Solving Recurrence Relations

A recurrence relation is an equation that recursively defines a sequence where the next term is a function of the previous terms (Expressing F_n as some combination of F_i with $i<n$).

How to solve linear recurrence relation
Homogeneous Recurrence Relations

- Suppose, a two ordered linear recurrence relation is $F_n = AF_{n-1} + BF_{n-2}$
  - where A and B are real numbers.
- The characteristic equation for the above recurrence relation is −

  $$x^2 - Ax - B = 0$$

- Three cases may occur while finding the roots −
- **Case 1** − If this equation factors as $(x - x_1)(x - x_1) = 0$ and it produces two distinct real roots x_1 and x_2, then $F_n = ax_1^n + bx_2^n$ is the solution. [Here, a and b are constants]
- **Case 2** − If this equation factors as $(x - x_1)^2$ and it produces single real root x_1, then $F_n = ax_1^n + bnx_1^n$

is the solution.

- **Case 3** − If the equation produces two distinct complex roots, x_1 and x_2 in polar form $x_1 = r\angle\Theta$ and $x_2 = r\angle(-\Theta)$, then following is the solution

$$F_n = r^n(a\cos(n\theta) + b\sin(n\theta))$$

Problem 1

Solve the recurrence relation $F_n = 5F_{n-1} - 6F_{n-2}$ where $F_0 = 1$ and $F_1 = 4$

Solution

- The characteristic equation of the recurrence relation is – $x^2 - 5x + 6 = 0$

So, $(x-3)(x-2) = 0$

Hence, the roots are –

$$x_1 = 3$$

and $x_2 = 2$

The roots are real and distinct. So, this is in the form of case 1

- Hence, the solution is –

$$F_n = ax_1^n + bx_2^n$$

- Here, $F = a3^n + b2^n$ (*As x*1=3 *and x*2=2)
- Therefore,

$$1 = F_0 = a3^0 + b2^0 = a + b$$

$$4 = F_1 = a3^1 + b2^1 = 3a + 2b$$

- Solving these two equations, we get *a*=2 and *b*=−1
- Hence, the final solution is −

$$F_n = 2.3^n + (-1).2^n = 2.3^n - 2^n$$

Problem 2

Solve the recurrence relation $F_n = 10F_{n-1} - 25F_{n-2}$

where $F_0 = 3$ and $F_1 = 17$

- **Solution**
- The characteristic equation of the recurrence relation is –

$$x^2 - 10x - 25 = 0$$

- So $(x-5)^2 = 0$
- Hence, there is single real root $x_1 = 5$
- As there is single real valued root, this is in the form of case 2
- Hence, the solution is –

$$F_n = ax_1^n + bnx_1^n$$

- $3 = F_0 = a.5^0 + b.5^0$ & $17 = F_1 = a.5^1 + b.1.5^1$

- Solving these two equations, we get *a*=3
- and *b*=2/5
- Hence, the final solution is $F_n = 3.5^n + (2/5).n.2^n$

Problem 3

Solve the recurrence relation $F_n = 2F_{n-1} - 2F_{n-2}$

where $F_0 = 1$ and $F_1 = 3$

- **Solution**
- The characteristic equation is $x^2 - 2x - 2 = 0$
  Hence, the roots are – $x_1 = 1+i$ and $x_2 = 1-i$
- In polar form, $x_1 = r\angle\theta$ And $x_2 = r\angle(-\theta)$
  where $r = \sqrt{2}$ and $\theta = \dfrac{\pi}{4}$

- The roots are imaginary. So, this is in the form of case 3.
- Hence,

$$F_n = (\sqrt{2})^n (a\cos(n.\Pi/4) + b\sin(n.\Pi/4))$$

$$1 = F_0 = (\sqrt{2})^0 (a\cos(0.\Pi/4) + b\sin(0.\Pi/4)) = a$$

$$3 = F_1 = (\sqrt{2})^1 (a\cos(1.\Pi/4) + b\sin(1.\Pi/4)) = \sqrt{2}(a/\sqrt{2} + b\sqrt{2})$$

Solving these two equations we get *a*=1 and *b*=2

- Hence, the final solution is –

$$F_n = (\sqrt{2})^n (\cos(n.\Pi/4) + 2\sin(n.\Pi/4))$$

Example: Fibonacci sequence

$$f_n = f_{n-1} + f_{n-2}$$

$$f_0 = 0, \quad f_1 = 1$$

Has solution: $$f_n = \lambda_1 r_1^n + \lambda_2 r_2^n$$

Characteristic roots: $$r_1 = \frac{1+\sqrt{5}}{2} \qquad r_2 = \frac{1-\sqrt{5}}{2}$$

$$\lambda_1 = \frac{f_1 - f_0 r_2}{r_1 - r_2} = \frac{1}{\sqrt{5}}$$

$$\lambda_2 = \frac{f_0 r_1 - f_1}{r_1 - r_2} = -\frac{1}{\sqrt{5}}$$

$$f_n = \lambda_1 r_1^n + \lambda_2 r_2^n$$

$$= \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n$$

Non-homogeneous Recurrence Relations

$F_n = AF_{n-1} + BF_{n-2} + f(n)$ where $f(n) \neq 0$

Its associated homogeneous recurrence relation is $F_n = AF_{n-1} + BF_{n-2}$

The solution (a_n) of a non-homogeneous recurrence relation has two parts.

First part is the solution (a_h) of the associated homogeneous recurrence relation and the second part is the particular solution (a_t).

$$a_n = a_h + a_t$$

Solution to the first part is done using the procedures discussed in the previous section.

To find the particular solution, we find an appropriate trial solution.

Let $f(n) = cx^n$; let $x^2 = Ax + B$ be the characteristic equation of the associated homogeneous recurrence relation and let x_1 and x_2 be its roots.

- If $x \neq x_1$ and $x \neq x_2$, then $a_t = Ax^n$
- If $x = x_1$, $x \neq x_2$, then $a_t = Anx^n$
- If $x = x_1 = x_2$, then $a_t = An^2x^n$

$$a_n = 5a_{n-1} - 6a_{n-2} + 7^n.$$

- This is a linear nonhomogeneous recurrence relation. The solutions of its associated
- homogeneous recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$

Are $a_n^{(h)} = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n,$ where *α1 and α2 are constants*

- Because $F(n) = 7^n,$ a reasonable trial solution is

$$a_n^{(p)} = C \cdot 7^n,$$

where *C is a constant. Substituting the terms of this sequence* into the recurrence relation implies that *C* $C \cdot 7^n = 5C \cdot 7^{n-1} - 6C \cdot 7^{n-2} + 7^n.$

- Factoring out 7^{n-2} this equation becomes $49C = 35C - 6C + 49,$
- which implies that $20C = 49$ $C = 49/20.$
- ***Hence,*** $a_n^{(p)} = (49/20)7^n$ Is a particular solution,
- ***All solutions are of the form.***

$$a_n = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n + (49/20)7^n.$$

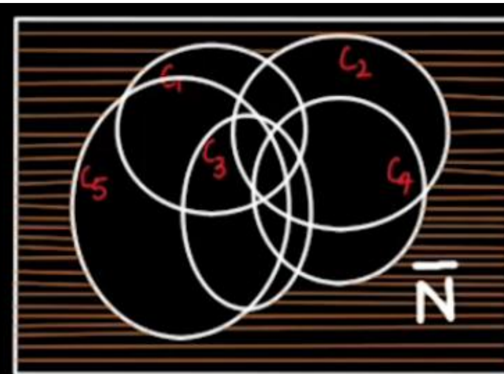
Counting Functions How many functions are there from a set with m elements to a set with n elements?

- *Solution:* A function corresponds to a choice of one of the n elements in the codomain for each of the m elements in the domain. Hence, by the product rule there are $n, n, n, \ldots . n = n^m$ functions from a set with m elements to one with n elements.
- For example, there are $5^3 = 125$ different functions from a set with three elements to a set with five elements.

Counting One-to-One Functions How many one-to-one functions are there from a set with m elements to one with n elements?

- First note that when $m > n$ there are no one-to-one functions from a set with m elements to a set with n elements.
- Now let $m \leq n$. Suppose the elements in the domain are $a1, a2, \ldots, am$. There are n ways to choose the value of the function at $a1$. Because the function is one-to-one, the value of the function at $a2$ can be picked in $n - 1$ ways (because the value used for $a1$ cannot be used again).
- In general, the value of the function at ak can be chosen in $n - k + 1$ ways.
- By the product rule, there are $n(n - 1)(n - 2) \cdots (n - m + 1)$ one-to-one functions from a set with m elements to one with n elements.
- For example, there are $5 \cdot 4 \cdot 3 = 60$ one-to-one functions from a set with three elements to a set with five elements.

Total number of graphs in which no one vertex is isolated



$$\bar{N} = N - [N(C_1) + N(C_2) + N(C_3) + N(C_4) + N(C_5)] + N(C_1C_2) + N(C_1C_3)$$
$$+ N(C_1C_4) + N(C_1C_5) + N(C_2C_3) + N(C_2C_4) + N(C_2C_5) + N(C_3C_4)$$
$$+ N(C_3C_5) + N(C_4C_5) - [N(C_1C_2C_3) + N(C_1C_2C_4) + N(C_1C_2C_5)$$
$$+ \dots + N(C_3C_4C_5)] + N(C_1C_2C_3C_4) + N(C_1C_2C_3C_5) + N(C_1C_2C_4C_5)$$
$$+ \dots + N(C_2C_3C_4C_5) - N(C_1C_2C_3C_4C_5)$$