Set Theory

A set is an unordered collection of objects

English alphabet vowels: $V = \{a, e, i, o, u\}$

$a \in V$ $\quad$ $b \notin V$

Odd positive integers less than 10:

$$O = \{1,3,5,7,9\}$$

elements of set
members of set

Other set representations

Set of positive integers less than 100:

$$\{1,2,3,\ldots,99\}$$

omitted
elements

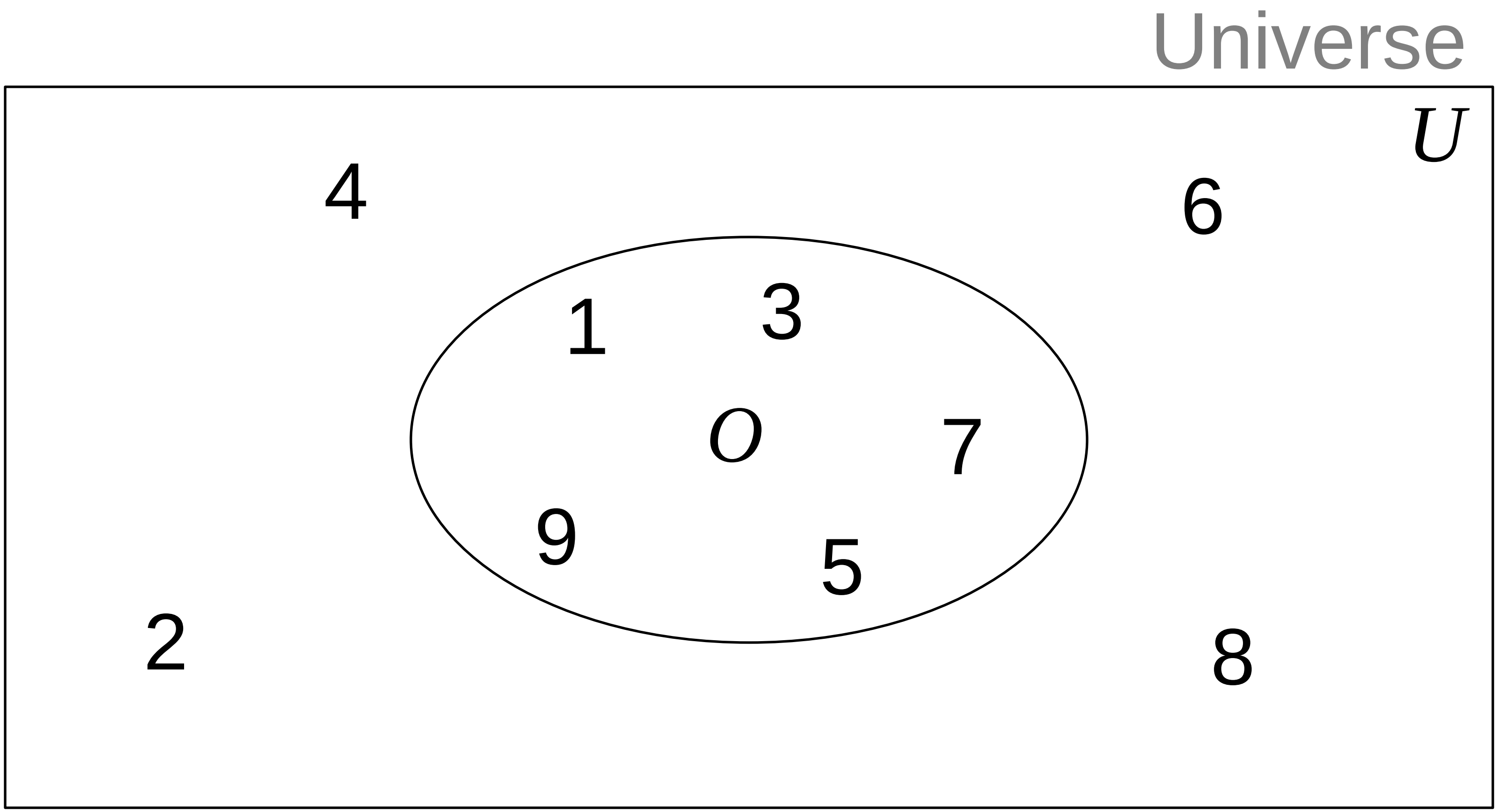
Odd positive integers less than 10:

$$O = \{1,3,5,7,9\}$$

$$O = \{x \mid x \text{ is an odd positive integer less than } 10\}$$

$$O = \{x \in Z^{+} \mid x \text{ is odd and } x < 10\}$$

Venn Diagram



$$U = \{x \mid x \text{ is a positive integer less than } 10\}$$

$$O = \{x \mid x \text{ is an odd positive integer less than } 10\}$$

Useful sets

$N = \{0,1,2,3,\ldots\}$ Natural numbers

$Z = \{\ldots,-2,1,0,1,2,\ldots\}$ Integers

$Z^{+} = \{1,2,3,\ldots\}$ Positive integers

$Q = \{p/q \mid p \in Z, q \in Z, q \neq 0\}$ Rational numbers

$R = \{\text{set of Real numbers}\}$ Real numbers

Empty set

$$\varnothing = \{\,\}$$

$$\varnothing \neq \{\varnothing\}$$

Cardinality (size) of set

Finite sets	Number of elements
$S_1 = \{a, e, i, o, u\}$	$\lvert S_1 \rvert = 5$
$S_2 = \{a, b, c, \ldots, z\}$	$\lvert S_2 \rvert = 26$
$S_3 = \{1, 2, 3, \ldots, 99\}$	$\lvert S_3 \rvert = 99$

$$\lvert \varnothing \rvert = \lvert \{\,\} \rvert = 0 \qquad \lvert \{\varnothing\} \rvert = 1$$

Infinite set $\quad N = \{0, 1, 2, 3, \ldots\}\quad$ infinite size

Equal sets

$$A = B$$

$$\forall x (x \in A \leftrightarrow x \in B)$$

Examples: $\{1,3,5\} = \{3,5,1\}$

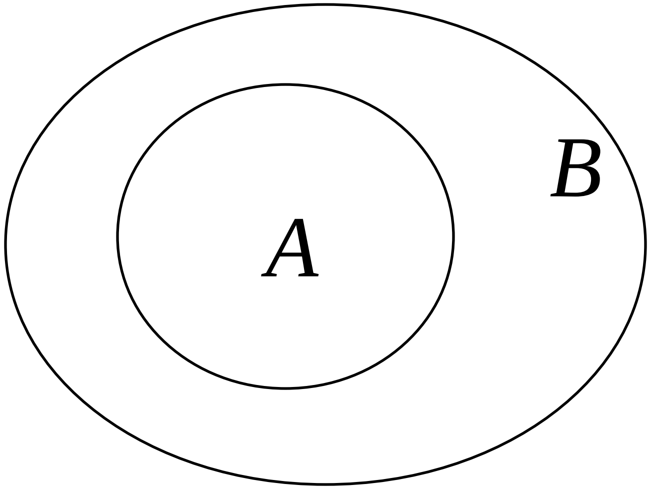
$$\{1,3,5\} = \{1,3,3,3,5,5,5,5\}$$

$$\{1,3,5,7,9\} = \{x \in Z^+ \mid x \text{ is odd and } x < 10\}$$

Subset

$$A \subseteq B$$

$$\forall x (x \in A \rightarrow x \in B)$$

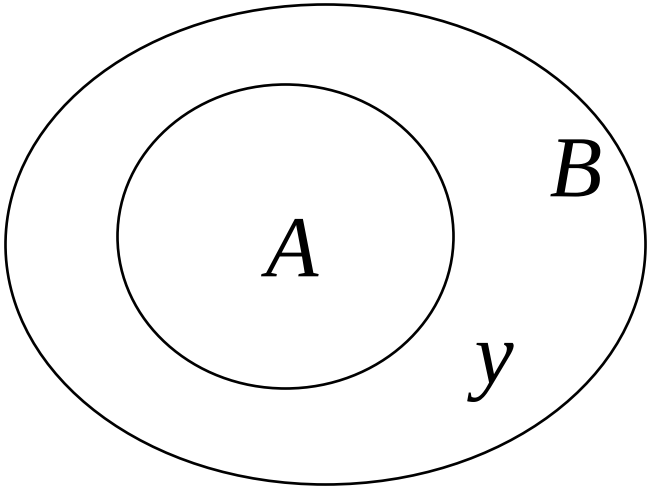


Examples: $\{1,3,5\} \subseteq \{0,1,3,5\}$ $\qquad N \subseteq Z$

For any set S: $\qquad S \subseteq S \qquad \varnothing \subseteq S$

Proper Subset

$$A \subset B$$



$$A \subseteq B \wedge A \neq B$$

$$\forall x(x \in A \rightarrow x \in B \wedge \exists y(y \in B \wedge y \notin A))$$

Examples: $\{1,3,5\} \subset \{0,1,3,5\}$ $\quad N \subset Z$

$$A = B$$

is equivalent to

$$A \subseteq B \quad \wedge \quad B \subseteq A$$

Notation

- $S=\{a, b, c\}$ refers to the set whose elements are a, b and c.
- $a \in S$ means "a is an element of set S".
- $d \notin S$ means "d is *not* an element of set S".
- $\{x \in S \mid P(x)\}$ is the set of all those x from S such that P(x) is true. *E.g.*, $T=\{x \in \mathbf{Z} \mid 0<x<10\}$.
- *Notes:*

  1) $\{a,b,c\}$, $\{b,a,c\}$, $\{c,b,a,b,b,c\}$ all represent the same set.

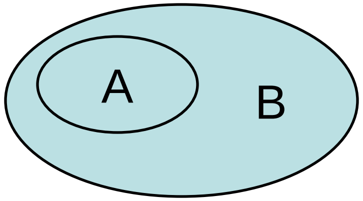
  2) Sets can themselves be elements of other sets, *e.g.*, S={ {Mary, John}, {Tim, Ann}, …}

Relations between sets

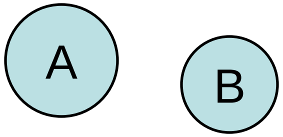
- **Definition:** Suppose A and B are sets. Then A is called a **subset** of B: **$A \subseteq B$**

  iff every element of A is also an element of B. Symbolically,

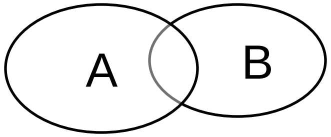
  $A \subseteq B \Leftrightarrow \forall x$, if $x \in A$ then $x \in B$.
- $A \not\subset B \Leftrightarrow \exists x$ such that $x \in A$ and $x \notin B$.



$A \subseteq B$



$A \not\subset B$



$A \not\subset B$

Relations between sets

- **Definition:** Suppose A and B are sets. Then
  A **equals** B: **$A = B$**
  iff every element of A is in B and
  every element of B is in A.
  Symbolically,
  $A=B \Leftrightarrow A\subseteq B$ and $B\subseteq A$.

- **Example:** Let $A = \{m\in \mathbf{Z} \mid m=2k+3 \text{ for some integer } k\}$;
  B = the set of all odd integers.
  Then $A=B$.

Operations on Sets

Definition: Let A and B be subsets of a set U.

1. Union of A and B: $A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$
2. Intersection of A and B:

$$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$$

3. Difference of B minus A: $B - A = \{x \in U \mid x \in B \text{ and } x \notin A\}$
4. Complement of A: $A^c = \{x \in U \mid x \notin A\}$

Ex.: Let $U = \mathbf{R}$, $A = \{x \in \mathbf{R} \mid 3 < x < 5\}$, $B = \{x \in \mathbf{R} \mid 4 < x < 9\}$. Then

1) $A \cup B = \{x \in \mathbf{R} \mid 3 < x < 9\}$.
2) $A \cap B = \{x \in \mathbf{R} \mid 4 < x < 5\}$.
3) $B - A = \{x \in \mathbf{R} \mid 5 \le x < 9\}$, $A - B = \{x \in \mathbf{R} \mid 3 < x \le 4\}$.
4) $A^c = \{x \in \mathbf{R} \mid x \le 3 \text{ or } x \ge 5\}$, $B^c = \{x \in \mathbf{R} \mid x \le 4 \text{ or } x \ge 9\}$

Properties of Sets

- **Theorem** 1 (*Some subset relations*):

  1) $A \cap B \subseteq A$

  2) $A \subseteq A \cup B$

  3) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

- To prove that $A \subseteq B$ use the "element argument":

  1. suppose that x is a particular but arbitrarily chosen element of A,
  2. show that x is an element of B.

Proving a Set Property

- **Theorem** 2 (*Distributive Law*):

  For any sets A,B and C:

  $$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) .$$

- **Proof:** We need to show that

  (I) $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ and

  (II) $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

Let's show (I).

Suppose $x \in A \cup (B \cap C)$ (1)

We want to show that $x \in (A \cup B) \cap (A \cup C)$ (2)

Proving a Set Property

- **Proof (cont.):**

$x \in A \cup (B \cap C) \Rightarrow x \in A$ or $x \in B \cap C$.

(a) Let $x \in A$. Then

$x \in A \cup B$ *and* $x \in A \cup C \Rightarrow x \in (A \cup B) \cap (A \cup C)$

(b) Let $x \in B \cap C$. Then $x \in B$ and $x \in C$.

$$\left.\begin{array}{l} x \in B \Rightarrow x \in A \cup B \\ x \in C \Rightarrow x \in A \cup C \end{array}\right\} \Rightarrow x \in (A \cup B) \cap (A \cup C)$$

Thus, (2) is true, and we have shown (I).

(II) is shown similarly (*left as exercise*). ■

Set Properties

- Commutative Laws:

$$(a)\ A \cap B = B \cap A$$

$$(b)\ A \cup B = B \cup A$$

- Associative Laws:

$$(a)\ (A \cap B) \cap C = A \cap (B \cap C)$$

$$(b)\ (A \cup B) \cup C = A \cup (B \cup C)$$

- Distributive Laws:

$$(a)\ A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$(b)\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Set Properties

- Double Complement Law:

$$(A^c)^c = A$$

- De Morgan's Laws:

$$(a)\ (A \cap B)^c = A^c \cup B^c$$

$$(b)\ (A \cup B)^c = A^c \cap B^c$$

- Absorption Laws:

$$(a)\ A \cup (A \cap B) = A$$

$$(b)\ A \cap (A \cup B) = A$$

Theorem: $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Proof: Show that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

Part 1: $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

$$x \in \overline{A \cap B}$$

$$\rightarrow x \notin A \cap B \rightarrow \neg(x \in A \cap B)$$

De Morgan's law from logic

$$\rightarrow \neg((x \in A) \wedge (x \in B)) \rightarrow \neg(x \in A) \vee \neg(x \in B)$$

$$\rightarrow (x \notin A) \vee (x \notin B) \rightarrow (x \in \overline{A}) \vee (x \in \overline{B})$$

$$\rightarrow x \in (\overline{A} \cup \overline{B})$$

Part 2: $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

$$x \in (\overline{A} \cup \overline{B})$$

$$\rightarrow (x \in \overline{A}) \vee (x \in \overline{B}) \rightarrow (x \notin A) \vee (x \notin B)$$

$$\rightarrow \neg(x \in A) \vee \neg(x \in B) \rightarrow \neg((x \in A) \wedge (x \in B))$$

$$\rightarrow \neg(x \in A \cap B)$$

De Morgan's law from logic

$$\rightarrow x \in \overline{A \cap B}$$

End of Proof

Showing that a set property is false

- **Statement:** For all sets A,B and C,

$$A-(B-C) = (A-B)-C \,.$$

The following counterexample shows that the statement is false.

- **Counterexample:**

Let $A=\{1,2,3,4\}$, $B=\{3,4,5,6\}$, $C=\{3\}$.

Then $B-C = \{4,5,6\}$ *and* $A-(B-C) = \{1,2,3\}$.

On the other hand,

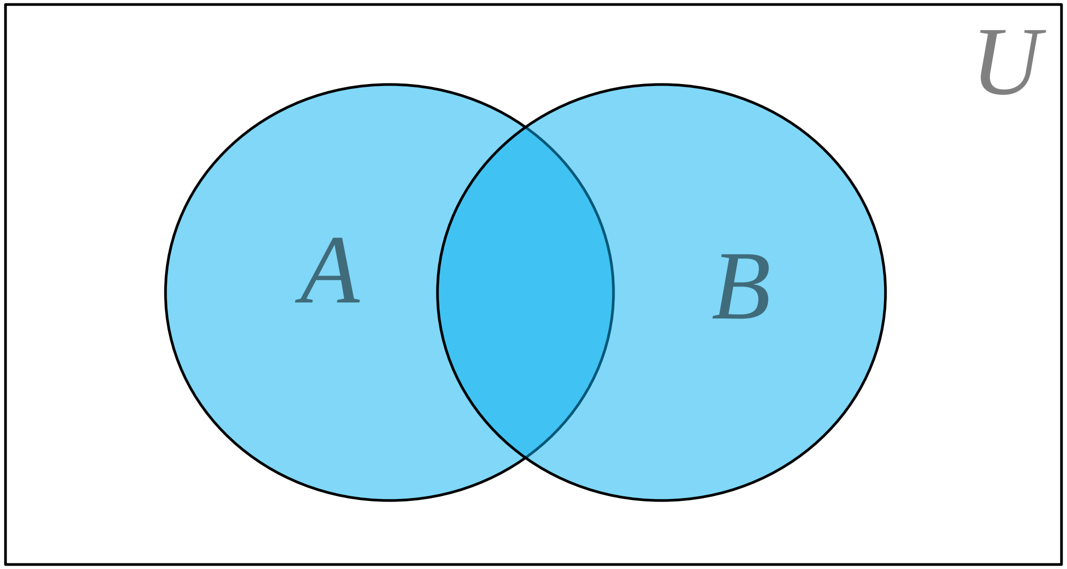
$$A-B = \{1,2\} \quad \textit{and} \quad (A-B)-C = \{1,2\} \,.$$

Thus, for this example

$$A-(B-C) \neq (A-B)-C \,.$$

Set operations

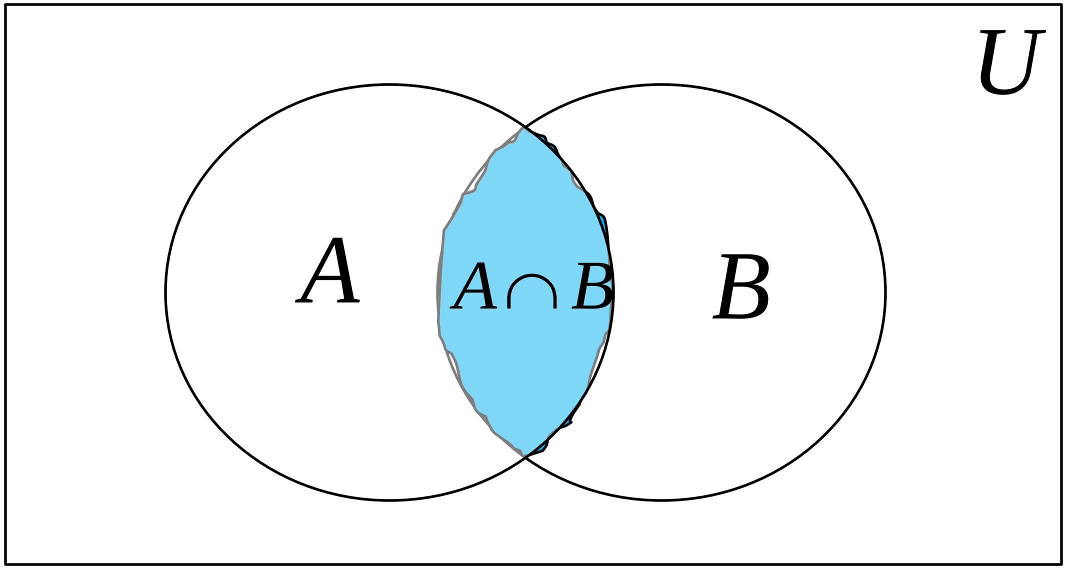
Union $A \cup B = \{x \mid x \in A \vee x \in B\}$



$A = \{1,3,5\}$ $B = \{1,2,3\}$ $A \cup B = \{1,2,3,5\}$

Intersection

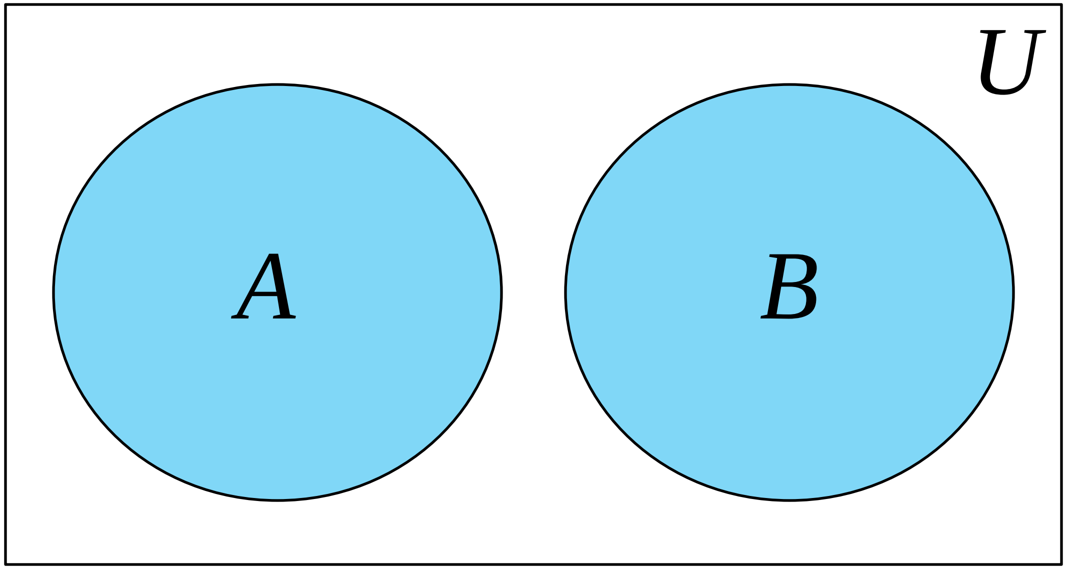
$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$



$A = \{1,3,5\}$ $\quad B = \{1,2,3\}$ $\quad A \cap B = \{1,3\}$

Disjoint sets A, B

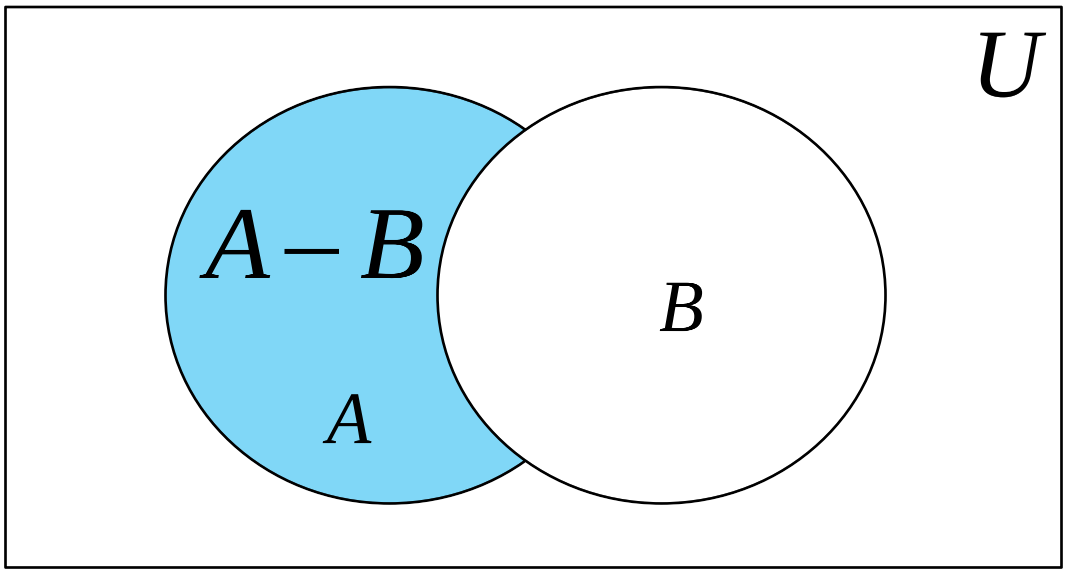
$$A \cap B = \varnothing$$



$A = \{1,3,5\}$ $B = \{2,9\}$ $A \cap B = \varnothing$

Set difference

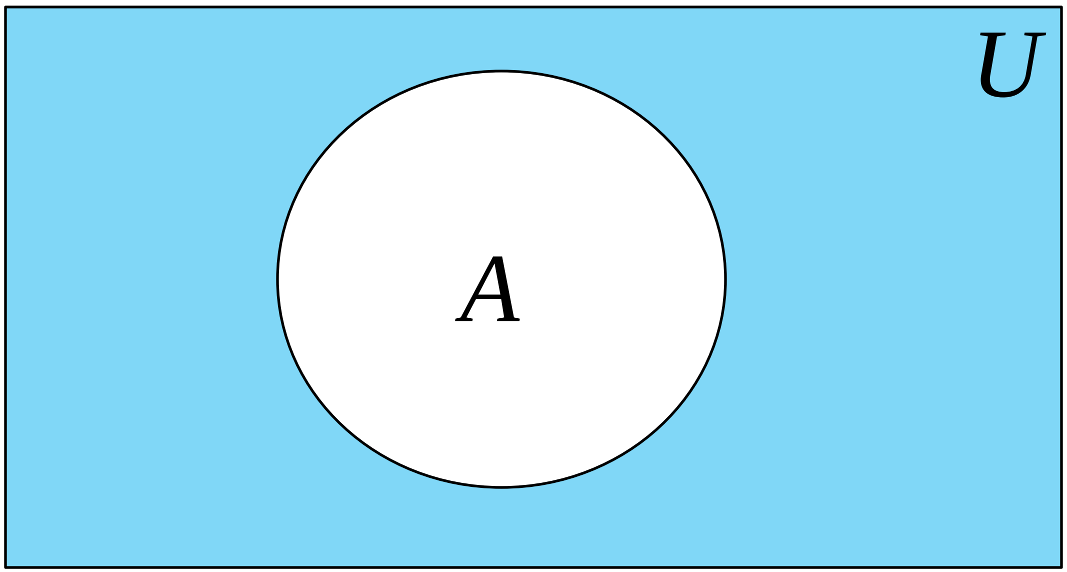
$$A - B = \{x \mid x \in A \wedge x \notin B\}$$



$A = \{1,3,5\}$ $B = \{1,2,3\}$ $A - B = \{5\}$

Complement

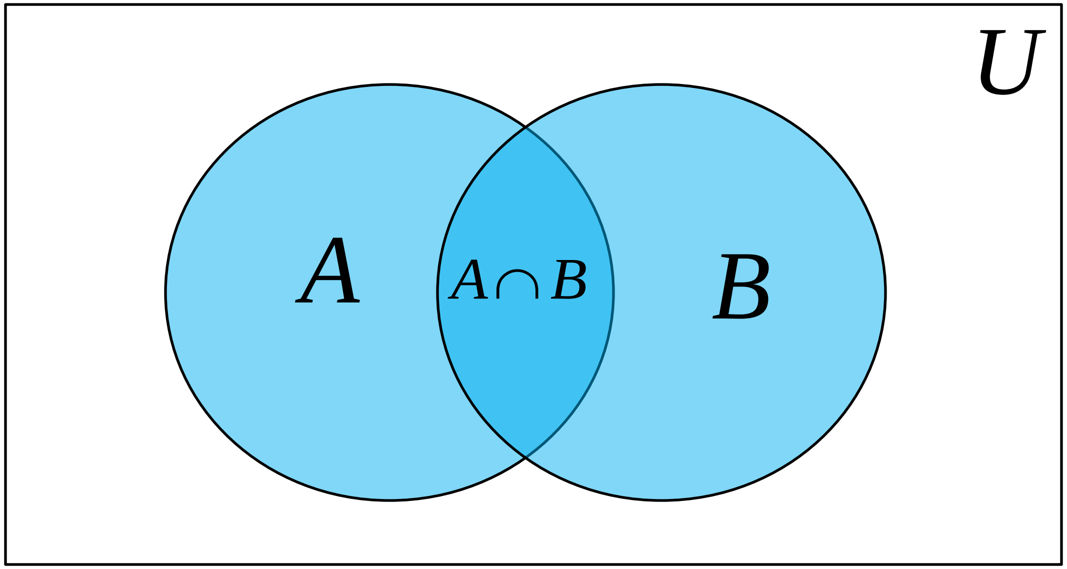
$$\overline{A} = \{x \mid x \notin A\}$$



$$A = \{1,3,5\} \qquad U = \{1,2,3,4,5\} \qquad \overline{A} = \{2,4\}$$

Size of union

$$|A \cup B| = |A| + |B| - |A \cap B|$$



$A = \{1,3,5\}$ $B = \{1,2,3\}$ $A \cup B = \{1,2,3,5\}$ $A \cap B = \{1,3\}$

$$|A \cup B| = |A| + |B| - |A \cap B| = 3 + 3 - 2 = 4$$

Empty Set

- The unique set with no elements is called **empty set** and denoted by $\varnothing$.
- Set Properties that involve $\varnothing$.

For all sets A,

1. $\varnothing \subseteq A$
2. $A \cup \varnothing = A$
3. $A \cap \varnothing = \varnothing$
4. $A \cap A^c = \varnothing$

Disjoint Sets

- A and B are called **disjoint** iff $A \cap B = \varnothing$.
- Sets $A_1, A_2, \ldots, A_n$ are called **mutually disjoint**

  iff for all i,j = 1,2,…, n

  $$A_i \cap A_j = \varnothing \text{ whenever } i \neq j .$$
- *Examples:*

  1) A={1,2} and B={3,4} are disjoint.
  2) The sets of even and odd integers are disjoint.
  3) A={1,4}, B={2,5}, C={3} are mutually disjoint.
  4) $A-B$, $B-A$ and $A \cap B$ are mutually disjoint.

Partitions

- **Definition:** A collection of nonempty sets $\{A_1, A_2, \ldots, A_n\}$ is a **partition** of a set A *iff*
  1. $A = A_1 \cup A_2 \cup \ldots \cup A_n$
  2. $A_1, A_2, \ldots, A_n$ are mutually disjoint.
- *Examples*:

  1) $\{\mathbf{Z}^+, \mathbf{Z}^-, \{0\}\}$ is a partition of **Z**.

  2) Let $S_0=\{n \in \mathbf{Z} \mid n=3k \text{ for some integer } k\}$

     $S_1=\{n \in \mathbf{Z} \mid n=3k+1 \text{ for some integer } k\}$

     $S_2=\{n \in \mathbf{Z} \mid n=3k+2 \text{ for some integer } k\}$

     Then $\{S_0, S_1, S_2\}$ is a partition of **Z**.

Power Sets

- **Definition:** Given a set A,

  the **power set** of A, denoted $\mathcal{P}$**(A)** ,

  is the set of all subsets of A.
- *Example:* $\mathcal{P}(\{a,b\}) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$.
- **Properties:**

  1) *If* $A \subseteq B$ *then* $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

  2) *If* a set A has n elements

     then $\mathcal{P}(A)$ has 2^n elements.

Power set

The power set of S contains all possible subsets of S (and the empty set)

$$S = \{1,2,3\}$$

Power set

$$P(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$$

$$\underbrace{|P(S)|}_{\text{Size of power set}} = 2^{|S|} = 2^3 = 8$$

Special cases

$$P(\emptyset) = \{\emptyset\}$$

$$P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$$

Cartesian product

Cartesian product of two sets A, B

$$A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$$

Example: $A = \{1,2\}$ $B = \{a,b,c\}$

$$A \times B = \{(1,a),(1,b),(1,c),(2,a),(2,b),(2,c)\}$$

$$B \times A = \{(a,1),(b,1),(c,1),(a,2),(b,2),(c,2)\}$$

For this case: $A \times B \neq B \times A$

Size: $|A \times B| = |A| \times |B| = 2 \times 3 = 6$

Cartesian product of sets $A_1, A_2, \ldots, A_n$

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i\}$$

Example: $A = \{1,2\}$ $B = \{a,b,c\}$ $C = \{x, y\}$

$$A \times B \times C = \{(1,a,x),(1,b,x),(1,c,x),(2,a,x),(2,b,x),(2,c,x),$$
$$(1,a,y),(1,b,y),(1,c,y),(2,a,y),(2,b,y),(2,c,y)\}$$

Size: $|A \times B \times C| = |A| \times |B| \times |C| = 2 \times 3 \times 2 = 12$

$$|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \times |A_2| \times \cdots \times |A_n|$$

Theorem (Inclusion–Exclusion Principle)

- **Suppose A and B are finite sets. Then A ∪ B and A ∩ B are** finite and

  $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ or

  $|A \cup B| = |A| + |B| - |A \cap B|$
- That is, we find the number of elements in A or B (or both) by first adding n(A) and n(B) (inclusion) and then
- subtracting $n(A \cap B)$ (exclusion) since its elements were counted twice.

- Let A,B,C be the finite sets . Then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

Example For Inclusion Exclusion Formula

- A computer company wants to hire 25 programmers to habdle systems programming jobs and 40 programmers for applications programming. Of those hired , ten will be expected to perform jobs of both types. How many programmers must be hired.
- Solutions:
  - Let A be the set of systems programmers hired and B be the set of applications programmers hired.
  - The company must have $|A|=25$, $|B|=40$, and $|A \cap B|=10$.
  - The number of programmers that must be hired is $|A \cup B|$, but

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$=25+40-10$$
$$= 55$$

Example 2

- Verify $|A \cup B \cup C| = |A| + |B| + |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$
  where A={1,2,3,4,5}, B={2,3,4,6}, C={3,4,6,8}.

Solution:

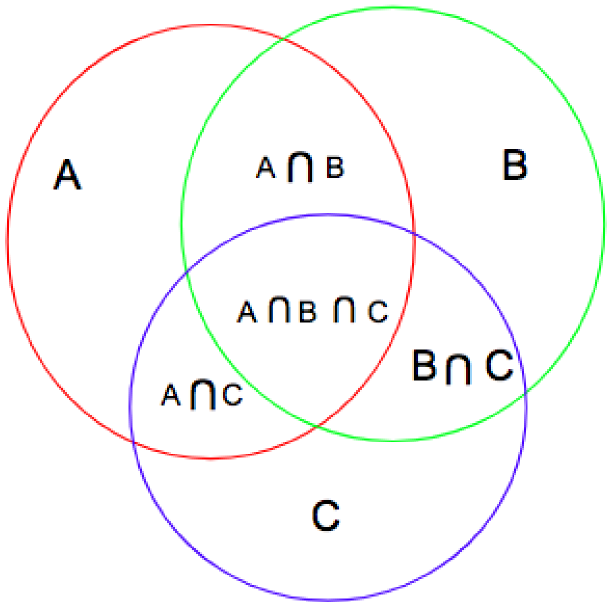
$A \cup B \cup C$ = {1,2,3,4,5,6,8}

$A \cap B = \{2,3,4\}, B \cap C = \{3,4,6\} and C \cap A = \{3,4\}$

$|A \cup B \cup C| = |A| + |B| + |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

= 5+4+4-3-3-2+2

= 7= $|A \cap B \cap C|$



$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Find the number of mathematics students at a college taking at least one of the languages French, German, and Russian given the following data:

65 study French	20 study French and German
45 study German	25 study French and Russian
42 study Russian	15 study German and Russian
	8 study all three languages

We want to find $n(F \cup G \cup R)$ where, F, G, and R denote the sets of students studying French, German, and Russian, respectively.

By the inclusion–exclusion principle,

$$\begin{aligned} n(F \cup G \cup R) &= n(F) + n(G) + n(R) - n(F \cap G) - n(F \cap R) - n(G \cap R) \\ &\quad + n(F \cap G \cap R) \\ &= 65 + 45 + 42 - 20 - 25 - 15 + 8 = 100 \end{aligned}$$

Thus 100 students study at least one of the languages.

Now, suppose we have any finite number of finite sets, say, $A_1, A_2, \ldots, A_m$. Let s_k be the sum of the cardinalities

$$n(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k})$$

of all possible k-tuple intersections of the given m sets. Then we have the following general inclusion–exclusion principle.

How many binary strings of length 8 either start with a ‘1’ bit or end with two bits ’00’?

- **Solution:** If the string starts with one, there are 7 characters left which can be filled in 2^7=128 ways.
  If the string ends with ’00’ then 6 characters can be filled in 2^6=64ways.
  Now if we add the above sets of ways and conclude that it is the final answer, then it would be wrong.
- This is because there are strings with start with ‘1’ and end with ’00’ both, and since they satisfy both criteria they are counted twice.
  So we need to subtract such strings to get a correct count.
  Strings that start with ‘1’ and end with ’00’ have five characters that can be filled in 2^5=32 ways.
- So by the inclusion-exclusion principle we get-
  Total strings = 128 + 64 – 32 = 160

- In case of the usage of three toothpastes A,B,C, It is fount that 60 people like A, 55 like, 40 like C, 20 like A and B, 35 like B and C, 15 like A and C , and 10 like all three toothpastes. Find the following
  - Number of persons included in the survey.
  - Number of persons who like A only
  - Number of persons who like A and B but not C

- In case of the usage of three toothpastes A,B,C, It is fount that 60 people like A, 55 like, 40 like C, 20 like A and B, 35 like B and C, 15 like A and C , and 10 like all three toothpastes. Find the following
  - Number of persons included in the survey.
  - Number of persons who like A only
  - Number of persons who like A and B but not C

Solution A,B,C, Denote set of people who like toothpastes A, B and C resp.

given, $|A|=60$, $|B|=55$, $|C|=40$, $|A\cap B|=20$, $|B\cap C|=35$, $|A\cap C|=15$, and $|A\cap B\cap C|=10$

Number of persons included in the survey.

$$|A\cup B\cup C|=|A|+|B|+|A\cap B|-|B\cap C|-|A\cap C|+|A\cap B\cap C|$$

$$= 60+55+40-20-35-15+10=95$$

Number of persons who like A only

$=|A|-(|A\cap B|+|A\cap C|- |A\cap B\cap C|)=60-(20+15-10)=35$

Number of persons who like A and B but not C

$=|A\cap B|+ |A\cap B\cap C|=20-10=10$

Countable Sets

Countable finite set:

Any finite set is countable by default

Countable infinite set:

An infinite set S is countable if there is a one-to-one correspondence from S to Z^+

Positive integers

Theorem: Even positive integers are countable

Proof:

Even positive integers: $2, \; 4, \; 6, \; 8, \; \ldots$

One-to-one Correspondence: $\updownarrow \; \updownarrow \; \updownarrow \; \updownarrow$

Positive integers: $1, \; 2, \; 3, \; 4, \; \ldots$

n corresponds to $2n$

End of Proof

Theorem: The set of rational numbers is countable

Proof:

We need to find a method to list

all rational numbers: $\frac{1}{2},\ \frac{3}{4},\ \frac{7}{8},\ \ldots$

Naïve Approach

Start with nominator=1

Rational numbers: $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots$

One-to-one correspondence:

Positive integers: $1, 2, 3, \ldots$

Doesn't work:

we will never list numbers with nominator 2: $\frac{2}{1}, \frac{2}{2}, \frac{2}{3}, \ldots$

Better Approach: scan diagonals

Nomin.=1 $\frac{1}{1}$ $\frac{1}{2}$ $\frac{1}{3}$ $\frac{1}{4}$ $\cdots$

Nomin.=2 $\frac{2}{1}$ $\frac{2}{2}$ $\frac{2}{3}$ $\cdots$

Nomin.=3 $\frac{3}{1}$ $\frac{3}{2}$ $\cdots$

Nomin.=4 $\frac{4}{1}$ $\cdots$

first diagonal

$$
\begin{array}{ccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\[2em]
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \cdots & \\[2em]
\frac{3}{1} & \frac{3}{2} & \cdots & & \\[2em]
\frac{4}{1} & \cdots & & &
\end{array}
$$

second diagonal

$$
\begin{array}{ccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\[2em]
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \cdots & \\[2em]
\frac{3}{1} & \frac{3}{2} & \cdots & & \\[2em]
\frac{4}{1} & \cdots & & &
\end{array}
$$

third diagonal

$$
\begin{array}{ccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\[2em]
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \cdots & \\[2em]
\frac{3}{1} & \frac{3}{2} & \cdots & & \\[2em]
\frac{4}{1} & \cdots & & &
\end{array}
$$

fourth diagonal…

$$
\begin{array}{cccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \cdots & \\
\frac{3}{1} & \frac{3}{2} & \cdots & & \\
\frac{4}{1} & \cdots & & &
\end{array}
$$

Every element will be eventually scanned

Diagonal listing

Rational Numbers: $\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{2}{2}, \ldots$

One-to-one correspondence:

Positive Integers: $1, \quad 2, \quad 3, \quad 4, \quad 5, \ldots$

End of Proof

We have proven: $(0,1) \subseteq R$ is uncountable

It can be proven: Every subset of a countable set is countable

It follows that the set of real numbers R is uncountable

Multisets

- Sets:
  - An unordered collection of distinct objects.
- Multisets:
  - Sets in which some elements occur more than once
  - $A=\{1,1,1,2,2,3\}$
- Notation to represent a multiset by:
  - $S=\{n_1.a_1,n_2.a_2,\ldots,n_i.a_i\}$
  - This denotes that a_1 occurs n_1 times
  - The number n_i=1,2,3,.. Are called multiplicities of the elements n_i.
  - $A=\{3.1,2.2,1.3\}$

Union of Multisets

- The union of the multisets A and B is the multiset where the multiplicity of an element is the maximum of its multiplicities in A and B

$$A = \{1, 1, 1, 2, 2, 3\} \text{ and}$$
$$B = \{1, 1, 4, 3, 3\}$$
$$A \cup B = \{1, 1, 1, 4, 2, 2, 3, 3\}$$

Intersection Multisets

- The Intersection of A and B is the multiset where the multiplicity of an element the minimum of its multiplicities in A and B.

$A = \{1, 1, 1, 2, 2, 3\}$ and

$B = \{1, 1, 4, 3, 3\}$

$A \cap B = \{1, 1, 3\}$

Difference of Multisets

- The difference of A and B is the multiset where the multiplicity of an element is the multiplicity of element in A less its multiplicity in B unless this difference is negative, in which case the multiplicity is zero.

$$A = \{1, 1, 1, 2, 2, 3, 4, 4, 5\}$$
$$B = \{1, 1, 2, 2, 2, 3, 3, 4, 4, 6\}$$
$$A - B = \{1, 5\}$$

Sum of Multisets

- The Sum of A and B is the multiset where the multiplicity of an element is sum of multiplicities in set A and set B denoted by A+B.

$$A = \{1, 1, 2, 3, 3\} \quad \text{and} \quad B = \{1, 2, 2, 4\},$$

$$A + B = \{1, 1, 1, 2, 2, 2, 3, 3, 4\}$$

Multiset Examples

Let A and B be multisets as $A = \{3.a, 2.b, 1.c\}$ and $B = \{2.a, 3.b, 4.d\}$

Find

(a) $A \cup B = \{3.a, 3.b, 1.c, 4.d\}$

(b) $A \cap B = \{2.a, 2.b\}$

(c) $A - B = \{1.a, 1.c\}$

(d) $B - A = \{1.b, 4.d\}$

(e) $A + B = \{5.a, 5.b, 1.c, 4.d\}$