$$= \int^{\frac{\pi}{2}} \sin^3\theta \cos^5\theta \cdot d\theta$$

$$= \frac{1}{2} \beta(2,3) = \frac{1}{2} \frac{\Gamma(2)+\Gamma(3)}{\Gamma(5)}$$

Tutorial :- 6

Q.1. Find Absolute maximum and minimum

1) $f(x) = \frac{-1}{x+3}, \quad -2 \leq x \leq 3$

$\rightarrow$ $f'(x) = \frac{1}{x+3}$ if $x = -3$, then the given eqn will be in not defined. but -3 is not in the range.

$\therefore$ $f'(x)$ is defined at $-2 \; \forall x \in -2, 3$.

$\therefore$ No critical point.

Boundary point $= -2, 3$

absolute minima $= \frac{-1}{-2+3} = \frac{-1}{+1} = -1$

absolute maxima $= \frac{-1}{3+3} = \frac{-1}{6}$

2) $f(x) = x^{\frac{1}{3}}, \quad -1 \leq x \leq 8$

$\rightarrow$ $f'(x) = \frac{1}{3} x^{-\frac{2}{3}} = \frac{1}{3} \frac{1}{x^{2/3}}$

If $x = 0$, then the given eqn will be not defined, & 0 is in the domain

$\therefore$ Hence 0 is the critical point.

$f'(x) \neq 0$ (for all) $\forall x \in -1, 8$

Absolute values occur at $-1, 0, 8$.

Boundary points, $(-1, \cancel{0}, 8)$

absolute maxima & minima.

$f(x) = x^{1/3} = (-1)^{1/3} = 1$

$f(x) = x^{1/3} = (0)^{1/3} = 0 \rightarrow$ minima

$f(x) = x^{1/3} = (8)^{1/3} = 2 \rightarrow$ maxima.

3) $f(x) = -3x^{\frac{2}{3}}, -1 \leq x \leq 1$

$\rightarrow$ $f'(x) = -3 \times \frac{2}{3} x^{-\frac{1}{3}} = 2x^{-\frac{1}{3}}$

If $x = 0$, then the given term will be not defined. Hence, 0 is the domain and it is critical point.

$f'(x) \neq 0, \forall x \in -1, 1$

Boundary points $(-1, \cancel{0}, 1)$

absolute values occur at $-1, 0, 1$

$f(x) = -3(-1)^{\frac{2}{3}} = -3 \times 1 = -3 \rightarrow$ minima

$f(x) = -3(0)^{2/3} = 0 \rightarrow$ maxima.

$f(x) = -3(1)^{2/3} = -3 \rightarrow$ minima

4) $f(x) = \sqrt{4-x^2}, -2 \leq x \leq 1$

$\rightarrow$ $f'(x) = \frac{1}{2\sqrt{4-x^2}} \times -2x$

$= \frac{-2x}{2\sqrt{4-x^2}}$

put $f'(x) = 0$

$0 = \frac{-x}{\sqrt{4-x^2}}$ $\therefore x = 0$

$f'(x) = 0 \quad x = 0$

$\therefore$ 0 is the only critical point.

absolute values occur at $-2, 0, 1$.

$f(x) = \sqrt{4-(-2)^2} = 0 \rightarrow$ minima.

$f(x) = \sqrt{4-(0)^2} = 2 \rightarrow$ maxima

$f(x) = \sqrt{4-(1)^2} = 1.73$

5) $f(\theta) = \sin\theta, -\pi/2 \leq \theta \leq \frac{5\pi}{6}$

$\rightarrow$ $f'(\theta) = \cos\theta$

When $\theta = \frac{\pi}{2}$, $f'(\theta) = 0$

$\cos\left(\frac{\pi}{2}\right) = 0 \quad \therefore \theta = \frac{\pi}{2}$

$\frac{\pi}{2}$ is the only critical point.

Absolute values occur at $\frac{-\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{6}$

$f(\theta) = \sin\left(-\frac{\pi}{2}\right) = -1 \rightarrow$ minima

$f(\theta) = \sin\left(\frac{\pi}{2}\right) = 1 \rightarrow$ maxima.

$f(\theta) = \sin\left(\frac{5\pi}{6}\right) = 0.5$

6) $3x^4 - 16x^3 + 18x^2$, $-1 \le x \le 4$

→ $f'(x) = 12x^3 - 48x^2 + 36x$

$= 12x[x^2 - 4x + 3]$

$12x(x^2 - 4x + 3) = 0$ $\quad 12x = 0$

$x^2 - 4x + 3 = 0$

$(x-3)(x-1) = 0$

$x = 3 \quad x = 1$

absolute values occur at , $-1, 0, 1, 3, 4$

$f(x) = 3(-1)^4 - 16(-1)^3 + 18(-1)^2$

$= 3 + 16 + 18$

$= 37$ ⟶ ~~maima~~ maxima.

$f(x) = 3(0)^4 - 16(0)^3 + 18(0)^2 = 0$

$f(x) = 3(1)^4 - 16(1)^3 + 18(1)^2$

$= 3 - 16 + 18$

$= 5$

$f(x) = 3(3)^4 - 16(3)^3 + 18(3)^2$

$= 3 \times 81 - 16 \times 27 + 18 \times 9$

$= 243 - 432 + 162$

$= -27$ → minima.

$f(x) = 3(4)^4 - 16(4)^3 + 18(4)^2$

$= 32$

Q.2 Identify the largest possible domain.

a) $f(x) = 2x^2 - 8x + 9$

→ Here Domain : $(-\infty, \infty)$

$\therefore f'(x) = 4x - 8$

$4x - 8 = 0$

$4x = 8$

$x = \frac{8}{4} = 2$

$\therefore$ critical point $= 2$

Extreme values at critical point

$= 2(2)^2 - 8(2) + 9$

$= 8 - 16 + 9$

$= 1$

b) $f(x) = x^3 - 2x + 4$

→ Here domain $(-\infty, \infty)$

$\therefore f'(x) = 3x^2 - 2$

$0 = 3x^2 - 2$

$3x^2 = 2$

$x = \pm\sqrt{\frac{2}{3}}$ = It is critical point.

Extreme value at critical point.

$= \left(\sqrt{\frac{2}{3}}\right)^3 - 2\left(\sqrt{\frac{2}{3}}\right) + 4$

$= \frac{2\sqrt{6}}{9} - \frac{2\sqrt{2}}{3} + 4$

$=$ 2.91 for $+\sqrt{\frac{2}{3}}$ and for $-\sqrt{\frac{2}{3}}$ = 5.088.

c) $f(x) = \sqrt{x^2-1}$

$\rightarrow$ $D = \{x \mid x^2 - 1 \geq 0\}$

$= \{x \mid x^2 \geq 1\}$ $\quad x^2 = \pm 1$

$\therefore (-\infty, -1] \cup [1, \infty)$

$f'(x) = (x^2-1)^{\frac{1}{2}}$

$= \frac{1}{2} \times (x^2-1)^{-\frac{1}{2}}$

$= \frac{1}{2\sqrt{x^2-1}} \times 2x$

$= \frac{x}{\sqrt{x^2-1}}$

put $f'(x) = 0$

$\frac{x}{\sqrt{x^2-1}} = 0$

$x = 0$

$\therefore$ 0 is not in the domain.

$\therefore f(-1) = \sqrt{(-1)^2-1} = 0$

$\therefore f(1) = \sqrt{(1)^2-1} = 0$

d) $f(x) = 1/\sqrt{x^2-1}$

$\rightarrow$ $D = \{x \mid x^2 - 1 > 0\}$

$= \{x \mid x^2 > 1\}$

$\therefore (-\infty, -1) \cup (\infty, 1)$

$f'(x) = \frac{1}{\sqrt{x^2-1}}$

$= \frac{1}{(x^2-1)^{1/2}} = -(x^2-1)^{-\frac{1}{2}}$

$= \frac{1}{2} \times -(x^2-1)^{-\frac{3}{2}} \times 2x$

$f'(x) = \frac{-x}{(x^2-1)^{3/2}}$

$\frac{-x}{(x^2-1)^{3/2}} = 0$

$-x = 0$ Not in domain

$\therefore$ No critical point.

e) $f(x) = \frac{x}{(x^2+1)}$

$\rightarrow$ $f'(x) = \frac{u}{v} = \frac{vu' - uv'}{v^2}$

$u = x \quad v = (x^2+1)$

$= \frac{(x^2+1)1 - x(2x)}{(x^2+1)^2}$

$= \frac{x^2+1-2x^2}{x^4+2x^2+1} = \frac{-x^2+1}{(x^2+1)^2}$

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put $f'(x) = 0$

$f'(x) = \dfrac{-x^2+1}{(x^2+1)^2}$

$0 = -x^2 + 1$

$x^2 = \pm 1$

Domain : $\mathbb{R}$

f) $f(x) = e^x$

$\rightarrow$ Domain : $\mathbb{R}$

$f'(x) = e^x$

put $f'(x) = 0$

$e^x = 0$

$\therefore$ no critical point.

Q.3 Find the set of critical points and determine the local extreme values.

1> $f(x) = x^{\frac{2}{3}}(x+2)$ $\qquad u = x^{\frac{2}{3}}$, $v = (x+2)$

Domain : $(-\infty, \infty)$

$f'(x) = u\dfrac{d}{dx}v + v\dfrac{d}{dx}u$

$= x^{\frac{2}{3}}(1) + (x+2)\,\dfrac{2}{3}\,x^{-\frac{1}{3}}$

$= x^{\frac{2}{3}} + \dfrac{2(x+2)}{3}\,x^{-\frac{1}{3}}$

$= \cancel{\dfrac{x^{-\frac{1}{3}}\,3x^{\frac{2}{3}} + 2(x+2)}{3x^{1/3}}}$

$= \cancel{\dfrac{x^{\frac{1}{3}} + 2(x+2)}{3}} = \cancel{x + 2x + 4}$

$= x^{\frac{2}{3}} + \dfrac{2(x+2)}{3x^{1/3}}$

$= \dfrac{3x^{\frac{1}{3}}\,x^{\frac{2}{3}} + 2(x+2)}{3x^{1/3}} = \dfrac{3x + (2x+4)}{3x^{1/3}}$

$f'(x) = 0$

$\dfrac{3x + 2x + 4}{3x^{1/3}} = 0$

$3x + 2x + 4 = 0$

$5x + 4 = 0$

$5x = -4$

$x = \dfrac{-4}{5}$

when $x = 0$, $f'(x)$ is not defined.

$\therefore$ $\dfrac{-4}{5}$ and 0 is the critical point.

For local extreme put $-\frac{4}{5}$ in $f(x)$;

2> $f(x) = x^2\sqrt{3-x}$

$\rightarrow$ Here for root, we have to check, that we don't get negative value.

$\therefore D = \{x \mid 3 - x \geq 0\}$

$D = \{ x \mid 3 \geq x \}$

$\therefore$ Domain $(-\infty, 3]$

put f

$f'(x) = uv' + vu'$

$= x^2\left(\frac{-1}{2\sqrt{3-x}}\right) + \sqrt{3-x}\,(2x)$

put $f'(x) = 0$

$= \frac{-x^2}{2\sqrt{3-x}} + 2x\sqrt{3-x}$

$= \frac{-x^2 + (2x\sqrt{3-x})(2\sqrt{3-x})}{2\sqrt{3-x}}$

$0 = \frac{-x^2 + 4x\,(3-x)}{2\sqrt{3-x}}$

$-x^2 + 12x - 4x^2 = 0$

$12x - 5x^2 = 0$

$(12 - 5x)x = 0 \quad \therefore x = 0$

$12 - 5x = 0$

$-5x = -12$

$x = \frac{12}{5}$

$\therefore$ 0 and $\frac{12}{5}$ is the critical point.

If we put $x = 3$ in $f'(x)$ the it will be not defined.

$\therefore$ 0, 3, $\frac{12}{5}$ are critical point.

$f(x) = x^2\sqrt{3-x} = (3)^2\sqrt{3-3} = 0$

$f\left(\frac{12}{5}\right) = \left(\frac{12}{5}\right)^2\sqrt{3-\left(\frac{12}{5}\right)} =$

c) $f(x) = x|x| - x$

$\rightarrow$ Domain : $\mathbb{R}$ $(-\infty, \infty)$

put $|x| = \pm x$

$f(x) = x(x) - x$	$f(x) = x(-x) - x$
$= x^2 - x$	$f(x) = -x^2 - x$
$f'(x) = 2x - 1$	$f'(x) = -2x - 1$
$0 = 2x - 1$	$1 = -2x$
$2x = 1$	$x = -\frac{1}{2}$
$x = \frac{1}{2}$	

$\therefore$ critical points $\frac{1}{2}, -\frac{1}{2}$

$f(x) = \frac{1}{2} \times \frac{1}{2} - \frac{1}{2} = \frac{1}{4} - \frac{1}{2} = \frac{1-2}{4} = -\frac{1}{4}$ local min

$f(x) = -\frac{1}{2} \times -\frac{1}{2} + \frac{1}{2} = \frac{1}{4} + \frac{1}{2} = \frac{3}{6} = \frac{1}{2}$ local max.

d) $f(x) = \begin{cases} 3-x & \text{if } x > 0 \\ 3+2x-x^2 & \text{if } x \leq 0 \end{cases}$

$f(x) = 3 - x$ $\qquad$ $f(x) = 3 + 2x - x^2$

$f'(x) = -1$ $\qquad$ $f'(x) = 2 - 2x$

Here $f'(x)$ is not zero $\qquad$ $0 = 2 - 2x$

∴ no critical point. $\qquad$ $-2x = -2$

$x = 1 \rightarrow$ critical point.

∴ ~~critical point~~

put values in $f(x)$

$3 + 2(1) - (1)^2 =$

$3 + 2 - 1 =$

$3 + 1 =$

4

∴

4. Show that equation $x^3 + x - 1 = 0$ has exactly one real root.

→ Step 1 :

When $x \rightarrow (-\infty)$ then $f'(x) \rightarrow (-\infty)$

When $x \rightarrow (\infty)$ then $f'(x) \rightarrow (\infty)$

Since f is continuous function, it will cut x-axis

∴ Function has real root.

By intermediate value theorem :

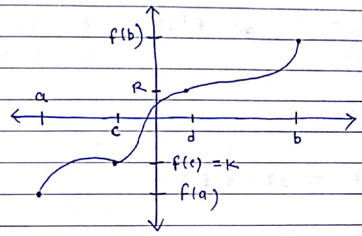
So f is continuous on closing interval $[a, b]$.

$\exists k$ such that $f(a) \leq K \leq f(b)$

K is the number.

$\exists$ $c \in [a, b]$ such that

$f(c) = k$



$f(x) = x^3 + x - 1$

$f'(x) = 3x^2 + 1$

→ suppose $[a, b]$ is $[-1, 1]$

$f(-1) = (-1)^3 + (-1) - 1 = -1 - 1 - 1 = -3$

$f(1) = (1)^3 + (1) - 1 = 1 + 1 - 1 = 1$

∴ range $[-1, 1]$

∴ $f(-1) < 0$ & $f(1) > 0$

∴ $f(c) = 0$

Step 2 :

Suppose $f(x)$ has two roots.

let c and d are the root.

then.

$\therefore f(c) = 0$ and $f(d) = 0$

$\because$ By LMVT

$\frac{f(d) - f(c)}{d - c} = f'(x)$

$\frac{0 - 0}{d - c} = f'(x)$

$\therefore f'(x) = 0$

But $f'(x) = 3x^2 + 21$

$\therefore$ Our supposition is wrong.

$f(x)$ does not have two roots.

It has exactly one root.

5) Let $f(x) = \tan x$ on $[0, \pi]$. Find the critical point if exist, and hence extreme values of f.

→ $f(x) = \tan x$

$f'(x) = \sec^2 x$

$\sec^2 x = 0$

$\frac{1}{\cos^2 x} = 0 \quad \therefore 1 \neq 0$

$\therefore$ 0 is not the critical point.

$\therefore$ Extreme values occur at $0, \frac{\pi}{2}$

$\therefore f(0) = \tan 0 = 0$

$f(\pi/2) = \tan\left(\frac{\pi}{2}\right) = \infty$

6) Show that the 5 is the critical point of the function $f(x) = 2 + (x-5)^3$ but f does not have a local extreme value at 5.

→ $\cancel{f'(x) = 2 + x^3}$

$f'(x) = 0 + 3(x-5)^2$

$0 = 3(x^2 - 25) \quad \cdots \quad (x-5)(x+5)$

$0 = 3x^2 - 25$

$0 = 3(x-5)(x+5)$

$0 = (x-5)$

$x = 5$

first $f'(x) = 0$, then $f'(x) = 3(x-5)^2$

put any value in x, the answer is +ve.

$f'(x) > 0$

$\therefore$ No extreme value at 5

7) ——?

8. —— ?

$f''(x) > 0 \;, \forall x \in \mathbb{R}$

$f'(0) = 1 \;, \; f'(1) = 1$

By LMVT,

$$\frac{f'(1) - f'(0)}{1-0} = f''(x)$$

$$\frac{1-1}{1} = f''(x)$$

$$\frac{0}{1} = f''(x)$$

$$0 = f''(x)$$

But it is given that $f''(x) > 0 , \forall x \in \mathbb{R}$

∴ our supposition is wrong

such function never exist.

9. —— ?

f is continous function on $[a,b]$.

∴ f $a \longrightarrow -\infty$ then $f(a) = -\infty$; $b \longrightarrow \infty$ then $f(b) = \infty$ } given

f is continous on closing interval $[a,b]$

~~such that~~ $\exists$ c

K is a number, such that $f(a) \leq K \leq f(b)$ but

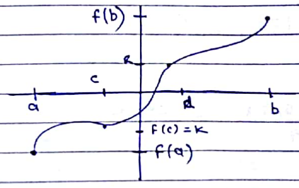
$f(a) \leq 0 \; f(b)$

∴ $K = 0$

$\exists \; c \in (a,b)$ such that

$f(c) = K$

$f(c) = 0$



c and d are the two root

∴ $f(c) = 0$

$f(d) = 0$

∴ By LMVT,

$$\frac{f(d) - f(c)}{d-c} = f'(x)$$

$$\frac{0-0}{d-c} = f'(x)$$

∴ $f'(x) = 0$

But $f' \neq 0$ is given in problem, so our supposition is wrong.

$f(x) = 0$ exactly once between a and b.

10) —— ?

$\sin f(x) = \sin x$

$f(a) = \sin a$

$f(b) = \sin b$

Applying LMVT

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$$\frac{\sin b - \sin a}{b - a} = \cos c$$

Apply mod on both side

$$\frac{|\sin b - \sin a|}{|b - a|} = |\cos c|$$

cos value ~~valu~~ is not greater than 1.

$$\therefore \frac{|\sin b - \sin a|}{|b - a|} = |\cos c| \leq 1$$

(It will multiply by 1)
(~~multiply~~ by 1.)

$$|\sin b - \sin a| \leq |b - a|$$

$$|\sin a - \sin b| \leq |a - b|$$

$\because$ (mod $b - a$) = (mod $a - b$)

11) $f(0) = -3$ and $f'(x) = -5$. how large can $f(2)$ possibly be.

→ Applying LMVT.

$$f'(x) = \frac{f(2) - f(0)}{2 - 0}$$

$$-5 = \frac{f(2) - (-3)}{2 - 0}$$

$$-5 = \frac{f(2) + 3}{2}$$

$$-10 = f(2) + 3$$

$$-10 - 3 = f(2)$$

$$-13 = f(2)$$

12. — ?

* Let $f(t)$ is the distance covered by p and $g(t)$ is the distance covered by s.

Initial, endpoint.

$[0, t_0]$

$f(0) = g(0)$

$f(t_0) = g(t_0)$

Let $h(t) = f(t) - g(t)$

such that

$h[0] = 0$

$h[t_0] = 0$

Now applying RT on $[0, t_0]$ such that $\exists$ c $\in (0, t_0)$. $h'(c) = 0$

$h'(c) = f'(c) - g'(c)$

$0 = f'(c) - g'(c)$

$f'(c) = g'(c)$

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13. —— ?

$f(a) = a \quad f(-a) = a \quad f'(x) \leq 1$

show $f(0) = 0$

$[-a, 0] \cup [0, a]$

→ Applying LMVT on $(0, a)$

$$\frac{f(a) - f(0)}{a - 0} = f'(x) \leq 1$$

$$\frac{a - f(0)}{a} = f'(x) \leq 1$$

$\cancel{a - f(0) = a}$

$\cancel{a - a = f(0)}$

$\cancel{0 = f(0)}$

$$\frac{a - f(0)}{a} \leq 1$$

$$a - f(0) \leq a$$

$$a - a \leq f(0)$$

$$0 \leq f(0)$$

$\therefore f(0) = 0$

14. —— ?

$$\frac{b-a}{\sqrt{1-a^2}} < \sin^{-1} b - \sin^{-1}(a) < \frac{b-a}{\sqrt{1-b^2}}$$

for $0 < a < 1$.

and $0 \leq 0 \leq a$

$\therefore f(x) = \sin^{-1}(x)$

$f(a) = \sin^{-1}(a)$

$f(b) = \sin^{-1}(b)$

By applying LMVT on $[a, b]$

$$\frac{f(b) - f(a)}{b - a} = f'(c) \quad \forall c \in [a, b]$$

$$\frac{\sin^{-1}(b) - \sin^{-1}(a)}{b - a} = \frac{1}{\sqrt{1-c^2}} \quad \text{—— (1)}$$

$$a < c < b$$

$$a^2 < c^2 < b^2$$

$$-a^2 > -c^2 > -b^2$$

$$1 - a^2 > 1 - c^2 > 1 - b^2$$

$$\sqrt{1-a^2} > \sqrt{1-c^2} > \sqrt{1-b^2}$$

$$\frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$$

$$\frac{1}{\sqrt{1-a^2}} < \frac{\sin^{-1}(b) - \sin(a)}{b - a} < \frac{1}{\sqrt{1-b^2}} \quad \{\text{from (1)}\}$$

Multiply by $b - a$

$$\frac{b-a}{\sqrt{1-a^2}} < \sin^{-1}(b) - \sin(a) < \frac{(b-a)}{\sqrt{1-b^2}}$$

(hence proved)

Prove that :

1. $\frac{b-a}{\sqrt{1-a^2}} < \sin^{-1}(b) - \sin^{-1}(b)$

→ consider the function

$\therefore f(x) = \sin^{-1}(x)$

$\therefore f(a) = \sin^{-1}(a)$

$\therefore f(b) = \sin^{-1}(b)$

Applying LMVT ,

$$\frac{f(b) - f(a)}{b-a} = f'(c) \quad \forall c \in [a,b]$$

$$\frac{\sin^{-1}(b) - \sin^{-}(a)}{b-a} = f'(c)$$

$$\frac{\sin^{-1}(b) - \sin(a)}{b-a} = \frac{1}{\sqrt{1-c^2}} \quad \text{— (1)}$$

$a < c < b$

$a^2 < c^2 < b^2$

$1-a^2 \cancel{<}> 1-c^2 >\cancel{<} 1-b^2$

$\frac{1}{1-a^2} \cancel{>}< \frac{1}{1-c^2} <\cancel{>} \frac{1}{1-b^2}$

$\frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$

put $\frac{1}{\sqrt{1-c^2}}$ by eqⁿ (1)

$$\frac{1}{\sqrt{1-a^2}} < \frac{\sin^{-1}(b) - \sin^{-1}(a)}{b-a} < \frac{1}{\sqrt{1-b^2}}$$

multiply by $b-a$

$$\frac{b-a}{\sqrt{1-a^2}} < \sin^{-1}(b) - \sin^{-1}(a) < \frac{b-a}{\sqrt{1-b^2}}$$

2. $\tan x > x$ for $0 < x < \frac{\pi}{2}$

→ consider $f(x) = \tan(x)$

$f(0) = \tan(0)$

$$\frac{\tan(x) - \tan(0)}{x-0} = f'(x)$$

$$\frac{\tan(x) - \tan(0)}{x \cdot 0} = \sec^2 x$$

$$\frac{\tan(x)}{x} = \sec^2 x$$

But $\cos^2 x = \frac{1}{\sec^2 x}$ and $\cos^2 x \le 1$

$\therefore$ ta $\frac{1}{\cos^2 x} \geq 1$

$\sec^2 x > 1$

but $\sec^2 x = \frac{\tan(x)}{x}$

$\frac{\tan(x)}{x} > 1$

$\boxed{\tan(x) > x}$

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derivative of $\log(1+x) = \frac{1}{1+x}$

3. $\frac{x}{1+x} < \log(1+x) < x; \; x > 0$

→ consider the function $[1, 1+x]$ $\boxed{x > 0}$

$f(x) = \log(1+x)$

$f(0) = \log(1+0)$

$$\frac{f(x) - f(0)}{x - 0} = f'(x)$$

$$\frac{\log(1+x) - \log(1)}{x} = \cancel{\frac{1}{1+x}(1)} \quad \frac{1}{c}$$

$$\frac{\log(1+x)}{x} = \frac{1}{1+x}$$

$x > 0$

$1+x > 0$

$\frac{1}{1+x} < 0$

→ consider interval $[1, 1+x]$

function will $F(t) = \log(t)$

$$\frac{\log(1+x) - \log(1)}{1+x-1} = f'(c)$$

$$\frac{\log(1+x)}{x} = \frac{1}{c}$$

$1 < c < 1+x$

$$\frac{1}{1} > \frac{1}{c} > \frac{1}{1+x}$$

$$\frac{1}{1} > \frac{\log(1+x)}{x} > \frac{1}{1+x}$$

multiply by x

$$x > \log(1+x) > \frac{x}{1+x}$$

$3(-2)^2 - 3$

$3 \cdot 4 \to$

$12 - 3$

$3(0)^2 - 3$

15) $f(x) = x^3 - 3x + 2$ $\quad c = 2$

→ $f'(x) = 3x^2 - 3$

$3 = 3x^2$

$x^2 = 1$

$x = \pm 1$

$+1 \;\; 1$

$(-\infty, -1) \; (-1, 1) \; (1, \infty)$

Interval	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
sign of f'	+	−	+
f is falling or rising	↑	↓	↑

put 1 and −1 in $f(x)$

$f(1) = (1)^3 - 3(1) + 2$

$= 1 - 3 + 2 = -2 + 2 = 0$

$f(-1) = (-1)^3 - 3(-1) + 2$

$= 4$

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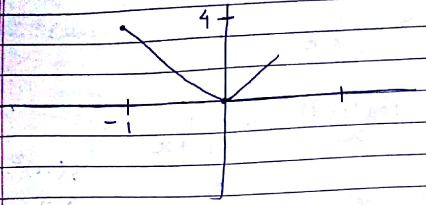
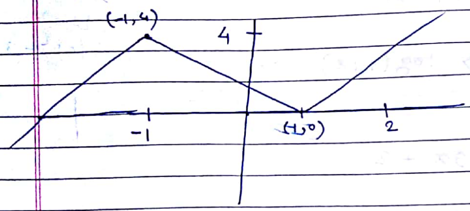




At point 2 graph is ~~falling~~ increasing

16. — ?

a) $f(x) = 4x^3 - x^4$

→ $f'(x) = 12x^2 - 4x^3$

put $f'(x) = 0$

$0 = 12x^2 - 4x^3$

$4x^3 = 12x^2$

$4x = 12$

$x = \frac{12}{4} = 3$

∴ critical point $x = 3, 0$

$f(3) = 4(3)^3 - (3)^4$

$= 27$

$f(0) = 4(0)^3 - (0)^4$

$f(0) = 0$

$(-\infty, 0)\ (0, 3)\ (3, \infty)$

Interval	$(-\infty,0)$	$(0,3)$	$(3,\infty)$
sign of f'	+ve	+ve	−ve
f is falling or rising	↑	↑	↓

$f''(x) = 24x - 12x^2$

put $f''(x) = 0$

$0 = 24x - 12x^2$

$\cancel{0 = (2 - 1x)12x}$

$\cancel{1x = 2}$

$\cancel{x = \frac{1}{2}}$

$0 = (2 - x)12x$

$x = 2, \quad x = 0$

c.p. 2, 0

$(-\infty, 0)\ (0, 2)\ (2, \infty)$

$f(2) = 4(2)^3 - (2)^4 = 32 - 16 = 16$

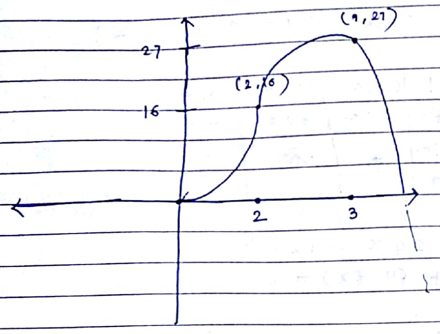
$f(0) = 0$

Interval	$(-\infty,0)$	$(0,2)$	$(2,\infty)$
sign of f''	−	+	−
f is U/D	D	U	D

0, 2, 3

(−∞, 0) (0, 2) (2, 3) (3, ∞)

Interval	(−∞,0)	(0,2)	(2,3)	(3,∞)
R or F	↑	↑	↑	↓
U or D	D	U	D	D



b) $f(x) = -x^3 + 6x^2 - 3$

→ $f'(x) = -3x^2 + 12x$

$f'(x) = 0$

$0 = -3x^2 + 12x$

$0 = (12x - 3x^2)$

$0 = 3x(4 - x)$

$\therefore 3x = 0 \quad 4 = x$

c.p. 0, 4

$f(0) = -(0)^3 + 6(0)^2 - 3$

$f(0) = 0 \quad 0 + 0 - 3 = -3$

$f(4) = -(4)^3 + 6(4)^2 - 3$

$= 29$

$f(4) = \cancel{0}$

(−∞, 0) (0, 4) (4, ∞)

Interval	(−∞,0)	(0,4)	(4,∞)
sign of f'	−	+	−
R or F	↓	↑	↓

$f''(x) = -6x + 12$

$f''(x) = 0$

$0 = -6x + 12$

$0 = 6x - 12$

$0 = 6(x - 2)$

$x = 2$

$f(2) = (-2)^3 + 6(2)^2 - 3$

$= 13$

~~(−∞, 0)~~ (0, 2) (2, ∞)

Interval	(∞,2)	(2,∞)		
sign of f''	+	−		
U / D	U	D		

0, 4, 2

(−∞, 0) (0, 2) (2, 4) (4, ∞)

Interval	(−∞,0)	(0,2)	(2,4)	(4,∞)
R/F	↓	↑	↑	↓
U/D	U	U	D	D

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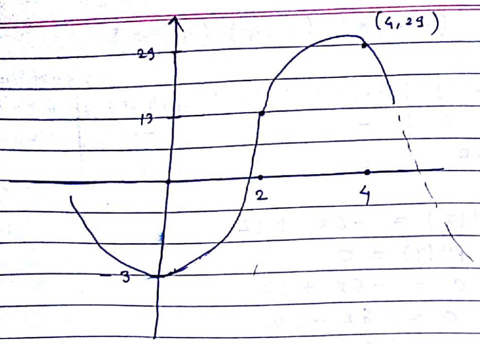

3) $f(x) = x^3 - 3x - 3$

$\rightarrow$ $f'(x) = 3x^2 - 3$

$f'(x) = 0$

$0 = 3x^2 - 3$ $\quad\quad$ $3x^2 = 3$

$\cancel{0} = \cancel{3(x-1)(x+1)}$ $\quad\quad$ $x^2 = \pm 1$

$\cancel{x} = \cancel{\pm 1}$

c.p. $= \pm 1$

$f(1) = (1)^3 - 3(1) - 3 \quad = -5$

$f(-1) = (-1)^3 - (3(-1)) - 3 \quad = -1$

$(-\infty, -1)$ $(-1, 1)$ $(1, \infty)$

Interval	$(-\infty,-1)$	$(-1,1)$	$(1,\infty)$
sign of f'	+	−	+
f falling or rising	↑	↓	↑

classmate
Date ____
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$f''(x) = 6x$

$f''(x) = 0$

$0 = 6x$

$x = 0$

c.p. $= 0$

$f(0) = -3$

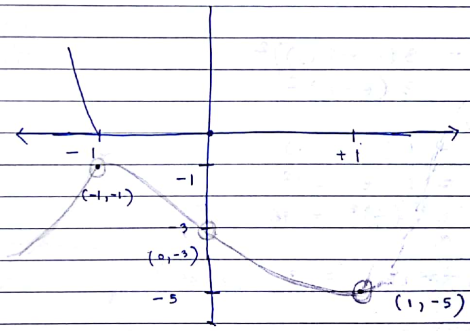
$(-\infty, 0)$ $(0, \infty)$

Interval	$(-\infty, 0)$	$(0, \infty)$
sign f''	−	+
U/D	D	U

$-1, 0, +1$

$(-\infty, -1)$ $(-1, 0)$ $(0, 1)$ $(1, \infty)$

Interval	$(-\infty,-1)$	$(-1,0)$	$(0,1)$	$(1,\infty)$
Falling / Rising	↑	↓	↓	↑
U/D	U	D	U	U



d) $f(x) = x(6-2x)^2$

→ $\cancel{f'(x) = x(36 - 24x + 4x^2)}$

$\cancel{= 36x - 24x^2 + 4x^2}$

$\cancel{f'(x) = 36 - 48x + 18x^2}$

$\cancel{f'(x) = 0}$ $24x$

$\cancel{0 = 36 - 24x + 8x^2}$

$\cancel{12x^2 - 48x + 36} = 0$

$\cancel{12x(x^2 - 2x + 3) = 0}$

$\cancel{x^2 - 2x + 3 = 0}$

$f(x) = x(36 - 24x + 4x^2)$

$= 36x - 24x^2 + 4x^3$

$f'(x) = 36 - 48x + 12x^2$

$f'(x) = 0$

$36 - 48x + 12x^2 = 0$

$12(3 - 4x + x^2) = 0$

$3 - 4x + x^2 = 0$

$x^2 - 4x + 3 = 0$

$x = 3, 1$

$f(3) = 3(6 - 2(3))^2$

$= 3(6-6)^2$

$f(3) = 0$

$f(1) = 1(6 - 2(1))^2$

$= 1(6-2)^2$

$= (4)^2 = 16$

$(-\infty, 1)\ (1, 3)\ (3, \infty)$

Interval	$(-\infty, 1)$	$(1, 3)$	$(3, \infty)$
f' sign	+	−	+
Fall or rise	↑	↓	↑

$f''(x) = 36 - 48x + 12x^2$

$= -48 + 24x$

$f''(x) = 0$

$0 = -48 + 24x$

$48 = 24x$

$x = \frac{48}{24} = 2$

$\therefore x = 2 \quad c.p = 2$

$f(2) = 2(6 - 2\times2)^2$

$= 2(6-4)^2$

$= 2(2)^2 = 8$

$(-\infty, 2)\ (2, \infty)$

Interval	$(-\infty, 2)$	$(2, \infty)$
sign f''	−	+
U / D	D	U

1, 2, 3

$(-\infty, 1)\ (1, 2)\ (2, 3)\ (3, \infty)$

Interval	$(-\infty, 1)$	$(1, 2)$	$(2, 3)$	$(3, \infty)$
Fall or rise	↑	↓	↓	↑
U/D	D	D	U	U

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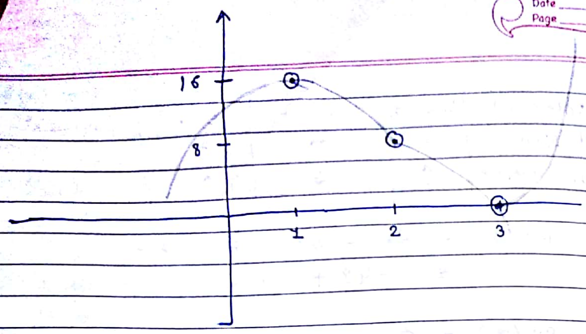


e) $f(x) = -2x^3 + 6x^2 - 3$

$\rightarrow$ $f'(x) = -6x^2 + 12x$

$f'(x) = 0$

$0 = -6x^2 + 12x$

$= 6x(-x+2)$

$6x = 0 \qquad -x+2 = 0$

$x = 0 \qquad x = +2$

c.p $= 0, -2$

$(-\infty, -2)\ (-2, 0)\ (0, \infty)$

Interval	$(-\infty,-2)$	$(-2,0)$	$(0,\infty)$
f' sign)	−	−	+
Fall/rise	↓	↓	↑

$f''(x) = -12x + 12$

$f''(x) = 0$

$0 = -12x + 12$

$12 = 12x$

$x = 1$

$(-\infty, 1)\ (1, \infty)$

$f(-2) = 3.5$

$f(0) = -3$

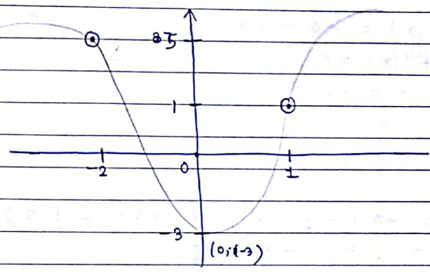
$f(1) = 1$

Interval	$(-\infty, 1)$	$(1,\infty)$
f'' sign	+	−
U/D	U	D

c.p. $-2, 0, 1$

$(-\infty, -2)\ (-2, 0)\ (0, 1)\ (1, \infty)$

Interval	$(-\infty,-2)$	$(-2,0)$	$(0,1)$	$(1,\infty)$
Fall/rise	↓	↓	↑	↑
U/D	U	U	U	D



f) $f(x) = 1 - 9x - 6x^2 - x^3$

$\rightarrow$ $f'(x) = -9 - 12x - 3x^2$

$f'(x) = 0$

$0 = -9 - 12x - 3x^2$

$0 = (-3 - 4x - x^2)12$

$= -3 - 4x - x^2$

$= x^2 + 4x + 3$

$x^2 + 3x + x + 3 \qquad x = -3$

$x(x+3)\ 1(x+3) \qquad x = -1$

$f(-3) = 1 - 9(-3) - 6(-3)^2 - (-3)^3$
$= 1$

$f(-1) = 1 - 9(-1) - 6(-1)^2 - (-1)^3$
$= 5$

$(-\infty, -3)$ $(-3, -1)$ $(-1, \infty)$

Interval	$(-\infty,-3)$	$(-3,-1)$	$(-1,\infty)$
f' sign	$-$	$+$	$-$
Fall/rise	$\downarrow$	$\uparrow$	$\downarrow$

$f''(x) = -4 - 2x$
$f''(x) = 0$
$0 = -4 - 2x$
$-2x = 4$
$x = -2$

$f(-2) = \cancel{-\infty}\ 1 - 9(-2) - 6(-2)^2 - (-2)^3$
$= 1 + 18 - 24 + 8$
$= 3$

Tutorial 4

1. let $T : V \longrightarrow W$

$\rightarrow$ show that

a) $T(0) = 0$

$\rightarrow$ $T(0) = T(0 + 0)$ we can write $(0) = (0 + 0)$
$= T(0) + T(0)$
$T(0) - T(0) = T(0)$
$\therefore 0 = T(0)$

b) $T(-v) = -T(v)$ for all $v \in V$

$\rightarrow$ i.e. $T(\alpha \cdot v) = T\,\alpha(v)$

assume $\alpha = -1$
$\therefore T(-1 \cdot v) = -1\,T(v)$
$T(-v) = -T(v)$

2. a) $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$
$F(x, y, z) = (x, z)$

$\rightarrow$ $F\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ z \end{pmatrix}$

$K(F) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \begin{matrix} x = 0 \\ z = 0 \end{matrix} \right\}$

$\therefore$ ~~Basis~~ Span $= \left\{ y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \middle| \; y \in \mathbb{R} \right\}$

$\therefore$ basis $= \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

$\therefore$ Dimension $= 1$

$\therefore$ Nulity $= 1$

$\mathrm{Im}(F) = \left\{ F\begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \right\}$

$= \left\{ \begin{pmatrix} x \\ z \end{pmatrix} \middle| \; x, z \in \mathbb{R} \right\}$

span (B) $= \left\{ x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 1 \end{pmatrix} \middle| \; x, z \in \mathbb{R} \right\}$

Basis $= \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

Dimen $(I) = 2$

$\therefore$ Image $(F) = 2$ $\quad \therefore$ $R(T) = 2$

By R.T. theorem

Dimension $= \cancel{R(T)}\; N(T) + R(T)$

$3 = 1 + 0$

$R(F) = 2.$

$\therefore$ $R(F) =$ dimension of co-domain

$I_{mage}(F)\; R(F) = W$

$\therefore$ Image $(F) = W$

b) $F : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$

$F \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \longrightarrow \begin{pmatrix} -x \\ -y \\ -z \\ -w \end{pmatrix}$

$\rightarrow$ $K(F) = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \middle| \; F\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

$= \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \middle| \begin{pmatrix} -x \\ -y \\ -z \\ -w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

$= \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -x = 0 \\ -y = 0 \\ -z = 0 \\ w = 0 \end{pmatrix} \right\}$

Basis $= \{ \phi \}$ $\quad$ Span $\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

$\therefore$ Nulity $= 0$

$I(F) = \left\{ F\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \middle| \; x, y, z, w \in \mathbb{R} \right\}$

$I(F) = \left\{ \begin{pmatrix} -x \\ -y \\ -z \\ -w \end{pmatrix} \middle| \; -x, -y, -z, -w \in \mathbb{R} \right\}$

Span $= \left\{ -x\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + -y\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + -z\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + -w\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

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basis $= \left\{ \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right\}$

$\therefore$ Dimension $= 4$

$I(F) = 4$

by Rank nulity theorem

$D = N(T) + R(T)$

$4 = 0 + R(T)$

$R(F) = 4$

$\therefore I(F) = 4.$

3) $F : \mathbb{R}^3 \to \mathbb{R}^3$

$F\begin{pmatrix}x\\y\\z\end{pmatrix} \to \begin{pmatrix}x\\y\\z\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix}$

$F\begin{pmatrix}x\\y\\z\end{pmatrix} \to \begin{pmatrix}x\\y+1\\z\end{pmatrix}$

$\to F(\bar{x} + \bar{y}) = F(\bar{x}) + F(\bar{y})$

$(x) = \begin{pmatrix}1\\2\\3\end{pmatrix}$ $\quad (y) = \begin{pmatrix}1\\1\\1\end{pmatrix}$

$F(\bar{x}) = \begin{pmatrix}1\\3\\3\end{pmatrix}$ $\quad F(y) = \begin{pmatrix}1\\2\\1\end{pmatrix}$

$F(x) + F(y) = \begin{pmatrix}1+1\\3+2\\3+1\end{pmatrix} = \begin{pmatrix}2\\5\\4\end{pmatrix}$

$(x+y) = \begin{pmatrix}2\\3\\4\end{pmatrix}$

$F(x+y) = \begin{pmatrix}2\\4\\4\end{pmatrix}$

$\therefore F(\bar{x}) + F(\bar{y}) \neq F(x+y)$

$\therefore$ It is not a linear map.

4) $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$F\begin{pmatrix}x\\y\end{pmatrix} \longrightarrow \begin{pmatrix}x-y\\2y\end{pmatrix}$

$\to K(F) = \left\{ \begin{pmatrix}x\\y\end{pmatrix} \middle| F\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \right\}$

$= \left\{ \begin{pmatrix}x\\y\end{pmatrix} \middle| \begin{pmatrix}x-y\\2y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \right\}$

$= \left\{ \begin{pmatrix}x\\y\end{pmatrix} \middle| \begin{matrix}x-y=0\\2y=0\end{matrix} \right.$

Span $= \left\{ \begin{pmatrix}0\\0\end{pmatrix} \right\}$

$\therefore$ Basis $= \{\phi\}$ $\quad \therefore$ Nulity $= 0$.

$N(F) = 0$.

$I(F) = \left\{ F\begin{pmatrix}x\\y\end{pmatrix} \middle| \begin{pmatrix}x\\y\end{pmatrix} \in \mathbb{R}^2 \right\}$

$= \left\{ \begin{pmatrix} x-y \\ 2y \end{pmatrix} \middle| \; x, y \in \mathbb{R} \right\}$

$= \left\{ x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} -1 \\ 2 \end{pmatrix} \middle| \; x, y \in \mathbb{R} \right\}$

$\therefore I(F) = 2 \quad \therefore R(F) = 2$

By Rank Nullity theorem.

$D = N(T) + R(T)$

$2 = 0 + R(T)$

$R(T) = 2$

$\therefore I(F) = W$. whole of co-domain.

e) $F : \mathbb{R}^2 \to \mathbb{R}^2$.

$F\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} xy \\ x+y \end{pmatrix}$

$\longrightarrow$ $x = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \quad y = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$

$F(x) = \begin{pmatrix} a_1b_1 \\ a_1+b_1 \end{pmatrix} \quad F(y) = \begin{pmatrix} a_2b_2 \\ a_2+b_2 \end{pmatrix}$

$x + y = \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \end{pmatrix}$

$F(x+y) = \begin{pmatrix} (a_1+a_2)(b_1+b_2) \\ a_1+a_2+b_1+b_2 \end{pmatrix}$

$F(x) + F(y) = \begin{pmatrix} a_1b_1 + a_2b_2 \\ a_1+b_1+a_2+b_2 \end{pmatrix}$

$\therefore F(x+y) \neq F(x) + F(y)$.

$\therefore$ It is not a linear map.

f) $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$\longrightarrow$ $F\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} y \\ x \end{pmatrix}$

(1) $F(x+y) = F(x) + F(y)$

$x = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \quad y = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$

$x + y = \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \end{pmatrix} \quad F(x+y) = \begin{pmatrix} b_1+b_2 \\ a_1+a_2 \end{pmatrix}$

$F(x) = \begin{pmatrix} b_1 \\ a_1 \end{pmatrix} \quad F(y) = \begin{pmatrix} b_2 \\ a_2 \end{pmatrix}$

$F(x) + F(y) = \begin{pmatrix} b_1 + b_2 \\ a_1 + a_2 \end{pmatrix}$

$\therefore F(x+y) = F(x) + F(y)$

(2) $F(\alpha \cdot u) = \alpha F(u)$

$u = \begin{pmatrix} a \\ b \end{pmatrix} \quad \alpha \cdot u = \begin{pmatrix} \alpha a \\ \alpha b \end{pmatrix}$

$F(\alpha \cdot u) = \begin{pmatrix} \alpha b \\ \alpha a \end{pmatrix} \quad \alpha F(u) = \alpha \begin{pmatrix} b \\ a \end{pmatrix}$

$= \begin{pmatrix} \alpha b \\ \alpha a \end{pmatrix}$

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$\ker(F) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;\middle|\; F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

$= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;\middle|\; \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

$= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;\middle|\; \begin{matrix} y = 0 \\ x = 0 \end{matrix} \right\}$

Span $= \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$ $\therefore$ Basis $= \{\phi\}$

~~Dimension~~ Nulity $(F) = 0$

$I(F) = \left\{ F\begin{pmatrix} x \\ y \end{pmatrix} \;\middle|\; \begin{pmatrix} x \\ y \end{pmatrix} \in R^2 \right\}$

$= \left\{ \begin{pmatrix} y \\ x \end{pmatrix} \;\middle|\; y, x \in R \right\}$

Span $= \left\{ y\begin{pmatrix} 1 \\ 0 \end{pmatrix} + x\begin{pmatrix} 0 \\ 1 \end{pmatrix} \;\middle|\; x \text{ and } y \in R \right\}$

Basis $= \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

~~Nulity (F) = 2~~

Dimen $(I(F)) = 2$

$I(F) = 2$

By Rank - N.

$D = N(T) + R(T)$

$2 = 0 + R(T)$

$R(T) = 2$

$\therefore I(F) = R^2$ whole of co-domain

g) $F : R^3 \to R$

$F\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow xy$

$\rightarrow$ (i) $x = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}$ $y = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$ ~~$z = \begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix}$~~

$F(x) = a_1 b_1$ $F(y) = a_2 b_2$ ~~$F(z) = a_3 b_3$~~

$x + y + \cancel{z} = \begin{pmatrix} a_1 + a_2 \cancel{+ a_3} \\ b_1 + b_2 \cancel{+ b_3} \\ c_1 + c_2 \cancel{+ c_3} \end{pmatrix}$

$F(x + y + \cancel{z}) = (a_1 + a_2 \cancel{+ a_3})(b_1 + b_2 \cancel{+ b_3})$

$F(x) + \cancel{a_1 b_1}\, F(y) \cancel{+ F(z)} = a_1 b_1 + a_2 b_2$

$a_1(b_1 + b_2)$

$a_1 b_1 + a_1 b_2$

$\therefore F(x + y) \neq F(x) + F(y)$

$\therefore$ It is not linear map.

h) $F : R^2 \to R^2$

$F\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ y+1 \end{pmatrix}$

$\rightarrow$ $x = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ $y = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$

$F(x) = \begin{pmatrix} a_1 \\ b_1+1 \end{pmatrix}$ $\quad F(y) = \begin{pmatrix} a_2 \\ b_2+1 \end{pmatrix}$

$x+y = \begin{pmatrix} a_1+a_2 \\ b_1+b_2 \end{pmatrix}$

$F(x+y) = \begin{pmatrix} a_1+a_2 \\ b_1+b_2+1 \end{pmatrix}$

$F(x)+F(y) = \begin{pmatrix} a_1+a_2 \\ b_1+1+b_2+1 \end{pmatrix}$

$\therefore\ f(x+y) \neq F(x)+F(y)$

i) $F: \mathbb{R}^3 \to \mathbb{R}$

$F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3x - 2y + z$

$x = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ $\quad y = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$

$F(x) = 3x_1 - 2y_1 + z_1$

$f(y) = 3x_2 - 2y_2 + z_2$

$x+y = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$

$F(x+y) = 3(x_1+x_2) - 2(y_1+y_2) + z_1 + z_2$

$F(x)+F(y) = 3x_1 - 2y_1 + z_1 + (3x_2 - 2y_2 + z_2)$

$= 3(x_1+x_2) - 2(y_1+y_2) + z_1 + z_2$

Let $a = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\quad \alpha \cdot a = \begin{pmatrix} \alpha x \\ \alpha y \\ \alpha z \end{pmatrix}$

$\alpha \cdot F(\alpha a) = \alpha \cdot F(a)$

$F(\alpha a) = \alpha 3x_1 - \alpha 2y_1 + \alpha z_1$

$\alpha \cdot F(a) = \alpha(3x_1 - 2y_1 + z_1)$

$= \alpha 3x - \alpha 2y + \alpha z$

$K(F) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle/ F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| 3x - 2y + z = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

= { Here we have 3 unknown factors. Hence, we get infinitely many solutions.

$\therefore$ put $y = t$ and $z = t$.

$\therefore\ 3x - 2t + t = 0$

$3x = 2t + t$

$x = \frac{2t+t}{3}$

$\therefore\ K = \left\{ \begin{matrix} \frac{2t+t}{3} \\ t \\ t \end{matrix} \right\}$

BASIS Span. $= \left\{ t\begin{pmatrix} 2/3 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1/3 \\ 0 \\ 1 \end{pmatrix} \middle/ t, t \in \mathbb{R} \right\}$

Basis $\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\therefore$ Nulity $= 2$

By Rank Nulity

$D = N(F) - R(F)$

$3 = 2 - R(F)$

$R(F) = 1$

2) $P : M \longrightarrow M$ $\quad$ A is $M_{2\times2}$ Matrics.

$\cancel{P(A)} \longrightarrow \cancel{I(A)}$

$\longrightarrow$ $\cancel{Ker(P)} =$

$P(A) = \dfrac{A + {}^tA}{2}$

$= P$

$Ker(P) = \left\{ A \mid P(A) = 0 \right\}$

$= \left\{ A \;\middle|\; \dfrac{A + {}^tA}{2} = 0 \right\}$

$= \left\{ A \mid A = -{}^tA \right\}$

$A = -{}^tA$ is skew symmetric matrix.

$\therefore$ Dimension $= 1$

Nulity $= 1$

$N(P) = 1$

$P : A \longrightarrow \dfrac{A + {}^tA}{2}$

by rank nulity theorem.

$D = N(P) + R(P)$

$4 = 1 + R(P)$

$\therefore R(P) = 3$

Let the following is the basis of matrix.

$B = \left\{ v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \; v_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \; v_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \; v_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \dfrac{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}{2} = \dfrac{\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}}{2} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$= 1 \cdot v_1 + 0 v_2 + 0 v_3 + 0 v_4$

$v_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \dfrac{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}{2} = \dfrac{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}{2} = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$

$= 0 \cdot v_1 + 1/2\, v_2 + 1/2\, v_3 + 0 \cdot v_4$

$v_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \dfrac{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}{2} = \dfrac{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}{2} = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$

$= 0 \cdot v_1 + 1/2\, v_2 + 1/2\, v_3 + 0 \cdot v_4$

$v_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \dfrac{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}{2} = \dfrac{\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}}{2} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$= 0 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3 + 1 \cdot v_4$

$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

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10) $P : M \longrightarrow M$

$P(A) = \dfrac{A - {}^tA}{2}$

$\ker(P) = \{ A \mid P(A) = 0 \}$

$= \left\{ A \;\middle|\; \dfrac{A - {}^tA}{2} = 0 \right\}$

$= \{ A \mid A = {}^tA \}$

$\therefore$ $A = {}^tA$ is the symmetric matrix

dimension of symmetric matrix = 3

Nullity.

$\therefore N(P) = 3$

$\therefore$ By Rank nullity theorem.

Dimension of domain $M_{2\times2} = 4$

$D = N(P) + R(P)$

$4 = 3 + R(P)$

$\therefore R(P) = 1$

$\therefore$ Rank $= 1$

$\therefore$

$\Rightarrow$ Generalize

$B = \left\{ v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \; v_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \; v_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \; v_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \dfrac{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}{2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$= 0v_1 + 0v_2 + 0v_3 + 0v_4$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \dfrac{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}{2} = \begin{bmatrix} 0 & 1/2 \\ -1/2 & 0 \end{bmatrix}$

$= 0\cdot v_1 + \frac{1}{2} v_2 + \left(-\frac{1}{2}\right) v_3 + 0\cdot v_4$

$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \dfrac{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}{2} = \begin{bmatrix} 0 & -1/2 \\ 1/2 & 0 \end{bmatrix}$

$= 0\cdot v_1 + (-1/2) v_2 + 1/2 v_3 + 0 v_4$

$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \dfrac{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}{2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$= 0\cdot v_1 + 0\cdot v_2 + 0\cdot v_3 + 0\cdot v_4$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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m) $P : M \longrightarrow M$

$P(A) \longrightarrow trace(A)$

$\longrightarrow$ $Ker(P) = \{ A \mid trace(A) = 0 \}$

$\therefore$ dimension of trace is 3.

$\therefore$ Nullity $= 3$

$N(P) = 3$

$I(P) = \{ P(A) \mid A \in M_{2\times 2} \}$

$= \{ trace(A) \mid A \in M_{2\times 2} \}$

~~Basis~~ $= \cancel{\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}}$

$\therefore$ ~~Dimension~~ (2/3

$I(P) = 3$

$\therefore$ By Rank nulity theorem.

Dimension of domain $= N(T) + R(T)$

$4 = 3 + R(T)$

$R(T) = 1$

$\Rightarrow$ Generalize

$P(A) = trace(A)$

$P(A) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 1 \cdot 1$ $P(C) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = 0 \cdot 1$

$P(B) = \begin{bmatrix} 0 & \phi \\ 0 & 0 \end{bmatrix} = 0 \cdot 1$ $P(D) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 1 \cdot 1$

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$\therefore$ Matrix $= [1 \ 0 \ 0 \ 1]$

Q. 4 —— ?

a) $2x + y - z = 0$

$2x + y + z = 0$

$$\begin{array}{c} R_1 \\ R_2 \\ \\ \end{array} \begin{bmatrix} 2 & 1 & -1 \\ 2 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\longrightarrow$ Perform Row operations

$R_2 = R_1 - R_2$

$= 2 - 2$

$= 0$

$R_2 = 1 - 1 = 0$

$R_2 = 1 - (-1) = 0$

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x + y - z = 0$

$z = 0$

$2x + y = 0$

$y = -2x$

$$\begin{bmatrix} x \\ -2x \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

Solution set

$$= \left\{ x\begin{pmatrix}1\\-2\\0\end{pmatrix} \;\middle|\; x \in \mathbb{R} \right\}$$

Basis $= \left\{ \begin{pmatrix}1\\2\\0\end{pmatrix} \right\}$

$\therefore$ Dimension $= 1$

2) $x + y + z = 0$

$x - y = 0$

$\cancel{x} \; y + z = 0$

$$\longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 = R_2 - R_3$

$= 1 - 0$

$= 1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$R_3 = 1 - 1 = 0$

$R_2 = R_1 + R_2$

$= 1 + 1 = 2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x + y + z = 0$

$2x + z = 0$

$x = 0$

$z = 0$

$y = 0$

Solution set $= \left\{ \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\}$

$\therefore$ Basis $= \{ \phi \}$

3) $2x - 3y + z = 0$

$x + y - z = 0$

$3x + 4y = 0$

$5x + y + z = 0$

$$\longrightarrow \begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & -1 \\ 3 & 4 & 0 \\ 5 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 = R_2 - R_1$

$= 1 - 1 = 0$

$= 1 - 1 = 0$

$$\begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & -1 \\ 3 & 4 & 0 \\ 0 & 4 & -6 \end{bmatrix}$$

$5 - 0$

$5 - 1 = 4$ $\quad R_4 = 5R_2 - R_4$

$-5 - 1 = -6$ $\quad = 5 - 5$

0 $\quad 5 - 1 = 4$

-5

$$\begin{bmatrix} 2 & -3 & 1 \\ 0 & 5 & -6 \\ 3 & 4 & 0 \\ 0 & 4 & -6 \end{bmatrix}$$

$R_2 = 2R_2 - R_1$

$= 2\times1 - 2$

$= 0$ $\quad R_4 = R_2 - R_4$

$= 2\times1 - (-3)$ $\quad = 5 - 4$

$= 2 + 3$ $\quad = 1$

$-5 - 1$ $\quad -6 + 6$

$= 0$

$$\begin{bmatrix} 2 & -3 & 1 \\ 0 & 5 & -6 \\ 3 & 4 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$R_4 = 5R_4 - R_2$

$= 5 \times 0 -$

$2x - 3y + z = 0$

$+5y - 6z = 0$

$3x + 4y = 0$

$y = 0$

$3x = 0$

$x = 0$

$2 \times 0 - 3(0) + z = 0$

$\underline{z = 0}$

$\therefore x = 0$

$y = 0$

$z = 0$

Solution set $= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \middle| \; x, y, z \in \mathbb{R} \right\}$

Basis $= \{ \phi \}$

d) $4x + 7y - \pi z = 0$

$2x - y + z = 0$

$\rightarrow \begin{bmatrix} 4 & 7 & -\pi \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_2 = R_1 - 2(\cancel{\pi}$ $-\pi - 2$

$R_2 = R_1 - 2R_2$

$= 4 - 2 \times 2$

$R_2 = 0$

$R_2 = 7 - 2(-1) = 7 + 2 = 9$

$\pi - 2(1)$

$R_2 (-\pi - 2)$

$\begin{bmatrix} 4 & 7 & -\pi \\ 0 & 9 & -\pi - 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$4x + 7y + \pi z = 0$

$9y + (\pi - 2)z = 0$

$z(-\pi - 2) = -9y$ put $(\pi - 2) = p$

$z = \dfrac{-9y}{(\pi - 2)}$

$z = \dfrac{-9y}{p}$

$4x + 7y + \pi\left(\dfrac{-9y}{p}\right) = 0$

$4x + 7y - \dfrac{9y\pi}{p} = 0$

$4x + 7y - 9y\cancel{N}$

$4x + \dfrac{7yp - 9yp\pi}{p} = 0$

$4x = \dfrac{-7yp + 9yp\pi}{4p} = \dfrac{\cancel{p}(-7y + 9y\pi)}{4\cancel{p}}$

$x = \cancel{\dfrac{-7y(\pi - 2) + 9y(\pi - 2)\pi}{4(\pi - 2)}}$

$\cancel{\dfrac{-7yp + 9yp\pi}{4p}}$ $x = \dfrac{-7y + 9y\pi}{4}$

$\begin{pmatrix} \dfrac{-7y + 9y\pi}{4} \\ y \\ \dfrac{-9y}{p} \end{pmatrix} = \dfrac{-7y}{4} + \dfrac{9y\pi}{4}$

$y \begin{pmatrix} \dfrac{-7 + 9\pi}{4} \\ 1 \\ \dfrac{-9}{\pi - 2} \end{pmatrix}$

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$4x + 7y - \pi z = 0$

$9y + (-\pi - 2)z = 0$

$9y = -(-\pi - 2)z$

$9y = (\pi + 2)z$

put $(\pi + 2) = K$

$9y = Kz$

$y = \frac{Kz}{9}$

$4x + 7\left(\frac{Kz}{9}\right) - \pi z = 0$

$4x + \frac{7Kz}{9} - \pi z = 0$

$4x \mp \frac{7Kz}{9} = \pi z$

$4x = \frac{\cancel{+}7Kz}{9} + \pi z$

$4x = 2\pi \quad \cancel{+} - \frac{7Kz}{9}$

$x = \frac{9\pi z - 7Kz}{9 \times 4}$

$x = \frac{9\pi z - 7Kz}{36}$

$= \frac{9\pi z - 7(\pi - 2)z}{36}$

$= \frac{9\pi z - 7\pi z + 14z}{36}$

$= \frac{2\pi z + 14z}{36}$

$= \frac{(2\pi + 14)z}{36}$

$$\left\{ \begin{pmatrix} \frac{(2\pi+14)z}{3} \\ \frac{(\pi+2)z}{9} \\ z \end{pmatrix} \Big/ z \in \mathbb{R} \right\}$$

$$B = \left\{ z \begin{pmatrix} \frac{2\pi+14}{36} \\ \frac{\pi+2}{9} \\ 1 \end{pmatrix} \right\}$$

5. $\boxed{T : \mathbb{R}^n \longrightarrow \mathbb{R}^m}$

$T(X) = AX$

$\cancel{\text{x}}$ i

$\longrightarrow$

$$\begin{bmatrix} \\ \\ \end{bmatrix}_{m \times 1} = \begin{bmatrix} & & \\ & & \end{bmatrix}_{m \times n} \begin{bmatrix} \\ \\ \end{bmatrix}_{m \times 1}$$

$\bar{x} = \cancel{AX}$

To prove T is a linear map.

Let x and $y \in \mathbb{R}$

1) $T(x + y) = A(x + y)$

$= A(x) + A(y)$

$\therefore \cancel{T}(x+y) = T(x) + T(y)$

2) $T(\alpha \cdot x) = \alpha T(x)$

Let α be any $\mathbb{R}$ no.

$X \in \mathbb{R}^n$

$\therefore T(\alpha \cdot x) = A(\cancel{\alpha \cdot x})$

$= \alpha AX$

$= \alpha T(X)$

$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

6. $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 7 \end{bmatrix}$

$\longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\ker(T) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

$R_2 = 3R_1 - R_2$ $\qquad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$3 \times 1 - 3$

$= 0$

$= 10 - 5$

$= 3 \times 2 - 5$

$= 6 - 5$

$9 - 7$ $\qquad \begin{pmatrix} 4 \\ y \\ z \end{pmatrix}$

$x + 2y + 3z = 0$

$y + 2z = 0$

$y = -2z$

$x + 2(-2z) + 3z$

$x - 4z + 3z = 0$

$x - z = 0$

$x = z$

$\begin{pmatrix} z \\ -2z \\ z \end{pmatrix} = z \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

Solution set $= \left\{ z \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \;\middle|\; z \in \mathbb{R} \right\}$

Basis $= \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$

Nullity $= 1$

R. T.

Dimension of domain $= N(T) + R(T)$

$3 = 1 + R(T)$

$R(T) = 2$

8) — ?

$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} 0 \\ 1 \end{pmatrix} + y\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$T\begin{pmatrix} x \\ y \end{pmatrix} = T\left(x\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) + T\left(y\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$

$= x\left(T\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) + y\left(T\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$

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$$T\begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} y$$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ 3x + y \end{pmatrix}$$

$$= \begin{pmatrix} 2(3) + 7 \\ 3(3) + 7 \end{pmatrix}$$

$$= \begin{matrix} 6 + 7 \\ 9 + 7 \end{matrix} = \begin{pmatrix} 13 \\ 16 \end{pmatrix}$$

⇒ Tutorial - 5

1. — ?

a) $F : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$

$F(x_1, x_2, x_3, x_4) = (x_1, x_2)$

→ Basis of $\mathbb{R}^2$

$$\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

$$F\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad F\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$F\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad F\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\text{Matrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

b) $F(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3)$

$$F\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad F\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad F\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$F\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{Matrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

c) $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$F(x, y) = (3x, 3y)$

(2) (1)

$$F\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \qquad F\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

d) $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n \qquad F(X) = 7X$

$$\begin{vmatrix}1 & -1\\ 2 & 4\end{vmatrix} - \begin{vmatrix}7 & 0\\ 0 & 7\end{vmatrix} = 0$$

$$\begin{vmatrix}-6 & -1\\ 2 & -3\end{vmatrix} = 0$$

$$|M - \lambda I| = 0$$

$$\begin{bmatrix}-6 & -1\\ 2 & -3\end{bmatrix} - \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix} = 0$$

$$\begin{bmatrix}-6-\lambda & -1-0\\ 2 & -3-\lambda\end{bmatrix} = 0$$

$$(-6-\lambda)(-3-\lambda) - (-1\times 2) = 0$$

$$18 + 6\lambda + 3\lambda + \lambda^2 + 2 = 0$$

$$\lambda^2 + 9\lambda + 20 = 0$$

$$\lambda^2 + 5\lambda + 4\lambda + 20 = 0$$

$$\lambda(\lambda+5) + 4(\lambda+5) = 0$$

$$(\lambda+5)(\lambda+4) =$$

$$\lambda = -5 \quad \lambda = -4$$

$$(M - \lambda I)\bar{u} = 0$$

$$\begin{bmatrix}-6 & -1\\ 2 & -3\end{bmatrix} - \begin{bmatrix}-5 & 0\\ 0 & -5\end{bmatrix}\bar{u} = 0$$

$$\begin{bmatrix}-1 & -1\\ 2 & 2\end{bmatrix}\bar{u} = 0.$$

$R_2 = 2R_1 + R_2$

$2 \ -2 + 2$

$$\begin{bmatrix}-1 & -1\\ 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = 0$$

$$-v_1 - v_2 = 0 \qquad \begin{bmatrix}-v_2\\ v_2\end{bmatrix} = v_2\begin{bmatrix}-1\\ 1\end{bmatrix}$$

$$v_1 = -v_2$$

$$\bar{V} = \begin{pmatrix}-1\\ 1\end{pmatrix} v_2 \quad \forall\ v_2 \in \mathbb{R}$$

$\lambda = -4$

$$\begin{bmatrix}-6 & -1\\ 2 & -3\end{bmatrix} - \begin{bmatrix}-4 & 0\\ 0 & -4\end{bmatrix}\bar{u} = 0$$

$$= \begin{bmatrix}-2 & -1\\ 2 & 1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = 0$$

$R_2 = R_1 + R_2$

$$= \begin{bmatrix}-2 & -1\\ 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = 0$$

$-2v_1 = v_2$

$v_1 = -\frac{v_2}{2}$

$$-2v_1 - v_2 = 0 \qquad \begin{bmatrix}-\frac{v_2}{2}\\ v_2\end{bmatrix} = v_2\begin{bmatrix}-\frac{1}{2}\\ 1\end{bmatrix}$$

$$-v_2 = 2v_1$$

$$v_1 = -\frac{v_2}{2}$$

$$\bar{u} = \begin{pmatrix}-\frac{1}{2}\\ 1\end{pmatrix} v_2 \quad \forall\ v_2 \in \mathbb{R}$$

$*$ $A = \begin{bmatrix}1 & -1\\ 2 & 4\end{bmatrix}$

$$|A - \lambda I| = 0$$

$$\begin{bmatrix}1 & -1\\ 2 & 4\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = 0$$

$$\begin{vmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

$(1-\lambda)(4-\lambda) - (-1)\times 2 = 0$

$4 - \lambda - 4\lambda + \lambda^2 + 2 = 0$

$6 - 5\lambda + \lambda^2 = 0$

$\lambda^2 - 5\lambda + 6 = 0$

$\lambda^2 - 3\lambda - 2\lambda + 6 = 0$

$\lambda(\lambda - 3) - 2(\lambda - 3)$

$(\lambda - 2)(\lambda - 3)$

eigenvalues = $\lambda = 2, 3$.

$(A - \lambda I)\bar{u} = 0$

For $\lambda = 2$ $\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\bar{u} = 0$

$= \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}\bar{u} = 0$

$= \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ v_2 \end{bmatrix} = 0$

$R_2 = 2R_1 + R_2$
$= -2 + 2$

$\begin{bmatrix} -1v_1 & -1v_2 \\ 0 & 0 \end{bmatrix} = 0$

$-v_1 = v_2$

$\begin{bmatrix} v_1 \\ -v_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} v_1$ or $\begin{bmatrix} -v_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} v_2$

$= \bar{u} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} v_2 \quad \forall v_2 \in \mathbb{R}$

For $\lambda = 3$

$\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = 0$

$\begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

$\begin{bmatrix} -2 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

$-2v_1 - v_2 = 0$

$-v_2 = 2v_1$

$-\frac{v_2}{2} = v_1$

$\begin{bmatrix} -\frac{v_2}{2} \\ v_2 \end{bmatrix} \quad \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} v_2$

$\bar{v} = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix} v_2 \quad \forall v_2 \in \mathbb{R}$

6. —— ?

$|{}^tA - \lambda I| = |{}^tA - \lambda I|$

L.H.S =

$= |A - \lambda I|$

$= (|A - \lambda I|)^t$

$= |{}^t(A) - {}^t(\lambda I)|$

$= |{}^tA - \lambda I|$

$\therefore$ LHS = RHS.

transpose of invertible matrix is invertible

5. — ?

~~suppose~~ clamim $\lambda_1 + \lambda_2 = trace(A)$

$\lambda_1 \cdot \lambda_2 = |A|$

suppose $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$

It is upper triangular matrix

$\therefore \lambda_1 = a, ~~\&~~ \lambda_2 = c$

$\therefore Trace(A) = a + c$

$\lambda_1 + \lambda_2 = a + c$ ✓

$\lambda_1 \cdot \lambda_2 = ac$

$|A| = ac - 0 \cdot b$

$|A| = ac$ ✓

~~7~~ 6. — ?

A is similar to B

$A \sim B$

$\therefore B = PAP^{-}$

claim : $|B - \lambda I| = |A - \lambda I|$

L.H.S $= |B - \lambda I|$

$= |PAP^{-1} - \lambda I|$ ~~put~~ $I = PP^{-1}$

$= |PAP^{-1} - \lambda PP^{-1}|$

$= |P(AP^{-1} - \lambda P^{-})|$

$= |P||(AP^{-1} - \lambda I P^{-})|$

$|P| (A - \lambda I)|P^{-1}| = 0$

$|P| (A - \lambda I) \frac{1}{|P|} = 0$

$A - \lambda I = 0$

$\therefore$ LHS = RHS.

suppose $\bar{v}$ is the eigenvector of B

$B\bar{v} = \lambda \bar{v}$

$(PAP^{-1})\bar{v} = \lambda \bar{v}$

multiply by p^{-1}

$\frac{1}{P} (P A P^{-1})\bar{v} = P^{-1}\lambda \bar{v}$

$A(P^{-1}\bar{v}) = \lambda(P^{-1}\bar{v})$

eigen vector of $A = (P^{-1}\bar{v})$

$\therefore$ eigenvectors are different.

$A\bar{v} = \lambda \bar{v}$

$B = PAP^{-}$

$P^{-}BP = A$

$(P^{-}BP)\bar{v} = \lambda \bar{v}$

multiply by P

$P P^{-1} B P \bar{v} = \lambda \bar{v} P$

$BP\bar{v} = \lambda \bar{v} P$

$B(P\bar{v}) = \lambda(\bar{v}P)$

eigenvectors B $= \bar{v}P$

8. — ?

$\rightarrow$

8. — ?

M has λ eigenvalue

$M\bar{v} = \lambda\bar{v}$ — defination.

mutiply by both side by M^{-1}

$M^{-1} \; M\bar{v} = \lambda\bar{v} M^{-1}$

$\frac{1}{\cancel{M}} \cancel{M}\bar{v} = \lambda\bar{v} M^{-1}$

$\bar{v} = \cancel{\lambda}/\cancel{\lambda} \cdot M^{-1}\lambda\bar{v}$

$\frac{\bar{v}}{\lambda} = M^{-1}\bar{v}$

$\lambda^{-1}\bar{v} = M^{-1}\bar{v}$

$\therefore$ eigenvalue of $M^{-1} = \lambda^{-1}$

M has $\bar{v}$ as eigen vectore
M^{-1} has $\bar{v}$ as eig envectore
$\therefore$ Both has same eigenvectore.

9. — ?

M has λ eigenvalue

$M\bar{v} = \lambda\bar{v}$ — def.

multiply by k $(\exists k \in \mathbb{R})$

$kM\bar{v} = k\lambda\bar{v}$

$\therefore$ kM has $k\lambda$ eigenvalue,
but $\bar{v}$ is same.

10. — ?

suppose $\begin{bmatrix} a_1 & b_2 \\ c & d \end{bmatrix}$

according to property

$\lambda_1 + \lambda_2 = 5$

$\lambda_1 = 5 - \lambda_2$

$\lambda_1 \cdot \lambda_2 = 6$

$(5-\lambda_2)\lambda_2 = 6$

$5\lambda_2 - \lambda_2^2 - 6 = 0$

$\lambda = 2, 3$

$-\lambda^2 + 5\lambda_2 - 6 = 0$

$\boxed{\lambda^2 - 5\lambda_2 + 6 = 0}$

11. — ?

1) $\begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$\rightarrow |A - \lambda I| = 0$

$\begin{vmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{vmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = 0$

$\begin{vmatrix} 3-\lambda & 4 & 2 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0$

$(3-\lambda)\left[(1-\lambda)(-\lambda)-0\right]-2(0-0)+4(0)$

$= (3-\lambda)\left[(1-\lambda)(-\lambda)\right]$

$= (3-\lambda)\left[-\lambda+\lambda^2\right]$

$= 3(-\lambda+\lambda^2)-\lambda(-\lambda+\lambda^2)$

$= -3\lambda+3\lambda^2+\lambda^2-\lambda^3$

$0 = -3\lambda+4\lambda^2-\lambda^3$

$\lambda_1 = 3 \quad \lambda_2 = 1 \quad \lambda_3 = 0$

put $\lambda_1 = 3$

$$\begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3-3 & 4 & 2 \\ 0 & 1-3 & 0 \\ 0 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$= 4y + 2z = 0$

$-2y = 0$

$-3z = 0$

$\therefore z = 0 \quad y = 0$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

put $\lambda_2 = 1$

$$\begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bar{v} = 0$$

$$\begin{bmatrix} 2 & 4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \bar{v} = 0$$

$\frac{4y}{-2} = x$

$-2y = x$

$-2y$
y
0
$\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

$2x + 4y + 2z = 0$

$-z = 0$

$2x + 4y = 0$

$4y = -2x$

$y = \frac{-2x}{4} = -\frac{1}{2}x$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -\frac{1}{2}x \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ 0 \end{pmatrix} x$$

$\lambda_3 = 0$

$$\begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$= \begin{pmatrix} 2z \\ -2z \\ z \end{pmatrix}$

$3x + 4y + 2z = 0$

$y + 2z = 0$

$y = -2z$

$3x + 4(-2z) + 2z = 0$

$3x - 8z + 2z = 0$

$3x - 6z = 0$

$3x = 6z$

$\therefore x = \frac{6z}{3} = 2z$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} z$

$\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

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$\lambda = 1$ For $\lambda = 2$

$A \cdot M = 1$ $A \cdot M = 1$

$G \cdot M = 1$ $G \cdot M = 1$

It is diagonisable.

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

2) $\begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$

$\longrightarrow$ $(A - \lambda I) = 0$

$$\begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\begin{bmatrix} -\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 2 & 0 & -\lambda \end{bmatrix} \begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$= -\lambda \left[(2-\lambda)(-\lambda) - 0 \right] - 2 \left[0 - 2(2-\lambda) \right] +$

$-\lambda [0 - 0]$

$=$

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$= -\lambda \left(-2\lambda + \lambda^2 \right) - 2 \left[0 - 4 + 2\lambda \right]$

$= 2\lambda^2 - \lambda^3 + 8 - 4\lambda$