U3:

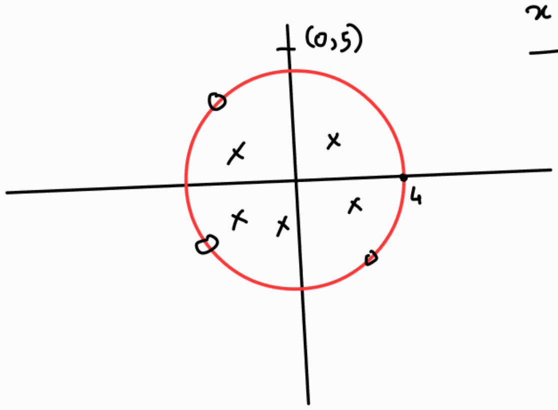
$f: \mathbb{R} \rightarrow \mathbb{R}$ ... UVC

$f: \mathbb{R}^n \rightarrow \mathbb{R}$ .... $n \geq 2$ MVC

e.g.) $f(x,y) = \sqrt{16 - x^2 - y^2}$

Domain $\Rightarrow D = \{(x,y) \mid 16 - x^2 - y^2 \geq 0$ i.e. $16 \geq x^2 + y^2\}$

$\underline{x^2+y^2 = 16}$ ← circle of radius 4



x— Interior points

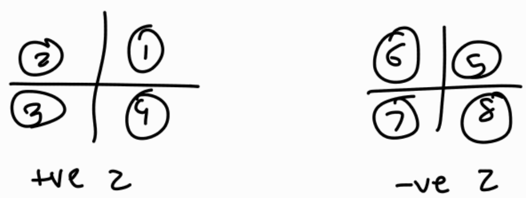
o – Boundary points.

when we have $f(x,y) = \frac{1}{\sqrt{16-x^2-y^2}}$ we get D as $\{f(x,y) \mid \underline{16 > x^2+y^2})$

this is to avoid $\frac{1}{0}$, ∴ we will consider only internal points here.

2) $f(x,y,z) \longleftrightarrow \log(xyz)$

$D = \{(x,y,z) \mid xyz > 0\}$



D will be the following octants:

$1^{st}, 3rd, 6^{th}, 8^{th}$

The 8 octants.

HW)

① $f(x,y) = \log(x^2+y^2)$

$D = \{f(x,y) \mid x^2+y^2 > 0\} \Rightarrow D = \{f(x,y) \mid (x,y) \in \mathbb{R}^2\}$

② $f(x,y,z) = \sqrt{16 - x^2 - y^2 - z^2}$

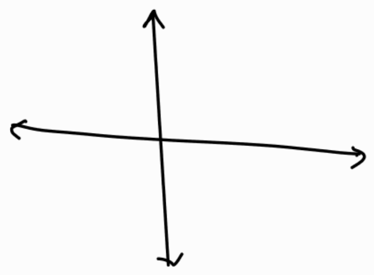
$D = \{f(x,y) \mid 16 \geq x^2+y^2+z^2\}$

③ $f(x,y) = \sin(xy) \Rightarrow D = \{f(x,y) \mid \mathbb{R}^2\}$

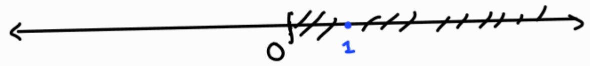
Level curve

All points in Domain D, that result in the same image on the Range of a function.

$f(x,y) = x^2+y^2$.



(D)



(R)

Curve of 1 is all points that result in 1 as image.

i.e. here the circle of radius 1.

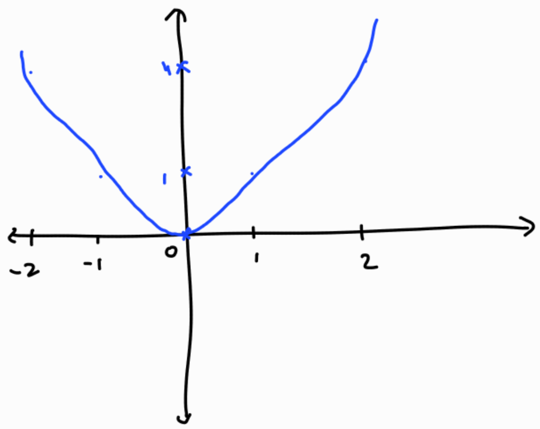
$x^2+y^2=1$. All points on boundary.

Curve of 0 has only (0,0).

Curve of any -ve number is $\{\emptyset\}$ as x^2+y^2 will never be -ve.

Graph of a function:

$f(x) = x^2$



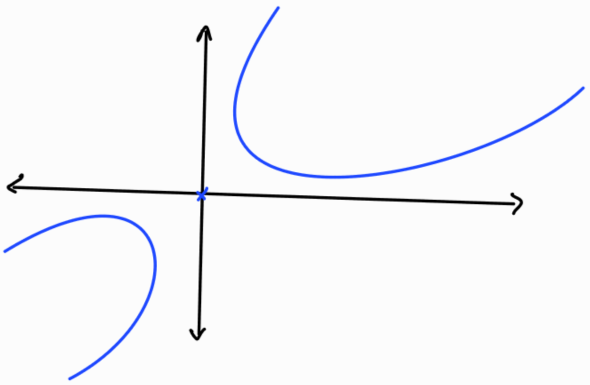
If f is a function from $\mathbb{R}$ to $\mathbb{R}$

then graph (F) $\subseteq \mathbb{R}^{n+1}$ $\qquad \downarrow \mathbb{R}^2$

Upward Parabolloid $(x,y,f(x,y))$ $\underline{\mathbb{R}^3}$.

H.W

$f(x,y) = xy$



Rectangular hyperbola.

Domain:

① Bounded

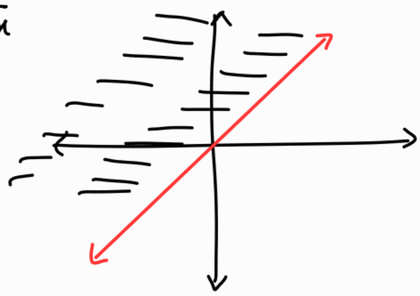
- If you're able to enclose a domain in a circle of fixed radius.

② Unbounded

- Unbounded close/open
  - If no circle of fixed radius possible then,

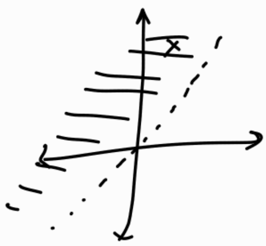
ⓧ if some boundary is included, unbounded closed

$(x,y) = \sqrt{y-x}$



ⓧ If boundary is not included, i.e. only interior points are considered.

$(x,y) = \dfrac{1}{\sqrt{y-x}}$



Unbounded Open.

Bounded

↳ open → boundary not included

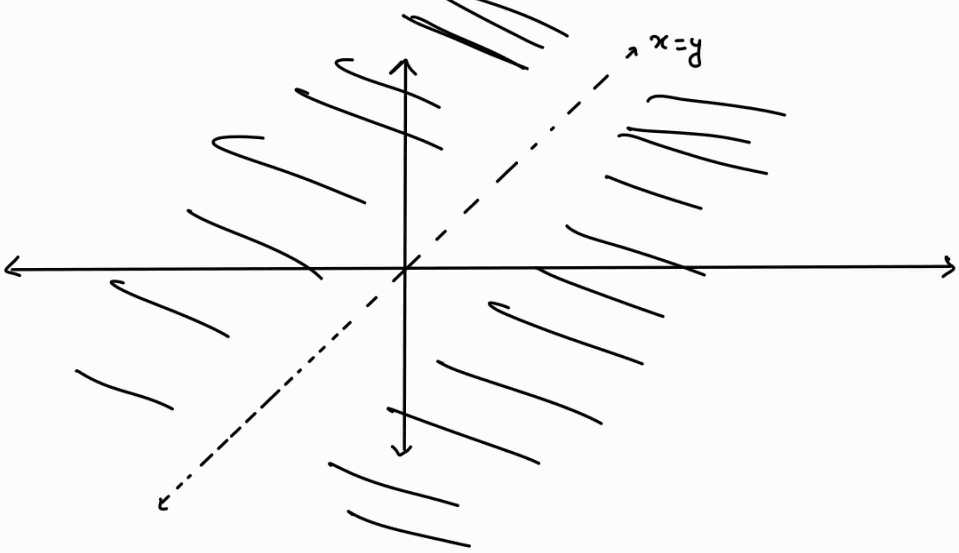
↳ closed → boundary included.

When in R^3, we try to enclose domain in a Sphere.

1) HW

① $(x,y) \longrightarrow \dfrac{1}{x-y}$

$D = \{ (x,y) \in \mathbb{R}^2 \mid x-y \neq 0 \text{ i.e. } x \neq y \}$



① Cannot be enclosed in a Circle.

∴ Unbounded.

② Does not include boundary

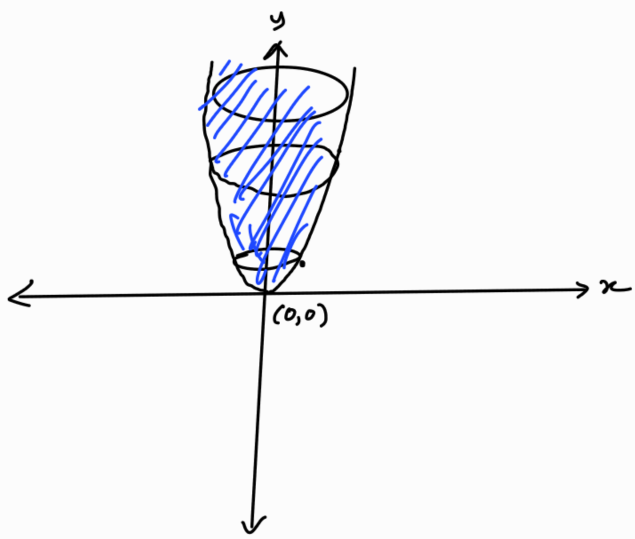
∴ open.

∴ Open Unbounded.

② $f(x,y) \longrightarrow \sqrt{y-x^2}$

$D ((x,y) \in \mathbb{R}^2 \mid y-x^2 \geq 0)$

$y \geq x^2$

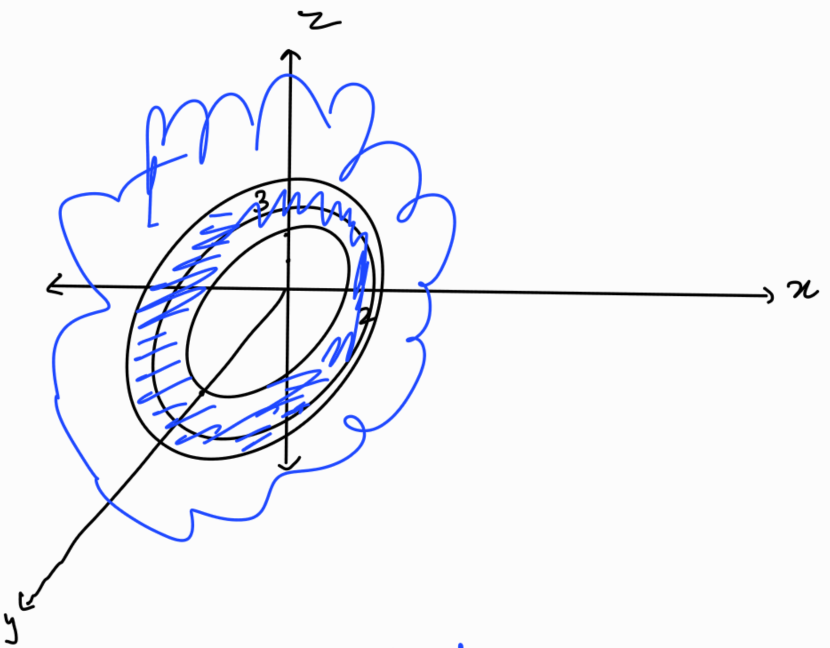


Unbounded Closed

③ $f(x,y) = \sqrt{\frac{x^2}{4}+\frac{y^2}{9}-1}$

$$D = \{(x,y) \in \mathbb{R}^2 \mid \frac{x^2}{4}+\frac{y^2}{9}-1 \geq 0\}$$

$$\frac{x^2}{4}+\frac{y^2}{9} \geq 1$$



Unbounded Closed

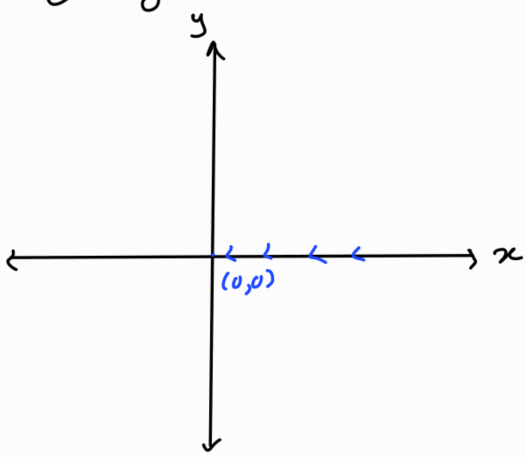
. Show Limit does not exist using different Paths:

Q) $f(x,y) = \frac{y}{\sqrt{x^2+y^2}}$

Show that,

$\lim_{(x,y)\to(0,0)} f(x,y)$ Does Not Exist .

① Along x- axis

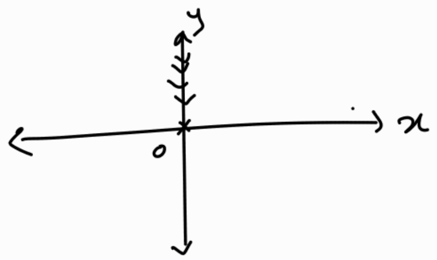


along x-axis, y=0

$\therefore \lim_{x\to(0,0)} f(x,0) = \frac{0}{\sqrt{x^2+0^2}} = \frac{0}{\sqrt{x^2}} = 0$

$\therefore \lim_{x\to(0,0)} F(x,0) = 0$

Along y-axis:



$$\lim_{\substack{y \to 0 \\ x = 0}} f(x,y) = f(0,y) = \frac{y}{\sqrt{0^2+y^2}} = \frac{y}{\sqrt{y^2}} = \frac{y}{y} = 1$$

$\because$ we are getting different values among different paths. We arrive at contradiction

$\therefore$ Limit Does not exist.

Find local Minima & Maxima:

$f(x,y) = \dots$

find f_x, f_y

If $f(x,y)$ has local max or min at (a,b) & partial derivative exists then $f_x(a,b) = 0$ and $f_y(a,b) = 0$.

Hessian Matrix:

$f(x,y) \rightarrow \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}_{2\times 2}$

for $\mathbb{R}^2 \rightarrow \mathbb{R}$

Δ_1 - Principal Minor $\rightarrow f_{xx}$

Δ_2 is 2×2 matrix's $|D|$.

① if $\Delta_1 > 0$ & $\Delta_2 > 0$

Local Minima

② $\Delta_1 < 0$ & $\Delta_2 < 0$

Local maxima

$\Delta_2 < 0$

Saddle point

$f(x,y,z) \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}$

for 3×3

$\Delta_1 = f_{xx}$

$\Delta_2 = |\quad|_{2\times2}$

$\Delta_3 = |\quad|_{3\times3}$

if $\Delta_1, \Delta_2, \Delta_3 > 0$ local Minima

if $\Delta_1 < 0$ Neg
$\Delta_2 > 0$ pos
$\Delta_3 < 0$ Neg
local Maxima

if

local

Order	Principal Minors	max / min / SP
2×2	$\Delta_1 > 0$, $\Delta_2 > 0$	Min
	$\Delta_1 < 0$, $\Delta_2 > 0$	Max
	$\Delta_2 < 0$	SP
3×3	$\Delta_1 > 0$, $\Delta_2 > 0$, $\Delta_3 > 0$	Min
	$\Delta_1 < 0$, $\Delta_2 > 0$, $\Delta_3 < 0$	Max
	$\Delta_2 < 0$	SP
4×4.	$\Delta_1, \Delta_2, \Delta_3, \Delta_4 > 0$	Min
	$\Delta_1 < 0$, $\Delta_2 > 0$, $\Delta_3 < 0$, $\Delta_4 > 0$	Max
	$-$ $+$ $-$ $+$	(Max)
	$D_2 < 0$	S.P.

if any of following are zero inconclusive.

$f(x,y) = x^3 + 3xy + y^3$

$f_x = 3x^2 + 3y = 0$ | $x^2 = -y \quad \ldots ① \Rightarrow y = -x^2$

$f_y = 3x + 3y^2 = 0$ | $y^2 = -x$

$(-x^2)^2 = -x$

$x^4 = -x$

$x^4 + x = 0$

$x(x^3+1) = 0$

$x = 0$ or $x^3 = 1$

i.e. $x = 1$

$\therefore x = 0, 1$

if $x = 0$, $y^2 = -0 \Rightarrow y = 0$

$x = 1$, $x^2 = -y$

$1^2 = -y$

$-y = 1$

$y = (0, -1)$

$(x, y) = (0,0), (-1,-1)$

<u>Hessian Matrix</u>:

$$\begin{bmatrix} 6x & 3 \\ 3 & 6y \end{bmatrix}$$

$\Delta_1 = 6x > 0$

$\Delta_2 = 36xy - 9$

for $(0,0)$, $\Delta_2 = 36(0) - 9$

$= -9.$

$\therefore \Delta_2$ is Negative, Saddle point at $(0,0)$

for $-1, -1$

$\Delta_1 = 6(-1)$

$\Delta_1 = -6$

$\Delta_2 = 36 - 9$

$= 25$

$\therefore \Delta_1 < 0$, $\Delta_2 > 25$

$\therefore$ local Max at $(-1, -1)$

① $f(x,y) = x^2 + xy + y^2 + 3x - 3y + 4$

$f_x = 2x + y + 3 = 0$

$f_y = x + 2y - 3 = 0$

$3x + 3y = 0$

$3(x - y = 6)$

$3x + 3y = 0$

$+ 3x - 3y = 18$

$6x = 18$

$x = 3$

$y = -3$

$\therefore (x, y) = (3, -3)$

$$\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$$

$\Delta_1 > 0$

$\Delta_2 = 2 \times 2 - 1$

$\Delta_2 = 3 > 0$

$\therefore$ local Minima at $(3, -3)$

$f(x,y,z) = x^2 - xy + y^2 + yz + z^2 - 2z$

$f_x = 2x - y = 0 \rightarrow y = 2x \quad \dots ①$

$f_y = -x + 2y + z = 0 \rightarrow -x + 4x + z \longrightarrow z = -3x \quad \dots ②$

$f_z = y + 2z - 2 = 0 \longrightarrow 2x + 2(-3x) - 2 = 0$

$2x - 6x = 2$

$-4x = 2$

$x = -\frac{2}{4}$

$\boxed{x = \frac{-1}{2}}$

$y = 2x$

$y = \cancel{2} \cdot \frac{-1}{\cancel{2}}$

$y = -1$

$z = -3x$

$= -3\left(-\frac{1}{2}\right)$

$\boxed{z = \frac{3}{2}}$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \Delta_1 = 2$$

$\Delta_2 = 4 - 1$

$\Delta_2 = 3$

$\Delta_3 = 2(3) - (-1(-2 - 0))$

$+ 0$

$= 6 - (2)$

$= 6$

$\Delta_1, \Delta_2, \Delta_3 > 0$

$\therefore$ at pt $\left(-\frac{1}{2}, -1, \frac{3}{2}\right)$ local minima

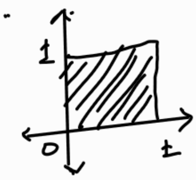
Absolute Maxima/Minima:

over closed & bounded domain

Q) $f(x,y) = 48xy - 32x^3 - 24y^2$

$0 \le x \le 1$

$0 \le y \le 1$



Case 1: Interior points

First derivative test

Case 2: Boundary points

Case 1:

$f_x = 48y - 96x^2 = 0$

$f_y = 48x - 48y = 0 \rightarrow 48x = 48y \Rightarrow x = y$

$-96x^2 + 48x = 0$

$-2x^2 + x = 0$

$x(-2x + 1) = 0$

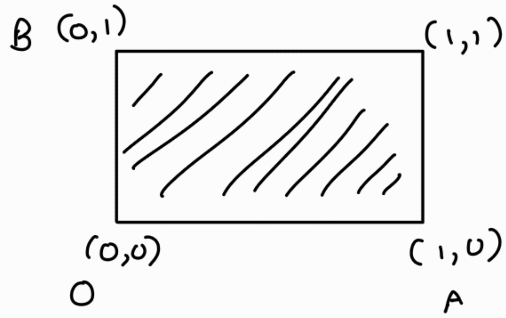
$x = 0$ or $x = \frac{-1}{-2}$

$x = 0$ or $x = \frac{1}{2}$ $\quad \therefore y = x$ or $\frac{1}{2}$

points $(0,0)$, $(1/2, 1/2)$.

$(0,0)$ is not critical point since its boundary point.

Case II) Boundary points:



for OA, $y=0$.

$f(x,y) = f(x,0) = 32x^3$

$f'(x) = 96x^2 = 0$

$= x^2 = 0$

$\Rightarrow x = 0$.

but (x,y) does not lie on the interior of OA. $\therefore$ It cannot be a critical point.

for OB, $x=0$

$f(0,y) = -24y^2$

$f'(0,y) = -48y = 0$

$\boxed{y=0}$

$\boxed{x=0}$

but $(0,0)$ is not interior point on this line.

on BC, $y=1$

$f(x,y) = 48x - 32x^3 - 24$

$f'(x,y) = 48 - 96x^2 = 0$

$= x^2 = \frac{48}{96}$

$= x^2 = 1/2$

$= x = \pm\sqrt{1/2}$ ... $-$ve not in domain.

$= x = +\frac{1}{\sqrt{2}}$

$\therefore$ C.P. $= \left(\frac{1}{\sqrt{2}}, 1\right)$, it is interior on BC (0,1) to (1,1).

on AC, $x = 1$,

$f(1,y) = 48y - 32 - 24y^2$

$f'(1,y) = 48 - 48y$

$= \boxed{y = 1}$

$\therefore (x,y) = (1,1)$ $\therefore$ Not CP as it is a boundary pt.

$\therefore$ CP are $(1/2, 1/2)$ and $(1/\sqrt{2}, 1)$
and corner points $(0,0), (1,0), (0,1), (1,1)$

$f(1/2, 1/2) = 12 - 4 - 6 = 2$

$f(1/\sqrt{2}, 1) = -1.37$

$f(0,0) = 0$

$f(1,0) = -32$

$f(1,1) = -8$

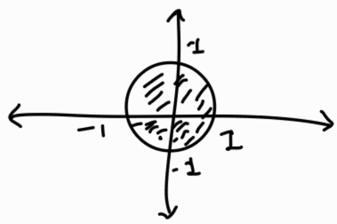
$f(0,1) = -24$

Absolute Maxima $= (1/2, 1/2)$
with value 2

Absolute Minima at $f(1,0)$
with value -32.

Q.2 $T(x,y) = x^2 + 2y^2 - x$

shape $x^2 + y^2 \leq 1$



$T_x = 2x - 1 = 0$

$T_y = 4y = 0$

$x = 1/2$

$y = 0$

CP $= (1/2, 0)$ in Interior

Boundary points

$$x^2+y^2=1$$
$$y^2 = 1-x^2$$

$$T(x) \Rightarrow x^2+2y^2-x = 0$$
$$= x^2+2(1-x^2)-x = 0$$
$$T(x) = -x^2-x+2$$
$$t'(x) = -2x-1 = 0$$
$$2x = -1$$
$$x = \frac{-1}{2}$$

$$y^2 = 1-x^2$$
$$= 1-\left(\frac{-1}{2}\right)^2$$
$$= 1-\frac{1}{4}$$
$$y^2 = 3/4$$
$$y = \pm\sqrt{3}/4$$

$\therefore$ C.P are $(1/2, 0)$, $(-1/2, \sqrt{3}/2)$, $(-1/2, -\sqrt{3}/2)$

$(1/2,0) \downarrow -1/4$

$(-1/2, \sqrt{3}/2) \downarrow 9/4$

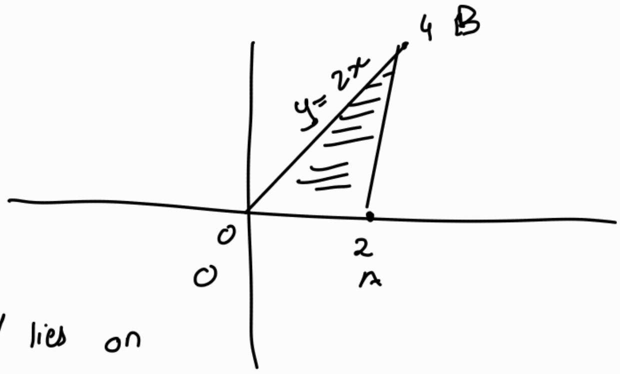
$(-1/2, -\sqrt{3}/2) \downarrow$

$$\frac{1}{4}+\cancel{2}\cdot\frac{3}{\cancel{4}2}-\left(\frac{-1}{2}\right)$$
$$\frac{1}{4}+\frac{4}{2}$$
$$9/4$$

Absolute minima at $-1/4$

No abs max as 2pts with local max values.

$f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$



$f_x = 4x - 4 \rightarrow 4x = 4 \rightarrow x = 1$

$f_y = 2y - 4 \rightarrow 2y = 4 \rightarrow y = 2$

$(x,y) = (1,2)$ not a C.P. as Y lies on boundary

Case 2: Along OA, $x = 0$

$f(0,y) = y^2 - 4y + 1$

$f_y = 2y - 4 \rightarrow \boxed{y = 2}$

$x = 0$

$(0,2)$ is boundary along OA.

Along OB, $y = 2x$

$f(x, 2x) = 2x^2 - 4x + (2x)^2 - 4(2x) + 1$

$= 2x^2 - 4x + 4x^2 - 8x + 1$

$= 6x^2 - 12x + 1 = 0$

$= 6x^2 - 12x = -1$

$x^2 - 2x = \frac{-1}{6}$

$x(x-2) = -\frac{1}{6}$

$x = -\frac{1}{6}$ or $x = -\frac{1}{6} + 2$

$x = \frac{-1}{6}$ $\quad x = \frac{11}{6} \rightarrow 1.9\ldots\ldots$

$y = -2/6$ or $y = \frac{22}{6}$

$\downarrow$ outside region $\quad \downarrow$ $3.67\ldots.$

$(11/6, 22/6)$ is an interior on OB. $\therefore$ CP along AB, $x = 2$.

$f(2,y) = 2(4) - 4(2) + y^2 - 4y + 1$

$= \cancel{8} - \cancel{8} + y^2 - 4y + 1$

$f' : 2y - 4 \Rightarrow y = 2 \quad (x,y) = \underline{2},2$

$(2,2)$ is interior of AB

$\therefore$ <u>c.P.</u>

$\left(\frac{11}{6}, \frac{22}{6}\right) \rightarrow -0.833$

$(2,2) \rightarrow -3$

$(0,0) \rightarrow 1$

$(0,2) \rightarrow -3$

$(2,4) \rightarrow 1$