Feedback Control System

Time Domain Analysis

Poles of a Transfer Function

The **poles** of a **transfer function** are

(1) the **values** of the **Laplace transform variable**, **s**, that **cause** the **transfer function** to become **infinite** or

(2) any roots of the **denominator** of the **transfer function** that are common to **roots** of the **numerator**.

Poles of a Transfer Function

- The first part completely **satisfy definition** of the **poles** of a **transfer function**

  i.e. the **roots** of the characteristic polynomial in the **denominator** are values of **‘s’** that make the transfer function **infinite**, so they are thus poles.

- However, **if a factor** of the **denominator** can be canceled by the **same factor** in the **numerator**, the **root** of this **factor** no longer **causes** the transfer function to **become infinite**.

- In control systems, we often **refer** to the **root** of the **canceled factor** in the **denominator** as a **pole** even though the **transfer function** will not be **infinite** at this **value**. Hence, the later part is included in the definition.

Zeros of a Transfer Function

The **zeros** of a **transfer function** are

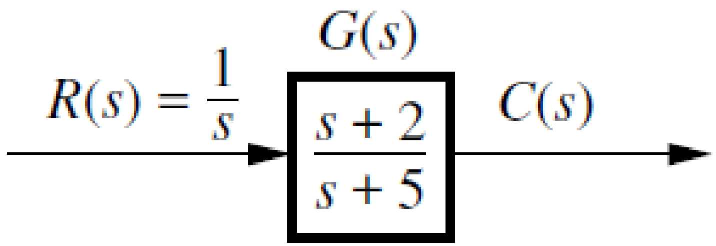
(1) the values of the **Laplace transform variable**, **s**, that **cause** the transfer function to **become zero**, or

(2) any roots of the **numerator** of the **transfer function** that are **common** to **roots** of the **denominator**.

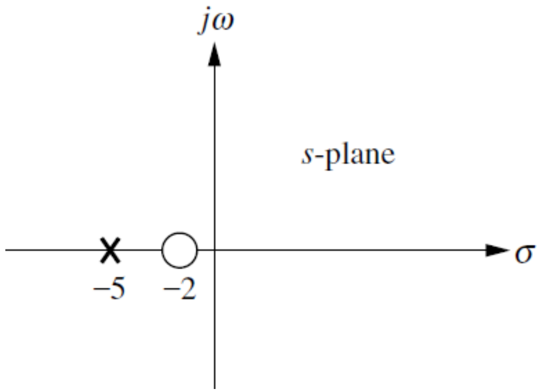
Zeros of a Transfer Function

- The first part completely satisfy definition of the poles of a transfer function i.e. the roots of the characteristic polynomial in the denominator are values of ‘**s**’ that make the transfer function zero, so they are thus zeros.
- However, if a factor of the numerator can be canceled by the same factor in the denominator, the root of this factor no longer causes the transfer function to become zero.
- In control systems, we often refer to the root of the canceled factor in the numerator as a zero even though the transfer function will not be zero at this value. Hence, the later part is included in the definition.

Poles and Zeros of a First-Order System



a



b

Given the transfer function G(s) = (s+2)/(s+5) in figure (a), a pole exists at s = - 5, and a zero exists at - 2. These pole and zero are are plotted on the complex s-plane in figure (b).

To show the properties of the poles and zeros, let us find the unit step response of the system.

Poles and Zeros of a First-Order System & System Response

$\frac{C(s)}{R(s)} = G(s) = \frac{s+2}{s+5}$ the step response means input $r(t) = u(t)$ i.e. $R(s) = \frac{1}{s}$

$$\therefore C(s) = G(s)R(s) = \frac{1}{s}\frac{s+2}{s+5} \ldots 1$$

$$\therefore C(s) = \frac{A}{s} + \frac{B}{s+5} = \frac{2/5}{s} + \frac{3/5}{s+5} \ldots 2$$

Where $A = \left.\frac{s+2}{s+5}\right|_{s\to 0} = \frac{2}{5}$ $and\ B = \left.\frac{s+2}{s}\right|_{s\to -5} = \frac{3}{5}$

Thus the output response of the first order system c(t) is

$$c(t) = \frac{2}{5} + \frac{3}{5}e^{-5t} \ldots 3$$

Poles and Zeros of a First-Order System & System Response

The output response of a system is the sum of two responses: the **forced response** and the **natural response**.

The **forced response** is also called the **steady-state response** or **particular solution**. The **natural response** is also called the **homogeneous solution**.

$= G(s) = \frac{s+2}{s+5}$ the step response means input $r(t) = u(t)$ i.e. $R(s) = \frac{1}{s}$

$$\therefore C(s) = G(s)R(s) = \frac{1}{s}\frac{s+2}{s+5} \ldots 1$$

$$\therefore C(s) = \frac{A}{s} + \frac{B}{s+5} = \frac{2/5}{s} + \frac{3/5}{s+5} \ldots 2$$

Where $A = \left.\frac{s+2}{s+5}\right|_{s\to 0} = \frac{2}{5}$ and $B = \left.\frac{s+2}{s}\right|_{s\to -5} = \frac{3}{5}$

Poles and Zeros of a First-Order System & System Response

Thus the output response of the first order system c(t) is

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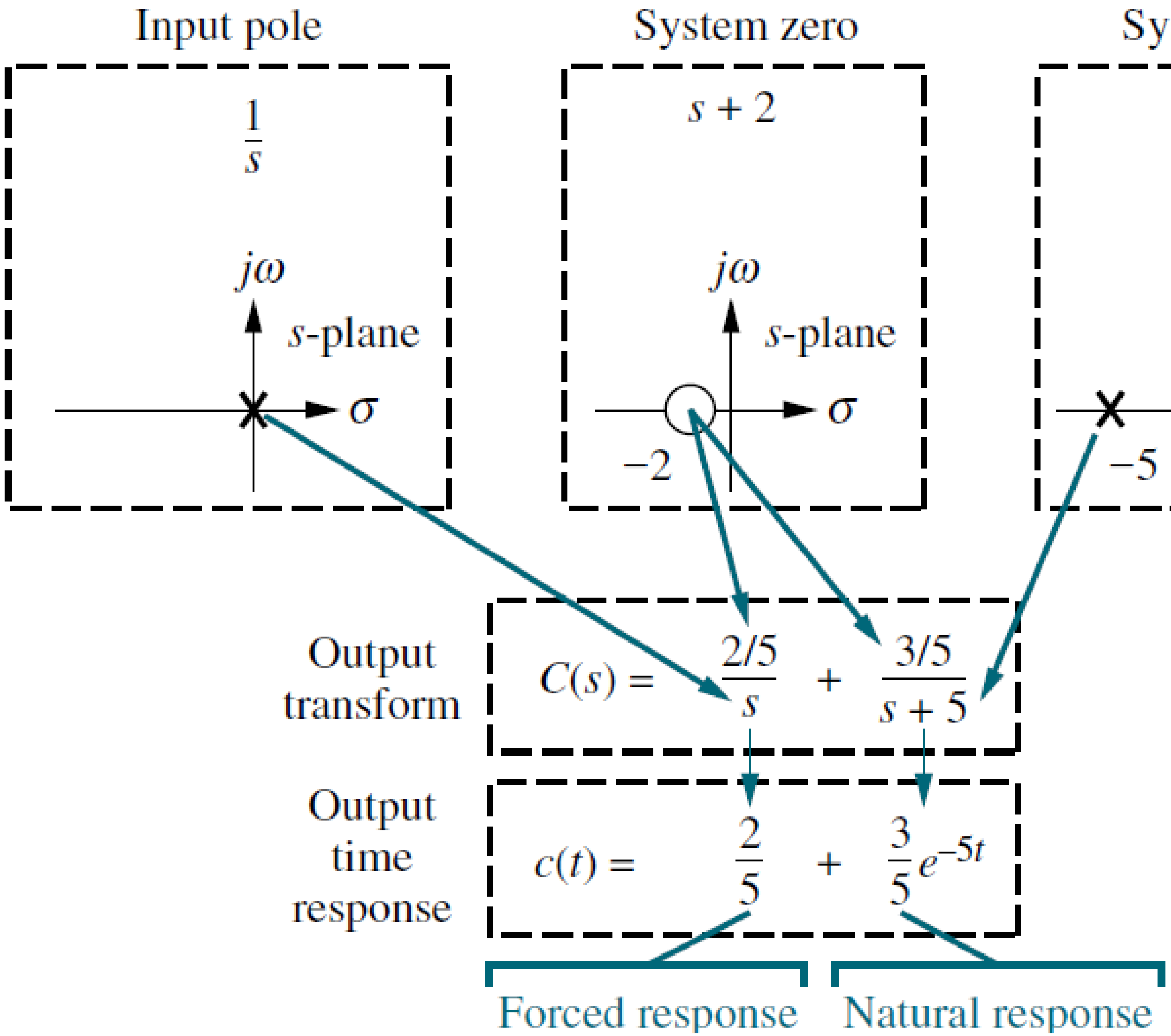
Poles and Zeros of a First-Order System & System Response



Evolution of a system response. The blue arrows show the evolution of the response component generated by the pole or zero.

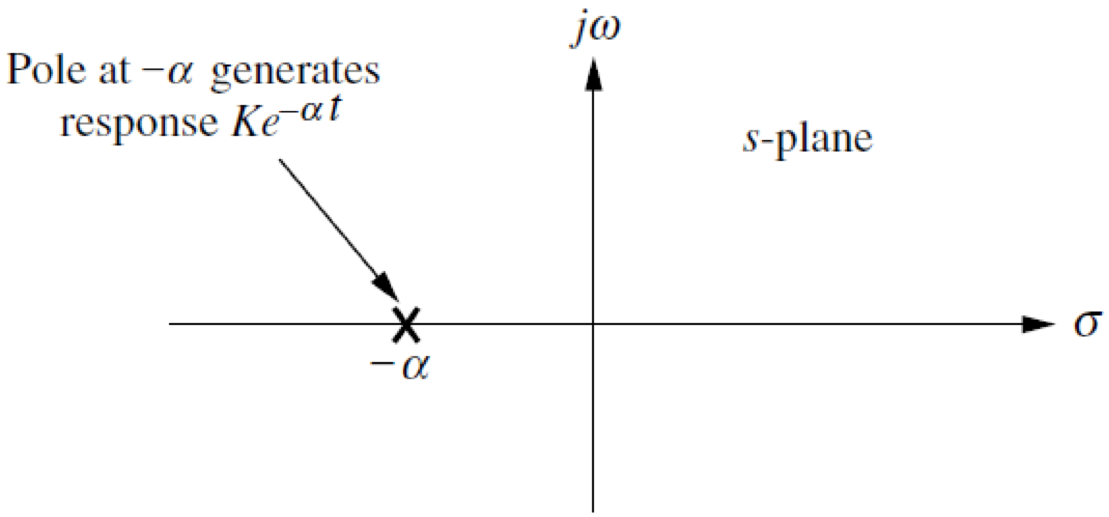
Poles and Zeros of a First-Order System & System Response

The conclusions are

1. A **pole** of the **input function** generates the form of the **forced response** (that is, the **pole at the origin generated a step function at the output**).
2. A **pole** of the **transfer function** generates the form of the **natural response** (that is, **the pole at -5 generated e^{-5t}**).
3. A pole on the real axis generates an exponential response of the form $e^{-\alpha t}$, where $-\alpha$ is the pole location on the real axis. Thus, the farther to the left a pole is on the negative real axis, the faster the exponential transient response will decay to zero.
4. The zeros and poles generate the amplitudes for both the forced and natural responses

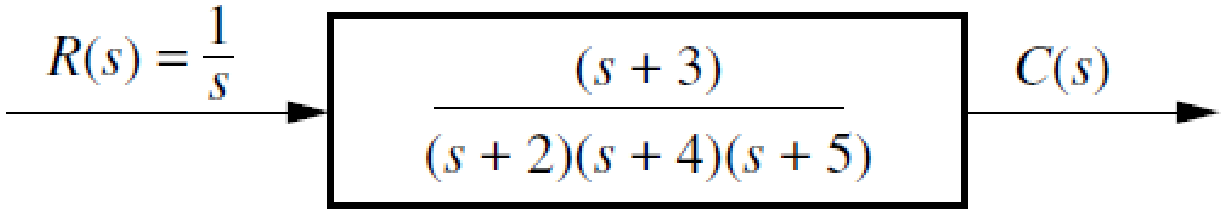
Poles and Zeros of a First-Order System & System Response

Effect of a real-axis pole upon transient response



Poles and Zeros of a First-Order System & System Response

Find the output c(t) Specify the forced and natural parts of the solution.



Poles and Zeros of a First-Order System & System Response

Find the output c(t) and specify the forced and natural parts of the solution if input is unit step.

$$G(s) = \frac{10(s+4)(s+6)}{(s+1)(s+7)(s+8)(s+10)}$$

First-Order Systems without zero

A first-order system without zeros can be described by the transfer function $G(s) = \frac{a}{(s+a)}$. If the input is a unit step, $R(s) = \frac{1}{s}$

the Laplace transform of the step response is C(s), where

$$C(s) = R(s)G(s) = \frac{a}{s(s+a)}$$

Taking the inverse transform, the step response is given by

$$c(t) = c_f(t) + c_n(t) = 1 - e^{-at} \quad \ldots 1$$

where the input pole at the origin generated the forced response **$c_f(t) = 1$**, and the system pole at – **a** generated the natural response **$c_n(t) = e^{-at}$**. Let us examine the significance of parameter a, the only parameter needed to describe the transient response.

First-Order Systems without zero

The parameter **a** is the only parameter needed to describe the transient response. When t = 1/a

$$e^{-at}\Big|_{t=1/a} = e^{-1} = 0.37 \quad \dots 2$$

$$c(t)\Big|_{t=1/a} = 1 - e^{-at}\Big|_{t=1/a} = 1 - 0.37 = 0.63 \quad \dots 3$$

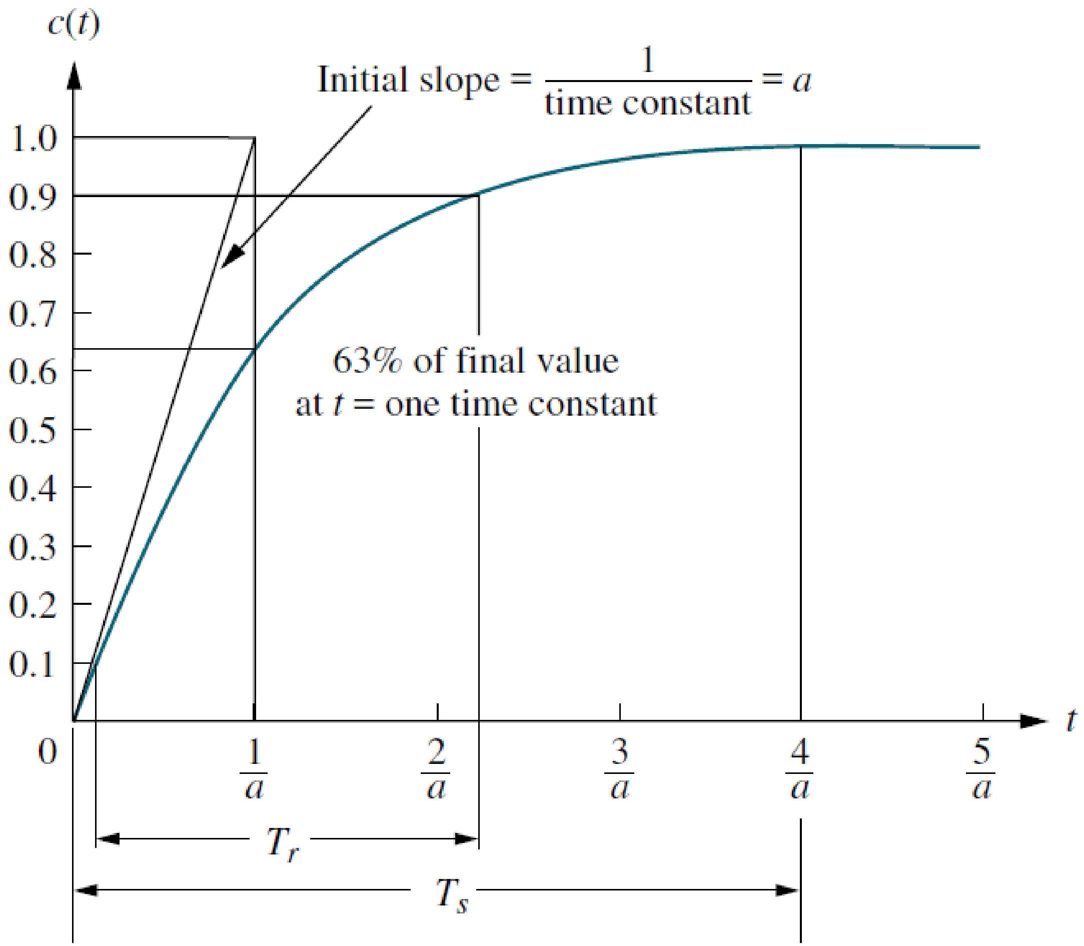
Time constant

The **1/a** is time constant of the response.

From equation $e^{-at}|_{t=1/a} = e^{-a} = 0.37$, the time constant can be described as the time for e^{-at} to decay to 37% of its initial value.

Alternately, from equation $c(t)|_{t=1/a} = 1 - e^{-at}|_{t=1/a} = 1 - 0.37 = 0.63$ the time constant is the time it takes for the step response to rise to 63% of its final value.

Time constant



First-order system response to a unit step

Time constant

The reciprocal of the time constant has the units (1/seconds), or frequency.

Thus, the parameter **'a'** the exponential frequency.

The derivative of $\mathbf{e^{-at}}$ is **'-a'** when t = 0, a is the initial rate of change of the exponential at t = 0.

Thus, the time constant can be considered a transient response specification for a first order system, since it is related to the speed at which the system responds to a step input.

The time constant can also be evaluated from the pole plot. Since the pole of the transfer function is at **'-a'**, we can say the pole is located at the reciprocal of the time constant, and the farther the pole from the imaginary axis, the faster the transient response.

Rise Time, T_r

Rise time is defined as the time for the waveform to go from 0.1 to 0.9 of its final value. Rise time is found by solving equation $c(t) = c_f(t) + c_n(t) = 1 - e^{-at}$ for the difference in time at c(t) = 0:9 and c(t) = 0:1. Hence

$$t_r = \frac{2.31}{a} - \frac{0.11}{a} = \frac{2.2}{a}$$

Settling Time, T_s

Settling time is defined as the time for the response to reach, and stay within, 2% of its final value.

Letting **c(t) = 0.98** in equation $c(t) = c_f(t) + c_n(t) = 1 - e^{-at}$ and solving for time, t, we find the settling time to be

$$T_s = \frac{4}{a}$$

Transfer function from laboratory test

It is not always possible or practical to obtain a system's transfer function analytically. Because the system is closed, and the component parts are not easily identifiable.

Since the transfer function is a representation of the system from input to output, the system's step response can help to find out the transfer function even though the inner construction is not known.

With a step input, we can measure the time constant and the steady-state value, from which the transfer function can be calculated

Transfer function from laboratory test

Consider a simple first-order system, $G(s) = \frac{K}{s+a}$, whose step response is

$$C(s) = \frac{K}{s(s+a)} = \frac{K/a}{s} - \frac{K/a}{(s+a)}$$

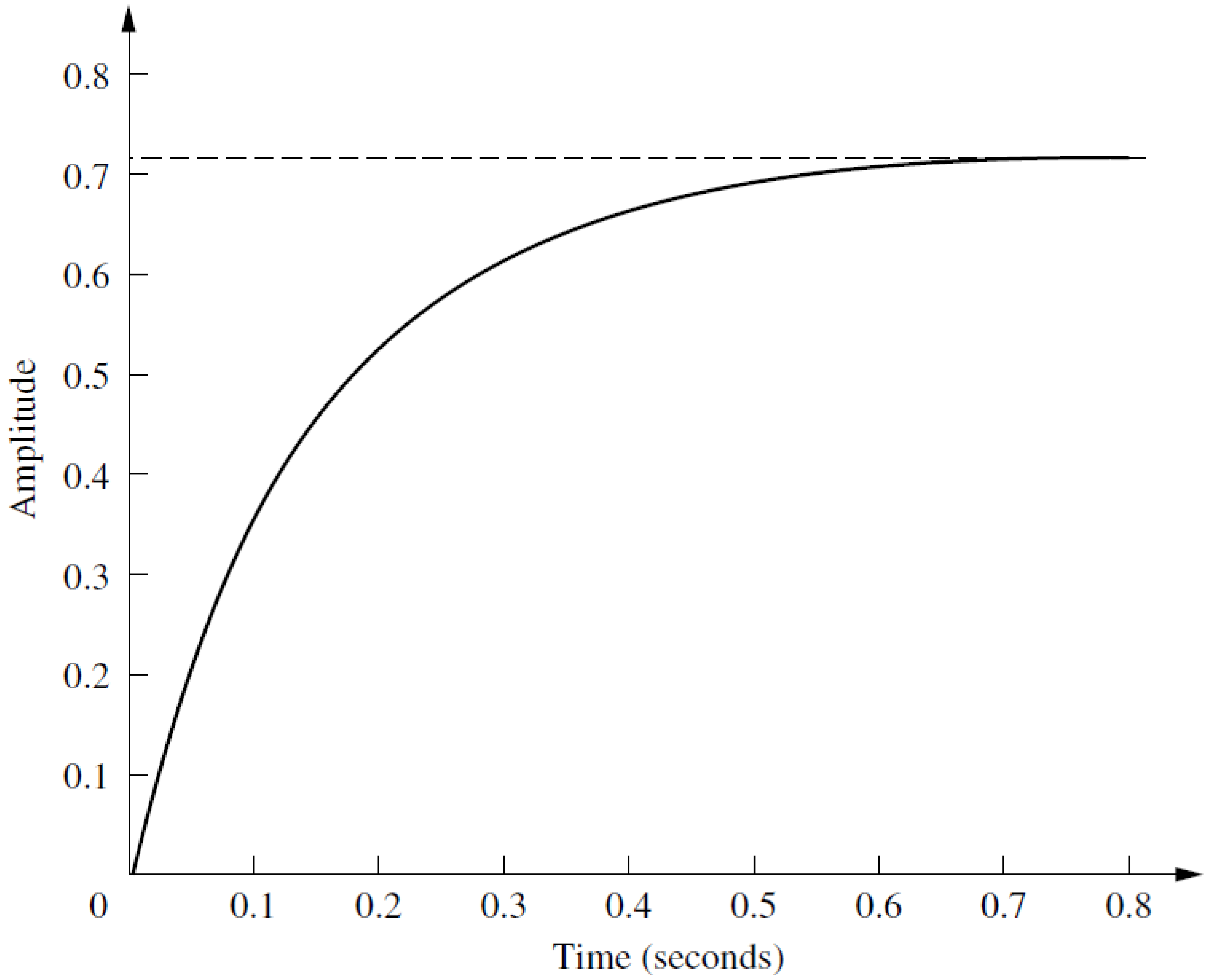
Now find out the values of identify **K** and **a** from laboratory testing, we can obtain the transfer function of the system.

Let the unit step response is given by the figure as shown. It is the first-order characteristics as no overshoot and nonzero initial slope.

From the response, we measure the time constant, that is, the time for the amplitude to reach 63% of its final value. Since the final value is about 0.72, the time constant is evaluated where the curve reaches 0.63 x 0.72 = 0.45, or about 0.13 second.

Hence, a = 1/0.13 = 7.7

Transfer function from laboratory test



Laboratory results of a system step response test

Transfer function from laboratory test

To find K, we realize from $C(s) = \frac{K}{s(s+a)} = \frac{K/a}{s} - \frac{K/a}{(s+a)}$ that the forced response reaches a steady state value of **K/a = 0.72**.

Substituting the value of **a**, we find **K = 5.54**.

Thus, the transfer function for the system is $G(s) = \frac{5.54}{(s+7.7)}$.

It is interesting to note that the response shown in the figure was generated using the transfer function $G(s) = \frac{5}{(s+7)}$

Transfer function from laboratory test

A system has a transfer function, $\mathrm{G}(\mathrm{s}) = \frac{50}{(s+50)}$. Find the time constant Tc, settling time Ts, and rise time Tr.

Second-Order Systems

Let us now extend the concepts of poles and zeros and transient response to second order systems.

Compared to the simplicity of a first-order system, a second-order system exhibits a wide range of responses

Whereas varying a first-order system's parameter simply changes the speed of the response, changes in the parameters of a second-order system can change the form of the response.

Second-Order Systems

Consider a general second order system of the form of $G(s) = \frac{b}{s^2+as+b}$

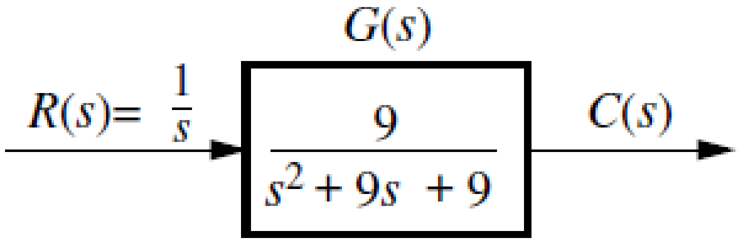
The term in the numerator is simply a scale or input multiplying factor that can take on any value without affecting the form of the derived results.

By assigning appropriate values to parameters **a** and **b**, we can show all possible second-order transient responses.

The unit step response then can be found using C(s) = R(s)G(s), where R(s) = 1/s.

The possible step response are based on the values of the parameters **a** and **b** are Overdamped, Underdamped, Undamped, and Critically damped.

Overdamped Response



For this second order system the transfer function $G(s) = \frac{9}{s^2+9s+9}$ and the response is

$$C(s) = \frac{1}{s}\frac{9}{s^2+9s+9} = \frac{9}{s(s+7.854)(s+1.146)}$$

The output response C(s) has a pole at the origin that comes from the unit step input and two real poles that come from the system.

The input pole at the origin **generates the constant forced response**;

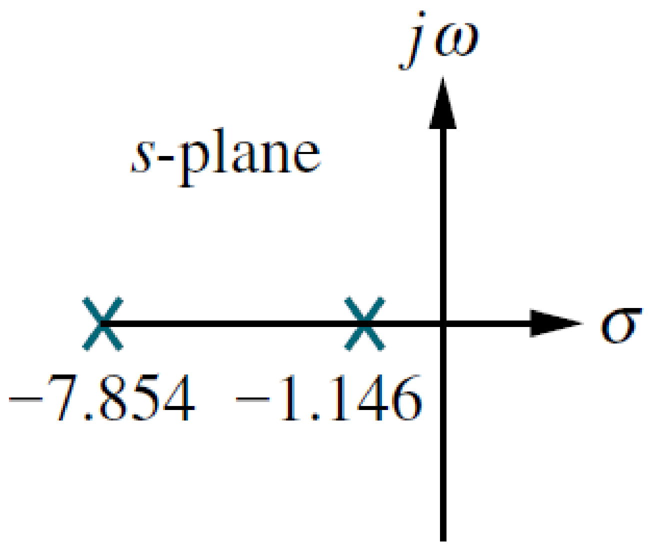
and the two system poles on the real axis **generates an exponential natural response** whose exponential frequency is equal to the pole location

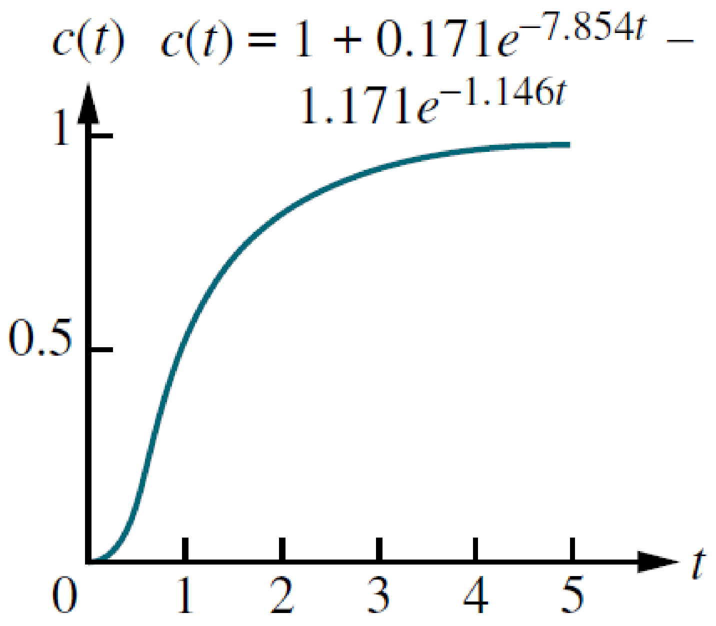
Overdamped Response

The output initially could have been written as $c(t) = K_1 + K_2 e^{-7.854t} + K_3 e^{-7.854t}$

This response, shown in following figure is called overdamped.

From the poles zero plot we can easily find out or visualized the nature of the response without the tedious calculation of the inverse Laplace transform.

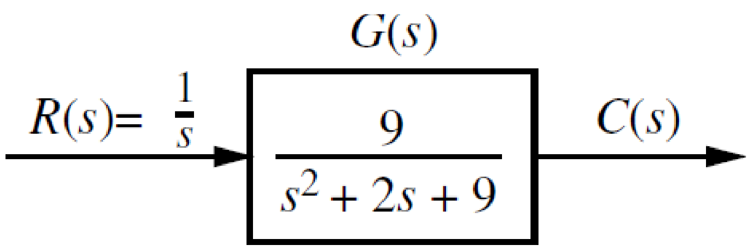




Pole zero plot

overdamped response

Underdamped Response



For this second order system the transfer function $G(s) = \frac{9}{s^2+2s+9}$ and the response is

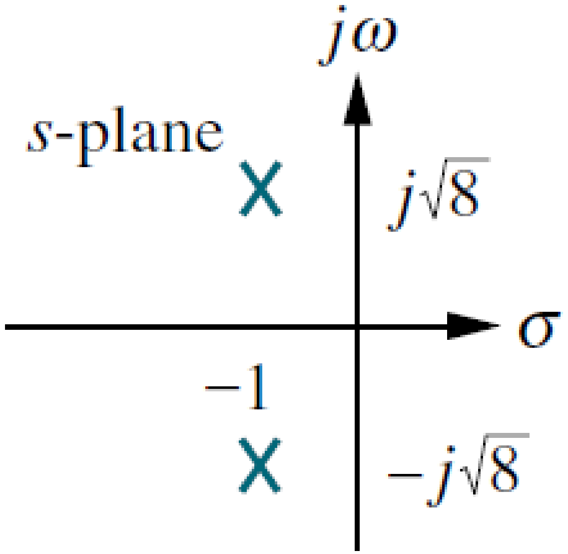
$$C(s) = \frac{1}{s}\frac{9}{s^2+2s+9}$$

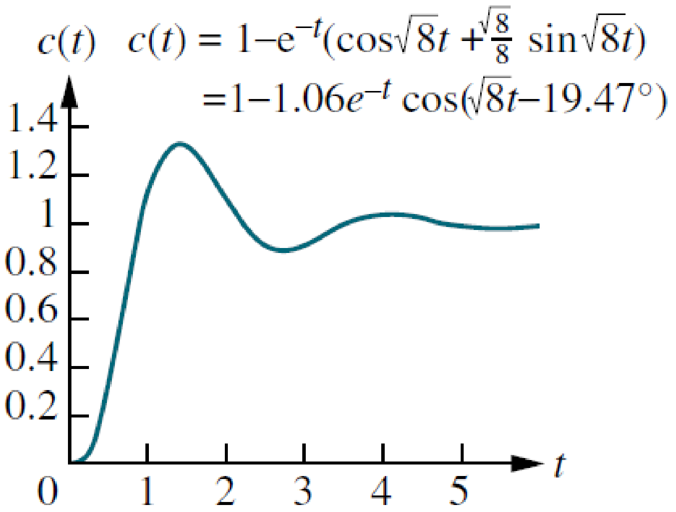
This function has a pole at the origin that comes from the unit step input and two complex poles that come from the system.

The input pole at the origin **generates the constant forced response**;

and the two system poles on the real axis **generates an exponential natural response** whose exponential frequency is equal to the pole location

Underdamped Response





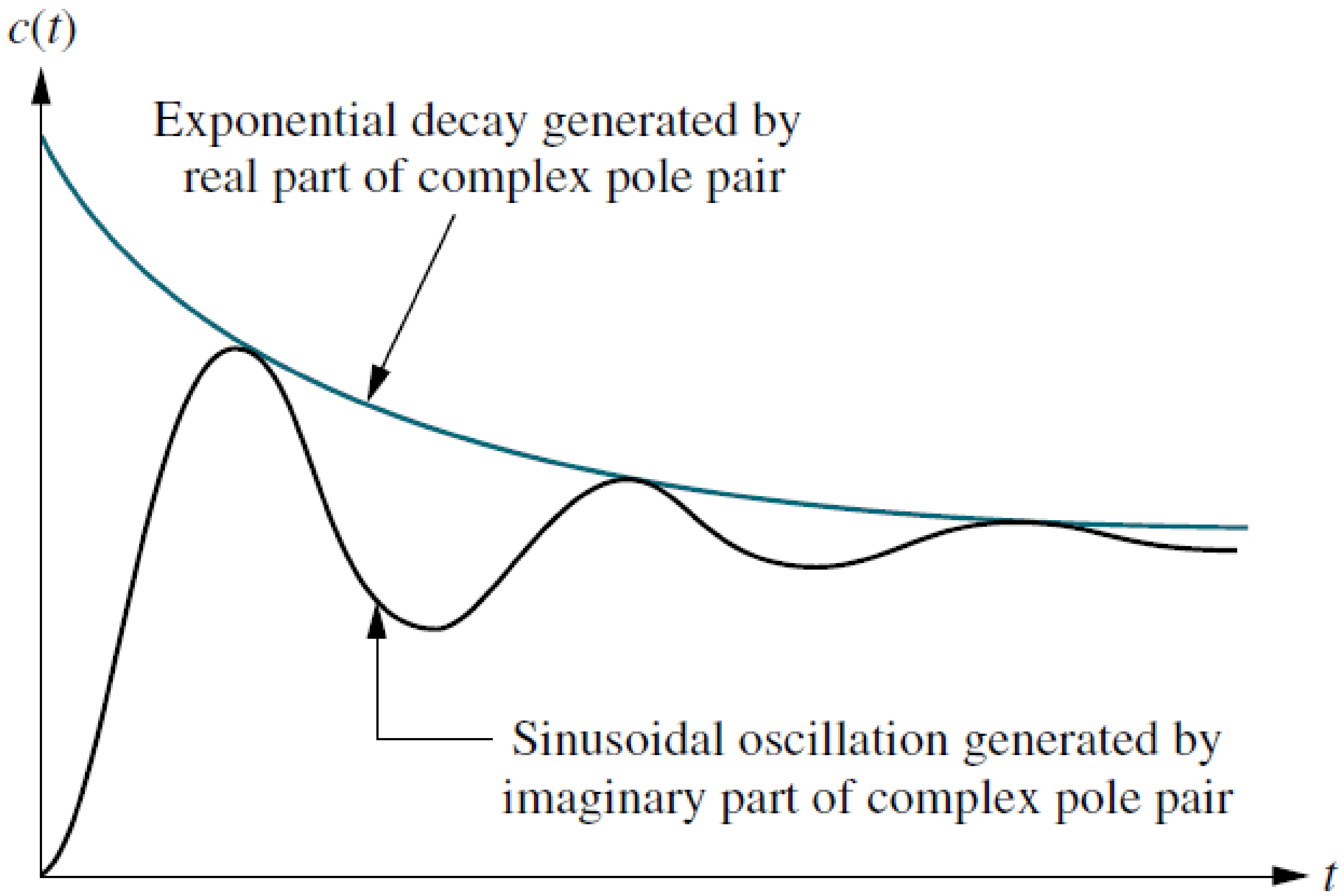
Pole zero plot

Underdamped response

The poles that generate the natural response are at $s = -1 \pm j\sqrt{8}$.

The real part of the pole matches the exponential decay frequency of the sinusoid's amplitude, while the imaginary part of the pole matches the frequency of the sinusoidal oscillation.

Underdamped Response



Second-order step response components generated by complex pole

Underdamped Response

Figure shows a general, damped sinusoidal response for a second order system.

The transient response consists of an **exponentially decaying amplitude generated by the real part** of the **system pole** times a sinusoidal waveform generated by the imaginary part of the system pole.

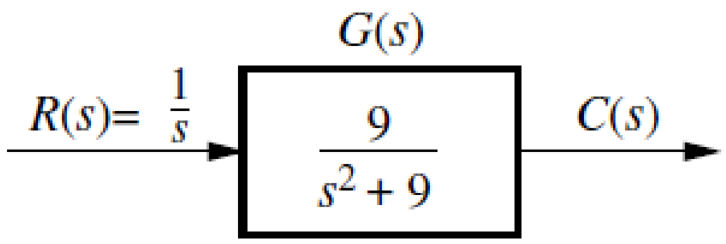
The time constant of the exponential decay is equal to the reciprocal of the real part of the system pole.

The value of the imaginary part is the actual frequency of the sinusoid.

This sinusoidal frequency is given the name damped frequency of oscillation, ω_d.

Finally, the steady-state response (unit step) was generated by the input pole located at the origin. This type of response is called as underdamped response, which approaches a steady-state value via a transient response that is a damped oscillation.

Undamped Response



For this second order system the transfer function $G(s) = \frac{9}{s^2+9}$ and the response is

$$C(s) = \frac{1}{s}\frac{9}{s^2 + 9}$$

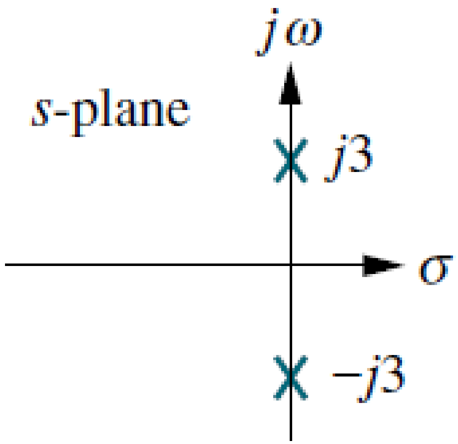
This function has a pole at the origin that comes from the unit step input and two imaginary poles that come from the system.

The input pole at the origin generates the constant forced response,

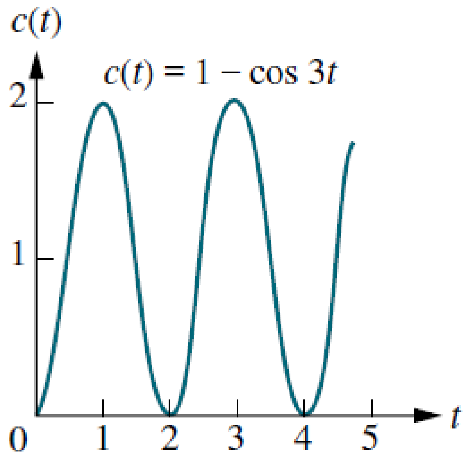
and the two system poles on the imaginary axis at $\pm$j3 generate a sinusoidal natural response whose frequency is equal to the location of the imaginary poles.

Undamped Response

Hence, the output can be estimated as $c(t) = K_1 + K_2\cos(3t - \varphi)$.



Pole zero plot

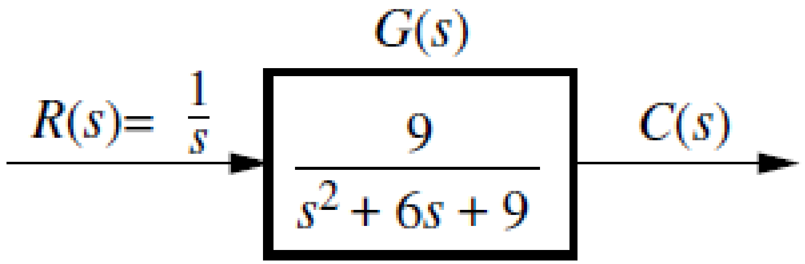


undamped response

This type of response is called undamped.

Note that the absence of a real part in the pole pair corresponds to an exponential that does not decay. Mathematically, the exponential is $e^{-0t} = 1$.

Critically damped Response



For this second order system the transfer function $G(s) = \frac{9}{s^2+6s+9}$ and the response is

$$C(s) = \frac{1}{s}\frac{9}{s^2 + 6s + 9}$$

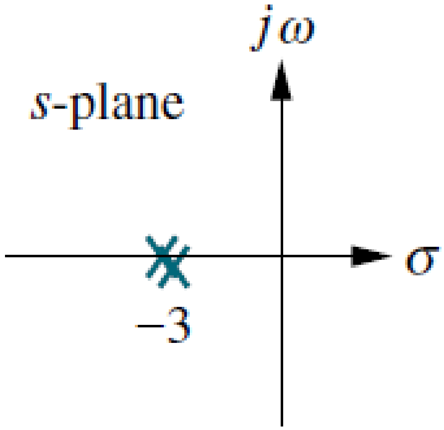
This function has a pole at the origin that comes from the unit step input and

Two multiple real poles that come from the system.

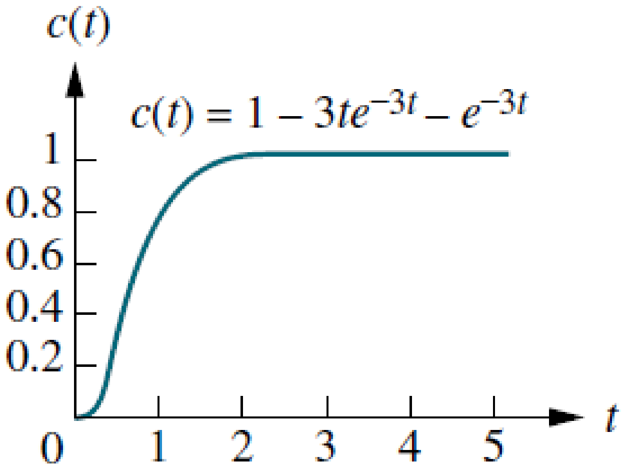
The input pole at the origin generates the constant forced response, and the two poles on the real axis at 3 generate a natural response consisting of an exponential and an exponential multiplied by time, where the exponential frequency is equal to the location of the real poles.

Critically damped Response

Hence, the output can be estimated as $c(t) = K_1 + K_2 e^{-\sigma_1 t} + K_3 t e^{-\sigma_1 t}$



Pole zero plot



critically damped response

The output can be estimated as $c(t) = K_1 + K_2 e^{-3t} + K_3 t e^{-3t}$.

This type of response is called critically damped.

Critically damped responses are the fastest possible without the overshoot that is characteristic of the underdamped response.

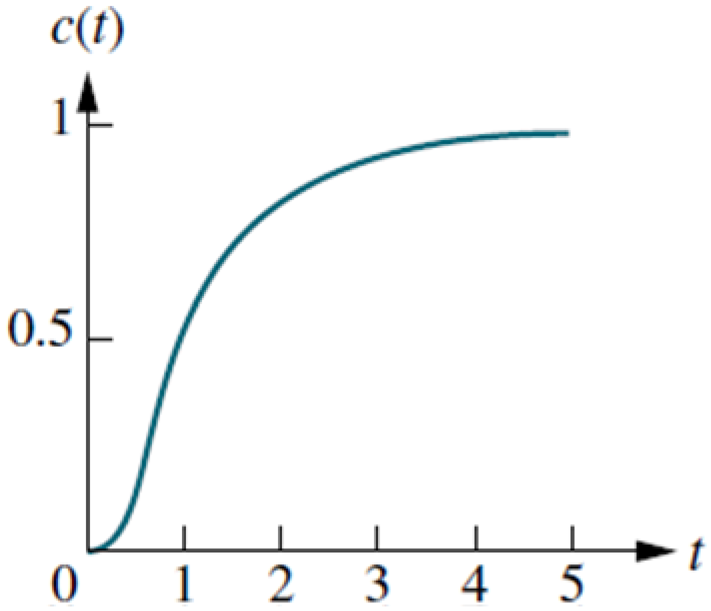
Step responses for second-order system damping cases: Conclusion

1. Overdamped responses

Poles: Two real at $-\sigma_1, -\sigma_2$

Natural response: Two exponentials with time constants equal to the reciprocal of the pole locations, or

$$c(t) = K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t}$$



Step responses for second-order system damping cases: Conclusion

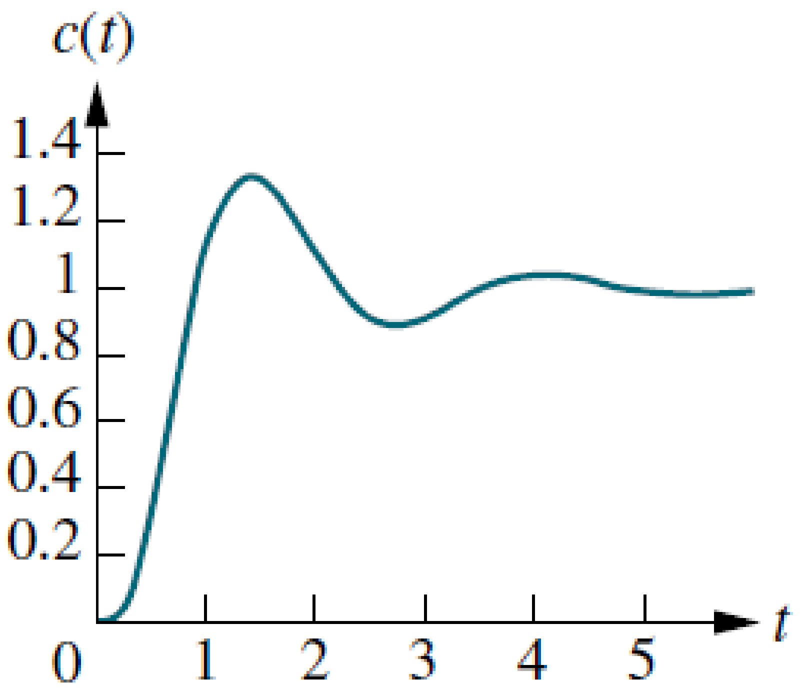
2. Underdamped responses

Poles: Two complex at $-\sigma_d \pm j\omega_d$

Natural response: Damped sinusoid with an exponential envelope whose time constant is equal to the reciprocal of the pole's real part.

The radian frequency of the sinusoid, the damped frequency of oscillation, is equal to the imaginary part of the poles, or

$$c(t) = Ae^{-\sigma_d t}\cos(\omega_d t - \phi)$$



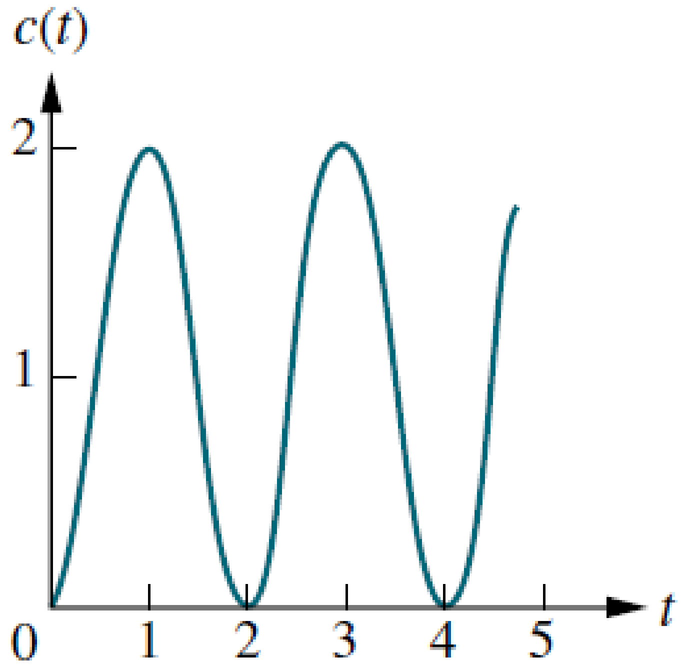
Step responses for second-order system damping cases: Conclusion

3. Undamped responses

Poles: Two imaginary at $\pm j\omega_1$

Natural response: Undamped sinusoid with radian frequency equal to the imaginary part of the poles, or

$c(t) = A\cos(\omega_1 t - \phi)$



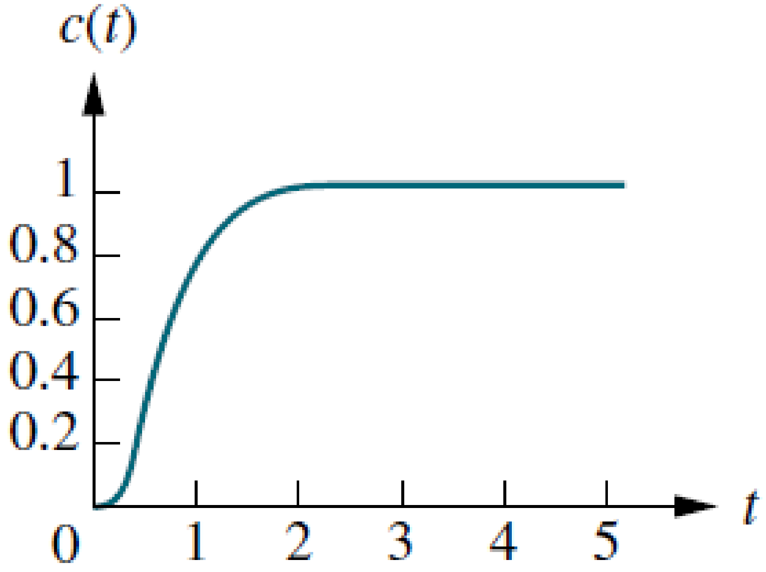
Step responses for second-order system damping cases: Conclusion

4. Critically damped responses

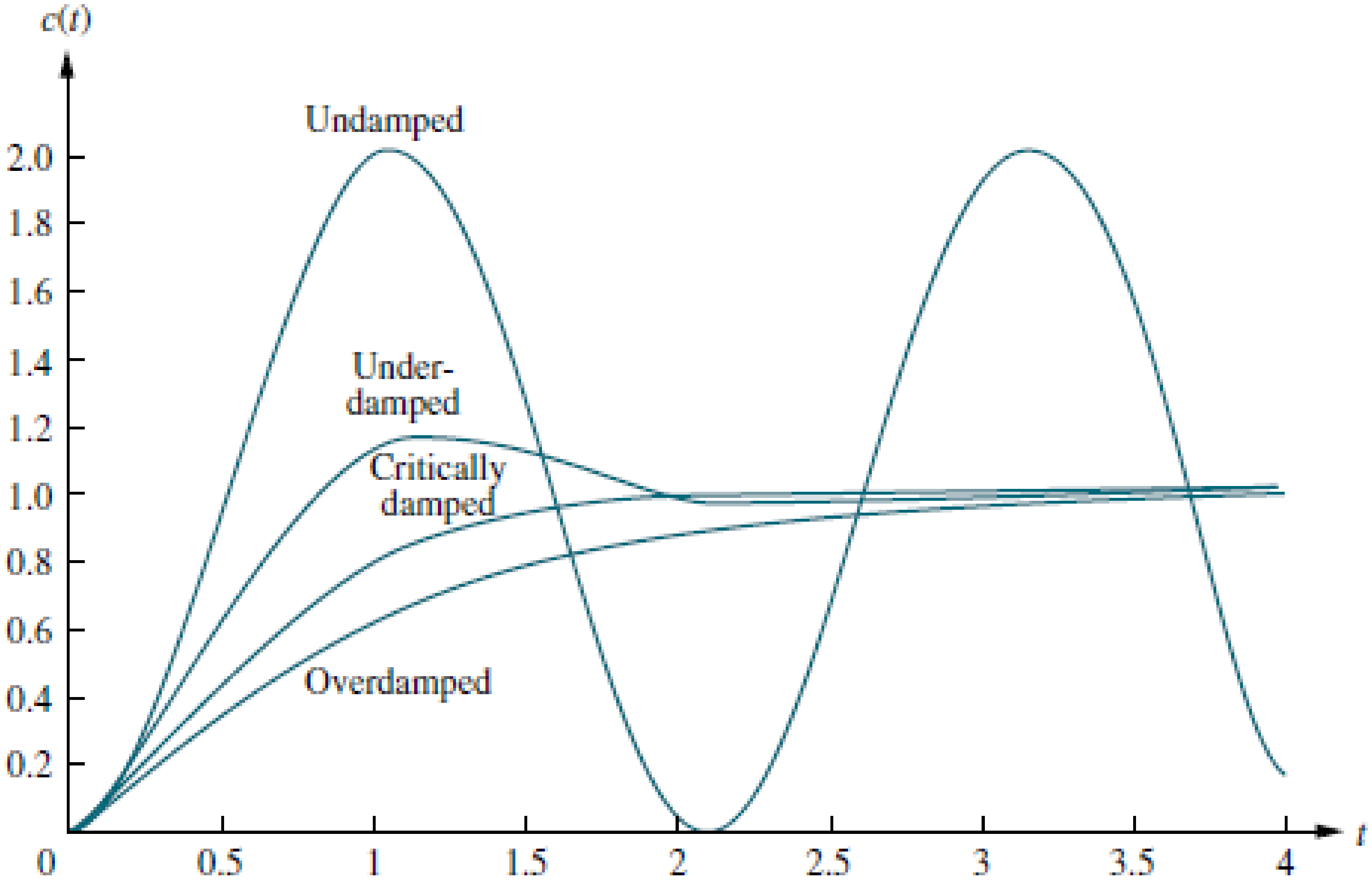
Poles: Two real at $-\sigma_1$

Natural response: One term is an exponential whose time constant is equal to the reciprocal of the pole location. Another term is the product of time, t, and an exponential with time constant equal to the reciprocal of the pole location, or

$$c(t) = K_1 e^{-\sigma_1 t} + K_2 t e^{-\sigma_1 t}$$



Step responses for second-order system damping cases: Conclusion



The step responses for the four cases of damping

Natural frequency and Damping ratio of second order system

The natural frequency and damping ratio can be used to describe the characteristics of the second-order transient response just as time constants describe the first-order system response.

Natural Frequency, ω_n

The natural frequency of a second-order system is the frequency of oscillation of the system without damping.

For example, the frequency of oscillation of a series RLC circuit with the resistance shorted would be the natural frequency.

Natural frequency and Damping ratio of second order system

Damping Ratio, ζ

This is the ratio of exponential decay frequency to Natural frequency (rad/second).

$$\zeta = \frac{\text{Exponential decay frequency}}{\text{Natual frequency (rad/second)}} = \frac{1}{2\pi}\frac{\text{Natural Period}}{\text{Exponential Period}}$$

Natural frequency and Damping ratio of second order system

Consider the general second order system

$$G(s) = \frac{b}{s^2 + as + b} \quad \dots 1$$

Without damping, the poles would be on the $j\omega$-axis, and the response would be an undamped sinusoid. For the poles to be purely imaginary, $a = 0$. Hence

$$G(s) = \frac{b}{s^2 + b} \dots 2$$

By definition, the natural frequency, ω_n, is the frequency of oscillation of this system. Since the poles of this system are on the $j\omega$ -axis at $\pm j\sqrt{b}$

$$\omega_n = \sqrt{b} \dots 3$$

Natural frequency and Damping ratio of second order system

$$b = \omega_n^2 \ldots 4$$

Assuming an underdamped system, where the complex poles have a real part, σ, equal to - a/2.

The magnitude of this value is then the exponential decay frequency. Hence,

$$\zeta = \frac{\text{Exponential decay frequency}}{\text{Natual frequency (rad/second)}} = \frac{|\sigma|}{\omega_n} = \frac{a/2}{\omega_n} \ldots 5$$

From which

$$a = 2\zeta\omega_n \ \ldots 6$$

Therefore, the transfer function of general second order system becomes

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \ \ldots 7$$

Natural frequency and Damping ratio of second order system

Find ω_n and ζ for the system described by the transfer function

$$G(s) = \frac{36}{s^2 + 4.2s + 36}$$

Use

$b = \omega_n^2$ and $a = 2\zeta\omega_n$

Natural frequency, Damping ratio and pole locations of second order system

The poles can be obtained by solving the following transfer function

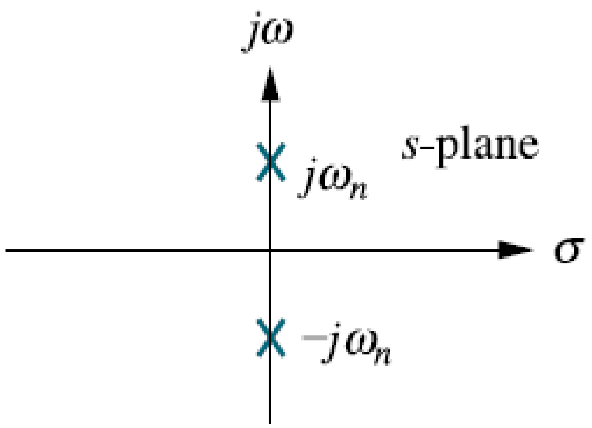
$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

We get

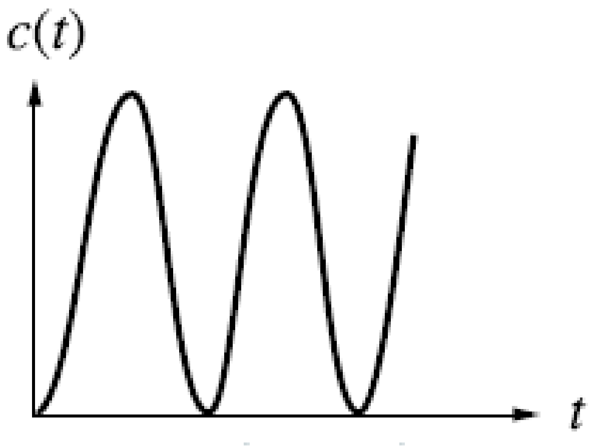
$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Natural frequency, Damping ratio and pole locations of second order system

For $\zeta = 0$



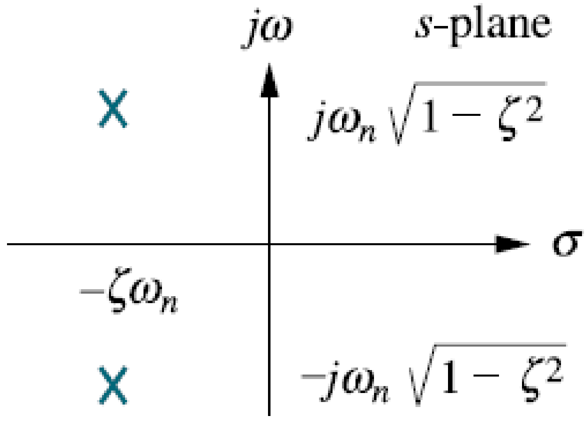
Poles



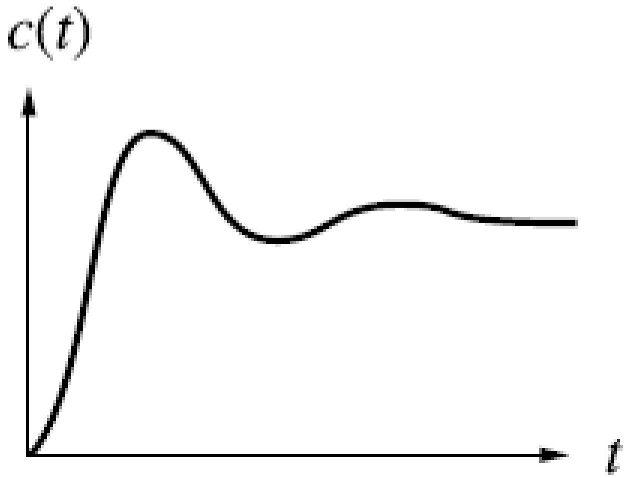
Step response: undamped

Natural frequency, Damping ratio and pole locations of second order system

For $0 < \zeta < 1$



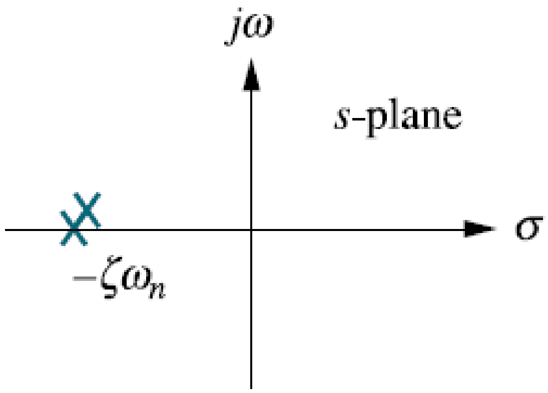
Poles



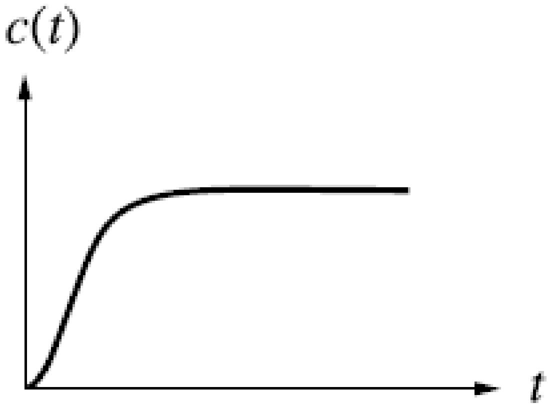
Step response: underdamped

Natural frequency, Damping ratio and pole locations of second order system

For $\zeta = 1$



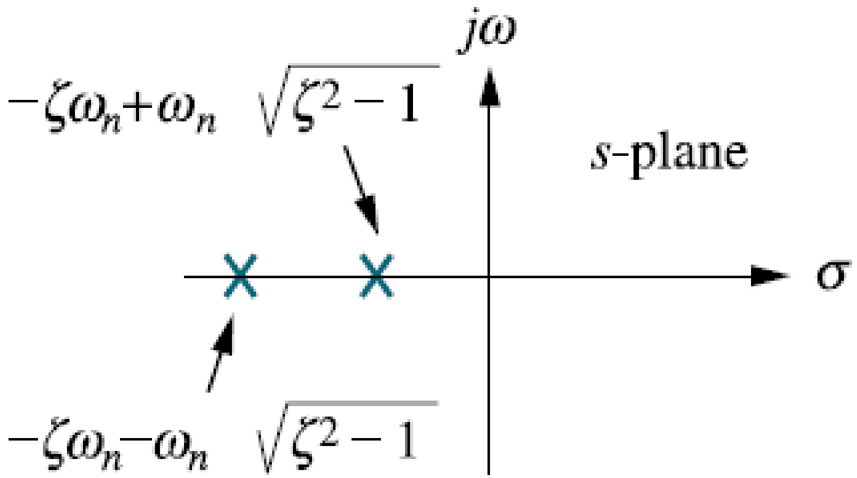
Poles



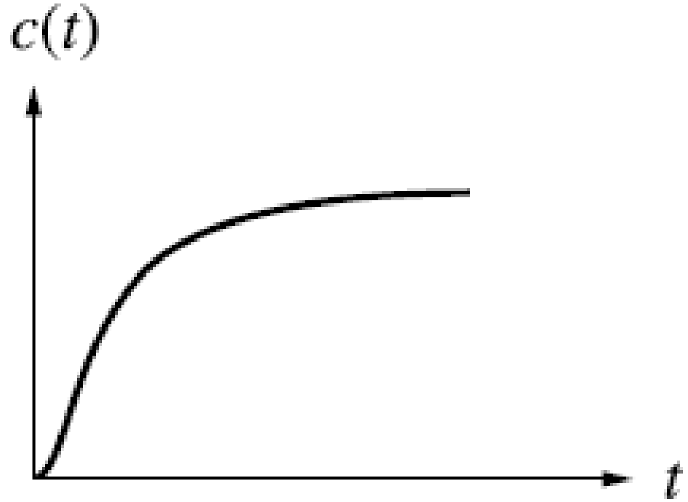
Step response: critically damped

Natural frequency, Damping ratio and pole locations of second order system

For $\zeta > 1$



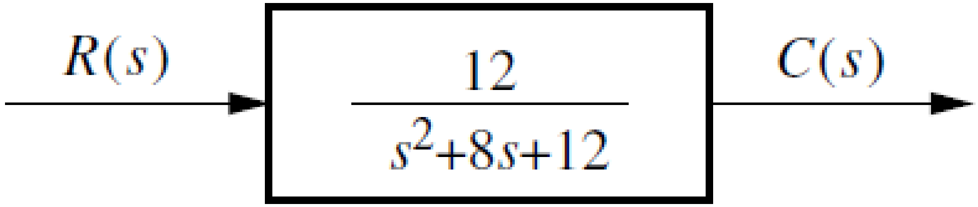
Poles

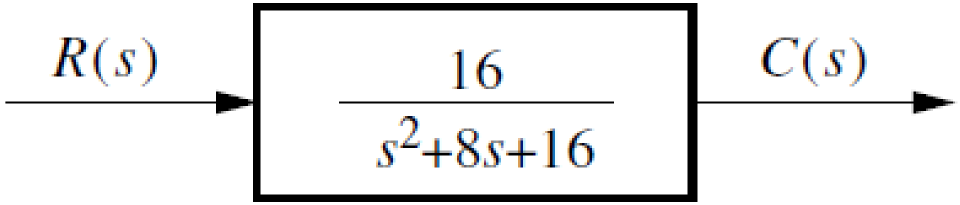


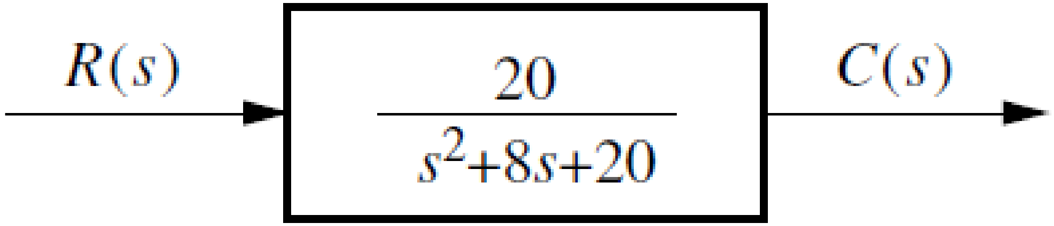
Step response: overdamped

Natural frequency, Damping ratio and pole locations of second order system

For each of the systems shown, find the value of ζ and report the kind of response expected.

A) 

B) 

C) 

Use $a = 2\,\zeta\omega_n$ and $\omega_n = \sqrt{b}$ therefore $\zeta = \frac{a}{2\sqrt{b}}$

Natural frequency, Damping ratio and pole locations of second order system

For each of the transfer functions

(1) Find the values of ζ and ω_n;

(2) characterize the nature of the response.

a. $\mathrm{G(s)} = \dfrac{400}{s^2 + 12s + 400}$

b. $\mathrm{G(s)} = \dfrac{900}{s^2 + 90s + 900}$

c. $\mathrm{G(s)} = \dfrac{225}{s^2 + 30s + 225}$

d. $\mathrm{G(s)} = \dfrac{625}{s^2 + 625}$

Natural frequency, Damping ratio and pole locations of second order system

For each of the transfer functions

(1) Find the values of ζ and ω_n;

(2) characterize the nature of the response.

a. $G(s) = \dfrac{400}{s^2 + 12s + 400}$ $\quad \zeta = 0.3 \; and \omega_n = 20;$ $\quad$ underdamped

b. $G(s) = \dfrac{900}{s^2 + 90s + 900}$ $\quad \zeta = 1.5 \; and \omega_n = 30;$ $\quad$ overdamped

c. $G(s) = \dfrac{225}{s^2 + 30s + 225}$ $\quad \zeta = 1 \; and \omega_n = 15;$ $\quad$ critically damped

d. $G(s) = \dfrac{625}{s^2 + 625}$ $\quad \zeta = 0 \; and \omega_n = 25;$ $\quad$ undamped

Underdamped Second-Order Systems

The underdamped second order system is a common model for physical problems.

A detailed description of the underdamped response is necessary for both analysis and design. The objectives of this study are

1. To define transient specifications associated with underdamped responses.
2. To relate these specifications to the pole location, drawing an association between pole location and the form of the underdamped second-order response.
3. to tie the pole location to system parameters.

Underdamped Second-Order Systems

Let us begin by finding the step response for the general second-order system.

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad \ldots 1$$

The transform of the response, C(s), is the transform of the input times the transfer function,

$$C(s) = \frac{1}{s}\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad \ldots 2$$

where it is assumed that $\zeta < 1$ (as it is the underdamped case). Expanding by partial fractions,

$$C(s) = \frac{1}{s} - \frac{(s + \zeta\omega_n) + \frac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s + \zeta\omega_n)^2 + \omega_n^2\left(\sqrt{1-\zeta^2}\right)^2} \quad \ldots 3$$

Underdamped Second-Order Systems

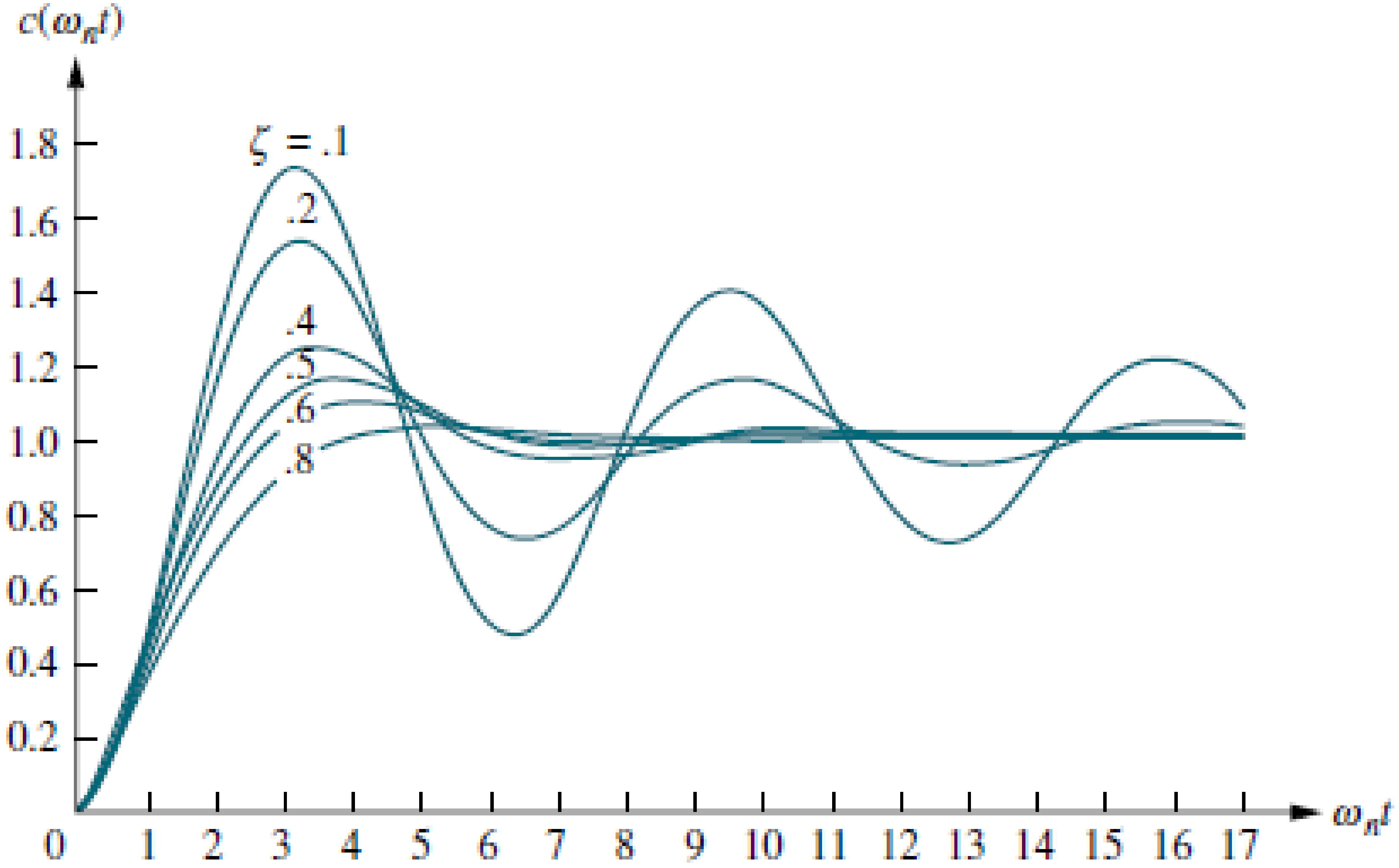
Taking the inverse Laplace transform

$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos \omega_n\sqrt{1-\zeta^2}\,t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin \omega_n\sqrt{1-\zeta^2}\,t\right) \ldots 4$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_n\sqrt{1-\zeta^2}t + \phi\right) \ldots 5$$

where $\phi = tan^{-1}\left(\sqrt{1-\zeta^2}/\zeta\right) \ldots 6$

Underdamped Second-Order Systems



Second-order underdamped responses for damping ratio (ζ) values

Underdamped Second-Order Systems

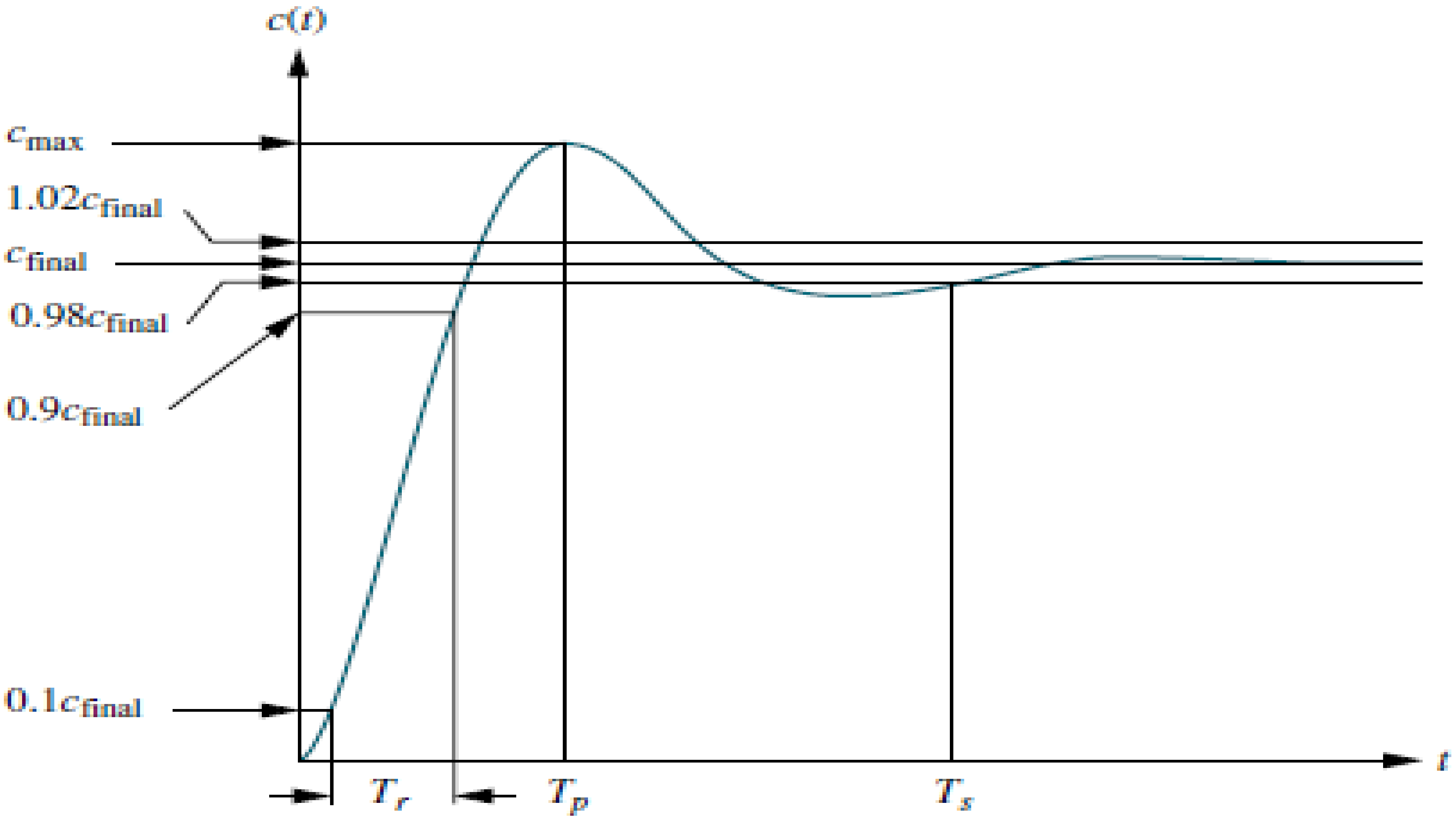
The relationship between the value of damping ratio (ζ) and the type of response obtained

The lower the value of damping ratio (ζ), the response is the more oscillatory.

The natural frequency (ω_n) is a time-axis scale factor and does not affect the nature of the response other than to scale it in time.

Other parameters associated with the underdamped response are **rise time**, **peak time**, **percent overshoot**, and **settling time**.

Underdamped Second-Order Systems



Second-order underdamped response specifications

Underdamped Second-Order Systems

1. **Rise time** (T_r) **:** The time required for the waveform to go from 0.1 of the final value to 0.9 of the final value.
2. **Peak time,** (T_p)**:** The time required to reach the first, or maximum, peak.
3. **Percent overshoot** $(\%OS)$ **:** The amount that the waveform overshoots the steady state, or final, value at the peak time, expressed as a percentage of the steady-state value.
4. **Settling time,** (T_s)**:** The time required for the transient's damped oscillations to reach and stay within $\pm 2\%$ of the steady-state value.

Underdamped Second-Order Systems

The definitions for settling time and rise time are basically the same as the definitions for the first-order response.

All definitions are also valid for systems of order higher than 2, although analytical expressions for these parameters cannot be found unless the response of the higher-order system can be approximated as a second-order system.

Underdamped Second-Order Systems

Rise time, peak time, and settling time yield information about the speed of the transient response. This information can help a designer determine if the speed and the nature of the response do or do not degrade the performance of the system.

For example, the speed of an entire computer system depends on the time it takes for a hard drive head to reach steady state and read data;

passenger comfort depends in part on the suspension system of a car and the number of oscillations it goes through after hitting a bump.

Underdamped Second-Order Systems: Evaluation of T_p

T_p : is found by differentiating c(t) and finding the first zero crossing after t = 0. This task is simplified by "differentiating" in the frequency domain by differentiation property of Laplace Transform assuming zero initial conditions, we get

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Underdamped Second-Order Systems: Evaluation of %OS

the percent overshoot, %OS, is given by

$$\%OS = \frac{c_{max} - c_{final}}{c_{final}} \times 100 \ldots 1$$

The term c_{max} is found by evaluating $c(t)$ at the peak time, $c(T_p)$.

$$\therefore \%OS = e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)} \times 100 \ldots 2$$

The percent overshoot is a function only of the damping ratio (ζ).

by equation 2 allows we can find %OS for given the damping ratio (ζ). Also we can find the damping ratio (ζ) for given %OS by following equation.

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}} \ldots 3$$

Underdamped Second-Order Systems: Evaluation of T_s

T_s, the settling time, is the time for which c(t) reaches and stays within 2% of the steady-state value, c_{final}.

The settling time is the time it takes for the amplitude of the decaying sinusoid in to reach 0.02,

$$T_s = \frac{4}{\zeta\omega_n}$$

Underdamped Second-Order Systems: Evaluation of T_r

T_r, the rise time, is the time for which c(t) reaches from 0.1 of c_{final} to o.9 of c_{final} for overdamped systems and 0 to 100 % is for underdamped system.

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_n\sqrt{1-\zeta^2}t + \phi\right)$$

where $\phi = tan^{-1}\left(\sqrt{1-\zeta^2}/\zeta\right)$

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_n\sqrt{1-\zeta^2}t + \phi\right) = 1$$

$$\therefore \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_n\sqrt{1-\zeta^2}t + \phi\right) = 0$$

Underdamped Second-Order Systems: Evaluation of T_r

$$\therefore \sin\left(\omega_n\sqrt{1-\zeta^2}t+\phi\right)=0$$

$$\therefore \omega_n\sqrt{1-\zeta^2}t_r+\phi=\pi$$

$$\therefore t_r=\frac{\pi-\phi}{\omega_n\sqrt{1-\zeta^2}}$$

where $\phi = tan^{-1}\left(\sqrt{1-\zeta^2}/\zeta\right)$

Underdamped Second-Order Systems:

Find the find T_p, %OS, T_s, and T_r For the system described by the transfer function

$$G(s) = \frac{100}{s^2 + 15s + 100}$$

Underdamped Second-Order Systems:

Find the find T_p, %OS, T_s, and T_r For the system described by the transfer function

$$G(s) = \frac{100}{s^2 + 15s + 100}$$

$\zeta = 0.75, \omega_n = 10$

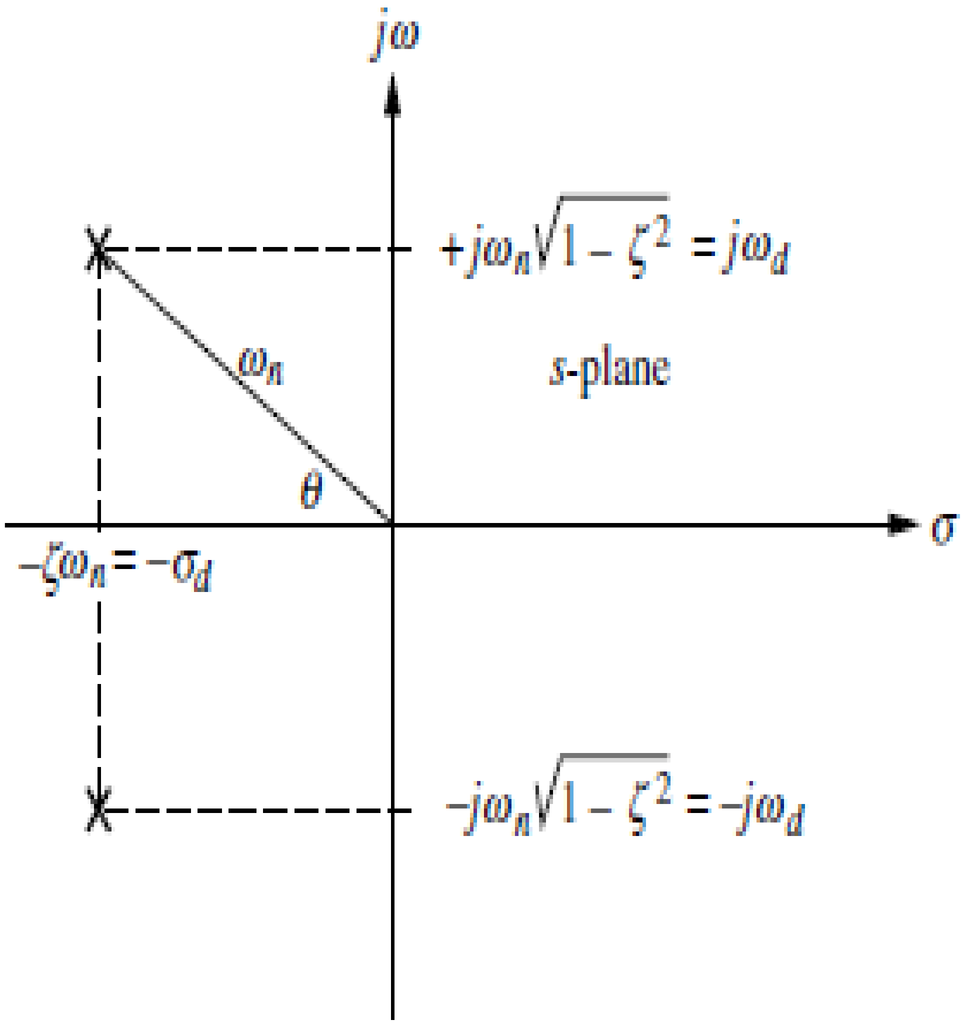
$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.475\ second$$

$$\%OS = e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)} \times 100 = 2.838$$

$$T_s = \frac{4}{\zeta\omega_n} = 0.533\ second$$

$$T_r = \frac{\pi - \phi}{\omega_n\sqrt{1-\zeta^2}} = 0.23\ second$$

Underdamped Second-Order Systems:



Pole plot for general underdamped second order system

The pole plot for a general, underdamped second-order system, shown in Figure.

We see from the Pythagorean theorem that the radial distance from the origin to the pole is the natural frequency ω_n and the $\cos\theta = \zeta$.

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d} \ldots 1$$

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma_d} \ldots 2$$

Underdamped Second-Order Systems:

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d} \dots 1$$

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma_d} \dots 2$$

where ω_d is the imaginary part of the pole and is called the damped frequency of oscillation, and

σ_d is the magnitude of the real part of the pole and is the exponential damping frequency.

Underdamped Second-Order Systems:

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d} \ldots 1$$

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma_d} \ldots 2$$

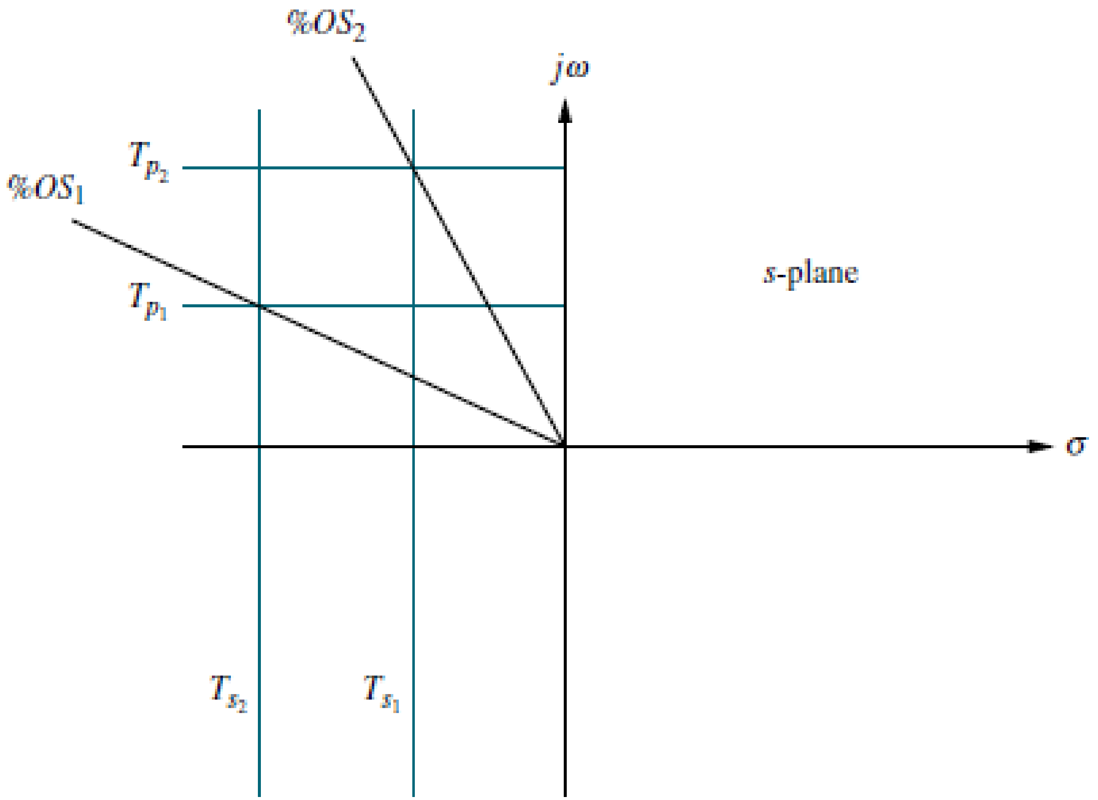
Equation (1) shows that T_p is inversely proportional to the imaginary part of the pole.

Since horizontal lines on the s-plane are lines of constant imaginary value, they are also lines of constant peak time.

Similarly, Eq. (2) tells us that T_s settling time is inversely proportional to the real part of the pole.

Since vertical lines on the s-plane are lines of constant real value, they are also lines of constant settling time.

Underdamped Second-Order Systems:

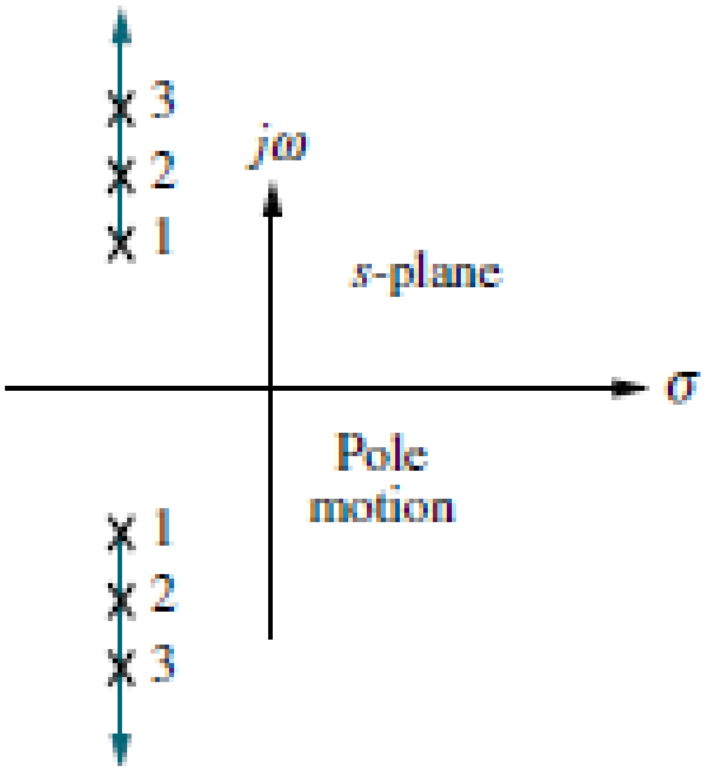


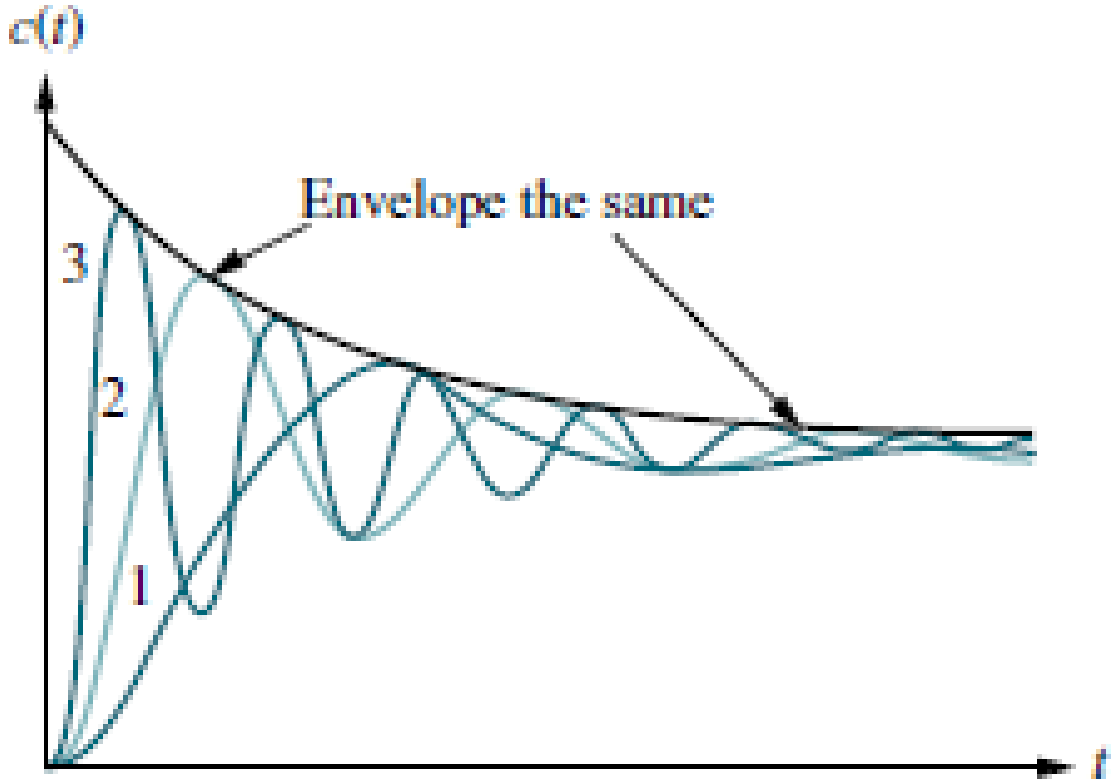
Lines of constant peak time, T_p, settling time, T_s, and percent overshoot, %OS.
Note: $T_{s2} < T_{s1}$;
$T_{p2} < T_{p1}$;
%OS1 < %OS2

Finally, since $\zeta = \cos\theta$, radial lines are lines of constant ζ. Since percent overshoot is only a function of ζ, radial lines are thus lines of constant percent overshoot, %OS.

The lines of constant Tp, Ts, and %OS are labeled on the s-plane.

Underdamped Second-Order Systems:





the poles are moved in a vertical direction, keeping the real part the same.

As the poles move in a vertical direction, the frequency increases, but the envelope remains the same since the real part of the pole is not changing.

Underdamped Second-Order Systems:

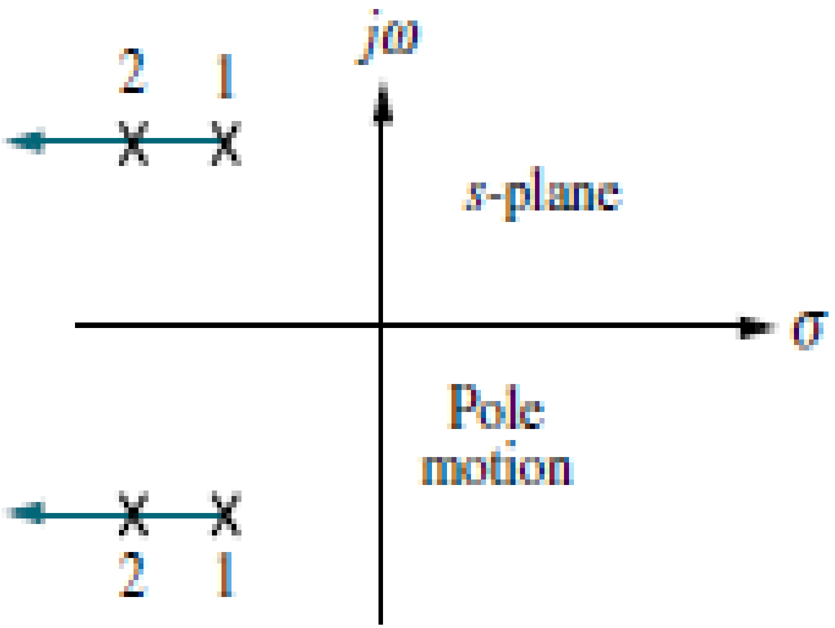
Figure shows the step responses as the poles are moved in a vertical direction, keeping the real part the same.

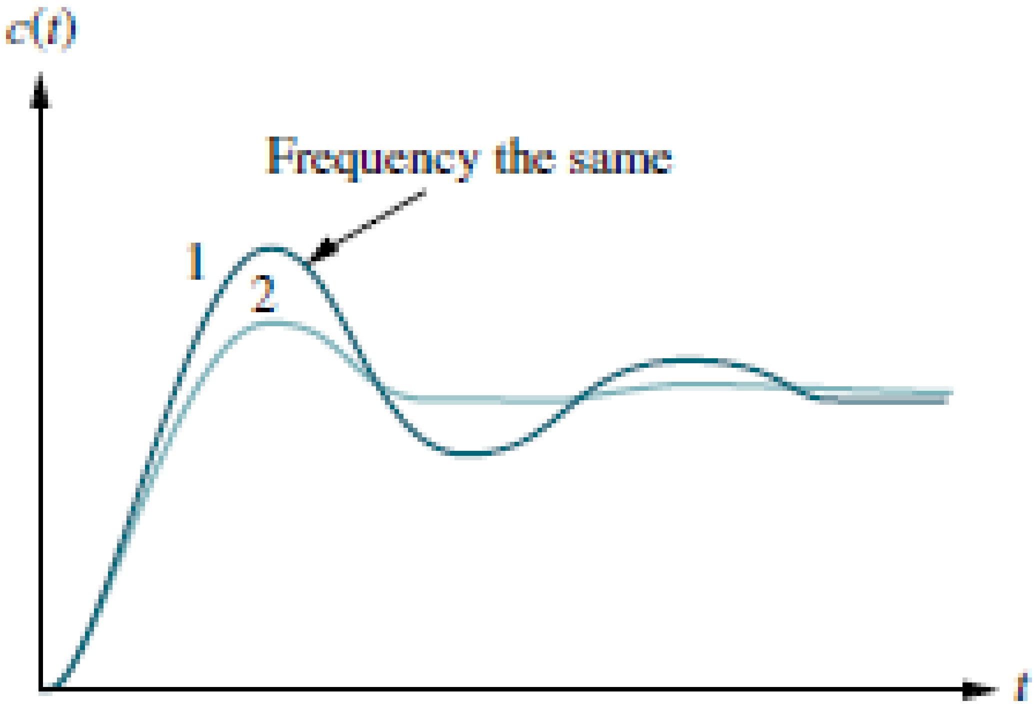
As the poles move in a vertical direction, the frequency increases, but the envelope remains the same since the real part of the pole is not changing.

The figure shows a constant exponential envelope, even though the sinusoidal response is changing frequency.

Since all curves fit under the same exponential decay curve, the settling time is virtually the same for all waveforms. Note that as overshoot increases, the rise time decreases

Underdamped Second-Order Systems:





the poles are moved in a horizontal direction, keeping the imaginary part the same.

As the poles move to the left, the response damps out more rapidly, while the frequency remains the same.

the peak time is the same for all waveforms because the imaginary part remains the same.

Underdamped Second-Order Systems:

Now move the poles to the right or left.

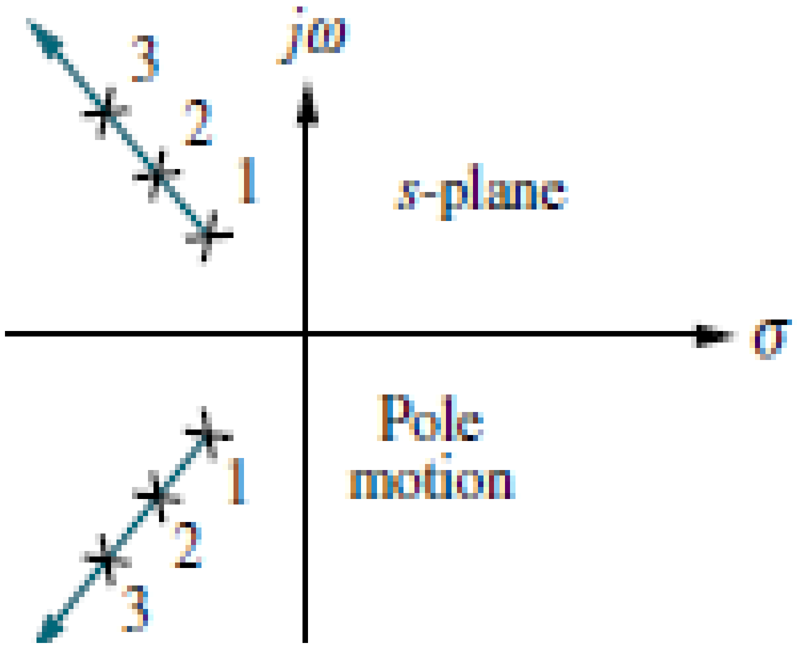
Since the imaginary part is now constant, movement of the poles yields the responses of as shown in figure.

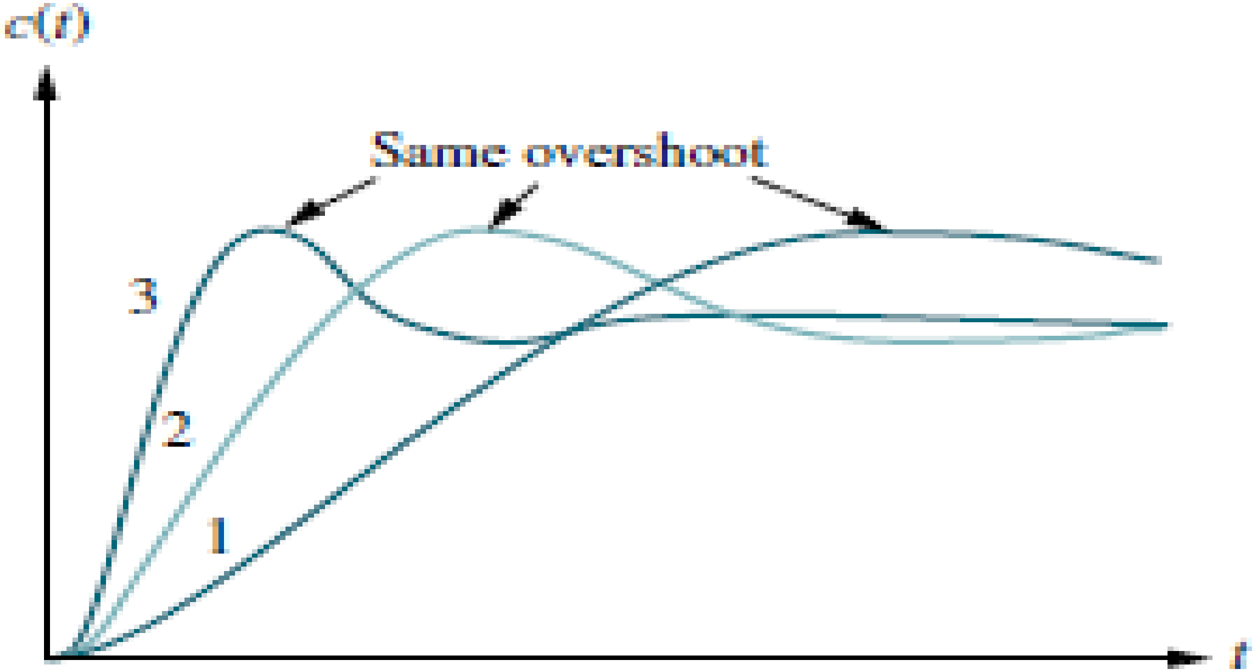
Here the frequency is constant over the range of variation of the real part.

As the poles move to the left, the response damps out more rapidly, while the frequency remains the same.

Notice that the peak time is the same for all waveforms because the imaginary part remains the same.

Underdamped Second-Order Systems:





the poles are moved in a along a constant radial line.

The responses look exactly alike, except for their speed.

The farther the poles are from the origin, the more rapid the response.

Underdamped Second-Order Systems:

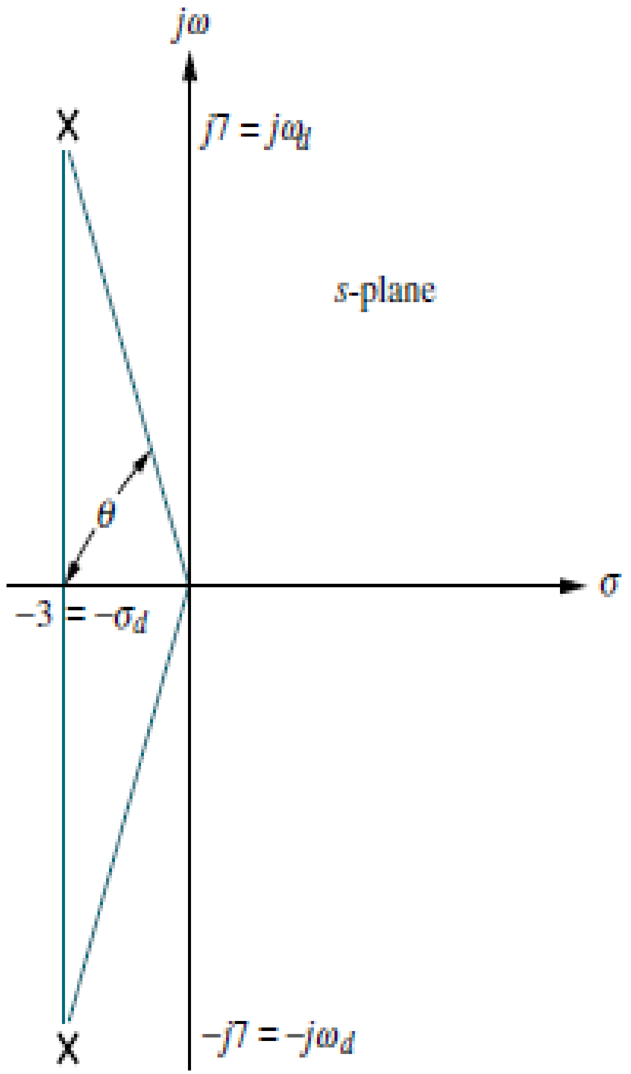
Now move the poles along a constant radial line yields the responses shown in figure.

Here the percent overshoot remains the same.

Notice also that the responses look exactly alike, except for their speed.

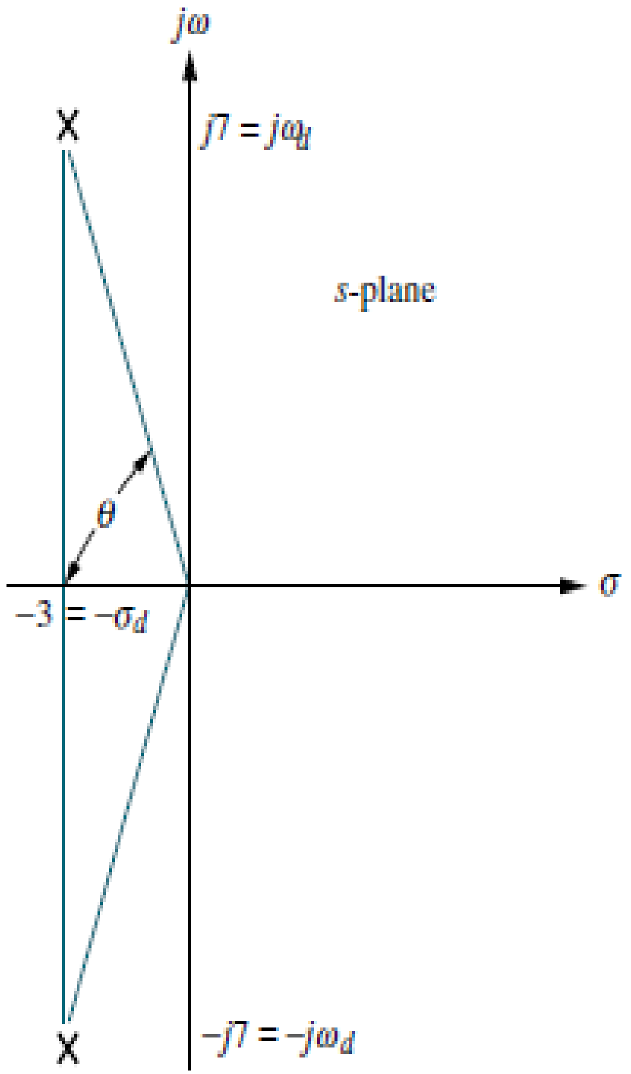
The farther the poles are from the origin, the more rapid the response.

Underdamped Second-Order Systems:



For the given the pole plot, find ζ, ω_n, Tp, %OS and Ts.

Underdamped Second-Order Systems:



The damping ratio is given by $\zeta = \cos\theta = 0.394$.

The natural frequency, ω_n, is the radial distance from the origin to the pole, or $\omega_n = \sqrt{7^2 + 3^2} = 7.616$

The peak time is

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{7} = 0.449\ seconds$$

The percent overshoot is

$$\%OS = e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)} \times 100 = 26\%$$

The approximate settling time is

$$T_s = \frac{4}{\sigma_d} = \frac{4}{3} = 1.333\ seconds$$

Underdamped Second-Order Systems:

Find $\zeta, \omega_n, Ts, Tp, Tr\ and\ \%Os$ for a system whose transfer function is

$$G(s) = \frac{361}{s^2 + 16s + 361}$$

Underdamped Second-Order Systems:

Find $\zeta, \omega_n, Ts, Tp, Tr\ and\ \%Os$ for a system whose transfer function is

$$G(s) = \frac{361}{s^2 + 16s + 361}$$

$\zeta = 0.421, \omega_n = 19, Ts = 0.5\ s, Tp = 0.182\ s, Tr = 0.079\ s,$ and $\%OS = 23.3\%$

Underdamped Second-Order Systems:

Find $\zeta, \omega_n, Ts, Tp, Tr\ and\ \%Os$ for a system whose transfer function is

$$G(s) = \frac{361}{s^2 + 16s + 361}$$

$\zeta = 0.421, \omega_n = 19, Ts = 0.5\ s, Tp = 0.182\ s, Tr = 0.079\ s,$ and $\%OS = 23.3\%$

System Response with Additional Poles

Consider a three-pole system with complex poles and a third pole on the real axis.

Assuming that the complex poles are at $s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}$ the real pole is at α_r, the step response of the system can be determined from a partial-fraction expansion. Thus, the output transform is

$$C(s) = \frac{A}{s} + \frac{B(s+\zeta\omega_n) + C\omega_d}{(s+\zeta\omega_n)^2 + \omega_d^2} + \frac{D}{s+\alpha_r} \quad \ldots 1$$

And the output response is

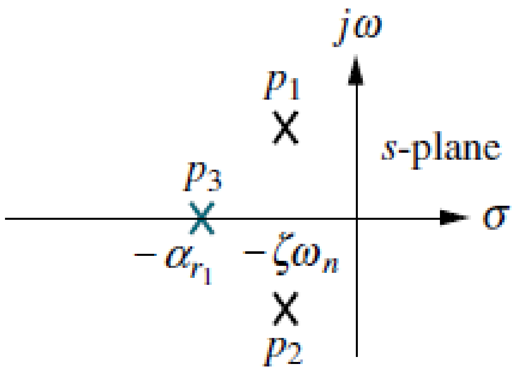
$$c(t) = Au(t) + e^{-\zeta\omega_n t}(B\cos\omega_d t + C\sin\omega_d t) + De^{-\alpha_r t} \quad \ldots 2$$

Consider the three cases of α_r.

Case I, $\alpha_r = \alpha_{r1}$ and is not much larger than $\zeta\omega_n$; for Case II, $\alpha_r = \alpha_{r2}$ and is much larger than $\zeta\omega_n$; and for Case III, $\alpha_r = \infty$.

System Response with Additional Poles

Case I, $\alpha_r = \alpha_{r1}$ and is not much larger than $\zeta\omega_n$



Case I

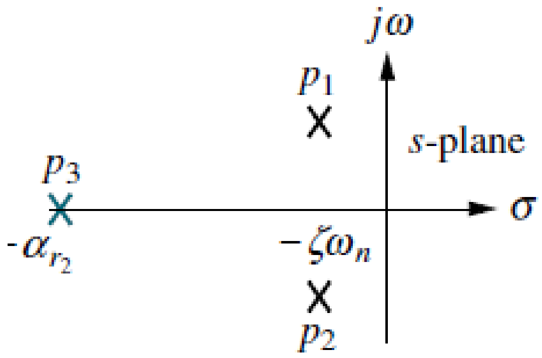
The real pole's transient response will not decay to insignificance at the peak time or settling time generated by the second-order pair.

In this case, the exponential decay is significant, and the system cannot be represented as a second-order system.

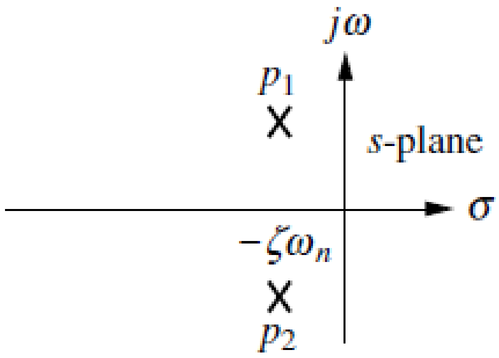
System Response with Additional Poles

Case II, $\alpha_r = \alpha_{r2}$ and is much larger than $\zeta\omega_n$ and

Case III, $\alpha_r = \infty$



Case II



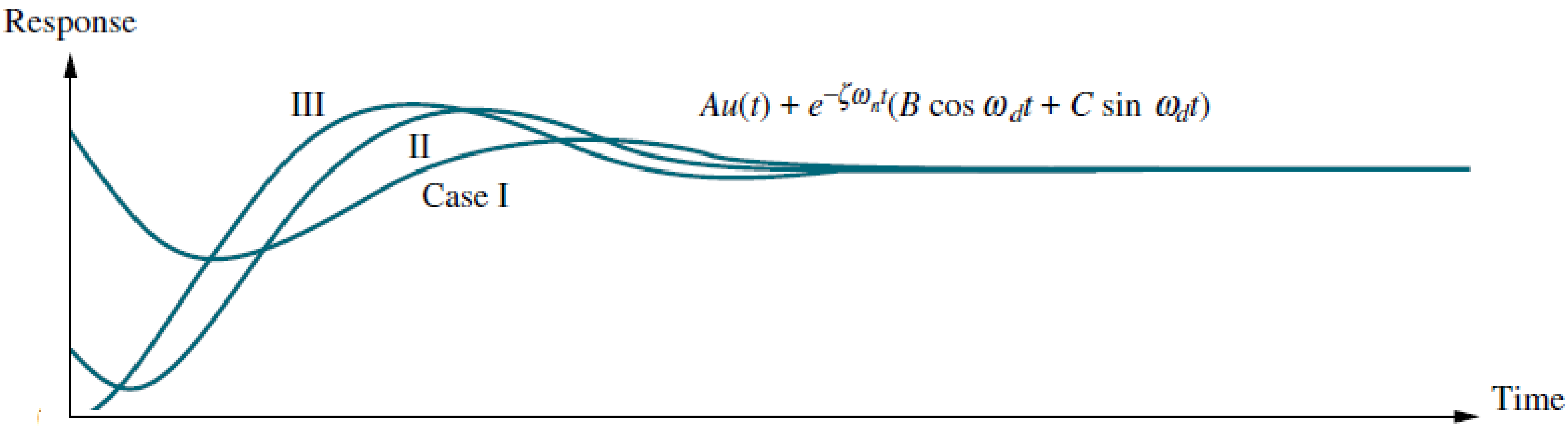
Case III

The pure exponential will die out much more rapidly than the second-order underdamped step response.

If the pure exponential term decays to an insignificant value at the time of the first overshoot, such parameters as percent overshoot, settling time, and peak time will be generated by the second-order underdamped step response component.

Thus, the total response will approach that of a pure second-order system.

System Response with Additional Poles



component responses: Nondominant pole is near dominant second-order pair (Case I), Far from the pair (Case II), and at infinity (Case III).

System Response with Additional Poles

In a three-pole system with dominant second-order poles and no zeros, will actually decrease in magnitude as the third pole is moved farther into the left half-plane.

Assume a step response, C(s), of a three-pole system

$$C(s) = \frac{bc}{s(s^2+as+b)(s+c)} = \frac{A}{s} + \frac{Bs+C}{(s^2+as+b)} + \frac{D}{s+c}$$

here we assume that the nondominant pole is located at c on the real axis and that the steady-state response approaches unity. Evaluating the constants in the numerator of each term

$$A = 1; \quad B = \frac{ca - c^2}{c^2 + b - ca}; \quad C = \frac{ca^2 - c^2a - bc}{c^2 + b - ca}; \quad \text{and } D = \frac{-b}{c^2 + b - ca}$$

System Response with Additional Poles

In a three-pole system with dominant second-order poles and no zeros, will actually decrease in magnitude as the third pole is moved farther into the left half-plane.

Assume a step response, C(s), of a three-pole system

$$C(s) = \frac{bc}{s(s^2+as+b)(s+c)} = \frac{A}{s} + \frac{Bs+C}{(s^2+as+b)} + \frac{D}{s+c}$$

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$$A = 1; \quad B = \frac{ca - c^2}{c^2+b-ca}; \quad C = \frac{ca^2 - c^2a - bc}{c^2+b-ca}; \quad \text{and } D = \frac{-b}{c^2+b-ca}$$

System Response with Additional Poles

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As the nondominant pole approaches ∞; or $c \to \infty$,

$A = 1; B = -1; C = -a;$ and $D = 0$.

Thus, for this example, D, the residue of the nondominant pole and its response, becomes zero as the nondominant pole approaches infinity.

System Response with Zero

We know that the zeros of a response affect the residue, or amplitude, of a response component but do not affect the nature of the response - exponential, damped sinusoid.

The zero will be added first in the left half-plane

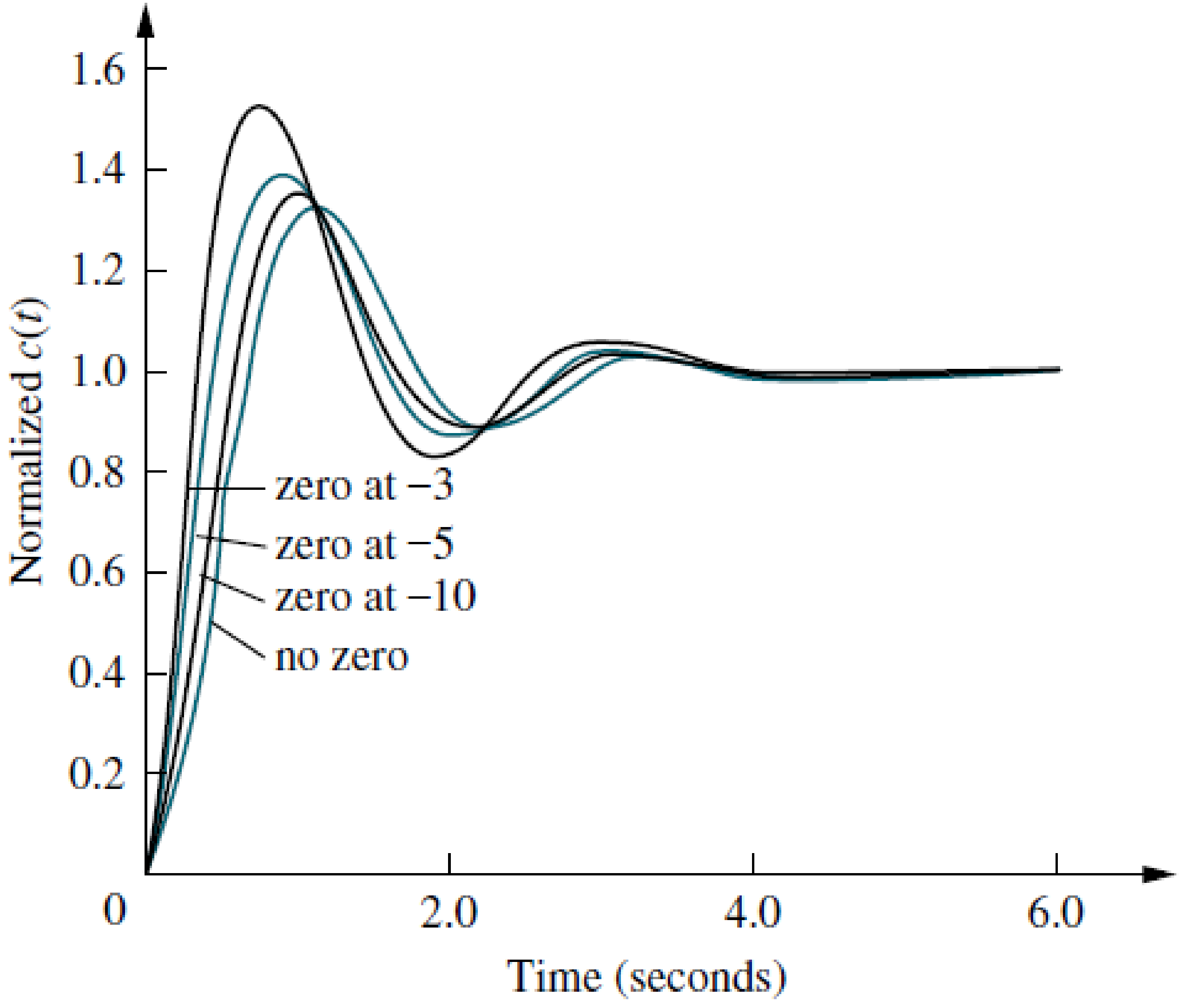
Starting with a two-pole system with poles at $-1 \pm j2.828$, we consecutively add zeros at $-3, -5$ and -10.

The results, normalized to the steady-state value, are plotted in Figure.

the closer the zero is to the dominant poles, the greater its effect on the transient response.

As the zero moves away from the dominant poles, the response approaches that of the two-pole system.

System Response with Zero



Effect of adding a zero to a two-pole system

System Response with Zero

Let's assume a system describes by the transfer function

$$G(s) = \frac{s+a}{(s+b)(s+c)} = \frac{A}{s+b} + \frac{B}{s+c}$$

$$= \frac{(-b+a)/(-b+c)}{s+b} + \frac{(-c+a)/(-c+b)}{s+c}$$

If the zero is far from the poles, then a is large compared to b and c.

$$G(s) \approx a\left[\frac{\frac{1}{-b+c}}{s+b} + \frac{\frac{1}{-c+b}}{s+c}\right] = \frac{a}{(s+b)(s+c)}$$

Hence, the zero looks like a simple gain factor and does not change the relative amplitudes of the components of the response.

System Response with Zero

Let $C(s)$ be the response of a system with unity in the numerator and $G(s)$ be the transfer function of the system

$$\therefore C(s) = G(s)R(s) = \frac{1}{D(s)}$$

If we add a zero to the transfer function i.e. now $\mathrm{G}(s) = (s + a)G(s)$

Now the Laplace transform response is $(\mathrm{s} + \mathrm{a})C(s)$

$$\therefore (\mathrm{s} + \mathrm{a})C(s) = sC(s) + aC(s)$$

Thus, the response of a system with a zero consists of two parts:

1. the derivative of the original response $sC(s)$ and

2. a scaled version of the original response $aC(s)$.

System Response with Zero

$(\mathrm{s}+\mathrm{a})C(s) = sC(s) + aC(s)$

If a is very large, the Laplace transform of the response is approximately $aC(s)$, or a scaled version of the original response.

$$\therefore (\mathrm{s}+\mathrm{a})C(s) \approx aC(s)$$

If a is not very large, the response has an additional component consisting of the derivative of the original response.

As a becomes smaller, the derivative term contributes more to the response and has a greater effect. For step responses, the derivative is typically positive at the start of a step response. Thus, for small values of a, we can expect more overshoot in second order systems because the derivative term will be additive around the first overshoot.

System Response with Zero

An interesting phenomenon occurs if a is negative, i.e. placing the zero in the right half-plane.

In the equation $(\mathrm{s}+\mathrm{a})C(s)=sC(s)+aC(s)$ the derivative term, which is typically positive initially, will be of opposite sign from the scaled response term.

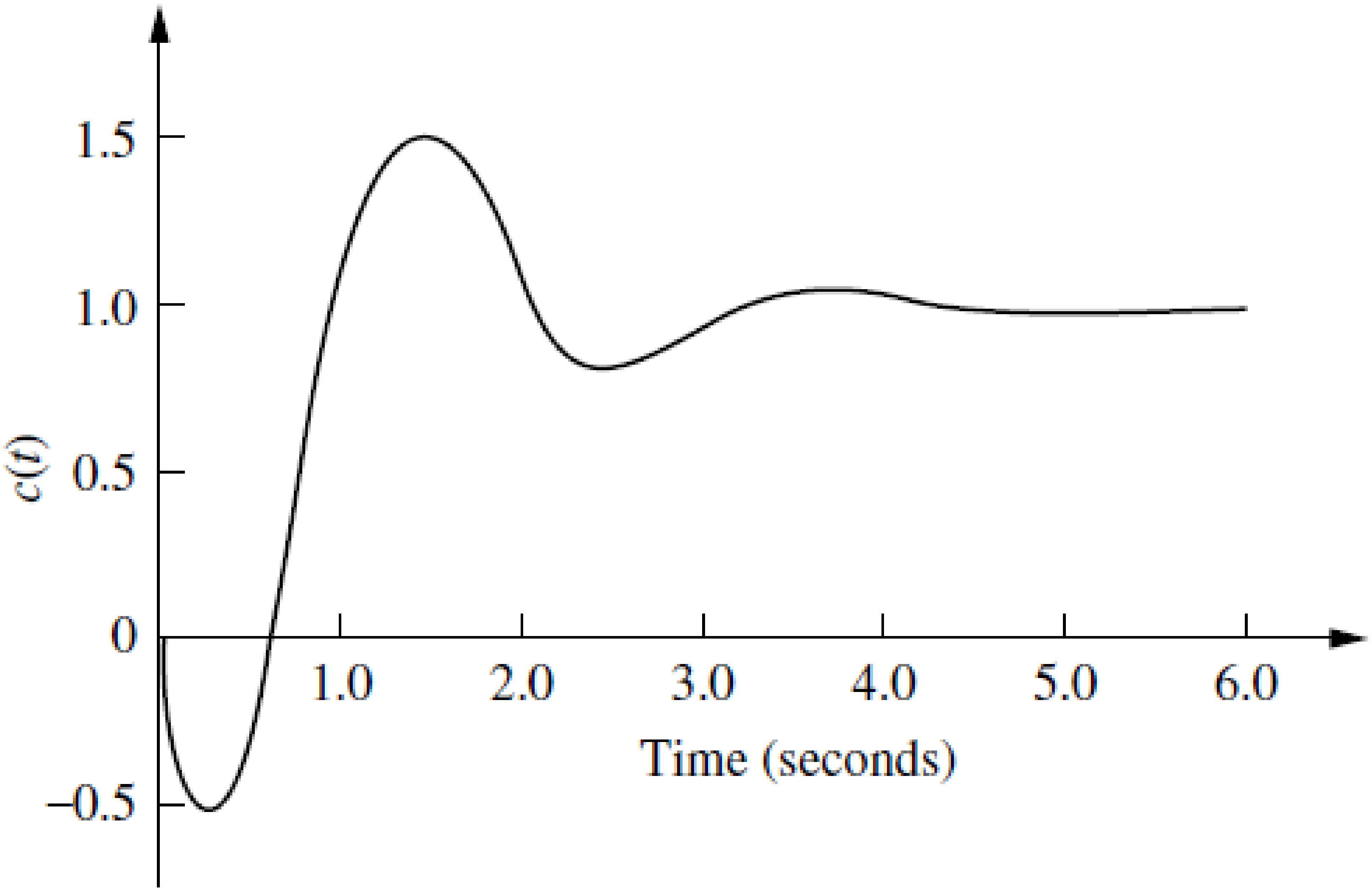
Thus, if the derivative term, $sC(s)$, is larger than the scaled response, $aC(s)$, the response will initially follow the derivative in the opposite direction from the scaled response. The result for a second-order system is shown in Figure.

where the sign of the input was reversed to yield a positive steady-state value.

Notice that the response begins to turn toward the negative direction even though the final value is positive.

A system that exhibits this phenomenon is known as a **nonminimum-phase** system.

System Response with Zero



Step response of a nonminimum-phase system

pole-zero cancellation

Assume a three pole system with a zero as represented by following equation.

$$G(s) = \frac{K(s+z)}{(s+p_3)(s^2+as+b)} \quad \ldots 1$$

If the pole term, $(s + p_3)$, and the zero term, $(s + z)$, cancel out, we get

$$G(s) = \frac{K}{(s^2+as+b)}$$

as a second-order transfer function.

From another perspective, if the zero at z is very close to the pole at p_3, then a partial-fraction expansion of Eq. 1 will show that the residue of the exponential decay is much smaller than the amplitude of the second-order response.

pole-zero cancellation

Determine whether there is cancellation between the zero and the pole closest to the zero. For any function for which pole-zero cancellation is valid, find the approximate response

$$C_1(s) = \frac{26.25(s+4)}{s(s+4.01)\,(s+5)(s+6)} \quad \ldots 1$$

If the partial fraction expansion for $C_1(s)$ is

$$C_1(s) = \frac{0.87}{s} - \frac{5.3}{s+5} + \frac{4.4}{s+6} + \frac{0.033}{s+4.01}$$

Here the pole at $s = -4.01$ is closer to zero at $s = -4$. The residue of the pole at $s = -4.01$ is equal to 0.033. It is too low as compared to other residues. Hence, we make a second-order approximation by neglecting the response generated by the pole at $s = -4.01$.

pole-zero cancellation

$$C_1(s) \approx \frac{0.87}{s} - \frac{5.3}{s+5} + \frac{4.4}{s+6}$$

and the output response is

$$c_1(t) \approx 0.87 - 5.3e^{-5t} + 4.4e^{-6t}$$

pole-zero cancellation

Determine whether there is cancellation between the zero and the pole closest to the zero. For any function for which pole-zero cancellation is valid, find the approximate response

$$C_2(s) = \frac{26.25(s+4)}{s(s+3.5)\,(s+5)(s+6)} \quad \ldots 1$$

If the partial fraction expansion for $C_1(s)$ is

$$C_2(s) = \frac{1}{s} - \frac{3.5}{s+5} + \frac{3.5}{s+6} - \frac{1}{s+3.5}$$

Here the pole at $s = -3.5$ is closer to zero at $s = -4$. The residue of the pole at $s = -4.01$ is equal to 1. It is not negligible as compared to other residues. Hence, we can not make a second-order approximation by neglecting the response generated by the pole at $s = 3.5$.

pole-zero cancellation

and the output response is

$$c_1(t) = 1 - 3.5\, e^{-5t} + 3.5\, e^{-6t} - e^{-3.5t}$$