Time Response Analysis of Systems

Motivation

- How to know the performance of a control system for any input signal?
- How to design a control system which meets the desired response and control requirements ?

Time Domain Analysis and Design Specifications

Time Domain Analysis

- Time domain analysis refers to the analysis of system performance in time i.e., the study of evolution of system variables (specifically output) with time
- There are two common ways of analyzing the response of systems:
    1. Natural response and forced response
    2. Transient response and steady state response
- In both cases, the complete response of the system is given by the combination of both responses i.e., natural and forced responses or transient and steady state responses

Natural and Forced Responses

- **Natural response (Zero input response) :**
  - System's response to initial conditions with all external forces set to zero
  - E.g. In RLC circuits, this would be the response of the circuit to initial conditions (inductor currents or capacitor voltages) with all the independent voltage and current sources set to zero
- **Forced response (Zero state response) :**
  - System's response to external forces with zero initial conditions
  - E.g. In RLC circuits, this would be the response of the circuit to only external voltage and current source, and zero initial conditions

Transient and Steady State Responses

- **Transient response $y_{tr}(t)$:**
  - Part of the time response that goes to zero as time tends to be large
  - Transient response can be tied to any event that affects the equilibrium of a system viz. switching, disturbance, change in input, etc.

$$\lim_{t \to \infty} y_{tr}(t) = 0$$

- **Steady state response $y_{ss}(t)$:**
  - Steady state response is the time response of a system after transient practically vanishes and as time goes to infinity

$$y(t) = y_{tr}(t) + y_{ss}(t)$$

Standard Test Inputs

- In most cases, the input signals to a control system are not known prior to design of control system
- Hence to analyse the performance of a control system, it is excited with standard test signals
- In general, control system design specifications are also based on the response of the system to such test signals
- Standard test signals include:
  - Unit impulse, unit step (sudden change), ramp (constant velocity), parabolic (constant acceleration) and sinusoidal
  - These inputs are chosen because they capture many of the possible variations that can occur in an arbitrary input signal

Initial Value Theorem (IVT)

- Relates the $s-$ domain expressions to the time domain behaviour as time approaches zero:

$$\lim_{t \to 0} x(t) = \lim_{s \to \infty} sX(s)$$

- E.g. $x(t) = 3 + 4\cos t$

$$\lim_{t \to 0} x(t) = 3 + 4 = 7$$

$$\lim_{s \to \infty} sX(s) = \lim_{s \to \infty} s\left(\frac{3}{s} + \frac{4s}{s^2 + 1}\right)$$

$$\Rightarrow \lim_{s \to \infty}\left(3 + \frac{4s^2}{s^2 + 1}\right) = \lim_{s \to \infty}\left(3 + \frac{4}{1 + \frac{1}{s^2}}\right) = 3 + 4 = 7$$

- **Note:** IVT is applicable only in the cases where the Laplace transform exists and its limit exists as $s \to \infty$

Final Value Theorem (FVT)

- Relates the $s-$ domain expressions to the time domain behaviour as time approaches infinity:

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

➢ E.g. $x(t) = \frac{1-e^{-2t}}{2}$

$$\lim_{t\to\infty} x(t) = \frac{1+0}{2} = \frac{1}{2}$$

$$\lim_{s\to 0} sX(s) = \lim_{s\to 0} \frac{s}{2}\left(\frac{1}{s} - \frac{1}{s+2}\right) = \frac{1}{2} - \lim_{s\to 0} \frac{1}{s+2} = \frac{1}{2}$$

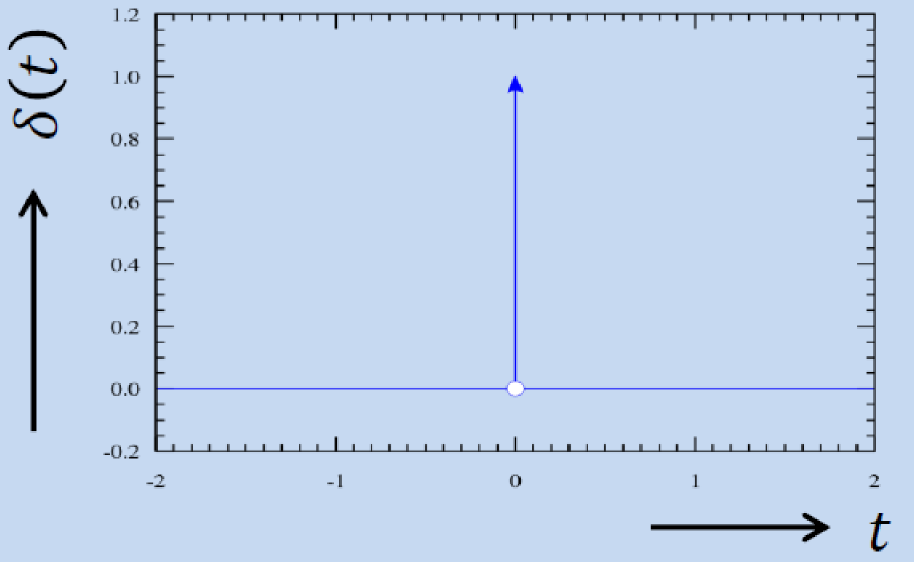
- **Note:** FVT is applicable only in the cases where the Laplace transform exists and its limit exists as $s \to 0$, and also final value should exist

Review: Standard Test Inputs

- **Unit impulse signal:**
  - A signal which is non-zero only at $t = 0$ and integrates to one

$$\int_{-\infty}^{\infty} \delta(t) = 1$$

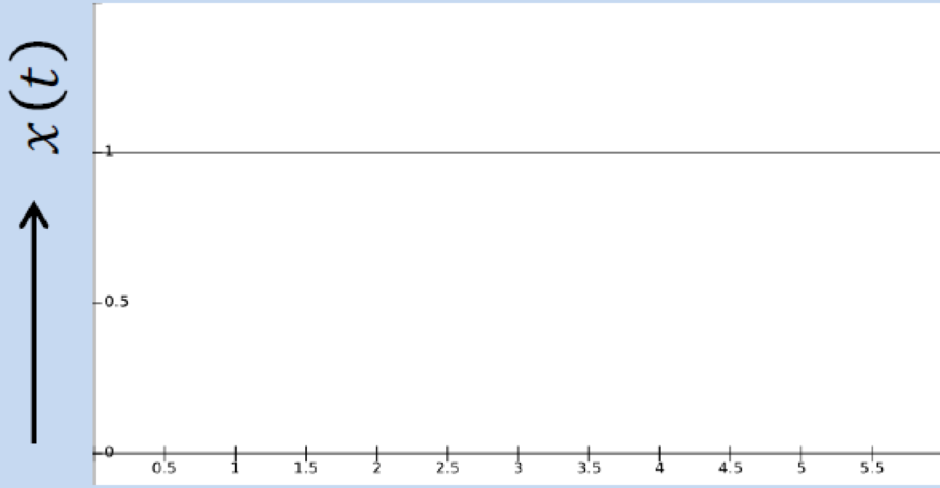
$$\mathcal{L}\{\delta(t)\} = 1$$



- **Unit step signal:**
  - A signal that switches to one at a time instant and stays there indefinitely

$$x(t) = \begin{Bmatrix} 1 \ \forall \ t > 0 \\ 0 \ \forall \ t < 0 \end{Bmatrix}$$

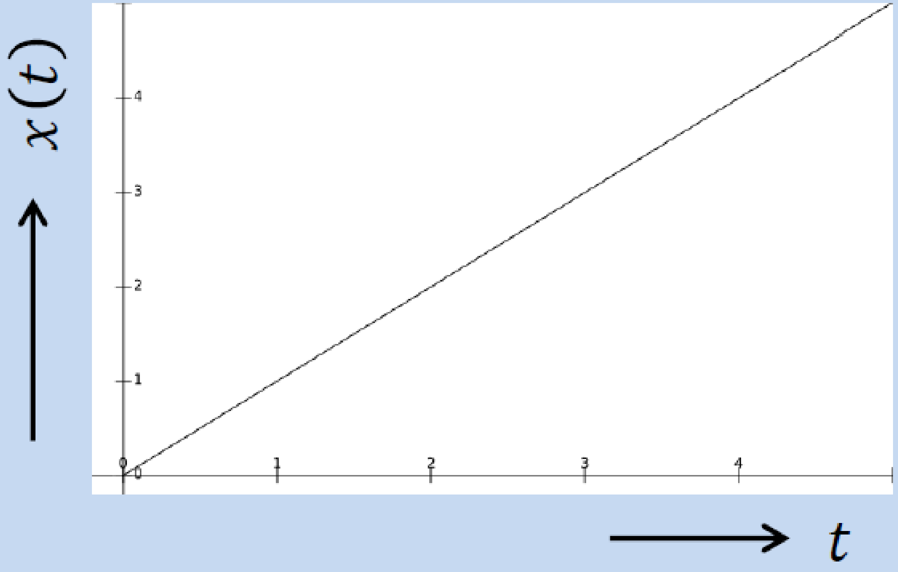
$$\mathcal{L}\{x(t)\} = \frac{1}{s}$$



- **Ramp signal:**
  - A signal which increases linearly with time

$$x(t) = \begin{Bmatrix} At \ \forall \ t \geq 0 \\ 0 \ \forall \ t < 0 \end{Bmatrix}$$

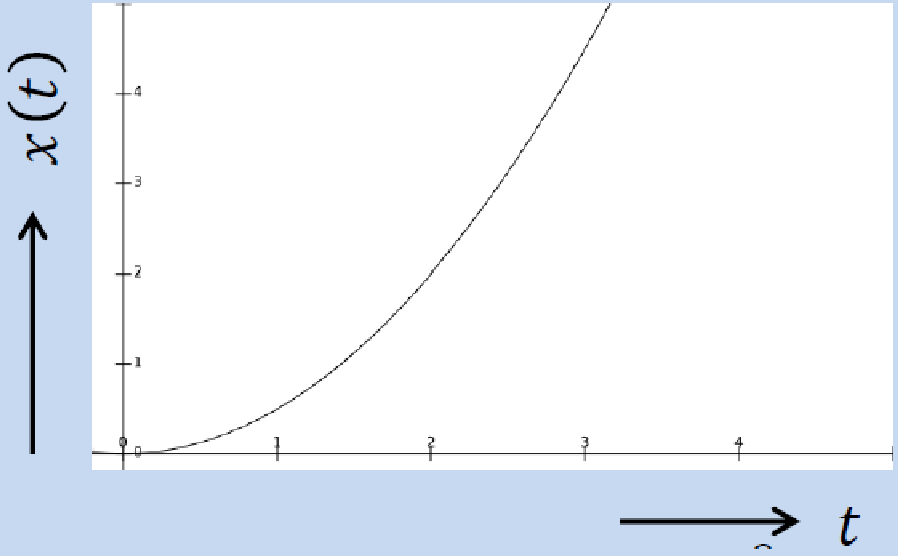
$$\mathcal{L}\{x(t)\} = \frac{A}{s^2}$$



- **Parabolic signal:**

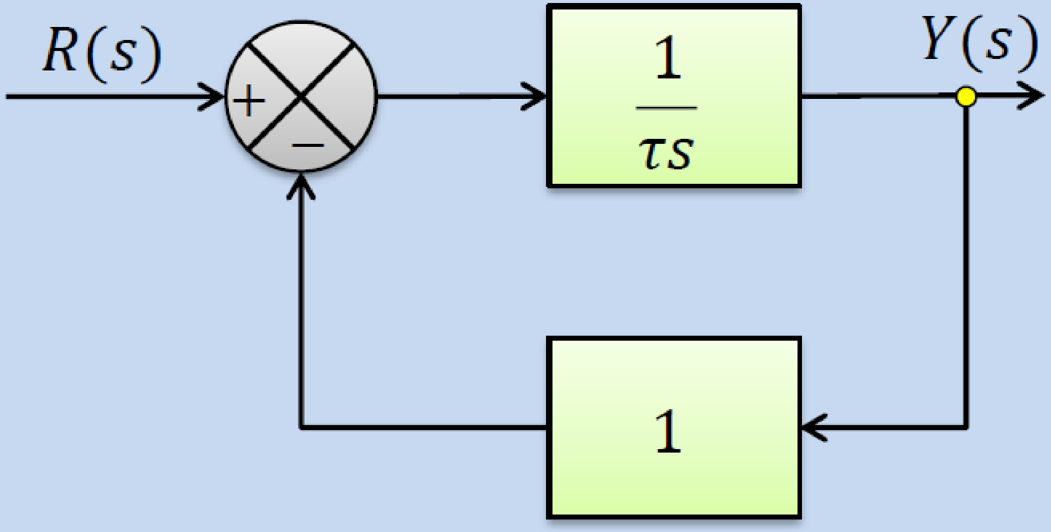
$$x(t) = \begin{Bmatrix} \frac{At^2}{2} \ \forall \ t \geq 0 \\ 0 \ \forall \ t < 0 \end{Bmatrix}$$

$$\mathcal{L}\{x(t)\} = \frac{A}{s^3}$$



1st Order Systems

- Systems with only one pole are called 1st order systems



Standard block diagram of a 1st order system

$$TF = \frac{Y(s)}{R(s)} = \frac{1}{\tau s + 1}$$

τ: System time constant

– It characterizes the speed of response of a system to an input

– Higher the time constant, slower the response and vice-versa

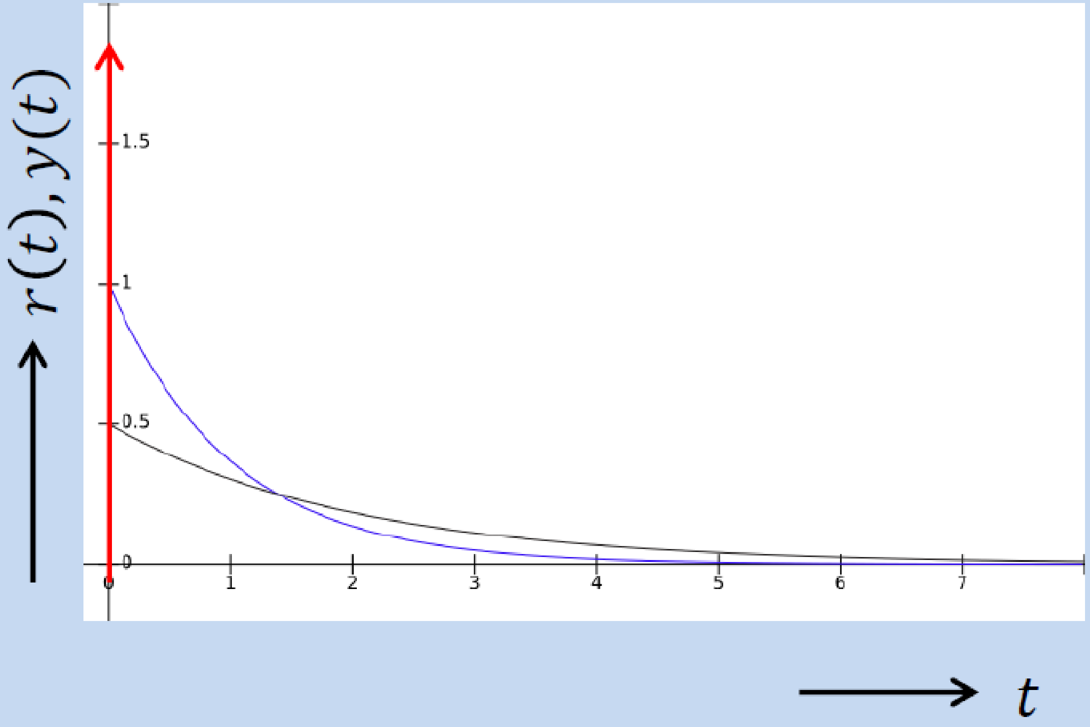
Impulse Response of 1st Order Systems

Unit impulse: $R(s) = 1$

$$Y(s) = \frac{1}{\tau s + 1} R(s) = \frac{1}{\tau s + 1}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{\tau} e^{-\frac{t}{\tau}}$$

- τ is the time constant of the system
- $\frac{1}{\tau} e^{-\frac{t}{\tau}}$ is the transient term $y_{tr}(t)$ while the steady state term $y_{ss}(t) = 0$



$r(t)$ and $y(t)$ when $\tau = 1, \tau = 2$

Step Response of 1st Order Systems

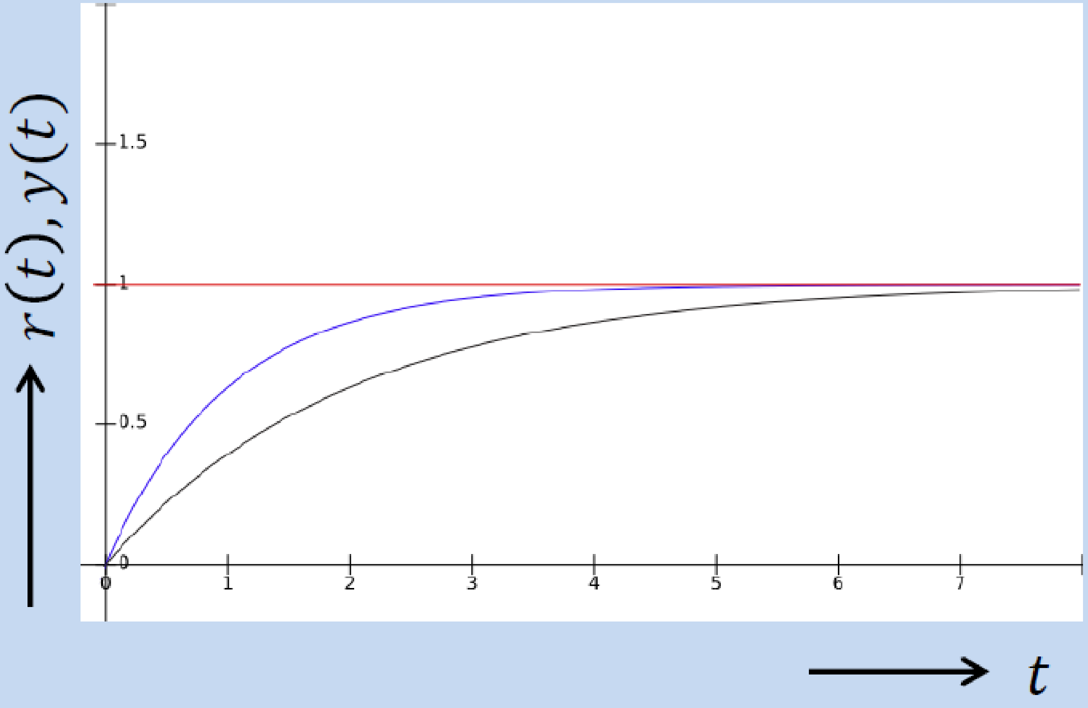
Unit step: $R(s) = \frac{1}{s}$

$$Y(s) = \frac{1}{\tau s + 1} R(s) = \frac{1}{s(\tau s + 1)}$$

$$= \frac{1}{s} - \frac{\tau}{\tau s + 1}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = 1 - e^{-\frac{t}{\tau}}$$

- In this case, $t_{tr}(t) = -e^{-\frac{t}{\tau}}$ and $t_{ss}(t) = 1$



$r(t)$ and $y(t)$ when $\tau = 1, \tau = 2$

Ramp Response of 1st Order Systems

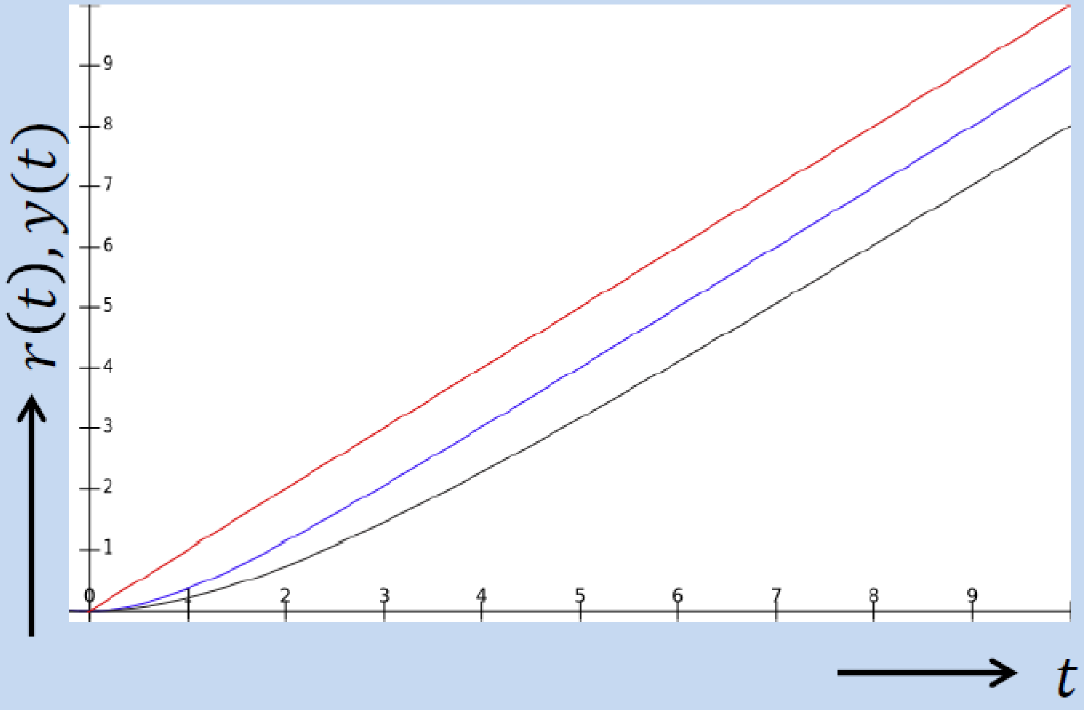
Unit ramp: $r(t) = t \implies R(s) = \frac{1}{s^2}$

$$Y(s) = \frac{1}{\tau s + 1} R(s) = \frac{1}{s^2(\tau s + 1)}$$

$$= \frac{1}{s^2} - \frac{\tau}{s} + \frac{\tau^2}{\tau s + 1}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = t - \tau + \tau e^{-\frac{t}{\tau}}$$

$$t_{tr}(t) = \tau e^{-\frac{t}{\tau}} \; ; \; t_{ss}(t) = t - \tau$$



$r(t)$ and $y(t)$ when $\tau = 1, \tau = 2$

Parabolic Response of 1st Order Systems

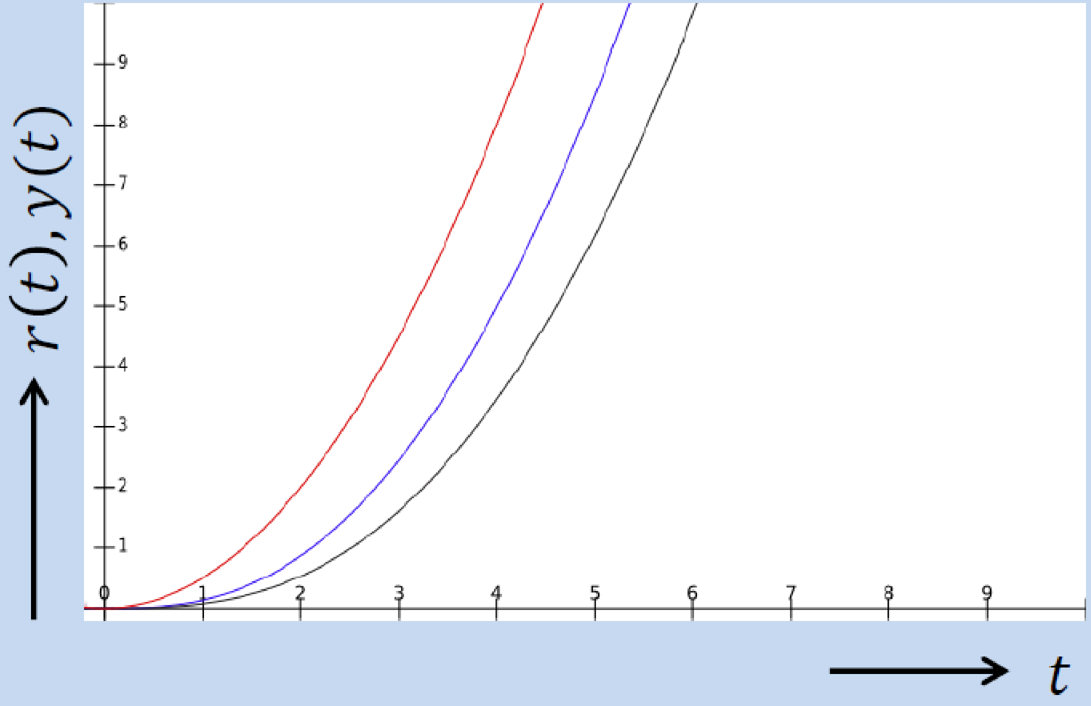
Unit parabolic: $r(t) = \frac{t^2}{2} \Longrightarrow R(s) = \frac{1}{s^3}$

$$Y(s) = \frac{1}{\tau s + 1} R(s) = \frac{1}{s^3(\tau s + 1)}$$

$$= \frac{1}{s^3} - \frac{\tau}{s^2} + \frac{\tau^2}{s} - \frac{\tau^3}{\tau s + 1}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{t^2}{2} - \tau t + \tau^2 - \tau^2 e^{-\frac{t}{\tau}}$$

$$t_{tr}(t) = -\tau^2 e^{-\frac{t}{\tau}}; t_{ss}(t) = \frac{t^2}{2} - \tau t + \tau^2$$

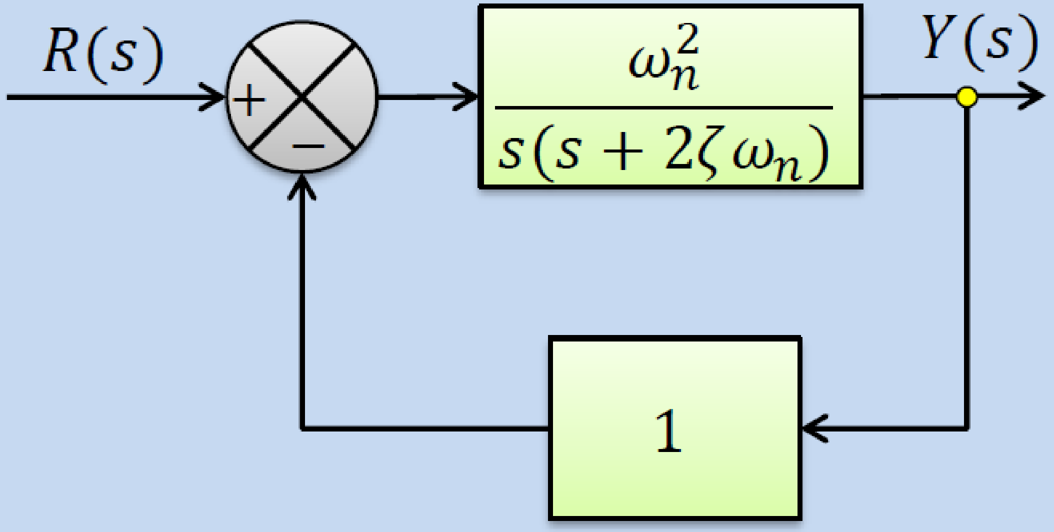


$r(t)$ and $y(t)$ when $\tau = 1, \tau = 2$

2nd Order Systems

- Systems with two poles are called 2nd order systems
- E.g. An RLC circuit or mass-spring-damper system
- For an RLC circuit : $TF = \frac{1}{s^2 LC + sRC + 1}$
- For an MSD system : $TF = \frac{1}{Ms^2 + Bs + K}$
- In general, the transfer function of a 2nd order system can be written as: $TF = \frac{b}{s^2 + as + b}$
- To study and understand the response of a 2nd order system, its transfer function is written in terms of certain system parameters

Standard Form of 2nd Order System



Block diagram of a 2nd order system

$$TF = \frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Standard form of transfer function

ω_n: System natural frequency

ζ: System damping ratio

Important System Parameters

- **System damping ratio ζ:** a dimensionless quantity describing the decay of oscillations during a transient response
- *It is the ratio of actual damping to critical damping of a system*
- **System natural frequency ω_n:** angular frequency at which system tends to oscillate in the absence of damping force
- **System damped frequency ω_d:** angular frequency at which system tends to oscillate in the presence of damping force

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

Response of 2nd Order Systems

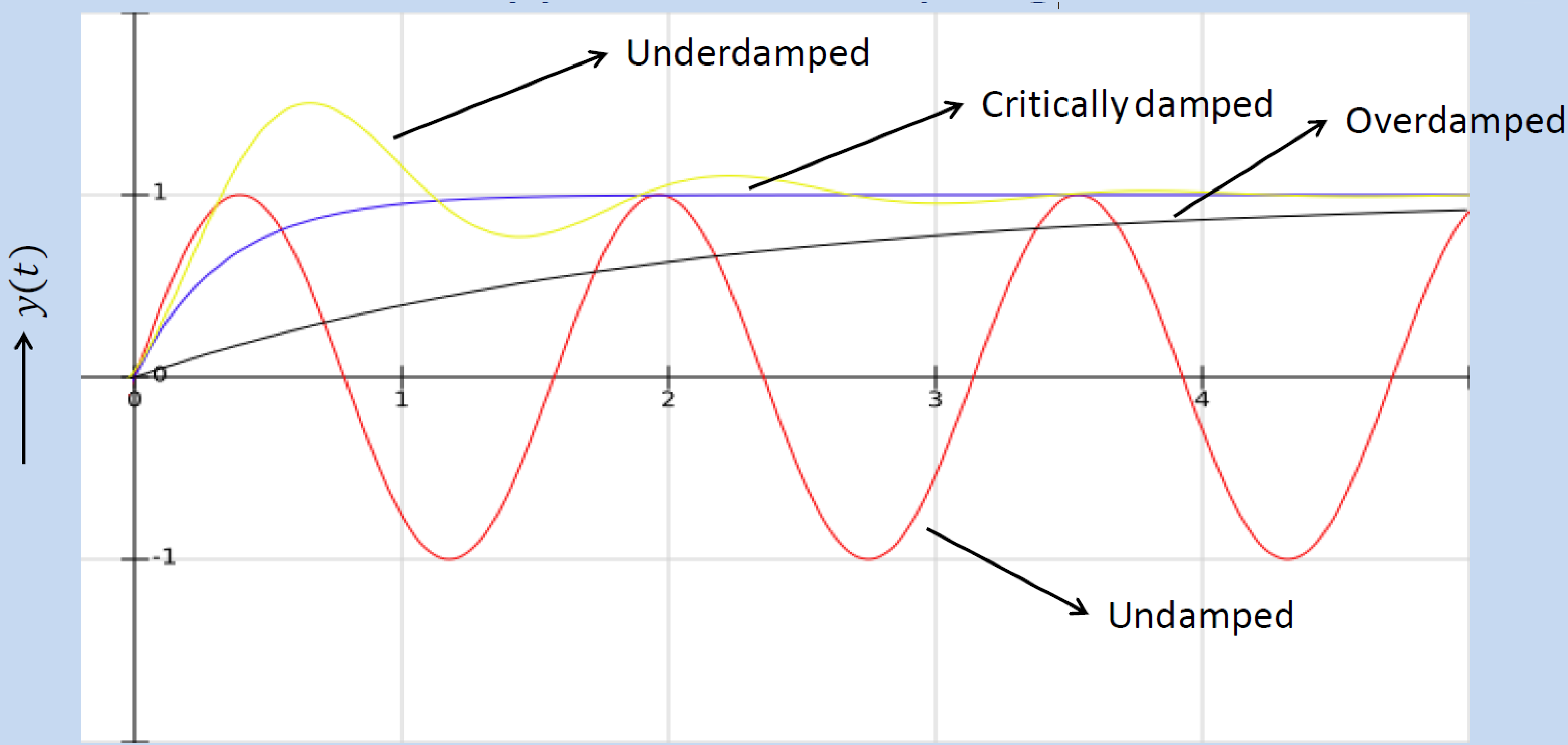
- Response of 2nd order systems mainly depends on the damping ratio ζ
- For any test input, the response of a 2nd order system can be studied in four cases depending on the damping effect created by value of ζ:
  1. $\zeta > 1$: Overdamped system
  2. $\zeta = 1$: Critically damped system
  3. $0 < \zeta < 1$: Underdamped system
  4. $\zeta = 0$: Undamped system

- **Note:** We do not consider negative damping ratio ζ because negative damping actually means the oscillations are increasing in amplitude which results in unstable systems

Damping and Types of Damping

- Damping is an effect created in an oscillatory system that reduces, restricts or prevents the oscillations in the system
- Systems can be classified as follows depending on damping effect:
  - **Overdamped systems:** Transients in the system exponentially decays to steady state without any oscillations
  - **Critically damped systems:** Transients in the system decay to steady state without any oscillations in shortest possible time
  - **Underdamped systems:** System transients oscillate with the amplitude of oscillation gradually decreasing to zero
  - **Undamped systems:** System keeps on oscillating at its natural frequency without any decay in amplitude

Types of Damping



Impulse Response of 2nd Order Systems

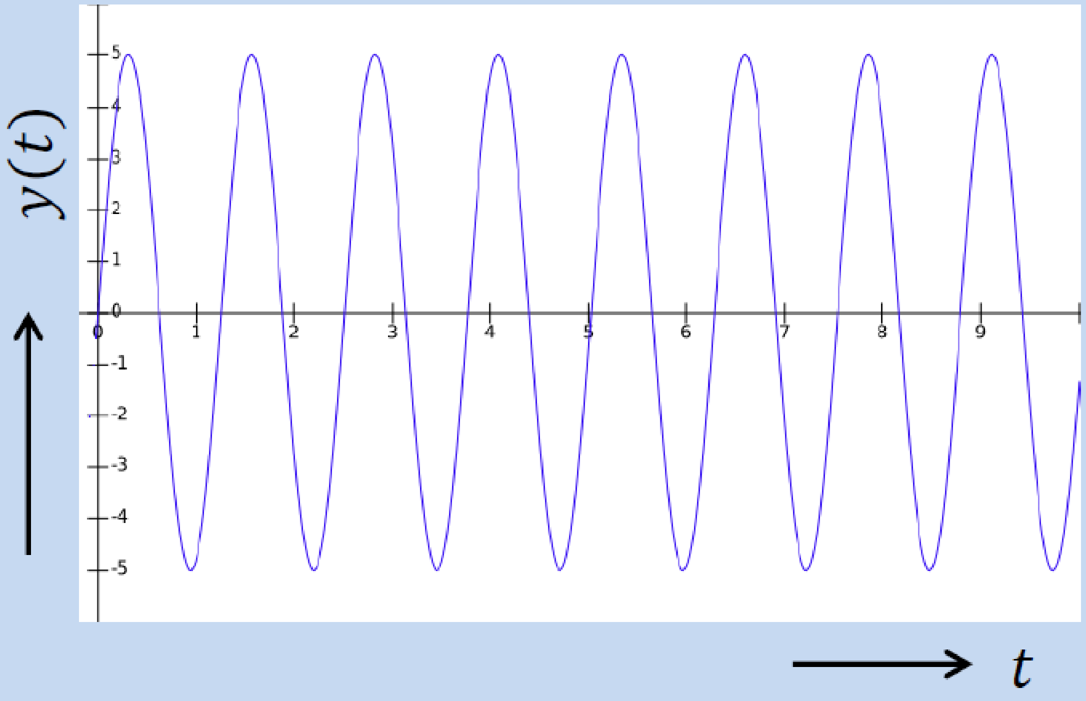
Unit impulse: $R(s) = 1$

$$Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Case 1: $\zeta = 0$ – Undamped system

$$Y(s) = \frac{\omega_n^2}{s^2 + \omega_n^2}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \omega_n \sin(\omega_n t)$$



$y(t)$ when $\zeta = 0, \omega_n = 5$

Impulse Response of 2nd Order Systems

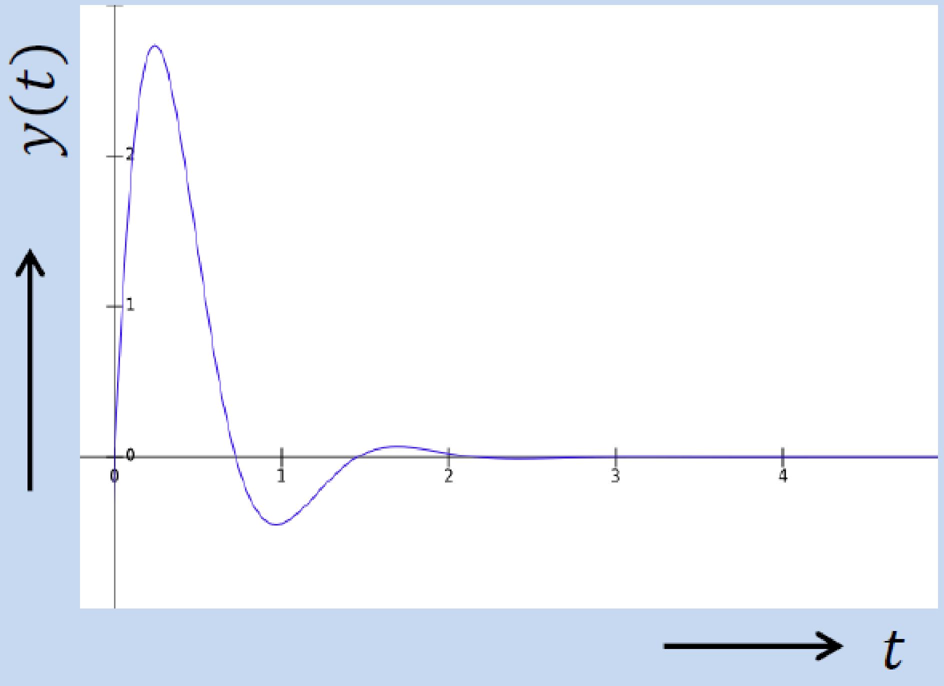
Case 2: $0 < \zeta < 1$ – Underdamped system

$$Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{\omega_n^2}{(s + \zeta\omega_n - j\omega_d)(s + \zeta\omega_n + j\omega_d)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{\left(\frac{\omega_n}{\sqrt{(1-\zeta^2)}}\right)\omega_d}{(s + \zeta\omega_n)^2 + (\omega_d^2)}\right\}$$

$$y(t) = \left(\frac{\omega_n}{\sqrt{(1-\zeta^2)}}\right) e^{-\zeta\omega_n t}\sin(\omega_d t)$$



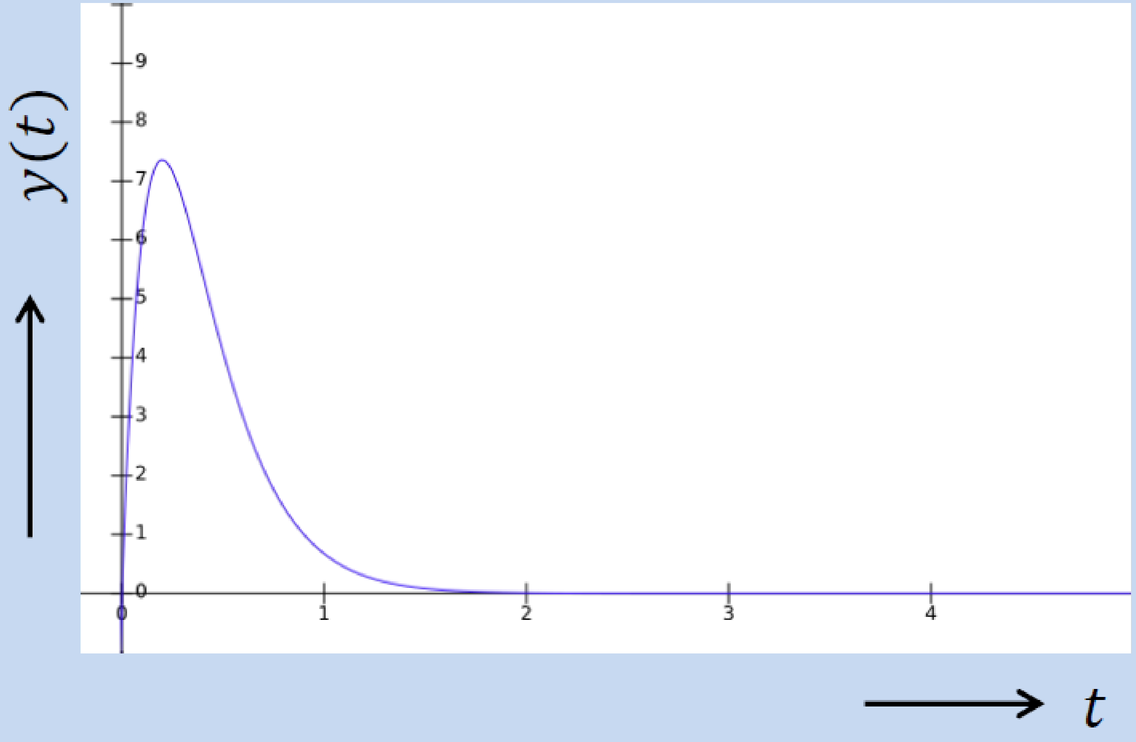
$y(t)$ when $\zeta = 0.5, \omega_n = 5$

Case 3: $\zeta = 1$ – Critically damped system

$$Y(s) = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{\omega_n^2}{(s+\omega_n)^2}\right\}$$

$$y(t) = \omega_n^2 t e^{-\omega_n t}$$



$y(t)$ when $\zeta = 1, \omega_n = 5$

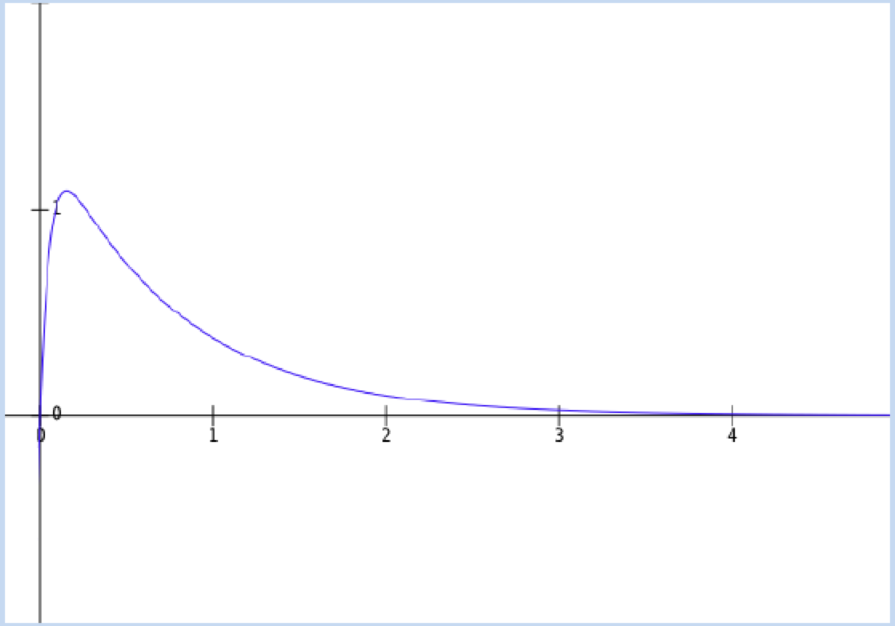
Case 4: $\zeta > 1$ – Overdamped system

$$Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{\omega_n^2}{\left(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}\right)\left(s + \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}\right)}\right\}$$

$$y(t) = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} e^{-\left(\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}\right)t} - \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} e^{-\left(\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}\right)t}$$

$y(t)$ $\uparrow$



$\longrightarrow t$

$y(t)$ when $\zeta = 2, \omega_n = 5$

Step Response of 2nd Order Systems

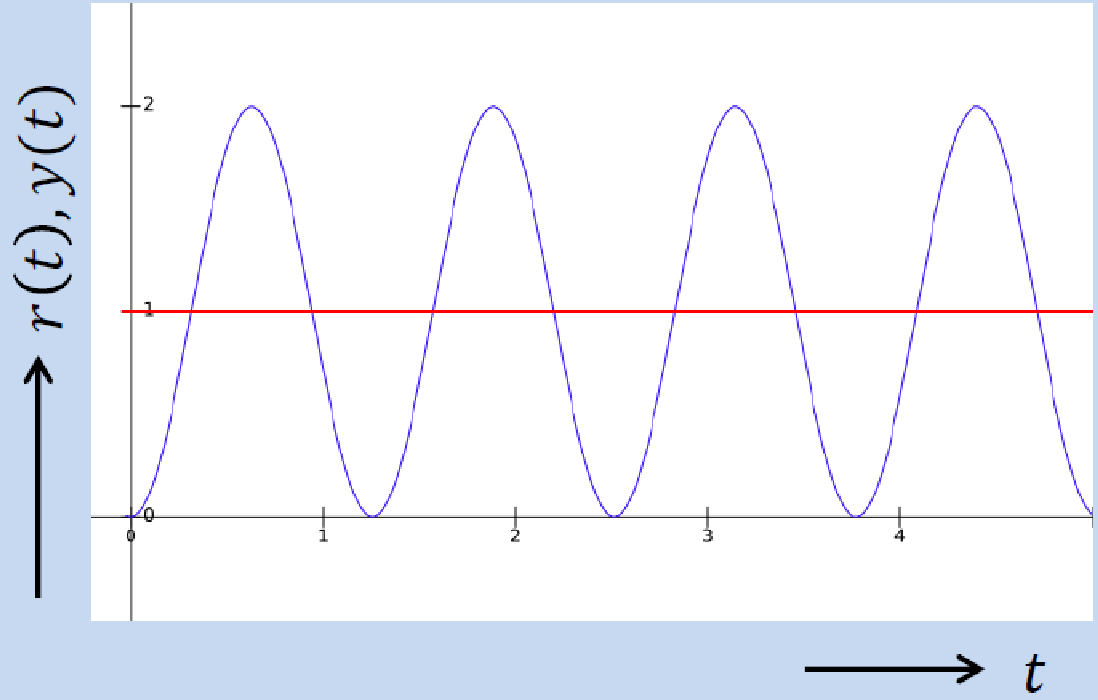
Unit step: $R(s) = \frac{1}{s}$

$$Y(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

Case 1: $\zeta = 0$ – Undamped system

$$Y(s) = \frac{\omega_n^2}{s(s^2 + \omega_n^2)} = \frac{1}{s} - \frac{s}{s^2 - \omega_n^2}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = 1 - \cos(\omega_n t)$$



$r(t)$ and $y(t)$ when $\zeta = 0, \omega_n = 5$

Case 2: $0 < \zeta < 1$ – Underdamped system

$$Y(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} \Longrightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + (\omega_d^2)} - \frac{\zeta\omega_n\left(\frac{\overbrace{\sqrt{(1-\zeta^2)}}^{\omega_d}}{\sqrt{(1-\zeta^2)}}\right)}{(s + \zeta\omega_n)^2 + (\omega_d^2)}\right\}$$

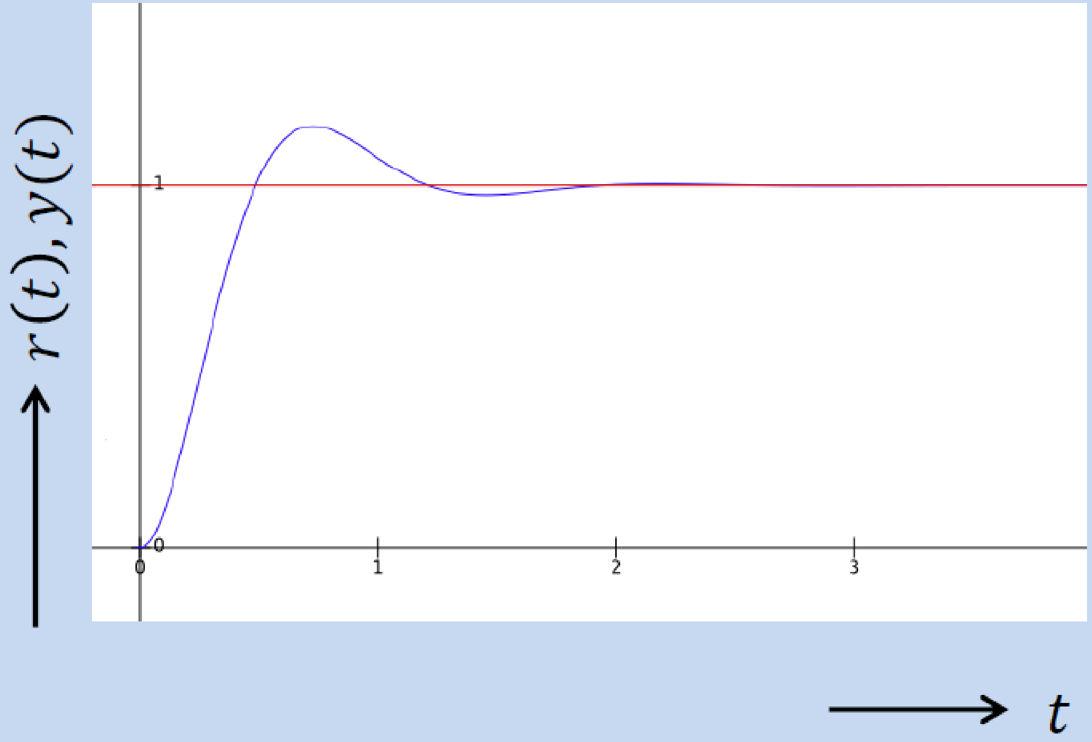
$$y(t) = 1 - e^{-\zeta\omega_n t}\cos(\omega_d t) - \left(\frac{\zeta}{\sqrt{(1-\zeta^2)}}\right)e^{-\zeta\omega_n t}\sin(\omega_d t)$$

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{(1-\zeta^2)}}\left[\sqrt{(1-\zeta^2)}\cos(\omega_d t) + \zeta\sin(\omega_d t)\right]$$

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{(1-\zeta^2)}}[\sin\theta\cos(\omega_d t) \ + \ \cos\theta\sin(\omega_d t)$$

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{(1-\zeta^2)}}\sin(\omega_d t + \theta)$$

where $\theta = \cos^{-1}\zeta = \sin^{-1}\sqrt{(1-\zeta^2)}$



$r(t)$ and $y(t)$ when $\zeta = 0.5, \omega_n = 5$

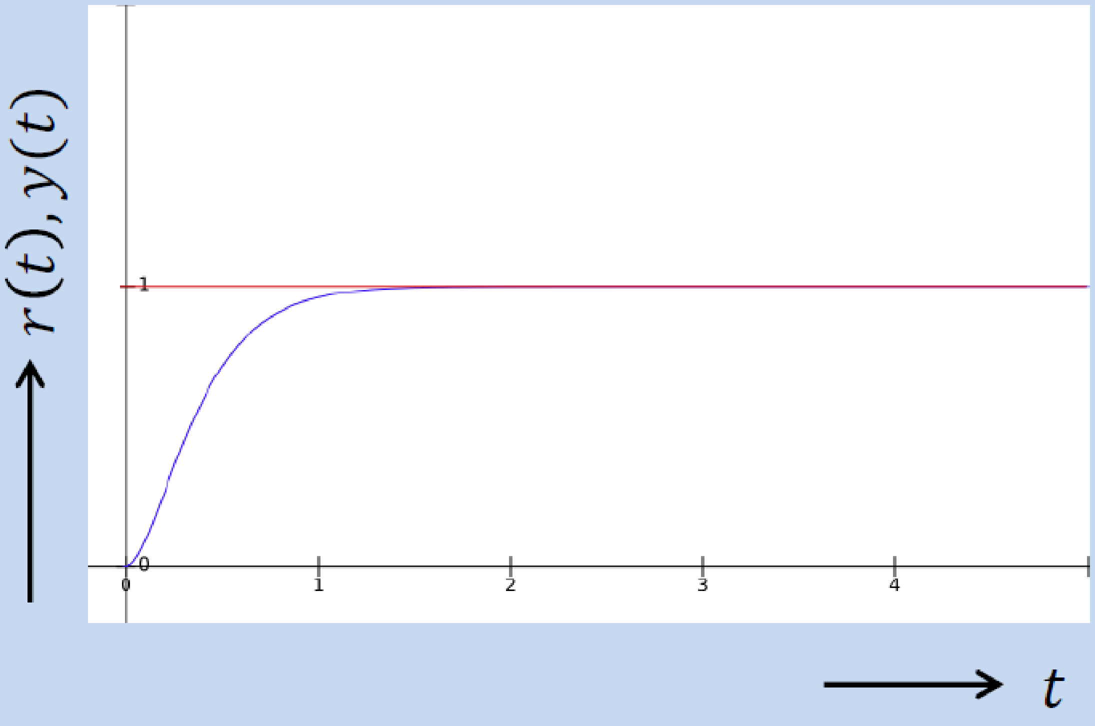
Case 3: $\zeta = 1$ – Critically damped system

$$Y(s) = \frac{\omega_n^2}{s(s^2 + 2\omega_n s + \omega_n^2)}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{\omega_n^2}{s(s+\omega_n)^2}\right\}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2}\right\}$$

$$y(t) = 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}$$



$r(t)$ and $y(t)$ when $\zeta = 1, \omega_n = 5$

Case 4: $\zeta > 1$ – Overdamped system

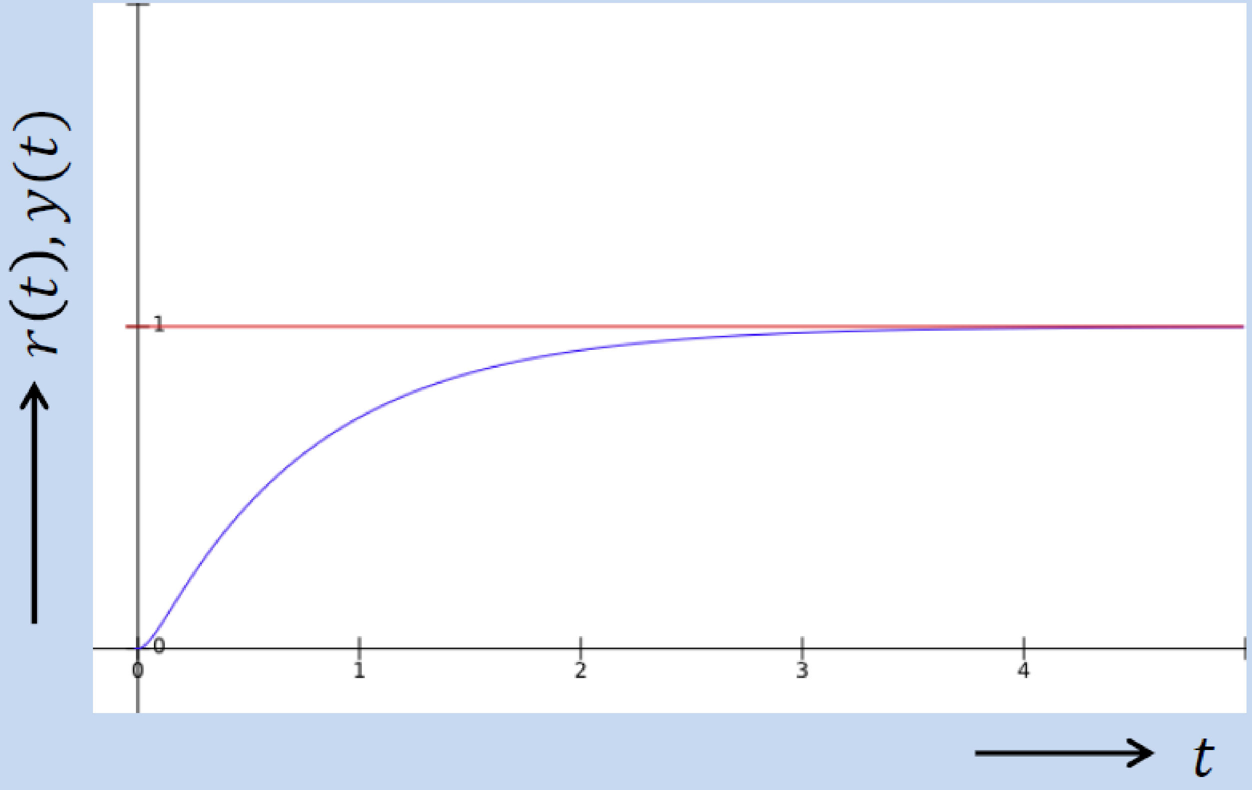
$$Y(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} \Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1 + \frac{\zeta}{\sqrt{(\zeta^2 - 1)}}}{2(s + \zeta\omega_n - \omega_n\sqrt{(\zeta^2 - 1)})} - \frac{\frac{\zeta}{\sqrt{(\zeta^2 - 1)}} - 1}{2(s + \zeta\omega_n + \omega_n\sqrt{(\zeta^2 - 1)})}\right\}$$

$$y(t) = 1 - \frac{1}{2}\left(1 + \frac{\zeta}{\sqrt{(\zeta^2 - 1)}}\right) e^{-\left(\zeta\omega_n - \omega_n\sqrt{(\zeta^2 - 1)}\right)t}$$

$$+ \frac{1}{2}\left(\frac{\zeta}{\sqrt{(\zeta^2 - 1)}} - 1\right) e^{-(\zeta\omega_n + \omega_n\sqrt{(\zeta^2 - 1)})t}$$

Case 4: $\zeta > 1$ – Overdamped system



$r(t)$ and $y(t)$ when $\zeta = 2, \omega_n = 5$

Time Response Specifications : Motivation

- These specifications refer to the performance indices of the step response of a system
- In general, these indices are specified as a part of the design requirements of control systems
- These indices answer the following questions pertaining to step response of a system:
  - How fast the system moves to follow the input ?
  - How oscillatory is the response (indicative of damping) ?
  - How long does it take to practically reach the final value ?

Time Response Specifications

- **Delay time t_d:**
  - Time required for the response to reach 50% of the final value at first instance

$$t_d = \frac{1 + 0.7\zeta}{\omega_n}$$

- **Rise time t_r:**
  - Time required for the response to rise from 10% to 90% of the final value for overdamped systems and 0 to 100% of the final value for underdamped systems, at first instance
- **Peak time t_p:**
  - Time required for the response to reach the peak value of time response

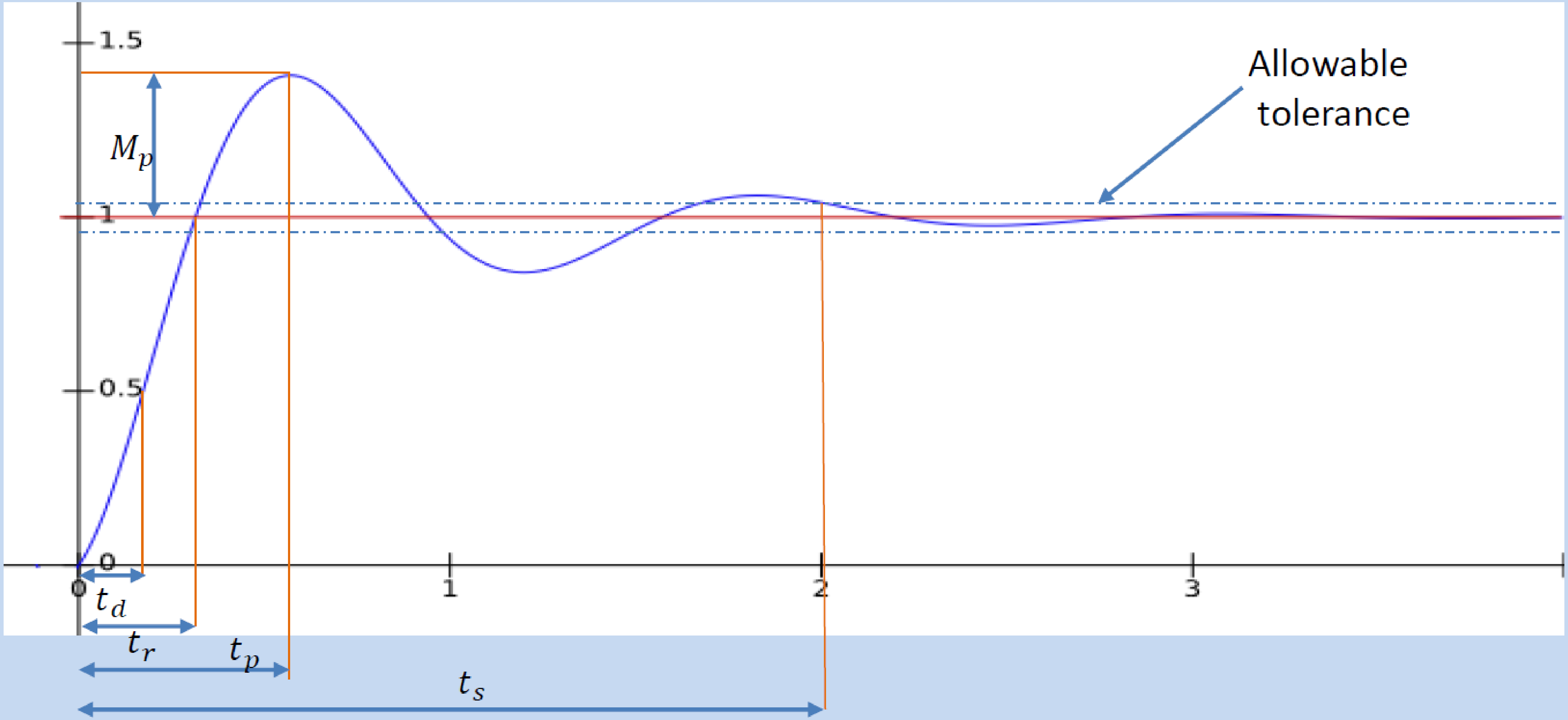
- **Peak overshoot M_p:**
  - It is the normalised difference between the peak value of time response and the steady state value

$$M_p = \frac{y(t_p) - y(\infty)}{y(\infty)} \times 100\%$$

- **Settling time t_s:**
  - Time required for the response to reach and stay within a specified tolerance band of its final value or steady state value
  - Usually the tolerance band is 2% or 5%

➢ Note: t_p and M_p are not defined for overdamped and critically damped systems

Response of an Underdamped system



Expression for Rise Time

- Consider a 2nd order underdamped system
- Rise time t_r is the time to taken by the step response to go from 0 to 100% of the final value i.e., one

- $y(t_r) = 1 = 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{(1-\zeta^2)}}\sin(\omega_d t_r + \theta)$ where $(\theta = \cos^{-1}\zeta)$

$$\Longrightarrow \sin(\omega_d t_r + \theta) = 0 \Longrightarrow \omega_d t_r + \theta = \pi$$

$$\boxed{t_r = \frac{\pi - \theta}{\omega_d} = \frac{\pi - \cos^{-1}\zeta}{\omega_n\sqrt{(1-\zeta^2)}}}$$

Expression for Peak Time

- Peak time t_p is the time taken by the step response to reach the peak value
- At peak, the time derivative of response is zero
- $\frac{dy}{dt}|_{t_p} = 0 = \frac{\zeta\omega_n e^{-\zeta\omega_n t_p}}{\sqrt{(1-\zeta^2)}} \sin(\omega_d t_p + \theta) - \frac{e^{-\zeta\omega_n t_p}}{\sqrt{(1-\zeta^2)}} \omega_d \cos(\omega_d t_p + \theta)$

$$\Longrightarrow \zeta \sin(\omega_d t_p + \theta) - \sqrt{(1-\zeta^2)} \cos(\omega_d t_p + \theta) = 0$$

$$\Longrightarrow \sin(\omega_d t_p + \theta) \cos\theta - \cos(\omega_d t_p + \theta) \sin\theta = 0$$

$$\Longrightarrow \sin(\omega_d t_p) = 0 \Longrightarrow \omega_d t_p = 0, \pi, 2\pi, \ldots$$

$$\boxed{t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{(1-\zeta^2)}}}$$

(corresponding to first peak)

Expression for Peak Overshoot

- As per definition, peak overshoot for a step response,

$$M_p = \frac{y(t_p) - 1}{1} = y(t_p) - 1$$

$$y(t_p) - 1 = -\frac{e^{-\zeta\omega_n t_p}}{\sqrt{(1-\zeta^2)}}\sin(\omega_d t_p + \theta) = -\frac{e^{-\frac{\zeta\omega_n\pi}{\omega_d}}}{\sqrt{(1-\zeta^2)}}\sin\left(\frac{\omega_d\pi}{\omega_d} + \theta\right)$$

$$\Longrightarrow M_p = \frac{e^{-\frac{\zeta\pi}{\sqrt{(1-\zeta^2)}}}}{\cancel{\sqrt{(1-\zeta^2)}}}\cancel{\sin(\theta)}$$

$$\boxed{M_p = 100e^{-\frac{\zeta\pi}{\sqrt{(1-\zeta^2)}}}\%}$$

(corresponding to first peak)

Expression for Settling Time

- Time required for the response to reach and stay within a specified tolerance band (take 2%) of its final value or steady state value

$$t_s = 4\tau = \frac{4}{\zeta\omega_n} \text{ (2\% tolerance)}$$

$$t_s = 3\tau = \frac{3}{\zeta\omega_n} \text{ (5\% tolerance)}$$

- Settling time is inversely proportional to the damping ratio
- As ζ increases, t_s decreases

Application of Damped Systems

- **Overdamped systems:**
  - –Push button water tap shut-off valves
  - –Automatic door closers (can be critically damped also)
- **Critically damped systems:**
  - –Elevator mechanism
  - –Gun mechanism (returns to neutral position in shortest possible time)
- **Underdamped systems:**
  - –All string instruments, bells are underdamped to make sound appealing
  - –Analog electrical or mechanical measuring instruments

Steady State Error

- It is the error between the actual output and the desired output as $t \to \infty$

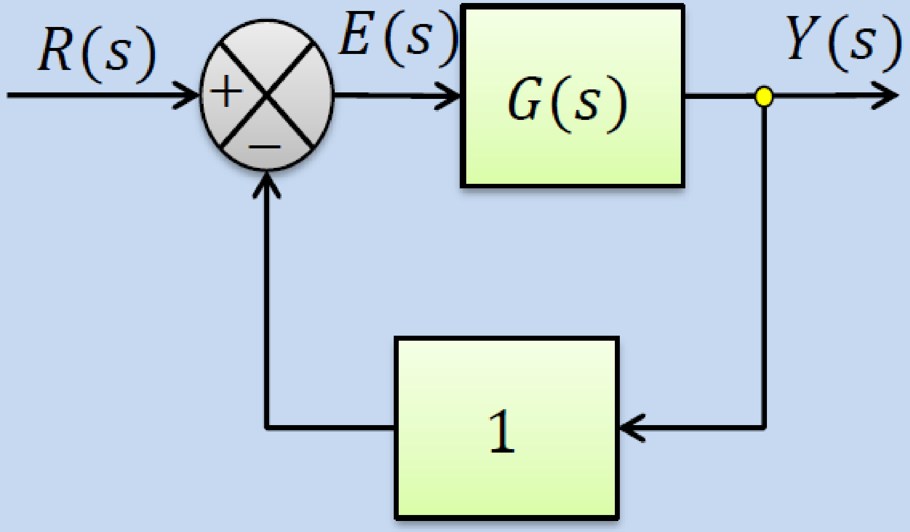
$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{t \to \infty} (r(t) - y(t))$$

By final value theorem,

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)$$

$$E = R - Y = R - \frac{GR}{1+G} = \frac{R}{1+G}$$

$$e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)}$$



Unity feedback system

Steady State Error for Standard Inputs

- **Unit step input:** $R(s) = \frac{1}{s}$

$$e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1 + G(s)} = \lim_{s\to 0} \frac{1}{1 + G(s)} = \frac{1}{1 + K_p}$$

where $K_p = \lim_{s\to 0} G(s)$ is called position error constant

- **Unit ramp (velocity) input:**

$$R(s) = \frac{1}{s^2}$$

$$e_{ss} = \lim_{s\to 0} \frac{1}{s + sG(s)} = \lim_{s\to 0} \frac{1}{sG(s)} = \frac{1}{K_v}$$

where $K_v = \lim_{s\to 0} sG(s)$ is called velocity error constant

Note: Velocity error is not error in the velocity but it is error in position due to ramp input

- **Unit parabolic (acceleration) input:**

$$R(s) = \frac{1}{s^3}$$

$$e_{ss} = \lim_{s \to 0} \frac{1}{s^2 + s^2 G(s)} = \lim_{s \to 0} \frac{1}{s^2 G(s)} = \frac{1}{K_a}$$

where $K_a = \lim_{s \to 0} s^2 G(s)$ is called acceleration error constant

- The error constants K_p, K_v and K_a describe the ability of a system to reduce or eliminate steady state errors
- These values mostly depend on the type of the system
- As the type of the system becomes higher, more steady-state errors are eliminated

Features of Steady State Error

- Steady state error is a measure of system accuracy
- In an ideal scenario, the system should match the reference input as time progresses
- It means the steady state error should be as low as possible and hence it is an important performance measure
- Steady state errors depend on two factors:
    1. Type of the reference input $Rs-$ step, ramp or parabolic
    2. Type of the system $G(s)$
- Steady state errors are calculated only for closed loop stable systems

Type of a System

- Consider following pole-zero form of open loop transfer function of a system:

$$G(s) = \frac{K'\big((s+z_1)(s+z_2)\ \dots\big)}{s^n(s+p_1)(s+p_2)\ \dots}$$

  - Term s^n in the denominator denotes the number of the poles (n) at origin
  - System with n poles at origin is called Type$-n$ system
  - n also indicates the number of integrations $\left(\frac{1}{s}\right)$ in the system
  - As $s \to 0$, s^n term dominates in determining the steady state error

Steady State Error for Different Systems

- **Type−0 system:**

$$G(s) = \frac{K'\left((s+z_1)(s+z_2)\dots\right)}{(s+p_1)(s+p_2)\dots}$$

- e_{ss}(position) $= \lim\limits_{s\to 0} \frac{1}{1+G(s)} = \frac{1}{1+K_p}$
- e_{ss}(velocity) $= \lim\limits_{s\to 0} \frac{1}{sG(s)} = \frac{1}{0} = \infty$
- e_{ss}(acceleration) $= \lim\limits_{s\to 0} \frac{1}{s^2 G(s)} = \frac{1}{0} = \infty$

- Constant position error, infinite velocity and acceleration errors at steady state

- **Type−1 system:**

$$G(s) = \frac{K'\big((s+z_1)(s+z_2)\ \ldots\big)}{s(s+p_1)(s+p_2)\ \ldots}$$

  - $e_{ss}(\text{position}) = \lim_{s\to 0} \frac{1}{1+G(s)} = \frac{1}{1+\infty} = 0$
  - $e_{ss}(\text{velocity}) = \lim_{s\to 0} \frac{1}{sG(s)} = \frac{1}{K_v}$
  - $e_{ss}(\text{acceleration}) = \lim_{s\to 0} \frac{1}{s^2G(s)} = \frac{1}{0} = \infty$

- Zero position error, a constant velocity error and infinite acceleration error at steady state

- **Type−2 system:**

$$G(s) = \frac{K'\big((s+z_1)(s+z_2)\ldots\big)}{s^2(s+p_1)(s+p_2)\ldots}$$

  - ➢ $e_{ss}(\text{position}) = \lim_{s\to 0}\frac{1}{1+G(s)} = \frac{1}{1+\infty} = 0$
  - ➢ $e_{ss}(\text{velocity}) = \lim_{s\to 0}\frac{1}{sG(s)} = \frac{1}{\infty} = 0$
  - ➢ $e_{ss}(\text{acceleration}) = \lim_{s\to 0}\frac{1}{s^2G(s)} = \frac{1}{K_a}$

- Zero position error, zero velocity error and a constant acceleration error at steady state

Summary

TABLE 7.2 Relationships between input, system type, static error constants, and steady-state errors

		Type 0		Type 1		Type 2	
Input	**Steady-state error formula**	**Static error constant**	**Error**	**Static error constant**	**Error**	**Static error constant**	**Error**
Step, $u(t)$	$\dfrac{1}{1+K_p}$	$K_p = \text{Constant}$	$\dfrac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\dfrac{1}{K_v}$	$K_v = 0$	∞	$K_v = \text{Constant}$	$\dfrac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\dfrac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a = \text{Constant}$	$\dfrac{1}{K_a}$

References

[1] N. S. Nise, *Control systems engineering*, 6th ed. Wiley, 2011.

[2] K. Ogata, *Modern Control Engineering*, 5th ed. Prentice Hall, 2010.

[3] M. Gopal, *Control systems : principles and design*, 3rd ed. Tata McGraw-Hill, 2012.