11 TRAJECTORY PLANNING FOR ROBOTS

11.1 INTRODUCTION

Given the position and the orientation of the tool at the initial instant t_0 and the final instant t_f, it is possible to determine the joint angles at t_0 and t_f. However, the way in which θ_f can be reached from θ_0 is purely arbitrary. We have to evolve a suitable procedure to bring the arm from the initial to the final position. We can have a straight line fit between θ_0 and θ_f w.r.t. time. However, this calls for infinite acclerations at the beginning and at the end of the path. To overcome this problem, a "cubic fit" could be attempted to generate the trajectory. *(See colour plate 5 and 6).*

11.2 JOINT SPACE SCHEME

Cubic polynomial fit

Let the initial joint angle be θ_0 and the final joint angle be θ_f. We can fit a cubic polynomial between θ_0 and θ_f.

Let
$$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \tag{11.1}$$
be the cubic fit.

$$\theta(t)\big|_{t=0} = \theta_0, \quad \theta(t)\big|_{t=t_f} = \theta_f; \quad \theta_0, \theta_f, t_f \text{ are given}$$

Clearly, at $t = 0$, $a_0 = \theta_0$

Now,
$$\theta'(t) = a_1 + 2a_2 t + 3a_3 t^2 \tag{11.2}$$

The initial velocity = final velocity = 0. From Eq. 11.2, we see that

$$a_1 = 0$$

and
$$2a_2 t_f + 3a_3 t_f^2 = 0 \tag{11.3}$$

Also
$$\theta_f = a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 \tag{11.4}$$

Using Eqs 11.3 and 11.4 and the fact that $a_1 = 0$ and $a_0 = \theta_0$, we can solve for a_2 and a_3:

$$a_2 = \frac{3(\theta_f - \theta_0)}{t_f^2}$$

$$a_3 = \frac{-2(\theta_f - \theta_0)}{t_f^3}$$

Example 1

Let $\theta_0 = 15°$ and $\theta_f = 75°$, $t_f = 3.0$ s. We can solve for the coefficients: $a_0 = 15.0$; $a_1 = 0.0$; $a_2 = 20.0$ and $a_3 = -4.44$.

Hence $\theta(t) = 15.0 + 20.0t^2 - 4.44t^3$

Then $\theta'(t) = 40t - 13.32t^2$

and $\theta''(t) = 40 - 26.64t$

The joint tuajectory is composed of $\theta(t)$, $\theta'(t)$ and $\theta''(t)$ and is shown in Fig. 11.1.

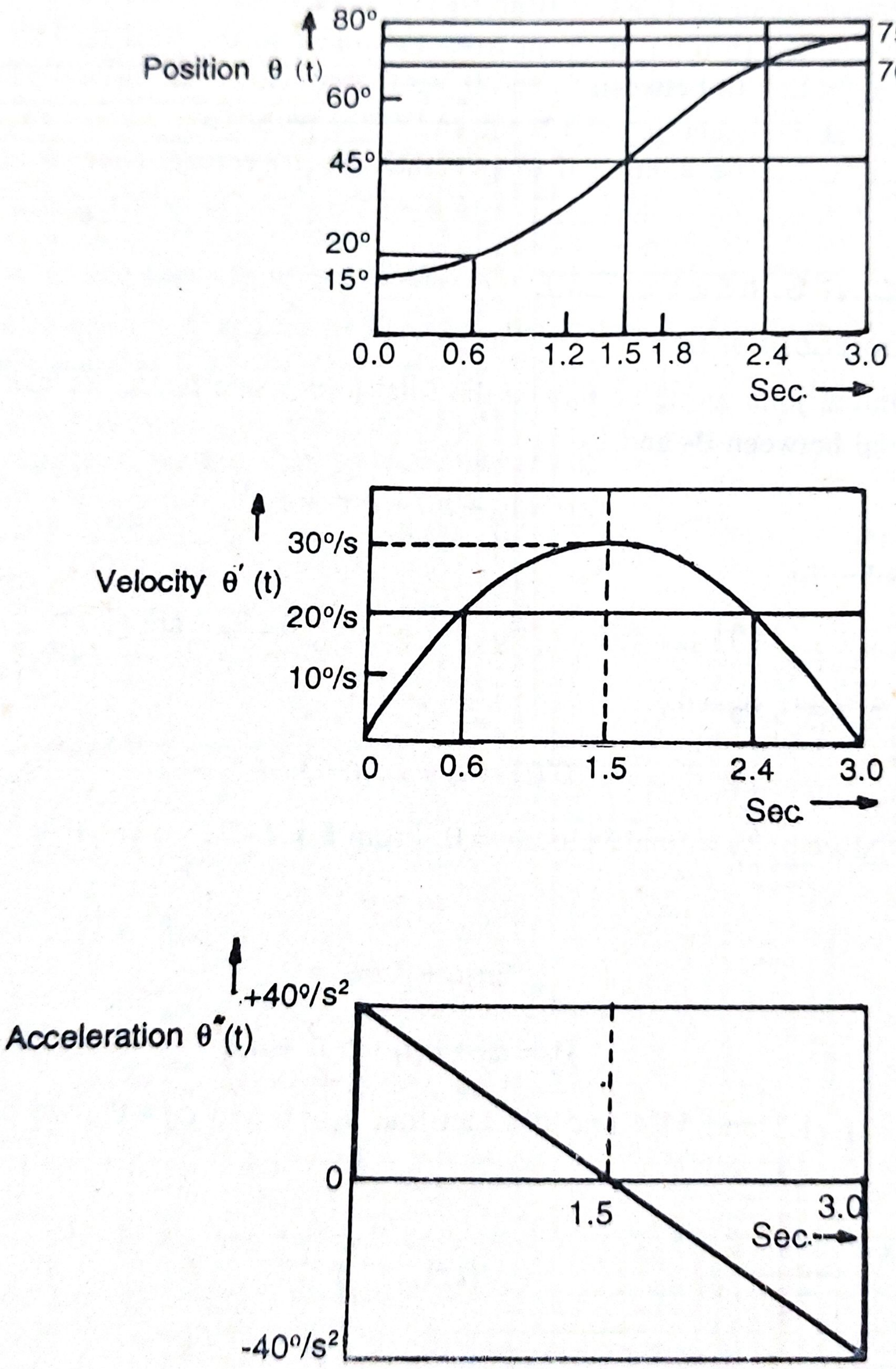

Fig. 11.1 Joint Trajectories

11.3 CUBIC POLYNOMIALS WITH VIA POINTS

When a robot moves from θ_0 to θ_f, it may encounter obstacles in its path. To avoid collision, the robot can be constrained to move through some specified "via points". In this section we shall consider the simplest case in which the robot is required to move from θ_0 to θ_f through a via point θ_v (Fig. 11.2).

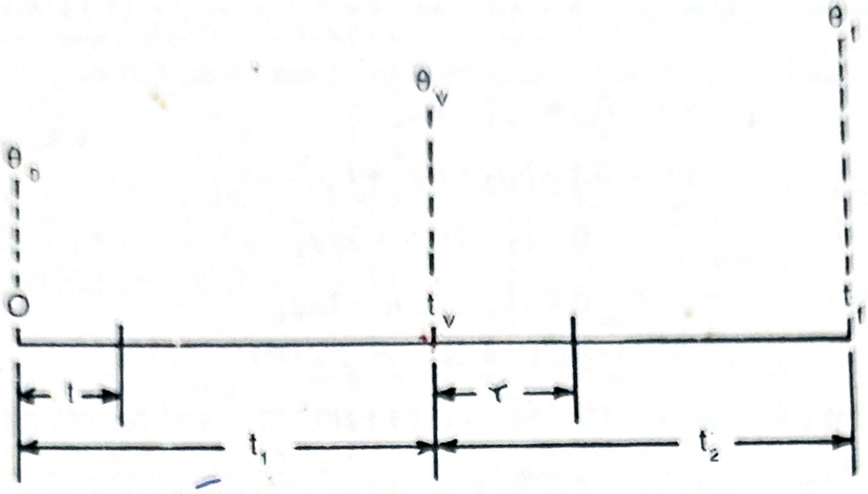

Fig. 11.2 Specifications for Via Point

We can fit two cubic curves for the intervals, (t_0 to t_v) and (t_v to t_f).
Further let us assume that $t_v = (t_f + t_0)/2$; by convention $t_0 = 0$.
Let the first curve be: $\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$ and the second curve: $\theta(\tau) = b_0 + b_1\tau + b_2\tau^2 + b_3\tau^3$ where τ is the time measured from the end of the first cubic curve (Fig. 11.2).

Using the first cubic, $$\theta_0 = a_0 \text{ (at } t = 0) \tag{11.5}$$

and $$\theta_v = a_0 + a_1 t_1 + a_2 t_1^2 + a_3 t_1^3 \tag{11.6}$$

From the second cubic, at $$\tau = 0,\ \theta_v = b_0 \tag{11.7}$$

and at $\tau = t_2$, $$\theta_f = b_0 + b_1 t_2 + b_2 t_2^2 + b_3 t_2^3 \tag{11.8}$$

The starting velocity is equal to zero; differentiating the first curve

$$\theta'(t) = a_1 + 2a_2 t + 3a_3 t^2 \tag{11.9}$$

Hence $$a_1 = 0 \tag{11.10}$$

The end velocity is also equal to zero; by differentiating the second curve and evaluating at $\tau = t_2$, we get

$$0 = b_1 + 2b_2 t_2 + 3b_3 t_2^2 \tag{11.11}$$

The robot must not come to a stop at the via point. Hence, the velocity and acceleration at the via point are assumed to be continuous.

Hence $$a_1 + 2a_2t_1 + 3a_3t_1^2 = b_1 \tag{11.12}$$

and $$2a_2 + 6a_3t_1 = 2b_2 \tag{11.13}$$

Eq. 11.12 is obtained by equating the velocity calculated from the first curve at $t = t_1$ with the velocity computed from the second curve at $\tau = 0$.

Similarly, equating the acceleration at $t = t_1$ and $\tau = 0$, we get Eq. 11.13. So the relevant equations are:

$$\begin{aligned}
(\theta_v - \theta_0) &= a_2t_1^2 + a_3t_1^3 \\
(\theta_f - \theta_v) &= b_1t_2 + b_2t_2^2 + b_3t_2^3 \\
0 &= b_1 + 2b_2t_2 + 3b_3t_2^2 \\
0 &= -b_1 + 2a_2t_1 + 3a_3t_1^2 \\
0 &= -2b_2 + 2a_2 + 6a_3t_1
\end{aligned} \tag{11.14}$$

Also $t_1 = t_2$ because the via point is located midway between t_0 and t_f. Solving the above equations, we get,

$$\begin{aligned}
a_0 &= \theta_0 \\
a_1 &= 0 \\
a_2 &= \frac{(12\theta_v - 3\theta_f - 9\theta_0)}{4t_1^2} \\
a_3 &= \frac{(-8\theta_v + 3\theta_f + 5\theta_0)}{4t_1^3} \\
b_0 &= \theta_v \\
b_1 &= \frac{(3\theta_f - 3\theta_0)}{4t_1} \\
b_2 &= \frac{(-12\theta_v + 6\theta_f + 6\theta_0)}{4t_1^2} \\
b_3 &= \frac{(8\theta_v - 5\theta_f - 3\theta_0)}{4t_1^3}
\end{aligned} \tag{11.15}$$

11.4 BLENDING SCHEME

If the joint angular velocity has to remain constant over a fairly large interval, then a blending technique can be employed. During the starting and stopping phases, to avoid an abrupt jump in the velocity, we constrain the robot to move in the constant acceleration mode. Parabolic blends are used to make the joint variable continuous. In essence, the problem can be stated as:

Given: $t_0 = 0$; $t_f = T$; $t_h = t_f/2$; t_b = blending time; a = constant initial and final acceleration. Determine the joint trajectory between θ_0 and θ_f so that the velocity is constant in the middle region (see Figs 11.3 and 11.4).

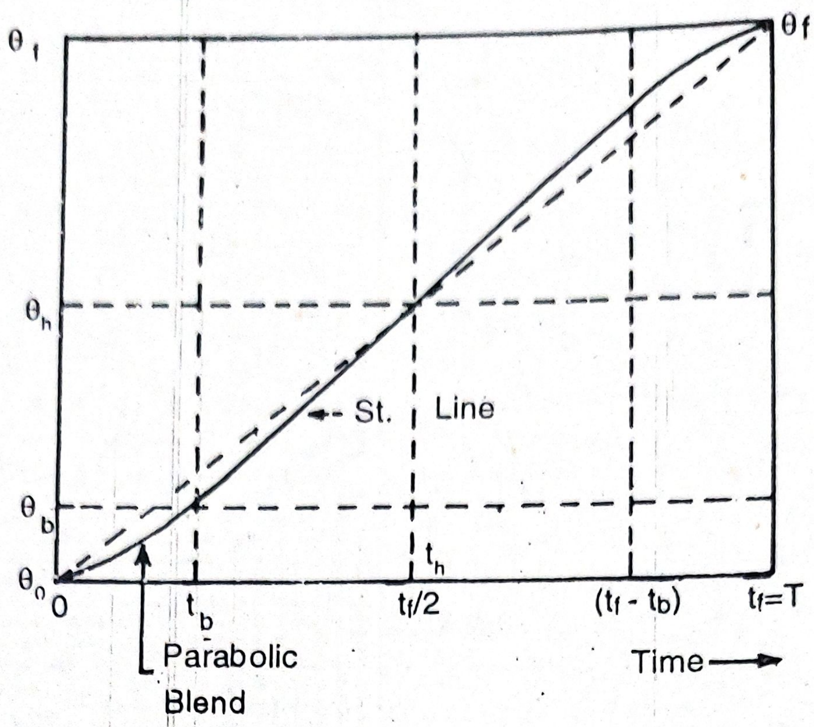

Fig 11.3 Paratonic Blending Scheme

Let the joint angle be equal to $(\theta_0 + \theta_f)/2$ when $t_h = t_f/2$. Between $t = 0$ and t_b, the acceleration a is a constant.

Hence
$$\theta'(t) = at$$
$$\theta'(t_b) = at_b \qquad (11.16)$$

Also during this interval, due to the parabolic nature of the blend,

$$\theta(t) = \theta_0 + \frac{1}{2}at^2$$

Hence
$$\theta(t_b) = \theta_0 + \frac{1}{2}at_b^2 \qquad (11.17)$$

Using the constraint that the velocity is continuous at t_b,

$$at_b = \frac{(\theta_h - \theta_b)}{(t_h - t_b)} \qquad (11.18)$$

In essence, we have the following equations:

$$t_h = \frac{T}{2}$$

$$at_bt_h - at_b^2 = \theta_h - \theta_b$$
$$2\theta_b = 2\theta_0 + at_b^2 \qquad (11.19)$$
$$2\theta_h = \theta_0 + \theta_f$$

Substituting for θ_h and t_h, we get

$$at_bT - 2at_b^2 = \theta_0 + \theta_f - 2\theta_b$$

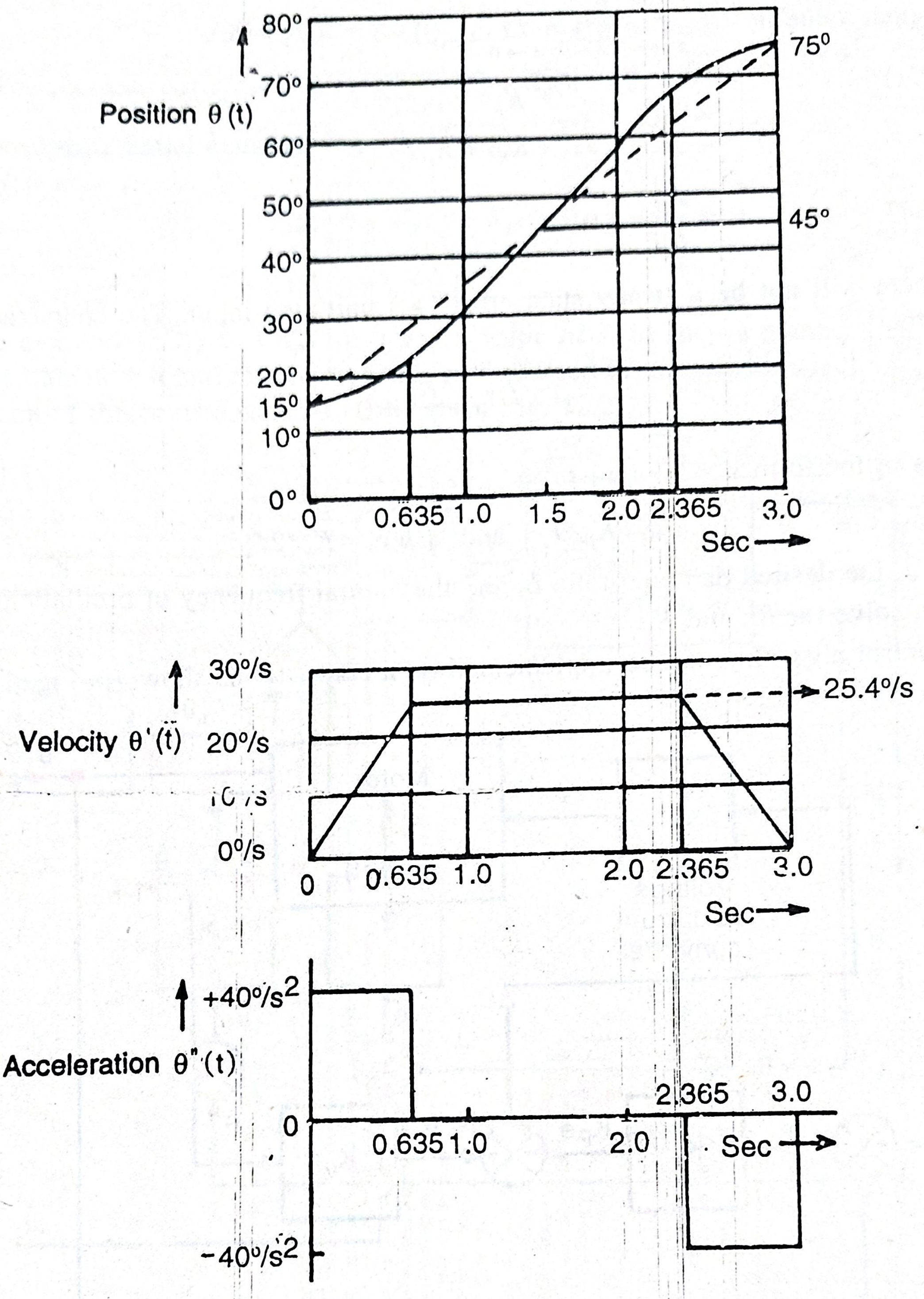

Fig 11.4 Spline Fit for Trajectories

and

$$at_b^2 + 2\theta_0 = 2\theta_b$$

Eliminating $2\theta_b$, we get

$$at_bT - at_b^2 = (\theta_f - \theta_0)$$

Solving this quadratic,

$$t_b = \frac{+\,aT \pm \sqrt{a^2T^2 - 4a(\theta_f - \theta_0)}}{2a} \qquad (11.20)$$

The blending time t_b is chosen to be minimum of the two possibilities. Further, for t_b to be real,

$$a^2T^2 \geq 4a(\theta_f - \theta_0)$$

i.e.,

$$a \geq \frac{4(\theta_f - \theta_0)}{T^2} \tag{11.21}$$

This inequality imposes a restriction on θ_f if a, T, θ_0 are given; or, if a and T are fixed, the job can be successfully done only if the above inequality is satisfied.

Example 2

Let the maximum available angular acceleration for a servomotor be $40°/s^2$, $\theta_f = 75°$, $\theta_0 = 15°$. Let $t_f = 3$.

The blending time

$$t_b = \frac{(40 \times 3) \pm \sqrt{(40 \times 3)^2 - 4 \times 40\,(75 - 15)}}{(2 \times 40)}$$

$$= \frac{120 \pm \sqrt{4800}}{80} = 0.635 \text{ or } 2.366 \text{ s}$$

We take $t_b = 0.635$ s as the reasonable value.

Now at $t = t_b$, the angular velocity is equal to $(40 \times 0.635) = 25.4°/s$. This is the maximum angular velocity.

θ at $t = t_b$ is given by:

$$\theta_b = 15 + 1/2 \times 40 \times (0.635)^2 = 15 + 8 = 23°$$

$$\theta \text{ at } (T - t_b) \text{ is } = 75° - 8° = 67°$$

$$\theta \text{ at } t_h \text{ is} = (75 + 15)/2 = 45°.$$

The acceleration of the motor is constant for the intervals, $0 \leq t \leq 0.635$ and $2.365 \leq t \leq 3.0$. It is zero for $0.635 < t < 2.365$. The results are shown in Fig. 11.4. The set of curves θ, θ', θ'' w.r.t. time is the trajectory of the arm under discussion.

FURTHER READING

Further reading on joint trajectories can be found in Paul [1972], Paul [1979] and Hollerbach [1984]. Cubic spline fits are explained in De Boor [1978], Craig [1986] and Fu, Gonzalez and Lee [1987]. Planning and execution of straight line trajectories are dealt with in Taylor [1979] and Taylor [1983]. Optimum path planning is discussed in Luh and Lin [1981]. Obstacle avoidance in multiple manipulator schemes is discussed in Lozano-Perez [1980] and Brooks [1983]. Minimum time control is discussed in Shin and McKay [1985].