Counting

Basic Counting Principles

Product Rule:

Suppose a procedure consists of 2 tasks

n_1 ways to perform 1st task

n_2 ways to perform 2nd task

$n_1 \cdot n_2$ ways to perform procedure

Example: 2 employees 10 offices

How many ways to assign employees to offices?

1st employee has 10 office choices

2nd employee has 9 office choices

Total office assignment ways: $10 \times 9 = 90$

Example: How many different variable names with 2 symbols?

$$XY$$

(e.g. A1, A2, AA)

1st symbol
letter
26 choices

2nd symbol
alphanumeric
26+10 = 36 choices

Total variable name choices: 26x36 = 936

Generalized Product Rule:

Suppose a procedure consists of k tasks

n_1 ways to perform 1st task

n_2 ways to perform 2nd task

$\vdots$

n_k ways to perform k th task

$\Downarrow$

$n_1 n_2 \cdots n_k$ ways to perform procedure

Example: How many different variable names with exactly $k \geq 1$ symbols?

$$XY_1 \cdots Y_{k-1}$$ (e.g. D1B…6)

1st symbol
letter
26 choices

Remaining symbols
alphanumeric
36 choices for each

Total choices: $26 \cdot \underbrace{36 \cdots 36}_{k-1} = 26 \cdot (36)^{k-1}$

Sum Rule:

Suppose a procedure can be performed with either of 2 different methods

n_1 ways to perform 1st method

n_2 ways to perform 2nd method

$$\Downarrow$$

$n_1 + n_2$ ways to perform procedure

Example: Number of variable names with 1 or 2 symbols

Variables with 1 symbol: 26

Variables with 2 symbols: 936

Total number of variables: 26+936=962

Principle of Inclusion-Exclusion:

Suppose a procedure can be performed with either of 2 different methods

n_1 ways to perform 1st method

n_2 ways to perform 2nd method

c common ways in both methods

$$\Downarrow$$

$n_1 + n_2 - c$ ways to perform procedure

Example:

Number of binary strings of length 8 that either start with 1 or end with 0

Strings that start with 1: $1x_1x_2\cdots x_7$ 128 choices

Strings that end with 0: $y_1y_2\cdots y_7 0$ 128 choices

Common strings: $1z_1\cdots z_7 0$ 64 choices

Total strings: 128+128-64=192

Selection (Ordered/ Unordered)

10 Boys and 8 girls
3 Boys and 3 Girls to be selected
How many ways?

Selecting 3 Boys out of 10	Selecting 3 Girls out of 8
(Order doesn't matter)	(Order doesn't matter)
10 Boys	8 Girls
S S S NS NS NS NS NS NS NS	S S S NS NS NS NS NS

$$\frac{10!}{3!\,7!} \times \frac{8!}{3!\,5!} = 6720$$

Counting Passwords

- Combining the sum and product rule allows us to solve more complex problems.

  Example: Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

Counting Passwords

- Combining the sum and product rule allows us to solve more complex problems.

  Example: Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

  Solution: Let P be the total number of passwords, and let P_6, P_7, and P_8 be the passwords of length 6, 7, and 8.

  - By the sum rule $P = P_6 + P_7 + P_8$.
  - To find each of P_6, P_7, and P_8 , we find the number of passwords of the specified length composed of letters and digits and subtract the number composed only of letters. We find that:

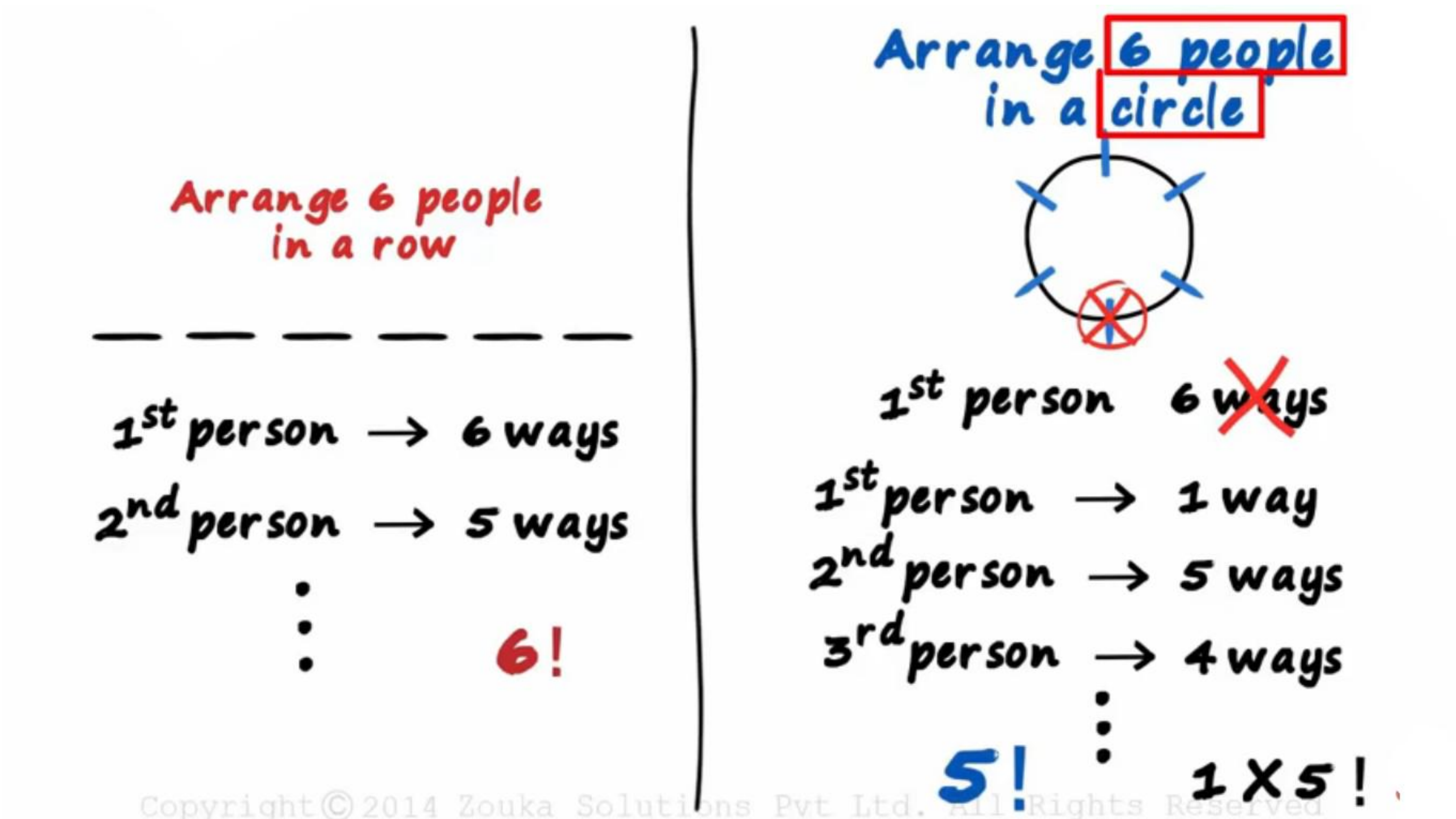
$$P_6 = 36^6 - 26^6 = 2{,}176{,}782{,}336 - 308{,}915{,}776 = 1{,}867{,}866{,}560.$$

$$P_7 = 36^7 - 26^7 = 78{,}364{,}164{,}096 - 8{,}031{,}810{,}176 = 70{,}332{,}353{,}920.$$

$$P_8 = 36^8 - 26^8 = 2{,}821{,}109{,}907{,}456 - 208{,}827{,}064{,}576 = 2{,}612{,}282{,}842{,}880.$$

Consequently, $P = P_6 + P_7 + P_8 = 2{,}684{,}483{,}063{,}360$.

Circular Arrangements



Pigeonhole Principle

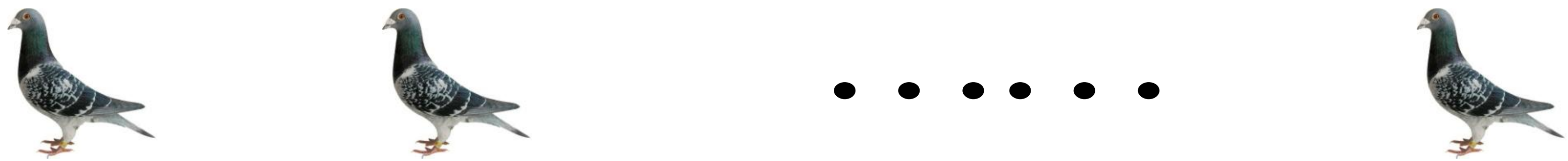
3 pigeons

2 pigeonholes

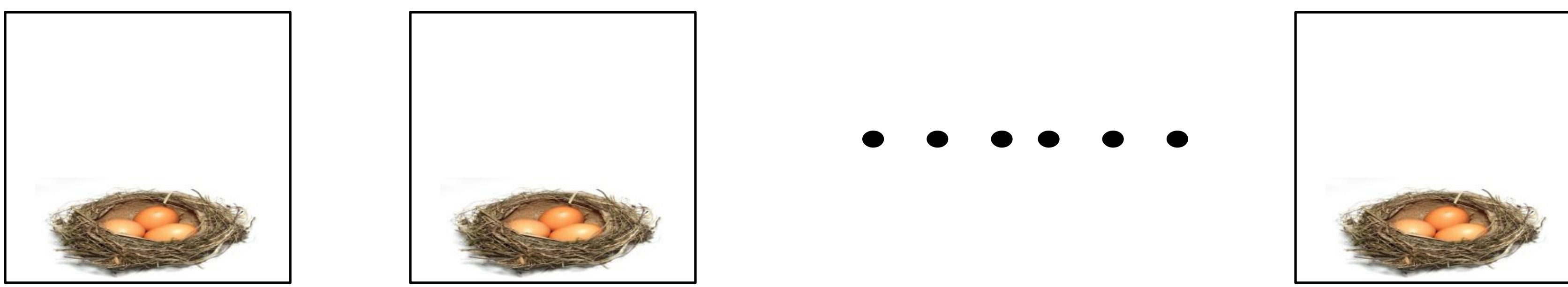
One pigeonhole contains 2 pigeons

3 pigeons

2 pigeonholes



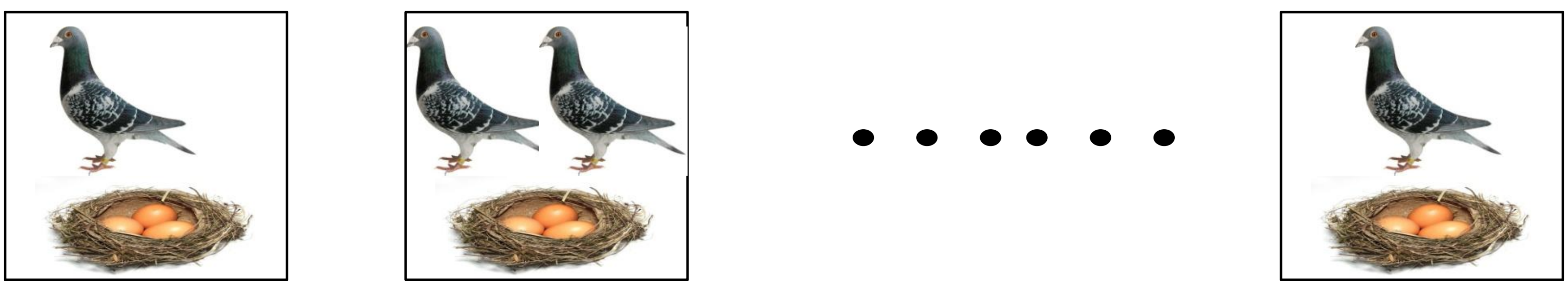
k+1 pigeons



k pigeonholes

At least one pigeonhole contains 2 pigeons

$k+1$ pigeons



k pigeonholes

Pigeonhole Principle:

If $k+1$ objects are placed into k boxes, then at least one box contains 2 objects

Examples:

- Among 367 people at least 2 have the same birthday (366 different birthdays)

- Among 27 English words at least 2 start with same letter (26 different letters)

Pigeonhole Principle

Example: How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

Solution: There are 101 possible scores on the final. The pigeonhole principle shows that among any 102 students there must be at least 2 students with the same score.

Pigeonhole Principle

Q. Assume there are n distinct pairs of shoes in a closet. Show that if you choose n + 1 single shoes at random from the closet, you are certain to have a pair.

Solution: The n distinct pairs constitute n pigeonholes. The n + 1 single shoes correspond to n + 1 pigeons.

Therefore, there must be at least one pigeonhole with two shoes and thus you will certainly have drawn at least one pair of shoes.

Pigeonhole Principle

Q. A student must take five classes from three areas of study. Numerous classes are offered in each discipline, but the student cannot take more than two classes in any given area. Using the pigeonhole principle, show that the student will take at least two classes in one area.

Solution: The three areas are the pigeonholes and the student must take five classes (pigeons). Hence, the student must take at least two classes in one area.

Generalized Pigeonhole Principle:

If N objects are placed into k boxes, then at least one box contains $\left\lceil \frac{N}{k} \right\rceil$ objects

Proof:

If each box contains less than $\left\lceil \frac{N}{k} \right\rceil$ objects:

$$\#\text{objects} \;\le\; k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) \;<\; k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N$$

contradiction

End of Proof

Example:

Among 100 people, at least $\left\lceil \frac{100}{12} \right\rceil = 9$ have birthday in same month

$N = 100$ people (objects)

$k = 12$ months (boxes)

Pigeonhole Principle

Q. Find the minimum number of students needed to guarantee that five of them belong to the same Class - 1st year, 2nd year, 3rd year, 4th year.

Solution: Here the $k = 4$ classes are the pigeonholes and $n + 1 = 5$ so $k = 4$. Thus among any $k.n + 1 = 17$ students (pigeons), five of them belong to the same class.

Example:

How many students do we need to have so that at least six receive same grade (A,B,C,D,F)?

$$N = ?$$ students (objects)

$$k = 5$$ grades (boxes)

$$\left\lceil \frac{N}{k} \right\rceil \geq 6$$ at least six students receive same grade

Smallest integer N with $\left\lceil \frac{N}{k} \right\rceil \geq r$

is smallest integer with $\frac{N}{k} > r - 1$

$$\frac{N}{k} > r - 1 \Rightarrow N > k(r-1) \Rightarrow N = k(r-1)+1$$

$$\left\lceil \frac{N}{k} \right\rceil \geq r \quad \Rightarrow \quad N = k(r-1)+1$$

For our example:

$$k = 5$$
$$r = 6$$
$$\Rightarrow \quad N = 5(6-1)+1 = 26 \quad \text{students}$$

We need at least 26 students

An elegant example:

In any sequence of n^2+1 numbers there is a sorted subsequence of length $n+1$

(ascending or descending)

$$n=3$$

$n+1=4$ $\qquad$ $n^2+1=10$ numbers

Ascending subsequence $\qquad$ 8, 11, 9, 1, 4, 6, 12, 10, 5, 7

Descending subsequence $\qquad$ 8, 11, 9, 1, 4, 6, 12, 10, 5, 7

Theorem:

In any sequence of n^2+1 numbers there is a sorted subsequence of length $n+1$

(ascending or descending)

Proof:

Sequence $a_1, a_2, a_3, \ldots, a_{n^2+1}$

$$(x_i, y_i)$$

Length of longest ascending subsequence starting from a_i

Length of longest descending subsequence starting from a_i

For example: $(x_1, y_1) = (3,3)$

Longest ascending subsequence from $a_1 = 8$

8, 11, 9, 1, 4, 6, 12, 10, 5, 7

$$x_1 = 3$$

Longest descending subsequence from $a_1 = 8$

8, 11, 9, 1, 4, 6, 12, 10, 5, 7

$$y_1 = 3$$

For example: $(x_2, y_2) = (2,4)$

Longest ascending subsequence from $a_2 = 11$

8, 11, 9, 1, 4, 6, 12, 10, 5, 7

$$x_2 = 2$$

Longest descending subsequence from $a_2 = 11$

8, 11, 9, 1, 4, 6, 12, 10, 5, 7

$$y_2 = 4$$

We want to prove that there is a (x_i, y_i) with:

$$x_i \geq n+1 \quad \text{or} \quad y_i \geq n+1$$

Assume (for sake of contradiction) that for every (x_i, y_i):

$$1 \leq x_i \leq n \text{ and } 1 \leq y_i \leq n$$

Number of unique pairs of form (x_i, y_i) with $1 \leq x_i \leq n$ and $1 \leq y_i \leq n$:

$$(x_i, y_i)$$

n choices $\qquad$ n choices

$n \cdot n = n^2$ unique pairs

For example: $(1,1), (1,2), (2,1), (1,3), (3,1), \ldots, (n,n)$

$n \cdot n = n^2$ unique pairs of form (x_i, y_i)

Since $a_1, a_2, a_3, \ldots, a_{n^2+1}$ has n^2+1 elements there are exactly n^2+1 pairs of form (x_i, y_i)

From pigeonhole principle, there are two equal pairs $(x_j, y_j) = (x_k, y_k), \quad j < k$

$$x_j = x_k \text{ and } y_j = y_k$$

Case $a_j \le a_k$:

$$(x_j, y_j) = (x_k, y_k), \quad j < k \quad \Rightarrow \quad x_j = x_k$$

Ascending subsequence with x_k elements

$$a_1, a_2, a_3, \ldots, \underbrace{a_j, \ldots, \overbrace{a_k, \ldots, a_{k_2}, \ldots, a_{k_{x_k}}}}, \ldots, a_{n^2+1}$$

Ascending subsequence with $x_k + 1 = x_j + 1 > x_j$ elements

Contradiction, since longest ascending subsequence from a_j has length x_j

Case $a_j > a_k$:

$$(x_j, y_j) = (x_k, y_k), \quad j < k \quad \Longrightarrow \quad y_j = y_k$$

Descending subsequence
with y_k elements

$$a_1, a_2, a_3, \ldots, \underbrace{a_j, \ldots, \overbrace{a_k, \ldots, a_{k_2}, \ldots, a_{k_{y_k}}}}, \ldots, a_{n^2+1}$$

Descending subsequence
with $y_k + 1 = y_j + 1 > y_j$ elements

Contradiction, since longest descending subsequence
from a_j has length y_j

Therefore, it is not true the assumption that for every (x_i, y_i): $1 \le x_i \le n$ and $1 \le y_i \le n$

Therefore, there is a (x_i, y_i) with:

$$x_i \ge n+1 \quad \text{or} \quad y_i \ge n+1$$

End of Proof

Home work

- During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

Permutations

Permutation: An ordered arrangement of objects

Example: Objects: a,b,c

Permutations:

a,b,c a,c,b

b,a,c b,c,a

c,a,b c,b,a

r-permutation: An ordered arrangement of n objects

Example: Objects: a,b,c,d

2-permutations:

a,b	a,c	a,d
b,a	b,c	b,d
c,a	c,b	c,d
d,a	d,b	d,c

How many ways to arrange 5 students in line?

1st position in line: 5 student choices

2nd position in line: 4 student choices

3rd position in line: 3 student choices

4th position in line: 2 student choices

5th position in line: 1 student choices

Total permutations: 5x4x3x2x1=120

How many ways to arrange 3 students in line out of a group of 5 students?

1st position in line: 5 student choices

2nd position in line: 4 student choices

3rd position in line: 3 student choices

Total 3-permutations: 5x4x3=60

Given n objects the number of r-permutations is denoted

$$P(n,r)$$

Examples:

$$P(5,5) = 120$$

$$P(5,3) = 60$$

$$P(4,2) = 12$$

$$P(3,3) = 6$$

Theorem: $P(n,r) = \dfrac{n!}{(n-r)!}$ $\quad 0 \leq r \leq n$

Proof:

$$P(n,r) = n \cdot (n-1) \cdot (n-2) \cdots (n-(r-1))$$

1st position object choices

2nd position object choices

3rd position object choices

r^{th} position object choices

$$P(n,r) = n \cdot (n-1) \cdot (n-2) \cdots (n-(r-1))$$

$$= \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-(r-1)) \cdot (n-r) \cdot (n-(r+1)) \cdots 2 \cdot 1}{(n-r) \cdot (n-(r+1)) \cdots 2 \cdot 1}$$

$$= \frac{n!}{(n-r)!}$$

Multiply and divide with same product

End of Proof

Example: How many different ways to order gold, silver, and bronze medalists out of 8 athletes?

$$P(8,3) = \frac{8!}{(8-3)!} = \frac{8!}{5!} = 8 \cdot 7 \cdot 6 = 336$$

Combinations

r-combination: An unordered arrangement of r objects

Example: Objects: a,b,c,d

2-combinations: a,b a,c a,d b,c b,d c,d

3-combinations: a,b,c a,b,d a,c,d b,c,d

Given n objects the number of r-combinations is denoted

$$C(n,r) \quad \text{or} \quad \binom{n}{r}$$

Also known as binomial coefficient

Examples: $C(4,2) = 6$

$C(4,3) = 4$

Combinations can be used to find permutations

3-combinations $C(4,3)$

a,b,c a,b,d a,c,d b,c,d

Objects: a,b,c,d

Combinations can be used to find permutations

3-combinations $C(4,3)$

3-permutations $P(3,3)$

a,b,c	a,b,d	a,c,d	b,c,d
a,c,b	a,d,b	a,d,c	b,d,c
b,a,c	b,a,d	c,a,d	c,b,d
b,c,a	b,d,a	c,d,a	c,d,b
c,a,b	d,a,b	d,a,c	d,b,c
c,b,a	d,b,a	d,c,a	d,c,b

Objects: a,b,c,d

Combinations can be used to find permutations

Total permutations: $P(4,3) = C(4,3) \cdot P(3,3)$

3-combinations $C(4,3)$

3-permutations $P(3,3)$

a,b,c	a,b,d	a,c,d	b,c,d
a,c,b	a,d,b	a,d,c	b,d,c
b,a,c	b,a,d	c,a,d	c,b,d
b,c,a	b,d,a	c,d,a	c,d,b
c,a,b	d,a,b	d,a,c	d,b,c
c,b,a	d,b,a	d,c,a	d,c,b

Objects: a,b,c,d

Theorem: $$C(n,r) = \frac{n!}{r!(n-r)!} \qquad 0 \le r \le n$$

Proof: $$P(n,r) = C(n,r) \cdot P(r,r)$$

$$\Downarrow$$

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{\dfrac{n!}{(n-r)!}}{\dfrac{r!}{(r-r)!}} = \frac{n!}{r!(n-r)!}$$

End of Proof

Example: Different ways to choose 5 cards out of 52 cards

$$C(52,5) = \frac{52!}{5!(47)!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 26 \cdot 17 \cdot 10 \cdot 49 \cdot 12 = 2{,}598{,}960$$

Observation: $C(n,r) = C(n,n-r)$

$$C(n,r) = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-(n-r))!(n-r)!} = C(n,n-r)$$

Example: $C(52,5) = C(52,47)$

Binomial Coefficients

$$C(n,r)=\binom{n}{r}$$

$$(x+y)^n=\binom{n}{0}x^n+\binom{n}{1}x^{n-1}y^1+\binom{n}{2}x^{n-2}y^2+\cdots+\binom{n}{n-1}xy^{n-1}+\binom{n}{n}y^n$$

$$(x+y)^n=\sum_{j=0}^{n}\binom{n}{j}x^{n-j}y^j$$

$$(x+y)^3 = (x+y)(x+y)(x+y)$$
$$= xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy$$
$$= \mathrm{x}^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$$

Possible ways to obtain product of 3 terms of x and 0 terms of y

Possible ways to obtain product of 2 terms of x and 1 terms of y

Possible ways to obtain product of 0 terms of x and 3 terms of y

$$(x+y)^n = \overbrace{(x+y)(x+y)(x+y)\cdots(x+y)}^{n \text{ times}}$$

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

Possible ways to obtain product of n terms of x and 0 terms of y

Possible ways to obtain product of $n-1$ terms of x and 1 terms of y

Possible ways to obtain product of 0 terms of x and n terms of y

Binomial Coefficients

We know that a *binomial* is a polynomial that has two terms. In this section, you will study a formula that provides a quick method of raising a binomial to a power.

To begin, look at the expansion of $(x + y)^n$

for several values of n

$$(x + y)^0 = 1$$

$$(x + y)^1 = (x + y)$$

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Binomial Coefficients

1. The sum of the powers of each term is n. For instance, in the expansion of

$(x + y)^5$

the sum of the powers of each term is 5.

$$(x + y)^5 = x^5 + 5\overbrace{x^4y^1}^{4 + 1 = 5} + 10\overbrace{x^3y^2}^{3 + 2 = 5} + 10x^2y^3 + 5x^1y^4 + y^5$$

2. The coefficients increase and then decrease in a symmetric pattern.

The coefficients of a binomial expansion are called **binomial coefficients.**

Binomial Coefficients

- To find them, use the **Binomial Theorem**.

The Binomial Theorem

In the expansion of $(x + y)^n$

$$(x + y)^n = x^n + nx^{n-1}y + \cdots + {}_nC_r x^{n-r} y^r + \cdots + nxy^{n-1} + y^n$$

the coefficient of $x^{n-r} y^r$ is

$${}_nC_r = \frac{n!}{(n-r)!r!}.$$

The symbol

$$\binom{n}{r}$$

is often used in place of ${}_nC_r$ to denote binomial coefficients.

Example 1 – Finding Binomial Coefficients

Find each binomial coefficient.

a. ${}_8C_2$

b. $\binom{10}{3}$

c. ${}_7C_0$

a. $${}_8C_2 = \frac{8!}{6! \cdot 2!} = \frac{(8 \cdot 7) \cdot \cancel{6!}}{\cancel{6!} \cdot 2!} = \frac{8 \cdot 7}{2 \cdot 1} = 28$$

b. $$\binom{10}{3} = \frac{10!}{7! \cdot 3!} = \frac{(10 \cdot 9 \cdot 8) \cdot \cancel{7!}}{\cancel{7!} \cdot 3!} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$$

c. $${}_7C_0 = \frac{\cancel{7!}}{\cancel{7!} \cdot 0!} = 1$$

Example 3 – *Expanding a Binomial*

Write the expansion of the expression $(x + 1)^3$.

Solution:

The binomial coefficients are

$$_3C_0 = 1,\ _3C_1 = 3,\ _3C_2 = 3, \text{ and } _3C_3 = 1.$$

Therefore, the expansion is as follows.

$$(x + 1)^3 = (1)x^3 + (3)x^2(1) + (3)x(1^2) + (1)(1^3)$$

$$= x^3 + 3x^2 + 3x + 1$$

Binomial Expansions

Sometimes you will need to find a specific term in a binomial expansion.

Instead of writing out the entire expansion, you can use the fact that, from the Binomial Theorem, the $(r + 1)$th term is

$$nCr\, x^{n-r}y^{r}.$$

Example: What is the coefficient for x^3 in $(2x+4)^8$

Solution: The exponents for x^3 are: $(2x)^3 * 4^5$

The coefficient is "8 choose 5". We can use Pascal's Triangle, or calculate directly:

$(n!/(k!(n-k)))! = 8! / (5!(8-5!)) = 8! / (5!3!) = (8\times7\times6) / (3\times2\times1) = 56$

And we get: 56(2x)345

Which simplifies to 458752 x3

Pascal's Triangle

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$

$$1$$

$$1 \quad 1$$

$$1 \quad 2 \quad 1$$

$$1 \quad 3 \quad 3 \quad 1$$

$$1 \quad 4 \quad 6 \quad 4 \quad 1$$

$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$

$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$

$$\binom{6}{4} = \binom{5}{3} + \binom{5}{4}$$

$$15 = 10 + 5$$

Pascal's Triangle

The first and last number in each row of Pascal's Triangle is 1. Every other number in each row is formed by adding the two numbers immediately above the number. Pascal noticed that the numbers in this triangle are precisely the same numbers as the coefficients of binomial expansions, as follows.

$$
\begin{array}{rcll}
(x+y)^0 &=& 1 & \text{0th row} \\
(x+y)^1 &=& 1x + 1y & \text{1st row} \\
(x+y)^2 &=& 1x^2 + 2xy + 1y^2 & \text{2nd row} \\
(x+y)^3 &=& 1x^3 + 3x^2y + 3xy^2 + 1y^3 & \text{3rd row} \\
(x+y)^4 &=& 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4 & \vdots \\
(x+y)^5 &=& 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5 & \\
(x+y)^6 &=& 1x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + 1y^6 & \\
(x+y)^7 &=& 1x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + 1y^7 &
\end{array}
$$

Pascal's Triangle

The top row of Pascal's Triangle is called the *zeroth row* because it corresponds to the binomial expansion

$$(x + y)^0 = 1.$$

Similarly, the next row is called the *first row* because it corresponds to the binomial expansion

$$(x + y)^1 = 1(x) + 1(y).$$

In general, the *nth row* of Pascal's Triangle gives the coefficients of $(x + y)^n$.

Example 8 – *Using Pascal's Triangle*

Use the seventh row of Pascal's Triangle to find the binomial coefficients.

8*C*0 8*C*1 8*C*2 8*C*3 8*C*4 8*C*5 8*C*6 8*C*7 8*C*8

Solution: