College of Engineering Pune
Ordinary Differential Equations and Multivariate Calculus
Tutorial-4 (2020-2021)

The main idea behind the Laplace Transformation is that we can solve an equation (or system of equations) containing differential and integral terms by transforming the equation in t-space to one in s-space. Usually t is time and s is frequency!

1. Laplace Transform and Inverse Laplace Transform of a function. State and prove the algebraic properties of Laplace Transform.
2. State the first shifting, second shifting and Convolution theorems.
3. When do we say that a function is of **exponential order**?
4. Why the limits of the integration in the definition of Laplace Transform is from 0 to ∞? Give the logical justification.
5. Is $L\{f(t)g(t)\} = L\{f(t)\}L\{g(t)\}$? Justify your answer!
6. Which of the following functions are of exponential order and why?
   a) $sin(e^{t^2})$
   b) e^{t^π}
7. Give an example of a function which of exponential order but its derivative is not of exponential order.
8. Give an example of a function whose Laplace transform exists, such that f is not piecewise continuous but has exponential order.
9. Give an example of a function whose Laplace transform exists, such that f is continuous but is not of exponential order.
10. Let f be a piecewise continuous function of exponential order and F be a Laplace transform of f then prove that:

$$\lim_{s\to\infty} F(s) = 0$$

11. Is it possible to find piecewise functions of exponential order whose Laplace transforms are:

    a) $F(s) = s, \qquad s \in R$

    b) $F(s) = \dfrac{s-1}{s+1}, \quad s > -1$

12. Is it possible to find functions (you may think of generalized functions such as Dirac delta function) whose Laplace transforms are:

    a) $F(s) = \dfrac{s^2}{s^2+1}, \quad s \in R$

    b) $F(s) = \dfrac{s^2}{s^2-1}, \quad s > 1$

13. Find the Laplace Transforms of the following functions:

    a) $(5e^{2t} - 3)^2$ Ans. $\dfrac{25}{s-4} - \dfrac{30}{s-2} + \dfrac{9}{s}$

    b) $\sin 3t - 2\cos 5t$ Ans. $\dfrac{3}{s^2+9} - 2\dfrac{s}{s^2+25}$

    c) $\cosh at - \cos at$ Ans.$\dfrac{2a^2 s}{s^4 - a^4}$

    d) $e^t(1+t)^2$ Ans. $\dfrac{s^2+1}{(s-1)^3}$

    e) $f(t) = \begin{cases} t, & 0 < t < 1 \\ e^{1-t}, & t > 1. \end{cases}$ Ans. $\dfrac{1}{s^2}\left[1 - e^{-s}\left(\dfrac{2s+1}{s+1}\right)\right]$

    f) $t^{7/2}e^{3t}$ Ans. $\dfrac{105\sqrt{\pi}}{16(s-3)^{9/2}}$

    g) $f(t) = t\cos at$ Ans. (Use $\mathcal{L}\{tf(t)\}$). $\dfrac{s^2-a^2}{(s^2+a^2)^2}$

    h) $\sin^2 t$ Ans. (Use $\mathcal{L}\{f'\}$). $\dfrac{2}{s(s^2+4)}$

    i) $\dfrac{e^{-at} - e^{-bt}}{t}$ Ans. (Use $\mathcal{L}\{f(t)/t\} = \int_s^\infty F(u)du$). $\ln \dfrac{s+b}{s+a}$

    j) $\dfrac{1}{2}\, t^2 \, \cos \dfrac{\pi}{2}\, t$ Ans. $16\dfrac{s(4s^2 - 3\pi^2)}{(4s^2+\pi^2)^3}$

    k) $e^{-t}\sinh 4t$ Ans. $\dfrac{4}{s^2+2s-15}$

l) $\dfrac{e^t\,\delta(t-2)}{t}$ Ans. $\dfrac{e^{-2(s-1)}}{2}$

m) $\delta(t-3)\,U(t-3)$ Ans. e^{-3s}

n) $t^2\sin 2t$ Ans. (Use $\mathcal{L}\left\{t^2 f(t)\right\} = F''(s)$). $\dfrac{-4(4-3s^2)}{(s^2+4)^3}$

o) $\displaystyle\int_0^t \frac{1-e^{-u}}{u}du$ Ans. (Use $\mathcal{L}\left\{\displaystyle\int_o^t f(u)du\right\} = \dfrac{\mathcal{L}\{f\}}{s}$). $\dfrac{1}{s}\ln\left(1+\dfrac{1}{s}\right)$

p) First sketch and express in terms of unit step:

$e^{-\pi t/2}$, $1 < t < 3$; 0 outside. Ans. $2\left(\dfrac{e^{-s-\pi/2} - e^{-3s-3\pi/2}}{2s+\pi}\right)$

q) $4t * e^{-2t}$, $*$ denotes the convolution. Ans. $\dfrac{8}{s^3(s+2)}$

13. Find the inverse Laplace transform of the following:

a) $\dfrac{0.1s+0.9}{s^2+3.24}$ Ans. $0.1\cos 1.8t + 0.5\sin 1.8t$

b) $\dfrac{-s-10}{s^2-s-2}$ Ans.$3e^{-t} - 4e^{2t}$

c) $\dfrac{1}{(s-1)(s^2+4)} + \dfrac{4}{s^5}$ Ans. $\dfrac{e^t}{5} - \dfrac{\cos 2t}{5} - \dfrac{\sin 2t}{10} + \dfrac{t^4}{6}$

d) $\dfrac{3s+1}{s^2+6s+13}$ Ans. $e^{-3t}(3\cos 2t - 4\sin 2t)$

e) $\dfrac{s^2}{(s-1)^4}$ Ans. $e^t\left(t + t^2 + \dfrac{t^3}{6})\right)$

f) $\dfrac{e^{-\pi s}}{s^2+9}$ Ans. $\dfrac{1}{3}\sin 3(t-\pi)U(t-\pi)$

g) $\dfrac{1-e^{-s}}{s^2}$ Ans. t, if $t < 1$ and 1 if $t > 1$.

h) $\cot^{-1}\left(\dfrac{s}{\omega}\right)$ Ans. (Let $f(t) = \mathcal{L}^{-1}F(s)$. Use $\mathcal{L}^{-1}F'(s) = -tf(t)$). $(\sin\omega t)/t$.

i) $\dfrac{1}{2}\ln\left(\dfrac{s^2-a^2}{s^2}\right)$ Ans. $\dfrac{1-\cosh at}{t}$

j) $\ln\sqrt{\dfrac{s^2+b^2}{s^2+a^2}}$ Ans. $\dfrac{\cos at - \cos bt}{t}$

k) $\dfrac{e^{-2s}}{s^6}$. Also sketch $f(t)$. Ans. $\dfrac{1}{120}(t-2)^5\;U(t-2)$

l) $\dfrac{s^3 - 3s^2 + 6s - 4}{(s^2 - 2s + 2)^2}$ Ans. $e^t\ (t \sin t + \cos t)$

m) $s \ln\left(\dfrac{s}{\sqrt{s^2+1}}\right)$ Ans.(Use $\mathcal{L}^{-1}F''(s) = t^2 f(t)$).

n) $\frac{e^{-s}}{s} \tan^{-1}(\frac{s-1}{4})$ Ans. Let $F(s) = e^{-s}/s$, $G(s) = \tan^{-1}\left(\dfrac{s-1}{4}\right)$. Then $\mathcal{L}^{-1}F(s) = U(t-1)$ and $\mathcal{L}^{-1}G(s) = \dfrac{-e^t \sin 4t}{t}$. By convolution thm, the required ans is $\mathcal{L}^{-1}F(s)G(s) = U(t-1) * \dfrac{-e^t \sin 4t}{t}$.

14. Solve using Laplace transform:

a) $y' + 2y = 4te^{-2t}, \quad y(0) = -3$
Ans. $y(t) = 2t^2e^{-2t} - 3e^{-2t}$

b) $y'' + y = r(t)$, $r(t) = t$ if $1 < t < 2$, 0 otherwise. $y(0) = y'(0) = 0$
Ans. $y = [t - \cos(t-1) - \sin(t-1)]U(t-1) + [-t + 2\cos(t-2) + \sin(t-2)]U(t-2)$

c) $y'' + y = e^{-2t} \sin t, y(0) = y'(0) = 0$.
Ans. $y = \frac{1}{8}[\sin t - \cos t + e^{-2t}(\sin t + \cos t)]$

d) $y'' + 2y' + 5y = 50t - 150, y(3) = -4, y'(3) = 14$.
Ans. $y = 10(t-3) - 4 + 2e^{-(t-3)} \sin 2(t-3)$

e) $y'' + 2y' + 5y = e^{-t} \sin t,\ y(0) = 0, y'(0) = 1$
Ans. $y = e^{-t}(\sin t + \sin 2t)/3$

f) Find the current $i(t)$ in an LC circuit assuming $L = 1$henry, $C = 1$ farad, zero initial current and charge on the capacitor and $v(t) = 1 - e^{-t}$ if $0 < t < \pi$ and 0, otherwise.
Ans. $\frac{1}{2}(e^{-t} - \cos t + \sin t)$, if $0 < t < \pi$ and $\frac{1}{2}[-(1+e^{-\pi})\cos t + (3 - e^{-\pi})\sin t]$, if $t > \pi$.

15. Solve the following linear integral equations:

    a) $y(t) = \sin 2t + \int_0^t y(\tau) \sin 2(t-\tau) d\tau.$ Ans. $\sqrt{2} \sin \sqrt{2}\, t$

    b) $y(t) = 1 - \sinh t + \int_0^t (1+\tau)\, y(t-\tau) d\tau.$ Ans. $\cosh t$

16. State and prove the theorem on existence of Laplace transforms. Does it give necessary and sufficient conditions for existence? Justify your answer.

17. Find Laplace transform of n^{th} derivative of a function $f(t)$ stating clearly the necessary conditions on the function and its derivatives.

18. Find the Laplace transform of $\int_0^t f(\tau) d\tau$ stating clearly the necessary conditions under which it exists.

19. Find the current in an RLC circuit if $R = 4\Omega, L = 1H, C = 0.05F$ and the applied voltage is $v = 34e^{-t}V, \quad 0 < t < 4; 0$ for $t > 4$. Assume that current and charge are 0 initially. Solve using Laplace transform method showing all the details.

20. Find the Laplace transform of a periodic function and hence find the Laplace transform of half wave rectification of $sin\omega t$.

21. Define convolution of two functions. Prove the commutative, associative and distributive properties of convolution of two functions.

22. State and prove the convolution theorem for Laplace transforms.

23. Write a summary on Laplace transforms in your own words not exceeding 500 words.

∗ Please report any mistakes in the problems and/or answers given here.

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