College of Engineering Pune
Tutorial on Laplace Transform

Questions on CO1

1. Define Laplace Transform and Inverse Laplace Transform of a function. Show that Laplace transform is a linear operator.
2. State the first shifting, second shifting and Convolution theorems.
3. Make a list of Laplace and Inverse Laplace Transforms of standard functions.

Questions on CO2 and CO3

1. Find the Laplace Transforms of the following functions:

   (a) $(5e^{2t}-3)^2$ Ans. $\frac{25}{s-4}-\frac{30}{s-2}+\frac{9}{s}$

   (b) $\sin 3t - 2\cos 5t$ Ans. $\frac{3}{s^2+9}-2\frac{s}{s^2+25}$

   (c) $\cosh at - \cos at$ Ans. $\frac{2a^2s}{s^4-a^4}$

   (d) $e^t(1+t)^2$ Ans. $\frac{s^2+1}{(s-1)^3}$

   (e) $f(t)=\begin{cases} t, & 0<t<1 \\ e^{1-t}, & t>1. \end{cases}$ Ans. $\frac{1}{s^2}[1-e^{-s}(\frac{2s+1}{s+1})]$

   (f) $t^{7/2}e^{3t}$ Ans. $\frac{105\sqrt{\pi}}{16(s-3)^{9/2}}$

   (g) $f(t)=t\cos at$ Ans. (Use $\mathcal{L}\{tf(t)\}$). $\frac{s^2-a^2}{(s^2+a^2)^2}$

   (h) $\sin^2 t$ Ans. (Use $\mathcal{L}\{f'\}$). $\frac{2}{s(s^2+4)}$

   (i) $\frac{e^{-at}-e^{-bt}}{t}$ Ans. (Use $\mathcal{L}\{f(t)/t\}=\int_s^{\infty} F(u)du$). $\ln\frac{s+b}{s+a}$

   (j) $\frac{\cos at-\cos bt}{t}$ Ans. $\frac{1}{2}\ln(\frac{s^2+b^2}{s^2+a^2})$

   (k) $\frac{\sin^2 t}{t}$ Ans. $\frac{1}{4}\ln\frac{s^2+4}{s^2}$

   (l) $\frac{e^t\,\delta(t-2)}{t}$ Ans. $\frac{e^{-2(s-1)}}{2}$

   (m) $\delta(t-3)\,U(t-3)$ Ans. e^{-3s}

   (n) $t^2\sin 2t$ Ans. (Use $\mathcal{L}\{t^2f(t)\}=F''(s)$). $\frac{-4(4-3s^2)}{(s^2+4)^3}$

   (o) $\int_0^t \frac{1-e^{-u}}{u}du$ Ans. (Use $\mathcal{L}\{\int_o^t f(u)du\}=\frac{\mathcal{L}\{f\}}{s}$). $\frac{1}{s}\ln(1+\frac{1}{s})$

2. Find the inverse Laplace transform of the following:

   (a) $\frac{0.1s+0.9}{s^2+3.24}$ Ans. $0.1\cos 1.8t+0.5\sin 1.8t$

   (b) $\frac{-s-10}{s^2-s-2}$ Ans. $3e^{-t}-4e^{2t}$

   (c) $\frac{1}{(s-1)(s^2+4)}+\frac{4}{s^5}$ Ans. $\frac{e^t}{5}-\frac{\cos 2t}{5}-\frac{\sin 2t}{10}+\frac{t^4}{6}$

(d) $\frac{3s+1}{s^2+6s+13}$ Ans. $e^{-3t}(3\cos 2t - 4\sin 2t)$

(e) $\frac{s^2}{(s-1)^4}$ Ans. $e^t(t + t^2 + \frac{t^3}{6})$

(f) $\frac{e^{-\pi s}}{s^2+9}$ Ans. $\frac{1}{3}\sin 3(t-\pi)U(t-\pi)$

(g) $\frac{1-e^{-s}}{s^2}$ Ans. t, if $t < 1$ and 1 if $t > 1$.

(h) $F(s) = \cot^{-1}\frac{s}{\omega}$ Ans. (Let $f(t) = \mathcal{L}^{-1}F(s)$. Use $\mathcal{L}^{-1}F'(s) = -tf(t)$). $(\sin\omega t)/t$.

(i) $\frac{1}{2}\ln(\frac{s^2-a^2}{s^2})$ Ans. $\frac{1-\cosh at}{t}$

(j) $\ln\sqrt{\frac{s^2+b^2}{s^2+a^2}}$ Ans. $\frac{\cos at - \cos bt}{t}$

(k) $\frac{s^3-3s^2+6s-4}{(s^2-2s+2)^2}$ Ans. $e^t(t\sin t + \cos t)$

(l) $F(s) = s\ln(\frac{s}{\sqrt{s^2+1}})$ Ans.(Use $\mathcal{L}^{-1}F''(s) = t^2 f(t)$).

(m) $\frac{e^{-s}}{s}\tan^{-1}(\frac{s-1}{4})$
Ans. Let $F(s) = e^{-s}/s$, $G(s) = \tan^{-1}(\frac{s-1}{4})$. Then $\mathcal{L}^{-1}F(s) = U(t-1)$ and $\mathcal{L}^{-1}G(s) = \frac{-e^t\sin 4t}{t}$. By convolution thm, the required ans is $\mathcal{L}^{-1}F(s)G(s) = U(t-1) * \frac{-e^t\sin 4t}{t}$.

3. Solve using Laplace transform:

(a) $y'' + y = r(t)$, $r(t) = t$ if $1 < t < 2$, 0 otherwise. $y(0) = y'(0) = 0$
Ans. $y = [t - \cos(t-1) - \sin(t-1)]U(t-1) + [-t + 2\cos(t-2) + \sin(t-2)]U(t-2)$

(b) $y'' + y = e^{-2t}\sin t, y(0) = y'(0) = 0.$
Ans. $y = \frac{1}{8}[\sin t - \cos t + e^{-2t}(\sin t + \cos t)]$

(c) $y'' + 2y' + 5y = 50t - 150, y(3) = -4, y'(3) = 14.$
Ans. $y = 10(t-3) - 4 + 2e^{-(t-3)}\sin 2(t-3)$

(d) $y'' + 2y' + 5y = e^{-t}\sin t,\ y(0) = 0, y'(0) = 1$
Ans. $y = e^{-t}(\sin t + \sin 2t)/3$

(e) Find the current $i(t)$ in an LC circuit assuming $L = 1$henry, $C = 1$ farad, zero initial current and charge on the capacitor and $v(t) = 1 - e^{-t}$ if $0 < t < \pi$ and 0, otherwise.
Ans. $\frac{1}{2}(e^{-t} - \cos t + \sin t)$, if $0 < t < \pi$ and $\frac{1}{2}[-(1+e^{-\pi})\cos t + (3 - e^{-\pi})\sin t]$, if $t > \pi$.

4. Solve the following linear integral equations:

(a) $y(t) = \sin 2t + \int_0^t y(\tau)\sin 2(t-\tau)d\tau.$ Ans. $\sqrt{2}\sin\sqrt{2}\,t$

(b) $y(t) = 1 - \sinh t + \int_0^t (1+\tau)\,y(t-\tau)d\tau.$ Ans. $\cosh t$