$R^n \longrightarrow R$ ... Scaler valued func

$R^n \longrightarrow R^m$ $m \geq 2$ ... vector valued function.

Unit Vectors:

- vector of length 1 is called unit vector.

Standard unit vectors:

$$\hat{i} = \langle 1, 0, 0 \rangle$$

$$\hat{j} = \langle 0, 1, 0 \rangle$$

$$\hat{k} = \langle 0, 0, 1 \rangle$$

The dot product $u \cdot v$ of vectors $u = \langle u_1, u_2, u_3 \rangle$ and $v = \langle v_1, v_2, v_3 \rangle$ is the scalar

$$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3 .$$

The Cross Product:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Parameterisation of curve :

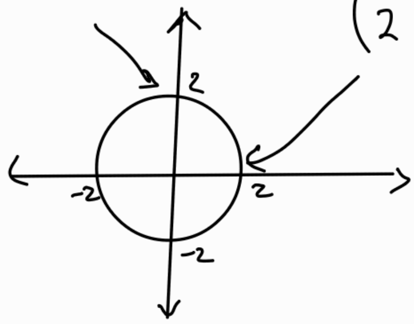
If you are able to define a function from a subset of real numbers to the curve. s.t. image of the function is the given curve.

Such a curve is called Parameterised curve.

$r(t) = (2\cos t, 2\sin t) \quad 0 \le t \le 2\pi$

$(2\cos(\pi/2), 2\sin \pi/2)$
$(0,2)$

$x \quad , \quad y$
$(2\cos 0, \; 2\sin 0) = (2,0)$



Simple Curve:

A curve which isn't cutting / touching itself.

∇
grad

Derivatives of Vector function:

$$r'(t) = \lim_{\Delta t \to 0} \frac{r(t+\Delta t) - r(t)}{\Delta t} \quad \text{OR} \quad \frac{r(t+\Delta t) - r(t)}{\Delta t}$$

tangent Vector :

Unit tangent vector = $\cancel{\frac{r' t}{|r' t|}}$ $\frac{r'}{|r'(t)|}$

to sketch tangent vector: at point 'P'.

$$q(w) = r(P) + w \cdot r'(P)$$

OR

$$q(w) = r + wr'$$

Gradient :

$f : \mathbb{R}^n \longrightarrow \mathbb{R}$

$(x_1, x_2 \ldots x_n) \longmapsto f(x_1, x_2 \ldots x_n)$

grad (f) aka ∇f (nabla) $= \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \ldots \right)$

e.g. given a scalar function $f(x, y, z)$

grad f or $\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right]$

$\nabla_f \leftarrow$ here nabla(∇) is a function taking another function (f) as input. (Whenever this happens original function is operator.)

$$\text{Scalar valued function} \xrightarrow{\quad \nabla \quad} \text{Vector valued function}$$

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \cdots \frac{\partial}{\partial x_n} \right)$$

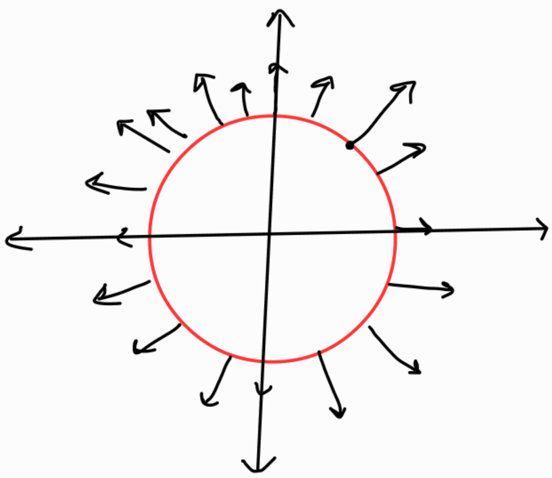
$$\nabla_f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \cdots \frac{\partial f}{\partial x_n} \right)$$

__Normal fields:__

- A vector field but drawn using the (∇_f) gradient vector.

e.g. $f(x,y) = x^2+y^2$

$\nabla_f = (2x, 2y)$



Laplacian Operator:

∇^2 ← Nabla²

$f: \mathbb{R}^n \longrightarrow \mathbb{R}$

$$\text{Scaler} \xrightarrow{\quad \nabla \quad} \text{Vector}$$

$$\text{Scaler} \xrightarrow{\quad \nabla^2 \quad} \text{Scaler}$$

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots \frac{\partial^2}{\partial x_n^2}$$

$$\Delta f = \nabla^2 f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_n^2}$$

e.g. $f(x,y) = x^2 + y^2$

$$\nabla f = (2x, 2y) \Longleftrightarrow 2x\hat{i} + 2y\hat{j}$$

$$\nabla^2 f = 2 + 2$$
$$= 4$$

e.g. ② $f(x,y,z) = \frac{xy}{z^2}$

$f_x = y/z^2$ | $f_y = x/z^2$ | $f_z = \frac{-2xy}{z^3}$

$f_{xx} = 0$ | $f_{yy} = 0$ | $f_{zz} = \frac{6xy}{z^4}$

$$\nabla_f^2 = 0 + 0 + \frac{6xy}{z^4} = \frac{6xy}{z^4}$$

If 'f and g' are scalar valued functions.

$$\nabla^2(f+g) = \frac{\partial^2 f}{\partial^2 x_1} + \frac{\partial^2 g}{\partial^2 x_1} + \frac{\partial^2 f}{\partial^2 x_2} + \frac{\partial^2 g}{\partial^2 x_2} \cdots\cdots \frac{\partial^2 f}{\partial^2 x_n} + \frac{\partial^2 g}{\partial^2 x_n}$$

$$\nabla^2(f+g) = \nabla^2 f + \nabla_g^2 \quad \text{AND} \quad \nabla^2(f-g) = \nabla^2 f - \nabla^2 g$$

$\therefore$ Laplacian is linear in Nature

$$\nabla^2(k \cdot f) = k \cdot \nabla^2 f$$

real number

$$\nabla^2(fg) = \frac{\partial^2(fg)}{\partial x^2} + \frac{\partial^2(fg)}{\partial y^2} + \frac{\partial^2(fg)}{\partial z^2}$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial(fg)}{\partial x}\right] + \frac{\partial}{\partial y}\left(\frac{\partial(fg)}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial(fz)}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\left(f_x \cdot g + f g_x\right] + \frac{\partial}{\partial y}\left(f_y g + f g_y\right] + \frac{\partial}{\partial z}\left[f_z g + f \cdot g_z\right)$$

$$= f_{xx} \cdot g + f_x g_x + f_x g_x + f g_{xx} + f_{yy} g + f_y g_y + f_y g_y + f g_{yy} + f_z g_z + f_{zz} g + f_z g_z + f g_{zz}$$

$$= g\left(f_{xx} + f_{yy} + f_{zz}\right) + f\left(g_{xx} + g_{yy} + g_{zz}\right) + 2\left(f_x, f_y, f_z\right) \cdot \left(g_x, g_y, g_z\right)$$

$$\nabla^2(fg) = g\nabla^2 f + f\nabla^2 g + 2\nabla f \cdot \nabla g$$

Divergence Operator:

$f : R^n \to R^n$

$$\operatorname{div}(f) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} \cdots \frac{\partial f_n}{\partial x_n}$$

e.g. $(x_1, x_2 \ldots x_n) \longrightarrow (v_1, v_2, v_3 \ldots v_n)$

$$\operatorname{div}(f) = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \cdots + \frac{\partial v_n}{\partial x_n}$$

$\operatorname{div}(\ )$

divergence operator → defined as sum of partial derivatives.

Divergence Meaning:

Net Outflow, i.e. at particular point 'output - input'

outflow > inflow → divergence > 0 (source)

outflow < inflow → divergence < 0 (sink)

outflow = inflow → divergence = 0 (incompressible / solenoidal).

$\text{div}(\text{grad } f) = \text{Laplacian } f$

$\text{div}(\nabla f) = \nabla^2 f$

$$\text{div}(f) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{y} + \frac{\partial f_3}{z}$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) \cdot (f_1, f_2, f_3)$$

$\text{div}(f) = \nabla \cdot \vec{f}$

$\nabla \times \nabla f = \nabla^2 f$

grad

Angle between two surfaces:

$$\theta = \cos^{-1}\left(\frac{\nabla f \cdot \nabla g}{|\nabla f| \, |\nabla g|}\right)$$

e.g. Find Angle between the surfaces

$x^2+y^2+z^2=9 \Rightarrow x^2+y^2+z^2-9=0$

$f=0$

AND

$z=x^2+y^2+3 \Rightarrow x^2+y^2-z+3$

$g=0$

at point $(2,-1,2)$

---

at given point

$\nabla f = (2x, 2y, 2z) = (4,-2,4)$

$\nabla g = (2x, 2y, -1) = (4,-2,-1)$

$x=2, y=-1, z=2$

$\nabla f \cdot \nabla g = 16+4-4 = 16$

$|\nabla f| = \sqrt{16+4+16}$

$= \sqrt{36}$

$= 6$

$|\nabla g| = \sqrt{16+4+1}$

$= \sqrt{21}$

$$\theta = \cos^{-1}\left(\frac{\nabla f \cdot \nabla g}{|\nabla f|\,|\nabla g|}\right)$$

$$= \cos^{-1}\left(\frac{16}{6\sqrt{21}}\right)$$

$$= \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

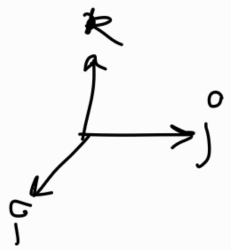
$$\theta = 54^\circ$$

⊛ Curl of a vector field:

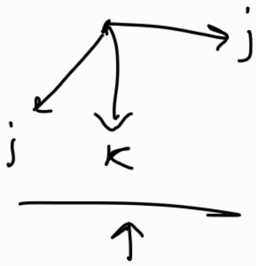
$$\vec{V} = [v_1, v_2, v_3]$$

at a given point how will a field rotate around the point.

$\text{curl}(\vec{V}) = 0$ irrotatational vector field. Particals are flowing straight.



$$\text{curl}(\vec{v}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} = \nabla \times \vec{v}$$



if $k < 0$
mul by -1

$$= \hat{i}\left((v_3)_y - (v_2)_z\right) - \hat{j}\left((v_3)_x - (v_1)_z\right) + \hat{k}\left((v_2)_x - (v_1)_y\right)$$

e.g.

$$\vec{V} = \left[\, y^n, z^n, x^n \,\right] \qquad n > 0$$

$$\operatorname{Curl}(\vec{V}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^n & z^n & x^n \end{vmatrix}$$

$$= \hat{i}\left(0 - nz^{n-1}\right) - \hat{j}\left(nx^{n-1} - 0\right) + k\left(0 - ny^{n-1}\right)$$

$$= -nz^{n-1}\hat{i} - nx^{n-1}\hat{j} - ny^{n-1}\hat{k}$$

$$= -n\left(x^{n-1}\hat{j} + y^{n-1}\hat{k} + z^{n-1}\hat{i}\right)$$

$$\operatorname{Curl}(\vec{V}) = -n \cdot \left[\, z^{n-1}, x^{n-1}, y^{n-1} \,\right]$$

Properties of curl:

if curl $(\vec{V}) = 0$, there exist Some potential field.

(i) curl (grad f) $= 0$

f is conservative means 'grad f'

$$\nabla f = [f_x, f_y, f_z] = F$$

$$\mathrm{Curl}\,(\nabla f) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} = \hat{i}(f_{zy} - f_{yz}) - \hat{j}(f_{zx} - f_{xz}) + \hat{k}(f_{yx} - f_{xy}) = 0 - 0 + 0 = 0$$

by mixed derivatives theorem,

$$f_{xy} = f_{yx}.$$

$$\therefore \ \mathrm{curl}\,(\nabla f) = 0$$

$$\mathrm{div}\,(\mathrm{curl}\,(\vec{V})) = 0$$

$$\text{divergence} = \nabla$$

$$\mathrm{curl}\,(\vec{V}) = (\nabla \times \vec{V})$$

$$= \hat{i}((V_3)_y - (V_2)_z) - \hat{j}((V_3)_x - (V_1)_z) + \hat{k}((V_2)_x - (V_1)_y)$$

$$\nabla \cdot (\nabla \times \vec{V}) = (V_3)_{yx} - (V_2)_{zx} - (V_3)_{xy} + \underline{(V_1)_{zy}} + (V_2)_{xz} - \underline{(V_1)_{yz}}$$

by mixed derivative theorem,

$$\nabla \cdot (\nabla \times \overline{v}) = 0$$

$$\therefore \text{div}(\text{curl}(\overline{v})) = 0$$

3. $\text{curl}(\overline{u} + \overline{v}) = \text{curl}(\overline{u}) + \text{curl}(\overline{v})$

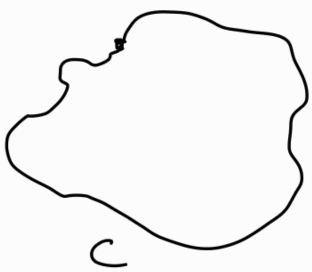
4. $\text{curl}(\alpha \overline{u}) = \alpha \, \text{curl}(\overline{u})$

5. $\text{div}(\overline{u} \times \overline{v}) = \overline{v} \cdot \text{curl}(\overline{u}) - \overline{u} \cdot \text{curl}(v)$

Applications of Line Integrals (Curve Integrals.):

Closed Curve : [Green's Theorem

[for simple curve starting & ending at same point but not integrating at all].



C

• Line integrals are path dependant. Values vary with paths.

e.g. F is scalar

$\mathbb{R}^n \longrightarrow \mathbb{R}$

Step 1: Parameterize the curve in $\mathbb{R}^n$

$$\int_C F \cdot ds = \int_{t=a}^{b} f(r(t)) \cdot |r'(t)| \, dt$$

if. F is vector Field

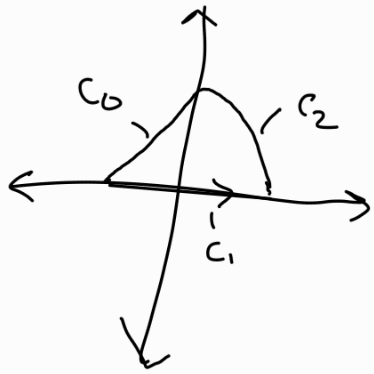
$\mathbb{R}^n \longrightarrow \mathbb{R}^m$

$$\int_C F \cdot ds = \int_{t=a}^{b} f(r(t)) \cdot r'(t) \, dt$$

over piecewise smooth curves:

$F = \langle y, x \rangle$

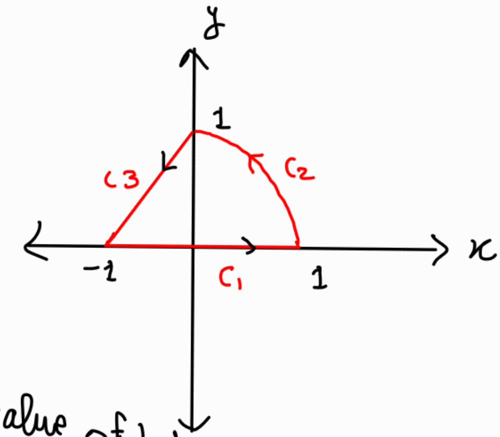
$C = C_1 \cup C_2 \cup C_3$



then,

$$\int_C F \cdot dr = \int_{C_1} F \cdot dr + \int_{C_2} F \, dr + \int_{C_3} F \cdot dr$$

e.g. $F = [y, x]$



C_1: $r(t) = (t, 0) \quad -1 \le t \le 1$

$F(r(t)) = [y, x]$

for y put r(t)'s y value and for x, value of 'x'.

$F(r(t)) = [0, t]$

$F(r(t)) \cdot r'(t) = [0, t] \cdot (1, 0) = 0$

$$\int_{C_1} F \cdot dr = 0$$

C_2: $r(t) = (\cos t, \sin t) \quad 0 \le t \le \pi/2$

$r'(t) = (-\sin t, \cos t)$

here $F = [y, x]$ ∵ given

$F(r(t)) = (\sin t, \cos t)$

$\therefore F(r(t)) = [\sin t, \cos t]$

$$= \int_0^{\pi/2} -\sin^2 t + \cos^2 t \; dt$$

$$= \frac{-1}{2}\int_0^{\pi/2} 1 - \cos 2t \; dt + \frac{1}{2}\int_0^{\pi/2} 1 + \cos 2t \; dt$$

$$= \frac{-1}{2}\left[t - \frac{\sin 2t}{2}\right]_0^{\pi/2} + \frac{1}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{\pi/2}$$

$$= \frac{-1}{2}\left(\pi/2 - \frac{0}{2}\right] + \frac{1}{2}\left\{\pi/2 + 0\right]$$

$$= -\pi/4 + \pi/4$$

$$= 0$$

C_3: $B = (-1, 0)$ $\quad A = (0, 1)$

$(1-t)A + Bt$ ... line parameterization

t

C_3: $(1-t)(-1,0) + (0,1)t$ $\quad 1 \le t \le 0$

$(-1+t, 0) + (0, t)$

$r(t) = (-1+t, t)$ $\quad F = \{y, x]$

$r'(t) = (1, 1)$

$F(r(t)) = [t, -1+t]$

$$\int_{1}^{0}{}_{C_3} F(r(t)) \cdot r'(t)$$

$$\int_{1}^{0} t - 1 + t \quad dt$$

$$\int_{0}^{1} 2t - 1 \quad dt$$

$$= \left[\frac{2t^2}{2} - t\right]_0^1$$

$$= \left[t^2 - t\right]_0^1$$

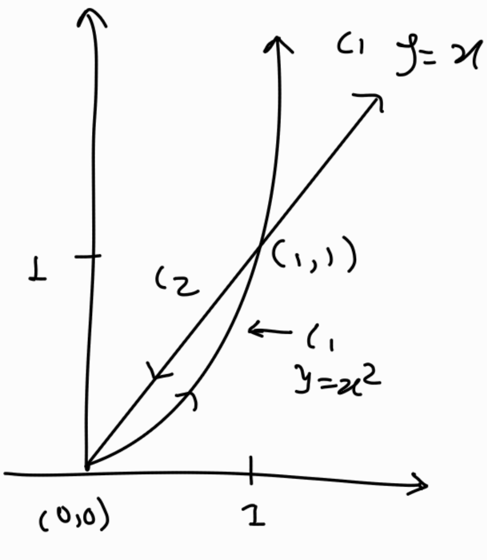
$$= \left[1 - 1\right]$$

$$\underline{C_3} = 0$$

$$C_1 + C_2 + C_3 = 0$$

Ex.2

$$F = [-y, -x]$$



Along C_1:

$r(t) =$ for $y = x^2$,

if $x = t$, $y = t^2$

$$\therefore r(t) = (t, t^2) \quad 0 \le t \le 1)$$

$$F(r(t)) = (-t^2, -t)$$

$$r'(t) = (1, 2t)$$

$$F(r(t)) \cdot r'(t) = -t^2 - 2t^2 = -3t^2$$

$$= \int_0^1 -3t^2$$

$$= -\left[\cancel{3}\frac{t^3}{\cancel{3}}\right]_0^1$$

$$= -\left[t^3\right]_0^1$$

$$= -\left[1 - 0\right]$$

$$C_1 = -1$$

C_2: $\qquad F = [-y, -x]$

$r(t) =$ for C_2, we have given $x = y$,

$\therefore r(t) = (t, t)$ s.t. $1 \leq t \leq 0$

$$r'(t) = (1, 1)$$

$$f(r(t)) = (-t, -t)$$

$$F(r(t)) \cdot r'(t) = -t - t$$

$$= -2t$$

$$\int F(r(t)) \cdot r'(t) = \int_1^0 -2t$$

$$= -\frac{2t^2}{2}$$

$$= \left[t^2\right]_1^0$$

$$= 0 - -1$$

$$= 0 + 1$$

$$\boxed{= 1}$$

$$\therefore C_2 = 1$$

$$\therefore C = C_1 + C_2$$

$$= -1 + 1$$

$$= 0$$

Path independence of line integral.

↓

whatever the path, always same resulting value

Line Integral is path independent iff $F = \nabla f$

$f =$ Potential Scalar field $\mathbb{R}^3 \to \mathbb{R}$

$F =$ Conservative vector field $\mathbb{R}^3 \to \mathbb{R}^3$

How to find if f exists?

Let $F = [F_1, F_2, F_3]$, if F_i is continuous and their 1st order PD's exist,

then, f exists iff $\underline{\text{curl}(F) = \bar{0}}$

How to Find 'f'?

find curl (F)

curl $(F) = 0$

No → Parameterization

Yes → f exists

↓

$$\int_C F \cdot dr = f(B) - f(A)$$

ex.
$$\int_{(0,0)}^{(4,\pi/8)} y\cos(xy)\,dx + x\cos(xy)\,dy$$

$F_1 = y\cos(xy)$

$F_2 = x\cos(xy)$

$F_3 = 0$

$F = [F_1, F_2, 0]$

$$\text{curl}(F) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y\cos(xy) & x\cos(xy) & 0 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}\left(\frac{\partial}{\partial x} x\cos xy - \frac{\partial}{\partial y} y\cos xy\right)$$

$$= 0 - 0 + \left(-x\sin(xy).\frac{\partial}{\partial x}xy \cancel{+\cos xy} + y\sin xy.\frac{\partial}{\partial y}xy \cancel{-\cos xy}\right)$$

$$= \left(\cancel{-xy\sin(xy)} \cancel{+xy\sin(xy)}\right)$$

$\text{curl}(F) = 0$

$\therefore \text{curl}(F) = 0,$

$\therefore f$ exists.

finding small f:

$\therefore$ since f exists, and we know $F = \nabla f$

but $F = [y\cos(xy), x\cos(xy)] = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right]$

$$\therefore \frac{\partial f}{\partial x} = y\cos(xy) \quad \text{-(i)}$$

$$\frac{\partial f}{\partial y} = x\cos(xy)$$

$$\int \frac{\partial f}{\partial x} = \int y\cos xy \, \partial x$$

$$= y\int \cos xy$$

$$= \cancel{y} \cdot \frac{\sin xy}{\cancel{y}}$$

$$\boxed{f = \sin xy + c(y)} \quad \ldots (2)$$

$$\frac{\partial f}{\partial y} = \frac{\partial c(y)}{\partial y} + \cos xy \cdot \frac{\partial}{\partial y} xy$$

$$\cancel{x \cos xy} = \frac{\partial c(y)}{\partial y} + \cancel{x \cos xy}$$

$$\frac{\partial c(y)}{\partial y} = 0$$

$$\int \frac{\partial c(y)}{\partial y} = \int 0$$

$$c(y) = 0 + k$$

$$\therefore f = \sin xy + k$$

let $k=0$

$\therefore f = \sin xy$

$$\therefore \int_c F = f(B) - f(A)$$

$$= f(4, \pi/8) - f(0,0)$$

$$= \sin\left(4 \times \frac{\pi}{8}\right) - \sin(0 \times 0)$$

$$= \sin(\pi/2) - \sin(0)$$

$$= 1 - 0$$

$$= 1$$

<u>3D curve:</u>

$$\int_{(2,3,0)}^{(0,1,2)} \left(z e^{xz}, 1, x e^{xz}\right) dr$$

$$\operatorname{curl}(F) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ze^{xz} & 1 & xe^{xz} \end{vmatrix} = \hat{i}(0-0) + \hat{j}\left(\frac{\partial}{\partial x} xe^{xz} - \frac{\partial}{\partial z} ze^{xz}\right)$$

$$+\hat{k}(0-0)$$

$$= \hat{j}\left(\cancel{1} + xe^{xz} \cdot z - \cancel{1} - ze^{xz} \cdot x\right.$$

$$= \hat{j}\left(xze^{xz} - xze^{xz}\right)$$

$$= \hat{j}(0) = 0$$

$\therefore \operatorname{curl}(F) = 0.$

$\therefore f$ exists.

$$F = \nabla f = \left[ze^{xz}, 1, xe^{xz}\right] = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]$$

$$\frac{\partial f}{\partial x} = ze^{xz} \quad \Bigg| \quad \frac{\partial f}{\partial y} = 1 \quad \Bigg| \quad \frac{\partial f}{\partial z} = xe^{xz}$$

$$\frac{\partial f}{\partial x} = ze^{xz}$$

$$\int \frac{\partial f}{\partial x} = z\int e^{xz}\, \partial x$$

$$f = \cancel{z} \cdot \frac{e^{xz}}{\cancel{z}} + c(y,z)$$

$$\frac{\partial f}{\partial y} = 0 + \frac{\partial c(y,z)}{\partial y}$$

$$\int 1 = \int \frac{\partial c(y,z)}{\partial y} \quad \ldots \because \frac{\partial f}{\partial y} = 1$$

$$y + k(z) = c(y,z)$$

$$f = e^{xz} + y + k(z)$$

$$\frac{\partial f}{\partial z} = x \cdot e^{xz} + 0 + \frac{\partial K(z)}{\partial z}$$

$$xe^{xz} - xe^{xz} = \frac{\partial K(z)}{\partial z}$$

$$\int \frac{\partial K(z)}{\partial z} = \int 0$$

$$K(z) = C$$

$$\therefore f = e^{xz} + y + C$$

let $C = 0$

$$\therefore f = e^{xz} + y.$$

$$\begin{aligned} \therefore \int_C F \cdot dr &= f(B) - f(A) \\ &= f(0,1,2) - f(2,3,0) \\ &= (e^{0 \cdot 2} + 1) - (e^{2 \cdot 0} + 3) \\ &= (1+1) - (1+3) \\ &= -2 \end{aligned}$$

Green's Theorem:

Relation between line integral and double integral.

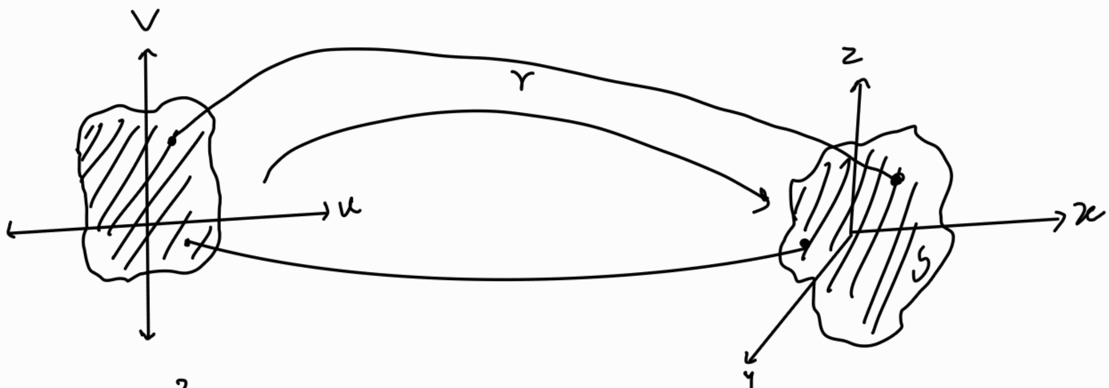
$$\oint_C F \cdot dr \longleftrightarrow \iint_R \left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right] \partial A$$

$F \to$ 2 dim vector field

$C \to$ closed bounded curve in $\mathbb{R}^2$

$$\oint_C F \cdot dr = \iint_R \left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right] \partial A$$

Parameterization of a surface:



$\mathbb{R}^2$

IF a function $r : R \to S$

such that, 1) image $(R) = S$ i.e. r: onto map.

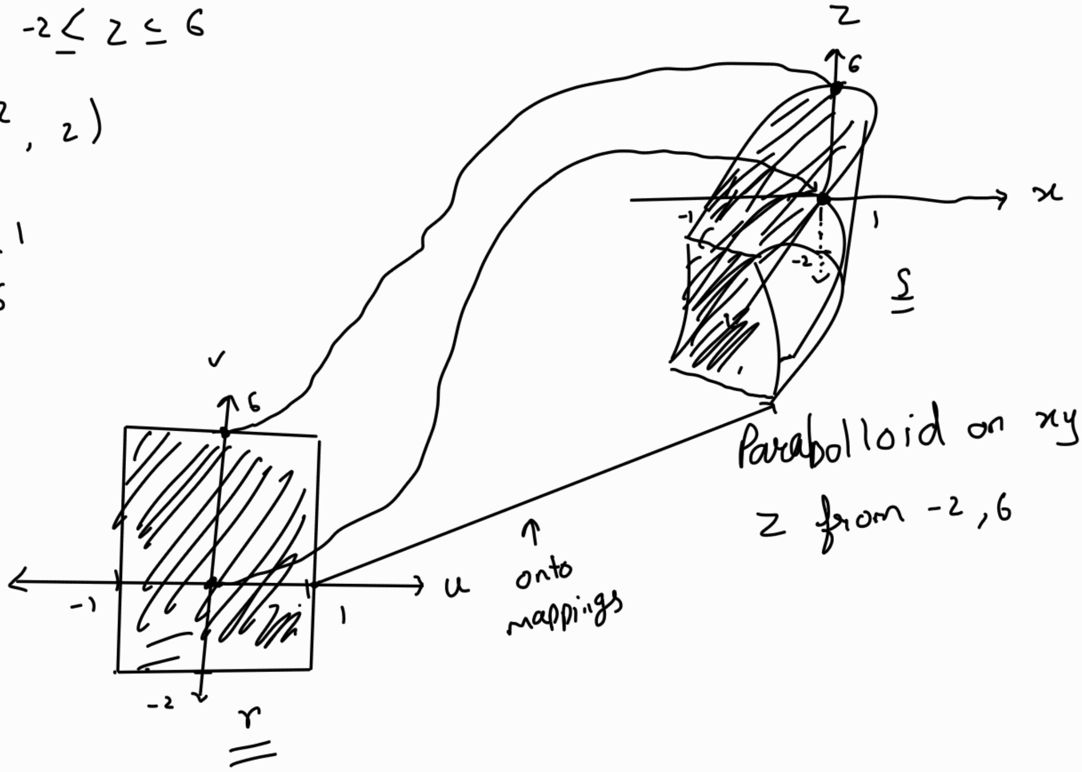
2) $r : (u, v) \mapsto [x(u,v), y(u,v), z(u,v)]$

e.g) $y = x^2, \ -2 \le z \le 6$

(u, u^2, z)

$-1 \le u \le 1$

$-2 \le z \le 6$



Surface Integrals:

$\text{Image}(r) = S$ .... parameterized surface

$r : R \to S$

then

$$\iint_R F \cdot n \, ds = \iint_R F(r(u,v)) \cdot N \, du \, dv$$

N : normal vector $\to$ a vector perpendicular to the tangent plane.

In question if mentioned '+ve orientation'/outward normal take +ve N, if -ve orientation/ inward normal then take N as -N.

e.g. $F = [2x, 2y, 2z]$

S : Cylinder $x^2 + y^2 = 9$, $0 \le z \le 5$

$r(u,v) = [3\cos u, 3\sin u, v]$ $\quad 0 \le u \le 2\pi$; $0 \le v \le 5$

$\therefore r_u = [-3\sin u, 3\cos u, 0]$

$$\therefore r_v = [0, 0, 1]$$

$$N = r_u \times r_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3\sin u & 3\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \hat{i}(3\cos u) - \hat{j}(-3\sin u) + \hat{k}(0-0)$$

$$N = (3\cos u, 3\sin u, 0)$$

$$F(r(u,v)) = [6\cos u, 6\sin u, 2v]$$

$$F(r(u,v)) \cdot N = 18\cos^2 u + 18\sin^2 u + 0$$

$$= 18(\sin^2 u + \cos^2 u)$$

$$= 18(1) = 18$$

$$\iint_S \overline{F} \cdot \hat{n}\, ds = \iint_R F(r(u,v)) \cdot \overline{N}\, du\, dv$$

$$= \int_{v=0}^{v=5} \int_0^{2\pi} 18\, du\, dv$$

$$= 18 \int_0^5 \left[u\right]_0^{2\pi}$$

$$= 18 \cdot 2\pi \left[v\right]_0^5$$

$$= 180\pi$$

Type 2: Scalar field, parameterized surface

if G is Scalar field,

we can't take dot product as G is scalar.

$$\iint_S G(r)\, \partial A = \iint_R G(r(u,v)) \cdot |N| \; \partial u \partial v$$

Gauss divergence Theorem:

Closed bounded region in space whose boundary is a piecewise smooth orientable surface S.

Let F be a vector function, then.

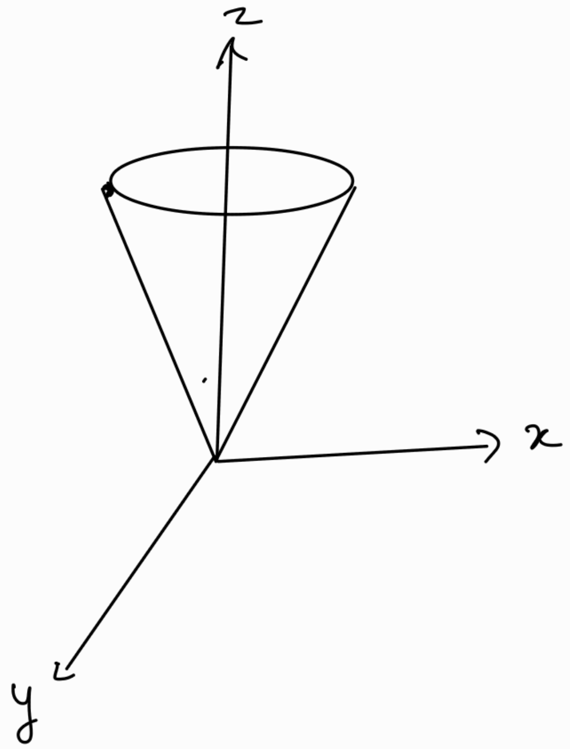
$$F = [F_1, F_2, F_3]$$

$$\iint_{\partial T} F \cdot n \, \partial A = \iiint_T \operatorname{div}(F)\, dv$$

e.g

1) $x^2+y^2 \le z^2$

$0 \le z \le 2$



$F = [4x, 3z, 5y]$

$div(F) = [4+0+0]$

$div(F) = 4$

$$\iint_S F\cdot\hat{n}\, ds = 4 \iiint_T 1 \cdot dv$$

$$= 4 \int_0^2\int_0^2\int_0^2 1 \, dx\, dy\, dz$$

$= 4 \text{ vol(cone)}$

$= 4 \times \frac{2}{3}\pi r^2 h$

$\because h = z = 2$

$\because r = 2$

$= \frac{4 \times \pi \times 4 \times 2}{3}$

$= \frac{32}{3}\pi$

$= 33.51$

② $S = x^2+y^2+z^2 = 9$ $F = [x, y, z]$

$\therefore$ for sphere of $r = 3$

$\therefore$

$\operatorname{div}(F) = (1+1+1) = 3$

$$\iint_S F\cdot\bar{n}\, dS = \iiint 3\, dV$$

$= 3 \operatorname{vol}(V)$

$= \cancel{3} \times \frac{4}{\cancel{3}} \pi r^3$

$= 4 \times 3^3 \times 3.14$

$= 339.30$

③ $F = [z-y, y^3, 2z^3]$

$S: y^2+z^2 \le 4, -3 \le x \le 3$

$\operatorname{div} F = 0 + 3y^2 + 6z^2$

$$\iint_S F\cdot n\, dA = 3\iiint (y^2 + 2z^2)\, dV$$

$$= \int_0^{2\pi}\int_0^{2}\int_{-3}^{3} (r^2\cos^2\theta + 2r^2\sin^2\theta)\, r \, dx\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{2}\int_{-3}^{3} r^3 \Big(\underbrace{\sin^2\theta + \cos^2\theta}_{1} + \underbrace{\sin^2\theta}_{\downarrow} \Big)$$

$$1 + \frac{1-\cos 2\theta}{2}$$

$$= \int_0^{2\pi}\int_0^{2} r^3 \int_{-3}^{3} 1 + \frac{1-\cos 2\theta}{2}$$

$\vdots$

#

Stoke's theorem:

• Piecewise smooth oriented surface in space and let $c = \partial S$

where C: piecewis smooth simple closed.

let F be vector function:

$$\iint_S (\text{curl } F) \cdot \bar{n}\, ds = \oint F \cdot r'(s)\, ds$$

Green's theorem — Line integral $\rightarrow$ double

Gauss's Divergence — surface integral $\rightarrow$ triple integral

Stoke's theorem — line integral $\rightarrow$ surface integral

---

e.g. $F = [\,xz, yz, xy\,]$

$\text{curl } F = [\,x-y, x-y, 0\,]$

$C: x^2+y^2=4 \quad \& \quad z=1$

Find: $\oint_C \bar{F} \cdot dr$

Soln: $r(t) = (2\cos t, 2\sin t, 1)$

$r'(t) = (-2\sin t, 2\cos t, 0)$

$F(r(t)) = [\,2\cos t\,(1), 2\sin t \cdot 1, 4\cos t \sin t\,]$

$F(r(t)) \cdot r'(t) = -4\cancel{\sin t}\cos t + \cancel{4\cos t}\sin t + 0$

$F(r(t)) \cdot r'(t) = 0$

$\therefore \oint_C F \cdot dr = 0$