1) Linear mapping , 2) Kernel , 3) Image , 4) Rank Nullity Theorem.

* Tutorial 04 *

Q.1. Let $T: V \to W$ be a linear transformation. Show that :

(a) $T(0)=0$

(b) $T(-v) = -T(v)$ for all $v \in V$.

→ Soln :-

(a) Let $u, v \in V$

$\therefore u, v = 0$

$\therefore T(u+v) = T(v) + T(v)$ .... (linear transformation)

$\therefore T(0+0) = T(0) + T(0)$

$= T(0) = 2T(0)$

$\therefore \underline{T(0) = 0}$

(b) Let $v \in V$ & $k \in \mathbb{R}$,

$T(kv) = kT(v)$ .... (linear transf)

Let, $k = -1$,

$T(-1v) = -1\,T(v)$

$\therefore T(-v) = -T(v)$

∴ Hence, proved.

Q.2. Determine which of the following mappings F are linear. If linear, then find its kernel and image space. Also, find Nullity and rank and hence verify rank nullity theorem.

(a) $F: \mathbb{R}^3 \to \mathbb{R}^2$ defined by $F(x, y, z) = (x, z)$.

→ Soln :-

Let $u, v \in \mathbb{R}^3$, $\Rightarrow u = (x_1, y_1, z_1)$ & $F(u) = (x_1, z_1)$

$v = (x_2, y_2, z_2)$ $F(v) = (x_2, z_2)$

~~$\therefore F(u+v) = T(x_1, y_1, z_1$~~

~~$\therefore T(u+v) = T(u) + T(v)$~~

~~$= (x_1, z_1) + (x_2, z_2$~~

$\therefore F(u+v) = F(u) + F(v)$ .... To prove

$\therefore F(u+v) = F(x_1 + x_2, z_1 + z_2)$

$= F(x_1, z_1) + F(x_2, z_2)$

$\therefore \underline{F(u+v) = F(u) + F(v)}$

Now, let, $k \in \mathbb{R}$,

$\Rightarrow F(kv) = F[k(x_1, z_1)]$

$= k\,F(x_1, z_2)$

$= k\,F(v)$

$\therefore \underline{F(kv) = kF(v)}$.

Hence, $F: \mathbb{R}^3 \to \mathbb{R}^2$ is a linear ~~function~~ Transformation.

$\therefore$ * $N(T) = \{(x, y, z) \in \mathbb{R}^3 \mid F(x, y, z) = (0, 0)\}$

$\therefore F(x, y, z) = F(x, z) = (0, 0)$

$\therefore x = z = 0$ & $y = \in \mathbb{R}$, let $y = t$

$\therefore N(T) \in$ y-axis.

(a) $F : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $F(x, y, z) = (x, z)$

→ Let, $u, v \in \mathbb{R}^3$, $u = (x_1, y_1, z_1) = \cancel{(x_1, z_1)}$

$\cancel{u(x} \Rightarrow f(u) = (x_1, z_1)$

$v = (x_2, y_2, z_2)$ $\therefore f(v) = (x_2, z_2)$

$F(u+v) = F(x_1 + x_2, y_1 + y_2, z_1 + z_2)$ ... (R.T.)

$= (x_1 + x_2, z_1 + z_2)$ ... (L.T.)

$= (x_1, z_1) + (x_2, z_2)$

$\therefore F(u+v) = F(u) + F(v)$

$Fk(u) = F[kx_1, ky_1, kz_1)$

$= kx_1, k(0), kz_1$

$= kx_1, kz_1$

$= k(x_1, z_1)$

$\therefore Fk(u) = k\,F(u)$

$\therefore$ F is a linear map

$\ker(F) = \{ v \in \mathbb{R}^3 \mid F(v) = 0 \}$

$F(x, y, z) = (0, 0)$

$\therefore (x, z) = (0, 0)$

$\therefore x = z = 0$, $y = t$ ... $t \in \mathbb{R}$,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix} = t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$\therefore \ker F = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} t \;\middle|\; t \in \mathbb{R} \right\}$

$\therefore \dim \ker(F) = 1$.

Now,

$Im(F) = \{ u \in \mathbb{R}^2 \mid \exists\, v \in \mathbb{R}^3$ for which $F(v) = u \}$

Consider, Basis for $\mathbb{R}^3$,

$\mathbb{B} = \{ (0,1,0), (0,0,1), (1,0,0) \}$

$F(0,1,0) = (0,0)$

$F(0,0,1) = (0,1)$

$F(1,0,0) = (1,0)$

Here, $(0,0)$ is L.D.

$\therefore$ Basis for $Im(F) = \{ (1,0), (0,1) \}$

$= \{ t(1,0) + s(0,1) \mid t, s \in \mathbb{R} \}$

$\dim(Im\,F) = 2$.

$\therefore \dim V = \dim \ker(F) + \dim Im(F)$ ... (R.N.T.)

$\therefore 3 = 1 + 2$

$\therefore \underline{\underline{3 = 3}}$

Hence, RNT is proved for $F : \mathbb{R}^3 \to \mathbb{R}^2$.

(b) $F : \mathbb{R}^4 \to \mathbb{R}^4$ defined by $F(x, y, z, w) = (-x, -y, -z, -w)$

→ Soln,

Let, u & $v \in \mathbb{R}^4$, $\Rightarrow u = (x_1, y_1, z_1, w_1)$

$F(u) = (-x_1, -y_1, -z_1, -w_1)$

$v = (x_2, y_2, z_2, w_2)$

$F(v) = (-x_2, -y_2, -z_2, -w_2)$

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$\therefore F(u+v) = F(x_1+x_2,\ y_1+y_2,\ z_1+z_2,\ w_1+w_2)$

$= (-x_1-x_2,\ -y_1-y_2,\ -z_1-z_2,\ -w_1-w_2)$

$= (-x_1, -y_1, -z_1, -w_1) + (-x_2, -y_2, -z_2, -w_2)$

$\therefore F(u+v) = F(u) + F(v)$

Now,

$F(ku) = F[k(x_1, y_1, z_1, w_1)]$

$= k(-x_1, -y_1, -z_1, -w_1)$

$\therefore F(ku) = k[F(u)]$

~~Now~~ Hence, F is a linear map

Now, $\ker(F) = \{ v \in \mathbb{R}^4 \mid F(v) = 0 \}$

$\Rightarrow F(x, y, z, w) = (0, 0, 0, 0)$

$\therefore -x, -y, -z, -w = (0, 0, 0, 0)$

$\therefore x = y = z = w = 0$

Hence, $\ker(F) = 0$

$\therefore \dim \ker(F) = 0$

$\Rightarrow Im(F)$ is a basis

Now, $Im(F) = \{ u \in \mathbb{R}^4 \mid \exists v \in \mathbb{R}^4$ for which $F(v) = u \}$

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Here,

$F(0,0,0,1) = (0,0,0,-1)$

$F(0,0,1,0) = (0,0,-1,0)$

$F(0,1,0,0) = (0,-1,0,0)$

$F(1,0,0,0) = (-1,0,0,0)$

~~$\Rightarrow (-x, -y, -z, -w) = (0,0,0,0)$~~

$\therefore$ Basis for $Im(F) = \{ (0,0,0,1), (0,0,1,0), (0,1,0,0), (1,0,0,0) \}$

$\therefore$ ~~dim Im~~

$\therefore Im\,F = \{ t(0,0,0,1) + s(0,0,1,0) + r(0,1,0,0) + q(1,0,0,0) \mid t, s, r, q \in \mathbb{R} \}$

$\therefore \dim Im(F) = 4.$

$\therefore \dim V = \dim k(F) + \dim Im(F)$ ... (R.N.T.)

$\therefore 4 = 0 + 4$

$\therefore \underline{\underline{4 = 4}}$

$\therefore$ Hence verified.

(c) $F : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $F(x, y, z) = (x, y, z) + (0, 1, 0)$

$\to$ Soln:-

Let, $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation

$\Rightarrow F(0) = 0$ $\quad$ $F(x, y, z) = (x, y, z) + (0, 1, 0)$

$\therefore F(0) = \cancel{F(0+0+0)}$ $\quad = (x, y+1, z)$

$=$

$\therefore F(0) = F(0, 0, 0)$ $\quad \ldots$ (L.T.P.)

$= 0, 0+1, 0$

$F(0) \neq (0, 1, 0)$

Hence, F is not a linear map

(d) $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x, y) = (x - y, 2y)$

$\to$ Soln:-

Let, $u, v \in \mathbb{R}^2$. $\Rightarrow u = (x_1, y_1) \Rightarrow F(u) = (x_1 - y_1, 2y_1)$

$v = (x_2, y_2) \Rightarrow F(v) = (x_2 - y_2, 2y_2)$

$\therefore F(u+v) = F(x_1 + x_2, y_1 + y_2)$

$\cancel{= (x_1 - y_1}$

$= F(x_1 + x_2, y_1 + y_2)$

$= (x_1 - y_1 + x_2 - y_2, 2y_1 + 2y_2)$

$= (x_1 - y_1, 2y_1) + (x_2 - y_2, 2y_2)$

$\therefore F(u+v) = F(u) + F(v)$

Now, $k \in \mathbb{R}$,

$F(ku) = F[k(x_1, y_1)]$

$= k[x_1 - y_1, 2y_1]$

$\therefore F(ku) = k[F(u)]$

Hence, F is linear map.

$\ker(F) = \{v \in \mathbb{R}^2 \mid F(v) = 0\}$

$F(x, y) = (0, 0)$

$(x - y, 2y) = (0, 0)$

$\Rightarrow x - y = 0$

$\Rightarrow 2y = 0 \Rightarrow y = 0$

$\Rightarrow x - 0 = 0 \Rightarrow x = 0$

Hence, $\ker(F) = 0$

$\therefore \dim \ker(F) = 0$.

$\Rightarrow \operatorname{Im}(F)$ is a basis $\quad \ldots$ (Theorem).

Now,

$\operatorname{Im}(F) = \{u \in \mathbb{R}^2 \mid \exists v \in \mathbb{R}^2 \text{ for which } F(v) = u\}$

$\therefore F(x, y) = (x - y, 2y)$

$\therefore$ Basis for $\mathbb{R}^2$,

$\mathbb{B} = \{(1, 0), (0, 1)\}$

$\therefore F(1, 0) = \cancel{x - y \Rightarrow x = y}$

$\therefore F(0, 1) = \cancel{2y}$

Let, $y = t$, $x =$

(e) $F: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x, y) = (xy, x+y)$

→ Soln:-

Let $u, v \in \mathbb{R}^2$, $u = (x_1, y_1) \Rightarrow F(u) = (x_1y_1, x_1+y_1)$
$v = (x_2, y_2) \Rightarrow F(v) = (x_2y_2, x_2+y_2)$

$F(u+v) = F(x_1+x_2, y_1+y_2)$
$= (x_1y_1 + x_2y_2, x_1+y_1+x_2+y_2)$ ... (L.T.)
$= (x_1y_1, x_1+y_1) + (x_2y_2, x_2+y_2)$
$\therefore F(u+v) = F(u) + F(v)$

Now, $F(ku) = \cancel{F[k(x_1+x_2} \; \cancel{F[k(x_1 y_1, x_1+y_1)]}$
$= F[k(x_1, y_1)]$
$= k(x_1y_1, x_1+y_1)$ ... (L.T.)
$\therefore F(ku) = k[F(u)]$

Hence, F is a linear map

Now,

$N(F) = \{v \in \mathbb{R}^2 \mid F(v) = 0\}$

$\Rightarrow F(u) = \Rightarrow F(x, y) = (0, 0)$
$(xy, x+y) = (0, 0)$

$\Rightarrow xy = 0 \Rightarrow$ Any one or ~~either~~ both x & y are zero.
$\Rightarrow x + y = 0 \Rightarrow x = -y$
$\Rightarrow$ x and y both are zero.

$\Rightarrow \cancel{x}\; N(F) = 0$
$\Rightarrow \dim N(F) = 0$.

(f) $F: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x, y) = (y, x)$

→ Soln:-

Let, $u, v \in \mathbb{R}$, $u = (x_1, y_1) \Rightarrow F(u) = (y_1, x_1)$
$v = (x_2, y_2) \Rightarrow F(v) = (y_2, x_2)$

$F(u+v) = \cancel{F(y_1, x_1} = F(x_1+x_2, y_1+y_2)$
$= (y_1+y_2, x_1+x_2)$
$= (y_1, x_1) + (y_2, x_2)$
$\therefore F(u+v) = F(u) + F(v)$

$F(ku) = F[k(x_1, y_1)]$

$= k(y_1, x_1)$
$\therefore F(ku) = k[F(u)]$

Hence, F is a linear map.

Now,

$N(F) = \{ u \in \mathbb{R}^2 \mid F(u) = 0 \}$

$\Rightarrow F(x,y) = (0,0)$
$(y, x) = (0,0)$

Hence, $N(F) = 0$
$\Rightarrow \dim N(F) = 0$

Now, $Im(F) = \{ u \in \mathbb{R}^2 \mid \exists\, v \in \mathbb{R}^2$ for which $F(v) = u \}$

$\therefore F(x,y) = (y, x)$

$\therefore$ Basis for $\mathbb{R}^2$,
$B = \{(1,0), (0,1)\}$

$\therefore F(1,0) = (1,0)$
$F(0,1) = (0,1)$

$\therefore$ Basis of $Im(F) = \{(1,0), (0,1)\}$

$= \{ t(1,0) + s(0,1) \}$

$\therefore \dim Im(F) = 2$

$\therefore \dim V = \dim N(F) + \dim Im(F)$
$\therefore 2 = 0 + 2$
$\therefore 2 = 2$

(g) $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x, y) = xy$

$\rightarrow$ Soln:-

Let u & $v \in \mathbb{R}^2$, $u = (x_1, y_1) \Rightarrow F(u) = x_1 y_1$
$v = (x_2, y_2) \Rightarrow F(v) = x_2 y_2$

$F(u+v) = F[(x_1, y_1) + (x_2, y_2)]$

$= F(x_1 + x_2, y_1 + y_2)$
$= (x_1 y_1, x_2 y_2)$

$F(u) + F(v) = F(x_1, y_1) + F(x_2, y_2)$
$= x_1 y_1 + x_2 y_2$
$= (x_1 + x_2) + (y_1 + y_2)$

$\Rightarrow F(u+v) \neq F(u) + F(v)$
$\Rightarrow$ This is not a linear map.

(h) $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x,y) = (x, y+1)$

$\rightarrow$ Soln:-

Let $u, v \in \mathbb{R}^2$, $u = (x_1, y_1) \Rightarrow F(u) = (x_1, y_1 + 1)$
$v = (x_2, y_2) \Rightarrow F(v) = (x_2, y_2 + 1)$

$\Rightarrow$ Let, F be a linear map

$\therefore F(0) = F(0+0)$
$= \cancel{F(0) + F(0)}$
$= (0, 0+1)$
$\therefore F(0) = (0, 1)$
$\therefore F(0) \neq 0$

$\Rightarrow F$ is not a linear map.

(f) $F: \mathbb{R}^3 \to \mathbb{R}$ defined by $F(x,y,z) = 3x - 2y + z$

$\to$ Sol^n:- Let, $u, v \in \mathbb{R}^3 \Rightarrow u = (x_1, y_1, z_1) \Rightarrow F(u) = 3x_1 - 2y_1 + z$

$v = (x_2, y_2, z_2) \Rightarrow F(v) = 3x_2 - 2y_2 + z$

$F(u+v) = F(x_1, y_1, z_1 + x_2, y_2, z_2)$

$= F(x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$= 3(x_1 + x_2) - 2(y_1 + y_2) + z_1 + z_2$

$F(u) + F(v) = F(x_1, y_1, z_1) + F(x_2, y_2, z_2)$

$= (3x_1 - 2y_1 + z_1) + (3x_2 - 2y_2 + z_2)$

$= 3(x_1 + x_2) - 2(y_1 + y_2) + z_1 + z_2$

$\therefore F(u+v) = F(u) + F(v)$.

Now,

$F(k(u)) = F[k(x_1, y_1, z_1)]$

$= \cancel{F[k(3x_1, 2y_1, z_1)]}$

$\cancel{= k(}$

$= F(kx_1, ky_1, kz_1)$

$= 3kx_1 - 2ky_1 + z_1$

$= k(3x_1 - 2y_1 + z_1)$

$\therefore F(ku) = k[F(u)]$

Hence, F is linear map

Now,

$N(F) = \{v \in \mathbb{R}^3 \mid F(v) = 0\}$

$\therefore F(v) = 0$

$\therefore 3x - 2y + z = 0$

Let, $z = t$ & $y = s$

$\therefore x = \dfrac{\cancel{2y+}\ 2y - z}{3} = \dfrac{2(s) - t}{3}$

~~Now~~ Now,

$\cancel{N(T) = \left\{\begin{matrix} x \\ y \\ z \end{matrix}\right\}}$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2s - t/3 \\ s \\ t \end{pmatrix}$

$= t\begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} 2/3 \\ 1 \\ 0 \end{pmatrix}$

$\therefore N(T) = \left\{ t\begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} 2/3 \\ 1 \\ 0 \end{pmatrix} \middle| t, s \in \mathbb{R} \right\}$

$\therefore \dim N(T) = 2$

Now, $Im\ F = \{u \in \mathbb{R} \mid \exists\ v \in \mathbb{R}^3$ for which $F(v) = u\}$

$\cancel{\therefore F(x,y,z) = 3x - 2y}$

Consider, basis for $\mathbb{R}^3$,

$F(0,0,1) = \cancel{0}\ 1$

$F(0,1,0) = \cancel{0}\ -2$

$F(1,0,0) = \cancel{3x-2y+z}\ 3$

$\Rightarrow 3, 1, -2 \in \mathbb{R}$

$Im(F) = \{t \mid t \in \mathbb{R}\}$

$\therefore \dim Im(F) = 1$

$\dim V = \dim N(P) + \dim Im(P)$ ... (R.N.T)

$\therefore 3 = 2+1$

$\therefore 3 = 3$

Hence, theorem is verified

(d) Let M be a space of all 2x2 matrices, Let $P: M \to M$ be a map such that $P(A) = \frac{A+A^t}{2}$. Generalise to nxn matrices.

→ Soln:-

Let, $A, B \in M_{2\times2}$ $\Rightarrow P(A) = \frac{A+A^t}{2}$

$\Rightarrow P(B) = \frac{B+B^t}{2}$

Now,

$$P(A+B) = \frac{(A+B)+(A+B)^t}{2}$$

$$= \frac{A+B+A^t+B^t}{2}$$

$$= \frac{A+A^t}{2} + \frac{B+B^t}{2}$$

$\Rightarrow P(A+B) = P(A) + P(B)$

Similarly,

$$P(kA) = \frac{kA+(kA)^t}{2} = \frac{kA+kA^t}{2}$$

$$= \frac{k(A+A)^t}{2} = k\,P(A)$$

$\Rightarrow P$ is linear map.

Now,

$\ker P = \{A \in M_{2\times2} \mid P(A) = 0\}$

$\Rightarrow \frac{A+A^t}{2} = 0$ $\Rightarrow A+A^t = 0$ $\Rightarrow A = -A^t$

$\Rightarrow \ker P$ is a skew symmetric 2x2 matric

$\therefore \ker P = \{$Set of all 2x2 skew symmetric matrix$\}$

Now, $\dim \ker P = \frac{n(n-1)}{2} = \frac{2(2-1)}{2} = 1$

$\boxed{\therefore \dim \ker P = 1}$

Now,

$Im P = \{B \in M_{2\times2} \mid \exists A \in M_{2\times2}$ for which $P(A) = B\}$

Consider basis for $M_{2\times2}$

$$\text{Basis} = \left\{\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$$

$P(A) = \frac{A+A^t}{2}$

$$P\begin{bmatrix}1&0\\0&0\end{bmatrix} = \frac{\begin{bmatrix}1&0\\0&0\end{bmatrix}+\begin{bmatrix}1&0\\0&0\end{bmatrix}}{2}$$

$$\therefore P\begin{bmatrix}1&0\\0&0\end{bmatrix} = \begin{bmatrix}1&0\\0&0\end{bmatrix}$$

Now,

$$P\begin{bmatrix}0&1\\0&0\end{bmatrix} = \frac{\begin{bmatrix}0&1\\0&0\end{bmatrix}+\begin{bmatrix}0&0\\1&0\end{bmatrix}}{2} = \begin{bmatrix}0&1/2\\1/2&0\end{bmatrix}$$

$\therefore$ ~~dim~~ ker P = { set of all 2×2 symmetric matrices }

$$\dim \ker(P) = \frac{n(n+1)}{2} = \frac{\cancel{2}(2+1)}{\cancel{2}} = 3$$

$$\therefore \dim[\ker(P)] = 3$$

Now,

Basis for Im(P) = $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$\mathrm{Im}(P) = \{ B \in M_{2\times 2} \mid \exists\, A \in M_{2\times 2}$ for which $P(A) = B \}$

$$\therefore P\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \frac{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \mp \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}{2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$P\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \frac{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}{2} = \begin{bmatrix} 0 & 1/2 \\ -1/2 & 0 \end{bmatrix}$$

$$P\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \frac{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}{2} = \begin{bmatrix} 0 & -1/2 \\ 1/2 & 0 \end{bmatrix}$$

$$P\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \frac{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}{2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Here, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1/2 \\ -1/2 & 0 \end{bmatrix}$ are L.D.

Removing L.D,

$$\mathrm{Im}(P) = \left\{ \begin{bmatrix} 0 & -1/2 \\ +1/2 & 0 \end{bmatrix} \right\}$$

$$\therefore \dim[\mathrm{Im}(P)] = 1$$

Here, $N(P) \neq 0$,

Hence, P is not 1-1 map

Also, ~~Im(P) ≠~~ dim Im(P) = 1 & dim (A) = 4

Hence, P is not an onto map

(n) Let M be the space of all 2×2 matrices. Let, $P: M \to \mathbb{R}$ be a map such that $P(A) = \mathrm{trace}(A)$. Generalise to n×n matrices.

→ Sol^n:-

$P: M \to \mathbb{R}$ & $P: M \to \mathbb{R}$ & $P(A) = \mathrm{trace}(A)$

Let, $M = \left\{ A \in \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, d, c, d \in \mathbb{R} \right\}$

$$P\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a + d$$

Let, $U, V \in M_{2\times 2}$ $\Rightarrow$ $U = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \Rightarrow P(U) = a_1 + d_1$

Now, $V = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = P(V) = a_2 + d_2$

$$P(U+V) = P\begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$$

$$= (a_1 + a_2) + (d_1 + d_2)$$

$$= (a_1 + d_1) + (a_2 + d_2)$$

$$\Rightarrow P(U+V) = P(U) + P(V)$$

Now,

$P(kU) = P\left[k\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right]$

$= P\begin{bmatrix} ka_1 & kb_1 \\ kc_1 & kd_1 \end{bmatrix}$

$= ka_1 + kd_1$

$= k(a_1 + d_1)$

$\therefore P(kU) = k[P(U)]$

Hence, P is linear map.

Now,

$\ker(P) = \{ \cancel{\forall} A \in M_{2\times2} \mid P(A) = 0 \}$

$\Rightarrow P(A) = 0$

$\cancel{P}(a+d) = 0$

$a = -d$

let, $d = t \Rightarrow a = -t$

here, let, $b = s$, $c_1 = q$

$\therefore \ker P = \left\{ t\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} + s\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + q\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \middle| t, s, q \in \mathbb{R} \right\}$

$\Rightarrow \dim \ker(P) = 3$

Now,

$Im(P) = \{ u \in \mathbb{R} \mid \exists A \in M_{2\times2} \text{ s.t. } P(A) = u \}$

$\Rightarrow P(A) =$

let,

~~Basis for Im P~~

Basis for $M_{2\times2} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow P\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = a$

$P\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0$

$P\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = 0$

$P\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = d$

Hence, $a, d \in \mathbb{R}$

Hence $Im(P) = \{ \text{Trace}(a) \mid \exists A \in M_{2\times2} \}$

$= \left\{ a+d \mid \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times2} \right\}$

$\therefore \dim Im(P) = 1$

$\therefore$ ~~Rank~~ $\dim Im(P) = 1$

Q.3. Using kernel classi.

Q.4. What is the dimension of the space of solutions of the following systems of linear equations? In each case, find basis for the space of solutions.

(a) $2x + y - z = 0$ — ①

$2x + y + z = 0$ — ②

→ Solⁿ:-

$A = \begin{bmatrix} 2 & 1 & -1 \\ 2 & 1 & 1 \end{bmatrix}$ & $B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$[A\,B] = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 2 & 1 & 1 & 0 \end{bmatrix}$

$R_2 - R_1$

$\begin{bmatrix} 2 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$.

Here,

$2z = 0 \Rightarrow z = 0$

$2x + y - z = 0$

$2z = 0$

Free variables $= 3 - 2 = 1$

Let, $y = t$ . . $t \in \mathbb{R}$,

$\therefore 2x = -t \quad \therefore x = \frac{-t}{2}$

$\therefore \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -t/2 \\ t \\ 0 \end{pmatrix} = t \begin{pmatrix} -1/2 \\ 1 \\ 0 \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}$

$\therefore$ The solution set $\{(-1/2, 1, 0)\}$

dim solution set $= 1$

Basis $= \{-1/2, 1, 0\}$

(b) $x + y + z = 0$ ....①

$x - y = 0$ ....②

$y + z = 0$ ....③

→ Solⁿ :-

$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ & $B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$[A\,B] = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

$R_2 - R_1$

$2R_3$ $= \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 2 & 2 & 0 \end{bmatrix}$.

$2R_3 + R_2$

$= \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$r(A) = r[AB]$

$= n$

$\therefore$ ~~Infinite~~ unique solⁿ

Here, $z = 0$, $y = 0$, $x = 0$

$\therefore x = 0,\ y = 0 \ \& \ z = 0$

Solution set is $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

dim (Solution set) $= 0$

(c) $2x - 3y + z = 0$ —— ①
$x + y - z = 0$ —— ②
$3x + 4y = 0$ —— ③
$5x + y + z = 0$ —— ④

$\rightarrow$ Soln :-

$$[AB] = \begin{bmatrix} 2 & -3 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 3 & 4 & 0 & 0 \\ 5 & 1 & 1 & 0 \end{bmatrix}$$

$R_2 \rightarrow R_1$

$$= \begin{bmatrix} 1 & 1 & -1 & 0 \\ 2 & -3 & 1 & 0 \\ 3 & 4 & 0 & 0 \\ 5 & 1 & 1 & 0 \end{bmatrix}$$

$R_2 - 2R_1$
$R_3 - 3R_1$
$R_4 - 5R_1$

$$= \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -5 & 3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & -4 & 6 & 0 \end{bmatrix}$$

$R_3 \leftrightarrow R_2$

$$= \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & -5 & 3 & 0 \\ 0 & -4 & 6 & 0 \end{bmatrix}$$

$R_3 + 5R_2 =$
$R_4 + 4R_2$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 18 & 0 \\ 0 & 0 & 18 & 0 \end{bmatrix}$$

$r(AB) = r(A)$
$= 3$
$\Rightarrow$ Unique soln

$\Rightarrow$ ~~z = 0~~ $R_4 - R_3 =$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 18 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x + y - z = 0$
$-y + 3z = 0$
$18z = 0$ $\Rightarrow z = 0,\ y = 0,\ x = 0.$

Solution set is $\{(0,0,0)\}$

dim (solution set is) $= 0$

(d) $4x + 7y - \pi z = 0$ —— ①
$2x - y + z = 0$ —— ②

$\rightarrow$

$$[AB] = \begin{bmatrix} 4 & 7 & -\pi & 0 \\ 2 & -1 & 1 & 0 \end{bmatrix}$$

$$2R_2 - R_1 = \begin{bmatrix} 4 & 7 & -\pi & 0 \\ 0 & -9 & 2+\pi & 0 \end{bmatrix}$$

$r(A) = r(AB) \neq 3$, $\Rightarrow$ Infinite soln.

Let, free variables $= 3 - 2 = 1$

Let, $z = t$, $t \in \mathbb{R}$

$-9y + (2-\pi)z = 0$

$\Rightarrow y = \frac{(2+\pi)}{9} t$

Now, $4x = -7y + \pi z$

$= -7\left(\frac{(2+\pi)t}{9}\right) + \pi t$

$= \frac{(-14 - 7\pi)t}{9} + \pi t$

$= \frac{-14t - 7\pi t + 9\pi t}{9}$

$= \frac{-14t + 2\pi t}{9}$

$x = \frac{\cancel{2}(-7t + \pi t)}{\cancel{4}_2 \times 9}$

$= \frac{-7t + \pi t}{18}$

$\therefore x = \frac{(-7+\pi)t}{18}$

$\therefore$ Solution set $=$

$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} (-7+\pi)/18 \\ (2+\pi)/9 \\ 1 \end{pmatrix} \middle| \; t \in \mathbb{R} \right\}$

$\therefore$ dim (solution set) $= 1$

$\therefore$ Basis for set $= \left\{ \frac{(-7+\pi)}{18}, \frac{(2+\pi)}{9}, 1 \right\}$

Q.5. Let A be a fixed $m \times n$ matrix. Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a map defined as : $T(X) = A(X)$ where X is a $n \times 1$ vector in $\mathbb{R}^n$. Show that T is a linear transformation.

$\rightarrow$ Soln :-

$T: \mathbb{R}^n \to \mathbb{R}^m$

$T(X) = A(X)$ .... $A = m \times n$ matrix

$\Rightarrow u, v \in \mathbb{R}^2 \longrightarrow u = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, v = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$T(u+v) = T \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}$

$= \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}$

$= \begin{bmatrix} a_{11}(x_1+y_1) + \cdots + a_{1n}(x_n+y_n) \\ \vdots \\ a_{m1}(x_1+y_1) + \cdots + a_{mn}(x_n+y_n) \end{bmatrix}_{m \times 1}$

Now,

$$T(u)+T(v) = A\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + A\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$= \underset{m\times n}{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}} \underset{n\times 1}{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}} + \underset{m\times n}{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}} \underset{n\times 1}{\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}}$$

$$= \underset{m\times 1}{\begin{bmatrix} a_{11}(x_1)+\dots+a_{1n}(x_n) \\ \vdots \\ a_{m1}(x_1)+\dots+a_{mn}(x_n) \end{bmatrix}} + \underset{m\times 1}{\begin{bmatrix} a_{11}(y_1)+\dots+a_{1n}(y_n) \\ \vdots \\ a_{m1}(y_1)+\dots+a_{mn}(y_n) \end{bmatrix}}$$

$$= \begin{bmatrix} a_{11}x_1+a_{11}y_1+\dots+a_{1n}x_n+a_{1n}y_n \\ \vdots \\ a_{m1}(x_1)+a_{m1}y_1+\dots+a_{mn}x_n+a_{mn}y_n \end{bmatrix}$$

$$T(u)+T(v) = \begin{bmatrix} a_{11}(x_1+y_1)+\dots+a_{1n}(x_n+y_n) \\ \vdots \\ a_{m1}(x_1+y_1)+\dots+a_{mn}(x_n+y_n) \end{bmatrix}$$

$\therefore T(u+v) = T(u)+T(v)$

Now,

$$T(ku) = A\begin{bmatrix} kx_1 \\ \vdots \\ kx_n \end{bmatrix}$$

$$= \underset{m\times n}{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}} \underset{n\times 1}{\begin{bmatrix} kx_1 \\ \vdots \\ kx_n \end{bmatrix}}$$

$$= \begin{bmatrix} a_{11}kx_1 + a_{1n} \dots + a_{1n}kx_n \\ \vdots \\ a_{m1}kx_1+\dots+a_{mn}kx_n \end{bmatrix}$$

$$= k\begin{bmatrix} a_{11}x_1+\dots+a_{1n}x_n \\ \vdots \\ a_{m1}x_1+\dots+a_{mn}x_n \end{bmatrix}$$

Now,

$$k(Tu) = k\,T(u)$$

$$= k\,[A]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= k\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= k\begin{bmatrix} a_{11}x_1+\dots+a_{1n}x_n \\ \vdots \\ a_{m1}x_1+\dots+a_{mn}x_n \end{bmatrix}$$

$\therefore$ T is a linear map.

Q6. In above example, Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 7 \end{bmatrix}$. Find Null space of T, Image of T. Hence, conclude N(T) and R(T). Further verify Rank-Nullity theorem.

→ $A = \underset{2\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 7 \end{bmatrix}}$ $T: \mathbb{R}^3 \to \mathbb{R}^2$

$x \in \mathbb{R}^3$, $x = (x, y, z)$

$$T(x) = Ax = T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 7 \end{bmatrix}_{2\times 3} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times 1}$$

$$= \begin{bmatrix} x+2y+3z \\ 3x+5y+7z \end{bmatrix}_{2\times 1}$$

$\therefore T(x,y,z) = (x+2y+3z,\ 3x+5y+7)$

$N(T) = \{ v \in \mathbb{R}^3 \mid T(v) = 0 \}$

$T(x,y,z) = (0,0)$

$\therefore (x+2y+3z,\ 3x+5y+7z) = (0,0)$

$\therefore x+2y+3z = 0$

$3x+5y+7z = 0$

Free variable $= 3-2 = 1$

Let $z = t \ \ldots \ t \in \mathbb{R}$

$x+2y+3z \times 3 = \quad 3x+6y+9t = 0$

$\quad 3x+5y+7t = 0$

$\quad (-) \ (-) \ (-)$

$\quad y+2t = 0$

$\quad \therefore y = -2t$

Now,

$x+2y+3z = 0$

$\therefore x = -2y-3z$

$= -2(-2t)-3(t)$

$= +4t-3t$

$\therefore x = t$

$$\therefore \ker(T) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \ \middle| \ t \in \mathbb{R} \right\}$$

$\therefore \dim \ker(T) = (1,-2,1)$

To $Im(T) = \{ u \in \mathbb{R}^2 \mid \exists v \in \mathbb{R}^3 \text{ s.t. } T(v) = u \}$

$$\Rightarrow T(v) = \begin{bmatrix} x+2y+3z \\ 3x+5y+7z \end{bmatrix}$$

$\mathbb{B}$ for $\mathbb{R}^3 = \{ (1,0,0), (0,1,0), (0,0,1) \}$

$$Im(T) = \left\{ T \begin{bmatrix} x \\ y \\ z \end{bmatrix} \ \middle| \ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in X \right\}$$

$$= \left\{ \begin{bmatrix} x+2y+3z \\ 3x+5y+7y \end{bmatrix} \ \middle| \ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in X \right\}$$

$\therefore \dim Im(T) = 2.$

By R.N.T,

$\dim \mathbb{R}^3 = \dim \ker(T) + \dim Im(T)$

$= 1+2$

$\therefore 3 = 3$

$\therefore$ Hence, theorem is verified.

Q.7. Take a 3×4 matrix of your choice and do above things. Try to take distinct entries.

→ Soln:-

Let, the matrix be

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \end{bmatrix}_{3\times4} \qquad T: \mathbb{R}^4 \to \mathbb{R}^3$$

$X \in \mathbb{R}^4$, $x = (x, y, z, w)$

$$T(X) = \cancel{T(A)} = A(X) = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \end{bmatrix}_{3\times4} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}_{4\times1}$$

$$= \begin{bmatrix} 0x+1y+2z+3w \\ 4x+5y+6z+7w \\ 8x+9y+10z+11w \end{bmatrix}_{3\times1}$$

$\therefore$ ~~$0x+y+z+3w$~~

Now,

$\ker(T) = \{ v \in \mathbb{R}^4 \mid T(v) = 0 \}$

$T(x, y, z, w) = (0, 0, 0, 0)$

$$\therefore \begin{bmatrix} \quad \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ $0x+1y+2z+3w=0$
$4x+5y+6z+7w=0$
$8x+9y+10z+11w=0$

Free variables $4-3=1$

Let $w=t$ ... $t \in \mathbb{R}$

→ Soln:-

Let, the matrix be

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -2 & -3 & -5 \\ 2 & 4 & -6 & 9 \end{bmatrix} \qquad T: \mathbb{R}^4 \to \mathbb{R}^3$$

$x \in \mathbb{R}^4$, $x = (x, y, z, w)$

$$T(X) = A(X) = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -2 & -3 & -5 \\ 2 & 4 & -6 & 9 \end{bmatrix}_{3\times4} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}_{4\times1}$$

$$= \begin{bmatrix} x+2y+3z+4w \\ -x-2y-3z-5w \\ 2x+4y-6z+9w \end{bmatrix}_{3\times1}$$

Now,

$\ker P = \{ v \in \mathbb{R}^4 \mid T(v) = 0 \}$

$T(x, y, z, w) = (0, 0, 0)$

$$\begin{bmatrix} \text{———} ① \\ \text{———} ② \\ \text{———} ③ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x+2y+3z+4w=0$ — (1)
$-x-2y-3z-5w=0$ — (2)
$2x+4y-6z+9w=0$ — (3)

F.V = 4 − 3 = 1

(1) + (2) → $-1w=0 \Rightarrow \underline{w=0}$

Now, $2\times$(2) + (3) $= \cancel{-2x-4y-6z=0}$
$\cancel{2x+4y-6z=0}$
(+) (−) (+) (−)
$\cancel{-4x-8y=0}$
$\therefore x+2y=0$

$2\times$(2) + (3) $-2x-4y-6z=0$
$2x+4y-6z=0$
(+) (+) (+) (+)
$-12z=0$
$\Rightarrow z=0$

Now, let
$x=t$,
$\therefore y = \frac{x}{2} = \frac{t}{2}$

$\therefore \ker(T) = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 1/2 \\ 0 \\ 0 \end{pmatrix} t \;\middle|\; t \in \mathbb{R} \right\}$

$\therefore$ dim ker(T) = 1

Now,
$\text{Im}\, T = \left\{ T\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} : x \in X \right\}$

$\therefore \left\{ \begin{bmatrix} x+2y+3z+4w \\ -x-2y-3z-5w \\ 2x+4y-6z+9w \end{bmatrix} \middle| \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in X \right\}$

$\therefore$ dim Im$(T) = 3$

$\therefore \dim \mathbb{R}^4 = \dim \ker(T) + \dim \text{Im}(T)$
$4 = 1+3$
$\therefore \underline{4=4}$

Hence, verified.

Q.8. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transf^n s.t. $T(1,0)=(1,1)$ & $T(0,1)=(2,3)$. Find $T(a,b)$ for any $(a,b) \in \mathbb{R}^2$. Hence calculate image of $(3,7)$.

→ Soln :- $T: \mathbb{R}^2 \to \mathbb{R}^2$

Consider $(0,1),(1,0)$ basis for $\mathbb{R}^2$ ... (observation)

$\Rightarrow$ Any element of $\mathbb{R}^2 = a_1(1,0)+a_2(0,1)$

$T(a,b) = T[a_1(1,0)+a_2(0,1)]$
$= a_1(1,1)+a_2(2,3)$
$\Rightarrow T(a,b) = (a_1+2a_2),(a_1+3a_2)$
here, $a=a_1$, $b=a_2$
Now,
$T(3,7) = (3+2(7)) + (3+3(7))$
$T(3,7) = (17, 24)$

Q.9. Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation s.t. $T(1,0,0) = (1,1,0)$, $T(0,1,0) = (2,3,0)$ and $T(0,0,1) = (1,0,5)$. Find $T(a,b,c)$ for any $(a,b,c) \in \mathbb{R}^3$. Hence calculate im of $(3,7,1)$

→ Soln:-

Here,

$(1,0,0), (0,1,0), (0,0,1)$ is basis for $\mathbb{R}^3$ ... (Observ

$\Rightarrow$ ~~T(a,b,c) = a(1,0,~~

$\Rightarrow$ Any element of $\mathbb{R}^3 = a(1,0,0) + b(0,1,0) + c(0,0,1)$

$$\Rightarrow T(a,b,c) = T[a(1,0,0) + b(0,1,0) + c(0,0,1)]$$

$$= aT(1,0,0) + bT(0,1,0) + cT(0,0,1)$$

$$= a(1,1,0) + b(2,3,0) + c(1,0,5)$$

$$= \cancel{a+2ba}$$

$$T(a,b,c) = (a+2b+3c), (a+3b), (5c)$$

for, $(a,b,c) = (3,7,1)$

$$T(3,7,1) = (3+14+3), (3+21), (5)$$

$$T(3,7,1) = (20, 24, 5)$$