College of Engineering Pune

Linear Algebra and Univariate Calculus(D.S.Y)

Tutorial 1

Basics of Matrices, System of linear equations and Determinants

1. Let
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$
Compute A^2, A^3, A^4 and B^2, B^3. Generalize A and B to 4x4 matrix.

2. Let A be a diagonal matrix with diagonal elements $a_1, a_2, ..., a_n$. What is A^2, A^3, A^k for any positive integer k.

3. Let A be a square matrix.
   (a) If $A^2 = 0$ show that $I - A$ is invertible.
   (b) If $A^3 = 0$ show that $I - A$ is invertible.
   (c) In general, If $A^n = 0$ for some positive integer n, show that $I - A$ is invertible.

4. If the inverse of A^2 is B, show that the inverse of A is AB. (Thus A is invertible whenever A^2 is invertible)

5. (a) If A is invertible and if $AB = BC$, then prove that $B = C$.
   (b) If A is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, find an example with $AB = BC$ but $B \neq C$.

6. Give examples of A and B such that
   (a) $A + B$ is not invertible although A and B are invertible.
   (b) $A + B$ is invertible although A and B are not invertible.
   (c) All of A, B and $A + B$ are invertible.

7. (a) Show that for any square matrix, the matrix $A +^t A$ is symmetric.
   (b) Show that for any square matrix, the matrix $A -^t A$ is skew-symmetric.
   (c) If a matrix is skew-symmetric then what can you say about its diagonal entries.
   (d) Show that any square matrix can always be written as sum of symmetric and skew-symmetric matrix.

8. Let A be a skew-symmetric matric with odd order then what can you say about its determinant?

9. True or false, with reason if true or counterexample if false
   (a) If A and B are symmetric then AB is symmetric.

(b) If A and B are invertible then BA is invertible.

10. Let A and B be two matrices of the same size. We say that A is **similar** to B if there exists an invertible matrix T such that $B = TAT^{-1}$. Suppose this is the case. Prove:

    (a) B is similar to A.

    (b) A is invertible iff B is invertible.

    (c) ${}^tA$ is similar to ${}^tB$.

    (d) Suppose $A^n = 0$ and B is an invertible matrix of the same size as A. Show that $(BAB^{-1})^n = 0$.

11. Find solutions to following systems using Gauss Elimination method.

    (a) $3x + y + z = 0$

    (b) $-2x + 3y + z + 4w = 0$
    $x + y + 2z + 3w = 0$
    $2x + y + z - 2w = 0$

    (c) $3x + 4y - 2z = 0$
    $x + y + z = 0$
    $-x - 3y + 5z = 0$

    (d) $-3x + y + z = 0$
    $x - y + z - 2w = 0$
    $-x + y - 3w = 0$

    (e) $x + y + z + w = 0$
    $x + y + z - w = 4$
    $x + y - z + w = -4$
    $x - y + z + w = 2$

    (f) $2x - 2y + 4z + 3w = 9$
    $x - y + 2z + 2w = 6$
    $2x - 2y + z + 2w = 3$
    $x - y + w = 2$

12. Determine the values of a and b for which the system has (i) No solution (ii) Infinite number of solutions (iii) Unique solution.

    (a) $x + 2y + 3z = 6$
    $x + 3y + 5z = 9$
    $2x + 5y + az = b$

    (b) $2x + 3y + 5z = 9$
    $7x + 3y - 2z = 8$
    $2x + 3y + az = b$

13. Find inverses of the following matrices, if exists.

    (a) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

    (b) $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$