College of Engineering Pune
Ordinary Differential Equations and Multivariate Calculus
Tutorial-4 (2021-2022)

The main idea behind the Laplace Transformation is that we can solve an equation (or system of equations) containing differential and integral terms by transforming the equation in "t-space" to one in "s-space". Usually t is time and s is frequency!

Questions on CO1

1. Define Laplace Transform and Inverse Laplace Transform of a function. State and prove the algebraic properties of Laplace Transform.
2. State the first shifting, second shifting and Convolution theorems.
3. When do we say that a function is of **exponential order**?
4. Why the limits of the integration in the definition of Laplace Transform is from 0 to ∞? Give the logical justification.
5. Is $L\{f(t)g(t)\} = L\{f(t)\}L\{g(t)\}$? Justify your answer!

Questions on CO2 and CO3

1. Which of the following functions are of exponential order and why?
   (a) $sin(e^{t^2})$
   (b) e^{t^π}
2. Give an example of a function which of exponential order but its derivative is not of exponential order.
3. Give an example of a function whose Laplace transform exists, such that f is not piecewise continuous but has exponential order.
4. Give an example of a function whose Laplace transform exists, such that f is continuous but is not of exponential order.
5. Let f be a piecewise continuous function of exponential order and F be a Laplace transform of f then prove that:
$$\lim_{s\to\infty} F(s) = 0$$
6. Is it possible to find piecewise functions of exponential order whose Laplace transforms are:
   (a) $F(s) = s, \quad s \in \mathbb{R}$
   (b) $F(s) = \frac{s-1}{s+1}, \quad s > -1$
7. Is it possible to find functions (you may think of generalized functions such as Dirac delta function) whose Laplace transforms are:
   (a) $F(s) = \frac{s^2}{s^2+1}, \quad s \in \mathbb{R}$

(b) $F(s) = \frac{s^2}{s^2-1}, \quad s > 1$

8. Find the Laplace Transforms of the following functions:

(a) $(5e^{2t} - 3)^2$ Ans. $\frac{25}{s-4} - \frac{30}{s-2} + \frac{9}{s}$

(b) $\sin 3t - 2\cos 5t$ Ans. $\frac{3}{s^2+9} - 2\frac{s}{s^2+25}$

(c) $\cosh at - \cos at$ Ans.$\frac{2a^2 s}{s^4-a^4}$

(d) $e^t(1+t)^2$ Ans. $\frac{s^2+1}{(s-1)^3}$

(e) $f(t) = \begin{cases} t, & 0 < t < 1 \\ e^{1-t}, & t > 1. \end{cases}$ Ans. $\frac{1}{s^2}[1 - e^{-s}(\frac{2s+1}{s+1})]$

(f) $t^{7/2}e^{3t}$ Ans. $\frac{105\sqrt{\pi}}{16(s-3)^{9/2}}$

(g) $f(t) = t\cos at$ Ans. (Use $\mathcal{L}\{tf(t)\}$). $\frac{s^2-a^2}{(s^2+a^2)^2}$

(h) $\sin^2 t$ Ans. (Use $\mathcal{L}\{f'\}$). $\frac{2}{s(s^2+4)}$

(i) $\frac{e^{-at}-e^{-bt}}{t}$ Ans. (Use $\mathcal{L}\{f(t)/t\} = \int\limits_s^\infty F(u)du$). $\ln\frac{s+b}{s+a}$

(j) $\frac{1}{2}t^2\cos\frac{\pi}{2}t$ Ans. $16\frac{s(4s^2-3\pi^2)}{(4s^2+\pi^2)^3}$

(k) $e^{-t}\mathrm{sinh}4t$ Ans. $\frac{4}{s^2+2s-15}$ $\frac{e^t\,\delta(t-2)}{t}$ Ans. $\frac{e^{-2(s-1)}}{2}$

(l) $\delta(t-3)\,U(t-3)$ Ans. e^{-3s}

(m) $t^2\sin 2t$ Ans. (Use $\mathcal{L}\{t^2 f(t)\} = F''(s)$). $\frac{-4(4-3s^2)}{(s^2+4)^3}$

(n) $\int\limits_0^t \frac{1-e^{-u}}{u}du$ Ans. (Use $\mathcal{L}\{\int\limits_o^t f(u)du\} = \frac{\mathcal{L}\{f\}}{s}$). $\frac{1}{s}\ln(1+\frac{1}{s})$

(o) First sketch and express in terms of unit step: $e^{-\pi t/2}$; $1 < t < 3$; 0 outside. Ans. $2\frac{e^{-s-\pi/2}-e^{-3s-3\pi/2}}{2s+\pi}$

(p) $4t * e^{-2t}$, $*$ denotes the convolution. Ans. $\frac{8}{s^3(s+2)}$

9. Find the inverse Laplace transform of the following:

(a) $\frac{0.1s+0.9}{s^2+3.24}$ Ans. $0.1\cos 1.8t + 0.5\sin 1.8t$

(b) $\frac{-s-10}{s^2-s-2}$ Ans. $3e^{-t} - 4e^{2t}$

(c) $\frac{1}{(s-1)(s^2+4)} + \frac{4}{s^5}$ Ans. $\frac{e^t}{5} - \frac{\cos 2t}{5} - \frac{\sin 2t}{10} + \frac{t^4}{6}$

(d) $\frac{3s+1}{s^2+6s+13}$ Ans. $e^{-3t}(3\cos 2t - 4\sin 2t)$

(e) $\frac{s^2}{(s-1)^4}$ Ans. $e^t(t + t^2 + \frac{t^3}{6})$

(f) $\frac{e^{-\pi s}}{s^2+9}$ Ans. $\frac{1}{3}\sin 3(t-\pi)U(t-\pi)$

(g) $\frac{1-e^{-s}}{s^2}$ Ans. t, if $t < 1$ and 1 if $t > 1$.

(h) $\cot^{-1}\frac{s}{\omega}$ Ans. (Let $f(t) = \mathcal{L}^{-1}F(s)$. Use $\mathcal{L}^{-1}F'(s) = -tf(t)$). $(\sin\omega t)/t$.

(i) $\frac{1}{2}\ln(\frac{s^2-a^2}{s^2})$ Ans. $\frac{1-\cosh at}{t}$

(j) $\ln\sqrt{\frac{s^2+b^2}{s^2+a^2}}$ Ans. $\frac{\cos at - \cos bt}{t}$

(k) $\frac{e^{-2s}}{s^6}$. Also sketch $f(t)$. Ans. $\frac{1}{120}(t-2)^5 U(t-2)$

(l) $\frac{s^3-3s^2+6s-4}{(s^2-2s+2)^2}$ Ans. $e^t(t\sin t+\cos t)$

(m) $s\ln(\frac{s}{\sqrt{s^2+1}})$ Ans.(Use $\mathcal{L}^{-1}F''(s)=t^2 f(t)$).

(n) $\frac{e^{-s}}{s}\tan^{-1}(\frac{s-1}{4})$ Ans. Let $F(s)=e^{-s}/s$, $G(s)=\tan^{-1}(\frac{s-1}{4})$. Then $\mathcal{L}^{-1}F(s)=U(t-1)$ and $\mathcal{L}^{-1}G(s)=\frac{-e^t\sin 4t}{t}$. By convolution thm, the required ans is $\mathcal{L}^{-1}F(s)G(s)=U(t-1)*\frac{-e^t\sin 4t}{t}$.

10. Solve using Laplace transform:

(a) $y'+2y=4te^{-2t}, \quad y(0)=-3$
Ans. $y(t)=2t^2e^{-2t}-3e^{-2t}$

(b) $y''+y=r(t)$, $r(t)=t$ if $1<t<2$, 0 otherwise. $y(0)=y'(0)=0$
Ans. $y=[t-\cos(t-1)-\sin(t-1)]U(t-1)+[-t+2\cos(t-2)+\sin(t-2)]U(t-2)$

(c) $y''+y=e^{-2t}\sin t, y(0)=y'(0)=0.$
Ans. $y=\frac{1}{8}[\sin t-\cos t+e^{-2t}(\sin t+\cos t)]$

(d) $y''+2y'+5y=50t-150, y(3)=-4, y'(3)=14.$
Ans. $y=10(t-3)-4+2e^{-(t-3)}\sin 2(t-3)$

(e) $y''+2y'+5y=e^{-t}\sin t,\ y(0)=0, y'(0)=1$
Ans. $y=e^{-t}(\sin t+\sin 2t)/3$

(f) Find the current $i(t)$ in an LC circuit assuming $L=1$henry, $C=1$ farad, zero initial current and charge on the capacitor and $v(t)=1-e^{-t}$ if $0<t<\pi$ and 0, otherwise.
Ans. $\frac{1}{2}(e^{-t}-\cos t+\sin t)$, if $0<t<\pi$ and $\frac{1}{2}[-(1+e^{-\pi})\cos t+(3-e^{-\pi})\sin t]$, if $t>\pi$.

(g) Solve for a common solution: $y_1'=u_1+y_2, y_2'=-y_1+3y_2, y_1(0)=1, y_2(0)=0$

11. Solve the following linear integral equations:

(a) $y(t)=\sin 2t+\int\limits_0^t y(\tau)\sin 2(t-\tau)d\tau.$ Ans. $\sqrt{2}\sin\sqrt{2}\,t$

(b) $y(t)=1-\sinh t+\int\limits_0^t(1+\tau)\,y(t-\tau)d\tau.$ Ans. $\cosh t$

Questions on CO4 and CO5

1. State and prove the theorem on existence of Laplace transforms. Does it give necessary and sufficient conditions for existence? Justify your answer.

2. Find Laplace transform of n^{th} derivative of a function $f(t)$ stating clearly the necessary conditions on the function and its derivatives.

3. Find the Laplace transform of $\int_0^t f(\tau)d\tau$ stating clearly the necessary conditions under which it exists.

4. Find the current in an RLC circuit if $R=4\Omega, L=1H, C=0.05F$ and the applied voltage is $v=34e^{-t}V, \quad 0<t<4; 0$ for $t>4$. Assume that current and charge are 0 initially. Solve using Laplace transform method showing all the details.

5. Find the Laplace transform of a periodic function and hence find the Laplace transform of half wave rectification of $sin\omega t$.

6. Define convolution of two functions. Prove the commutative, associative and distributive properties of convolution of two functions.

7. State and prove the convolution theorem for Laplace transforms.

8. Write a summary on Laplace transforms in your own words not exceeding 500 words.

9. Note that any problem similar to the problems in CO3 in a new or unknown situation can be treated as a problem of CO4 or CO5. Hence you should try to solve all problems in the exercises from the text book.

Please report any mistakes in the problems and/or answers given here.