Stability and Routh-Hurwitz Criterion

UNIT – III

Introduction

- Stability is the most desired property in designing of control systems.

Why stability is important?

Let us see some examples.

A.

Fig 4.1.1 - Not stable at road bumps

B.

 [1]

Fig 4.1.2 - Unstable Aeroplane

1 – https//www.giphy.com

What is Stability?

- We restrict ourselves to linear time invariant systems.
- A system is stable if the system eventually comes back to the equilibrium state when the system is subjected to an initial condition.
- A system is unstable if the output diverges without bound from its equilibrium state when the system is subjected to an initial condition.
- A system is marginally stable if the system tends to oscillate about it's equilibrium state subjected to an initial condition.

Bounded Signals

- What is bounded input/output?

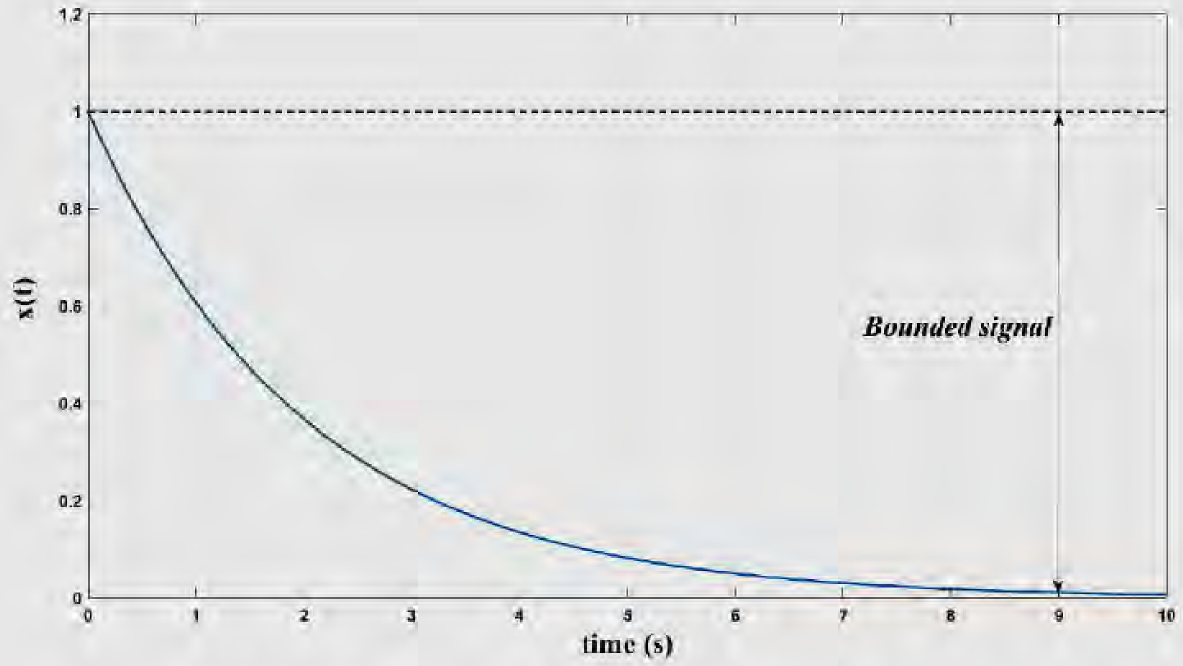


Fig 4.1.3 – Bounded Signal (exponential)

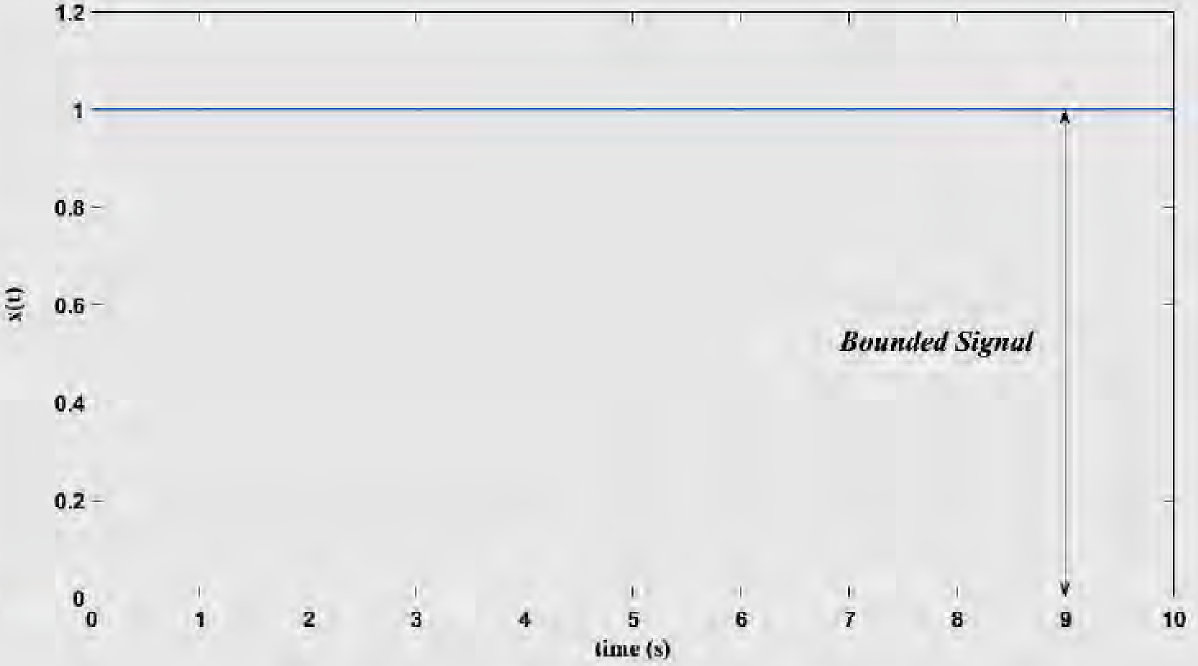


Fig 4.1.4 – Bounded Signal (constant)

Bounded and Unbounded Signals (contd.)

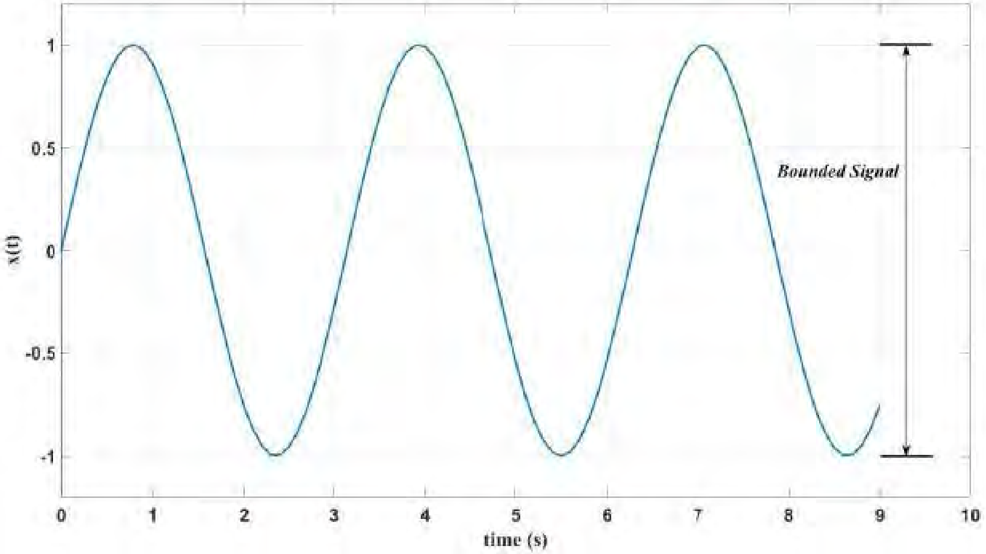


Fig 4.1.5 – Bounded Signal (sinusoid)

Unbounded Signal

Fig 4.1.6

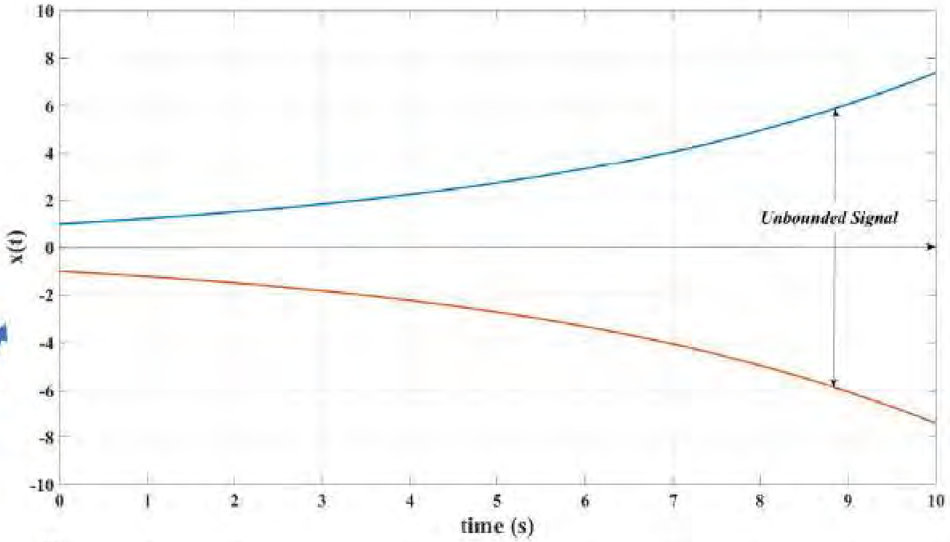
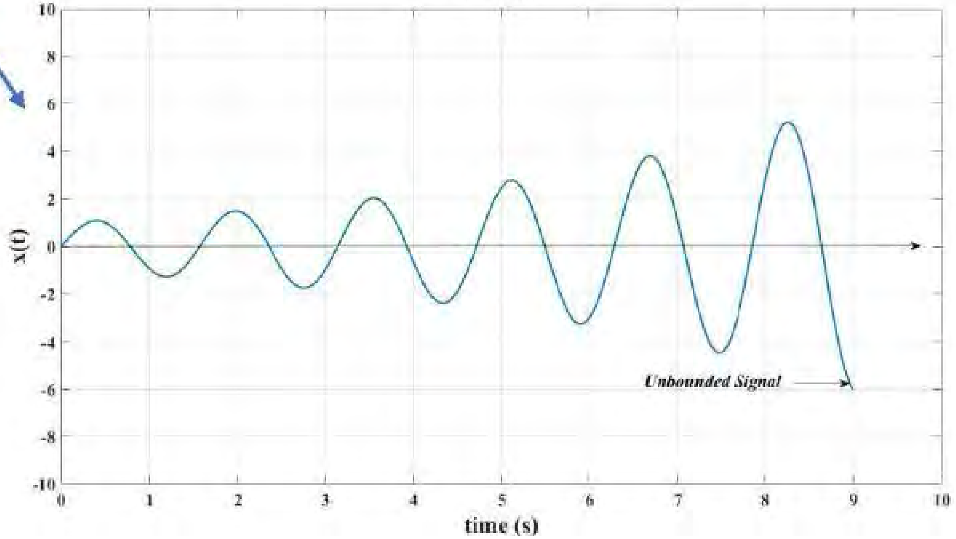


Fig 4.1.7



BIBO Stability

BIBO Stability criterion:

- A system is stable if for every bounded input signal the system response is bounded.
- A system is unstable if for any bounded input signal the system response is unbounded.

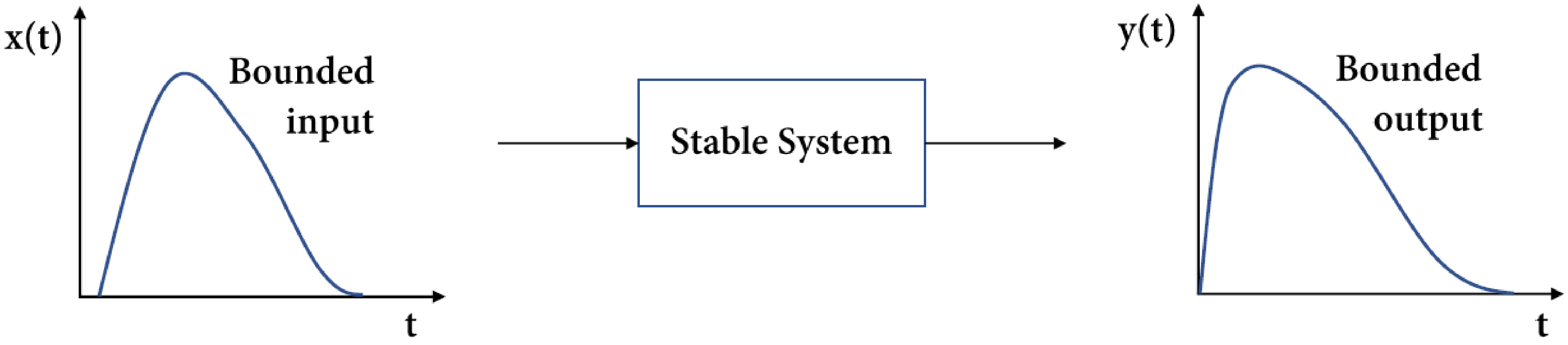


Fig 4.1.8 – Stable Response

BIBO Stability (contd.)

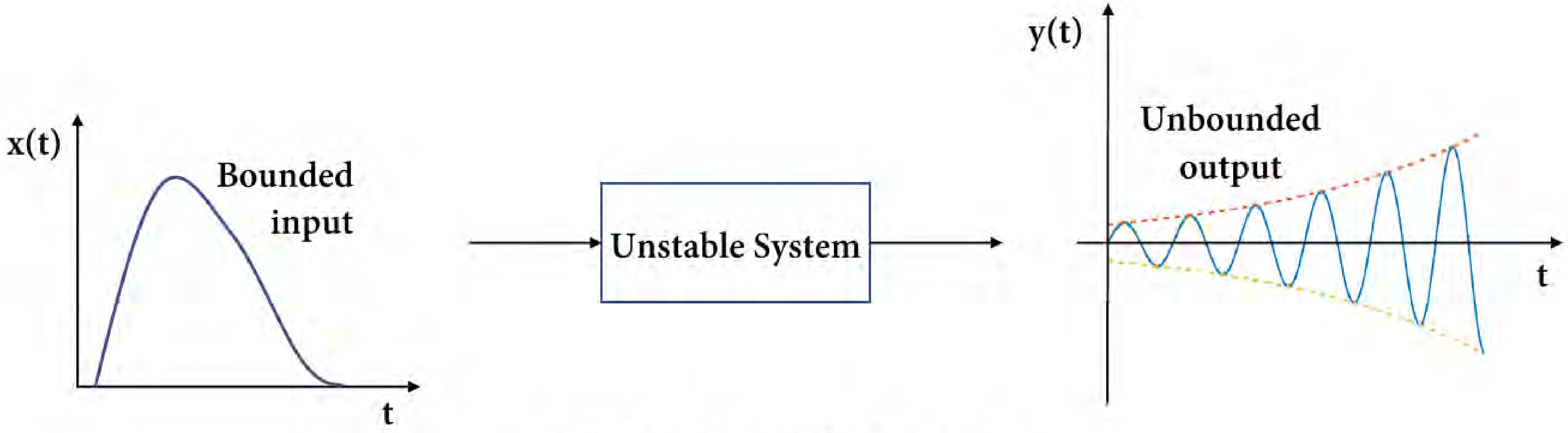


Fig 4.1.9 – Unstable Response

Exponential Stability:

For zero input to a system, if the system response asymptotically decays to zero within a decaying exponential envelope then the system is called exponentially stable.

Note: For linear time invariant systems, exponential stability and BIBO stability are identical.

BIBO Stability - Mathematical form

- Let us consider the system with input $r(t)$, output $c(t)$, and impulse response $g(t)$ as shown in *Fig 4.1.10*.

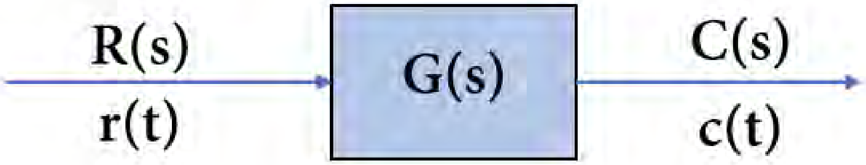


Fig 4.1.10 – General open loop system

- The output of the system in *Fig 4.1.10* is given by

$$C(s) = G(s)R(s) \tag{4.1.1}$$

- Using convolution property of Laplace transform equation (4.1.1) can be written as

$$c(t) = \int_{0}^{\infty} g(\tau)r(t-\tau)d\tau \tag{4.1.2}$$

BIBO Stability - Mathematical form (contd.)

- If $r(t)$ is bounded such that

$$|r(t)| \leq M < \infty \quad 4.1.3$$

then the magnitude of the output satisfies

$$|c(t)| \leq M \int_0^\infty |g(\tau)| d\tau \quad 4.1.4$$

- Therefore, the output is bounded when

$$\int_0^\infty |g(\tau)| d\tau < \infty \quad 4.1.5$$

- The system is BIBO stable if and only if the impulse response of the system is absolutely integrable.

Pole and Zeros

- The transfer function of a general system $G(s)$ can be represented as the ratio of two polynomials as

$$G(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \qquad 4.1.6$$

Poles: Poles of the system $G(s)$ are the roots of the denominator polynomial $D(s)$ i.e. values of s for which $D(s) = 0$.

Zeroes: Zeroes of the system $G(s)$ are the roots of the numerator polynomial $N(s)$ i.e. values of s for which $N(s) = 0$.

- Therefore, the system $G(s)$ represented in equation (4.1.6) has 'n' poles and 'm' zeroes.

Pole and Zeros example

- Let us consider the system with transfer function

$$G(s) = \frac{s^2 + 7s + 12}{s(s^2 + 2s + 5)} \qquad 4.1.7$$

- The zeroes of the system is obtained from

$$s^2 + 7s + 12 = 0 \qquad 4.1.8$$

and, the poles are obtained from

$$s\left(s^2 + 2s + 5\right) = 0 \qquad 4.1.9$$

- Therefore, the zeroes are $z_1 = -3$ and $z_2 = -4$ and the poles are $p_1 = 0$ and $p_{2,3} = -1 \pm j2$.

Poles and Zeros – Example (contd.)

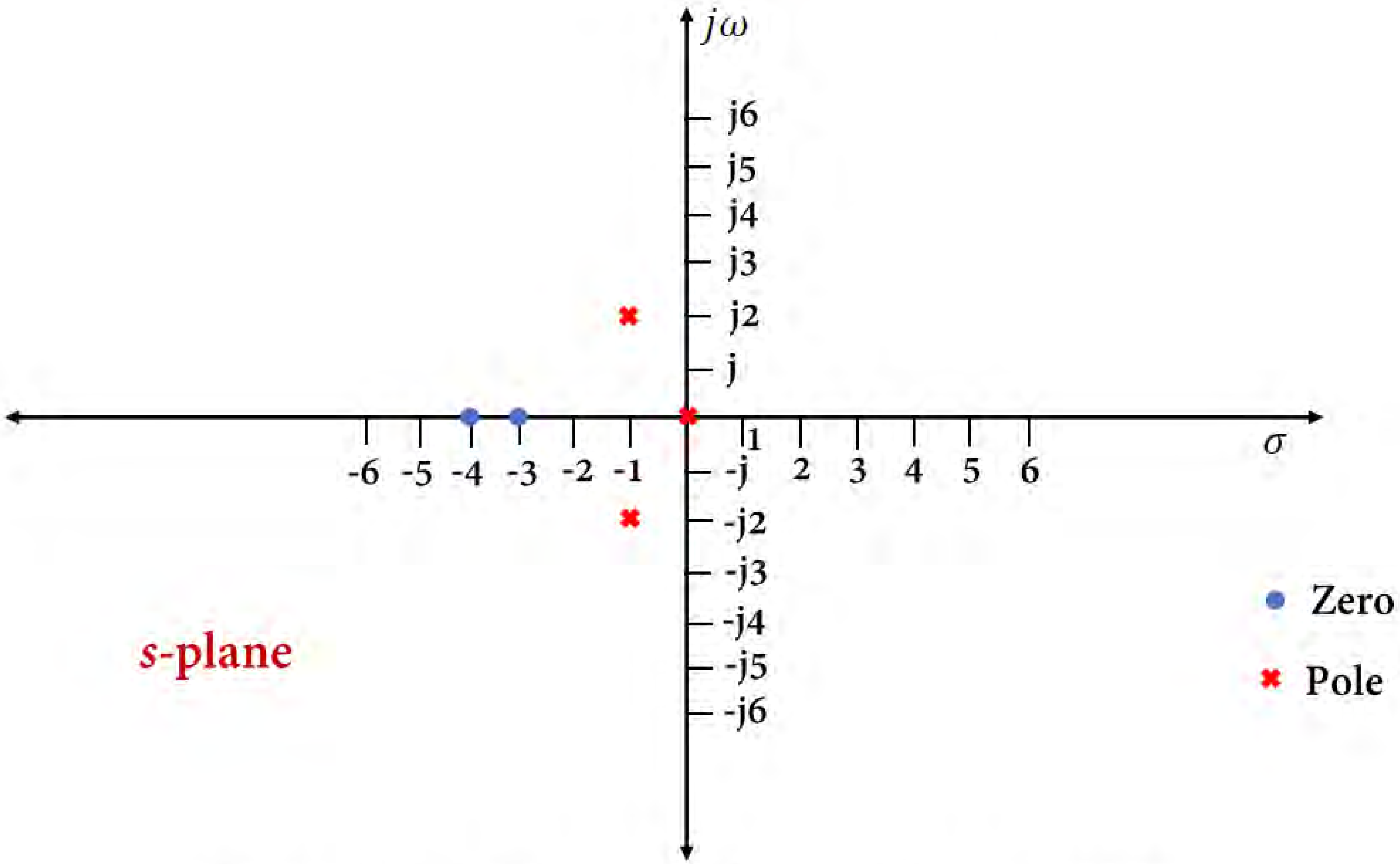


Fig 4.1.11 – Pole-Zero map of system in equation (4.7)

Stability in Frequency domain

- Let us consider the linear time invariant system whose transfer function is given by

$$G(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \qquad 4.1.10$$

- The denominator polynomial $D(s)$ equal to zero i.e. $D(s) = 0$ is called the '*characteristic equation*' of the system.
- The roots of the characteristic equation are known as '*poles*'.
- Assuming the poles of the system are distinct the transfer function can be expressed in partial fraction form as

$$G(s) = \frac{A_1}{s - s_1} + \frac{A_2}{s - s_2} + \cdots + \frac{A_n}{s - s_n} \qquad 4.1.11$$

- The poles can be real, imaginary or complex.

Real Poles

- Let us consider one partial fraction term from the expression of equation (4.11) i.e. $\frac{A_k}{s-s_k}$
- If the pole s_k is real, say $s_k = a$, then the pole lies in the left half plane for $a < 0$, in the right hand plane for $a > 0$ and at the origin for $a = 0$.
- The inverse Laplace transform of the term $\frac{A_k}{s-a}$ is $A_k e^{at} u(t)$.
- When $a < 0$ i.e. the pole lies in the left hand plane the impulse response dies out to zero after finite time, indicating bounded output response. Hence, the system is stable for $a < 0$.

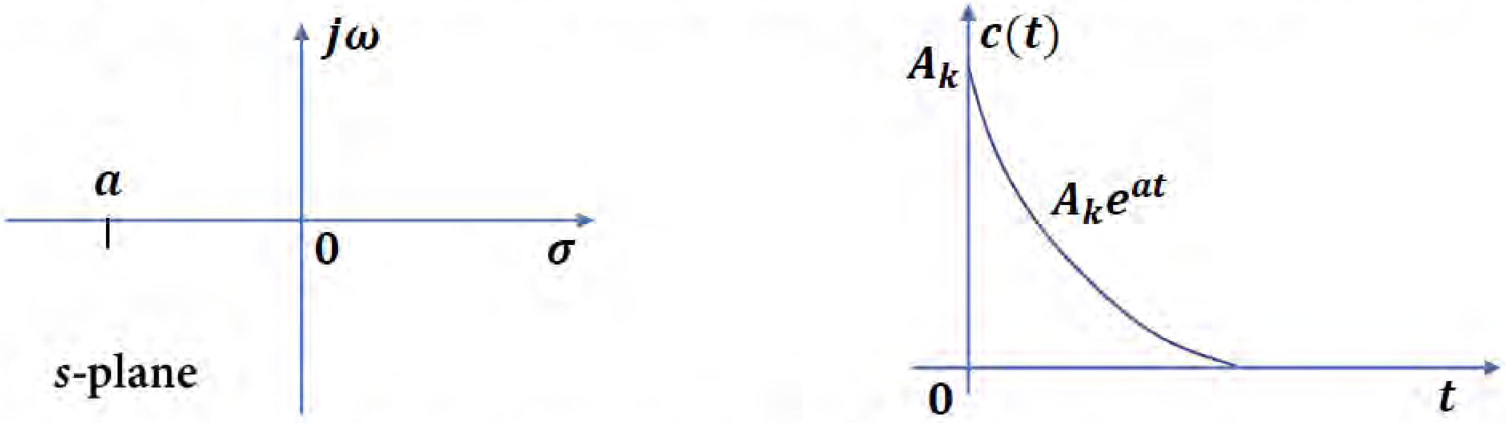


Fig 4.1.12 – System response for $a < 0$

Real Poles (contd.)

- When $a > 0$ i.e. the pole lies in the right hand plane the impulse response increases without any bound as time increases. Hence, the system is unstable for $a > 0$.
- When $a = 0$ i.e. the pole lies at the origin the impulse response is constant with time, indicating bounded output response. The system is not absolutely integrable.

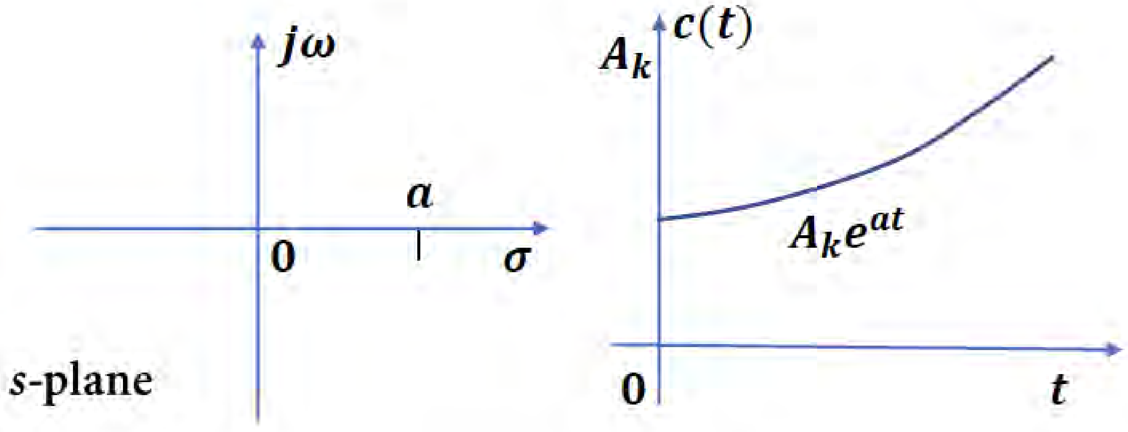


Fig 4.1.13 – System response for $a > 0$

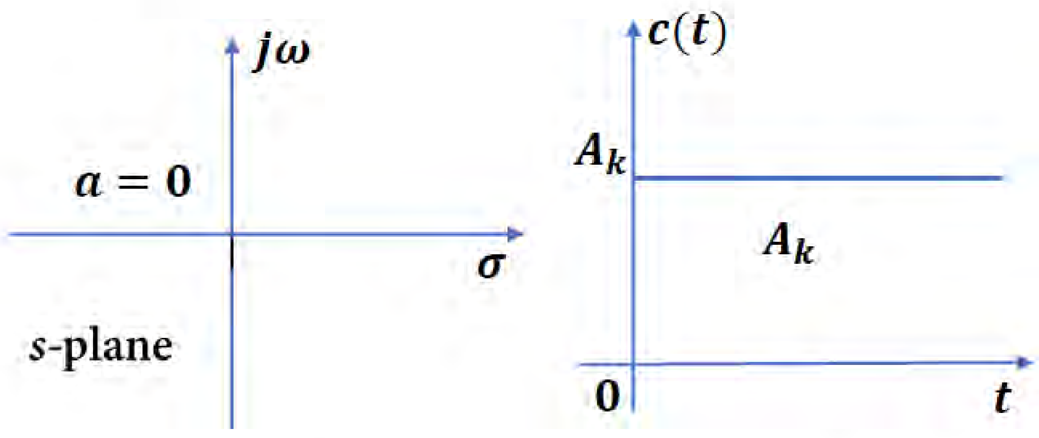


Fig 4.1.14 – System response for $a = 0$

Complex Poles

- If the pole s_k is complex, say $s_k = a \pm jb$, then the poles lie in the left half plane for $a < 0$, in the right hand plane for $a > 0$.
- The partial fraction form of the complex pole is given by

$$G(s) = \frac{A_k}{s - a - jb} + \frac{A_k^*}{s - a + jb} \qquad 4.1.12$$

- Using inverse Laplace transform, the impulse response for the partial fraction form is given by $Ae^{at}\cos(bt + \varphi)$.
- For $a < 0$ i.e. the pole lies in the left hand plane the impulse response decreases exponentially to zero, indicating absolutely integrable impulse response implying BIBO stability. Hence, the system is stable for $a < 0$.
- For $a > 0$ i.e. the pole lies in the right hand plane the impulse response increases without any bound. Hence, the system is unstable for $a > 0$.

Complex Poles (contd.)

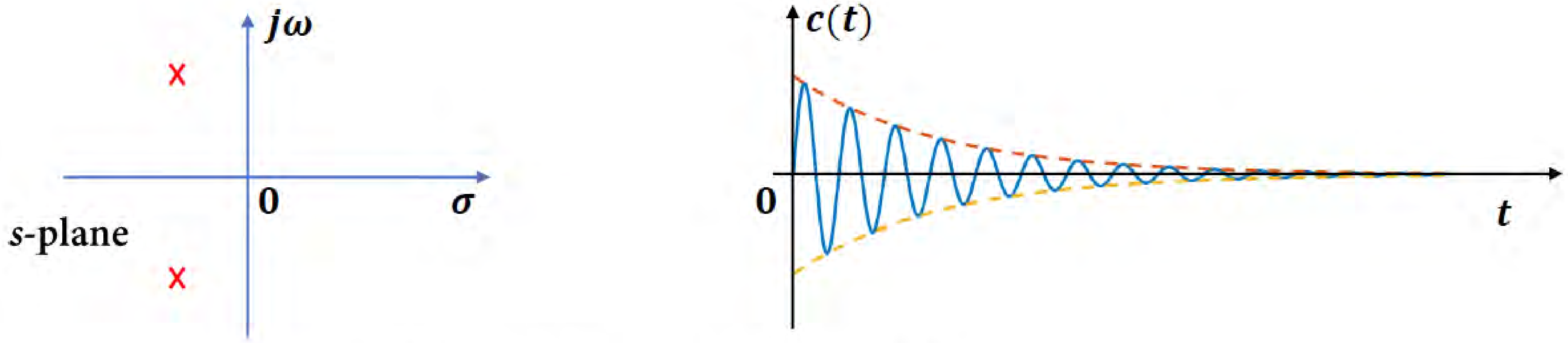


Fig 4.14 – System response for $a < 0$

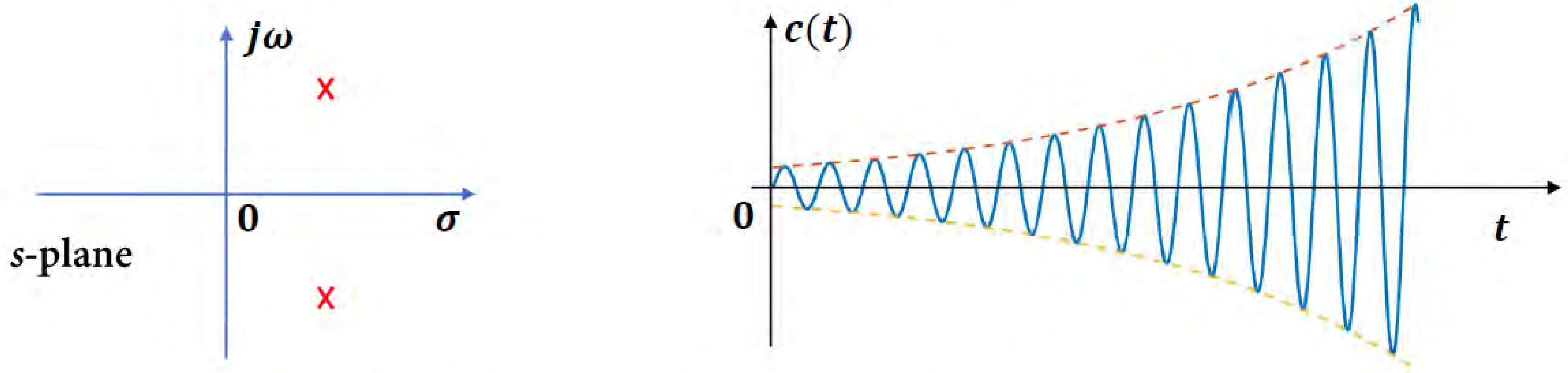


Fig 4.1.15 – System response for $a > 0$

Imaginary Poles

- If the pole s_k is imaginary, say $s_k = \pm jb$, then the poles lie on the imaginary axis jw.
- The partial fraction form of the imaginary poles is given by

$$G(s) = \frac{A_k}{s - jb} + \frac{A_k^*}{s + jb} \quad 4.1.13$$

- The impulse response is of the form $A\cos(bt + \varphi)$ i.e. there is no damping and the response continuously oscillates between the limits.

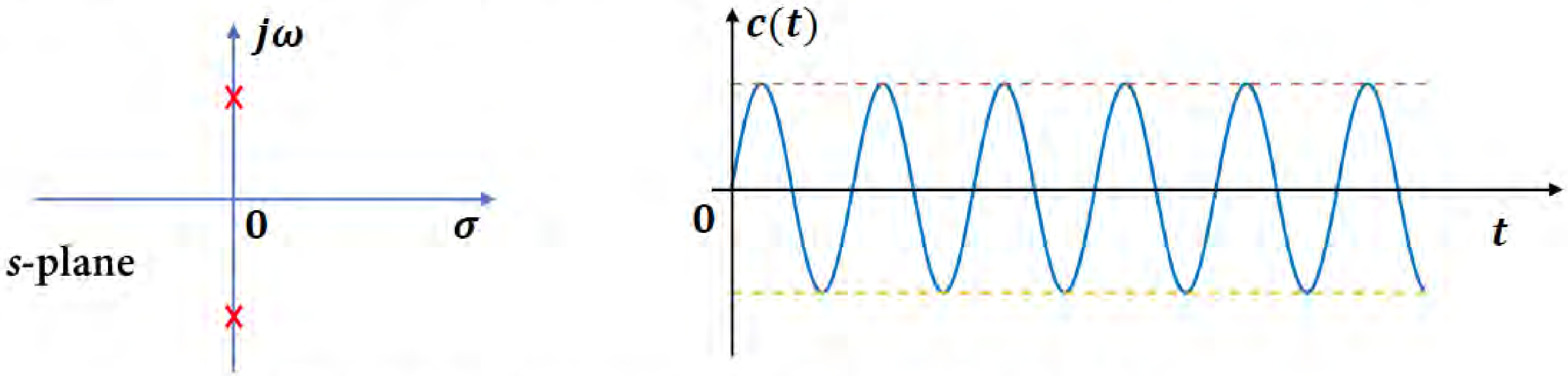


Fig 4.1.16 – System response for imaginary poles

Repeated Poles

- For repeated real poles, say 'r' repeated poles at $s_{k,k+1\cdots,k+r-1} = a$, the impulse response is of the form

$$(A_k + A_{k+1}t + \cdots + A_{k+r-1}t^{r-1})e^{at}$$

- For repeated complex poles, say 'r' repeated poles at $s_{k,k+1\cdots,k+r-1} = a \pm jb$, the impulse response is of the form

$$[A_1 \cos(bt + \varphi_1) + A_2 t \cos(bt + \varphi_2) + \cdots + A_r t^{r-1} \cos(bt + \varphi_r)]e^{at}$$

- For repeated poles at the origin the impulse response is of the form

$$(A_k + A_{k+1}t + \cdots + A_{k+r-1}t^{r-1})$$

- For repeated imaginary poles on the imaginary axis, the impulse response is of the form

$$A_1 \cos(bt + \varphi_1) + A_2 t \cos(bt + \varphi_2) + \cdots + A_r t^{r-1} \cos(bt + \varphi_r)$$

- Presence of repeated poles on imaginary axis makes the system unstable since the response becomes unbounded with increase in time (t).

Stability relations with Poles

- If all the roots of the characteristic equation have negative real parts, then the impulse response is bounded and eventually decreases to zero. Therefore, $\int_0^\infty |g(\tau)|d\tau$ is finite and the system is BIBO stable.

- If any root of the characteristic equation has a positive real part, $g(t)$ is unbounded and $\int_0^\infty |g(\tau)|d\tau$ is infinite. Thus the system is a BIBO unstable system.

- If the characteristic equation has repeated roots on the imaginary axis, $\int_0^\infty |g(\tau)|d\tau$ is infinite and $g(t)$ is unbounded. Hence, the system is a BIBO unstable system.

- If one or more non-repeated roots of the characteristic equation are on the imaginary axis, then $g(t)$ is bounded. However, if the input signal have a common pole on the imaginary axis then the output $c(t)$ becomes unbounded. In absence of any common pole the output is bounded and has sustained oscillations. These kind of systems are called '*marginally stable*'.

Stability relationship with Poles (contd.)

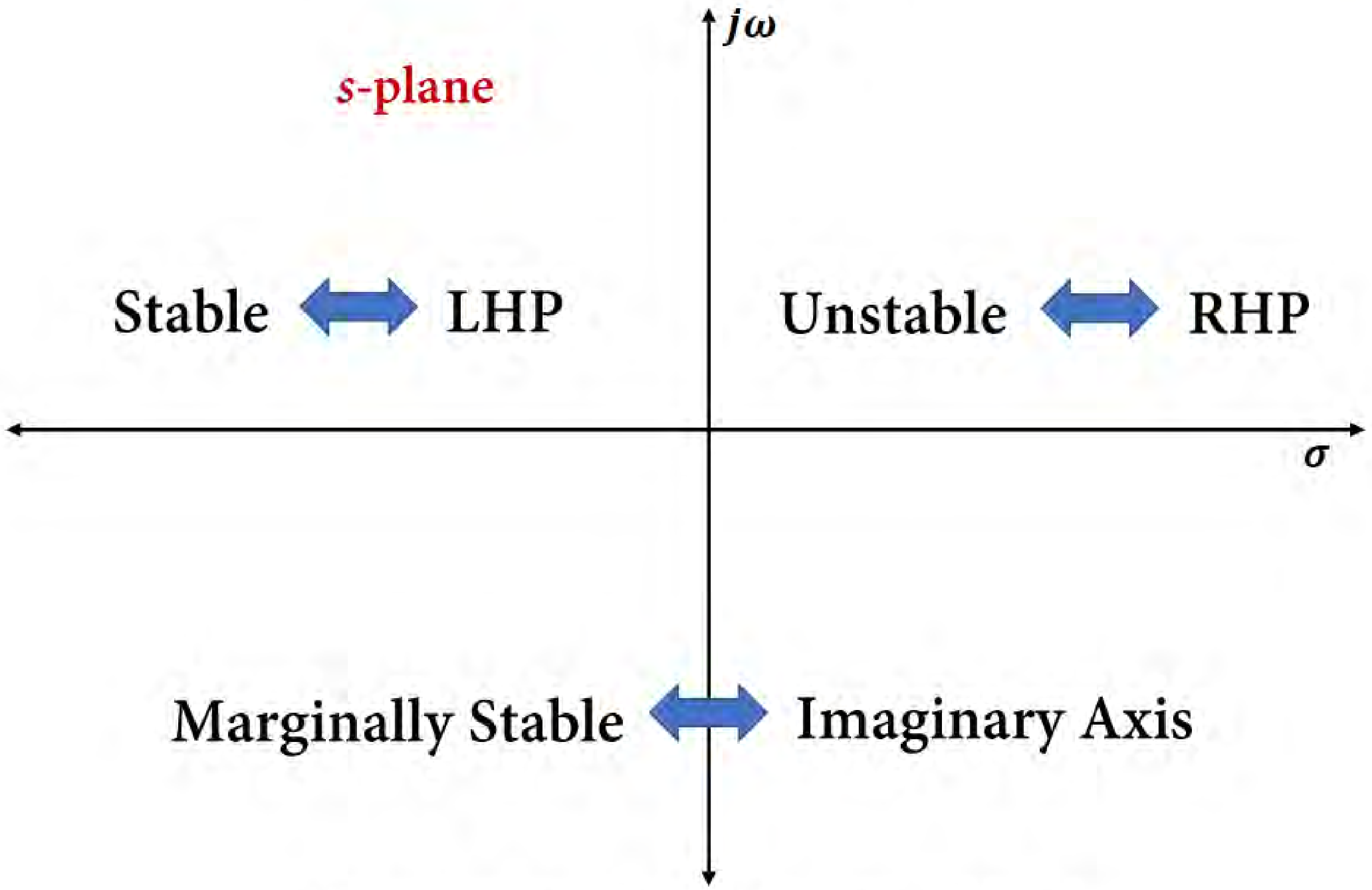


Fig 4.1.17 – Stability relationship with poles

Routh-Hurwitz Criterion: Introduction

- Consider a system with general form of transfer function

$$T(s) = \frac{p(s)}{q(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \qquad 4.2.1$$

- The characteristic equation of the system is given by

$$q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0 \qquad 4.2.2$$

- For stability it is necessary to determine whether any roots of the system lies in the RHP of the *s*-plane.

- The characteristic equation is represented in factored form as

$$q(s) = a_n (s - p_1)(s - p_2) \cdots (s - p_n) = 0 \qquad 4.2.3$$

$$\Rightarrow q(s) = a_n \prod_{i=1}^{n} (s - p_i) = 0 \qquad 4.2.4$$

Introduction (contd.)

- Multiplying all the factors in equation (4.2.4) we get

$$q(s) = a_n s^n - a_n\left(\sum_{i=1}^{n} p_i\right)s^{n-1} + a_n\left(\sum_{i=1,j=1}^{n} p_i p_j\right)s^{n-2} - \cdots + a_n(-1)^n\left(\prod_{i=1}^{n} p_i\right) = 0 \quad 4.2.5$$

- To ensure the roots of the characteristic equation, in equation (4.2.5), lie in the LHP of the *s*-plane, it is necessary that
  I. All the coefficients are non-negative
  II. All the coefficients of the characteristic equation have the same sign
- The above conditions follow from the general properties of a polynomial.

Introduction (contd.)

- It can be shown that

$$\frac{a_{n-1}}{a_n} = -\left(\sum_{i=1}^{n} p_i\right); \ \frac{a_{n-2}}{a_n} = \left(\sum_{i=1,j=1}^{n} p_i p_j\right); \ \cdots; \ \frac{a_0}{a_n} = (-1)^n \left(\prod_{i=1}^{n} p_i\right)$$

- All the ratios should be non-negative and of the same sign for the system to be stable.
- But these conditions are not sufficient to ensure stability.
- Consider a system with characteristic equation given by

$$s^3 + s^2 + 2s + 8 = 0 \qquad 4.2.6$$

- The roots of equation (4.2.6) are $s = -2, 0.5 \pm j1.936$, indicating the system to be unstable while all the coefficients are positive.

Routh-Hurwitz Stability Criterion: A Brief History

- Has origins in the nineteenth century when J. C. Maxwell and others, became interested in the stability of dynamical systems. Maxwell was especially interested in the theoretical analysis of centrifugal governors.
- Invented in 1788 by James Watt for the precise control of his steam engine in the presence of variable loads and variable fuel supply conditions.
- Maxwell was the first to provide a theoretical analysis of such feedback systems using linearized differential equations.

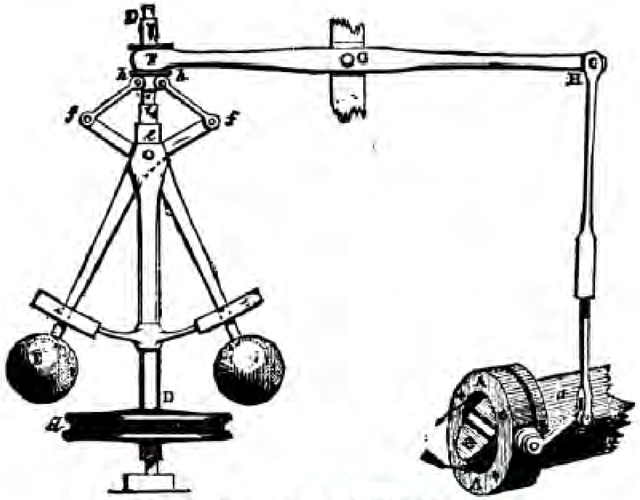


FIG. 4.—*Governor and Throttle-Valve.*

Photos Courtesy : Wikipedia.

"...

[The condition for stability of the governor] is mathematically equivalent to the condition that all the possible roots, and all the possible parts of the impossible roots, of a certain equation shall be negative.

I have not been able completely to determine these conditions for equations of a higher degree than the third; but I hope that the subject will obtain the attention of mathematicians.

..."

- J.C. Maxwell, "On governors," in *Proc. Royal Society of London*, vol 16, pp. 270-283, 1868.

Two Independent historical paths to the stability criterion

- J. C. Maxwell's "On Governors" (1868). → Subject for the Adams prize 1877 with Maxwell as one of the examiners – "The criterion for dynamical stability". → Edward Routh's essay *'A treatise on the stability of a given state of motion'* contained the stability criterion and won the prize.
- A. I. Vyshnegradskii's independent analysis of the steam engine with a governor in 1876. → Results used by A. Stodola (1893) to design water turbine governors. → Adolf Hurwitz at ETH, Zurich arrived at the stability criterion independently of Routh, using different methods.
- To honor these independent efforts their result is known as Routh- Hurwitz stability criterion.

J.C. Maxwell, 1831-1979

Edward Routh, 1831-1907

Adolf Hurwitz, 1859-1919

Routh-Hurwitz Criterion

- The Routh-Hurwitz criterion is a *necessary* and *sufficient* condition for the stability of linear time invariant systems.
- The method requires two step
  i. Generating Routh array
  ii. Interpreting the Routh array for location of poles in the *s*-plane.
- The Routh-Hurwitz criterion states that the number of roots of the characteristic equation with positive real parts is equal to the number of changes in sign of the first column of the Routh array

Routh array

- Consider the characteristic equation as in equation (4.2.2)

$$q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 \qquad 4.2.7$$

- The coefficients of the characteristic equation are arranged as rows in an array as follows

$$\begin{array}{c|cccc} s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \end{array}$$

- The remaining rows are formed by using the following procedure

$$\begin{array}{c|cccc} s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\ s^{n-2} & b_{n-1} & b_{n-2} & b_{n-3} & \cdots \end{array}$$

$$b_{n-1} = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}, b_{n-2} = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}, \cdots$$

Routh array (contd.)

- **Similarly**

$$
\begin{array}{c|cccc}
s^{n} & a_{n} & a_{n-2} & a_{n-4} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\
s^{n-2} & b_{n-1} & b_{n-2} & b_{n-3} & \cdots \\
s^{n-3} & c_{n-1} & c_{n-2} & c_{n-3} & \cdots
\end{array}
$$

$$
c_{n-1} = \frac{b_{n-1}a_{n-3} - a_{n-1}b_{n-2}}{b_{n-1}}, c_{n-2} = \frac{b_{n-1}a_{n-5} - a_{n-1}b_{n-3}}{b_{n-1}}, \cdots
$$

Routh array (contd.)

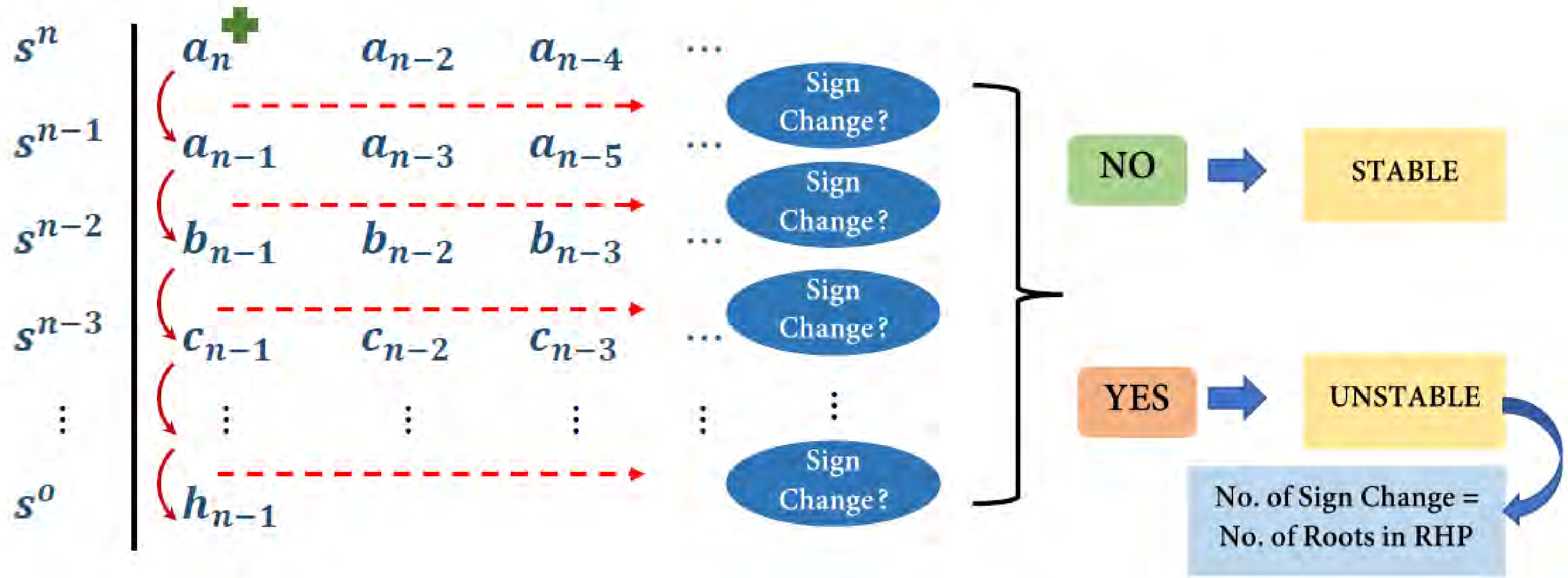
- The process is continued till s^0 and the complete table of array is obtained as shown below

$$
\begin{array}{c|ccccc}
s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\
s^{n-2} & b_{n-1} & b_{n-2} & b_{n-3} & \cdots \\
s^{n-3} & c_{n-1} & c_{n-2} & c_{n-3} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
s^o & h_{n-1} & & &
\end{array}
$$

Interpretation of Routh array

- For a system to be stable it is sufficient that all elements of the first column in the Routh array is positive.

- If the condition is not met, then the system is unstable and the number of roots with positive real part is equal to the number of changes in the sign of the elements of the first column of the array.



Stability analysis- Example 1

- Consider a system with characteristic equation of 3rd order given by

$$q(s) = s^3 + 4s^2 + 9s + 10 = 0 \qquad 4.2.8$$

- The roots of equation (4.2.8) are $s = -2, -1 \pm j2$. All the roots have negative real parts, hence system is stable.

- Let us examine using Routh-Hurwitz criterion

$$\begin{array}{c|cc} s^3 & 1 & 9 \\ s^2 & 4 & 10 \\ s^1 & \frac{26}{4} & \\ s^0 & 10 & \end{array}$$

No Sign Change

No Sign Change

No Sign Change

STABLE

- All the elements in the first column of the Routh Table are positive. Hence, the system is stable.

Stability analysis- Example 2

- Consider a system with characteristic equation of 3rd order given by

$$q(s) = s^3 + s^2 + 3s - 5 = 0 \tag{4.2.9}$$

- The roots of equation (4.2.9) are $s = 1, -1 \pm j2$. One root lie in the RHP. Hence the system is unstable.

- Let us examine using Routh-Hurwitz criterion

- Not all the elements in the first column of the Routh Table are positive. Hence, the system is unstable. There is one sign change which indicates one pole in the RHP.

Special Case 1 – Zero in the first column

- Consider a system with characteristic equation

$$s^5 + s^4 + 2s^3 + 2s^2 + 3s + 15 = 0 \qquad 4.2.10$$

- Form the Routh table of equation (4.2.10) as

$$\begin{array}{c|ccc} s^5 & 1 & 2 & 3 \\ s^4 & 1 & 2 & 15 \\ s^3 & 0 & -12 & \\ s^2 & \text{Undefined} & & \end{array}$$

- If the first element of any row in the Routh array is zero, the zero is replaced by a small positive number, say ϵ.
- The value of ϵ is allowed to approach zero and the sign of the entries in the Routh table is interpreted for stability.

Special Case 1

- Consider a system with characteristic equation

$$s^5 + s^4 + 2s^3 + 2s^2 + 3s + 15 = 0 \tag{4.2.10}$$

- Form the Routh table of equation (4.2.10) as

s^5	1	2	3
s^4	1	2	15
s^3	0	−12	
s^2	Undefined		

- First term becomes undefined.
- The zero in the first column is replaced with a small positive number ϵ.

Special Case 1- Example (contd.)

- The Routh table is modified as

s^5	$+$	1	2	3
s^4	$+$	1	2	15
s^3	$+$	ϵ	-12	
s^2	$+$	$\frac{2\epsilon+12}{\epsilon}$	15	
	Sign Change			
s^1	$-$	$-\frac{15\epsilon^2+24\epsilon+144}{2\epsilon+12}$		
	Sign Change			
s^0	$+$	15		

- When $\epsilon \to 0$, the first element of 4^{th} row is positive while the first element of 5^{th} row is negative.
- One element of the first column is negative. Hence the system is unstable.
- There are two sign changes in the first column. So two poles of the system lie in the RHP.

Special Case 2- Entire row is zero

- Consider a system with characteristic equation

$$s^3 + 5s^2 + 6s + 30 = 0 \tag{4.2.11}$$

- Form the Routh table of equation (4.2.11) as

$$\begin{array}{c|cc} s^3 & 1 & 6 \\ s^2 & 5 & 30 \\ s^1 & \boxed{0 \quad\quad 0} & \end{array}$$

- If an entire row is zero an auxiliary polynomial is formed with the entries of the row immediately above.
- The auxiliary polynomial is differentiated w.r.t '*s*'.
- The row of zeros is replaced with the coefficients of the derivative of the auxiliary polynomial and the Routh table is interpreted along with the roots of the auxiliary polynomial.

Special Case 2- Example

- Consider a system with characteristic equation

$$s^3 + 5s^2 + 6s + 30 = 0 \tag{4.2.11}$$

- Form the Routh table of equation (4.2.11) as

$$\begin{array}{c|cc} s^3 & 1 & 6 \\ s^2 & 5 & 30 \\ s^1 & \boxed{0 \quad\quad 0} \end{array}$$

- Entire row for s^1 is zero.
- The auxiliary equation is given by

$$A(s) = 5s^2 + 30 = 0 \tag{4.2.12}$$

$$\therefore \frac{dA(s)}{ds} = 10s + 0 \tag{4.2.13}$$

Special Case 2- Example (contd.)

- Form the Routh table of equation (4.2.11) as the third row is replaced with coefficients 10 and 0.

- All the elements in the first column of the Routh Table are positive. Hence, the system is marginally stable since the roots of the auxiliary equation (4.2.12) lie on the imaginary axis.

Auxiliary Polynomial

- If the auxiliary equation is an even polynomial in which the exponents of s are even integers or zero, the roots are symmetric about the origin.
- The symmetry occurs under the conditions
  i. the roots are symmetric and real
  ii. the roots are symmetric and imaginary
  iii. the roots are quadrantal about the two axes of the s-plane
- Roots of the auxiliary equation are the roots of the characteristic equation

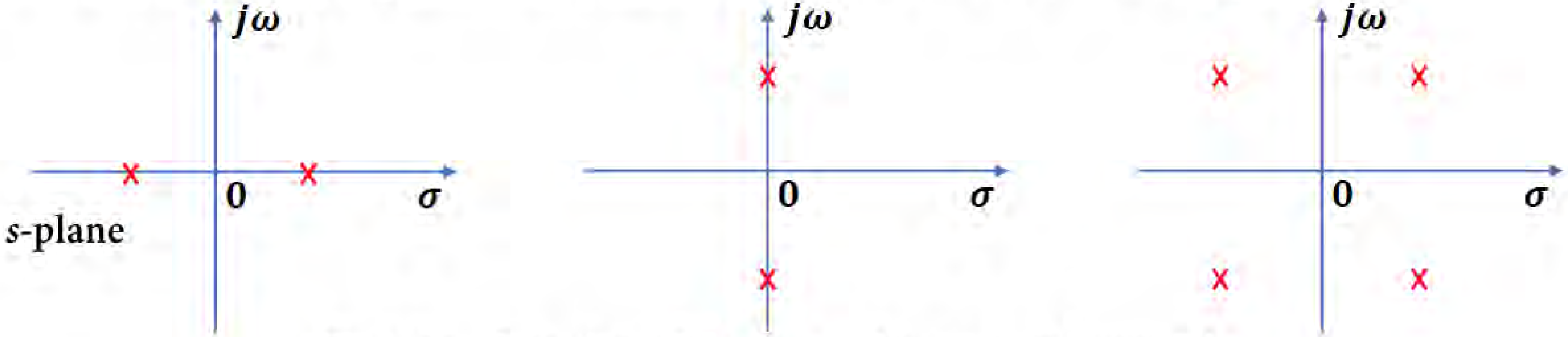


Fig 4.2.1 – Location of the roots of an even polynomial

Auxiliary Equation - Example

- Consider a system with characteristic equation

$$s^5 + 2s^4 + 2s^3 + 4s^2 + s + 2 = 0 \tag{4.2.14}$$

- Form the Routh table of equation (4.2.14) as

$$\begin{array}{c|ccc} s^5 & 1 & 2 & 1 \\ s^4 & 2 & 4 & 2 \\ s^3 & 0 & 0 & 0 \end{array}$$

- Entire row for s^3 is zero.
- The auxiliary equation is given by

$$A(s) = s^4 + 2s^2 + 1 = 0 \tag{4.2.15}$$

$$\therefore \frac{dA(s)}{ds} = 4s^3 + 4s + 0 \tag{4.2.16}$$

Auxiliary Equation - Example (Contd.)

- The 3rd row is replaced with the coefficients of the derivative of auxiliary equation $A(s) = 0$
- The Routh table of equation (4.2.14) becomes

$$\begin{array}{c|ccc} s^5 & 1 & 2 & 1 \\ s^4 & 2 & 4 & 2 \\ s^3 & 1 & 1 & 0 \\ s^2 & 1 & 1 & \\ s^1 & 0 & 0 & \end{array}$$

- Again the entire row for s^1 is zero.
- The auxiliary equation is given by

$$A'(s) = s^2 + 1 = 0 \qquad 4.2.17$$

$$\therefore \frac{dA'(s)}{ds} = 2s^1 + 0 \qquad 4.2.18$$

Auxiliary Equation - Example (Contd.)

- The 5^{th} row is replaced with the coefficients of the derivative of auxiliary equation $A'(s) = 0$
- The Routh table of equation (4.2.14) becomes

s^5	1	2	1
s^4	2	4	2
s^3	1	1	0
s^2	1	1	
s^1	2		
s^0	1		

- There is no sign change in the first column. The roots of the auxiliary equation (4.2.15) are $+j, -j, +j, -j$ i.e. repeated roots on the imaginary axis. Hence, the system is unstable.

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