College of Engineering Pune.
MVC and DE
Tutorial on Vector Calculus and Integration
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1. Define a vector function and a scalar function. Give an example of each.
2. Define the derivative of a vector function. What is its significance in mechanics and in geometry?
3. Define gradient of a scalar function, divergence and curl of a vector function and explain their physical significance.
4. Sketch the following curves and identify them:
   (a) $\overline{r(t)} = [2 + 4cost, 2sint, 0]$
   (b) $\overline{r(t)} = [-2, 2 + 5cost, -1 + 5sint]$
5. Find the parametric representation of the circle in the $yz-$ plane with center $(4, 0)$ and passing through $(0, 3)$. Sketch it.
6. Find the parametric representation of the helix $x^2 + y^2 = 25, z = arctan(y/x)$. Sketch it.
7. Find the tangent and the unit tangent vector to the given curve at the given point:
   (a) $\overline{r(t)} = [cost, sint, 9t]$ Point $P(1, 0, 18\pi)$
   (b) $\overline{r(t)} = [t, 4/t, 0]$ Point $P(4, 1, 0)$
8. Let $f = xy - yz$, $\overline{v} = [2y, 2z, 4x + z]$, $\overline{w} = [3z^2, 2x^2 - y^2, y^2]$. Find
   (a) div(grad f) (b) grad(div $\overline{w}$) (c) div(curl$\overline{v}$) (d) $D_{\overline{w}}f$ at (1,1,0) (e) [(curl $\overline{v}$) $\times$ $\overline{w}$] $\cdot$ $\overline{w}$
9. For $f = x^2 - y^2$ and $g = e^{x+y}$, verify div $(f\nabla g)$−div $(g\nabla f) = f\nabla^2 g - g\nabla^2 f$.
10. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point (2,-1,2).
11. Find potential field f for given $\overline{v}$ or state that $\overline{v}$ has no potential.
    (a) $[xy, 2xy, 0]$ (b) $[x^2 - yz, y^2 - zx, z^2 - xy]$
12. If $\overline{u}$ and $\overline{v}$ are irrotational, then show that $\overline{u} \times \overline{v}$ is incompressible.
13. The velocity vector $\overline{v} = [x, y, -z]$ of a fluid motion is given. Is the flow irrotational? Incompressible?
14. Define line integral and surface integral of a vector valued function along a curve and along surface resp.
15. State the theorems which link line integral with surface integral and surface integral with volume integral
16. What is path independence of a line integral?
17. State the role of gradient in vector integral calculus

18. Calculate $\int_C \overline{F}(r) \cdot \overline{dr}$ for the following data -

(a) $\overline{F} = [xy,\ x^2y^2]$, C is the quarter circle from $(2,0)$ to $(0,2)$ with center at $(0,0)$.Ans: 8/5

(b) $\overline{F} = [xy,\ x^2y^2]$, C is the straight line from $(2,0)$ to $(0,2)$. Ans: -4/15

(c) $\overline{F} = [x-y,\ y-z,\ z-x]$, C : $[2\cos t,\ t,\ 2\sin t]$ from $(2,0,0)$ to $(2,2\pi,0)$. Ans: $2\pi^2 - 8\pi$

(d) $\overline{F} = [coshx,\ sinhy,\ e^z]$, C : $[t, t^2, t^3]$ from $(0,0,0)$ to $(2,4,8)$.Ans: $sinh2 + cosh4 + e^8 - 2$

(e) $\overline{F} = [ze^{xz},\ 2\sinh(2y), xe^{xz}]$, C is the parabola $y = x, z = x^2, -1 \leq x \leq 1$.Ans: $e - e^{-1}$

19. Check the following integrals for path independence. In the case of independence evaluate them.

(a) $\int_{(\pi/2,-\pi)}^{(\pi/4,0)} (\cos x \cos 2y dx - 2\sin x \sin 2y dy)$ Ans: path independent. $\frac{\sqrt{2}-2}{2}$

(b) $\int_{(0,0,0)}^{(1,1,1)} (ye^z dy - ze^y dz)$ Ans: path dependent

(c) $\int_{(\pi,\pi/2,2)}^{(0,\pi,1)} (-z\sin(xz)dx + \cos y dy - x\sin(xz)dz)$ Ans: -1

20. Using Green's theorem, evaluate the line integral $\oint_C \overline{F}(r) \cdot \overline{dr}$ counterclockwise around the boundary C of the region R, where

(a) $\overline{F} = [3y^2, x - y^4]$, R is the square with vertices (1,1), (-1,1), (-1,-1), (1,-1). Ans: 4

(b) $\overline{F} = [2xy^3, 3x^2y^2], C : x^4 + y^4 = 1$. (Sketch the curve).Ans: 0

(c) $\overline{F} = [xcosh2y, 2x^2sinh2y]$, $R : x^2 \leq y \leq x$. Ans: $(sinh2 - cosh2 + 1)/4$

(d) $\overline{F} = [xe^{-y^2}, -x^2ye^{-y^2}]$, R : region that is bounded by the square of side $2a$ determined by the inequalities $|x| \leq a, |y| \leq a$ Ans: 0

21. Find a parametric representation of the following surfaces :

(a) Plane $3x + 4y + 6z = 24$ Ans: one out of many $\mathbf{r}(u,v) = [u, v, (24 - 3u - 4v)/6]$

(b) Elliptic cylinder $9x^2 + 4y^2 = 36$. Ans: one out of many $\mathbf{r}(u,v) = [2cosu, 3sinu, v]$

22. Evaluate $\int\int_S \overline{F} \cdot \hat{n}\ \ dA$. Indicate the kind of surface in (a) and (b).

(a)$\bar{F} = [3x^2, y^2, 0]$; S : $\bar{r} = [u, v, 2u + 3v]$; $0 \leq u \leq 2$, $-1 \leq v \leq 1$. Ans: -36, plane$z = 2x + 3y; 0 \leq x \leq 2, -1 \leq y \leq 1$

(b) $\bar{F} = [\sinh(yz), 0, y^4]$; S : $\bar{r} = [u, \cos v, \sin v]$; $-4 \leq u \leq 4$, $0 \leq v \leq \pi$. Ans:$-16/5$, Cylinder $y^2 + z^2 = 1, -4 \leq x \leq 4$

(c) $\bar{F} = [ax, by, cz]$; S : entire surface of the sphere $x^2 + y^2 + z^2 = d^2$, Ans: $-4\pi d^3(a+b+c)/3$

(d) $\bar{F} = [x, z^2 - zx, -xy]$; S : the triangular surface with vertices $(2,0,0), (0,2,0)$ and $(0,0,4)$. Ans: 92/3

23. Evaluate $\displaystyle\int\int_S \overline{F}\cdot\hat{n}\ \ dA$ by using the divergence theorem.

(a)$\bar{F} = [e^x, e^y, e^z]$; S : the surface of the cube $|x| \leq 1, \quad |y| \leq 1, \quad |z| \leq 1$

(b) $\bar{F} = [x^3, y^3, z^3]$; S : the surface of the sphere $x^2 + y^2 + z^2 = 4$

24. Verify Stoke's Theorem for the following data.

(a)$\bar{F} = [y^2, z^2, x^2]$; S : the portion of the paraboloid $x^2 + y^2 = z, \quad y \geq 0, \quad z \leq 1$ Ans: -4/3

(b) $\bar{F} = [y^2, -x^2, 0]$; S : the circular semidisk $x^2 + y^2 \leq 4; \quad y \geq 0, \quad z = 1$ Ans: -32/3

25. (a) Evaluate $\displaystyle\int_c x^2y dx + 2xy^2 dy$ from (0,0) along a straight line segment to (1,1/2), and then along a straight line segment to (1,1).

(b) Evaluate $\displaystyle\int_c x^2y dx + 2xy^2 dy$ from (0,0) along a straight line segment (1/2,1), and then along a straight line segment to (1,1).

Is $I = \displaystyle\int_c x^2y dx + 2xy^2 dy$ path dependent?

Ans: 37/48 ; 55/96 ; Yes

26. Verify Green's theorem for $F_1 = 3x^2 - 8y^2, F_2 = 4y - 6xy$ and C is the boundary of the region defined by $y = \sqrt{x}$ and $y = x^2$. Ans: 3/2