U1

① Variable Separation Method

$$x + y\frac{dy}{dx} = 0$$

$$y\,dy = -x\,dx$$

$$\int y\,dy = -\int x\,dx$$

$$\frac{y^2}{2} + C_1 = -\frac{x^2}{2} + C_2$$

$$\frac{y^2+x^2}{2} = C \quad \ldots C = C_2 - C_1$$

$$\frac{y^2+x^2}{2} = K \quad \ldots K = 2C.$$

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③ Exact differential equation.

① Total derivative :

$$\frac{\partial}{\partial x} \Rightarrow e^{xy} \cdot \frac{\partial\, xy}{\partial x} = e^{xy} \cdot y$$

$$\frac{\partial}{\partial y} \Rightarrow e^{xy}\, \frac{\partial\, xy}{\partial y} = e^{xy} \cdot x$$

$$F(x,y) = x^2y^3 + e^{xy}$$

$$df = \frac{\partial F}{\partial x}\,\partial x + \frac{\partial F}{\partial y}\,dy$$

$$2xy^3 + ye^{xy}\,dx + 3x^2y^2 + xe^{xy}\,dy$$ ← total derivative

② Exact differential equation:

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad \ldots ①$$ ← must be homogeneous. [form]

if $u(x,y) = Mdx + Nxy$ then exact.

$Mdx + Ndy$ is exact iff,

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}} \longleftarrow$$ necessary condition of

find $u(x,y)$

$$\frac{\partial u}{\partial x}\partial x + \frac{\partial u}{\partial y}\partial y = M\partial x + N\partial y.$$

$$\frac{\partial u}{\partial x}\partial x = M\partial x \quad \ldots (1)$$

$$\frac{\partial u}{\partial y}\partial y = N\partial y \quad \ldots (2)$$

Integrate (1) to get $u = \int M\partial x + \kappa(y)$ ... (3)

differentiate (3) w.r.t y to get $\kappa(y)$'s value

$u = \int M dx + \kappa(y)$

hogaya solve

④ Integrating Factors:

when equations are non-exact i.e. $M_y \neq N_x$,

I

$$R(x) = \frac{1}{N}\left(M_y - N_x\right)$$

If the above term contains y, this won't be integrating factor.

else

$$IF = e^{\int R(x)\,dx}$$

II

$$R(x) = \frac{1}{M}\left(N_x - M_y\right)$$

IF contains any x term, this won't be IF.

else

$$IF = e^{\int R(y)\,dy}$$

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⑤ LDE Linear differential Equations:

First order ODE is linear if it can brought down to the form

$$y' + p(x)y = r(x)$$

Case 1: $r(x) = 0$ homogeneous

Case 2: $r(x) \neq 0$ heterogeneous

Case 1:

if, $r(x) = 0$

$$y = k \cdot e^{-\int p(x)\,dx}$$

case 2:

when $r(x) \neq 0$

$$y = e^{-\int p(x)\,dx} \cdot \int e^{\int p(x)\,dx}\, \underbrace{q(x)}_{r(x)} + C$$

⑥ Bernoulli's D.E

- First order LDE.

$$\frac{\partial y}{\partial x} + P(x) \cdot y = q(x) \cdot y^n \quad \cdots ①$$

divide both sides by 'y^n'

$$y^{-n} \frac{\partial y}{\partial x} + p(x)\, y^{(1-n)} = q(x)\,(1-n)$$

let $y^{(1-n)} = u$

$$\underbrace{\cancel{(1-n)}\, y^{-n} \cdot \frac{\partial y}{\partial x}}_{\text{replace this in ① with } \frac{\partial u}{\partial x}} = \frac{\partial u}{\partial x} \times \frac{1}{(1-n)}$$

$$\frac{\partial u}{\partial x} + (1-n)\, p(x)\, u = (1-n)\, q(x)$$

Method using Reduction to first order:

formule:

$$(\ast) \quad y_2 = u y_1 \; ; \quad \underline{u = \int U dx}_{\;(\ast)} \; ; \quad U = \frac{1}{y_1^2} \cdot e^{-\int p(x)\, dx}$$

another approach

if (x, y'', y')

if $(y, y', y'') \longrightarrow y' = k y' \Rightarrow \frac{\partial^2 y}{\partial x^2} = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial x}$

$$\therefore y'' = \partial z / \partial y \cdot z$$

Wronskain & Linear dependance:

if (y_1, y_2) are solutions and $W(y_1, y_2) = 0$

then (y_1, y_2) are linearly dependant.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

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② Second Order Homogeneous Linear differential equations with constant coefficients.

$$y'' + ay' + by = 0 \quad \ldots ①$$

let $y = e^{\lambda x}$, $\therefore y' = \lambda e^{\lambda x}$; $y'' = \lambda^2 e^{\lambda x}$

$$\therefore ① \Rightarrow \lambda^2 e^{\lambda x} + \lambda e^{\lambda x} + e^{\lambda x} = 0$$

$$(\lambda^2 + \lambda + 1) e^{\lambda x} = 0$$

$e^{\lambda x}$ can never be zero,

$\therefore \lambda^2 + \lambda + 1 = 0$ ... Auxillary equation

λ_1, λ_2 .. roots

$y_1(x) = e^{\lambda_1 x}$, $y_2(x) = e^{\lambda x}$

Solutions

Case I: real & distinct

$\lambda_1 \neq \lambda_2$

then $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$; $\therefore y = c_1 y_1 + c_2 y_2$

$y_1 = e^{-5x}$ & $y_2 = e^{-2x}$ ... if $\lambda = -5, -2$

Case 2: real and repeated

$\lambda = \lambda_1 = \lambda_2$

$y_1 = e^{\lambda x}$ $\qquad$ $y_2 = x e^{\lambda x}$

Case 3: Complex

$e^{i\theta} = \cos\theta + i\sin\theta$

let $\lambda = -2x \pm \sqrt{5}i$

$y_1 = \left(e^{(-2+\sqrt{5}i)x}\right) c_1$ $\qquad$ $y_2 = c_2\left(e^{(-2-\sqrt{5}i)x}\right)$ $\quad e^{\lambda x}$

$y = c_1 y_1 + c_2 y_2 \Rightarrow y = e^{-2x}\left[c_1 e^{(\sqrt{5}x)i} + c_2 e^{(-\sqrt{5}x)i}\right)$

$= e^{-2x}\left[c_1\left(\cos\sqrt{5}x + i\sin\sqrt{5}x\right) + c_2\cos(\sqrt{5}x)\right.$

$\left. - c_2 i\sin\sqrt{5}x\right]$

$A = c_1 + c_2 \qquad B = c_1 - c_2$

$= e^{-2x}\left[A\cos\sqrt{5}x + B\sin\sqrt{5}x\right]$

$$y = e^{\text{real-part}}\left[A\cos(\text{imaginary} \times x) + B\sin(\text{Imaginary} \times x)\right)$$

$$y_1 = e^{\text{real}}\cos(\text{img} \times x)$$

$$y_2 = e^{\text{real}}\sin(\text{img} \times x)$$

<u>Euler Cauchy</u> $\quad x^n y^{(n)} + a x^{(n-1)} y^{(n-1)} \ldots\ldots + x^0 y = 0$

e.g. $x^2 y'' + a x y' + b y$

$$y(x) = x^m$$

$$\therefore\ y'(x) = m x^{(m-1)} \quad ; \quad y''(x) = m(m-1)\, x^{(m-2)}$$

$$\cancel{x^2}\, m(m-1)\, x^{(m-\cancel{2})} + a\cancel{x}\, m x^{(m-\cancel{1})} + b x^m$$

$$(m^2 - m + am + b)\, x^m$$

$$(m^2 - m(a-1) + b)\, x^m$$

roots $\quad m_1 \quad m_2$

soln $\Rightarrow \quad x^{m_1} \quad x^{m_2}$

Case I:
Real & distinct

$y_1 = x^{m_1}$

$y_2 = x^{m_2}$

$y = C_1 x^{m_1} + C_2 x^{m_2}$

Case 2:
real and repeated

$m = m_1 = m_2$

$y_1 = x^m$

$y_2 = \ln(x)\, x^m$

$(\ln(x))^2 \ldots (\ln(x))^n$

Case 3:
Complex

$y = x^{\text{Real}}\left(A\cos(\ln(x)\cdot \text{img}) + B\sin(\ln(x)\cdot \text{img})\right)$

$y_1 = x^{\text{real}}\cos(\ln(x)\cdot \text{img})$

$y_2 = x^{\text{real}}\sin(\ln(x)\cdot \text{img})$

Non-homogeneous Differential equations:

RHS is non-zero.

① Method of undetermined coefficients:

$$y'' + ay' + by = r(x) \quad \cdots ①$$

$$y(x) = y_h(x) + y_p(x)$$

$$\underbrace{y'' + ay' + by = 0}_{\text{the solution here will be } y_h(x)} \quad \cdots ② \quad \cdots \text{homogeneous equation for ①.}$$

for $y_p(x)$:

$r(x)$	choice of $y_p(x)$
$e^{\alpha x}$	$k e^{\alpha x}$
Polynomial	$k_n x^n + k_{n-1} x^{n-1} \cdots + k_0$
$\cos(\omega x)$, $\sin(\omega x)$	$K\cos(\omega x) + M\sin(\omega x)$
$e^{\alpha x}\cos\omega x$, $e^{\alpha x}\sin\omega x$	$e^{\alpha x}(K\cos(\omega x) + M\sin(\omega x))$

Variation of Parameters

$$y'' + p(x)y' + q(x)y = r(x)$$

$$y(x) = y_h + y_p$$

$$y_h(x) = c_1 y_1 + c_2 y_2$$

$$y_p(x) = u_1 y_1 + u_2 y_2 \, ,$$

where,

$$u_1 = -\int \frac{y_2 r}{W} \quad \text{and} \quad u_2 = \int \frac{y_1 r}{W}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

for more than 2 roots, it might get hard

$$W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ 1 & y_2'' & y_3'' \end{vmatrix} \qquad W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y_2' & 0 & y_3' \\ y_3' & 1 & y_3'' \end{vmatrix}$$

$$u_1 = \int \frac{W_1 r}{W} \qquad u_2 = \int \frac{W_2 r}{W} \qquad u_3 = \int \frac{W_3 r}{W} \quad \ldots$$

Laplace

$$f(t) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \quad \dots \quad s > a$$

$$\mathcal{L}\{1\} = \frac{1}{s}$$

$$\mathcal{L}\{t\} = \frac{1}{s^2}$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}\{\sin\omega t\} = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}\{\cos\omega t\} = \frac{s}{s^2+\omega^2}$$

First Shift theorem:

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

Second shift theorem:

$$\mathcal{L}^{-1}\{e^{-as} F(s)\} = \mathcal{L}\{f(t-a) \cdot u(t-a)\}$$

$\boxed{\begin{array}{l}\text{* When dividing by } t \text{, we have to integrate numerator by } \displaystyle\int_s^{\infty} F(s)\,ds\end{array}}$

$\rightarrow \quad t^n f(t) \longleftrightarrow (-1)^n \dfrac{d^n}{ds^n}\left[F(s)\right]$

$$\Rightarrow \quad \delta(t-a) = e^{-as}$$

$$\delta(t) = \delta(t-0) = e^{-0s}$$

$$= 1$$

$$\Rightarrow \quad \int_0^t f(t) = \frac{F(s)}{s}$$

US

Dot Product : $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 \ldots$

Cross Product : $\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

$r(t) \leftarrow$ parameterised curve

$r'(t)$

$|r(t)|$ ← Mod operation $\Rightarrow \sqrt{r_1^2 + r_2^2 + \ldots r_n^2}$

Unit tangent vector: $\dfrac{r'}{|r'|}$

to sketch tangent vector:

$$q(\omega) = r + \omega r'$$

Gradient:

∇
nebla $\longrightarrow \left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z} \right)$

$$\nabla_f = \left(\frac{\partial F_1}{\partial x}, \frac{\partial F_2}{\partial y}, \frac{\partial F_3}{\partial z} \right)$$

Laplacian:

$$\nabla^2 f = \frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_2}{\partial y^2} + \frac{\partial^2 f_3}{\partial z^2}$$

$$\nabla^2 (f \pm g) = \nabla^2 f \pm \nabla^2 g$$

$$\nabla^2 (fg) = g \nabla^2 f + f \nabla^2 g + 2 \nabla f \nabla g$$

Divergence operator:

$$\text{div}(f) = \frac{\partial f_1}{x_1} + \frac{\partial f_2}{x_2} \cdots + \frac{\partial f_n}{x_n}$$

Sum of partial derivatives

$\text{div}(f)$	> 0	Source
	< 0	Sink
	$= 0$	incompressible

$$\text{div}(\nabla f) = \nabla^2 f$$

Angle between two surfaces:

$$\theta = \cos^{-1}\left(\frac{\nabla f \cdot \nabla g}{|\nabla f| \cdot |\nabla g|} \right)$$

Curl of a Vector:

$$\vec{v} = [v_1, v_2, v_3]$$

$$\text{curl}(\vec{v}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

- $\text{curl}(\text{grad } f) = \text{div}(\text{curl}(\vec{v})) = 0$
- $\text{curl}(\vec{v} + \vec{v}) = \text{curl}(\vec{u}) + \text{curl}(\vec{v})$
- $\text{curl}(\alpha \vec{v}) = \alpha \, \text{curl}(\vec{u})$

$$\text{div}(\vec{u} \times \vec{v}) = \vec{v} \cdot \text{curl}(\vec{u}) - \vec{u} \cdot \text{curl}(\vec{v})$$

Line Integrals:

for scalar fields: $\mathbb{R}^n \to \mathbb{R}$

$$\int_C F \cdot ds = \int_{t=a}^{b} f(r(t)) \cdot |r'(t)| \, dt$$

for vector fields $\mathbb{R}^n \to \mathbb{R}^m$

$$\int_C F \cdot ds = \int_{t=a}^{b} f(r(t)) \cdot r'(t)$$

Steps:

1. Write down given: F, $r(t)$ [if given], all equations
2. Parameterize curve, write $r(t)$
3. Calculate $F(r(t))$:

   replace F's componen'ts with $r(t)$'s values
4. $r'(t)$ calculation
5. $F(r(t)) \cdot r'(t)$
6. integrate $\uparrow$ for range of t

Path Independence:

$f \rightarrow$ Potential field (scalar)

$F \rightarrow$ Conservative Vector field.

if curl $(\vec{F}) = 0$, then f exists:

$$F = \nabla f$$

$$[F_1, F_2] = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] \cdot f$$

equate F's components with ∇ to find f.

$$\frac{\partial f}{\partial x} = F_1 \quad \cdots (1) \qquad \Bigg| \qquad \frac{\partial f}{\partial y} = F_2 \quad \cdots (2)$$

$$\int \frac{\partial f}{\partial x} = \int F_1 \, \partial x$$

$$f = \int F_1 \, \partial x + C(y) \quad \cdots (1)$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial y}$$

(2) $\rightarrow$ = $\underline{\qquad\qquad}$

find $c(y)$ substitute.

if f exists, find f! then,

$$\int_C F \cdot dr = f(B) - f(A)$$ ← Points of co-ords

Point B co-ords

Green's theorem:

Relation betwn double integral & line integral

$$\iint_R [\ ] \partial A \qquad \oint_C F \cdot dr$$

F = 2 dim vector field

C = Closed bounded curve.

$$\iint_R \left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right] \partial A$$

Surface Integrals:

$$\iint_R F\cdot n\, ds = \iint_R F(r(u,v))\cdot N\, du dv$$

N: Normal Vector

→ if $-ve$ orientation, $N = (-1)N$

① Parameterize the structure given.

e.g. cylinder $x^2+y^2=9$, $z \leq 5$, $z \geq 0$

$$r(u,v) = (3\cos u,\ 3\sin u,\ z)$$

$$0 \leq u \leq 2\pi$$

$$0 \leq v \leq 5$$

② find r_u, r_v, and N.

$N = r_u \times r_v$

↓

$$\begin{bmatrix} i & j & k \\ r_{u_1} & r_{u_2} & r_{u_3} \\ r_{v_1} & r_{v_2} & r_{v_3} \end{bmatrix}$$

③ find $F(r(u,v))$

④ $F(r(u,v)) \cdot N$

$$\iint_S \bar{F} \cdot \vec{n} \, ds = \iint_R F(r(u,v)) \cdot \vec{N} \, \partial u \, \partial v$$

Gauss Divergence

- Closed bounded region in space!!

$$\iint_{\partial T} F \cdot n \, \partial A = \iiint_T \operatorname{div}(F) \, \partial v$$

① Calculate $\operatorname{div}(F)$.

② Put in above formula

volume of cone $= 4 \times \frac{2}{3} \pi r^2 h$ | Volume of sphere $= \frac{4}{3} \pi r^3$

Stoke's theorem:

$$\iint_S \text{curl}(F) \cdot \vec{N} \, ds = \oint F \cdot r'(s) \, ds$$

$$\uparrow$$

Stoke's theorem.

Partial Derivation Classification:

Order - Highest derivative that occur in PDE

$u_{xxx} + u_{yy} + u_t \quad \Rightarrow \text{Order} = 3 \qquad u_{xxx} = \frac{\partial^3 u}{\partial x^3}$

Quasi-Linear: Degree of highest order in equation are linear (1)

Semi-Linear PDE - (has to be Quasi) Co-efficient of highest order must not have dependent variable

e.g. $u^2 x \, u_{tt} \ldots$ ⓧ Not semi linear.

Linear: (Semi linear): All dependent variables and their derivatives have degree 1.

Non-linear (Not Quassi)

Homogeneous PDE : All terms contain dependent variable or its derivatives.