Combinatorics

Combinatorics is a special branch of Mathematics. It has many applications not only in several interesting fields such as enumerative combinatorics (the classical application), but also in the other fields.

eg: The person would like to buy a shirt & a pants. There are ~~In shop~~ 4 types of shirts ~~available~~ & 3 types of pants available in the shop.

- From a statistical perspective, the customer is evaluating the possible combinations before making a decision. Depending on their preference, the order will be placed by choosing one of the combinations.
- In this example, it is easy to calculate the number of possible combinations.

(4x3) - Combinations.

What if the person decides to buy ~~order~~ two shirts ~~cakes~~, how will it affect the number of possible combinations of choices? It will be a tedious task to count all these possibilities. So we need a systematic approach to count such possible combinations.

- Combinatorics deals with the counting of diffrent possibilities in a systematic approach.

There are n balls in the basket and we want to draw n/k balls. (- Replacement & Without replacement).

- How many possibilities exist to draw m out of n balls (thus determining the number of distinguishable combinations).

- Total number of ways in which the chosen set of balls can be arranged in a distinguishable order (which we will define as permutations) to

<u>Def^n</u> A group of elements is said to be <u>ordered</u> if the order in which these elements are drawn is of relevance. Otherwise is called unordered.

• <u>Ordered sample:</u>

<u>exa:</u> ① The first three places in an Olympic 100 m race are determined by the order in which the athletes arrive at the finishing line. If 9 athletes are competing with each other, the number of possible results for the first three places is of interest. In the urn language we are taking draws without replacement (since every athlete can only have one distinct place)

<u>Unordered sample:</u>

e.g. ① The selected member for a national football team. The order in which the selected names are announced is irrelevant.

② Fishing 20 fish from a lake.

③ A bunch of 10 Flowers from 25 Flowers of 5 diffrent colors.

defⁿ: The factorial function $n!$ defined as

$$n! = \begin{cases} 1 & \text{for } n=0 \\ 1 \cdot 2 \cdots n & \text{for } n>0 \end{cases}$$

Permutations:

Consider a set of n elements. Each ordered composition of these n elements is called a permutation.

∴ We distinguish between two case:

- If all elements are distinguishable, then we speak of permutation without replacement. However, if some or all of the elements are not distinguishable, then we speak of permutation with replacement.

replacement for only convention.

- Permutation without Replacement:

If all the n elements are distinguishable then there are $\underline{\underline{n!}}$ diffrent combinations of these elements.

e.g: Ranking of three players.

- Permutation with Replacement:

Assume that not all n elements are distinguishable. The elements are divided into groups & these groups distinguishable. Suppose, there are s groups of sizes $n_1, n_2, \cdots, n_s$. The total number of diffrent way to arrange the n elements in s groups is combination

$$\frac{n!}{n_1!\, n_2! \cdots n_s!}$$

Combinations:

e.g: How many distinguishable arrangement of letters of word "Permutations".

Defn: The Binomial coefficient for any integer m & n with $n \geq m \geq 0$ is ~~defined~~ denoted & defined as

$$\binom{n}{m} = \frac{n!}{m!\,(n-m)!}$$

It is read as "n choose m" & can be calculated in R. Choose (n,m).

Calculation rules for the Binomial coefficient:

$$\binom{n}{0} = 1 \qquad \binom{n}{1} = n \qquad \binom{n}{m} = \binom{n}{n-m}.$$

$$\binom{n}{m} = \frac{n \times (n-1) \times \cdots \times (n-m+1)}{m \times (m-1) \times \cdots \times 1.}$$

Q: How many diffrent possibilities exist to draw m out of n elements.

① Combination without replacement & without consideration of order of the elements.

② Without & with

③ with replacement & without

④ with replacement & with order.

① Combinations without replacement and without consideration of the order:

When there is no replacement and the order of the elements is also not relevant, then total number of distinguishable combinations in drawing m out of n elements is $\binom{n}{m}$.

e.g: Suppose a company elects a new board of directors. The board consist 6 members & 20 people are eligible to be selected. How many combinations for the board of directors?

$$\binom{20}{6}$$

② Combinations without replacement and with consideration of the order.

The total number of diffrent combinations for the setting without replacement and with consideration of the order is $\frac{n!}{(n-m)!} = \binom{n}{m} \cdot m!$

e.g: 100 m race, 20 player.

The total number of diffrent combinations for the players in the first three places is $\frac{20!}{(20-3)!} = 20 \times 19 \times 18$

③ Combinations with replacement and without consid. of the order.

xa ① At a / hands be

The total number of diffrent combinations with replacement and without consideration of the order is
$$\binom{n+m-1}{m} = \frac{(n+m-1)!}{m!\,(n-1)!} = \binom{n+m-1}{n-1}$$

e.g: A farmer has 2 fields and aspires to cultivate one out of 5 diffrent organic product per field. Then the total number of choices he has is
$$\binom{5+2-1}{2} = \binom{6}{2} = \frac{6!}{2!\,4!} = \frac{6\times5}{2} = \underline{\underline{15}}$$

④ Combinations with replacement & with consideration of the order.

The total number of diffrent combinations for the integers m and n with replacement and when the order is of relevance is n^m.

e.g: Consider a credit card with a four digit ~~pers~~ PIN code. The total number of ~~possibilities~~ possible combinations for PIN is 10^4.

Exa ① At a party with 15 guests, every guest shakes hands with each other guest. How many handshakes can be counted in total?

Solⁿ: There are $n=15$ guests & $m=2$ guests shakes hands with each other. The order is not important: two guests shakes hands no matter who is drawn first. It is also not possible to shake hands with onself.
$\Rightarrow$ no replacement case.
$\therefore$ We obtain the solution as $\binom{n}{m} = \binom{15}{2} = \frac{15 \cdot 14}{2} = \underline{\underline{105}}$ handshakes in total.

② The Cricket World Cup 2023 in India consists of 10 teams. How many combinations for the top 4 teams exist when

ⓐ taking into account the order of these 4 teams &
ⓑ Without taking into account of order of these top 4 teams.

Solⁿ: Since each team has exactly one final place, we have a without replacement situation. Using $n=10$ & $m=4$

ⓐ $\frac{10!}{(10-4)!} = \binom{10}{4} 4! = 10 \times 9 \times 8 \times 7 = 5040$

ⓑ $\binom{10}{4} = \frac{10!}{4!\,(10-4)!} = \frac{5040}{4 \times 3 \times 2} = \underline{\underline{210}}$

③ An online book store assigns membership code to each member. For administrative reasons, these codes consist of 4 letter (A-Z). A special discount
26
period increased the total number of members from 4,00,000 to 5,00,000. Are there enough combinations of codes left to be assigned for the new membership codes?

chess Russian Gore

Soln: There are $n = 26$ diffrent letters for $m = 4$ positions of membership code.

Each letter can be used more than once if desired and we have obtain $n^m = 26^4$ = ~~20,736~~ possible combinations
4,56,976

∴ We conclude that there is no codes left to assign to the new members. (sufficient membership code are left).

(ii) $26 \times 10^3 = 26,000$
22,000 to 24,000.

(iii) $26 \times 10^5 = 26,00,000$
20,00,000 to 28,00,000. (6 digi[t]
first Alphabate.

(iv) $26 \times 26 \times 10^2 = 67600$
50,000 to 65,000.

④ How many diffrent letter arrangements can be formed from the letters PEPPER?

3-P's, 2-E's & 1-R.

Answer: $\dfrac{6!}{3! \cdot 2!}$

A chess tournament has 10 competitors of which 4 are Russian, 3 are from the United States, 2 are from Great Britain and 1 is from Brazil. If the tournament result list just the nationalities of the players in order in which they placed, how many outcomes are possible?

Sol^n: There are $\dfrac{10!}{4!\,3!\,2!\,1!} = \dfrac{\cancel{10}^{5}\times7\times8\times9\times6\times5}{3!\times\cancel{2}}$.

$= 12,600.$

⑥ How many diffrent signals, each consisting of 9 flags hung in a line, can be made from a set of 4 white flags, 3 red flags and 2 blue flags of the same color all flags of the same color are identical?

Sol^n: There are $\dfrac{9!}{4!\,3!\,2!} = 1260$

⑦ Form a group of 5 women & 7 men, how many diffrent committees consisting of 2 women and 3 men can be formed? What if 2 of the men are feuding & refuse to serve on the committee together?

As there are $\binom{5}{2}$ possible groups of 2 women & $\binom{7}{3}$ possible groups of 3 men, it follows from the basic principle there are $\binom{5}{2}\binom{7}{3} = \frac{5.4}{2} \cdot \frac{7.6.5}{3.2.1} = 350$ possible committees consists 2 women & 3 men.

Now, suppose that 2 of the men refuse to serve together. Because a total of $\binom{2}{2} \cdot \binom{5}{1} = 5$ out of the $\binom{7}{3} = 35$ possible groups of 3 men contain both of the feuding men, it follows that there are $35 - 5 = 30$ groups that does not contain both ~~the~~ of the feuding men. Because there are still $\binom{5}{2} = 10$ ways to choose the 2 women, there are $30 \cdot 10 = 300$ possible committees in the case.

(8) Customers are allowed to choose (20 numbers ~~of piece~~ tray of fruits) from 6 variety of fruits.

(a) What are the number of possible combinations to ~~fil~~ choose 20 ~~piece~~ fruits

(b) A customer insist of having at least one fruit from each variety in his 20 piece. How many option does he have to ~~fill~~ choose 20 ~~piece~~ fruits.

$m = 20$ $\quad n = 6$ $\quad$ (a) $\binom{n+m-1}{m} = \binom{25}{20} = 53{,}130$ (b) $\binom{19}{14} = 11{,}628.$

$m = 14$

(a) How many 5-card hands (subsets) can be formed from a standard 52-card deck?

(b) If a 5-card hand chosen at random, How many flushes (all 5 cards in the hand are in the same suit)?

(c) How many 5-card hand containing 3, but not 4, Aces?

Soln: (a) $\binom{52}{5}$ (b) $\binom{4}{1}\binom{13}{5}$ (c) $\binom{4}{3}\cdot\binom{48}{2}$.

Suits: Clubs, diamonds, hearts, spades.

exa: (2) A committee of k people is to be chosen from a set of 7 women & 4 men. How many ways are there to form the committee if

(a) The committee consists of 3 women & 2 men?

(b) The committee can be any positive size but must have equal numbers of women & men?

(c) The committee has 4 people and one of them must be Mr. ~~Amit~~ Patil?

(d) The committee has 4 people and at least 2 are women?

(e) The committee has 4 people, two of each sex & Mr. & Ms. Patil cannot both be on the committee.

Soln: (a) $\binom{7}{3}\binom{4}{2} = 35 \times 6 = 210.$

(b) $\binom{7}{1}\binom{4}{1}+\binom{7}{2}\binom{4}{2}+\binom{7}{3}\binom{4}{3}+\binom{7}{4}\binom{4}{4} = \underline{\underline{329}}$

(c) Mr. Patil must be on the committee means the problem reduces to picking 3 other people on 4-person committee from the remaining 10 people (7 women & 3 men).

$\therefore \binom{10}{3}.$

(d) $\binom{7}{2}\binom{4}{2}+\binom{7}{3}\binom{4}{1}+\binom{7}{4} = 301.$

Suppose one approch is to pick 2 women first, $\binom{7}{2}$ ways then pick any 2 of the remaing set of 9 people (5 women & 4 men). However counting all committees in this fashion counts some outcomes more than once, since any woman in ~~this fashion~~ one of these committees could be chosen as either one of the first 2 women or one of the remaing people.

e.g: W_i denotes i^{th} woman & M_i the ith man then (W_1, W_3) composed with the 2 remaining people (W_2, M_3) yeids the same set as (W_1, W_2) composed with (W_3, M_3).

Choo... wa... Min...

(e) Complementary approch:

$\binom{7}{2}\cdot\binom{4}{2}$ - 2 women - 2 men committees.

Mr. & Mrs Patil on committee so $\binom{6}{1}\binom{3}{1}$

$\therefore$ Answer: $\binom{7}{2}\binom{4}{2} - \binom{6}{1}\binom{3}{1}$.

exa: (3) How many arrangements of the seven letters in the word SYSTEMS have the E occurring somewhere before M?. How many arrangements have the E somewhere before M & the three S's grouped consecutively.

3. Choose two positions out of 7. (for E & M)
(pair)
$=\binom{7}{2}$ ways. Put in the first in this pair position M in second one. Then 5 positions for 3-S, 1-Y, 1-T.

$\therefore \dfrac{5!}{3!\times 1!\times 1!} = 5\times 4 = 20.$

Answer is $21\times 20 = 420.$

④ First take <u>SSS</u> as a letter.
So 5 positions & 1-Y, 1-T, 1-E & 1-M, 1-SSS are letters.

For E & M. $\binom{5}{2} = 10$ ways & Then 3! ways
$\therefore 10\times 6 = \underline{\underline{60}}.$

⑤ How many arrangements of 5 αs, 5 βs & 5 γs are there with at least one β and at least one γ between each pair of α's?

<u>Soln</u>: There are three cases:
① Exactly one β and exactly one γ between each pair of αs: Between each of the four pairs of αs, the β or the γ can be first 2^4 ways. The fifth β and fifth γ along with the sequence of the rest of the letters can be considered as 3 objects to be arranged 3! ways. Altogether $2^4\times 3! = 96$ ways.

② Exactly one β between each pair of α's and two γs between some pair of αs (or two βs between some pair of αs and exactly one γ between each pair of αs):

There are 4 choices for between which pair of αs the two γs go and 3 ways to arrange two γs & one β there. There are 2^3 choices for whether the β or the γ goes first between the other 3 pairs of αs and 2 choices for at which end of the arrangement the fifth β goes.

There is $(4 \times 3 \times 2^3 \times 2) = 192$ ways.

Multiplying by 2 for the case of two βs between some pair of αs, we obtain $(4 \times 3 \times 2^3 \times 2) = 192$ ways.

In total $= 384$ ways.

(3) Two βs between some pair of αs and two γs between some pair of αs: There are two subcases. If the two βs and two γs are between the same pair of αs, there are 4 choices for which pair of αs, $\binom{4}{2} = \frac{4!}{2!2!}$ ways to arrange them between this pair of αs, and 2^3 ~~to~~ choices for whether the β or γ goes first between the other 3 pairs of αs. ~~If two~~

If two βs and two γs are between the diffrent pairs of αs, there are 4×3 ways to pick between which αs the two βs and then between which αs the two γs go, 3^2 ways to arrange the two γs and one β and to arrange the one γ and two βs, & 2^2 choices for whether the β or the γ goes first between the other 2 pairs of αs. Together $4 \times \binom{4}{2} \times 2^3 + 4 \times 3 \times 3^2 \times 2^2 = 1056$ ways.

All together, $96 + 384 + 1056 = 1536$ arrangements.