Predicates & Quantifiers

Universal and Existential

Predicate Logic

- A predicate is an expression of one or more variables defined on some specific **domain**. A predicate with variables can be made a proposition by either assigning a value to the variable or by quantifying the variable.
- The following are some examples of predicates −
  - Let E(x, y) denote "x = y"
  - Let X(a, b, c) denote "a + b + c = 0"
  - Let M(x, y) denote "x is married to y“
  - Let P(x) denote “x is greater than 3”
    - In last statement first part variable x ,is the subject of the statement, the second part is predicate “is greater than 3”, P(x) is a propositional function P at x.

Example

- Let P(x) is x>3 what are the truth values for P(2) and P(4)? Unary
- Let Q(x,y) denote “x=y+3” what are the truth values for Q(1,2) and Q(3,0)? Binary
- Let R(x,y,x) denote “x+y=z” what are the truth values for R(1,2,3) & R(0,0,1)?
- Similarly for P(x1,x2,…xn) can be a value for n tuple, and P is also known as Predicate. N-ary predicate

Example

- Let $P(x; y; z)$ denote that $x + y = z$ and U (Universe of Discourse) be the integers for all three variables.
  - $P(-4; 6; 2)$ is true.
  - $P(5; 2; 10)$ is false.
  - $P(5; x; 7)$ is not a proposition.

Quantifiers

- We need quantifiers to formally express the meaning of the words "all" and "some".
- The two most important quantifiers are:
  - Universal quantifier, "For all". Symbol: $\forall$
  - Existential quantifier, "There exists". Symbol: $\exists$
- $\forall x\, P(x)$ asserts that $P(x)$ is true for **every x in the domain.**
- $\exists x\, P(x)$ asserts that $P(x)$ is true for **some x in the domain.**
- The quantifiers are said to bind the variable x in these expressions.
- Variables in the scope of some quantifier are called **bound variables**. All other variables in the expression are called **free variables.**
- A propositional function that does not contain any free variables is a proposition and has a truth value.

Quantifiers

- The variable of predicates is quantified by quantifiers. There are two types of quantifier in predicate logic −
  - Universal Quantifier and
  - Existential Quantifier.

Universal Quantifier

- Universal quantifier states that the statements within its scope are true for every value of the specific variable. It is denoted by the symbol $\forall$
- $\forall x P(x)$ is read as for every value of x, P(x) is true.
- **Example** –
  - "Man is mortal" can be transformed into the propositional form $\forall x P(x)$
  - where P(x) is the predicate which denotes x is mortal and the universe of discourse is all men.

Existential Quantifier

- Existential quantifier states that the statements within its scope are true for some values of the specific variable. It is denoted by the symbol $\exists$
- $\exists x P(x)$ is read as for some values of x, P(x) is true.
  - **Example** − "Some people are dishonest" can be transformed into the propositional form $\exists x P(x)$
  - where P(x) is the predicate which denotes x is dishonest and the universe of discourse is some people.

Uniqueness Quantifier

- $\exists!\, x\, P(x)$ means that there exists one and only one x in the domain such that P(x) is true.
- $\exists_1!\, x\, P(x)$ is an alternative notation for $\exists!x\, P(x)$.
- This is read as

  There is one and only one x such that P(x).

  There exists a unique x such that P(x).
- **Example: Let P(x) denote x + 1 = 0 and U are the integers.**
  - Then $\exists!x\, P(x)$ is true.
- **Example: Let P(x) denote x > 0 and U are the integers.**
  - **Then** $\exists!x\, P(x)$ is false.
- The uniqueness quantifier can be expressed by standard operations. $\exists!x\, P(x)$ is equivalent to

$$\exists x\, (P(x) \wedge \forall y\, (P(y) \rightarrow y = x)).$$

- Quantifiers $\forall$ and $\exists$ have higher precedence then all logical operators.
- $\forall x\, P(x) \wedge Q(x)$ means $(\forall x\, P(x)) \wedge Q(x)$ In particular, this expression contains a free variable.
- $\forall x\, (P(x) \wedge Q(x))$ means something different.

Example

- Translate the following sentence into predicate logic: "Every student in this class has taken a course in Java."
- Solution:
  - First decide on the domain U (Universe of discourse).
  - Solution 1: If U is all students in this class, define a propositional function J(x) denoting "x has taken a course in Java" and translate as $\forall x\, J(x)$.
  - Solution 2: But if U is all people, also define a propositional function S(x) denoting "x is a student in this class" and translate as $\forall x\ (S(x) \rightarrow J(x))$
- Note: $\forall x\ (S(x) \wedge J(x))$ is not correct. What does it mean?

- Some student in this class has visited Mexico
  - means that
  - “There is a student in this class with the property that the student has visited Mexico.”
  - We can introduce a variable x, so that our statement becomes
  - “There is a student x in this class having the property that x has visited Mexico.”
  - $M(x)$, which is the statement “x has visited Mexico
  - If the domain for x consists
  - of the students in this class, we can translate this first statement as $\exists x M(x)$.
  - if we are interested in people other than those in this class,
  - “There is a person x having the properties that x is a student in this class and x has visited Mexico.”
  - $S(x)$ to represent “x is a student in this class.”
  - Solution: $\exists x(S(x) \wedge M(x))$

- "Every student in this class has visited either Canada or Mexico"
  - $C(x)$ be "x has visited Canada."
  - domain for x consists of
  - the students in this class, this second statement can be expressed as $\forall x(C(x) \vee M(x))$.
  - if the domain for x consists of all people
  - "For every person x, if x is a student in this class, then x has visited Mexico or x has visited Canada."
  - $\forall x(S(x) \rightarrow (C(x) \vee M(x)))$.