Discrete Structures and Graph Theory

Connectives

1. Negation (Not)
2. Conjunction (and)
3. Disjunction (or)
4. Conditional (if…then) /implication
5. Bi-conditional (if and only if)

Negation (NOT)

- Statements Formed by introducing “not” word
- “P” is Statement then negation of p is written as “not p“ or It is not case that P.
- $\neg p$
- Unary Connective
- If P is true then $\neg p$ is false and vice versa.

P	**¬P**
T	F
F	T

P:London is a city.

Then

¬ p: London is not a city.

OR

¬ p: It is not the case that London is a city.

Q: I went to my class yesterday

Then

¬ Q:I did not go to my class yesterday

Conjunction (and)

- **Statements Formed by introducing "and" word**
- **Binary Connective**
- **Used to combine two or more statements.**
- **Denote by** $\wedge$
- If both the statements are true then **p** $\wedge$ **Q** is true otherwise false.

P	Q	P$\wedge$Q
T	T	T
T	F	F
F	T	F
F	F	F

P:London is a capital of India.

Q: India is country.

London is a capital of India and India is country.

$P \wedge Q$

Disjunction (OR)

- Statements Formed by introducing "OR" word
- Binary Connective
- Denote by $\vee$
- If one statement is true then $p \vee Q$ is true otherwise false.

P	Q	P∨Q
T	T	T
T	F	T
F	T	T
F	F	F

P:London is a capital of India.

Q: India is country.

London is a capital of India or India is country.

$P \vee Q$

Conditional (if..then)

- Statements Formed by introducing "if…then" word
- Binary Connective
- Denote by $\rightarrow$
- If First statement is true and second statement is false then $p \rightarrow Q$ is false otherwise true.

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

"If elephants were red, then they could hide in cherry trees.".

$$P \to Q$$

P is known as Antecedent

Q is known as consequent

For Q $\to$ P, vice versa

Implication

- If you study regularly you then you will get grade ‘A’

Case 1 : You did regular study , you got A grade.

$(P \rightarrow Q)$: True

Case 2: You did regular study ,by chance you didn’t get grade A. $(P \rightarrow Q)$: False

Case 3: You didn’t study regularly, you may get grade A. $(P \rightarrow Q)$: True

Case 4: You didn’t study regularly, you didn’t get grade A. $(P \rightarrow Q)$: True

Some reading for P->Q

- "*p* implies *q*"
- "if *p*, then *q*"
- "if *p*, *q*"
- "when *p*, *q*"
- "whenever *p*, *q*"
- "*q* if *p*"
- "*q* when *p*"
- "*q* whenever *p*"
- "*p* only if *q*"
- "*p* is sufficient for *q*"
- "*q* is necessary for *p*"
- "*q* follows from *p*"
- "*q* is implied by *p*"

We will see some equivalent logic expressions later.

Bi-conditional (if and only if)

- **Statements Formed by introducing "if and only if " word**
- **Binary Connective**
- **Denote by ↔**
- If both the statement has same truth value then **p ↔ Q** is true otherwise false.

P	Q	P↔Q
T	T	T
T	F	F
F	T	F
F	F	T

"$x < y$ if and only if $y > x$."

$$P \leftrightarrow Q$$

EX-OR (Either-Or)

- Statement formed by “Either Or” word.
- Exclusive Or
- P x Q proposition will be true, if exactly one of two propositions of both is true. Otherwise false

P	Q	P $\oplus$ Q
T	T	F
T	F	T
F	T	T
F	F	F

Inclusive or OR Exclusive or

- In order to get a job in this multinational company , experience with C++ **or** Java is mandatory.

Inclusive or OR Exclusive or

- In order to get a job in this multinational company , experience with C++ **or** Java is mandatory.

Inclusive OR

Disjunction

Inclusive or OR Exclusive or

- “When you buy a mobile of xyz company, you get Rs.500 cashback or a mobile cover of worth Rs.500.”

Inclusive or OR Exclusive or

- “When you buy a mobile of xyz company, you get Rs.500 cashback **or** a mobile cover of worth Rs.500.”

Exclusive OR

Statement Formula and Truth Table

- Atomic statements/proposition
- Compound statements/proposition

$\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$, $\neg(P \wedge Q)$, $\neg(P \wedge Q)$

- Statement formula
- Truth Table
- 2^n where n is number of distinct statement variable
- $P \wedge \neg P$

    2 rows, n=1, 2^1

- $(P \wedge Q)$

    4 rows, n=2, 2^2

- Statements and operators (Connectives and parenthesis) can be combined in any way to form new statements.
- $(\neg P) \vee (\neg Q)$

P	Q			
T	T			
T	F			
F	T			
F	F			

- Statements and operators can be combined in any way to form new statements.
- $(\neg P)\vee(\neg Q)$

P	Q	$\neg P$		
T	T	F		
T	F	F		
F	T	T		
F	F	T		

- Statements and operators can be combined in any way to form new statements.
- $(\neg P) \vee (\neg Q)$

P	Q	$\neg P$	$\neg Q$	
T	T	F	F	
T	F	F	T	
F	T	T	F	
F	F	T	T	

- Statements and operators can be combined in any way to form new statements.
- $(\neg P)\vee(\neg Q)$

P	Q	$\neg P$	$\neg Q$	$(\neg P)\vee(\neg Q)$
T	T	F	F	
T	F	F	T	
F	T	T	F	
F	F	T	T	

- Statements and operators can be combined in any way to form new statements.
- $(\neg P)\vee(\neg Q)$

P	Q	$\neg P$	$\neg Q$	$(\neg P)\vee(\neg Q)$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

$$\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$$

P	Q		
T	T		
T	F		
F	T		
F	F		

$$\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$$

P	Q	$\neg$P	$\neg$Q
T	T	F	F
T	F	F	T
F	T	T	F
F	F	T	T

$$\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$$

P	Q	$\neg P$	$\neg Q$
T	T	F	F
T	F	F	T
F	T	T	F
F	F	T	T

$(P \wedge Q)$			
T			
T			
T			
F			

$\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$

P	Q	$\neg P$	$\neg Q$
T	T	F	F
T	F	F	T
F	T	T	F
F	F	T	T

$(P \wedge Q)$	$\neg(P \wedge Q)$		
T	F		
T	F		
T	F		
F	T		

$\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$

P	Q	$\neg P$	$\neg Q$
T	T	F	F
T	F	F	T
F	T	T	F
F	F	T	T

$(P \wedge Q)$	$\neg(P \wedge Q)$	$(\neg P) \vee (\neg Q)$	
T	F	F	
T	F	T	
T	F	T	
F	T	T	

$\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$

P	Q	$\neg P$	$\neg Q$
T	T	F	F
T	F	F	T
F	T	T	F
F	F	T	T

$(P \wedge Q)$	$\neg(P \wedge Q)$	$(\neg P) \vee (\neg Q)$	$\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$
T	F	F	T
F	T	T	T
F	T	T	T
F	T	T	T

$\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$

P	Q	$\neg P$	$\neg Q$
T	T	F	F
T	F	F	T
F	T	T	F
F	F	T	T

$(P \wedge Q)$	$\neg(P \wedge Q)$	$(\neg P) \vee (\neg Q)$	$\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$
T	F	F	T
F	T	T	T
F	T	T	T
F	T	T	T

Example

- Using the statements:

  R:Mark is Rich.

  H:Mark is happy
- Write the following statements in **symbolic form**:
- *(a) Mark is poor but happy.*

  $\neg R \wedge H$
- *(b) Mark is rich or unhappy;*

  $R \vee \neg H$
- *(c) Mark is neither rich nor happy.*

  $\neg R \wedge \neg H$
- *(d) Mark is poor or he is both rich and unhappy.*

  $\neg R \vee (R \wedge \neg H)$

Example

- Let *p be "It is cold"* and let *q be "It is raining". Give a simple* ***verbal sentence*** *which describes each of the following statements:*
- *(a)* $\neg p$; *(b)* $p \wedge q$; *(c)* $p \vee q$; *(d)* $q \vee \neg p$.
- *(a)* $\neg p$;

  It is not cold.
- *(b)* $p \wedge q$;

  It is cold and raining.
- *(c)* $p \vee q$;

  It is cold or it is raining
- *(d)* $q \vee \neg p$.

  It is raining or it is not cold.

Example 1.17 There are two restaurants next to each other. One has a sign that says, "Good food is not cheap," and the other has a sign that says, "Cheap food is not good." Are the signs saying the same thing?

Using the statements:

P:Food is good.

H:Food is cheap.

Good food is not cheap.

$P \to \neg H$

Cheap food is not good.

$H \to \neg P$

$H \to \neg P$

P	H	$\neg P$	$\neg H$	$P \to \neg H$	$H \to \neg P$
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

WFF (well formed formula)

- Now consider the proposition : $P \vee \sim Q \rightarrow P \wedge R$

Trying to construct a truth table for this is quite confusing. Which is to be assumed?

$$(P \vee \sim Q) \rightarrow (P \wedge R) \text{ or } P \vee (\sim Q \rightarrow P) \wedge R$$

Which part is calculated first?

for such cases we have order of precendence for these operators.

WFF (well formed formula)

- A statement formula is said to be WFF if it has :

1. Every Atomic statement is wff
2. If P is wff then ~ p is also wff
3. If P and Q are wff then $(P \wedge Q)$, $(P \vee Q)$, and $(P \rightarrow Q)$ are wff
4. Nothing else is wff

For example: $((P \wedge Q) \vee R)$ is wff w, where as $P \vee Q \wedge R$ is not a wff

Precedence of the operators

- $\sim$
- $\wedge$
- V, $\oplus$
- $\rightarrow$
- $\longleftrightarrow$

For example ,

$\sim P \wedge Q \rightarrow R \vee Q$ is not a wff,

can be converted to wff by using rules of precedence as $((\sim P) \wedge Q) \rightarrow (R \vee Q))$

Equivalent Statements

- If truth values of statement formula/proposition A is equal to the truth values of statement formula/proposition B for every possible truth values then A and B are logically equivalent to each other.

P	Q	$\neg P$	$\neg Q$	$\neg(P \wedge Q)$	$(\neg P) \vee (\neg Q)$
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

Denoted by symbol $\Leftrightarrow$

- *Let P be "Roses are red" and Q be "Violets are blue." Let S be the statement:*

"It is not true that roses are red and violets are blue."

- *Then S can be written in the form* $\neg(p \wedge q)$.
- *Accordingly, S* has the same meaning as the statement:

"Roses are not red, or violets are not blue."

Then S can be written in the form $\neg p \vee \neg q$.

However, as noted above, $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$.

Equivalent Statements

- The statements $\neg(P \wedge Q)$ and $(\neg P) \vee (\neg Q)$ are logically equivalent, since $\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$ is always true.

P	Q	$\neg P$	$\neg Q$	$\neg(P \wedge Q)$	$(\neg P) \vee (\neg Q)$	$\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$
T	T	F	F	F	F	T
T	F	F	T	T	T	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Convert the following English statements in symbolic form.

- You can access the internet from campus if you are computer science major or you are not a freshman.

Solution: P: You can access the internet from campus.

Q: you are computer major.

R: you are a freshman.

$P \rightarrow (Q \vee \neg R)$

- You can ride on roller coaster if you are under 4 feet tall unless you are older than 16 years old.

Solution :

P: You can ride on roller coaster

Q: You are under 4 feet

R: You are older than 16 years old.

$(Q \vee \neg R) \rightarrow P$

Logical Equivalence

The easiest way to check for logical equivalence is to see if the truth tables of both variants have *identical last columns*:

p	q	$p \rightarrow q$

p	q	$\neg q$	$\neg p$	$\neg q \rightarrow \neg p$

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p	q	$p \rightarrow q$
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

p	q	$\neg q$	$\neg p$	$\neg q \rightarrow \neg p$

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T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

p	q	$\neg q$	$\neg p$	$\neg q \rightarrow \neg p$
T	T			
T	F			
F	T			
F	F			

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p	q	$p \rightarrow q$
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

p	q	$\neg q$	$\neg p$	$\neg q \rightarrow \neg p$
T	T	F		
T	F	T		
F	T	F		
F	F	T		

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T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

p	q	$\neg q$	$\neg p$	$\neg q \rightarrow \neg p$
T	T	F	F	
T	F	T	F	
F	T	F	T	
F	F	T	T	

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p	q	$p \rightarrow q$
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

p	q	$\neg q$	$\neg p$	$\neg q \rightarrow \neg p$
T	T	F	F	**T**
T	F	T	F	**F**
F	T	F	T	**T**
F	F	T	T	**T**

Exercises

- Prove that:

1) $(P \rightarrow Q) \Leftrightarrow \neg P \vee Q$

2) $P \rightarrow \mathbf{(}Q \rightarrow \mathbf{R)} \Leftrightarrow (P \wedge Q) \rightarrow \mathbf{R}$.

Tautologies and Contradictions

- Some propositions P *contain only T in the last column of their truth tables or, in other words,* they are true for any truth values of their variables. Such propositions are called *tautologies.* A tautology is a statement that is always true.

Examples:

- $R \vee (\neg R)$

∀ $\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$

- If $S \rightarrow T$ is a tautology, we write $S \Rightarrow T$.
- If $S \leftrightarrow T$ is a tautology, we write $S \Leftrightarrow T$.

$$\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$$

P	Q	$\neg P$	$\neg Q$
T	T	F	F
T	F	F	T
F	T	T	F
F	F	T	T

$(P \wedge Q)$	$\neg(P \wedge Q)$	$(\neg P) \vee (\neg Q)$	$\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$
T	F	F	T
F	T	T	T
F	T	T	T
F	T	T	T

Tautology by truth table

p	q	$\neg p$	$p \vee q$	$\neg p \wedge (p \vee q)$	$[\neg p \wedge (p \vee q)] \rightarrow q$
T	T				
T	F				
F	T				
F	F				

Tautology by truth table

p	q	$\neg p$	$p \vee q$	$\neg p \wedge (p \vee q)$	$[\neg p \wedge (p \vee q)] \to q$
T	T	F			
T	F	F			
F	T	T			
F	F	T			

Tautology by truth table

p	q	$\neg p$	$p \vee q$	$\neg p \wedge (p \vee q)$	$[\neg p \wedge (p \vee q)] \rightarrow q$
T	T	F	T		
T	F	F	T		
F	T	T	T		
F	F	T	F		

Tautology by truth table

p	q	$\neg p$	$p \vee q$	$\neg p \wedge (p \vee q)$	$[\neg p \wedge (p \vee q)] \rightarrow q$
T	T	F	T	F	
T	F	F	T	F	
F	T	T	T	T	
F	F	T	F	F	

Tautology by truth table

p	q	$\neg p$	$p \vee q$	$\neg p \wedge (p \vee q)$	$[\neg p \wedge (p \vee q)] \rightarrow q$
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

Tautologies and Contradictions

- a proposition *P is called* *a contradiction* *if it contains only* *F* *in the* *last column* *of its truth table* or, *in other words*, if it is false for any truth values of its variables.
- *A* contradiction is a statement that is always false.

Examples:

- $R \wedge (\neg R)$

∀ $\neg(\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q))$

- The negation of any tautology is a contradiction, and the negation of any contradiction is a tautology.

- Two way to finding the Equivalences, Tautology and Contradiction
- Truth Table
- Without Truth Table Using Substitution (by formulas)

P	Q	$\neg P$	$\neg Q$	$\neg P \vee Q$	$P \to Q$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Logical Equivalences

- Identity Laws: $p \wedge \mathbf{T} \Leftrightarrow p$ and $p \vee \mathbf{F} \Leftrightarrow p$.
- Domination Laws: $p \vee \mathbf{T} \Leftrightarrow \mathbf{T}$ and $p \wedge \mathbf{F} \Leftrightarrow \mathbf{F}$.
- Idempotent Laws: $p \wedge p \Leftrightarrow p$ and $p \vee p \Leftrightarrow p$.
- Double Negation Law: $\neg(\neg p) \Leftrightarrow p$.
- Commutative Laws:
    - $(p \vee q) \Leftrightarrow (q \vee p)$ and $(p \wedge q) \Leftrightarrow (q \wedge p)$.
- Associative Laws: $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$
    - and $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$.

Logical Equivalences

- Distributive Laws:
  - $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ and
  - $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$.
- DeMorgan's Laws:
  - $\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$ and
  - $\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$.
- Absorption Laws:
  - $p \vee (p \wedge q) \Leftrightarrow p$ and $p \wedge (p \vee q) \Leftrightarrow p$.
- Negation Laws: $p \vee \neg p \Leftrightarrow \mathbf{T}$ and $p \wedge \neg p \Leftrightarrow \mathbf{F}$.

Logical Equivalences for Implication

$$p \to q \equiv \neg p \lor q$$

$$p \to q \equiv \neg q \to \neg p$$

$$p \lor q \equiv \neg p \to q$$

$$p \land q \equiv \neg(p \to \neg q)$$

$$\neg(p \to q) \equiv p \land \neg q$$

$$(p \to q) \land (p \to r) \equiv p \to (q \land r)$$

$$(p \to r) \land (q \to r) \equiv (p \lor q) \to r$$

$$(p \to q) \lor (p \to r) \equiv p \to (q \lor r)$$

$$(p \to r) \lor (q \to r) \equiv (p \land q) \to r$$

Logical Equivalences for Double Implication

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$

$$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$

$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

Substitution instance

- A formula A is called substitution instance of formula B if A can be obtained from B by substituting formulas for some variable of B.

Examples:

- B:$P \rightarrow (J \wedge P)$
- If P be $R \leftrightarrow S$
- A:$(R \leftrightarrow S) \rightarrow (J \wedge (R \leftrightarrow S))$
- **As like we can substitute the formula with another formula if both have same truth values**
- $(R \rightarrow S) \wedge (R \leftrightarrow S)$
- $(\neg R \vee S) \wedge (R \leftrightarrow S)$
- Equivalent formula can be substitute for each other.

- Prove that $P \rightarrow \mathbf{(}Q \rightarrow \mathbf{R)} \Leftrightarrow (P \wedge Q) \rightarrow \mathbf{R}$.
- $P \rightarrow \mathbf{(}Q \rightarrow \mathbf{R)} \Leftrightarrow P \rightarrow \mathbf{(}\neg Q \vee R\mathbf{)}$ **…..implication law**

$\Leftrightarrow \neg P \vee (\neg Q \vee R)$ **..implication law**

$\Leftrightarrow (\neg P \vee \neg Q) \vee R$ …Associative **law**

$\Leftrightarrow \neg(P \wedge Q) \vee R$ …Associative **law**

$\Leftrightarrow (P \wedge Q) \rightarrow \mathbf{R}$.

Prove: $(p \wedge \neg q) \vee q \Leftrightarrow p \vee q$

$(p \wedge \neg q) \vee q$ Left-Hand Statement

$\Leftrightarrow q \vee (p \wedge \neg q)$ Commutative

$\Leftrightarrow (q \vee p) \wedge (q \vee \neg q)$ Distributive

$\Leftrightarrow (q \vee p) \wedge T$ Or Tautology

$\Leftrightarrow q \vee p$ Identity

$\Leftrightarrow p \vee q$ Commutative

Prove: $(p \wedge \neg q) \vee q \Leftrightarrow p \vee q$

$(p \wedge \neg q) \vee q$	Left-Hand Statement
$\Leftrightarrow q \vee (p \wedge \neg q)$	Commutative
$\Leftrightarrow (q \vee p) \wedge (q \vee \neg q)$	Distributive

Why did we need this step?

Prove: $p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$

Contrapositive

$p \rightarrow q$

$\Leftrightarrow \neg p \vee q$ Implication Equivalence

$\Leftrightarrow q \vee \neg p$ Commutative

$\Leftrightarrow \neg(\neg q) \vee \neg p$ Double Negation

$\Leftrightarrow \neg q \rightarrow \neg p$ Implication Equivalence

If $p \rightarrow q$ is a statement then $q \rightarrow p$ is called converse.

$\neg p \rightarrow \neg q$ is inverse and

$\neg q \rightarrow \neg p$ is contrapositive.

Prove: $p \rightarrow p \vee q$ is a tautology

Must show that the statement is true for any value of p,q.

$p \rightarrow p \vee q$

$\Leftrightarrow \neg p \vee (p \vee q)$ Implication Equivalence

$\Leftrightarrow (\neg p \vee p) \vee q$ Associative

$\Leftrightarrow (p \vee \neg p) \vee q$ Commutative

$\Leftrightarrow T \vee q$ Or Tautology

$\Leftrightarrow q \vee T$ Commutative

$\Leftrightarrow T$ Domination

This tautology is called the addition rule of inference.