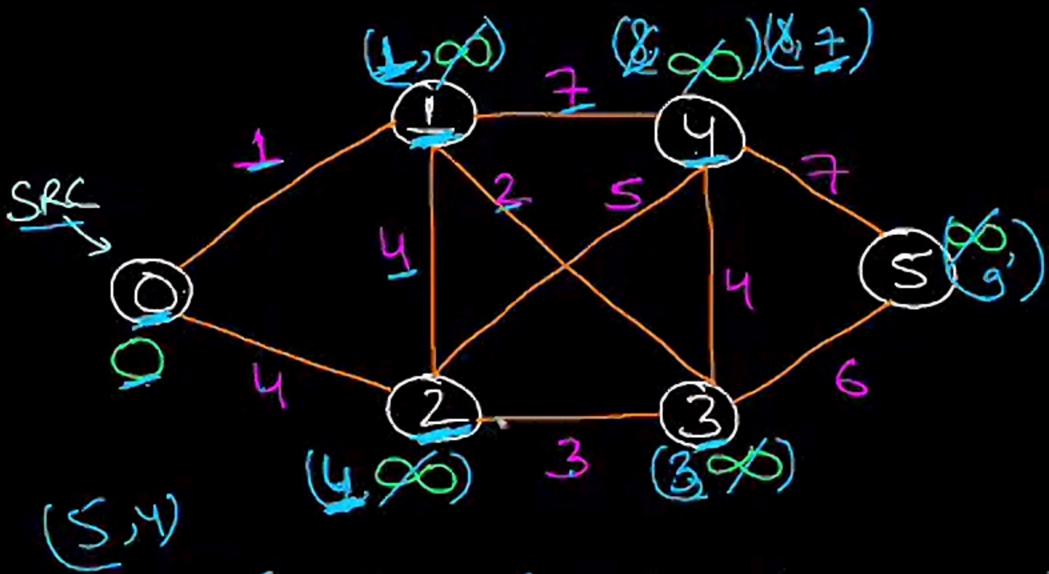
# Equivalent Statements

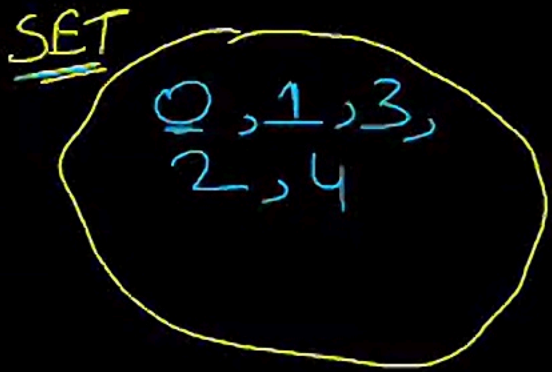
- The statements $\neg(P \wedge Q)$ and $(\neg P) \vee (\neg Q)$ are logically equivalent, since $\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$ is always true.

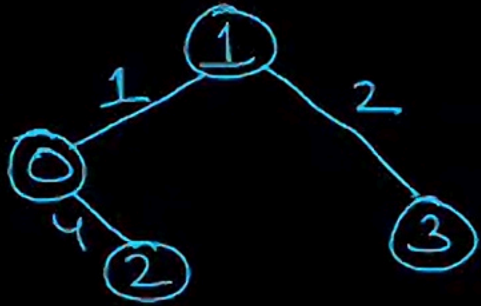
| $P$ | $Q$ | $\neg P$ | $\neg Q$ | $\neg(P \wedge Q)$ | $(\neg P) \vee (\neg Q)$ | $\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$ |
|---|---|---|---|---|---|---|
| T | T | F | F | F | F | T |
| T | F | F | T | T | T | T |
| F | T | T | F | T | T | T |
| F | F | T | T | T | T | T |

# Tautology by truth table

| $p$ | $q$ | $\neg p$ | $p \vee q$ | $\neg p \wedge (p \vee q)$ | $[\neg p \wedge (p \vee q)] \rightarrow q$ |
|---|---|---|---|---|---|
| T | T | F | T | F | T |
| T | F | F | T | F | T |
| F | T | T | T | T | T |
| F | F | T | F | F | T |



| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| Value | 0 | ~~∞~~ 1 | ~~∞~~ 4 | ~~∞~~ 3 | ~~∞~~ ~~8~~ 7 | ~~∞~~ 9 |

NOTE: We Need only (V-1) STEPS like in MST algo.

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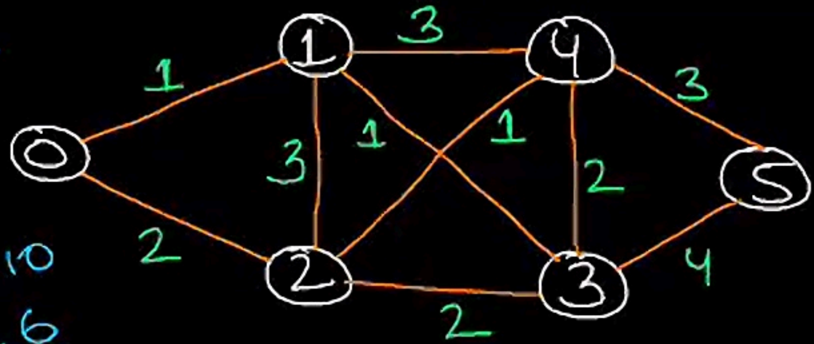


$E = 10$

$V = 6$

EL

| SRC | DST | WT |
|---|---|---|
| 0 | 1 | 1 |
| 1 | 3 | 1 |
| 2 | 4 | 1 |
| 0 | 2 | 2 |
| 2 | 3 | 2 |
| 3 | 4 | 2 |
| 1 | 2 | 3 |
| 1 | 4 | 3 |
| 4 | 5 | 3 |
| 3 | 5 | 4 |

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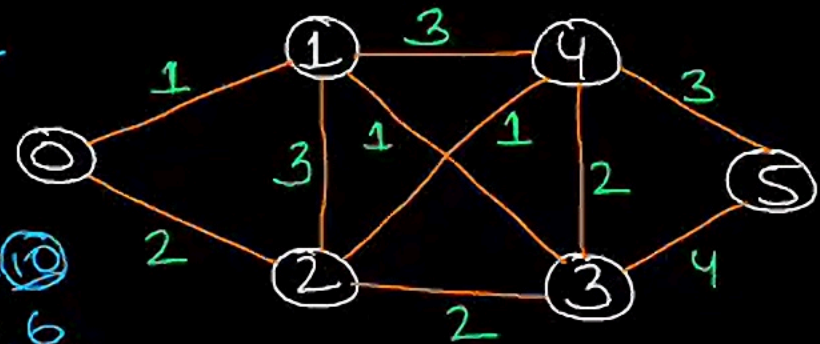
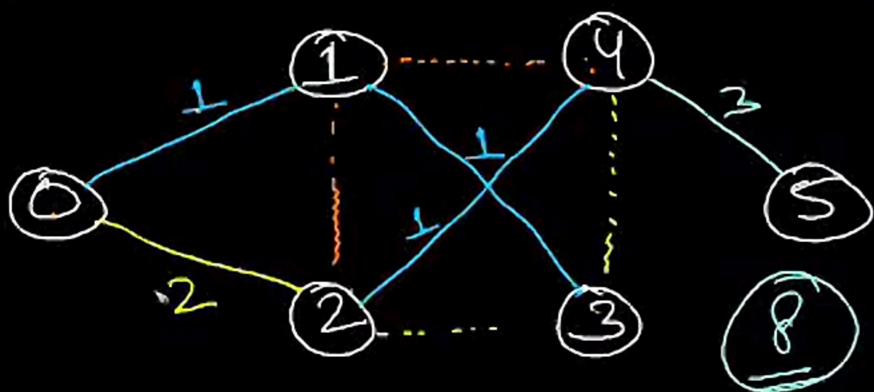



$E = 10$

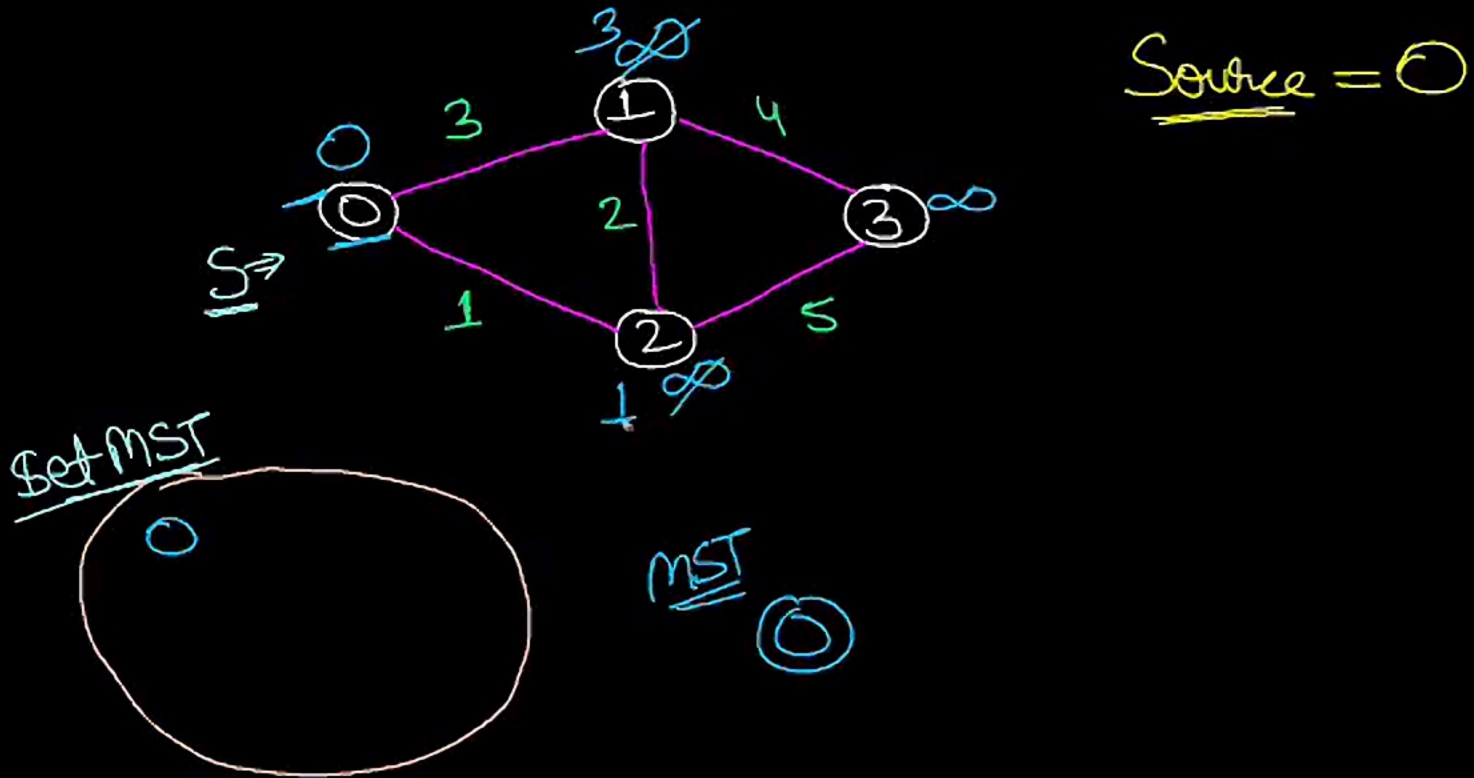
$V = 6$

$(V-1) = 5$

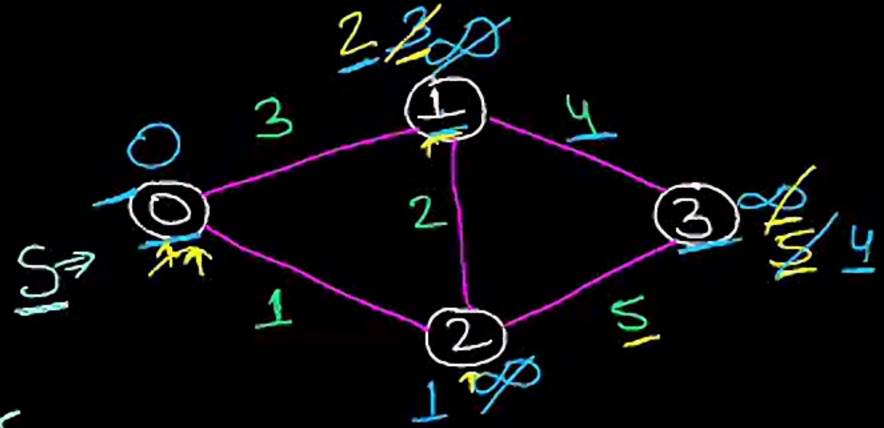
MST




| EL | SRC | DST | WT |
|---|---|---|---|
| | 0 | 1 | 1 |
| 1 | 1 | 3 | 1 |
| 2 | 2 | 4 | 1 |
| 3 | 0 | 2 | 2 |
| 4 | 2 | 3 | 2 |
| 5 | 3 | 4 | 2 |
| | 1 | 2 | 3 |
| | 1 | 4 | 3 |
| | 4 | 5 | 3 |
| | 3 | 5 | 4 |



Source = 0
S
Set MST
0
MST
0



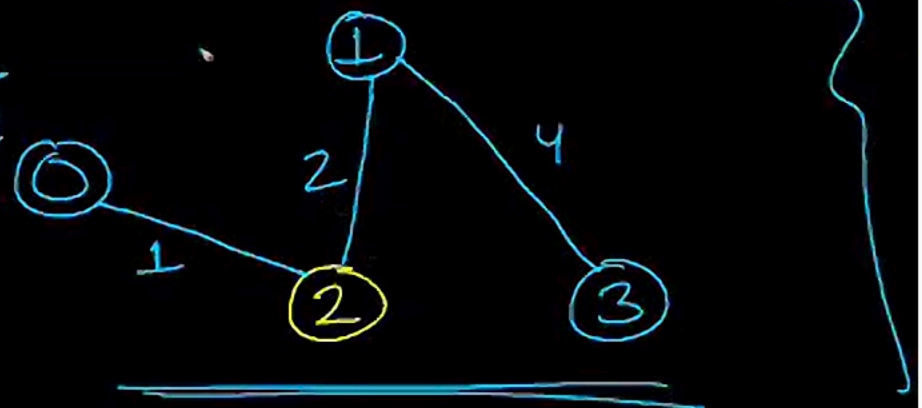

Source = 0

Set MST

0, 2,

1, 3

MST



# Monoid

Ex. Show that the set 'N' is a monoid with respect to multiplication.

- Solution: Here, $N = \{1,2,3,4,\ldots\ldots\}$

  1. Closure property : We know that product of two natural numbers is again a natural number.

  i.e., $a.b \in N$ for all $a,b \in N$

  $\therefore$ Multiplication is a closed operation.

  2. Associativity : Multiplication of natural numbers is associative.

  i.e., $(a.b).c = a.(b.c)$ for all $a,b,c \in N$

  3. Identity : We have, $1 \in N$ such that

  $a.1 = 1.a = a$ for all $a \in N$.

  $\therefore$ Identity element exists, and 1 is the identity element.

# Monoid

Ex. Show that the set 'N' is a monoid with respect to multiplication.

- Solution: Here, N = {1,2,3,4,......}

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3. Identity : We have, $1 \in N$ such that

   $a.1 = 1.a = a$ for all $a \in N$.

   $\therefore$ Identity element exists, and 1 is the identity element.

Hence, N is a monoid with respect to multiplication.

# Subsemigroup & submonoid

closed Associative.

**Subsemigroup** : Let (S, * ) be a semigroup and let T be a subset of S. If T is closed under operation * , then (T, * ) is called a subsemigroup of (S, * ).

Ex: (N, .) is semigroup and T is set of multiples of positive integer m then (T,.) is a sub semigroup.

**Submonoid :** Let (S, * ) be a monoid with identity e, and let T be a non- empty subset of S. If T is closed under the operation * and e $\in$ T, then (T, * ) is called a submonoid of (S, * ).

# Inverse Element

- Let (S, *) be an Algebraic Structure and let $e$ be the identity element of S. An element $a$ is said to be *left invertible w.r.t. ** if there exists an element $b$ in S such that $b * a = e$ and $b$ is called the *left inverse of a*
- Similarly an element $a$ is said to be *right invertible* w.r.t. * if there exists an element $c$ in S such that $a * c = e$ and $c$ is called the *right inverse of a*
- If $a$ is both left and right invertible then we say that $a$ is invertible. If * is an associative operation, then the inverse of $a$, if it exists, is unique and is denoted by $a^{-1}$
- The identity element $e$ is its own inverse $e^{-1} = e$

**To show that the inverse of a is unique. Let us assume that x and y are two inverses of a. Then**

$$
\begin{aligned}
y &= y * e \\
&= y * (a * x) \\
&= (y * a) * x \\
&= e * x \\
&= x
\end{aligned}
$$

**Thus the two inverses are equal, i.e. inverse of a is unique and we denote it as $a^{-1}$**

$a \leftarrow x$

$y$

$a * x = e$

$y * a = e$

# Group

A Group $< G , * >$ is an algebraic system in which $*$ on G satisfies four condition

- **Closure Property**

  For all $x , y \in G$

  $x * y \in G$
- **Associative Property**

  For all $x , y , z \in G$

  $x * (y * z) = (x * y) * z$
- **Existence of Identity element**

  There exists an element $e \in G$ such that for any $a \in G$

  $x * e = x = e * x$
- **Existence of Inverse Element**

  For every $x \in G$ ,there exists an element denoted by $a^{-1} \in G$ such that

  $x^{-1} * x = x * x^{-1} = e$

# Abelian Group

A Group $< G , * >$in which the operation $*$ is commutative is called abelian Group i.e. $\forall a,b \in G , a * b = b * a$

**Example**

1. $< Z , + >$ is Abelian Group
2. $< Q , + >$ is abelian Group

# Group Properties

Theorem : The identity element in a group is unique.

Proof : Suppose e and e′ are two identity elements of a group (G,*)

$e, e' \in G.$

∴ e and e′ are the elements of G

If e is the identity element, then

$$e * e' = e' \quad ......(1)$$

If e′ is the identity element, then

$$e * e' = e \quad ......(2)$$

From eq. (1) and (2),

$$e = e'$$

Hence the identity element of a group is unique.

Theorem 2 : Inverse of each element of a group < G , $*$ > is unique

$a \in G$

Proof :

$b, c \in G.$

⇒ Let a be any element of G and e the identity of G

⇒ Suppose b and c are two different inverse of a in G.

⇒ $a * b = e = b * a$ (if b is an inverse of a)

⇒ $a * c = e = c * a$ (if c is an inverse of a)

⇒ Now , $b = b * e$

$$= b * (a * c)$$

$$= (b * a) * c$$

$$= e * c = c$$

Thus a has unique inverse

Theorem 3 : if $a^{-1}$ is the inverse of an element $a$ of group $< G , * >$ then $(a^{-1})^{-1} = a$

Proof :

$e \in G.$

$\Rightarrow$ Let $e$ be the identity of Group $< G , * >$

$\Rightarrow a^{-1} * a = e$

$\Rightarrow (a^{-1})^{-1} * (a^{-1} * a) = (a^{-1})^{-1} * e$

$\Rightarrow ((a^{-1})^{-1} * a^{-1}) * a = (a^{-1})^{-1}$

$\Rightarrow e * a = (a^{-1})^{-1}$

$\Rightarrow (a^{-1})^{-1} = a$

Cancellation Property : if a , b and c be any three elements of a group $< G, \bullet >$ then

$ab = ac \Rightarrow b = c$ left cancellation

$ba = ca \Rightarrow b = c$ right cancellation

Proof :

$\Rightarrow$ Let $a \in G$ and also $a^{-1} \in G$

$\Rightarrow aa^{-1} = e = a^{-1}a$

$\Rightarrow$ where e is identity of G

$\Rightarrow$ Now , $ab = ac$

$\Rightarrow \quad a^{-1}(ab) = a^{-1}(ac)$

$\Rightarrow \quad (a^{-1}a)b = (a^{-1}a)c$

$\Rightarrow \quad e . b = e . c$

$\Rightarrow \quad b = c$

$\Rightarrow$ similarly , $ba = ca$

$\Rightarrow \quad b = c$

$a * a = a$

only one idempotent element in the group & that is $e$

Ex. If $(G, *)$ is a group and $a \in G$ such that $a * a = a$, then show that $a = e$, where $e$ is identity element in $G$.

Proof: Given that, $a * a = a$

$\Rightarrow a * a = a * e$ ( Since, $e$ is identity in $G$)

$\Rightarrow a = e$ ( By left cancellation law)

Hence, the result follows.

Note: $a^2 = a * a$

$a^3 = a * a * a$ etc.

Ex. In a group $(G, *)$, if $(a * b)^2 = a^2 * b^2 \quad \forall a, b \in G$

then show that G is abelian group.

Proof: Given that $(a * b)^2 = a^2 * b^2$

$\Rightarrow (a * b) * (a * b) = (a * a) * (b * b)$

$\Rightarrow a * (b * a) * b = a * (a * b) * b$ ( By associative law)

$\Rightarrow (b * a) * b = (a * b) * b$ ( By left cancellation law)

$\Rightarrow (b * a) = (a * b)$ ( By right cancellation law)

Hence, G is abelian group.