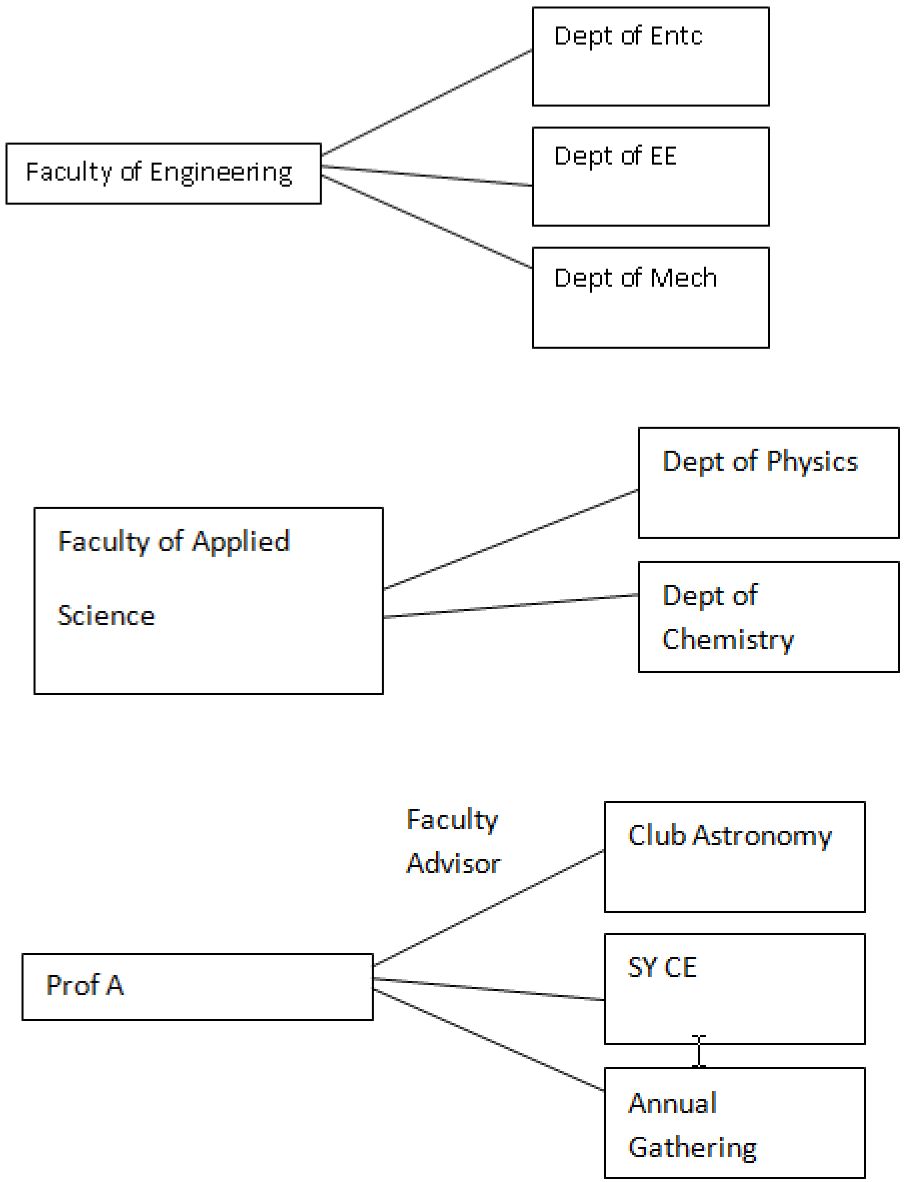
Unit 2:
Relations, Functions and Recurrence Relations

Relations


Faculty of Engineering
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Relations

•If we want to describe a relationship between elements of two sets *A* and B, we can use **ordered pairs** with their first element taken from *A* and their second element taken from B.

•Since this is a relation between **two sets**, it is called a **binary relation**.

•**Definition:** Let *A* and B be sets. *A* binary relation from *A* to B is a subset of $A \times B$.

•In other words, for a binary relation R we have $R \subseteq A \times B$. We use the notation aRb to denote that $(a, b) \in R$ and a<u>R</u>b to denote that $(a, b) \notin R$.

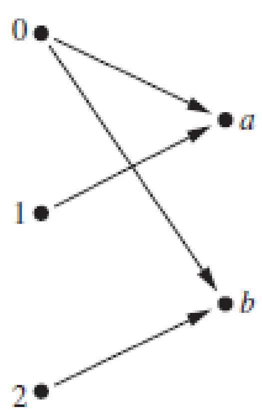
- When (a, b) belongs to R, a is said to be **related** to b by R.
- **Example:** Let P be a set of people, C be a set of cars, and D be the relation describing which person drives which car(s).
- P = {Carl, Suzanne, Peter, Carla},
- C = {Mercedes, BMW, tricycle}
- D = {(Carl, Mercedes), (Suzanne, Mercedes), (Suzanne, BMW), (Peter, tricycle)}
- This means that Carl drives a Mercedes, Suzanne drives a Mercedes and a BMW, Peter drives a tricycle, and Carla does not drive any of these vehicles.

Relations

sets of ordered pairs are called binary relations.

- *Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then $\{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B.*

This means, for instance, that $0 R a$, but that $1 \not R b$.



R	a	b
0	×	×
1	×	
2		×

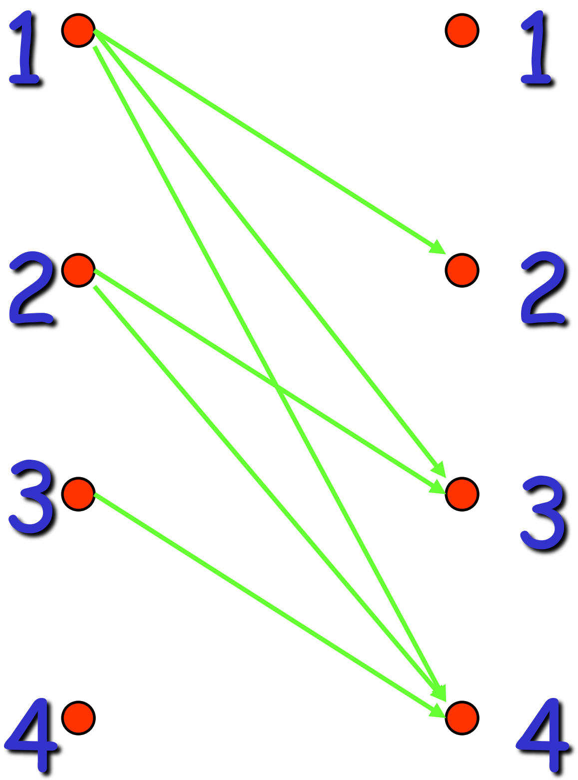
Relations on set

- **Definition:** A relation on the set A is a relation from A to A.

- In other words, a relation on the set A is a subset of $A \times A$.

- **Example:** Let $A = \{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a < b\}$?

- Solution: R = {(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)}



R	1	2	3	4
1		X	X	X
2			X	X
3				X
4				

Relations and Their Properties

A binary relation from set A to B
is a subset of Cartesian product $A \times B$

Example: $A = \{0,1,2\}$ $B = \{a,b\}$

A relation: $R = \{(0,a),(0,b),(1,a),(2,b)\}$

- **How many different relations can we define on a set A with n elements?**
- A relation on a set A is a subset of $A \times A$.
- How many elements are in $A \times A$?

- There are n^2 elements in $A \times A$, so how many subsets (= relations on A) does $A \times A$ have?

- The number of subsets that we can form out of a set with m elements is 2^m, here $m = n^2$ for $A \times A$ .Therefore, 2^{n^2} subsets can be formed out of $A \times A$.
- **Answer:** We can define 2^{n^2} different relations on A.
- For example, there are 2^{3^9} = 512 relations on the set $\{a, b, c\}$.

- **Example: Let A = {1,2}**
- • What is $A \times A$ = **{(1,1),(1,2),(2,1),(2,2)}**
- • **List of possible relations (subsets of $A \times A$):**
  - $\varnothing$ 1
  - {(1,1)} {(1,2)} {(2,1)} {(2,2)} 4
  - {(1,1), (1,2)} {(1,1),(2,1)} {(1,1),(2,2)} 6
    {(1,2),(2,1)} {(1,2),(2,2)} {(2,1),(2,2)}
  - {(1,1),(1,2),(2,1)} {(1,1),(1,2),(2,2)} 4
    {(1,1),(2,1),(2,2)} {(1,2),(2,1),(2,2)}
  - {(1,1),(1,2),(2,1),(2,2)} 1
  - Use formula: 2^4 = **16**

Types of Relations

A relation on set A is a subset of $A \times A$

Example:

A relation on set $A = \{1,2,3,4\}$:

$$R = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$$

Reflexive relation R on set A :

$$\forall a \in A, \quad (a,a) \in R$$

Example: $A = \{1,2,3,4\}$

$$R = \{(1,1),(1,2),(2,1),(2,2),(3,4),(3,3),(4,3),(4,4)\}$$

Is the "divides" relation on the set of positive integers reflexive?

Solution: *Because $a \mid a$ whenever a is a positive integer, the "divides" relation is reflexive.*

(Note that if we replace the set of positive integers with the set of all integers the relation is not reflexive

because by definition 0 does not divide 0.)

Example

- *Solution:* Consider the following relations on {1, 2, 3, 4}:

R1 = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)},

R2 = {(1, 1), (1, 2), (2, 1)},

R3 = {(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)},

R4 = {(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)},

R5 = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)},

R6 = {(3, 4)}. Which of these relations are reflexive?

The relations R3 and R5 are reflexive because they both contain all pairs of the form (a, a), namely, (1, 1), (2, 2), (3, 3), and (4, 4).

The other relations are not reflexive because

they do not contain all of these ordered pairs. In particular, *R1, R2, R4, and R6 are not reflexive*

because *(3, 3) is not in any of these relations.*

- Are the following relations on {1, 2, 3, 4} reflexive?
- R = {(1, 1), (1, 2), (2, 3), (3, 3), (4, 4)} No
- R = {(1, 1), (2, 2), (2, 3), (3, 3), (4, 4)} yes
- R = {(1, 1), (2, 2), (3, 3)} No

Symmetric relation R :

$$(a,b) \in R \to (b,a) \in R$$

Example: $A = \{1,2,3,4\}$

$$R = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,3),(4,4)\}$$

Asymmetric relation R :

$$(a,b) \in R \rightarrow (b,a) \notin R$$

A relation is said to be asymmetric if it is both <u>antisymmetric</u> and <u>irreflexive</u> or else it is not.

Example: $A = \{1,2,3,4\}$

$$R = \{(1,2),(3,4)\}$$

Antisymmetric relation R :

$$(a,b) \in R \wedge (b,a) \in R \rightarrow a = b$$

Example: $A = \{1,2,3,4\}$

$$R = \{(1,1),(1,2),(2,2),(3,4),(4,4)\}$$

Properties of Relations

- Are the following relations on $\{1, 2, 3, 4\}$ symmetric, antisymmetric, or asymmetric?

$R = \{(1, 1), (1, 2), (2, 1), (3, 3), (4, 4)\}$	symmetric
$R = \{(1, 1)\}$	sym. and antisym.
$R = \{(1, 3), (3, 2), (2, 1)\}$	antisym. and asym.
$R = \{(4, 4), (3, 3), (1, 4)\}$	antisym.

In last example

- ✓ The relations *R2 and R3 are symmetric, because in each case (b, a) belongs to the* relation whenever *(a, b) does.*
- ✓ *For R2, the only thing to check is that both (2, 1) and (1, 2) are* in the relation.
- ✓ For *R3, it is necessary to check that both (1, 2) and (2, 1) belong to the relation,* and *(1, 4) and (4, 1) belong to the relation.*
- ✓ This is done by finding a pair *(a, b) such that it is in the relation* but *(b, a) is not.*

In last example

- ✓ *R4, R5, and R6 are all antisymmetric. For each of these relations there is no pair of elements a and b with a = b such that both (a, b) and (b, a) belong to the relation.*
- ✓ This is done by finding a pair *(a, b)* with *a = b* *such that (a, b) and (b, a) are both in the relation.*

Is the "divides" relation on the set of positive integers symmetric? Is it antisymmetric?

Solution: *This relation is not symmetric because 1| 2, but 2 | 1. It is antisymmetric, for if a*

and b are positive integers with a |b and b |a, then a = b

Transitive relation R :

$$(a,b) \in R \wedge (b,c) \in R \rightarrow (a,c) \in R$$

Example: $A = \{1,2,3,4\}$

$$R = \{(1,1),(1,2),(2,3),(3,4)(1,3),(1,4),(2,4)\}$$

In same Example

- *R4, R5, and R6 are transitive. For each of these relations, we can show that it is* transitive by verifying that if *(a, b) and (b, c) belong to this relation, then (a, c) also does.*
- *For* instance, *R4 is transitive, because (3, 2) and (2, 1), (4, 2) and (2, 1), (4, 3) and (3, 1), and (4, 3)* and *(3, 2) are the only such sets of pairs, and (3, 1), (4, 1), and (4, 2) belong to R4.*
- *R1 is not transitive because (3, 4) and (4, 1) belong to R1, but (3, 1) does not.*
- *R2 is* not transitive because *(2, 1) and (1, 2) belong to R2, but (2, 2) does not.*
- *R3 is not transitive* because *(4, 1) and (1, 2) belong to R3, but (4, 2) does not.*

Is the "divides" relation on the set of positive integers transitive?

Solution: Suppose that a divides b and b divides c. Then there are positive integers k and l

such that *b = ak and c = bl. Hence, c = a(kl), so a divides c. It follows that this relation is*

transitive.

How many reflexive relations are there on a set with *n elements*?

Solution:

A relation R on a set A is a subset of A × A.

Consequently, a relation is determined by specifying whether each of the *ordered pairs in A × A is in R.*

However, if R is reflexive, each of the *n ordered* pairs *(a, a) for a ∈ A must be in R.*

Each of the other n(n − 1) ordered pairs of the form *(a, b), where a ≠ b, may or may not be in R.*

Hence, by the product rule for counting, $2^{n(n-1)}$ *reflexive relations [this is the number of ways to choose whether each element (a, b), with a ≠ b, belongs to R].*

Relations of Relations summary

	=	<	>	≤	≥
Reflexive	X			X	X
Irreflexive		X	X		
Symmetric	X				
Asymmetric		X	X		
Antisymmetric	X			X	X
Transitive	X	X	X	X	X

Combining Relations

$$R_1 = \{(1,1),(2,2),(3,3)\}$$

$$R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$$

$$R_1 \cup R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)\}$$

$$R_1 \cap R_2 = \{(1,1)\}$$

$$R_1 - R_2 = \{(2,2),(3,3))$$

n-ary relations

An n-ary relation on sets $A_1, A_2, \ldots, A_n$
is a subset of Cartesian product $A_1 \times A_2 \times \cdots \times A_n$

The sets $A_1, A_2, \ldots, A_n$ *are called the domains of the relation, and* ***n*** *is called its degree.*

Example: A relation on $N \times N \times N$

All triples of numbers (a, b, c) with $a < b < c$

$$R = \{(1,2,3), (1,2,4), (1,2,5), \ldots\}$$

Relational data model

n-ary relation R is represented with table

fields

R: Teaching assignments

records

Professor	Department	Course-number
Cruz	Zoology	335
Cruz	Zoology	412
Farber	Psychology	501
Farber	Psychology	617
Rosen	Comp. Science	518
Rosen	Mathematics	575

primary key
(all entries are different)

Selection operator: $s_C(R)$

keeps all records that satisfy condition C

Example: $C: \text{Department} = \text{Psychology}$

Result of selection operator $s_C(R)$

Professor	Department	Course-number
Farber	Psychology	501
Farber	Psychology	617

Projection operator: $P_{i_1, i_2, \ldots, i_m}(R)$

Keeps only the fields $i_1, i_2, \ldots, i_m$ of R

Example:

$$P_{\text{Professor, Department}}(R)$$

Professor	Department
Cruz	Zoology
Farber	Psychology
Rosen	Comp. Science
Rosen	Mathematics

Join operator: $J_k(R, S)$

Concatenates the records of R and S
where the last k fields of R
are the same with the first k fields of S

S: Class schedule

Department	Course-number	Room	Time
Comp. Science	518	N521	2:00pm
Mathematics	575	N502	3:00pm
Mathematics	611	N521	4:00pm
Psychology	501	A100	3:00pm
Psychology	617	A110	11:00am
Zoology	335	A100	9:00am
Zoology	412	A100	8:00am

$J_2(R,S)$

Professor	Department	Course Number	Room	Time
Cruz	Zoology	335	A100	9:00am
Cruz	Zoology	412	A100	8:00am
Farber	Psychology	501	A100	3:00pm
Farber	Psychology	617	A110	11:00am
Rosen	Comp. Science	518	N521	2:00pm
Rosen	Mathematics	575	N502	3:00pm

Representing Relations with Matrices

$$A = \{a_1, a_2, a_3\} \qquad B = \{b_1, b_2, b_3, b_4, b_5\}$$

$$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}$$

Relation Matrix

$$M_R \qquad \begin{array}{cc} & B \\ & \begin{array}{ccccc} b_1 & b_2 & b_3 & b_4 & b_5 \end{array} \\ A \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

Reflexive relation R on set A :

$$\forall a \in A, \quad (a,a) \in R$$

Diagonal elements must be 1

Example: $A = \{1,2,3,4\}$

$$R = \{(1,1),(1,2),(2,1),(2,2),(3,4),(3,3),(4,3),(4,4)\}$$

$$\begin{array}{cc} & \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} & \begin{bmatrix} 1 & 1 & & \\ 1 & 1 & & \\ & & 1 & 1 \\ & & 1 & 1 \end{bmatrix} \end{array}$$

Symmetric relation R : $(a,b) \in R \rightarrow (b,a) \in R$

Matrix is equal to its transpose: $M_R = M_R^T$

Example: $A = \{1,2,3,4\}$

$$R = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,3),(4,4)\}$$

For all i, j

$M_R[i,j] = M_R[j,i]$

$$\begin{array}{cc} & \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} & \begin{bmatrix} 1 & 1 & & \\ 1 & 1 & & \\ & & & 1 \\ & & 1 & 1 \end{bmatrix} \end{array}$$

Antisymmetric relation R :

$$(a,b) \in R \wedge (b,a) \in R \rightarrow a = b$$

Example: $A = \{1,2,3,4\}$

$$R = \{(1,1),(2,2),(2,1),(3,4),(4,1),(4,4)\}$$

For all $i \neq j$

$M_R[i,j] \neq M_R[j,i]$

$$\begin{array}{cc} & \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} & \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & & & 1 \\ 1 & & & 1 \end{bmatrix} \end{array}$$

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Union $R \cup S$:

$$M_{R \cup S} = M_R \vee M_S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Intersection $R \cap S$:

$$M_{R \cap S} = M_R \wedge M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A=\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$AUB=\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$A=\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A\cap B=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Composite Relation

- *Let R be a relation from a set A to a set B and S a relation from B to a set C. The composite of R and S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$.*
- *We denote the composite of R and S by $S \circ R$.*

Composite relation: $S \circ R$

$$(a,b) \in S \circ R \leftrightarrow \exists x : (a,x) \in R \wedge (x,b) \in S$$

Note: $(a,b) \in R \wedge (b,c) \in S \rightarrow (a,c) \in S \circ R$

Example:

$$R = \{(1,1),(1,4),(2,3),(3,1),(3,4)\}$$

$$S = \{(1,0),(2,0),(3,1),(3,2),(4,1)\}$$

$$S \circ R = \{(1,0),(1,1),(2,1),(2,2),(3,0),(3,1)\}$$

Combining Relations

- **Example:** Let D and S be relations on A = {1, 2, 3, 4}.
- $D = \{(a, b) \mid b = 5 - a\}$ "b equals (5 – a)"
- $S = \{(a, b) \mid a < b\}$ "a is smaller than b"

- $D = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$
- $S = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$
- $S \circ D = \{$ (2, 4), (3, 3), (3, 4), (4, 2), (4, 3), (4, 4)}

D maps an element a to the element (5 – a), and afterwards S maps (5 – a) to all elements larger than (5 – a), resulting in $S \circ D = \{(a,b) \mid b > 5 - a\}$ or $S \circ D = \{(a,b) \mid a + b > 5\}$.

Power of relation: R^n

$$R^1 = R \qquad\qquad R^{n+1} = R^n \circ R$$

Example: $R = \{(1,1),(2,1),(3,2),(4,3)\}$

$$R^2 = R \circ R = \{(1,1),(2,1),(3,1)(4,2)\}$$

$$R^3 = R^2 \circ R = \{(1,1),(2,1),(3,1)(4,1)\}$$

$$R^4 = R^3 \circ R = R^3$$

Theorem: A relation R is transitive if an only if $R^n \subseteq R$ for all $n = 1,2,3,\ldots$

Proof: 1. If part: $R^2 \subseteq R$

2. Only if part: use induction

1. If part: We will show that if $R^2 \subseteq R$ then R is transitive

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Assumption: $R^2 \subseteq R$

Definition of power: $R^2 = R \circ R$

Definition of composition:

$$(a,b) \in R \wedge (b,c) \in R \rightarrow (a,c) \in R \circ R$$

$$\Rightarrow (a,c) \in R$$

Therefore, R is transitive

2. Only if part:

We will show that if R is transitive then $R^n \subseteq R$ for all $n \geq 1$

Proof by induction on n

Inductive basis: $n = 1$

It trivially holds $R^1 = R \subseteq R$

Inductive hypothesis:

Assume that $R^k \subseteq R$

for all $1 \leq k \leq n$

Inductive step: We will prove $R^{n+1} \subseteq R$

Take arbitrary $(a,b) \in R^{n+1}$

We will show $(a,b) \in R$

$$(a,b) \in R^{n+1}$$

$\Downarrow$ definition of power

$$(a,b) \in R^{n} \circ R$$

$\Downarrow$ definition of composition

$$\exists x : (a,x) \in R \wedge (x,b) \in R^{n}$$

$\Downarrow$ inductive hypothesis $R^{n} \subseteq R$

$$\exists x : (a,x) \in R \wedge (x,b) \in R$$

$\Downarrow$ R is transitive

$$(a,b) \in R$$

End of Proof

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Composition $S \circ R$: Boolean matrix product

$$M_{S \circ R} = M_R \circ M_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

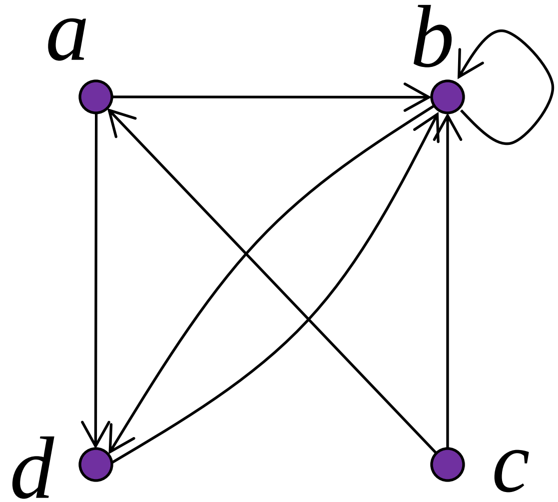
$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Power $R^2 = R \circ R$: Boolean matrix product

$$M_{R^2} = M_R \circ M_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Digraphs (Directed Graphs)

$$R = \{(a,b),(a,d),(b,b),(b,d),(c,a),(c,b),(d,b)\}$$



Theorem: $(a,b) \in R^n$

if and only if

there is a path of length n

from a to b in R

Connectivity relation:

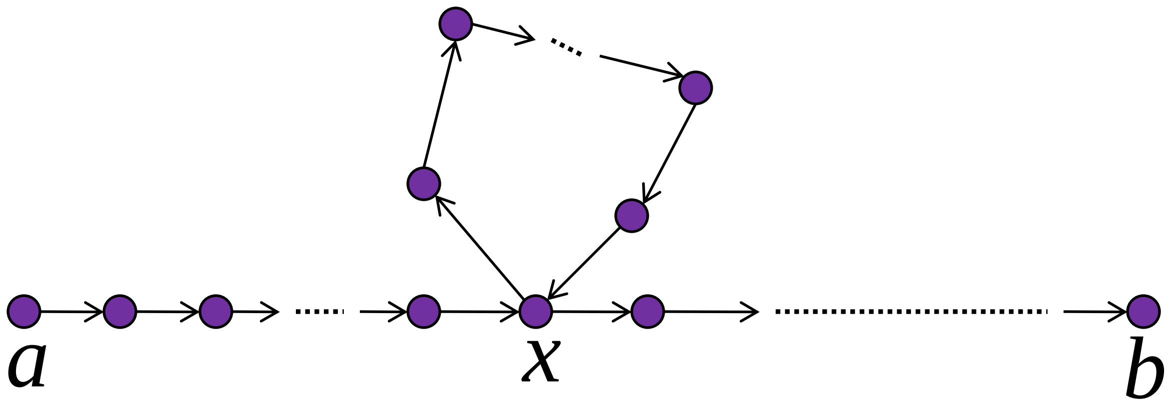
$$R^{*} = R^{1} \cup R^{2} \cup R^{3} \cup \cdots = \bigcup_{i=1}^{\infty} R^{i}$$

$$(a,b) \in R^{*}$$

if and only if
there is some path (of any length)
from a to b in R

Theorem: $R^* = R^1 \cup R^2 \cup R^3 \cup \cdots \cup R^n$

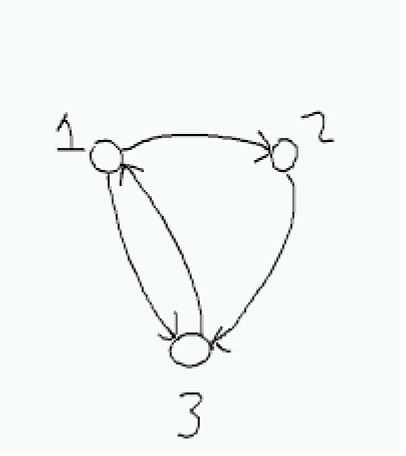
Proof: if $(a,b) \in R^{n+1}$ then $(a,b) \in R^i$ for some $i \in \{1,\ldots,n\}$



Repeated node

Example

- *Let R be the relation on the set of all people in the world that contains (a, b) if a has met b. What is R^n, where n is a positive integer greater than one? What is R^*?*
- Soluion:
  - The relation R^2 *contains (a, b) if there is a person c such that (a, c) ∈ R and (c, b) ∈ R,*
  - that is, if there is a person *c such that a has met c and c has met b.*
  - *Similarly,* R^n *consists of* those pairs *(a, b) such that there are people x1, x2, . . . , xn−1 such that a has met x1, x1 has* met *x2, . . . , and xn−1 has met b.*
  - The relation R^* contains *(a, b) if there is a sequence of people, starting with a and ending* with *b, such that each person in the sequence has met the next person in the sequence.*


1
2
3

Closures and Relations

Reflexive closure of R:

Smallest size relation that contains R and is reflexive

Easy to find

- The relation $R = \{(1, 1), (1, 2), (2, 1), (3, 2)\}$ on the set $A = \{1, 2, 3\}$ is not reflexive
- To make R reflexive we add $(2, 2)$ and $(3, 3)$ to R,
- *any* reflexive relation that contains R must also contain $(2, 2)$ and $(3, 3)$.
- Because this relation contains R, is reflexive, and is contained within every reflexive relation that contains R, it is called the **reflexive closure** of R.

- What is the reflexive closure of the relation $R = \{(a, b) \mid a < b\}$ on the set of integers?
- *Solution*: The reflexive closure of R is

$R \cup \ = \{(a, b) \mid a < b\} \cup \{(a, a) \mid a \in \mathbf{Z}\} = \{(a, b) \mid a \leq b\}$.

Symmetric closure of R :

Smallest size relation that contains R and is symmetric

Easy to find

- $\{(1, 1), (1, 2), (2, 2), (2, 3), (3, 1), (3, 2)\}$ on $\{1, 2, 3\}$ is not symmetric
- we need only add $(2, 1)$ and $(1, 3)$, because these are the only pairs of the form (b, a) with $(a, b) \in R$ that are not in R.
- This new relation is symmetric and contains R.
- new relation is called the **symmetric closure** of R.

- What is the symmetric closure of the relation $R = \{(a, b) \mid a > b\}$ on the set of positive integers?
- *Solution:* The symmetric closure of R is the relation

$R \cup R^{-1} = \{(a, b) \mid a > b\} \cup \{(b, a) \mid a > b\} = \{(a, b) \mid a = b\}$.

$$R \cup R^{-1} = \{(a, b) \mid a > b\} \cup \{(b, a) \mid a > b\} = \{(a, b) \mid a \neq b\}.$$

- What is the symmetric closure of the relation $R = \{(a, b) \mid a > b\}$ on the set of positive integers?
- *Solution:* The symmetric closure of R is the relation

$RUR^{-1} = \{(a, b) \mid a > b\} \cup \{(b, a) \mid a > b\} = \{(a, b) \mid a = b\}$.

$$R \cup R^{-1} = \{(a, b) \mid a > b\} \cup \{(b, a) \mid a > b\} = \{(a, b) \mid a \neq b\}.$$

Transitive closure of R:

Smallest size relation that contains R and is transitive

More difficult to find

- $R = \{(1, 3), (1, 4), (2, 1), (3, 2)\}$ on the set $\{1, 2, 3, 4\}$. This relation is not transitive because it does not contain all pairs of the form (a, c) where (a, b) and (b, c) are in R.
- The pairs of this form not in R are (1, 2), (2, 3), (2, 4), and (3, 1). Adding these pairs does *not* produce a transitive relation, because the resulting relation contains (3, 1) and (1, 4) but does not contain (3, 4). This shows that constructing the transitive closure of a relation is more complicated than constructing either the reflexive or symmetric closure.

Closure

- $(R^+)^\# = (R^\#)^+$
- $(R^*)^+ = (R^+)^*$
- $(R^*)^\# = (R^\#)^*$

R^+ Reflexive closure

$R^\#$ Symmetric closure

R^* Transitive Closure

Theorem: R^* is the transitive Closure of R

Proof: Part 1: R^* is transitive

Part 2: If $R \subseteq S$ and S is transitive

Then $R^* \subseteq S^* \subseteq S$

Directed Graph

- *A directed graph, or digraph, consists of a set V of vertices (or nodes) together with a set*
- *E of ordered pairs of elements of V called edges (or arcs). The vertex a is called the initial*
- *vertex of the edge (a, b), and the vertex b is called the terminal vertex of this edge.*

- a relation is reflexive if and only if there is a loop at every vertex of the directed graph, so that every ordered pair of the form *(x, x) occurs in the relation.*
- *A* relation is symmetric if and only if for every edge between distinct vertices in its digraph there is an edge in the opposite direction, so that *(y, x) is in the relation whenever (x, y) is* in the relation.
- *A* relation is symmetric if and only if for every edge between distinct vertices in its digraph there is an edge in the opposite direction, so that *(y, x) is in the relation whenever (x, y) is* in the relation.

Theorem: $(a,b) \in R^n$

if and only if

there is a path of length n

from a to b in R

Connectivity relation:

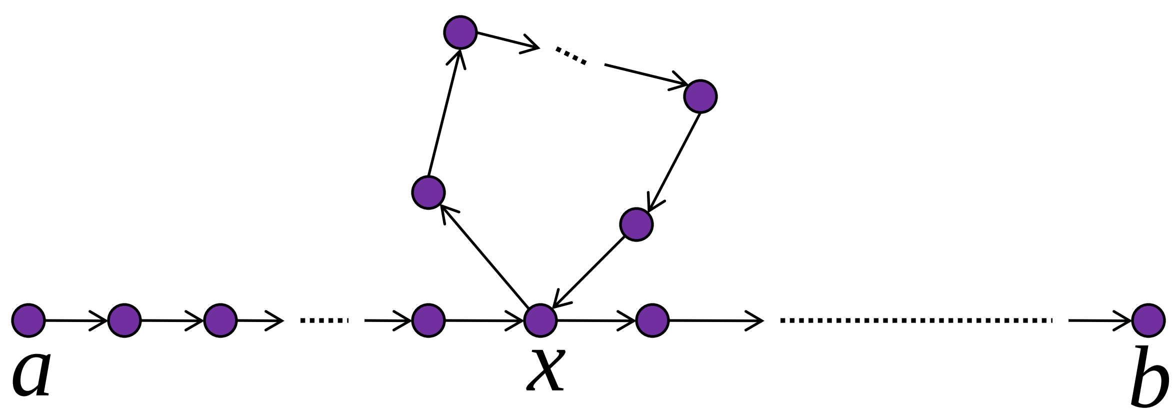
$$R^* = R^1 \cup R^2 \cup R^3 \cup \cdots = \bigcup_{i=1}^{\infty} R^i$$

$$(a,b) \in R^*$$

if and only if
there is some path (of any length)
from a to b in R

Theorem: $R^* = R^1 \cup R^2 \cup R^3 \cup \cdots \cup R^n$

Proof: if $(a,b) \in R^{n+1}$ then $(a,b) \in R^i$
for some $i \in \{1,\ldots,n\}$



Repeated node

Example

- *Let R be the relation on the set of all people in the world that contains (a, b) if a has met b. What is* R^n*, where n is a positive integer greater than one? What is* R^**?*
- Soluion:
  - The relation R^2 *contains (a, b) if there is a person c such that*

  (a, c) ∈ R and (c, b) ∈ R,
  - that is, if there is a person *c such that a has met c and c has met b.*
  - *Similarly,* R^n *consists of* those pairs *(a, b) such that there are people x1, x2, . . . , xn−1 such that a has met x1, x1* has met *x2, . . . , and xn−1 has met b.*
  - The relation R^* contains *(a, b) if there is a sequence of people, starting with a and ending with b, such that each person in the sequence has met the next person in the sequence.*

Theorem

- Let M_R ***be the zero–one matrix of the relation R on a set with n elements. Then the zero–one*** matrix of the transitive closure R^* is $\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]} \vee \cdots \vee \mathbf{M}_R^{[n]}.$

Example

- Find the zero-one matrix of the transitive closure of the relation *R where*

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

By Theorem 3, it follows that the zero-one matrix of $R*$ is

$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]}$$

Because

$$\mathbf{M}_R^{[2]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_R^{[3]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

Is follows that

$$\mathbf{M}_{R^*} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Algorithm for the transitive closure

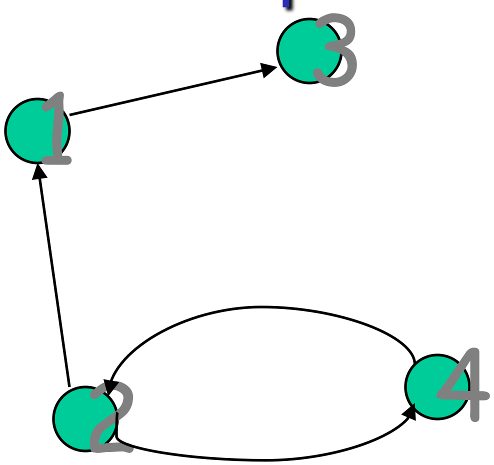
procedure *transitive closure* ($\mathbf{M}_R$: zero–one $n \times n$ matrix)
$\mathbf{A} := \mathbf{M}_R$
$\mathbf{B} := \mathbf{A}$
for $i := 2$ **to** n
    $\mathbf{A} := \mathbf{A} \odot \mathbf{M}_R$
    $\mathbf{B} := \mathbf{B} \vee \mathbf{A}$
return B{**B** is the zero–one matrix for R^*}

- *requires n–1.*
- Boolean products can be found using *bit operations*
- Hence, these products can be computed using $n^2(2n-1)(n-1)$ *bit operations.*
- To find $\mathbf{M}_{R^*}$ need
- *Therefore,*
- *bit operations needed*

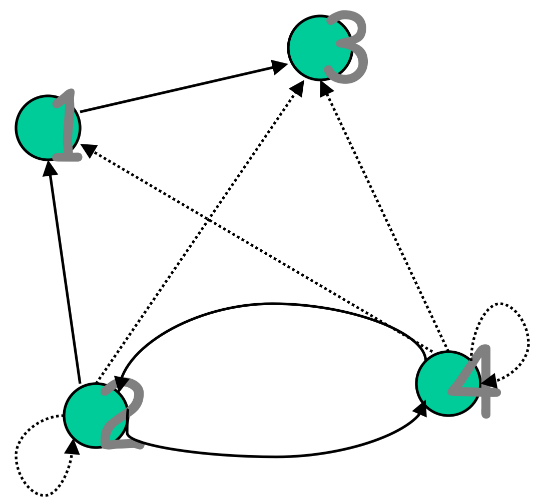
Warshall's algorithm: transitive closure

- Computes the transitive closure of a relation
- (Alternatively: all paths in a directed graph)

- Example of transitive closure:



$$\begin{matrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix}$$



$$\begin{matrix} 0 & 0 & 1 & 0 \\ 1 & \mathbf{1} & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \\ \mathbf{1} & 1 & \mathbf{1} & \mathbf{1} \end{matrix}$$

Warshall's algorithm

- Main idea: a path exists between two vertices i, j, iff there is an edge from i to j;

or

- there is a path from i to j going through intermediate vertices which are drawn from set {vertex 1}; or

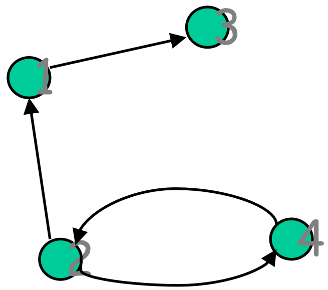
- there is a path from i to j going through intermediate vertices which are drawn from set {vertex 1, 2}; or

- …

Warshall's algorithm

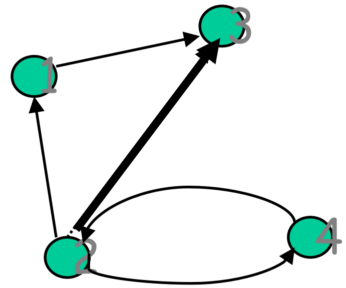
- Main idea: a path exists between two vertices i, j, iff
  - there is a path from i to j going through intermediate vertices which are drawn from set {vertex 1, 2, … k-1}; or
  - there is a path from i to j going through intermediate vertices which are drawn from set {vertex 1, 2, … k}; or
  - ...
  - there is a path from i to j going through any of the other vertices

Warshall's algorithm



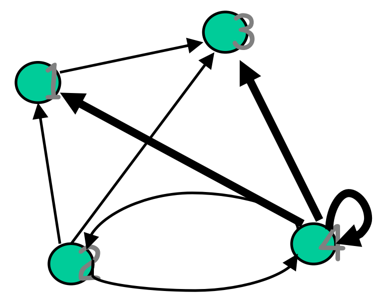
R^0

$$\begin{matrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix}$$



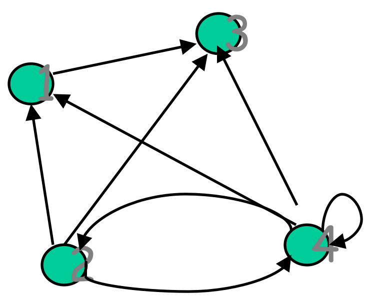
R^1

$$\begin{matrix} 0 & 0 & 1 & 0 \\ 1 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix}$$



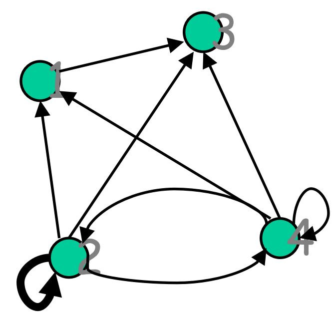
R^2

$$\begin{matrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ \mathbf{1} & 1 & \mathbf{1} & \mathbf{1} \end{matrix}$$



R^3

$$\begin{matrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{matrix}$$



R^4

$$\begin{matrix} 0 & 0 & 1 & 0 \\ 1 & \mathbf{1} & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{matrix}$$

Warshall's algorithm

$R^0 = A$

$$\begin{matrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix}$$

R^1

$$\begin{matrix} 0 & 0 & 1 & 0 \\ 1 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix}$$

R^2

$$\begin{matrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ \mathbf{1} & 1 & \mathbf{1} & \mathbf{1} \end{matrix}$$

R^3

$$\begin{matrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{matrix}$$

R^4

$$\begin{matrix} 0 & 0 & 1 & 0 \\ 1 & \mathbf{1} & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{matrix}$$

In-class exercises

- Apply Warshall's algorithm to find the transitive closure of the digraph defined by the following adjacency matrix

$$\begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{matrix}$$

Solution

M_R

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

M_R^2

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

M_R^3

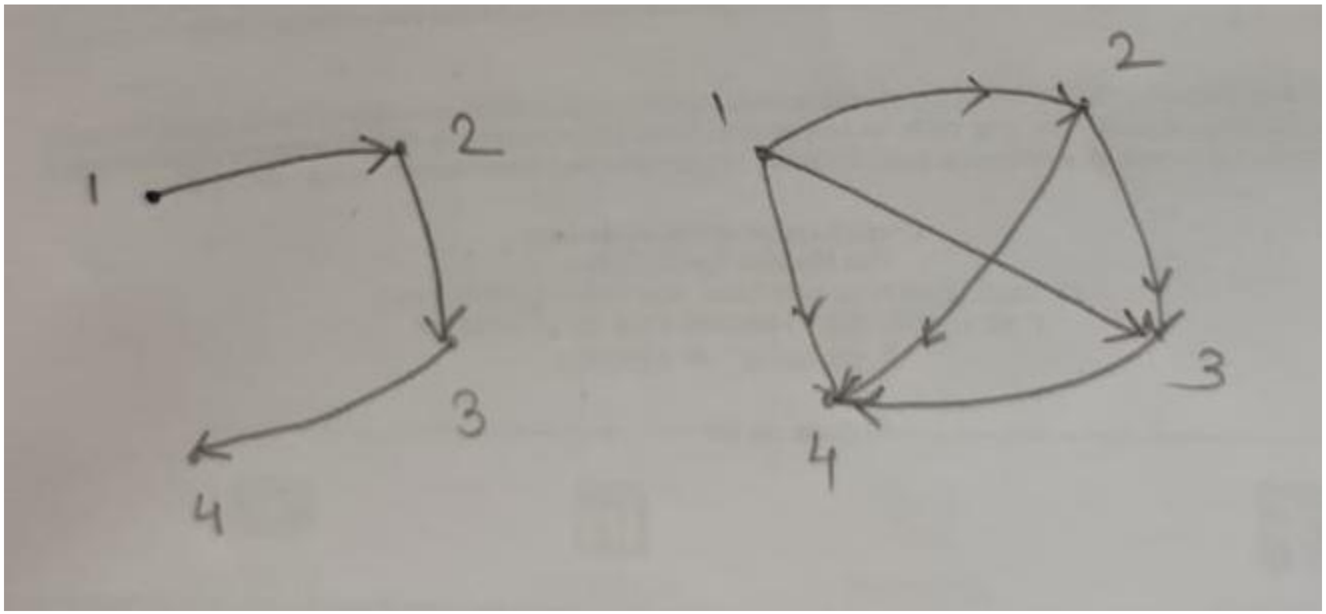
$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

M_R^4

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

M_R^5

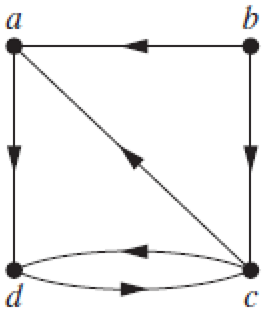
$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$



1
2
3
4
1
2
3
4

Let R be the relation with directed graph shown in Figure 3. Let a, b, c, d be a listing of the elements of the set. Find the matrices**W0,*W1*,*W2*,*W3*, *andW4*. *The matrixW4 is the transitive*** a b closure of R.

Let v1 = a, v2 = b, v3 = c, and v4 = d.**W0 is the matrix of the**



Hence,

$$\mathbf{W}_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

- ***W1 has 1 as its (i, j)th entry if there is a path from vi to vj that has only v1 = a as an interior*** vertex. Note that all paths of length one can still be used because they have no interior vertices.
- *Also*, there is now an allowable path from *b to d, namely, b, a, d. Hence,*

$$\mathbf{W}_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

- ***W2 has 1 as its (i, j)th entry if there is a path from vi to vj that has only v1 = a and/or v2 = b*** as its interior vertices, if any. Because there are no edges that have *b as a terminal vertex, no* new paths are obtained when we permit *b to be an interior vertex. Hence,* ***W2 = W1 .***
- ***W3 has 1 as its (i, j)th entry if there is a path from vi to vj that has only v1 = a, v2 = b,*** and/or *v3 = c as its interior vertices, if any. We now have paths from d to a, namely, d, c, a,* and from *d to d, namely, d, c, d. Hence,*

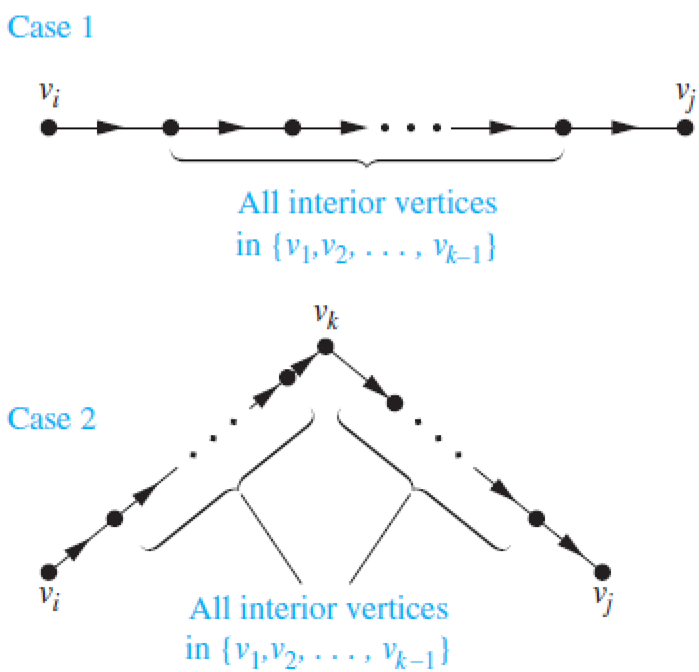
$$\mathbf{W}_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

- Finally, ***W4 has 1 as its (i, j)th entry if there is a path from vi to vj that has v1 = a, v2 = b,*** *v3 = c, and/or v4 = d as interior vertices, if any. Because these are all the vertices of the graph,* this entry is 1 if and only if there is a path from *vi to vj . Hence,*

$$\mathbf{W}_4 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

- This last matrix, **W4, is the matrix of the transitive closure**

- The first type of path exists if and only if *w[k–1]ij*= 1, and the second type of path exists ifand only if both *w[k–1]ik and w[k–1]kj are 1. Hence, w[k]ij is 1 if and only if either w[k–1]ij is 1 or both w[k–1]ik and w[k–1]kj are 1.*



Lemma

Let $\mathbf{W}_k = [w_{ij}^{[k]}]$ be the zero–one matrix that has a 1 in its (i, j)th position if and only if there is a path from v_i to v_j with interior vertices from the set $\{v_1, v_2, \ldots, v_k\}$. Then

$$w_{ij}^{[k]} = w_{ij}^{[k-1]} \vee (w_{ik}^{[k-1]} \wedge w_{kj}^{[k-1]}),$$

whenever i, j, and k are positive integers not exceeding n.

procedure *Warshall* ($\mathbf{M}_R$: $n \times n$ zero–one matrix)
$\mathbf{W} := \mathbf{M}_R$
for $k := 1$ **to** n
  for $i := 1$ **to** n
    for $j := 1$ **to** n
      $w_{ij} := w_{ij} \vee (w_{ik} \wedge w_{kj})$
return W$\{\mathbf{W} = [w_{ij}]$ is $\mathbf{M}_{R^*}\}$

- The computational complexity of Warshall's algorithm can easily be computed in terms of bit operations. To find the entry *w[k]ij from the entries w[k−1]ij ,w[k−1]ik , and w[k−1]kj using Lemma* requires two bit operations. To find all n^2 *entries of* ***Wk from those of Wk−1 requires*** $2n^2$ bit operations. Because '$n \cdot 2n^2 = 2n^3$ algorithm begins with **W0 =** ***MR and computes the sequence*** of *n zero–one matrices**W1 , W2, . . . , Wn = MR**∗* , the total number of bit operations used

Which of the following is the transitive closure?

$$R = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$a)\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$b)\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$c)\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Equivalence Relations

Equivalence relations are used to relate objects that are similar in some way.

Definition: A relation on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive.

Two elements that are related by an equivalence relation R are called **equivalent**.

Since R is **symmetric**, a is equivalent to b whenever b is equivalent to a.

Since R is **reflexive**, every element is equivalent to itself.

Since R is **transitive**, if a and b are equivalent and b and c are equivalent, then a and c are equivalent.

Obviously, these three properties are necessary for a reasonable definition of equivalence.

Example: Suppose that R is the relation on the set of strings that consist of English letters such that aRb if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x. Is R an equivalence relation?

Solution:

- R is reflexive, because $l(a) = l(a)$ and therefore aRa for any string a.
- R is symmetric, because if $l(a) = l(b)$ then $l(b) = l(a)$, so if aRb then bRa.
- R is transitive, because if $l(a) = l(b)$ and $l(b) = l(c)$, then $l(a) = l(c)$, so aRb and bRc implies aRc.

R is an equivalence relation.

Example

- *Let R be the relation on the set of real numbers such that aRb if and only if a – b is an integer. Is R an equivalence relation?*
- *Solution:*
  - Because $a - a = 0$ is an integer for all real numbers a, aRa for all real numbers a. Hence, R is reflexive.
  - Now suppose that aRb. Then $a - b$ is an integer, so $b - a$ is also an integer. Hence, bRa. It follows that R is symmetric.
  - If aRb and bRc, then $a - b$ and $b - c$ are integers.
  - Therefore, $a - c = (a - b) + (b - c)$ is also an integer. Hence, aRc. Thus, R is transitive. Consequently, R is an equivalence relation

Equivalence Classes

Definition: Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the **equivalence class** of a.

The equivalence class of a with respect to R is denoted by $\mathbf{[a]_R}$.

When only one relation is under consideration, we will delete the subscript R and write **[a]** for this equivalence class.

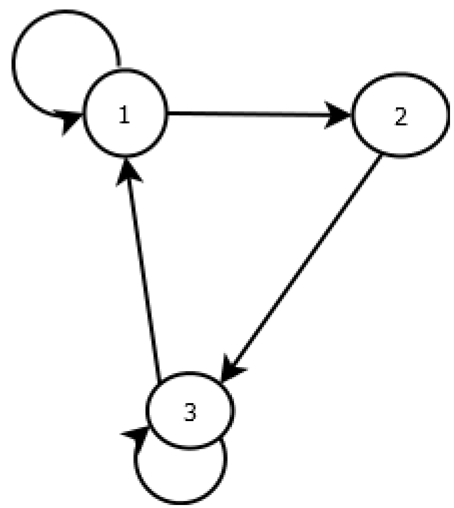
If $b \in [a]_R$, b is called a **representative** of this equivalence class.

Example: In the previous example (strings of identical length), what is the equivalence class of the word mouse, denoted by [mouse] ?

Solution: [mouse] is the set of all English words containing five letters.

For example, 'horse' would be a representative of this equivalence class.

Is the Diagram represents an Equivalence Relation?



Equivalence classes

Theorem: Let R be an equivalence relation on a set A. The following statements are equivalent:

- aRb
- [a] = [b]
- $[a] \cap [b] \neq \emptyset$

Equivalence classes

- **Example:**
- The relation $R=\{(a,b) \mid |a+1|=|b+1|\}$ is defined on the set of integers Z. Find the equivalence classes for R.
- Solution:
  - It's easy to make sure that R is an equivalence relation. The equivalence classes of R are defined by the expression $\{-1-n,-1+n\}$, where n is an integer.
  - Below are some examples of the classes E_n for specific values of n and the corresponding pairs of the relation R for each of the classes:

Equivalence classes

n=0:E0=[−1]={−1},R0={(−1,−1)}
n=1:E1=[−2]={−2,0},R1={(−2,−2),(−2,0),(0,−2),(0,0)}
n=2:E2=[−3]={−3,1},R2={(−3,−3),(−3,1),(1,−3),(1,1)}
n=−2:E−2=[1]={1,−3},R−2={(1,1),(1,−3),(−3,1),(−3,−3)}
n=10:E10=[−11]={−11,9},R10={(−11,−11),(−11,9),(9,−11),(9, 9)}
n=−10:E−10=[−11]={9,−11},R−10={(9,9),(9,−11),(−11,9),(−1 1,−11)} As it can be seen, E2=E−2, E10=E−10. It follows from here that we can list all equivalence classes for R by using non-negative integers n.

Partition

Definition: *A* **partition** of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets A_i, $i \in I$, forms a partition of S if and only if

(i) $A_i \neq \emptyset$ for $i \in I$

- $A_i \cap A_j = \emptyset$, if $i \neq j$
- $\cup_{i \in I} A_i = S$

Examples: Let S be the set {u, m, b, r, o, c, k, s}. Do the following collections of sets partition S ?

{{m, o, c, k}, {r, u, b, s}}	yes.
{{c, o, m, b}, {u, s}, {r}}	no (k is missing).
{{b, r, o, c, k}, {m, u, s, t}}	no (t is not in S).
{{u, m, b, r, o, c, k, s}}	yes.
{{b, o, o, k}, {r, u, m}, {c, s}}	yes ({b,o,o,k} = {b,o,k}).
{{u, m, b}, {r, o, c, k, s}, ∅}	no (∅ not allowed).

- **Congruence Modulo** *m* Let *m* be an integer with *m* > 1. Show that the relation $R = \{(a, b) \mid a \equiv b \pmod{m}\}$ is an equivalence relation on the set of integers
  - $a \equiv b \pmod{m}$ if and only if *m* divides $a - b$. Note that $a - a = 0$ is divisible by *m*, because $0 = 0 \cdot m$. Hence, $a \equiv a \pmod{m}$, so congruence modulo *m* is **reflexive**.
  - suppose that $a \equiv b$ (mod m). Then $a - b$ is divisible by m, so $a - b = km$, where k is an integer. It follows that $b - a = (-k)m$, so $b \equiv a$ (mod m). Hence, congruence modulo m is symmetric.
  - Next, suppose that $a \equiv b$ (mod m) and $b \equiv c$ (mod m).
  - Then m divides both $a - b$ and $b - c$. Therefore, there are integers k and l with $a - b = km$ and $b - c = lm$.
  - Adding these two equations shows that

$a - c = (a - b) + (b - c) = km + lm = (k + l)m$. Thus, $a \equiv c$ (mod m). Therefore, congruence modulo m is transitive.

Equivalence Class

- What are the equivalence classes of 0 and 1 for congruence modulo 4?
  - equivalence class of 0 contains all integers a such that $a \equiv 0 \pmod 4$. The integers in this class are those divisible by 4. Hence, the equivalence class of 0 for this relation is

$[0] = \{. . . ,-8,-4, 0, 4, 8, . . . \}$.

  - The equivalence class of 1 contains all the integers a such that $a \equiv 1 \pmod 4$. The integers in this class are those that have a remainder of 1 when divided by 4. Hence, the equivalence class of 1 for this relation is

$[1] = \{. . . ,-7,-3, 1, 5, 9, . . . \}$.

- let $n = 3$ and let S be the set of all bit strings. Then $sR3\ t$ either when $s = t$ or both s and t are bit strings of length 3 or more that begin with the same three bits. For instance, $01\ R_3\ 01$ and $00111\ R_3\ 00101$, but $01\ \cancel{R}_3\ 010$ and $01011\ \cancel{R}_3\ 01110$. Show that for every set S of strings and every positive integer n, Rn is an equivalence relation on S.
  - The relation Rn is reflexive because s = s, so that sRn s whenever s is a string in S.
  - If s Rn t , then either s = t or s and t are both at least n characters long that begin with the same n characters.

t RnS ∴ Rn is symmetric.

- suppose that s Rn t and t Rn u.
- Then either s = t or s and t are at least n characters long and s and t begin with the same n characters,
- And either t = u or t and u are at least n characters long and t and u begin with the same n characters.
- From this, either s = u or both s and u are n characters long and s and u begin with the same n characters
- Consequently, Rn is transitive.
- It follows that Rn is an equivalence relation.

- What is the equivalence class of the string 0111 with respect to the equivalence relation $R3$ from on the set of all bit strings? (Recall that $sR3\ t$ if and only if s and t are bit strings with $s = t$ or s and t are strings of at least three bits that start with the same three bits.)
  - The bit strings equivalent to 0111 are the bit strings with at least three bits that begin with 011. These are the bit strings 011, 0110, 0111, 01100, 01101, 01110, 01111, and so on.
  - Consequently, $[011]_{R_3} = \{011, 0110, 0111, 01100, 01101, 01110, 01111, \ldots\}$.