathbb{R}^2$.

   (b) vector $(1,0)$ and $(1,1) \in \mathbb{R}^2$.

   (c) the two vectors $(1,1,-1)$ and $(-1,-1,1) \in \mathbb{R}^3$

   (d) the three vectors $(0,1,1)$, $(1,1,0)$ and $(0,0,0)$.

   (e) the vector $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

   (f) the vectors $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

   (g) the vector $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

   (h) the vectors $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

2. What is the dimension of the following spaces.

   (a) Subspace spanned by $B = \{(1,1,0),(0,1,1),(1,0,1)\}$ in $\mathbb{R}^3$ 3

   (b) Subspace spanned by $B = \{(1,1),(2,1)\}$ in $\mathbb{R}^2$ 2

   (c) Subspace spanned by $B = \{(1,1,0,0),(0,0,1,1),(1,0,1,0)\} \subset \mathbb{R}^4$ 3

   (d) all 2x2 matrices. 4

   (e) all mxn matrices.

   (f) all 2x2 symmetric, n(n+1)/2 skew-symmetric, upper n(n+1)/2 triangular, lower triangular,trace 0, scalar, diagonal matrices. Generalize this to nxn matrices. n2-1 2

3. Find a basis for each of these subspaces of $\mathbb{R}^4$:

(a) All vectors whose components are equal.

(b) All vectors whose components add to zero.

4. Find rank of the following matrix:

(a) $\begin{bmatrix} 2 & 1 & 3 \\ 7 & 2 & 0 \end{bmatrix}$ 2

(b) $\begin{bmatrix} -1 & 2 & -2 \\ 3 & 4 & -5 \end{bmatrix}$ 2

(c) $\begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 7 \end{bmatrix}$ 3

(d) $\begin{bmatrix} 2 & 0 & 0 \\ -5 & 1 & 2 \\ 3 & 8 & -7 \end{bmatrix}$ 3

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Tutorial 4

Linear Mappings, Kernel and image of a linear map, Rank nullity theorem

1. Let $T : V \to W$ be a linear transformation. Show that:

   (a) $T(0) = 0$.

   (b) $T(-v) = -T(v)$ for all $v \in V$

2. Determine which of the following mappings F are linear. If linear, then find its kernel and image space. Also find Nullity and rank and hence verify Rank-Nullity theorem.

   Onto (a) $F : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $F(x, y, z) = (x, z)$ Dim ker-1 img-2

   1-1; onto (b) $F : \mathbb{R}^4 \to \mathbb{R}^4$ defined by $F(x, y, z, w) = (-x, -y, -z, -w)$ Dim ker-0 img-4

   (c) $F : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $F(x, y, z) = (x, y, z) + (0, 1, 0)$

   1-1; onto (d) $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x, y) = (x - y, 2y)$ Dim ker-0 img-2

   (e) $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x, y) = (xy, x + y)$ Dim ker-0 img-2

   1-1; onto (f) $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x, y) = (y, x)$ Dim ker-0 img-2

   (g) $F : \mathbb{R}^2 \to \mathbb{R}$ defined by $F(x, y, z) = xy$

   (h) $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x, y) = (x, y + 1)$

   Onto (i) $F : \mathbb{R}^3 \to \mathbb{R}$ defined by $F(x, y, z) = 3x - 2y + z$ Dim ker-2 img-1

   (j) Let D be a derivative map from set of differentiable functions to set of differentiable functions i.e., $D(f) = \frac{df}{dx}$.

   (k) Let D^2 be a double derivative map from set of twice differentiable functions to set of twice differentiable functions i.e., $D^2(f) = \frac{d^2 f}{dx^2}$.

   (l) Let M be the space of all 2 x 2 matrices. Let, $P : M \to M$ be a map such that $P(A) = \frac{A + {}^t A}{2}$. Generalize to nxn matrices. Dim ker-1 img-3

   (m) Let M be the space of all 2 x 2 matrices. Let, $P : M \to M$ be a map such that $P(A) = \frac{A - {}^t A}{2}$. Generalize to nxn matrices. Dim ker-3 img-1

   Onto (n) Let M be the space of all 2 x 2 matrices. Let, $P : M \to M$ be a map such that $P(A) = trace(A)$. Generalize to n x n matrices.

   Dim ker-3 img-1

3. Using Kernel classify whether above functions are one-one or not. Further, using Rank-Nullity theorem conclude whether function is onto or not.

4. What is the dimension of the space of solutions of the following systems of lineat equations? In each case, find a basis for the space of solutions.

enter in the dimention etc cal without including the zeros

(a) Basis : {1, -2 , 0}
Dim : 1

$$2x + y - z = 0$$
$$2x + y + z = 0$$

(b) Solution set : {0,0,0}
Dim : 0

$$x + y + z = 0$$
$$x - y = 0$$
$$y + z = 0$$

(c) Dim : 0

$$2x - 3y + z = 0$$
$$x + y - z = 0$$
$$3x + 4y = 0$$
$$5x + y + z = 0$$

(d) Basis : {(-7+pi)/18, (2+pi)/9, 1
Dim : 1

$$4x + 7y - \pi z = 0$$
$$2x - y + z = 0$$

5. Let A be a fixed mxn matrix. Let. $T : \mathbb{R}^n \to \mathbb{R}^m$ be a map defined as: $T(X) = AX$ where X is a n x 1 vector in $\mathbb{R}^n$. Show that T is a linear transformation.

6. In above example, Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 7 \end{bmatrix}$. Find Null space of T, Image space of T. Hence conclude Nullity(T) and Rank(T). Further verify Rank-Nullity theorem. Dim ker-1 img-2

7. Take a 3 x 4 matrix of your choice and do the above things. (Don't take a null matrix :) . Also try to take distinct entries!)

8. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $T(1, 0) = (1, 1)$ and $T(0, 1) = (2, 3)$. Find $T(a, b)$ for any $(a, b) \in \mathbb{R}^2$. Hence calculate image of $(3, 7)$. (17, 24)

9. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $T(1, 0, 0) = (1, 1, 0)$, $T(0, 1, 0) = (2, 3, 0)$ and $T(0, 0, 1) = (1, 0, 5)$. Find $T(a, b, c)$ for any $(a, b, c) \in \mathbb{R}^3$. Hence calculate image of $(3, 7, 1)$. (18,24,5)

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Tutorial 5

Matrices associated to a linear map, Eigenvalues and Eigenvectors.

1. Find the matrix associated with the following linear maps with respect to standard basis.
   (a) $F : \mathbb{R}^4 \to \mathbb{R}^2$ given by $F(x_1, x_2, x_3, x_4) = (x_1, x_2)$.(the projection.)
   (b) The projection from $\mathbb{R}^4$ to $\mathbb{R}^3$.
   (c) $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x, y) = (3x, 3y)$.
   (d) $F : \mathbb{R}^n \to \mathbb{R}^n$ given by $F(X) = 7X$.
   (e) $F : \mathbb{R}^4 \to \mathbb{R}^2$ given by $F(X) = cX$, where $c \in \mathbb{R}$.
   (f) Find matrices with respect to standard basis for the tansformations given in Question 1 of tut 4.

2. Let V be the vector space generated by the three functions $f_1(t) = 1, f_2(t) = t, f_3(t) = t^2$. Let $D : V \to V$ be the derivative. What is the matrix of D with respect to the basis $\{f_1, f_2, f_3\}$.

3. Let V be the vector space generated by two functions $f_1(t) = cos\ t$ and $f_2(t) = sin\ t$. Let D be the derivative. What is the matrix of D with respect to the basis $\{f_1, f_2\}$.

eigen values of 7I = (-4,-5)
eigen vectors of 7I = (-½ , 1) (-1,1)
They are related bcoz they are same.

4. Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$. Compute eigenvalues and eigenvectors of $A - 7I$. How are they related to those of A.

5. Verify that sum of eigenvalues of A(above) is equal to trace of A and product of eigenvalues of A is equal to determinant of A. Is this true in general?

6. Prove that eigenvalues of a matrix and its transpose are always same.

7. Prove that similar matrices have same eigenvalues. What can you say about eigenvectors?

M^{-1} = 1/lamda
They have same eigen vectors

8. If a matrix M has λ as an eigenvalue then what can say about eigenvalue of M^{-1}. What about eigenvectors of M and M^{-1}?

9. If a matrix M has λ as an eigenvalue then what can say about eigenvalue of kM where k is some real number. What about eigenvectors of M and kM?

KM has eigen value K.lamda
both have same eigen vectors.

Lamda1= (3,2)
Lamda2= (2,3)

10. Consider a 2×2 matrix whose trace is 5 and determinant is 6. Find its eigenvalues.

11. For the following matrices:

   (a) Compute real eigenvalues and eigenvectors.
   (b) Write down algebraic and geometric multiplities for each eigenvalues.
   (c) Are the matrices diagonalizable? Justify. Further write down the diagonal matrix D and the invertible matrix P.

(a) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $(Rotation)$

(b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $(Projection)$

(c) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $(Reflection)$

(d) $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ $(stretching\ in\ x-direction)$

(e) $\begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$

(f) $\begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix}$

(g) $\begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$

(h) $\begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

(i) $\begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$

Put directly in the diagonal or not calculator to ge the values

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Tutorial 6

Application of derivatives.

1. Find absolute maximum and minimum values of each function on the given interval. Then graph the function. Identify the points on the graphs where the absolute extreme occurs.

just put in the calculator directly to get the values

-1,-1/6 (a) $f(x) = \frac{-1}{x+3}, -2 \leq x \leq 3$

0,2 (b) $f(x) = x^{1/3}, -1 \leq x \leq 8$

0,-3 (c) $f(x) = -3x^{2/3}, -1 \leq x \leq 1$

(d) $f(x) = \sqrt{4 - x^2}, -2 \leq x \leq 1$

(e) $f(\theta) = sin\theta, -\pi/2 \leq \theta \leq 5\pi/6$

(f) $3x^4 - 16x^3 + 18x^2, -1 \leq x \leq 4$

2. Identify the largest possible domain of the following functions. Find the extreme values of the functions and where they occur.

cp at 2 val 1 (a) $f(x) = 2x^2 - 8x + 9$

cp at sqrt(2/3), -sqrt 2/3 (b) $f(X) = x^3 - 2x + 4$

(c) $f(x) = \sqrt{x^2 - 1}$

(d) $f(x) = 1/\sqrt{x^2 - 1}$

(e) $f(x) = x/(x^2 + 1)$

(f) $f(X) = e^x$

just put in critical calculator
x = critical point
f(x)= val of CP

3. Find the set of critical points and determine the local extreme values.

0,-4/5 (a) $f(x) = x^{2/3}(x + 2)$

0,3,12/5 (b) $f(x) = x^2\sqrt{3 - x}$

(c) $f(x) = x|x| - x$

(d) $f(x) = \{ \begin{array}{cc} 3 - x & if\ x > 0 \\ 3 + 2x - x^2 & if\ x \leq 0 \end{array}$

4. Show that equation $x^3 + x - 1 = 0$ has exactly one real root.

5. Let $f(x) = tanx$ on $[0, \pi]$. Find the critical point if exist; and hence extreme values of f.

6. Show that the 5 is a critical point of the function $f(x) = 2 + (x - 5)^3$ but f does not have a local extreme value at 5.

7. Sketch a graph of a function

   (a) has local maximum at 2 and is differentiable at 2.

(b) has local maximum at 2 and it is continuous but not differentiable at 2.

(c) has local maximum at 2 but not continuous at 2.

8. Use LMVT to conclude that the given function f which satisfies all the conditions do not exists: $f''(x) > 0, \forall x \in \mathbb{R}$ and $f'(0) = 1, f'(1) = 1$.

9. Assume that f is continuous on $[a, b]$ and differentiable on (a, b). Also assume that $f(a)$ and $f(b)$ have opposite sign and $f' \neq 0$ between a and b. Show that $f(x) = 0$ exactly once between a and b.

10. Use the Mean Value theorem to prove $|Sin\ a - Sin\ b| \leq |a - b|, \forall a, b \in \mathbb{R}$.

11. Suppose that $f(0) = -3$ and $f'(x) = -5$ for all values of x. How large can $f(2)$ possibly be?

12. Two runners start the race at the same time and finish in a tie. Prove that at some time during the race they have the same velocity.

13. Let $a > 0$ and f be continuous on $[-a, a]$. Suppose that $f'(x) \leq 1, \forall x \in (-a, a)$, if $f(a) = a$ and $f(-a) = -a$. Show that $f(0) = 0$.

14. Prove the following inequalities

    (a) $\frac{b-a}{\sqrt{1-a^2}} < Sin^{-1}b - Sin^{-1}a < \frac{b-a}{\sqrt{1-b^2}}$ for $0 < a < 1$ and $0 \leq \theta \leq 2\pi$

    (b) $tan\ x > x$ for $0 < x < \pi/2$

    (c) $\frac{x}{1+x} < log(1 + x) < x; x > 0$

15. As x moves from left to right through point $c = 2$ is the graph of $f(x) = x^3 - 3x + 2$ rising or falling?

16. For the following functions:

    (a) Find Critical points.

    (b) Find Extreme values of f.

    (c) Find intervals where f is increaing or decreasing.

    (d) Find inflection points.

    (e) Find intervals where f is concave up or concave down.

    (f) Sketch the graph of f.

(a) $f(x) = 4x^3 - x^4$.

(b) $f(x) = -x^3 + 6x^2 - 3$

(c) $f(x) = x^3 - 3x - 3$

(d) $f(x) = x(6 - 2x)^2$

(e) $f(x) = -2x^3 + 6x^2 - 3$

(f) $f(x) = 1 - 9x - 6x^2 - x^3$

(g) $f(X) = (x - 2)^3 + 1$

(h) $f(x) = x^4 - 2x^2$

find critical points

find f'

find domain

use domain and f' to find increases or decreases

find f''

use f'' = 0 to find critical points or use f' for CP of f'' on cal

use domain and f'' CP to find if converges or diverges

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Tutorial on Unt IV.

(1) Suppose that $\int_0^x f(t)dt = x^2 - 2x + 1$. Find $f(x)$. (x-1)x+c

(2) Find $f(4)$ if $\int_0^x f(t)dt = x cos \pi x$. -16

(3) Find the volumes of the solids generated by revolving the regions bounded by the lines and curves about the x- axis:

Formula :integration(pi (y)^2)

(a) $y = x^2, y = 0, x = 2.$ 32/5 pi
(b) $y = x^3, y = 0, x = 2.$ 128/7 pi
(c) $y = \sqrt{(9 - x^2)}, y = 0.$ 36 pi
(d) $y = x - x^2, y = 0.$ pi/30 Solve that to get the values for the upper and lower bound
(e) $y = \sqrt{cos\ x}, 0 \le x \le \pi/2, y = 0, x = 0.$ pi
(f) $y = sec\ x, y = 0, x = -\pi/4, x = \pi/4.$ 2pi

(4) Find the volumes of the solids generated by revolving the regions bounded by the lines and curves about the y- axis.

Formula :same as above

(a) The region enclosed by $x = \sqrt{5}y^2, x = 0, y = 1, y = -1.$ 2pi
(b) The region enclosed by $x = y^{3/2}, x = 0, y = 2.$ 4pi
(c) The region enclosed by $x = \sqrt{2sin\ 2y}, 0 \le y \le \pi/2, x = 0.$ 2pi
(d) The region enclosed by $x = \sqrt{cos(\pi y/4)}, -2 \le y \le 0, x = 0$ 4
(e) $x = 2/(y + 1), x = 0, y = 0, y = 3.$ 3pi

(5) The region in the first quadrant bounded above by the line y = 2, below by the curve $y = 2sin\ x, 0 \le x \le \pi/2$ and on the left by the y-axis, about the line $y = 2$.

(6) Find the volumes of the solids generated by revolving the regions bounded by the lines and curves about the x- axis:

Formula :integration(pi(R(y)^2 - r(x)^2))

(a) $y = x, y = 1, x = 0.$ 2pi/3
(b) $y = 2\sqrt{x}, y = 2, x = 0.$ 2pi
(c) $y = x^2 + 1, y = x + 3.$
(d) $y = 4 - x^2, y = 2 - x.$
(e) $y = sec\ x, y = \sqrt{2}, -\pi/4 \le x \le \pi/4.$
(f) $y = sec\ x, y = tan\ x, x = 0, x = 1.$

(7) The disk $x^2 + y^2 \le a^2$ is revolved about the line $x = b, (b > a)$ to generate a solid shaped like a doughnut and called a torus. Find its volume.

(8) A bowl has a shape that can be generated by revolving the graph of $y = x^2/2$ between $y = 0$ and $y = 5$ about the y- axis. Find the volume of the bowl.

(9) Find the lengths of the following curves.

Enter directly in the arc calculator

(a) $y = (1/3)(x^2 + 2)$ from $x = 0$ to $x = 3$. 4.43...
(b) $y = x^{3/2}$ from $x = 0$ to $x = 4$. 9.07..
(c) $x = (y^3/3) + (1/(4y))$ from $y = 1$ tp $y = 3$. 8.833....
(d) $x = (y^{3/2}/3) - y^{1/2}$ from $y = 1$ to $y = 9$.
(e) $y = \int_0^x tan\ t\ dt, 0 \leq x \leq \pi/6$.

(10) The graph of the equation $x^{2/3} + y^{2/3} = 1$ is one of a family of curves called astroids (not "asteroids"!) because of their starlike appearance (Use Grapher to plot). Find the length of this particular astroids.

(11) Find the area of the surface generated by revolving the given curve about the indicated axis.
(a) $y = x^2, 0 \leq x \leq 2$, x axis. 53.22
(b) $xy = 1, 1 \leq y \leq 2, y-$ axis. 2pi
(c) $x = 2\sqrt{(4 - y)}, 0 \leq y \leq 15/4$, y- axis. 81.95

(12) Find the area of the surface geoerated by revolving about the x-axis the portion of the astroid $x^{2/3} + y^{2/3} = 1$ lying in upper half plane.

(13) Evaluate the following improper integrals:

Put directly into the calculator

(a) $\displaystyle\int_0^\infty \frac{dx}{x^2+1}$
(b) $\displaystyle\int_0^4 \frac{dx}{\sqrt{(4-x)}}$
(c) $\displaystyle\int_{-1}^1 \frac{dx}{x^{2/3}}$
(d) $\displaystyle\int_0^1 \frac{dx}{\sqrt{(1-x^2)}}$
(e) $\displaystyle\int_{-\infty}^\infty \frac{2x\ dx}{(x^2+1)^2}$
(f) $\displaystyle\int_{-1}^4 \frac{dx}{|x|}$
(g) $\displaystyle\int_{-1}^\infty \frac{dx}{x^2+5x+6}$
(h) $\displaystyle\int_0^\infty \frac{dx}{(x+1)(x^2+1)}$

(14) Test the convergence of follwoing integrals:

convergent (a) $\displaystyle\int_0^\pi \frac{sin\ \theta\ d\theta}{\sqrt{(\pi-\theta)}}$
convergent (b) $\displaystyle\int_0^\pi \frac{dt}{\sqrt{t}+sin\ t}$
Divergent (c) $\displaystyle\int_0^1 \frac{dt}{t - sin\ t}$
convergent (d) $\displaystyle\int_0^2 \frac{dt}{1-t^2}$
(e) $\displaystyle\int_0^2 \frac{dt}{1-t}$ convergent
(f) $\displaystyle\int_1^\infty \frac{dt}{t^3+1}$ Divergent
(g) $\displaystyle\int_4^\infty \frac{dt}{\sqrt{t}-1}$ Divergent
(h) $\displaystyle\int_2^\infty \frac{dt}{\sqrt{(t-1)}}$ Divergent