Stability Analysis using Root Locus

Unit- III

Introduction

- Primary aim of a control engineer: To design a control system that meets the desired specifications.
- Stability of the system is also a necessary condition.
- Analytical approach can be followed till second order systems while Routh-Hurwitz criterion can be followed for higher order systems.
- Drawbacks of Routh-Hurwitz criterion -

  (a) does not provide sufficient information about relative stability, which may make the system unstable,

  (b) does not help in designing a control system where tuning of parameters is important.

Example:

- The open loop gain 'K' of the system in *Fig 5.1* can be tuned from $-\infty$ to $+\infty$. System performance varies with 'K'. Routh-Hurwitz criterion can neither provide information about amount of overshoot, settling time nor can it tell about the system response w.r.t the open loop gain.
- The poles of the closed loop system is obtained from the characteristic equation

$$1 + K\big(G(s)H(s)\big) = 0 \tag{5.1.1}$$

- It is important to know how the closed loop move in the s-plane as the gain 'K' is varied.

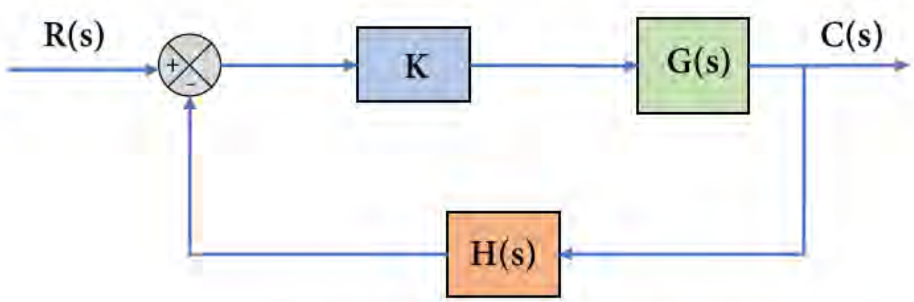


Fig 5.1.1 General feedback system

Root Locus Technique

- The locus of the migration of the roots, of the characteristic equation, in the *s*-plane is called '*Root Locus*'.
- The root locus technique was introduced by W.R. Evans.
- The root locus technique is a graphical method for sketching the locus of roots of the characteristic equation in the *s*-plane as the design parameters of the corresponding system is varied.

References:

- Graphical Analysis of Control System, AIEE Trans. Part II,67(1948),pp.547-551.
- Control System Synthesis by Root Locus Method, AIEE Trans. Part II,69(1950),pp.66-69

Root Locus: Example

- Let us consider a second order system, $G(s) = \frac{1}{s(s+a)}$, and $H(s) = 1$ *i.e.* the poles of the open loop system is at $s = 0, -a$ and it is an unity feedback system.
- The characteristic equation of the system is

$$s^2 + as + K = 0 \tag{5.1.2}$$

- The roots of the equation are

$$s_{1,2} = \frac{-a \pm \sqrt{a^2 - 4K}}{2} \tag{5.1.3}$$

- For positive values of p and K the system the system is always stable and the roots lie in the left half of the s-plane.
- The relative stability of the system depends on the location of the roots.
- Desired transient response can be obtained by varying the open loop gain K.

Root Locus: Example (contd.)

- As K is varied from 0 to ∞, the loci of the roots s_1 and s_2 in the s-plane is explained considering three cases:
  - Case 1 – For $0 \leq K \leq \frac{a^2}{4}$ the roots are distinct. When $K = 0$, the roots are at $s_1 = 0$ and $s_2 = -a$, which are the poles of the open loop system.
  - Case 2 – For $K = \frac{a^2}{4}$, the roots are real and equal *i.e.* $s_1 = s_2 = -\frac{a}{2}$. As K is varied from 0 to ∞, one root start moving from $s_1 = 0$ and the other starts moving from $s_2 = -a$ in opposite directions and at $K = \frac{a^2}{4}$ both roots meet at $s = -\frac{a}{2}$.
  - Case 3 – For $K \geq \frac{a^2}{4}$ the roots are complex and conjugate, given by

$$s_{1,2} = -\frac{a}{2} \pm j\frac{\sqrt{a^2 - 4K}}{2} \tag{5.1.4}$$

- The real parts remain constant and the imaginary part of the roots vary as K varies. Thus, the roots move along the vertical line, one upwards and one downwards.

Root Locus: Example (contd.)

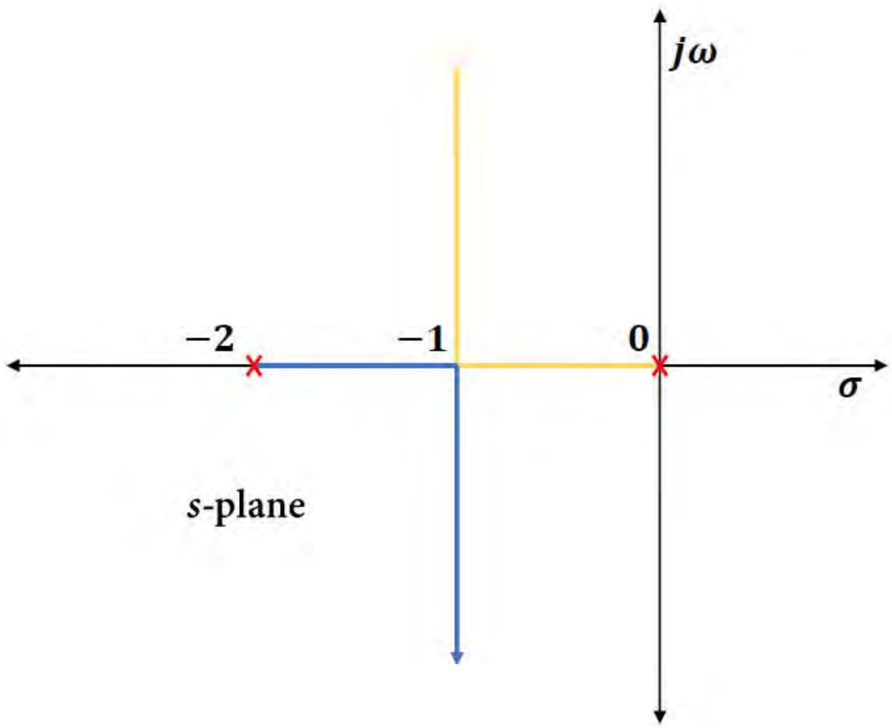
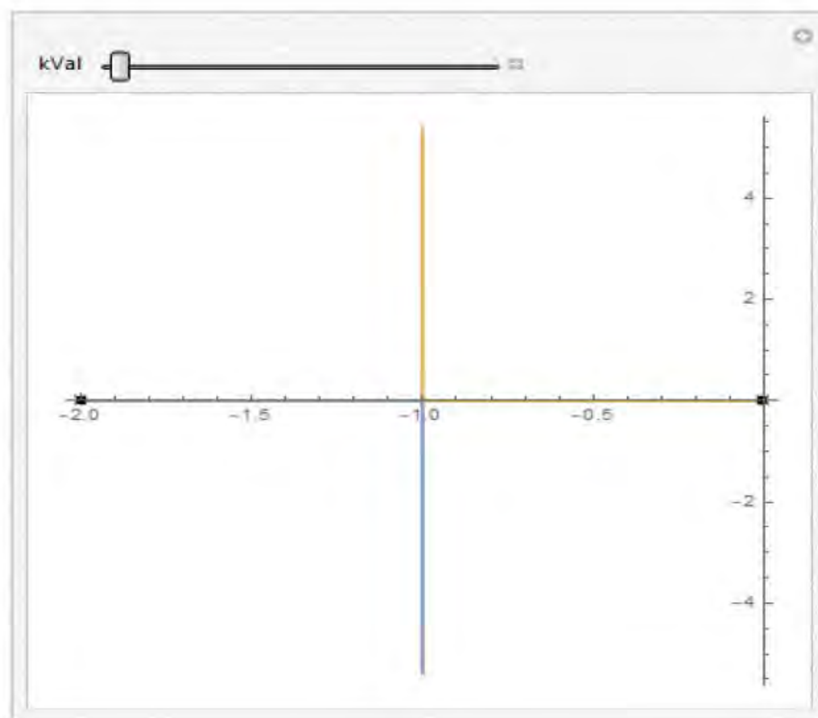


Fig 5.1.2 Root Locus plot of equation (5.1.2)



Advantages

- The advantages of root locus technique are:
- Clearly showing the contributions of each loop poles or zeros to the location of the closed loop poles.
- Indicating the manner in which the loop poles and zeros should be modified so that the response meets system performance specifications.
- Knowledge of the open loop system is sufficient to analyse the behaviour of the system, detailed study of the closed loop system is not required.

System Parameters and Pole Locations

First Order System

- General form of transfer function for a first order system is given by

$$G(s) = \frac{K}{\tau s + 1} \tag{5.1.5}$$

where K and τ are steady state gain and time constant respectively.

- Pole of the first order system is at $s = -\frac{1}{\tau}$; influences the speed of response of the output.

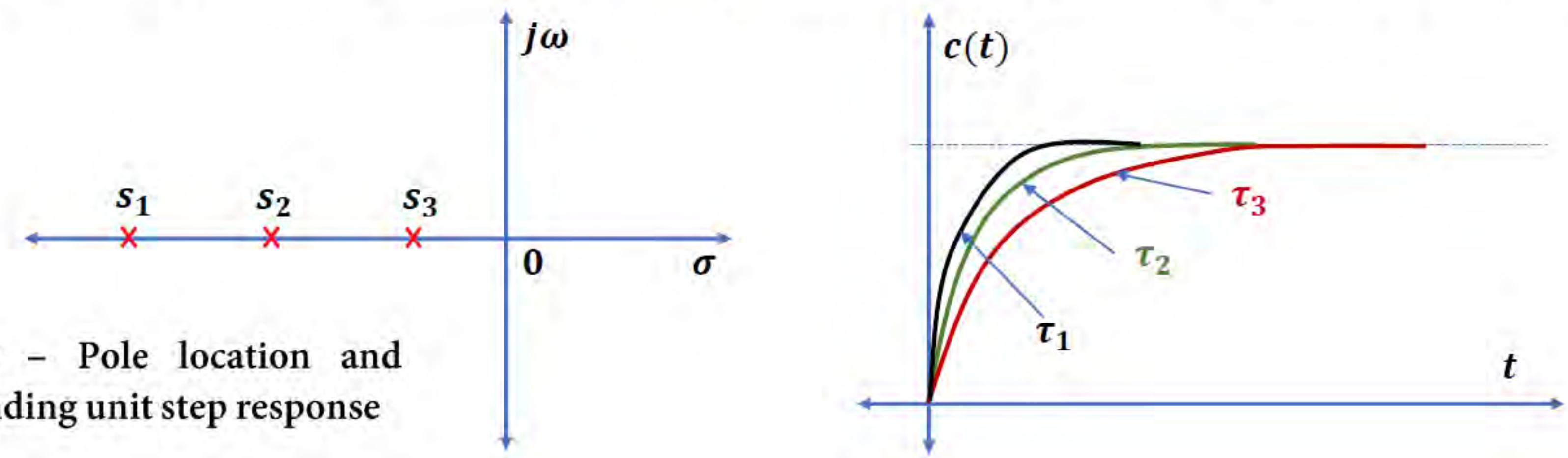


Fig 5.1.3 – Pole location and corresponding unit step response

System Parameters and Pole Locations

Second Order System

- General form of transfer function for a second order system is given by

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2} \tag{5.1.6}$$

where ω_n and ζ are natural frequency and damping ratio respectively.

- For $\zeta = 0$, the system is undamped.
- For $0 < \zeta < 1$, the system is under damped.
- For $\zeta = 1$, the system is critically damped.
- For $\zeta > 1$, the system is over damped.

System Parameters and Pole Locations

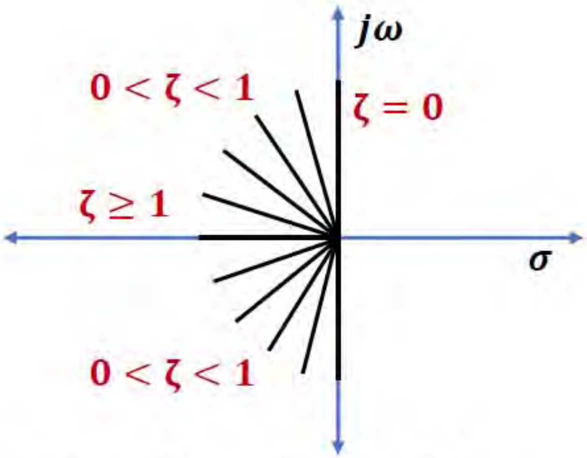


Fig 5.1.4 – Locus of poles with constant ζ

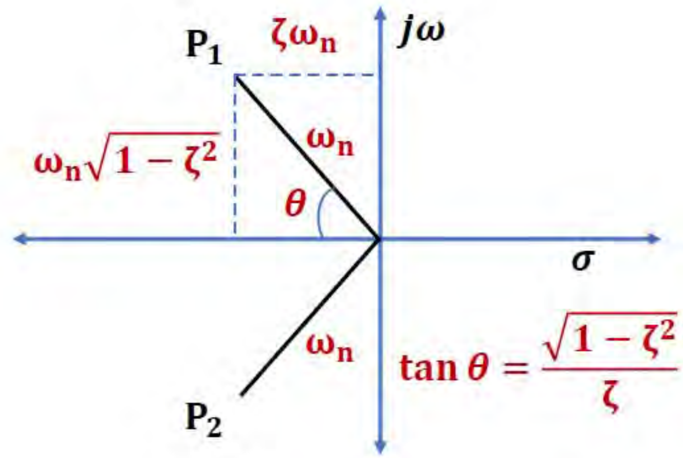


Fig 5.1.5 – Locus of poles with constant ω_n

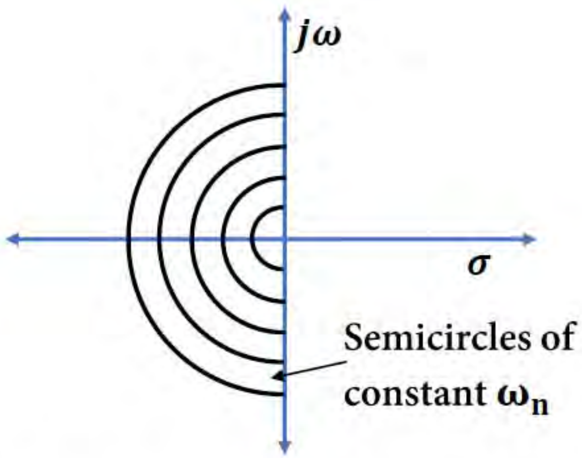


Fig 5.1.6 – Locus of poles with constant ω_n

Evans' Conditions

Let us consider the system as in *Fig 5.2.1*.

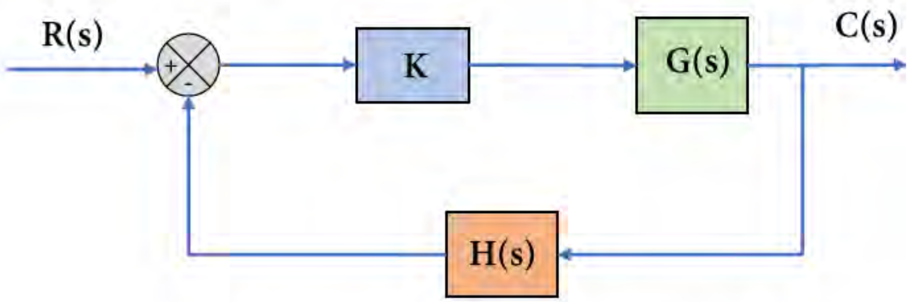


Fig 5.2.1 General feedback system

The closed loop transfer function of the system is given by

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)} = \frac{p(s)}{q(s)} \tag{5.2.1}$$

where $p(s)$ and $q(s)$ are polynomials in s and K is a variable parameter, and $0 \leq K \leq \infty$.

Evans' Conditions (contd.)

The characteristic equation of the system is given by,

$$q(s) = 1 + KG(s)H(s) = 0 \tag{5.2.1}$$

$$\therefore KG(s)H(s) = P(s) = -1 \tag{5.2.2}$$

Since '*s*' is a complex variable equation (5.2.2) can be written in polar form as,

$$|P(s)|\angle P(s) = -1 + j0 \tag{5.2.3}$$

It is necessary that

$$|P(s)| = 1 \; \textit{(magnitude criterion)} \tag{5.2.4}$$

$$\angle P(s) = \pm(2q+1)180^\circ; \quad q = 0, 1, 2, \ldots \; \textit{(angle criterion)} \tag{5.2.5}$$

Equation (5.2.4) and (5.2.5) are known as Evans' Conditions.

Points on Root locus

Consider a system with open loop transfer function in pole-zero form

$$KG(s)H(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} \tag{5.2.6}$$

The Evans' conditions for the existence of a point on the root locus are

$$|P(s)| = 1 \text{ (magnitude criterion)} \tag{5.2.7}$$

$$\angle P(s) = \pm(2q+1)180^{\circ}; \quad q = 0, 1, 2, \ldots \text{ (angle criterion)} \tag{5.2.8}$$

Then applying conditions in equation (5.2.4) and (5.2.5) in equation (5.2.6)

$$K|G(s)H(s)| = \frac{K|s+z_1||s+z_2|\cdots|s+z_m|}{|s+p_1||s+p_2|\cdots|s+p_n|} = \frac{K\prod_{i=1}^{m}|s+z_i|}{\prod_{i=1}^{n}|s+p_i|} = 1 \tag{5.2.9}$$

and

$$\angle KG(s)H(s) = \sum_{i=1}^{m}\angle(s+z_i) - \sum_{i=1}^{n}\angle(s+p_i) = \pm(2q+1)180^{\circ} \tag{5.2.10}$$

- Every point s in the s-plane that satisfies equation (5.2.7) lies on the root locus of the system with open loop transfer function in equation (5.2.6).

- For every point s on the root locus, there exists a K satisfying

$$K = \frac{\prod_{i=1}^{n} |s + p_i|}{\prod_{i=1}^{m} |s + z_i|} \tag{5.2.11}$$

Example

- Let us consider the open loop transfer function as

$$KG(s)H(s) = \frac{K(s+2)}{s(s+1)} \tag{5.2.12}$$

- Let, s_0 be a point on the root locus. Therefore, the point s_0 satisfies the angle criterion given by equation (5.2.5), *i.e.*

$$\angle(s_0+2) - \angle s_0 - \angle(s_0+1) = \pm(2q+1)180^{\circ} \tag{5.2.13}$$

for some 'q' such that $q \in \mathbb{N}$, where $\mathbb{N}$ is the set of Natural numbers.

- Then, the parameter K is given by

$$K = \frac{|s_0||s_0+1|}{|s_0+2|} = \frac{AB}{C} \tag{5.2.14}$$

as shown in *Figure 5.2.2*.

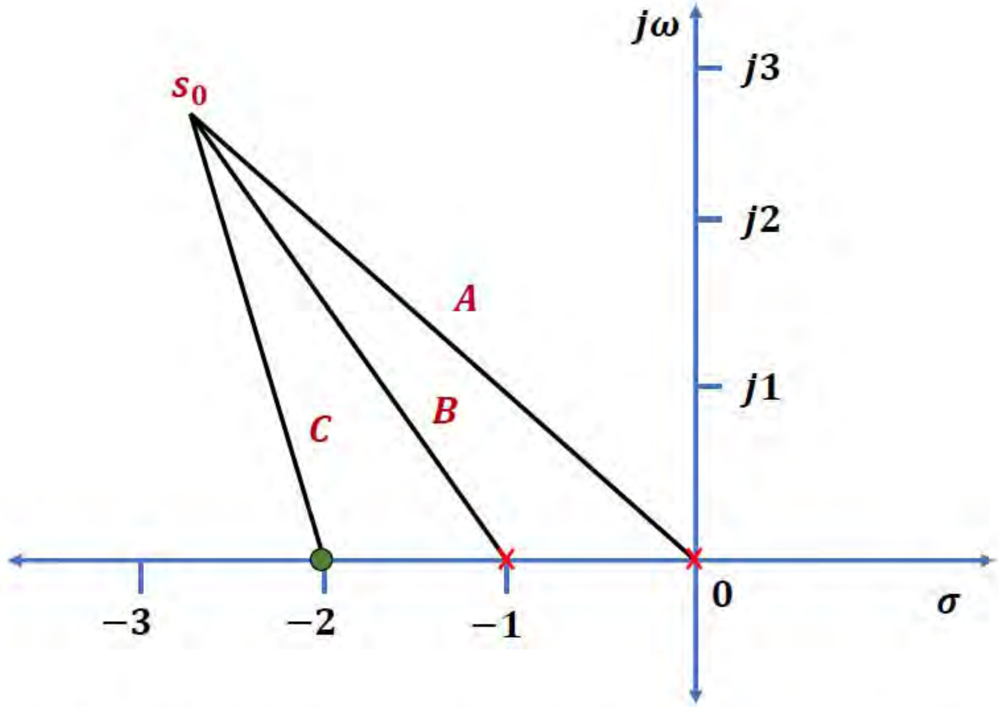


Fig 5.2.2 – Pole-zero plot of equation (5.2.12)

Rules for constructing Root Loci

Rule 1 Number of Root Loci *The root locus plot consists of n root loci as K varies from 0 to ∞. The loci are symmetric with respect to the real axis.*

The characteristic equation (7.22a) can be written as

$$\prod_{j=1}^{n}\left(s+p_j\right) + K\prod_{i=1}^{m}\left(s+z_i\right) = 0$$

This equation has degree n. Thus for each real K, there are n roots. As the roots of this equation are continuous functions of its coefficients, the n roots form n continuous loci as K varies from 0 to ∞.

Since the complex roots of the characteristic equation always occur in conjugate pairs, the n root loci, must be symmetrical about the real axis.

Rule 2 Starting and Terminal Points of Root Loci *As K increases from zero to infinity, each root locus originates from an open-loop pole with K = 0 and terminates either on an open-loop zero or on infinity with K = ∞. The number of loci terminating on infinity equals the number of open-loop poles minus zeros.*

The characteristic equation (7.22a) can be written as

$$\prod_{j=1}^{n}\left(s+p_j\right) + K\prod_{i=1}^{m}\left(s+z_j\right) = 0 \qquad (7.25)$$

When $K = 0$, this equation has roots at $-p_j$ ($j = 1, 2, \ldots, n$), which are the open-loop poles. The root loci, therefore, start at the open-loop poles.

When $K = \infty$, this equation has roots at $-z_i$($i = 1, 2, \ldots, m$), which are the open-loop zeros. Therefore, m root loci terminate on the open-loop zeros.

In case $m < n$, the open-loop transfer function has $(n - m)$ zeros at infinity.

Rule 3 Asymptotes to Root Loci *The $(n-m)$ root loci which tend to infinity do so along straight line asymptotes radiating out from a single point $s = -\sigma_A$ on the real axis (called the centroid), where*

$$-\sigma_A = \frac{\sum (\text{real parts of open-loop poles}) - \sum (\text{real parts of open-loop zeros})}{n-m} \tag{7.26a}$$

These $(n-m)$ asymptotes have angles

$$\phi_A = \frac{(2q+1)180^\circ}{n-m}; \; q = 0, 1, \ldots, (n-m-1) \tag{7.26b}$$

Rule 4 On-Locus Segments of the Real Axis *A point on the real axis lies on the locus if the number of open-loop poles plus zeros on the real axis to the right of this point is odd.*

Rule 5 On-Locus Points of the Imaginary Axis *The intersections (if any) of root loci with the imaginary axis can be determined by use of the Routh criterion.*

- Segment of root loci can exist in the right half of s-plane. This signifies instability, the points at which the root loci cross the imaginary axis define the stability limits.
- Basic application of the Routh array determines the gains at the satability limits. By substituting this value of gain in the auxiliary equation, the value of s=jw at the stability limit is evaluated.

For the fourth-order example under consideration (refer Eqns (7.28)), the characteristic equation is

$$1+\frac{K(s+2)}{(s+1+j4)(s+1-j4)(s+3)(s+4)}=0.$$

which is equivalent to

$$s^4+9s^3+43s^2+(143+K)s+204+2K=0 \tag{7.29}$$

The Routh array for the characteristic polynomial is

s^4	1	43	$204+2K$
s^3	9	$143+K$	
s^2	$\dfrac{244-K}{9}$	$204+2K$	
s^1	$\dfrac{18368-61K-K^2}{244-K}$		
s^0	$204+K$		

For Eqn. (7.29) to have no roots on $j\omega$-axis or in the right-half s-plane, the elements in the first column of Routh array must all be of the same sign. Therefore, the following inequalities must be satisfied:

$$244-K>0$$
$$18368-61K-K^2>0$$
$$204+K>0$$

These inequalities are satisfied if K is less than 108.4, which means that the critical value of K which corresponds to the roots on the $j\omega$-axis is 108.4.

The value of $K = 108.4$ makes all the coefficients of s^1-row of the Routh array zero. For this value of K, the auxiliary equation formed from the coefficient of the s^2-row, is given by

$$\left(\frac{244 - K}{9}\right) s^2 + (204 + 2K) = 0$$

For $K = 108.4$, the roots of the above equation lie on the $j\omega$-axis and are given by

$$s = \pm j5.28$$

Thus the root loci intersect the $j\omega$-axis at $s = \pm j5.28$ and the value of K corresponding to these roots is 108.4.

Rule 6 Angle of Departure from Complex Poles *The angle of departure, ϕ_p, of a locus from a complex open-loop pole is given by*

$$\phi_p = 180° + \phi \tag{7.30}$$

where ϕ is the net angle contribution at this pole of all other open-loop poles and zeros.

Rule 7 Angle of Arrival at Complex Zeros The angle of arrival, ϕ_z, of a locus at a complex zero is given by

$$\phi_z = 180° - \phi \tag{7.31}$$

where ϕ is the net angle contribution at this zero of all other open-loop poles and zeros.

Rule 8 Locations of Multiple Roots *Points at which multiple roots of the characteristic equation occur (breakaway points of root loci) are the solutions of*

$$\frac{dK}{ds} = 0 \tag{7.33a}$$

where

$$K = -\frac{\prod_{j=1}^{n}(s + p_j)}{\prod_{i=1}^{m}(s + z_i)} \tag{7.33b}$$

(ix) The gain K at any point s_0 on a root locus is given by

$$K = \frac{\prod_{j=1}^{n}|s_0 + p_j|}{\prod_{i=1}^{m}|s_0 + z_i|} = \frac{\text{[Product of phasor lengths (read to scale) from } s_0 \text{ to poles of } F(s)]}{\text{[Product of phasor lengths (read to scale) from } s_0 \text{ to zeros of } F(s)]}$$

Rules

(i) The root locus plot consists of n root loci as K varies from 0 to ∞. The loci are symmetric with respect to the real axis.

(ii) As K increases from zero to infinity, each root locus originates from an open-loop pole with $K = 0$ and terminates either on an open-loop zero or on infinity with $K = \infty$. The number of loci terminating on infinity equals the number of open-loop poles minus zeros.

(iii) The $(n - m)$ root loci which tend to infinity, do so along straight-line asymptotes radiating out from a single point $s = -\sigma_A$ on the real axis (called the centroid), where

$$-\sigma_A = \frac{\begin{array}{c}\sum(\text{real parts of open-loop poles})\\ \sum(\text{real parts of open-loop zeros})\end{array}}{n-m}$$

These $(n - m)$ asymptotes have angles

$$\phi_A = \frac{(2q+1)180^\circ}{n-m}; q = 0, 1, \ldots, (n-m-1)$$

(iv) A point on the real axis lies on the locus if the number of open-loop poles plus zeros on the real axis to the right of this point is odd. By use of this fact, the real axis can be divided into segments *on-locus* and *not-on-locus*; the dividing points being the real open-loop poles and zeros.

(v) The intersections (if any) of root loci with the imaginary axis can be determined by use of the Routh criterion.

(vi) The angle of departure, ϕ_p, of a root locus from a complex open-loop pole is given by

$$\phi_p = 180^\circ + \phi$$

where ϕ is the net angle contribution at this pole of all other open-loop poles and zeros.

(vii) The angle of arrival, ϕ_z, of a locus at a complex zero is given by

$$\phi_z = 180^\circ - \phi$$

where ϕ is the net angle contribution at this zero of all other open-loop poles and zeros.

(Contd.)

Rules
(viii) Points at which multiple roots of the characteristic equation occur (breakaway points of root loci) are the solutions of $\frac{dK}{ds} = 0$ where $$K = -\frac{\prod_{j=1}^{n}(s+p_j)}{\prod_{i=1}^{m}(s+z_i)}$$
(ix) The gain K at any point s_0 on a root locus is given by $$K = \frac{\prod_{j=1}^{n}\lvert s_0+p_j\rvert}{\prod_{i=1}^{m}\lvert s_0+z_i\rvert} = \frac{\text{[Product of phasor lengths (read to scale) from } s_0 \text{ to poles of } F(s)]}{\text{[Product of phasor lengths (read to scale) from } s_0 \text{ to zeros of } F(s)]}$$

Illustrative example:

Example 7.3 Consider a feedback system with the characteristic equation

$$1 + \frac{K}{s(s+1)(s+2)} = 0;\ K \geq 0 \tag{7.38}$$

The open-loop poles are located at $s = 0, -1$ and -2, while there are no finite open-loop zeros. The pole-zero configuration is shown in Fig. 7.18.

Rule (i) (refer Table 7.1) tells us that the root locus plot consists of three root loci as K is varied from 0 to ∞.

Rule (ii) tells us that the three root loci originate from the three open-loop poles with $K = 0$ and terminate on infinity with $K = \infty$.

Rule (iii) tells us that the three root loci tend to infinity along asymptotes radiating out from

$$s = -\sigma_A = \frac{\sum \text{real parts of poles} - \sum \text{real parts of zeros}}{\text{number of poles} - \text{number of zeros}} = \frac{-2-1}{3} = -1$$

with angles $$\phi_A = \frac{(2q+1)180^\circ}{\text{number of poles} - \text{number of zeros}};\ q = 0, 1, \ldots$$

$$= \frac{(2q+1)180°}{3}; q = 0, 1, 2$$

$$= 60°, 180°, 300°$$

The asymptotes are shown by dotted lines in Fig. 7.18.

Rule (iv) tells us that the segments of the real axis between 0 and -1, and between -2 and $-\infty$ lie on the root locus. On-locus segments of the real axis are shown by thick lines in Fig. 7.18.

From Fig. 7.18, it is seen that out of the three loci, one is a real-root locus which originates from $s = -2$ and terminates on $-\infty$. The other two loci originate from $s = 0$ and $s = -1$, and move on the real axis approaching each other as K is increased. These two loci must therefore meet on the real axis. The characteristic equation has a double root at such a point. As the gain K is further increased, the root loci breakaway from the real axis to give a complex conjugate pair of roots.

Rule (v) tells us to check by use of the Routh criterion if the two root loci breaking away from the real axis will intersect the imaginary axis. The system characteristics equation is

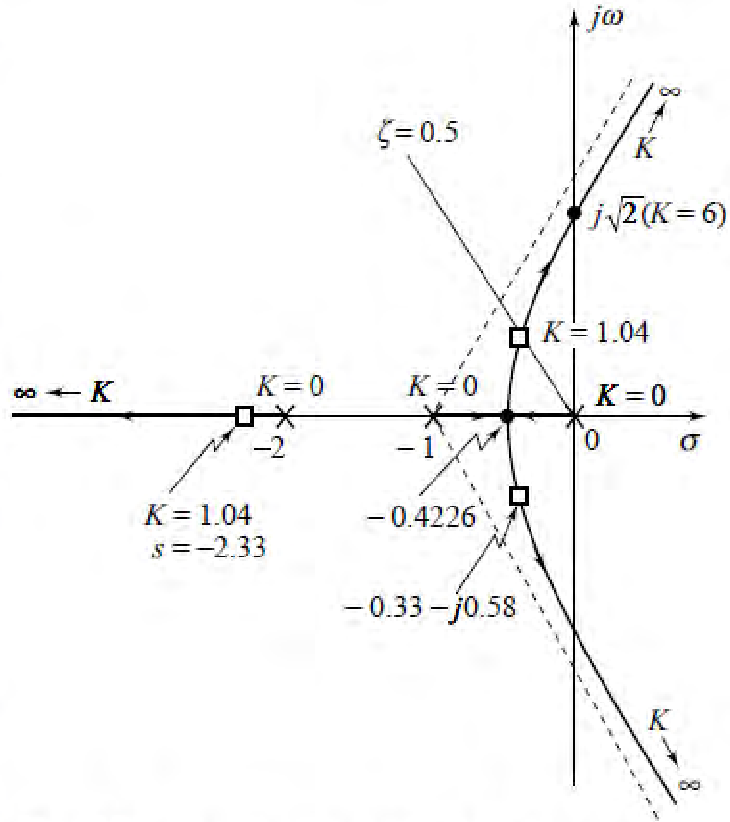


Fig. 7.18 Root locus plot for the characteristic equation (7.38)

$$s(s+1)(s+2) + K = 0$$

or
$$s^3 + 3s^2 + 2s + K = 0$$

The Routh array becomes

s^3	1	2
s^2	3	K
s^1	$(6 - K)/3$	
s^0	K	

For all the roots of the characteristic equation to lie to the left of the imaginary axis, the following conditions must be satisfied:

$$K > 0; (6 - K)/3 > 0$$

Therefore the critical value of K which corresponds to the location of roots on the $j\omega$-axis is 6. This value of K makes all the coefficients of s^1-row of Routh array zero. The auxiliary equation formed from the coefficients of the s^2-row is given by

$$3s^2 + K = 3s^2 + 6 = 0$$

The roots of this equation lie on the $j\omega$-axis and are given by

$$s = \pm j\sqrt{2}$$

Thus the complex-root branches intersect the $j\omega$-axis at $s = \pm j\sqrt{2}$, and the value of K corresponding to these roots is 6.

From the characteristic equation of the system,

$$K = -s(s + 1)(s + 2) = -(s^3 + 3s^2 + 2s)$$

Differentiating this equation, we get

$$\frac{dK}{ds} = -(3s^2 + 6s + 2)$$

The roots of the equation $dK/ds = 0$, are $s_{1,2} = -0.4226, -1.5774$

From Fig. 7.18, we see that the breakaway point must lie between 0 and – 1. Therefore $s = -0.4226$ corresponds to the breakaway point. The other solution of the equation $dK/ds = 0$ is $s = -1.5774$. This solution is not on the root locus (it does not satisfy Eqn. (7.38) for any $K \geq 0$) and therefore does not represent a breakaway point. The derivative condition $dK/ds = 0$ is therefore necessary but not sufficient to indicate a breakaway, or multiple-root situation.

If two loci breakaway from a breakaway point, then their tangents will be 180° apart. In general, as we shall see in other illustrative examples, if r loci breakaway from a breakaway point, then their tangents will be $360°/r$ apart, i.e., the tangents will equally divide 360°.

With the information obtained through the use of these rules, the root locus is sketched in Fig. 7.18 from where it is seen that for $K > 6$, the system has two closed-loop poles in the right half of the s-plane and is thus unstable.

It is important to note that the root locus plot given in Fig. 7.18 is only a rough sketch; it gives qualitative information about the behaviour of the closed-loop system when the parameter K is varied. For quantitative analysis, the loci or specific segments of the loci need be constructed with sufficient accuracy.

Suppose it is required to determine dominant closed-loop poles with damping ratio $\zeta = 0.5$. A closed-loop pole with $\zeta = 0.5$ lies on a line passing through the origin and making an angle $\cos^{-1} \zeta = \cos^{-1} 0.5 = 60°$ with the negative real axis. The point of intersection of the ζ-line with the rough sketch of the root locus plot gives the first guess. In the region where the rough root locus sketch intersects the ζ-line, a trial-and-error procedure is adapted along the ζ-line by applying the angle criterion to determine the point of intersection accurately. From Fig. 7.18, we see that the point $-0.33 + j0.58$, which lies on the ζ-line, satisfies the angle criterion. Therefore the dominant closed-loop poles of the system are $s_{1,2} = -0.33 \pm j0.58$.

The value of K that yields these poles is found from the magnitude criterion as follows:

$$K = \{|s|\ |s+1|\ |s+2|\}_{s=-0.33+j0.58}$$

$$= \text{Product of the distances from the open-loop poles to the point } -0.33 + j0.58$$

$$= 0.667 \times 0.886 \times 1.768 = 1.04$$

The third closed-loop pole will lie on the third locus which is along the negative real axis. We need to guess a test point, compute a trial gain, and correct the guess until we found the point where $K = 1.04$. Using this trial-and-error procedure, we find from Fig. 7.18 that the point $s = -2.33$ corresponds to $K = 1.04$. Therefore, the third closed-loop pole is at $s = -2.33$.

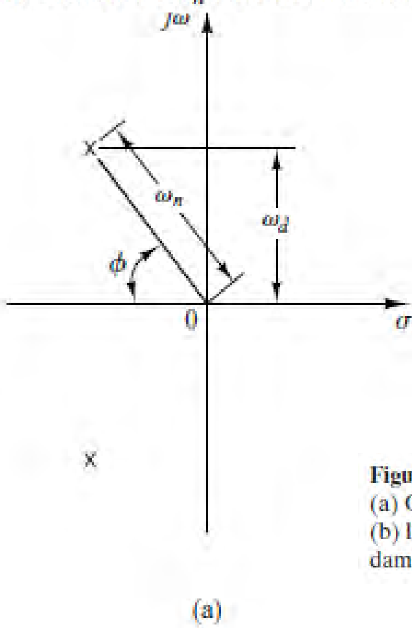
The closed-loop transfer function of the system under consideration is

$$M(s) = \frac{1.04}{(s + 0.33 - j0.58)(s + 0.33 + j0.58)(s + 2.33)}$$

Constant ζ Loci and Constant ω_n Loci. Recall that in the complex plane the damping ratio ζ of a pair of complex-conjugate poles can be expressed in terms of the angle ϕ, which is measured from the negative real axis, as shown in Figure 6–21(a) with

$$\zeta = \cos\phi$$

In other words, lines of constant damping ratio ζ are radial lines passing through the origin as shown in Figure 6–21(b). For example, a damping ratio of 0.5 requires that the complex-conjugate poles lie on the lines drawn through the origin making angles of $\pm 60^\circ$ with the negative real axis. (If the real part of a pair of complex-conjugate poles is positive, which means that the system is unstable, the corresponding ζ is negative.) The damping ratio determines the angular location of the poles, while the distance of the pole from the origin is determined by the undamped natural frequency ω_n. The constant ω_n loci are circles.



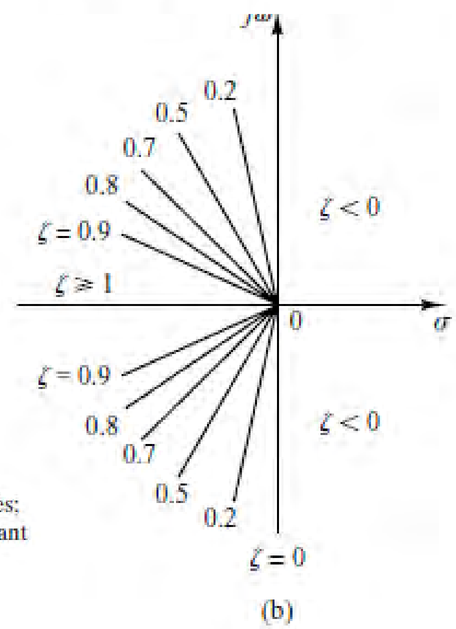


Figure 6–21
(a) Complex poles;
(b) lines of constant damping ratio ζ.

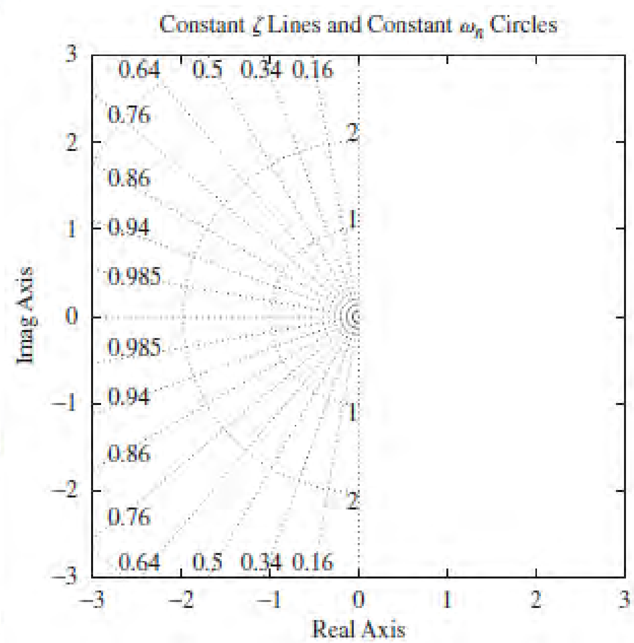
Plotting Polar Grids in the Root-Locus Diagram. The command

sgrid

overlays lines of constant damping ratio ($\zeta = 0 \sim 1$ with 0.1 increment) and circles of constant ω_n on the root-locus plot. See MATLAB Program 6–5 and the resulting diagram shown in Figure 6–22.

MATLAB Program 6–5

```
sgrid
v = [-3  3  -3  3]; axis(v); axis('square')
title('Constant \zeta Lines and Constant \omega_n Circles')
xlabel('Real Axis')
ylabel('Imag Axis')
```



If only particular constant ζ lines (such as the $\zeta = 0.5$ line and $\zeta = 0.707$ line) and particular constant ω_n circles (such as the $\omega_n = 0.5$ circle, $\omega_n = 1$ circle, and $\omega_n = 2$ circle) are desired, use the following command:

sgrid([0.5, 0.707], [0.5, 1, 2])

If we wish to overlay lines of constant ζ and circles of constant ω_n as given above to a root-locus plot of a negative feedback system with

num = [0 0 0 1]
den = [1 4 5 0]

Example 7.4 Consider a feedback system with the characteristic equation

$$1+\frac{K}{s(s+3)\left(s^2+2s+2\right)}=0;\ K \geq 0 \tag{7.39}$$

The open-loop poles are located at $s = 0, -3, -1+j1$ and $-1-j1$, while there are no finite open-loop zeros. The pole-zero configuration is shown in Fig. 7.19.

Rule (i) tells us that the root locus plot consists of four root loci as K is varied from 0 to ∞.

Rule (ii) tells us that the four root loci originate from the four open-loop poles with $K = 0$ and terminate on infinity with $K = \infty$.

Rule (iii) tells us that the four root loci tend to infinity along asymptotes radiating out from

$$s=-\sigma_A=\frac{-3-1-1}{4}=-1.25$$

with angles

$$\phi_A=\frac{(2q+1)180^\circ}{4};\ q=0, 1, 2, 3$$

$$= 45^\circ, 135^\circ, 225^\circ, 315^\circ$$

The asymptotes are shown by dotted lines in Fig. 7.19.

Rule (iv) tells us that root loci exist on the real axis for $-3 \leq s \leq 0$; shown by thick lines on the real axis in Fig. 7.19.

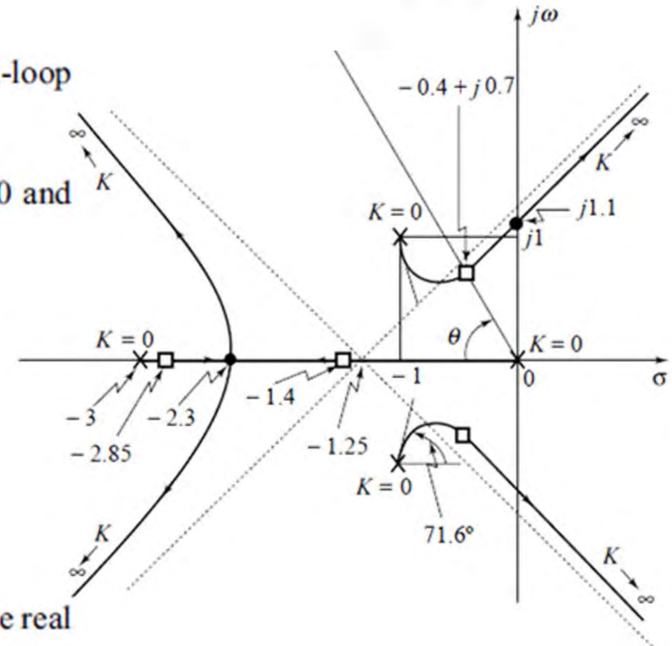


Fig. 7.19 Root locus plot for the characteristic equation (7.39)

Rule (v) tells us to check by use of the Routh criterion if the root loci will intersect the imaginary axis. The system characteristic equation is

$$s(s+3)(s^2+2s+2)+K=0$$

or

$$s^4+5s^3+8s^2+6s+K=0$$

The Routh array becomes

s^4	1	8	K
s^3	5	6	
s^2	34/5	K	
s^1	$\dfrac{\{(204/5)-5K\}}{34/5}$		
s^0	K		

Examination of the elements in the first column of the Routh array reveals that the root loci will intersect the imaginary axis at a value of K given by

$$(204/5) - 5K = 0$$

from which $K = 8.16$.

The auxiliary equation, formed from the coefficients of the s^2-row when $K = 8.16$, is

$$(34/5)s^2 + 8.16 = 0$$

from which $s = \pm j1.1$.

Therefore, purely imaginary closed-loop poles of the system are located at $s = \pm j1.1$ as shown in Fig. 7.19; the value of K corresponding to these poles is 8.16.

Rule (vi) tells us that a root locus leaves the pole at $s = -1 + j1$ at angle ϕ_p, given by (refer Fig. 7.19)

$$\phi_p = 180° + [-135° - 90° - 26.6°] = -71.6°$$

The characteristic equation under consideration does not require application of rule (vii).

Rule (viii) is used below to determine the breakaway points.

From the characteristic equation of the system,

$$K = -s(s+3)(s^2+2s+2) = -(s^4+5s^3+8s^2+6s)$$

Differentiating this equation, we get

$$\frac{dK}{ds} = -4(s^3+3.75s^2+4s+1.5)$$

The solutions to the cubic

$$s^3+3.75s^2+4s+1.5=0 \tag{7.40}$$

are the possible breakaway points.

From Fig. 7.19, we see that a breakaway point must lie between 0 and -3 on the real axis. By trial-and-error procedure, we find that $s = -2.3$ satisfies Eqn. (7.40) to a reasonable accuracy. The CAD software (MATLAB, for example) may be used to obtain the plot whenever the problem becomes a little complex. Experience of hand-sketching will be helpful in interpreting the CAD results. The other two solutions of Eqn. (7.40) are $s_{1,2} = -0.725 \pm j0.365$.

It can easily be checked that $-0.725 \pm j0.365$ are not the root locus points as the angle criterion is not met at these points.

The root locus plot, therefore, has only one breakaway point at $s = -2.3$. Tangents to the two loci breaking away from this point will be 180° part.

With the information obtained through the use of these rules, the root locus plot is sketched in Fig. 7.19 from where it is seen that for $K > 8.16$, the system has two closed-loop poles in the right half of s-plane and is thus unstable.

Dominant roots of the characteristic equation with damping $\zeta = 0.5$ are determined as follows. A ζ-line is drawn in the second quadrant at an angle of $\theta = \cos^{-1}0.5 = 60°$ with the negative real axis. By trial-and-error procedure, it is found that the point $s = -0.4 + j0.7$ which lies on the ζ-line, satisfies the angle criterion. Therefore the dominant roots of the characteristic equation are $s_{1,2} = -0.4 \pm j0.7$. The value of gain K at the dominant root is

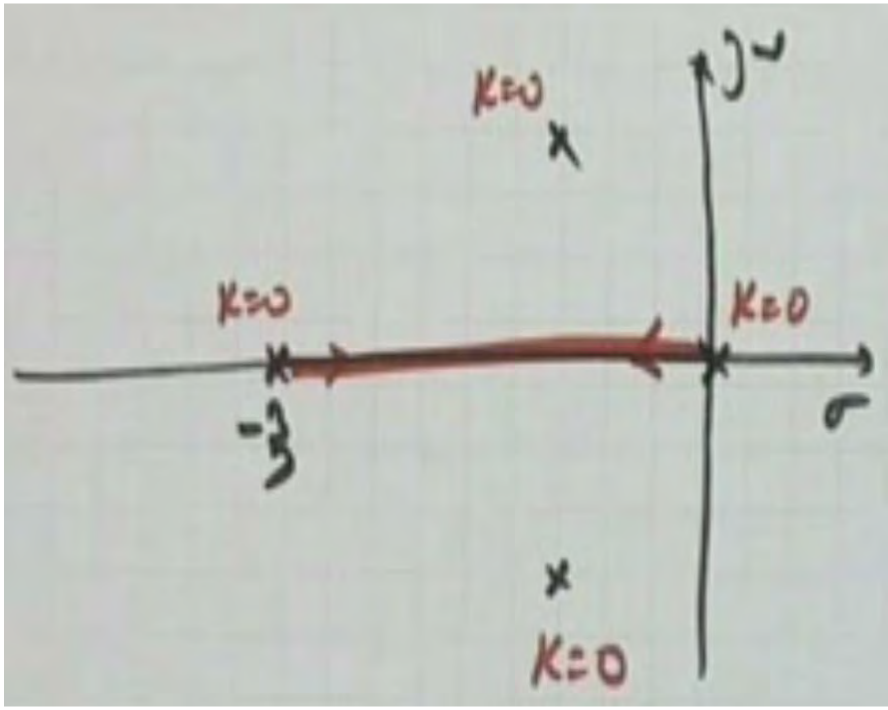
$$K = \text{Product of distances from open-loop poles to the point } -0.4 + j0.7$$
$$= 0.84 \times 1.86 \times 2.74 \times 0.68 = 2.91$$

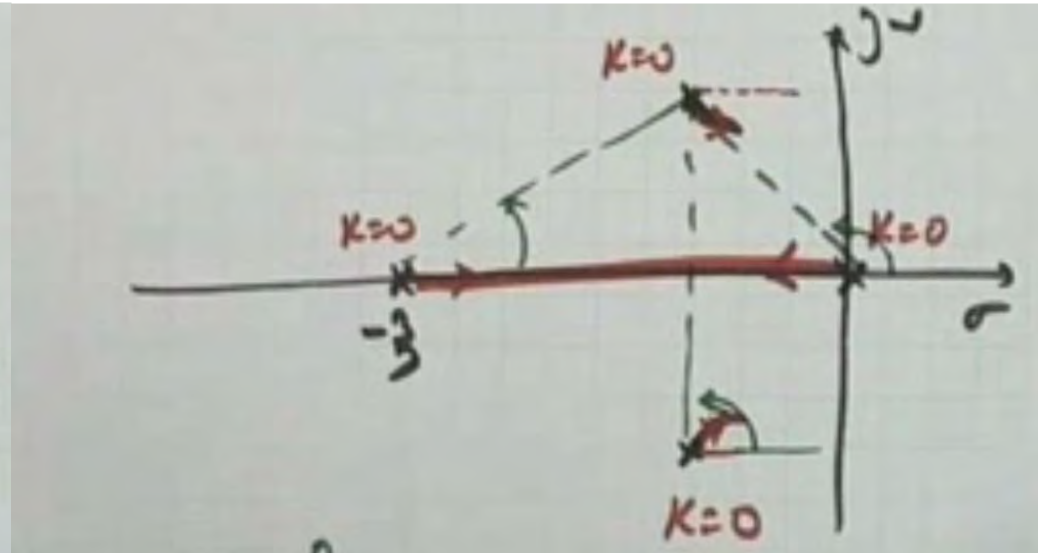
The other two roots of the characteristic equation are obtained from the loci originating from the poles at $s = 0$ and $s = -3$. From Fig. 7.19, it is seen that at the double-root location $s = -2.3$, the value of K is 4.33:

$$K = \{|s||s+3|\,|s+1+j1|\,|s+1-j1|\}_{s=-2.3}$$
$$= 2.3 \times 0.7 \times 1.64 \times 1.64 = 4.33$$

Therefore, the points corresponding to $K = 2.91$ must lie on the real-root segments of these loci. Using the trial-and-error procedure, it is found that the points $s = -1.4$ and $s = -2.85$ have gain $K = 2.91$. Thus the closed-loop transfer function of the system under consideration is

$$M(s) = \frac{2.91}{(s+0.4+j0.7)(s+0.4-j0.7)(s+1.4)(s+2.85)}$$





$\phi_p = -71.6^{\circ}$ $\quad \phi_A = 45, 135, 225, 315$

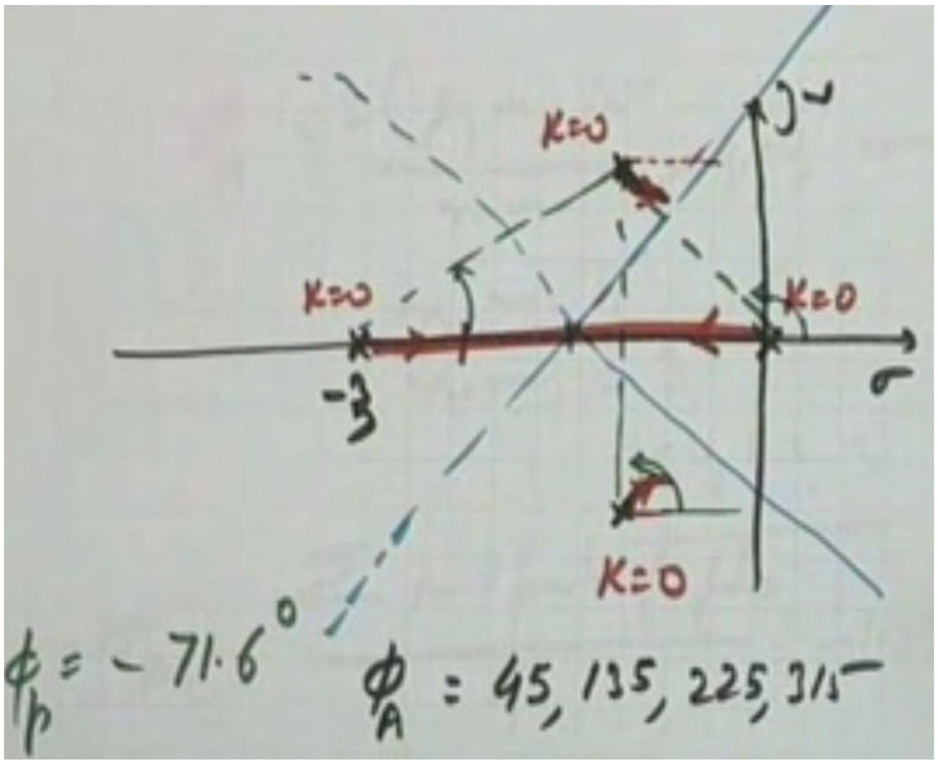
$$-\sigma_A = \frac{\sum \text{real parts of poles}}{4} = -1.25$$

$$\phi_A = \pm\frac{(2q+1)\,180}{n-m}\,;\ q=0, \ldots, n-m-1$$

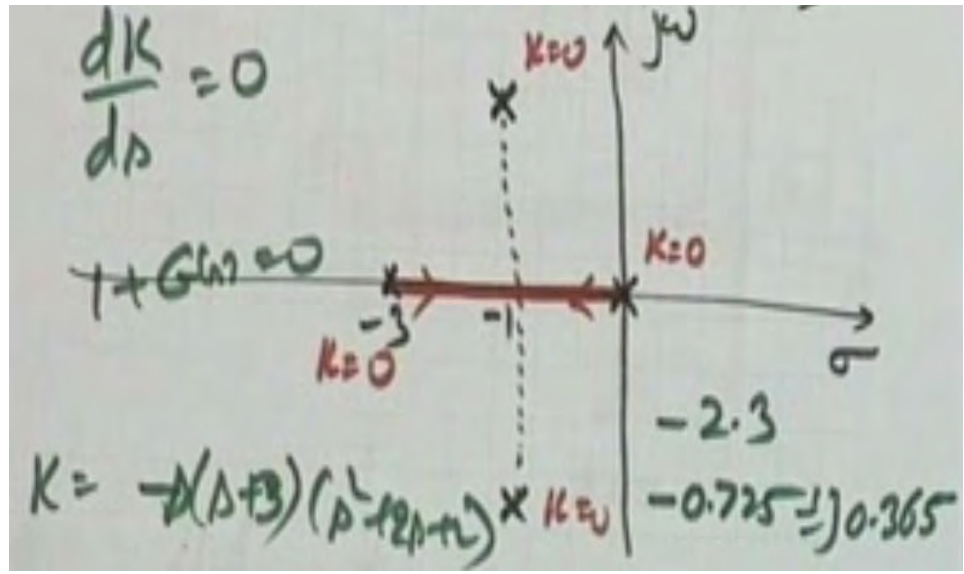
$n-m=4$

$n-m-1=3$

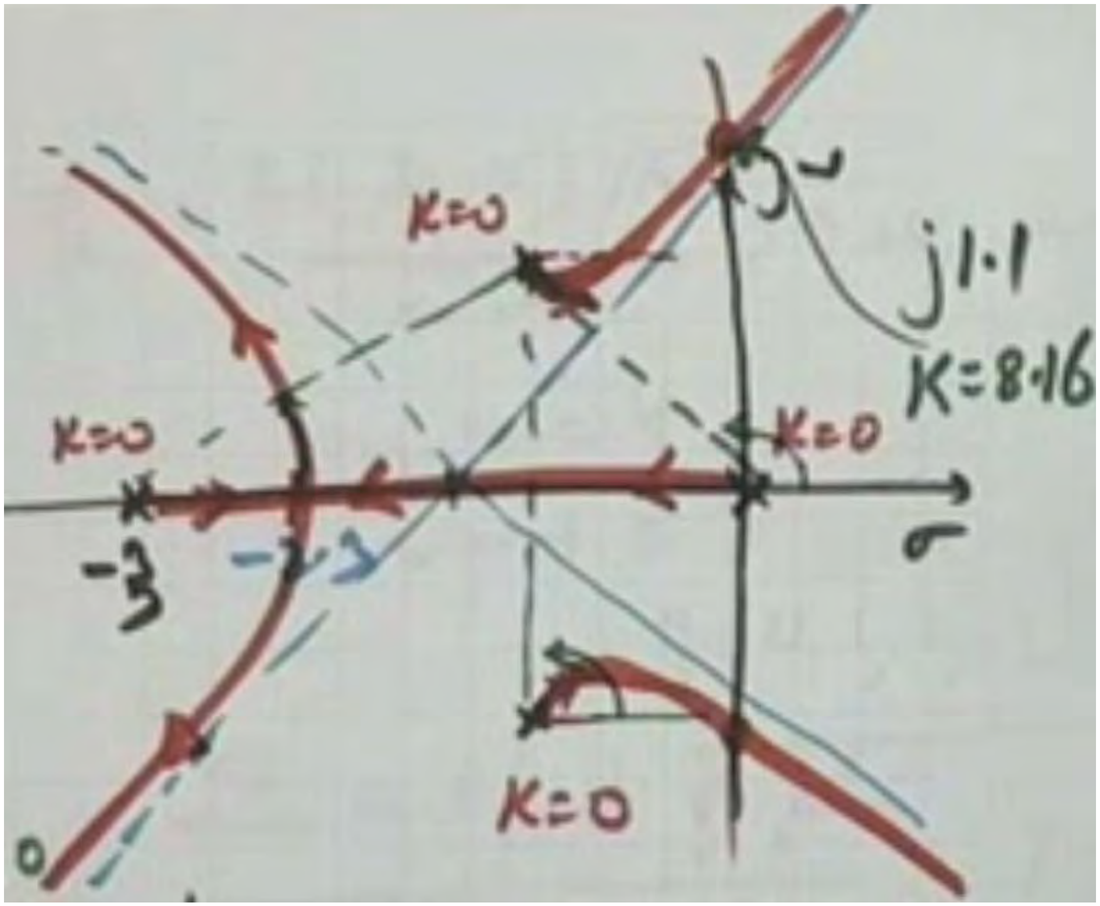
$q=0,1,2,3$



K=∞
K=∞
K=0
jω
σ
-3
K=0
$\phi_p = -71.6^0$
ϕ_A : 45, 135, 225, 315

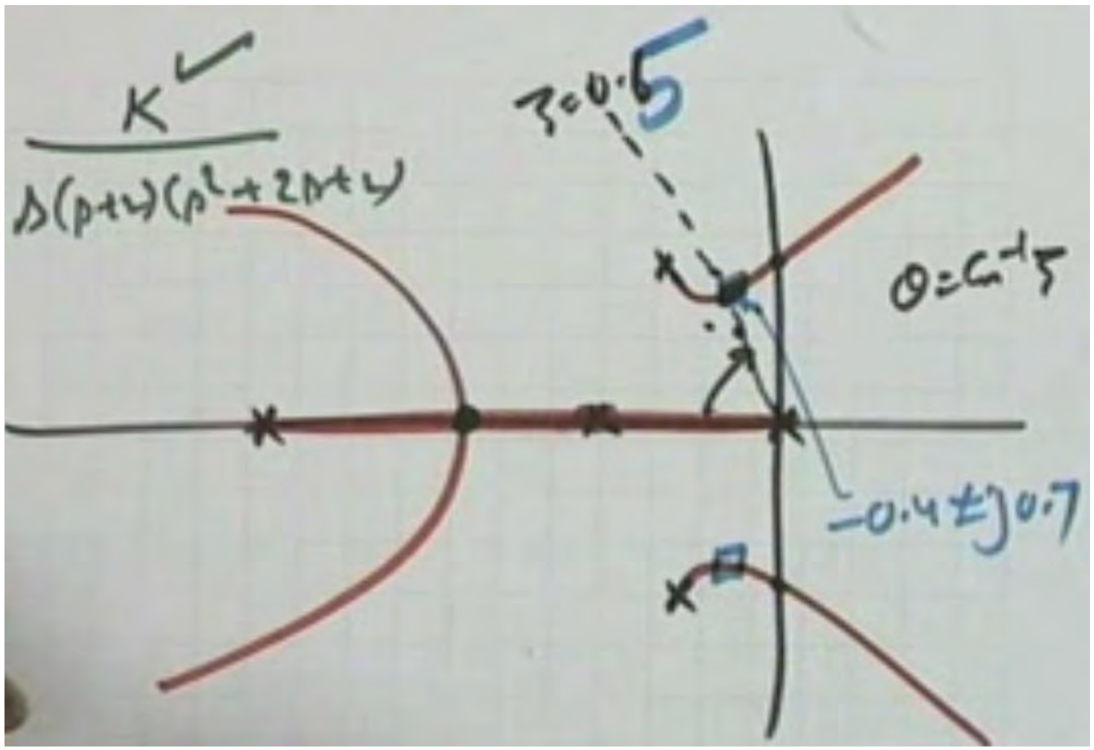


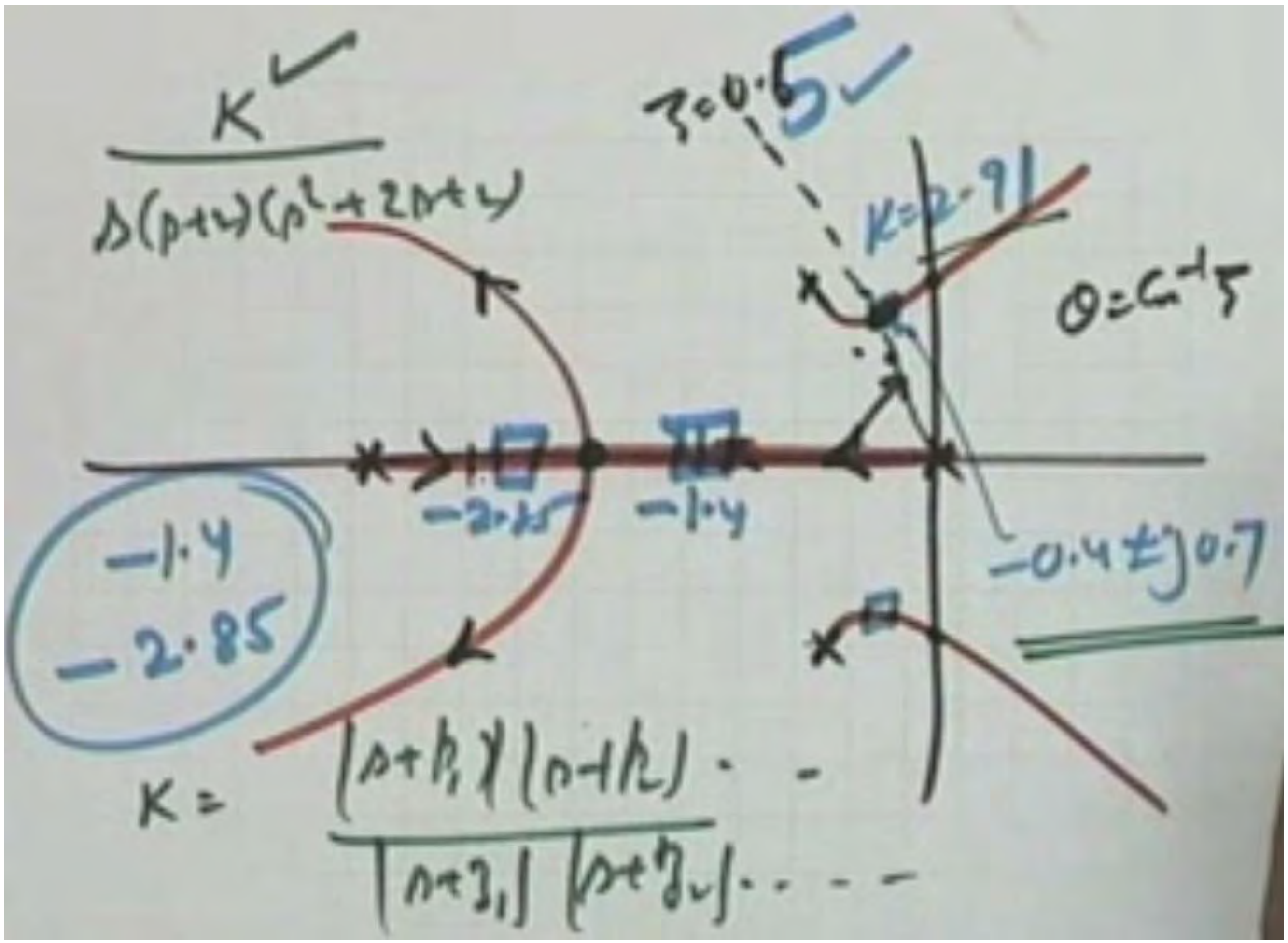
$\frac{dK}{ds} = 0$
K=∞
jω
K=0
1+G(s)=0
-3
-1
K=0
σ
-2.3
K= -s(s+3)(s²+2s+2)
K=∞
-0.725 ± j0.365



K=0
j1.1
K=8.16
K=0
K=0
-3
-2
σ
K=0

Design problem:



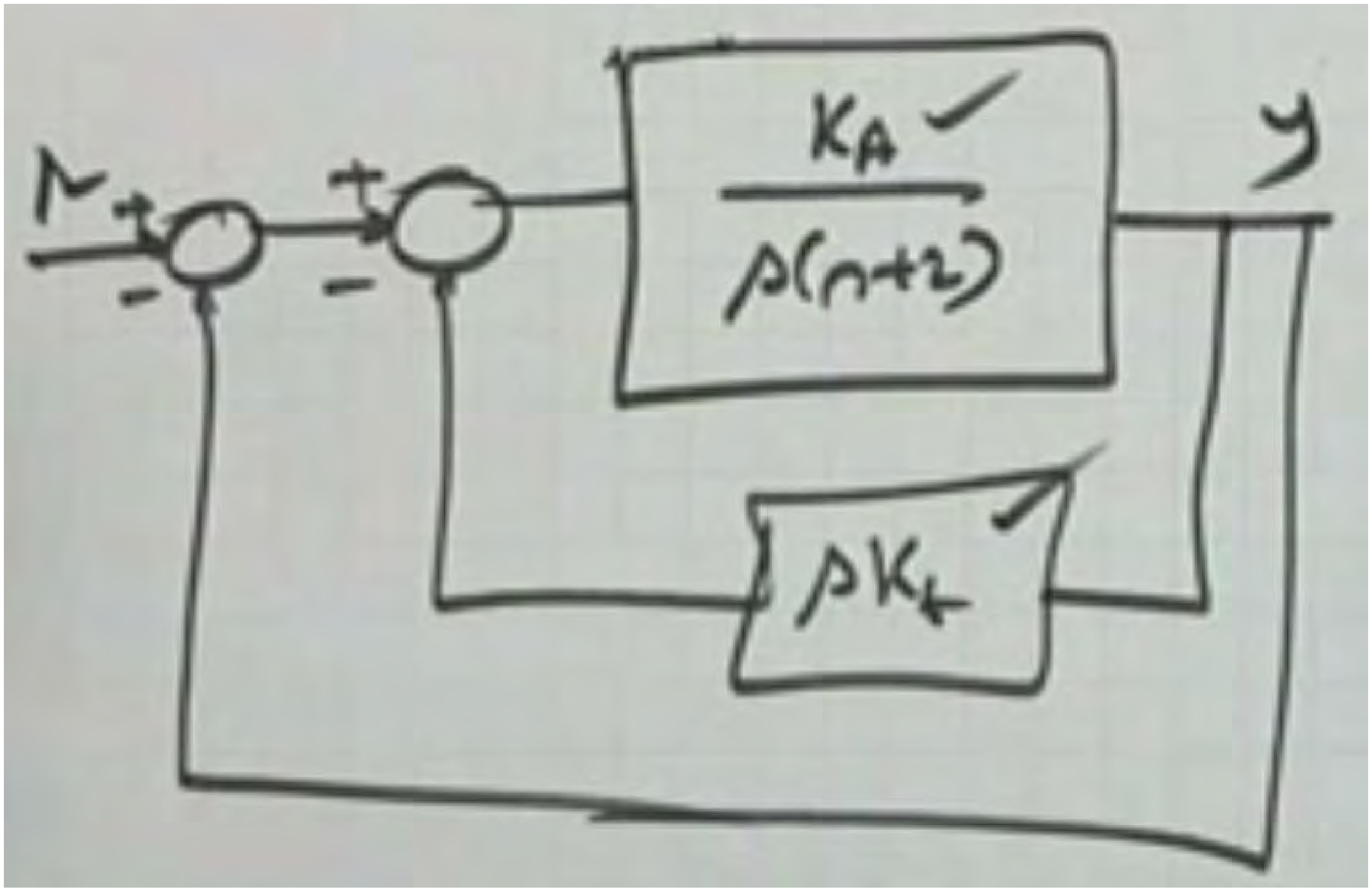


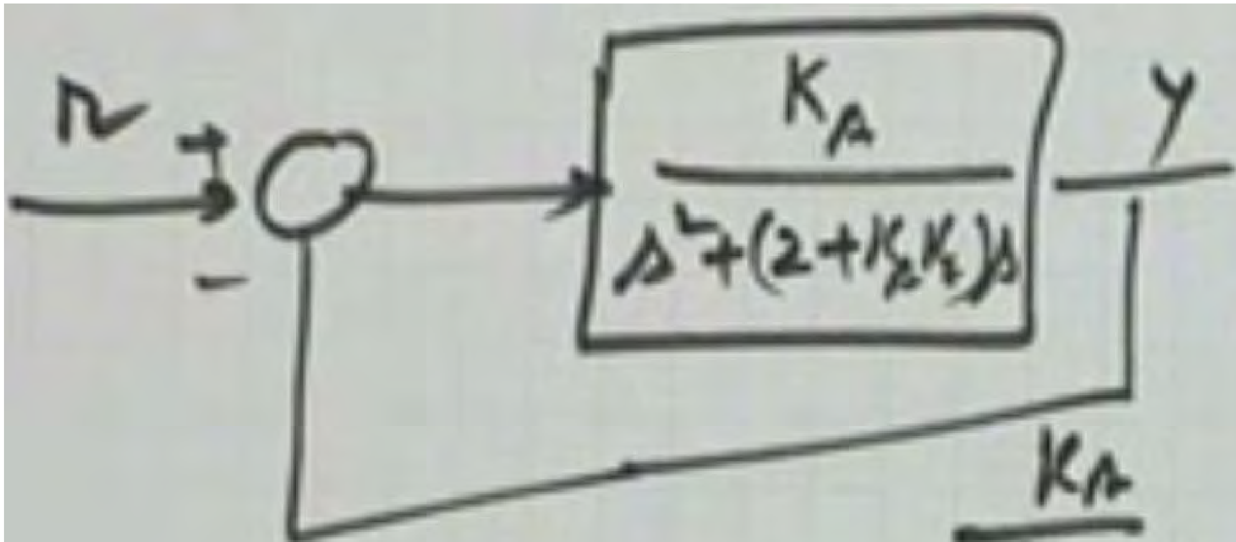
K
s(s+4)(s^2+2s+4)
ζ=0.5
K=2.91
θ=cos^-1 ζ
-2.85
-1.4
-0.4±j0.7
-1.4
-2.85
K = |s+p1||s+p2|.... / |s+z1||s+z2|....

Over all Closed loop transfer function:

$$\frac{2.91}{(s+1.4)(s+2.85)(s+.4+j.7)(s+.4-j.7)}$$

Example 2:





$$\frac{\frac{K_A}{s(s+2)}}{1+\frac{sK_aK_t}{s(s+2)}}$$

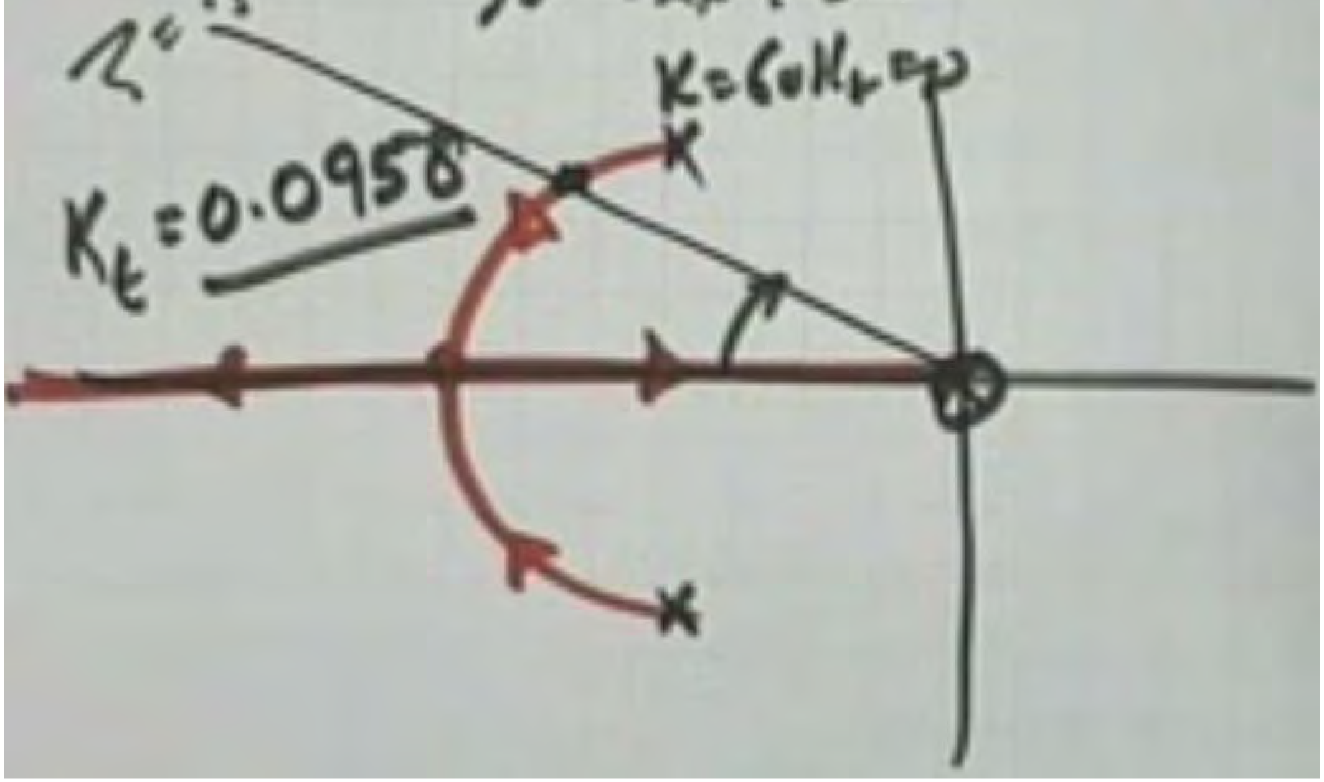
$$s^2 + (2 + K_A K_t) s + K_A = 0$$

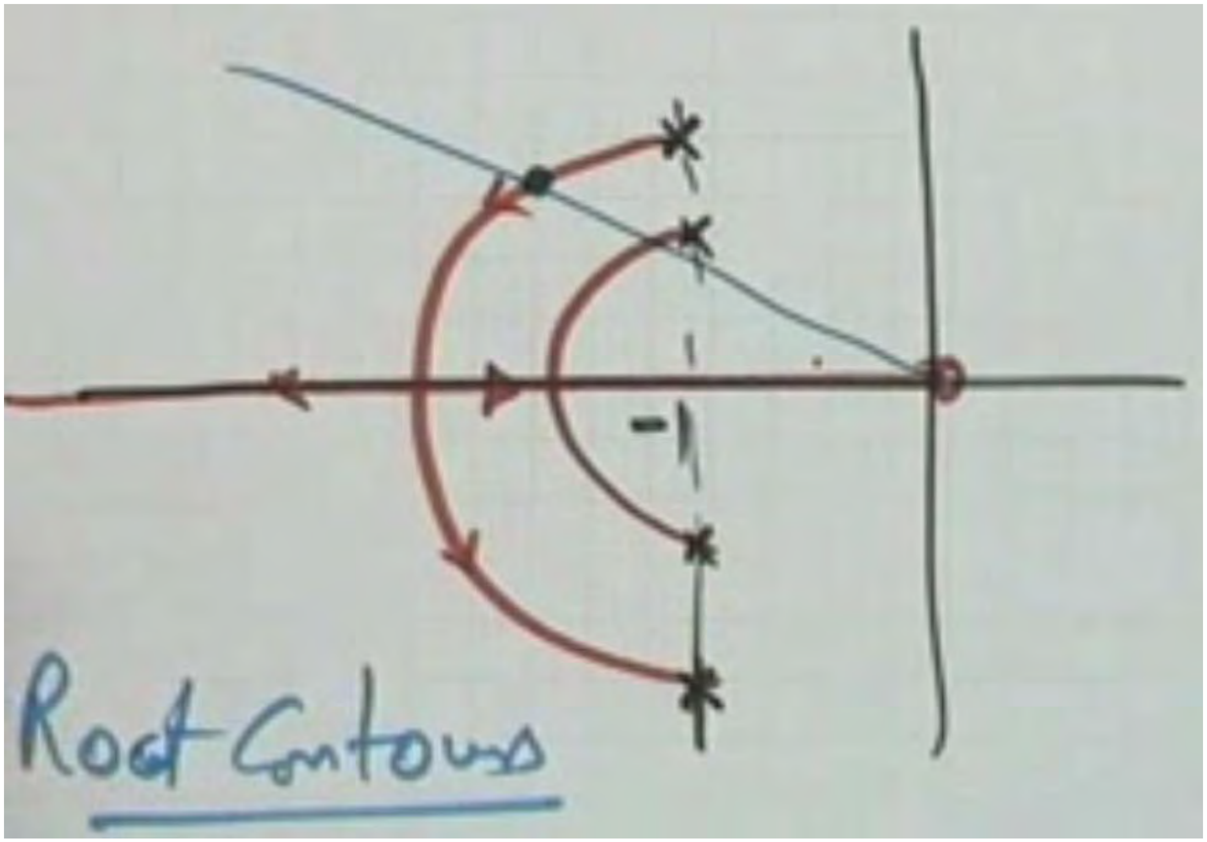
$$s^2 + 2s + K_A + K_A K_t s = 0$$

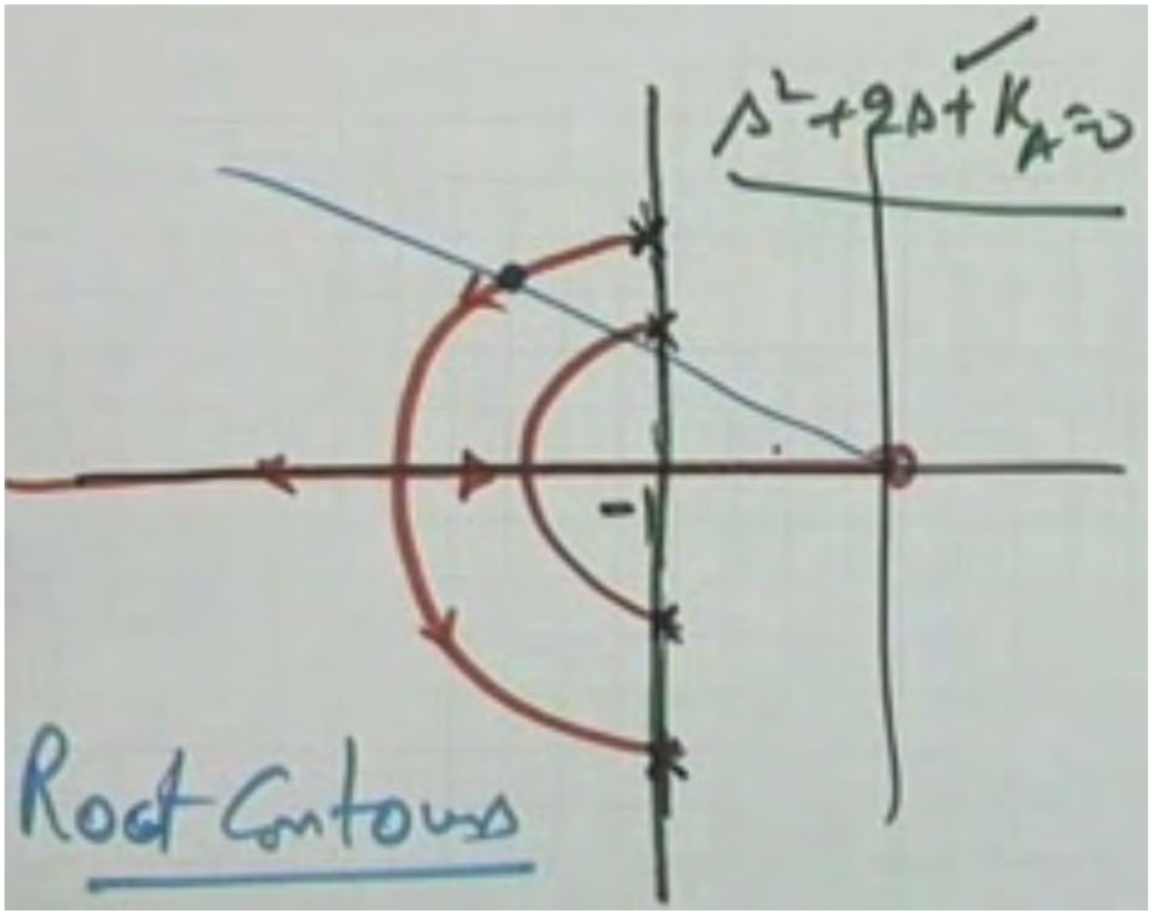
$$1 + \frac{K_A K_t s}{s^2 + 2s + K_A} = 0$$

$$1 + K F(s) = 0 \qquad K = K_A K_t$$

$$1 + \frac{60 K_t s}{s^2 + 2s + 60} = 0$$




-1
Root Contours

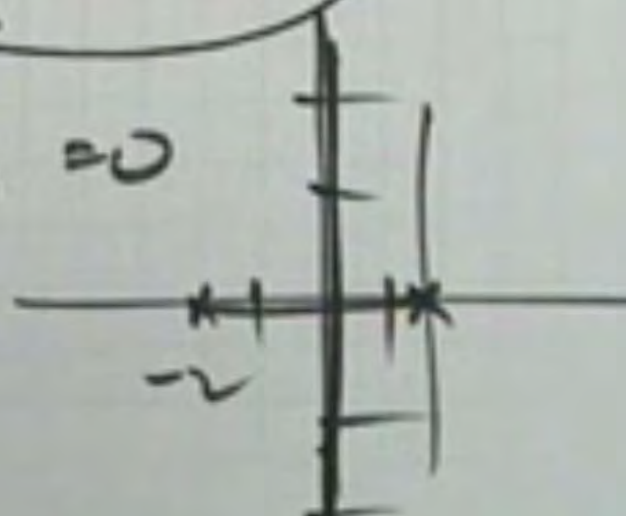


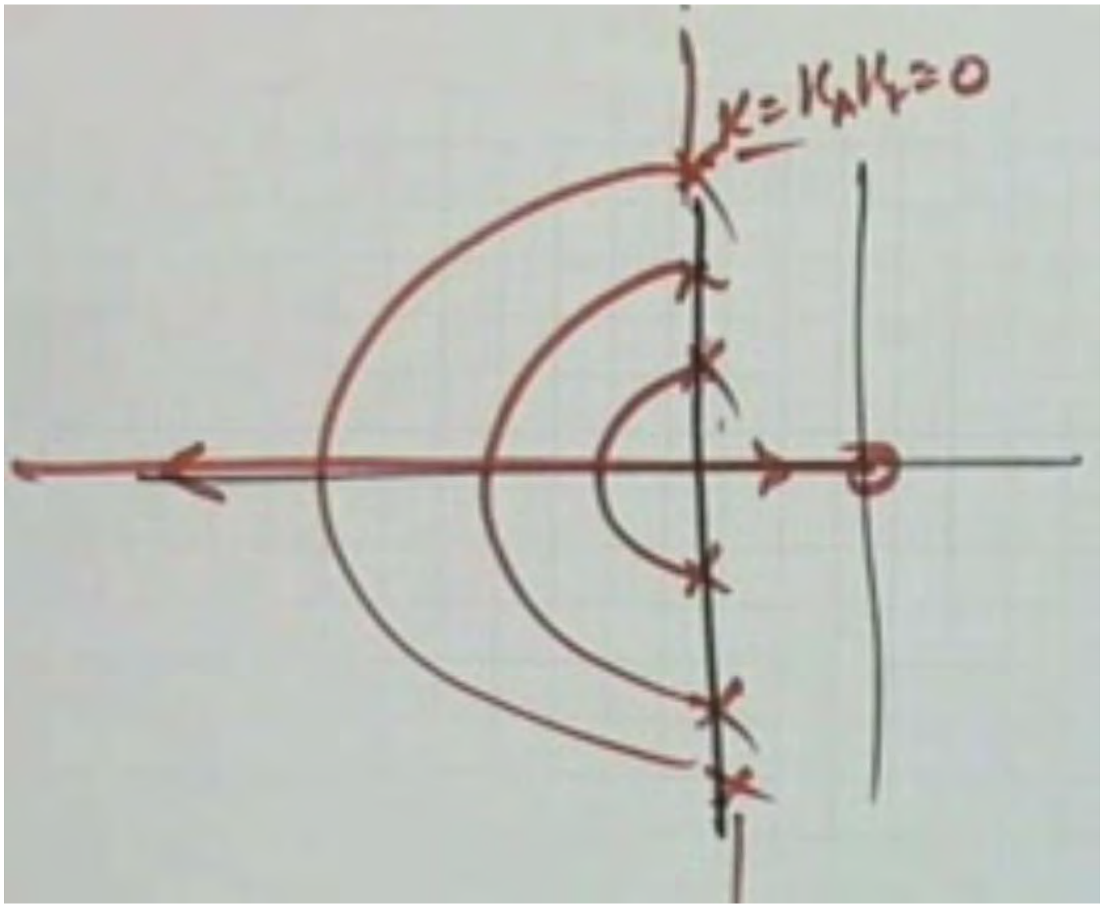
$s^2+2s+K_A=0$
-1
Root Contours

$$1 + \frac{K_A K_t s}{s^2 + 2s + K_A} = 0$$

$$s^2 + 2s + K_A = 0$$

$$1 + \frac{K_A}{s(s+2)}$$




K=KAKr=0

Reference

Books:

[1] M. Gopal, Control systems: principles and design, 3rd ed. Tata McGraw-Hill, 2012.

[2] K. Ogata, Modern Control Engineering, 5th ed. Prentice-Hall, 2010.

[3] N. S. Nise, Control systems engineering, 6th ed. Wiley, 2011.

Or

NPTEL Video lecture:

1. Lec1
2. Lec2
3. Lec3