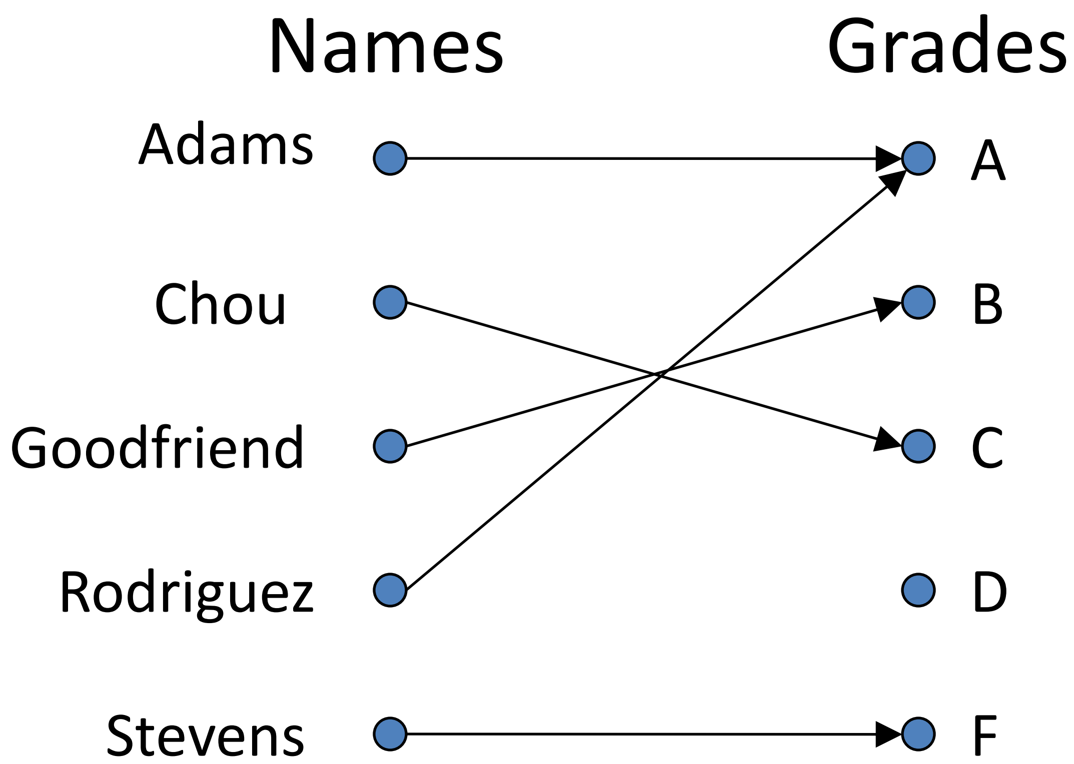
Functions

Function Definition

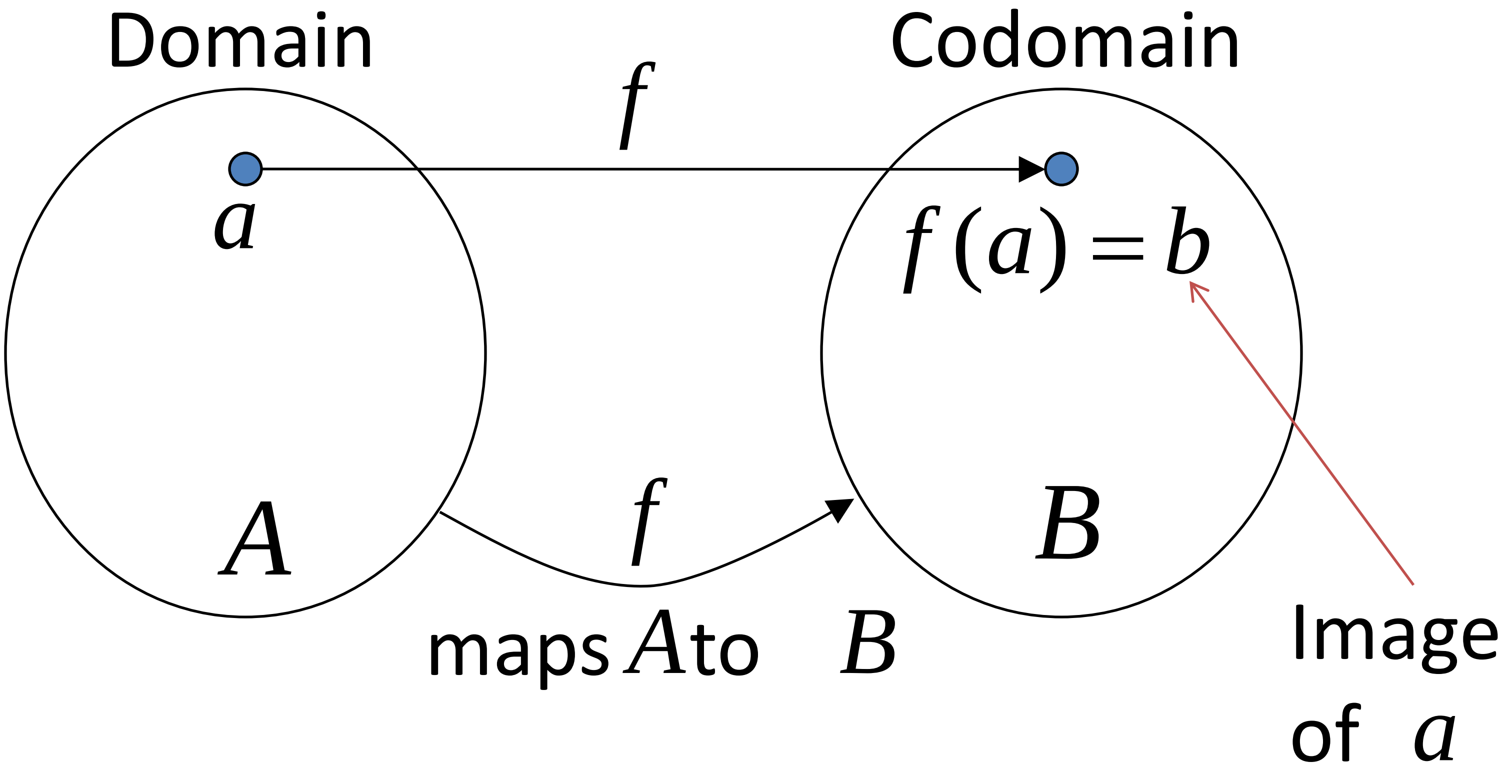
- Let *A and B be nonempty sets. A function f from A to B is an assignment of exactly one* element of *B to each element of A. f (a) = b if b is the unique element of B* assigned by the function *f to the element a of A. If f is a function from A to B, we write f : A → B.*

Functions

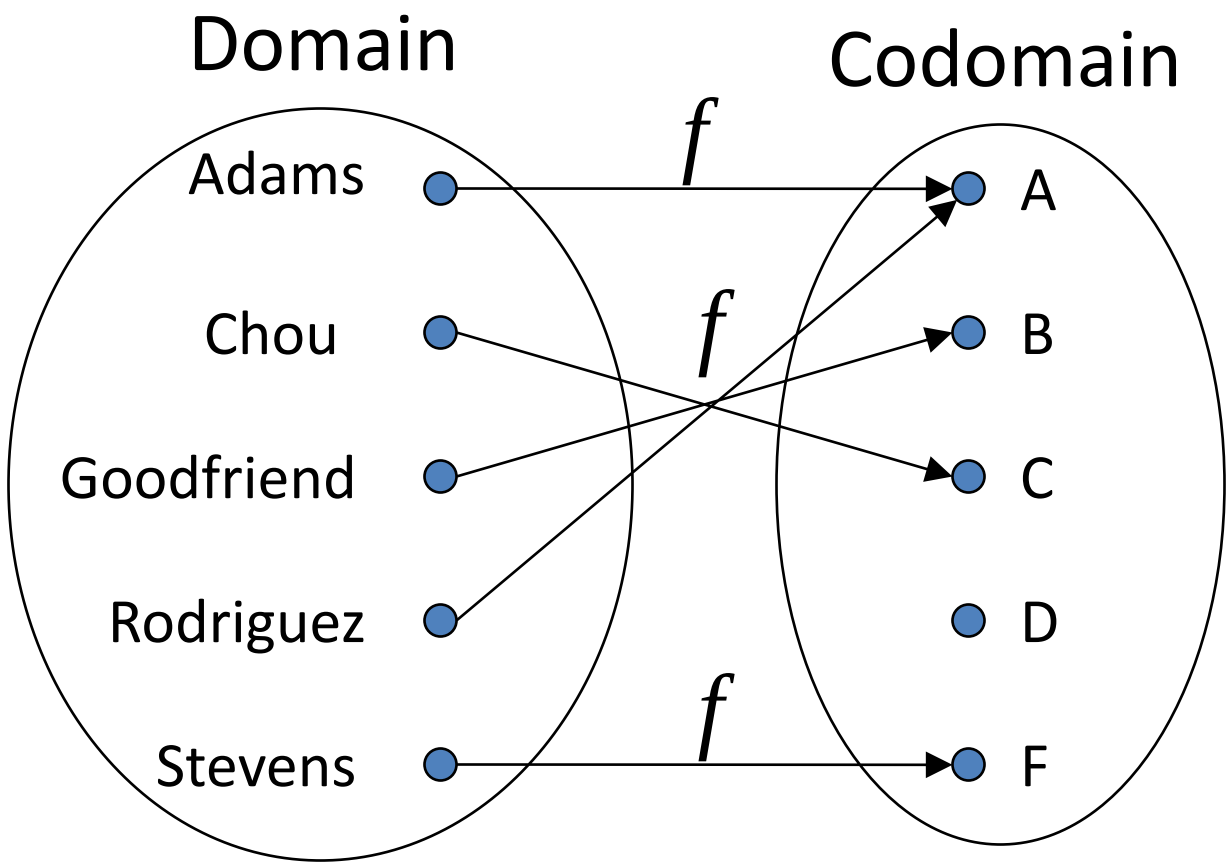


$$f(\text{Chou}) = C \qquad f(\text{Rodriguez}) = A$$

$$f : A \to B$$



Every element of domain has exactly one image



$$\text{Domain} = \{\text{Adams}, \text{Chou}, \text{Goodfriend}, \text{Rodriguez}, \text{Stevens}\}$$

$$\text{Codomain} = \{\text{A}, \text{B}, \text{C}, \text{D}, \text{F}\}$$

$\text{Range} = \{\text{A}, \text{B}, \text{C}, \text{F}\}$ set of all images

$$f : Z \to Z$$

$$f(x) = x^2$$

$$\text{Domain} = Z$$

$$\text{Codomain} = Z$$

$$\text{Range} = \{0,1,4,9,\ldots\}$$

Equal functions

$f : A \to B$ $\qquad\qquad$ $g : C \to D$

$$f = g$$

$A = C$ same domain

$B = D$ same codomain

$\forall x \in A,\ f(x) = g(x)$ same mapping

In some programming languages, domain and codomain are explicitly defined

```
int f(int a) {
        return a*a;
}
```

Its domain is set of all real numbers and codomain is the set of integers.

Add and multiply functions

Real numbers

$$f_1 : A \to R \qquad (f_1 + f_2)(x) = f_1(x) + f_2(x)$$

$$f_2 : A \to R \qquad (f_1 f_2)(x) = f_1(x) f_2(x)$$

Example: $f_1(x) = x^2 \qquad f_2(x) = x - x^2$

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + (x - x^2) = x$$

$$(f_1 f_2)(x) = f_1(x) f_2(x) = x^2(x - x^2) = x^3 - x^4$$

Image of set

Set S $$f(S) = \{t \mid \exists x \in S(t = f(x))\}$$
$$= \{f(x) \mid x \in S\}$$

Example:

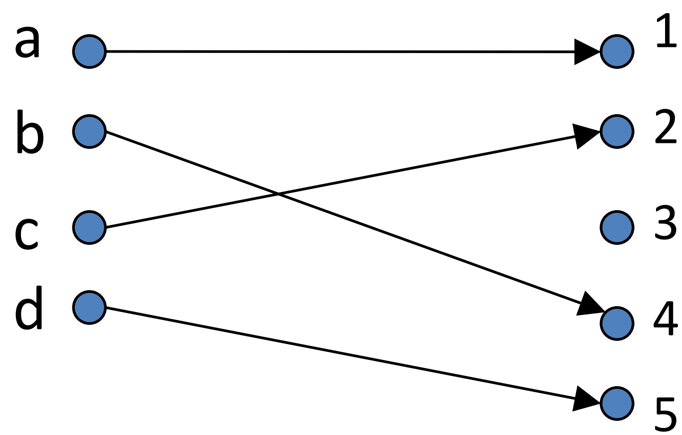
$$f(x) = x^2$$
$$f(\{1,2,3\}) = \{1,4,9\}$$

Ex.2. Let *A = {a, b, c, d, e} and B = {1, 2, 3, 4} with f (a) = 2, f (b) = 1, f (c) = 4, f (d) = 1, and f (e) = 1.*
The image of the subset S = {b, c, d} is the set f (S) = {1, 4}.

One-to-one (injection) function

For every x, y in domain

$$f(x) = f(y) \quad \text{implies} \quad x = y$$



Each element of range is image of one element of domain

Examples: $f(x) = x + 1$ is one-to-one

$g(x) = x^2$ is not one-to-one: $g(-1) = g(1) = 1$

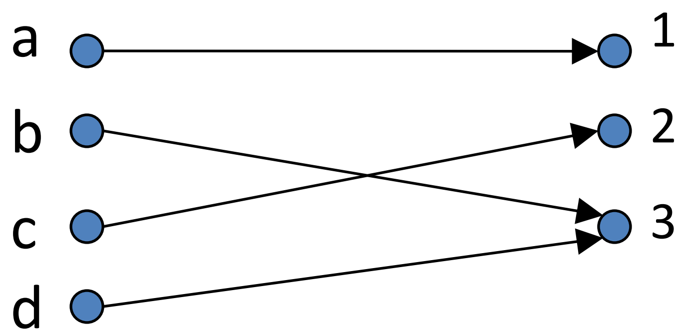
Increasing function: $x < y \rightarrow f(x) \leq f(y)$

Strictly increasing: $x < y \rightarrow f(x) < f(y)$

Strictly increasing functions are one-to-one

Onto (surjection) function $f: A \to B$

For every $y \in B$ there is $x \in B$ such that $f(x) = y$



Range = Codomain

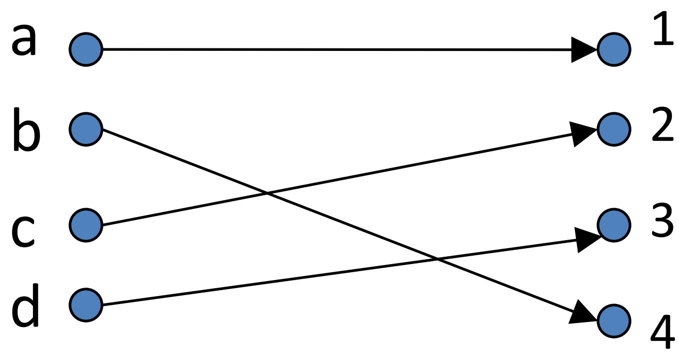
Examples: $f(x) = x + 1$

This function is onto, because for every integer *y there is an integer x such that f (x) = y. To see this, note that f (x) = y if and only if x + 1 = y, which holds if and only if x = y − 1.*

$g(x) = x^2$ is not onto: $\forall x \in Z, g(x) \neq -1$

One-to-one correspondence (bijection) function

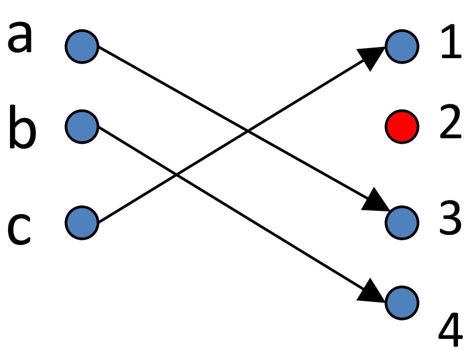
a function which is one-to-one and onto



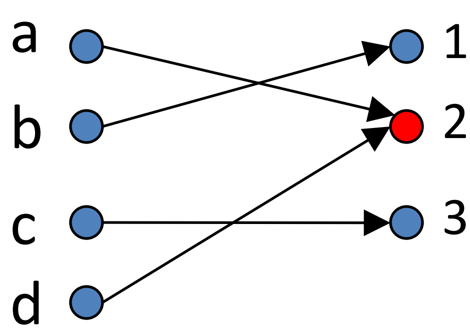
Examples: $f(x) = x + 1$ is bijection

$g(x) = x^2$ is not bijection

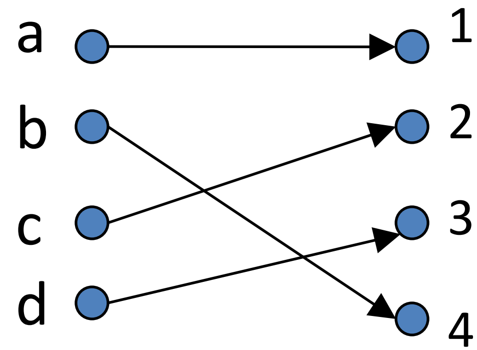
one-to-one
not onto



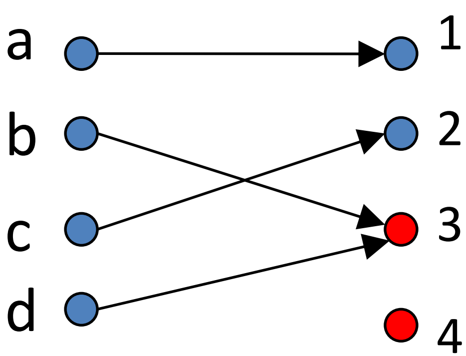
not one-to-one
onto



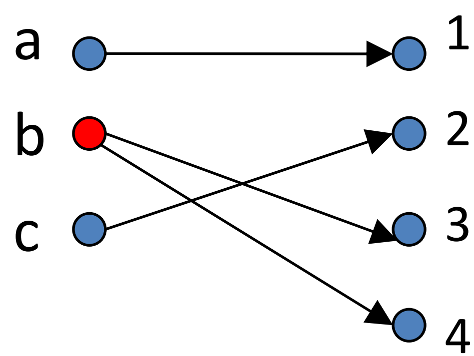
one-to-one
onto



not one-to-one
not onto

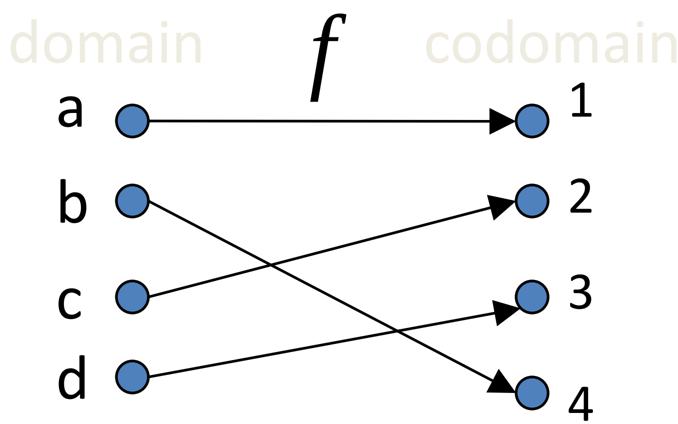


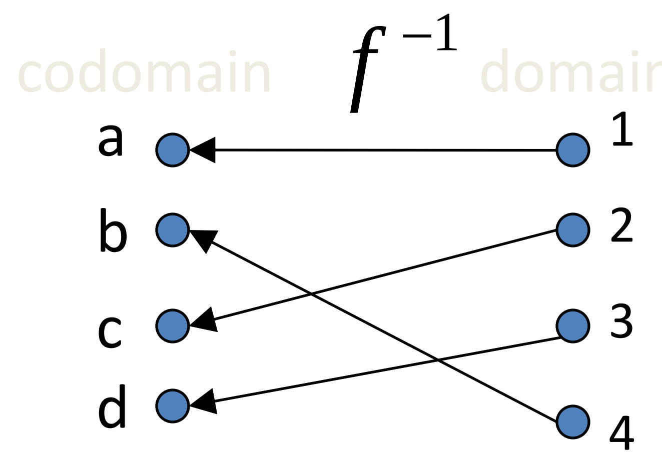
not a function



Inverse f^{-1} of a bijection function f

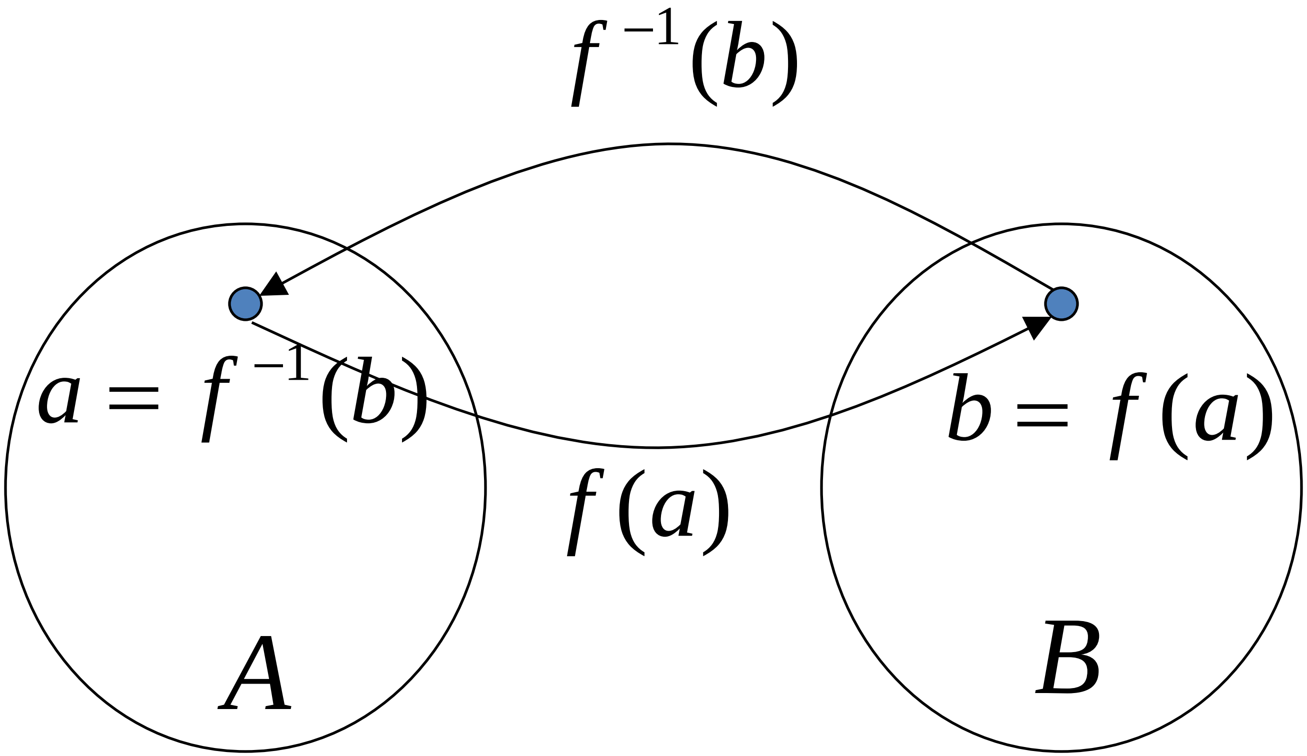
$$f^{-1}(y) = x \quad \text{when} \quad f(x) = y$$





f is invertible function

Example: $f(x) = x + 1$ $\qquad$ $f^{-1}(y) = y - 1$


$f^{-1}(b)$
$a = f^{-1}(b)$
$b = f(a)$
$f(a)$
A
B

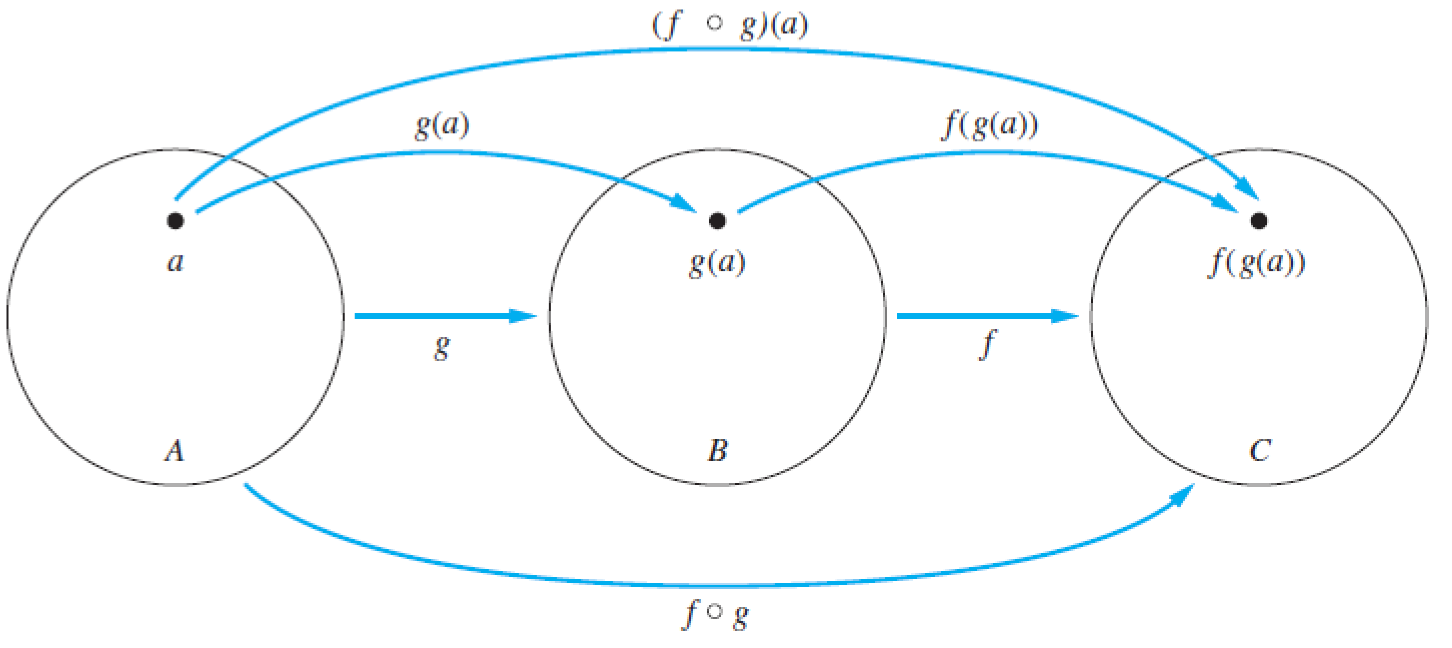
Composition of functions

$$f : B \to C$$

$$g : A \to B$$

$$f \circ g : A \to C$$

$$(f \circ g)(x) = f(g(x))$$



Composition of function

Example: $f(x) = 2x$ $g(x) = x^2$

$$(f \circ g)(x) = f(g(x)) = f(x^2) = 2x^2$$

$$(g \circ f)(x) = g(f(x)) = g(2x) = (2x)^2 = 4x^2$$

- Let *g be the function from the set {a, b, c} to itself such that g(a) = b, g(b) = c, and g(c) = a.* Let *f be the function from the set {a, b, c} to the set {1, 2, 3} such that f (a) = 3, f (b) = 2, and f (c) = 1. What is the composition of f and g, and what is the composition of g and f ?*

Solution: The composition $f \circ g$ *is defined by*

$(f \circ g)(a) = f(g(a)) = f(b) = 2,$

$(f \circ g)(b) = f(g(b)) = f(c) = 1,$ *and*

$(f \circ g)(c) = f(g(c)) = f(a) = 3.$

- Note that $g \circ f$ *is not defined, because the range of f is not a subset of the domain of g.*

- Let *f and g be the functions from the set of integers to the set of integers defined by* $f(x) = 2x + 3$ *and* $g(x) = 3x + 2$. *What is the composition of f and g? What is the composition* of *g and f ?*

Solution: Both the compositions $f \circ g$ *and* $g \circ f$ *are defined. Moreover,*

$(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$

- and

$(g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11.$

identity function

$$f \circ f^{-1} = f^{-1} \circ f = i$$

Suppose $f(x) = y$

$$(f \circ f^{-1})(y) = f(f^{-1}(y)) = f(x) = y$$

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x$$

Floor and Ceiling

Let x be real

Floor function: $\lfloor x \rfloor$ largest integer less or equal to x

Ceiling function: $\lceil x \rceil$ smallest integer greater or equal to x

Examples: $\left\lfloor \frac{1}{2} \right\rfloor = 0$ $\left\lceil \frac{1}{2} \right\rceil = 1$ $\lfloor -3.1 \rfloor = -4$ $\lceil -3.1 \rceil = -3$

Factorial function

$$f : N \to Z^{+} \qquad f(n) = n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$$

$$f(0) = 0! = 1$$

$$1! = 1 \qquad 2! = 1 \cdot 2 = 2 \qquad 6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 760$$

$$20! = 1 \cdot 2 \cdot 3 \cdots 19 \cdot 20 = 2{,}432{,}902{,}008{,}176{,}640{,}000$$

Stirling's formula: $n! \approx \sqrt{2\pi n} \cdot \left(\dfrac{n}{e}\right)^{n}$

Sequences

Sequence: function from a subset of integers to a set S

Finite sequence

2, 4, 6, 8, 10

$$a_1, a_2, a_3, a_4, a_5$$

Infinite sequence

1,3,9,27,81,...

$$f(n) = a_n$$

$$f(1) = a_1 = 2$$

$$f(5) = a_5 = 10$$

Alternate representation

$$a_n = 3^k, \quad k \geq 0$$

$$\{a_n\} = a_1, a_2, a_3, a_4, a_5, \ldots$$

$$= 1,3,9,27,81,\ldots$$

finite sequence: $a_1, a_2, \ldots, a_n$

String: $a_1 a_2 a_3 \cdots a_n$

all elements of sequence concatenated

Length of string: $|a_1 a_2 \cdots a_n| = n$

Empty string (null): λ $|\lambda| = 0$

Arithmetic progression

$$a, a+d, a+2d, \ldots, a+nd, \ldots$$

Initial term a

Common difference d

Example: $\{s_n\} = -1 + 4n$ start with $n = 0$

$$-1, 3, 7, 11, \ldots$$

Geometric progression

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

Initial term a

Common ratio r

Example: $\{c_n\} = 2 \cdot 5^n$ start with $n = 0$

$$2, 10, 50, 250, 1250, \ldots$$

Summations

Sequence: $a_m, a_{m+1}, a_{m+2}, \ldots, a_n$

Sum: $a_m + a_{m+1} + a_{m+2} + \cdots + a_n = \sum_{i=m}^{n} a_i$

Example: $\sum_{i=1}^{5} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$

Theorem: $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Proof:

$$\begin{array}{lccccccc}
\{a_n\} = & 1 & 2 & 3 & 4 & \cdots & n-1 & n \\
\{b_n\} = & n & n-1 & n-2 & n-3 & \cdots & 2 & 1 \\
\{c_n\} = & n+1 & n+1 & n+1 & n+1 & \cdots & n+1 & n+1
\end{array}$$

$$\left.\begin{array}{l}
S = \sum_{i=1}^{n} i = \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i \\
n(n+1) = \sum_{i=1}^{n} c_i = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i = 2S
\end{array}\right\} \Rightarrow S = \frac{n(n+1)}{2}$$

End of Proof

Theorem: If a, r are real numbers and $r \notin \{0,1\}$ then

$$\sum_{i=0}^{n} ar^{i} = \frac{ar^{n+1} - a}{r - 1}$$

Proof: Let $S = \sum_{i=0}^{n} ar^{i}$

$$rS$$

$$= r\sum_{i=0}^{n} ar^{i}$$

$$= \sum_{i=0}^{n} ar^{i+1}$$

$$= \sum_{k=1}^{n+1} ar^{k}$$

$$= \left(\sum_{k=0}^{n} ar^{k}\right) + (ar^{n+1} - a)$$

$$= S + (ar^{n+1} - a)$$

$$\Rightarrow \quad rS = S + (ar^{n+1} - a)$$

$$\Downarrow$$

$$S = \frac{ar^{n+1} - a}{r - 1}$$

End of Proof

Useful Summation Formulas

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=0}^{n} ar^i = \frac{ar^{n+1} - a}{r-1}, \quad r \notin \{0,1\}$$

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}, \quad |x| < 1$$

Recurrence Relations

- A *recurrence relation for the sequence* $\{a_n\}$ *is an equation that expresses* a_n *in terms of one or* more of the previous terms of the sequence, namely, $a_0, a_1, \ldots a_n$ *for all integers n with n* ≥ n_0, *where* n_0 *is a nonnegative integer. A sequence is called a solution of a recurrence* relation if its terms satisfy the recurrence relation.

- Let $\{a_n\}$ *be a sequence that satisfies the recurrence relation* $a_n = a_{n-1} + 3$ *for* $n = 1, 2, 3, . . .$, and suppose that $a_0 = 2$. *What are* a_1, a_2, *and* a_3?

Solution: We see from the recurrence relation that

$a_1 = a_0 + 3 = 2 + 3 = 5.$

$a_2 = 5 + 3 = 8$ *and*

$a_3 = 8 + 3 = 11.$

- Let $\{a_n\}$ *be a sequence that satisfies the recurrence relation* $a_n = a_{n-1} - a_{n-2}$ *for* $n = 2, 3, 4, \ldots,$ *and suppose that* $a_0 = 3$ *and* $a_1 = 5$*. What are* a_2 *and* a_3*?*

Solution: *We see from the recurrence relation that* $a_2 = a_1 - a_0 = 5 - 3 = 2$

and $a_3 = a_2 - a_1 = 2 - 5 = -3.$

We can find a_4, a_5, *and each successive term in a similar way.*

Creating Recurrence Relation

```
Fib(a)                                   T(n)
{
 if(a==1 || a==0)                        1
return 1;
 return Fib(a-1) + Fib(a-2);             T(n-1)+T(n-2)
 }
```

(comparison, comparison, addition) and also calls itself recursively.

$$f_n = f_{n-1} + f_{n-2}$$

The *Fibonacci sequence*, $f_0, f_1, f_2, \ldots,$ *is defined by the initial conditions* $f_0 = 0$, $f_1 = 1$, and the recurrence relation $f_n = f_{n-1} + f_{n-2}$ for $n = 2, 3, 4, \ldots.$

- Find the Fibonacci numbers $f_2, f_3, f_4, f_5,$ *and* f_6.

Solution:. Because the initial conditions tell us that $f_0 = 0$ *and* $f_1 = 1$, *using the recurrence relation in the definition we find that*

- $f_2 = f_1 + f_0 = 1 + 0 = 1,$
- $f_3 = f_2 + f_1 = 1 + 1 = 2,$
- $f_4 = f_3 + f_2 = 2 + 1 = 3,$
- $f_5 = f_4 + f_3 = 3 + 2 = 5,$
- $f_6 = f_5 + f_4 = 5 + 3 = 8.$