Predicates and Quantifiers

Puzzle

Brown, Jones and Smith are suspected of income tax evasion. They testify under oath as follows:

Brown: Jones is guilty and Smith is innocent.

Jones: If Brown is guilty, then so is Smith.

Smith: I am innocent but at least one of the others is guilty.

Assume,

Brown

innocent — guilty

B	In	×
J	gui	
S	I	×

B	G	G	G	G
J	I	I	G	G
S	I	G	I	G
	×	✓	×	×

$$\Downarrow$$

$$\neg(\neg J \land S)$$

$$J \lor \neg S$$

$$\left\{ \begin{array}{l} B \rightarrow \text{guilty} \\ J \rightarrow \text{innocent} \\ S \rightarrow \text{guilty.} \end{array} \right.$$

Real use

- An important type of programming language is designed to reason using the rules of predicate logic. Prolog (from *Pro*gramming in *Log*ic), developed in the 1970s by computer scientists working in the area of artificial intelligence, is an example of such a language. Prolog programs include a set of declarations consisting of two types of statements, **Prolog facts** and **Prolog rules**.
- Prolog facts define predicates by specifying the elements that satisfy these predicates.
- Prolog rules are used to define new predicates using those already defined by Prolog facts.

Quantifiers as Conjunctions/Disjunctions

- If the domain is finite then universal/existential quantifiers can be expressed by conjunctions/ disjunctions.
- If U consists of the integers 1,2, and 3, then

$$\begin{aligned}\forall x\ P(x) &\equiv P(1) \wedge P(2) \wedge P(3)\\ \exists x\ P(x) &\equiv P(1) \vee P(2) \vee P(3)\end{aligned}$$

- Even if the domains are infinite, you can still think of the quantifiers in this fashion, but the equivalent expressions without quantifiers will be infinitely long.

Negation for Quantifiers

- The rules for negating quantifiers are:
- We can say, De Morgan's Law for Quantifiers

$$\neg \forall x\ P(x) \equiv \exists x\ \neg P(x)$$

$$\neg \exists x\ P(x) \equiv \forall x\ \neg P(x)$$

Negating Quantifiers

- Consider the quantified statement:
  - “Every student has at least one course where the lecturer is a teaching assistant.”
  - Its negation is the statement:
  - “There is a student such that in every course the lecturer is not a teaching assistant.”

Negate each of the following statements

(*a) All students live in the dormitories.*

(b) All mathematics majors are males.

(*c) Some students are 25 years old or older.*

solution

(*a) At least one student does not live in the dormitories. (Some students do not live in the dormitories.)*

(*b) At least one mathematics major is female. (Some mathematics majors are female.)*

(*c) None of the students is 25 years old or older. (All the students are under 25.)*

Negate each of the following statements:

(a) $\exists x\ \forall y,\ p(x, y);$

(b) $\exists x\ \forall y,\ p(x, y);$

(c) $\exists y\ \exists x\ \forall z,\ p(x, y, z).$

Use $\neg\forall x\, p(x) \equiv \exists x\, \neg p(x)$ *and* $\neg\exists x\, p(x) \equiv \forall x\, \neg p(x);$

Solution

(*a*) $\neg(\exists x \forall y,\ p(x, y)) \equiv \forall x \exists y\, \neg p(x, y)$

(*b*) $\neg(\forall x \forall y,\ p(x, y)) \equiv \exists x \exists y\, \neg p(x, y)$

(*c*) $\neg(\exists y\ \exists x\ \forall z,\ p(x, y, z)) \equiv \forall y\ \forall x\ \exists z\, \neg p(x, y, z)$

- Express the statement "Every student in this class has studied calculus" using predicates and quantifiers.
  - rewrite the statement
  - "For every student in this class, that student has studied calculus."
  - "For every student *x* in this class, *x* has studied calculus."

    C(x): "x has studied calculus."
  - domain for *x* consists of the students in the class
  - we can translate our statement as $\forall x C(x)$
  - If we change the domain to consist of all people
  - "For every person *x*, if person *x* is a student in this class then *x* has studied calculus."

    S(x): person *x* is in this class

    $\forall x(S(x) \rightarrow C(x))$.
  - Our statement cannot be expressed as $\forall x(S(x) \wedge C(x))$ because this statement says that all people are students in this class and have studied calculus!
  - *As this property,* $P \longrightarrow Q \equiv \sim P \vee Q$

- Express the statements “Some student in this class has visited Mexico” and “Every student in this class has visited either Canada or Mexico” using predicates and quantifiers
  - “There is a student in this class with the property that the student has visited Mexico.”
  - “There is a student *x* in this class having the property that *x* has visited Mexico.”
  - $M(x)$: *x* has visited Mexico
- domain for x consists of the students in this class, then $\exists x M(x)$.
  - Domain: all people.
  - “There is a person x having the properties that x is a student in this class and x has visited Mexico.”
  - $S(x)$:“*x* is a student in this class.”
  - Now, $\exists x(S(x) \land M(x))$

  Means: there is a person *x* who is a student in this class and who has visited Mexico.

- *Our statement cannot be expressed as $\exists x(S(x) \rightarrow M(x))$, which is true when there is someone not in the class because, in that case, for such a person x, $S(x) \rightarrow M(x)$ becomes either $F \rightarrow T$ or $F \rightarrow F$, both of which are true.*
- *Statement becomes,*
- *"For every x in this class, x has the property that x has visited Mexico or x has visited Canada."*

Example to transfer from English to Logical

- Consider these statements. The first two are premises and the third is the conclusion.
  - "All lions are fierce."
  - "Some lions do not drink coffee."
  - "Some fierce creatures do not drink coffee."
- Solution
  - Let P(x),Q(x) and R(x) be the statements "x is a lion", "x is fierce" and "x drinks coffee." respectively. Let the domain consists of all creatures. Now the statements are:
  - $\forall x\ (P(x) \rightarrow Q(x))$.
  - $\exists x\ (P(x) \wedge \neg R(x))$.
  - $\exists x\ (Q(x) \wedge \neg R(x))$.
- Not okay:
  - $\exists x\ (P(x) \rightarrow \neg R(x))$ here ,if creature is not lion then also they drink coffee.
  - $\exists x\ (Q(x) \rightarrow \neg R(x))$
- Not exact -- both are true even if P(x) and Q(x) both are not true!

- Consider these statements. The first three are premises and the fourth is a valid conclusion.
  - "All hummingbirds are richly colored."
  - "No large birds live on honey."
  - "Birds that do not live on honey are dull in color."
  - "Hummingbirds are small."
- Solution
  - Let P(x): "x is a hummingbird" ,
  - Q(x): "x is large",
  - R(x): "x lives on honey",
  - S(x) : "x is richly colored."
  - Let the domain consists of all birds. So the statements are:
  - $\forall x\ (P(x) \rightarrow S(x))$.
  - $\neg\exists x\ (Q(x) \wedge R(x))$.
  - $\forall x\ (\neg R(x) \rightarrow \neg S(x))$.
  - $\forall x\ (P(x) \rightarrow \neg Q(x))$.

Propositions for More than one variable

Let $B = \{1, 2, 3, \ldots, 9\}$ and let $p(x, y)$ denote "$x + y = 10$"
Then $p(x, y)$ is a propositional function.

- The following is a statement since there is a quantifier for each variable:
  - $\forall x \exists y, p(x, y)$, *that is, "For every x, there exists a y such that x + y = 10"*
  - This statement is true. For example, if *x = 1, let y = 9; if x = 2, let y = 8, and so on.*
- The following is also a statement:
  - $\exists y \forall x, p(x, y)$, *that is, "There exists a y such that, for every x, we have x + y = 10"*
  - No such *y exists; hence this statement is false.*
- *Note: Change of order for different quantifiers can change the meaning.*

Quantifications of Two Variables **Statement**

Statement	When True?	When False
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	$P(x,y)$ is true for every pair x,y.	There is a pair x, y for which $P(x,y)$ is false.
$\forall x \exists y P(x,y)$	For every x there is a y for which $P(x,y)$ is true.	There is an x such that $P(x,y)$ is false for every y.
$\exists x \forall y P(x,y)$	There is an x for which $P(x,y)$ is true for every y.	For every x there is a y for which P(x,y) is false.
$\exists x \exists y P(x,y)$ $\exists y \exists x P(x,y)$	There is a pair x, y for which $P(x,y)$ is true.	P(x,y) is false for every pair x,y

Examples

- Determine the truth value of each of the following statements where **U = {1, *2, 3} is the universal set:***

(a) $\exists x \forall y, x2 < y + 1;$

(b) $\forall x \exists y, x2 + y2 < 12;$

Solution

(a) True. For if $x = 1$*, then 1, 2, and 3 are all solutions to* $1 < y + 1$*.*

(b) True. For each $x0$*, let* $y = 1$*; it is a true statement.*

If we change order meaning can get changed.

- Examples:
- $\forall x \exists y$ [x is married to y] is true,

  however, $\exists y\ \forall x$[x married to y] asserts that there is some person in the universe who married to everyone, this is false .
- $\forall x\ \exists y\ [x+y=0]$ (for all x, there exists a y such that $x+y=0$ is true, since for any value of s there is a value of y (i.e, -x) which makes it true.

However,

- $\exists y\ \forall x\ [x+y=0]$ (There exists a y such that for all x, $x+y=0$) asserts that value of y can be chosen independently of the value of x, since no y exists which yields zero when added to arbitrary integer x , this is false.

Examples in Mathematics Nested Quantifiers

- Translate the logical statement into Logical.

1. The sum of two integers is always positive.
   – To solve this, Read "For every two integers, if these integers are both positive, then the sum of these integers is positive".

$$\forall x \forall y((x > 0) \wedge (y > 0) \to (x + y > 0))$$

$$\forall x \forall y(x + y > 0)$$

2. “Every real number expect zero has a multiplicative inverse” (A multiplicative index of a real number x is a real number y such that xy=1.)

Solution:

We can rewrite as, “For every real number x expect 0, x has a multiplicative inverse.”

“For every real number x, if x ≠ 0”, then there exists a real number y such that xy=1”

$$\forall x((x \neq 0) \longrightarrow \exists y(xy = 1))$$

Valid, Satisfiable and unsatisfiable

- If $P(x_1,x_{2,}\ldots x_n)$ is true for all values $C_1,C_{2,}\ldots C_n$ from the universe U, then $P(x_1,x_{2,}\ldots x_n)$ is **valid** in U.
- If $P(x_1,x_{2,}\ldots x_n)$ is true for some values of $C_1,C_{2,}\ldots C_n$ from the universe U, then $P(x_1,x_{2,}\ldots x_n)$ is **Satisfiable** in U.
- If $P(x_1,x_{2,}\ldots x_n)$ is not true for any values of $C_1,C_{2,}\ldots C_n$ from the universe U, then $P(x_1,x_{2,}\ldots x_n)$ is **Unsatisfiable** in U.

Nested Quantifiers

- Complex meanings require nested quantifiers.
  - "Every real number has an inverse w.r.t. addition."
    - Let the domain U be the real numbers. Then the property is expressed by

$$\forall x\ \exists y\ (x + y = 0)$$

  - "Every real number except zero has a multiplicative inverse."
    - Let the domain U be the real numbers. Then the property is expressed by

$$\forall x\ (x \neq 0 \rightarrow \exists y\ (x * y = 1))$$

Examples on Negation

- Negate the following :
- "There does not exist a woman who has taken a flight on every airline in the world "

Solution:

- "There is a woman who has taken a flight on every airline in the world " we can express,

$$\neg\exists w \forall a \exists f (P(w,f) \wedge Q(f,a))$$

Where, P(w, f) is "w has taken f " $Q(f,a)$ is " f is a flight on a".

By applying Demorgon's law for quantifiers we can move negation inside successive quantifiers and by applying this in last step we will get the equation equivalent this.

$$\forall w \neg\forall a \exists f (P(w,f) \wedge Q(f,a))$$
$$\forall w \exists a \neg\exists f (P(w,f) \wedge Q(f,a))$$
$$\forall w \exists a \forall f \neg(P(w,f) \wedge Q(f,a))$$
$$\forall w \exists a \forall f (\neg P(w,f) \vee \neg Q(f,a))$$