Discrete Structures

Mathematical Logic- Rules of Inference

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Rules of Inference

- ***An argument:*** a sequence of statements that end with a conclusion.
  - **argument ("valid") :** never lead from correct statements to an incorrect conclusion**.**
  - **argument ("fallacies")** : can lead from true statements to an incorrect conclusion.

- ***A logical argument*** consists of premises/hypotheses and a single proposition called the conclusion.

- ***Logical rules of inference:*** Templates for constructing valid arguments.

Valid Arguments

- Example: A logical argument

  If I dance all night, then I get tired.

  I danced all night.

  Therefore I got tired.

- Logical representation of underlying variables:

  p: I dance all night. q: I get tired.

- Logical analysis of argument:

$$
\begin{array}{ll}
p \rightarrow q & \text{premise 1} \\
\underline{p} & \underline{\text{premise 2}} \\
\therefore q & \text{conclusion}
\end{array}
$$

Inference Rules: General Form

- An *Inference Rule is*

  A pattern establishing that if we know that a set of *premise statements of certain* forms are all true, then we can validly deduce that a certain related *conclusion* statement is true.

$$\boxed{\begin{array}{l} \quad \textit{premise 1} \\ \quad \textit{premise 2} \\ \quad \ldots \\ \hline \therefore\ \textit{conclusion} \end{array}}$$

"$\therefore$" means "therefore"

Valid Arguments

- A form of logical argument is ***valid*** if whenever every premise is true, the conclusion is also true.

- A form of argument that is not valid is called a ***fallacy.***

Inference Rules Summary

- Each valid logical inference rule corresponds to an implication that is a tautology.

$$\begin{array}{l} \textit{premise 1} \\ \textit{premise 2} \\ \ldots \\ \hline \therefore\ \textit{conclusion} \end{array}$$

Inference rule

- Corresponding tautology:

$$((\textit{premise 1}) \wedge (\textit{premise 2}) \wedge \ldots) \rightarrow \textit{conclusion}$$

Rules of inference

- Modus ponens
- Modus tollens
- Hypothetical syllogism
- Disjunctive syllogism
- Resolution
- Addition
- Simplification
- Conjunction

Modus Ponens

$$\begin{array}{l} \quad p \\ \quad p \rightarrow q \\ \hline \therefore q \end{array}$$

Rule of ***Modus ponens***
(a.k.a. *law of detachment*)

"the mode of affirming"

$(p \wedge (p \rightarrow q)) \rightarrow q$ is a tautology

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$	$(p \wedge (p \rightarrow q)) \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

- Notice that the first row is the only one where premises are all true

Modus Ponens: Example

$$\text{If}\ \left.\begin{cases} p \to q : \text{"If it snows today then we will go skiing"} \\ p \quad : \text{"It is snowing today"} \end{cases}\right\}\ \text{assumed TRUE}$$

$$\text{Then}\ \therefore q \quad : \text{"We will go skiing"} \qquad \text{is TRUE}$$

$$\text{If}\ \left.\begin{cases} p \to q : \text{"If } n \text{ is divisible by 3 then } n^2 \text{ is divisible by 3"} \\ p \quad : \text{"} n \text{ is divisible by 3"} \end{cases}\right\}\ \text{assumed TRUE}$$

$$\text{Then}\ \therefore q \quad : \text{"} n^2 \text{ is divisible by 3"} \qquad \text{is TRUE}$$

Modus Tollens

$$\begin{array}{r} \neg q \\ p \rightarrow q \\ \hline \therefore \neg p \end{array}$$

Rule of ***Modus tollens***

"the mode of denying"

$(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$ is a tautology

Modus Tollens: Example

- Example

If $\left\{ \begin{array}{l} p \to q : \text{“If this jewel is really a diamond then it will scratch glass”} \\ \neg q \quad : \text{“The jewel doesn't scratch glass”} \end{array} \right\}$ assumed TRUE

Then $\therefore \neg p$: “The jewel is not a diamond” is TRUE

More Inference Rules

- $$\frac{p}{\therefore\ p \lor q}$$ Rule of ***Addition***

  Tautology: $p \to (p \lor q)$

- $$\frac{p \land q}{\therefore\ p}$$ Rule of ***Simplification***

  Tautology: $(p \land q) \to p$

- $$\begin{array}{c} p \\ q \\ \hline \therefore\ p \land q \end{array}$$ Rule of ***Conjunction***

  Tautology: $[(p) \land (q)] \to p \land q$

Examples

- State which rule of inference is the basis of the following arguments:
  - It is below freezing and raining now. Therefore, it is below freezing now.

Examples

- State which rule of inference is the basis of the following arguments:
  - It is below freezing and raining now. Therefore, it is below freezing now.

- *p: It is below freezing now.*
- *q: It is raining now.*
  - ***$(p \land q) \rightarrow p$ (rule of simplification)***

Quick Quiz 4.1:

- State which rule of inference is the basis of the following argument:

  "It is below freezing now. Therefore, it is below freezing or raining now."

  Visit moodle to submit answer

Quick Quiz 4.1:

- State which rule of inference is the basis of the following argument:

  "It is below freezing now. Therefore, it is below freezing or raining now."

Answer:

$$\begin{array}{l} p \\ \hline \therefore p \vee q \end{array}$$

This is an argument that uses **the addition rule.**

Examples

- State which rule of inference is the basis of the following arguments:
  - It is below freezing, It is raining now. Therefore, it is below freezing and raining now.

- *p: It is below freezing now.*
- *q: It is raining now.*
  - $(p) \land (q) \rightarrow (p \land q)$ ***(rule Conjunction)***

Hypothetical Syllogism

$$\boxed{\begin{array}{l} \quad p \to q \\ \quad q \to r \\ \hline \therefore p \to r \end{array}}$$

Rule of ***Hypothetical syllogism***

Tautology:

$$[(p \to q) \wedge (q \to r)] \to (p \to r)$$

Example:

- State the rule of inference used in the argument:

  "If it rains today, then we will not have a barbecue today. If we do not have a barbecue today, then we will have a barbecue tomorrow. Therefore, if it rains today, then we will have a barbecue tomorrow."

"If it rains today, then we will not have a barbecue today. If we do not have a barbecue today, then we will have a barbecue tomorrow. Therefore, if it rains today, then we will have a barbecue tomorrow."

$$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$$

Disjunctive Syllogism

$$\boxed{\begin{array}{l} \quad p \lor q \\ \quad \neg p \\ \hline \therefore\ q \end{array}}$$

Rule of ***Disjunctive syllogism***

Tautology: $[(p \lor q) \land (\neg p)] \rightarrow q$

Example

Ed’s wallet is in his back pocket or it is on his desk. Ed’s wallet is not in his back pocket. Therefore, Ed’s wallet is on his desk.

Example

- Ed's wallet is in his back pocket or it is on his desk. ($p \vee q$)
  p q
- Ed's wallet is not in his back pocket. ($\neg p$)
- Therefore, Ed's wallet is on his desk. (q)

Resolution

$$\begin{array}{r} p \lor q \\ \neg p \lor r \\ \hline \therefore\ q \lor r \end{array}$$

Rule of ***Resolution***

Tautology:

$$[(p \lor q) \land (\neg p \lor r)] \rightarrow (q \lor r)$$

- When $q = r$:

  $$[(p \lor q) \land (\neg p \lor q)] \rightarrow q$$

- When $r = \mathbf{F}$:

  $$[(p \lor q) \land (\neg p)] \rightarrow q \quad \text{(Disjunctive syllogism)}$$

Resolution: Example

- Use resolution to show that the hypotheses

  "Jasmine is skiing or it is not snowing" and "It is snowing or Bart is playing hockey" imply that "Jasmine is skiing or Bart is playing hockey"

"*Jasmine is skiing* or *it is not snowing*" and "*It is snowing* or *Bart is playing hockey*" imply that "*Jasmine is skiing* or *Bart is playing hockey*"

$$\underbrace{\text{Jasmine is skiing}}_{r} \quad \underbrace{\text{it is not snowing}}_{\neg p} \quad \underbrace{\text{It is snowing}}_{p} \quad \underbrace{\text{Bart is playing hockey}}_{q}$$

$$(p \vee q) \wedge (\neg p \vee r) \rightarrow (q \vee r)$$

Formal Proofs

- A formal proof of a conclusion *C, given premises*

  p1, p2,…, pn

  consists of a sequence of steps, each of which applies some inference rule to premises or previously-proven statements to yield a new true statement (the *conclusion).*

- A proof demonstrates that *if the premises are* true, *then the conclusion is true.*

Formal Proof Example

- Suppose we have the following premises:
  "It is not sunny and it is cold."
  "We will swim only if it is sunny."
  "If we do not swim, then we will take a trip."
  "If we take a trip, then we will be home by sunset."

- Given these premises, prove the conclusion **"We will be home by sunset" using** inference rules.

- **Step 1:** Identify the propositions (Let us adopt the following abbreviations)

sunny = ***"It is sunny";***

cold = ***"It is cold";***

swim = ***"We will swim";***

canoe = ***"We will take a trip"*****;**

sunset = ***"We will be home by*** **sunset".**

- **Step 2:** Identify the argument. (Build the argument form)

It is not sunny and it is cold.

We will swim only if it is sunny.

If we do not swim, then we will take a trip.

If we take a trip, then we will be home by sunset.

We will be home by sunset.

- **Step 2:** Identify the argument. (Build the argument form)

It is not sunny and it is cold.

We will swim only if it is sunny.

If we do not swim, then we will canoe.

If we take a trip, then we will be home by sunset.

We will be home by sunset.

$$\begin{array}{l} \neg sunny \wedge cold \\ swim \rightarrow sunny \\ \neg swim \rightarrow canoe \\ canoe \rightarrow sunset \\ \hline \therefore\ sunset \end{array}$$

- **Step 3:** Verify the reasoning using the rules of inference

$$\begin{array}{l} \neg sunny \wedge cold \\ swim \rightarrow sunny \\ \neg swim \rightarrow canoe \\ canoe \rightarrow sunset \\ \hline \therefore sunset \end{array}$$

Step	Proved by
1. $\neg sunny \wedge cold$	Premise #1.
2. $\neg sunny$	Simplification of 1.
3. $swim \rightarrow sunny$	Premise #2.
4. $\neg swim$	Modus tollens on 2 and 3.
5. $\neg swim \rightarrow canoe$	Premise #3.
6. $canoe$	Modus ponens on 4 and 5.
7. $canoe \rightarrow sunset$	Premise #4.
8. $sunset$	Modus ponens on 6 and 7.

Quick Quiz 4.2:

- **True or False?**
- The premises $(p \land q) \lor r$ *and* $r \to s$ *imply the conclusion* $p \lor s$.

- *Visit Moodle to Submit Answer.*

Quick Quiz 4.2:

True or False?

- The premises $(p \wedge q) \vee r$ *and* $r \rightarrow s$ *imply the conclusion* $p \vee s$.

Solution: True

- rewrite the premises $(p \wedge q) \vee r$ as two clauses, $p \vee r$ and $q \vee r$.
- replace $r \rightarrow s$ by the equivalent clause $\neg r \vee s$.
- $(\neg r \vee s) \wedge (p \vee r) \wedge (q \vee r)$
- $(p \vee s) \wedge (q \vee r)$
- Using the two clauses $p \vee r$ and $\neg r \vee s$, we can use resolution to conclude $p \vee s$.

Common Fallacies

- A ***fallacy*** is an inference rule or other proof method that is not logically valid.
- **A fallacy may yield a false conclusion!**

- **Ex.** $((p \vee q) \wedge p) \rightarrow \sim q$ is not a tautology.
- **Fallacy of Disjunction**

- ***Fallacy of affirming the conclusion:***
- "*$p \rightarrow q$ is true, and q is true, so p must be* true."
  (No, because **$\mathbf{F} \rightarrow \mathbf{T}$ is true.)**

- **Example**
- If David Cameron (DC) is president of the US, then he is at least 40 years old. *($p \rightarrow q$)*
- DC is at least 40 years old. *(q)*
- Therefore, DC is president of the US. *(p)*

- ***Fallacy of denying the hypothesis:***
- "*$p \rightarrow q$ is true, and p is false, so q must be* false."

  (No, again because **F → T is true.)**

  Or
- The proposition $((p \rightarrow q) \land \neg p) \rightarrow \neg q$ is not a tautology

- **Example**
- If a person does arithmetic well then his/her checkbook will balance. ($p \rightarrow q$)
- I cannot do arithmetic well. ($\neg p$)
- Therefore my checkbook does not balance. ($\neg q$)

Quick Quiz 5.1

- Is the following argument valid?

  "If you do every problem in this book, then you will learn discrete mathematics." "You learned discrete mathematics.“ Therefore, "you did every problem in this book."

- **Visit Moodle : To Submit Your Answer.**

Quick Quiz 5.1

- Is the following argument valid?

  "If you do every problem in this book, then you will learn discrete mathematics." "You learned discrete mathematics." Therefore, "you did every problem in this book."

- **Solution:**

  Let

  p : "You did every problem in this book."

  q : "You learned discrete mathematics."

  Argument is of the form: if $p \rightarrow q$ and q, then p.

  This is an example of an incorrect argument using the **fallacy of affirming the conclusion.**

Inference Rules for Quantifiers

- $$\frac{\forall x\, P(x)}{\therefore P(c)}$$ **Universal instantiation** (substitute *any* specific member *c* in the domain)

- $$\frac{P(c)}{\therefore \forall x\, P(x)}$$ (for an arbitrary element *c* of the domain) **Universal generalization**

- $$\frac{\exists x\, P(x)}{\therefore P(c)}$$ **Existential instantiation** (substitute an element *c* for which $P(c)$ is true)

- $$\frac{P(c)}{\therefore \exists x\, P(x)}$$ (for some element *c* in the domain) **Existential generalization**

Example

Show that the premises imply the conclusion

- *“Every animal has brain.” and “Human is a animal.” Therefore, “Human has brain.”*

Example

- *Every animal has brain. Human is a animal. Therefore, Human has brain.*
- **Proof**
- Define the predicates:

  M(x): x is a animal

  L(x): x has brain

  J: Human, a member of the universe
- The argument becomes

$$
\begin{array}{l}
1.\ \forall x\,[M(x) \to L(x)] \\
2.\ \ M(J) \\
\hline
\therefore\ L(J)
\end{array}
$$

- The proof is

$$\begin{array}{l} \forall x\,(M(x) \rightarrow L(x)) \\ M(J) \\ \hline \therefore L(J) \end{array}$$

- **Note:** Using the rules of inference requires lots of practice.
  - Try example problems in the textbook.

$$\begin{array}{l} \forall x\,(M(x) \rightarrow L(x)) \\ M(J) \\ \hline \therefore L(J) \end{array}$$

- The proof is

1. $\forall x\,[M(x) \rightarrow L(x)]$ *Premise 1*
2. $M(J) \rightarrow L(J)$ *U. I. from (1)*
3. $M(J)$ *Premise 2*
4. $L(J)$ *Modus Ponens from (2) and (3)*

- **Note:** Using the rules of inference requires lots of practice.
  - Try example problems in the textbook.

Combining Rules of Inference for Propositions and Quantified Statements

Universal Modus Ponens:

- If $\forall x(P(x) \to Q(x))$ is true, and if $P(a)$ is true for a particular element a in the domain of the universal quantifier, then $Q(a)$ must also be true.

$$\begin{array}{l} \forall x(P(x) \to Q(x)) \\ \underline{P(a), \text{ where } a \text{ is a particular element in the domain}} \\ \therefore Q(a) \end{array}$$

EXAMPLE

- Assume that

"For all positive integers *n, if n is greater than 4, then* n^2 *is less than* 2^n"

is true. Use universal modus ponens to show that $100^2 < 2^{100}$.

EXAMPLE

Solution:

P(n): "n > 4"

Q(n): "n^2<2"

"For all positive integers n, if n is greater than 4, then n^2 is less than 2^n"

Represented by $\forall n(P(n) \rightarrow Q(n))$*, Assumed True*

- *P(100) is true because 100 > 4.*
- *It follows by universal modus ponens that Q(100)* is true, namely, that $100^2 < 2^{100}$.

Universal Modus tollens

$$\forall x(P(x) \rightarrow Q(x))$$
$$\neg Q(a), \text{ where } a \text{ is a particular element in the domain}$$
$$\therefore \neg P(a)$$

Universal Modus tollens

$$\forall x(P(x) \rightarrow Q(x))$$

$$\neg Q(a), \text{ where } a \text{ is a particular element in the domain}$$

$$\therefore \neg P(a)$$

Discrete Structures

Introduction to Proofs

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Quick Quiz 5.2:

- **Correct or incorrect:**

"At least one of the 20 students in the class is intelligent. John is a student of this class. Therefore, John is intelligent."

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- **Step 1:**

  Separate premises from conclusion

  Premises:

  1. *At least one of the 20 students in the class is intelligent.*
  2. *John is a student of this class.*

  Conclusion:

  John is intelligent.

- **Step 2:**

  Translate the example in logic notation.

- ***Premise 1:*** *At least one of the 20 students in the* class is intelligent. (Let the domain = all people)

  $C(x)$ = *"x is in the class"*

  $I(x)$ = *"x is intelligent"*

  Then *Premise 1 says:* $\exists x(C(x) \land I(x))$

- ***Premise 2:*** *John is a student of this class.*

  Then *Premise 2 says:* ***C(John)***

- ***Conclusion:*** ***John is intelligent.***

  And the *Conclusion says:* ***I(John)***

- **Step 2:**

  Translate the example in logic notation.

- ***Premise 1:*** *At least one of the 20 students in the* class is intelligent. (Let the domain = all people)

  C(x) = "x is in the class"

  I(x) = "x is intelligent"

  Then *Premise 1 says:* $\exists \boldsymbol{x(C(x) \wedge I(x))}$

- ***Premise 2:*** *John is a student of this class.*

  Then *Premise 2 says:* ***C(John)***

- ***Conclusion:*** ***John is intelligent.***

  And the *Conclusion says:* ***I(John)***

$$\boxed{\begin{array}{l} \exists x\,(C(x) \wedge I(x)) \\ C(John) \\ \hline \therefore\ I(John) \end{array}}$$

$$\begin{array}{l} \exists x\,(C(x) \wedge I(x)) \\ C(John) \\ \hline \therefore I(John) \end{array}$$

- No, the argument is invalid;

  we can disprove it with a counter-example, as follows:
- Consider a case where there is only one intelligent student A in the class, and A ≠ John.
  - Then by existential instantiation of the premise
    ∃x (C(x) ⋀ I(x)) ⋀ C(A) -> I(A) is true,
  - But the conclusion ***I(John) is false,*** *since A is* the only intelligent student in the class, and John ≠ A.
- Therefore, the premises *do not imply the* conclusion.

Proof Terminology

- A ***proof*** is a valid argument that establishes the truth of a mathematical statement
- ***Axiom (**or **postulate):*** a statement that is assumed to be true
- ***Theorem***

  A statement that has been proven to be true
- ***Hypothesis, premise***

  An assumption (often unproven) defining the structures about which we are reasoning

More Proof Terminology

- ***Lemma***

  A minor theorem used as a stepping-stone to proving a major theorem.

- ***Corollary***

  A minor theorem proved as an easy consequence of a major theorem.

- ***Conjecture***

  A statement whose truth value has not been proven. (A conjecture may be widely believed to be true, regardless.)

Methods of Proving Theorems

Example:

To prove a theorem of the form $\forall x(P(x) \rightarrow Q(x))$

Note:

"If $x > y$, where x and y are positive real numbers, then $x^2 > y^2$"

really means

"For all positive real numbers x and y, if $x > y$, then $x^2 > y^2$."

Proof Methods

- For proving implications $p \to q$, *we have:*
- ***Trivial proof:*** *Prove q by itself.*
- ***Direct proof:*** *Assume p is true, and prove q.*
- ***Indirect proof:***

  Proof by Contraposition ($\neg q \to \neg p$):

  Assume $\neg q$, *and prove* $\neg p$.

  Proof by Contradiction:

  Assume $p \land \neg q$, *and show this leads to a* contradiction. (i.e. prove $(p \land \neg q) \to \boldsymbol{F}$)
- ***Vacuous proof:*** *Prove* $\neg p$ *by itself.*

Direct Proof Example

- Starting with the hypothesis and leading to the conclusion.
- **E.g.**
- **Definition:** An integer *n is called odd iff n=2k+1* for some integer *k; n is even iff n=2k for some k.*

- **Theorem:** Every integer is either odd or even, but not both.

  This can be proven from even simpler axioms.

- **Theorem:**

(For all integers *n) If n is odd, then* n^2 *is odd.*

- **Proof: To prove P(n)->Q(n) assume P(n) is true.**

If *n is odd, then* $n = 2k + 1$ *for some integer k.*

Thus, $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.

Therefore n^2 *is of the form* $2j + 1$ *(with j the integer* $2k^2 + 2k$*), thus* n^2 *is odd.*

Quick Quiz 6.1

Let the statement be “If n is not an odd integer then square of n is not odd.”, then if P(n) is “n is an not an odd integer” and Q(n) is “(square of n) is not odd.” For direct proof we should prove ________

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Indirect Proof : Proof by Contraposition

- That do not start with the premises and end with the conclusion, are called **indirect proofs.**

- An extremely useful type of indirect proof is known as **proof by contraposition.**

- The conditional statement $p \rightarrow q$ *can be proved by* showing that its contrapositive, $\sim q \rightarrow \sim p$, *is true.*

- ***Note:*** we take ¬q as a premise

Indirect Proof Example: Proof by Contraposition

- **Theorem: (For all integers *n*)**

  If $3n + 2$ *is odd, then* n *is odd.*

- **Proof:**

  (Contrapositive: If n is even, then $3n + 2$ is even)

  Suppose that the conclusion is false, *i.e.*, that n is even.

  Then $n = 2k$ for some integer k.

  Then $3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$.

  Thus $3n + 2$ is even, because it equals $2j$ for an integer $j = 3k + 1$. So $3n + 2$ is not odd.

  We have shown that $\neg(n \text{ is odd}) \rightarrow \neg(3n + 2 \text{ is odd})$, thus its contrapositive $(3n + 2 \text{ is odd}) \rightarrow (n \text{ is odd})$ is also true. ■

Vacuous Proof Example

- Show $\neg p$ *(i.e. p is false) to prove* $p \rightarrow q$ *is true.*

- ***E.g.***
- **Theorem:** (For all n) If n is both odd and even, then $n^2 = n + n$.

- **Theorem:** (For all n) If n is both odd and even, then $n^2 = n + n$.
- **Proof:**

  The statement "n is both odd and even" is necessarily false, since no number can be both odd and even. So, the theorem is vacuously true. ■

Trivial Proof Example

- Show *q (i.e. q is true) to prove* $p \rightarrow q$ *is true.*

- **Theorem:** (For integers n) If n is the sum of two prime numbers, then either *n is odd or n* is even.

- **Proof:**

  Any integer n is either odd or even. So the conclusion of the implication is true regardless of the truth of the hypothesis. Thus the implication is true trivially.

Proof by Contradiction

A method for proving *p.*

- Assume $\neg p$, *and prove both q and* $\neg q$ *for* some proposition *q. (Can be anything!)*
- Thus $\neg p \rightarrow (q \land \neg q)$
- $(q \land \neg q)$ *is a trivial contradiction, equal to* ***F***
- Thus $\neg p \rightarrow$ ***F,*** *which is only true if* $\neg p$ **= *F***
- Thus *p is true*

Rational Number

- **Definition:**

The real number *r is rational if there exist* integers *p and q with q ≠ 0 such that r = p/q. (p/q is in lowest terms i.e. no common factors)* A real number that is not rational is called *irrational.*

Proof by Contradiction: Example

Theorem: $\sqrt{2}$ is irrational.

Proof by Contradiction: Example

Prove that $\sqrt{2}$ is irrational by giving a proof by contradiction.

Solution:

- *Let p :"$\sqrt{2}$ is irrational."*
- Suppose that $\sim p$= "$\sqrt{2}$ *is rational" is true.* (leads to a contradiction.)
- *So $\sqrt{2} = a/b$, where $b \neq 0$ and a and b have no common factors.*
- Both sides of this equation are squared $2b^2 = a^2$.
- It follows that *a is even. so assume $a = 2c$*
- $2b^2 = 4c^2$ this means that *b2 is even.*
- our assumption of $\sim p$ *leads to the* contradiction , So$\sim p$ *must be false.*
- *"$\sqrt{2}$ is irrational." is true.*

Quick Quiz 6.2

A proof that p → q is true based on the fact that q is true, such proofs are known as __________

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Summary: Proof by Contradiction

- Proving implication $p \rightarrow q$ *by contradiction*
- Assume $\neg q$*, and use the premise* p *to* arrive at a contradiction, i.e. $(\neg q \wedge p) \rightarrow \boldsymbol{F}$

  $(p \rightarrow q \equiv (\neg q \wedge p) \rightarrow \boldsymbol{F})$
- How does this relate to the proof by contraposition?
- ***Proof by Contraposition*** $(\neg q \rightarrow \neg p)$**:**

  Assume $\neg q$*, and prove* $\neg p$.

Mathematical Induction

- A powerful, rigorous technique for proving that a statement $P(n)$ is true for ***every*** positive integers n, no matter how large.
- Essentially a "domino effect" principle.
- Based on a predicate-logic inference rule:

$$\begin{array}{l} P(1) \\ \forall k \geq 1\ [P(k) \rightarrow P(k+1)] \\ \hline \therefore \forall n \geq 1\ P(n) \end{array}$$

"The First Principle of Mathematical Induction"

Mathematical Induction

- **PRINCIPLE OF MATHEMATICAL INDUCTION:**

  To prove that a statement $P(n)$ is true for all positive integers n, we complete two steps:

  - **BASIS STEP**: Verify that $P(1)$ is true
  - **INDUCTIVE STEP**: Show that the conditional statement $\underbrace{P(k)}_{\text{Inductive Hypothesis}} \rightarrow P(k+1)$ is true for all positive integers k

Induction Example

- Show that, for $n \geq 1$

$$\boxed{1 + 2 + \cdots + n = \frac{n(n+1)}{2}}$$

- Proof by induction
  - $P(n)$: the sum of the first n positive integers is $n(n+1)/2$, i.e. $P(n)$ is
  - ***Basis step***: Let $n = 1$. The sum of the first positive integer is 1, i.e. $P(1)$ is true.

$$1 = \frac{1(1+1)}{2}$$

- ***Inductive step***: Prove $\forall k \geq 1: P(k) \rightarrow P(k+1)$.
  - Inductive Hypothesis, $P(k)$:

$$1+2+\cdots+k=\frac{k(k+1)}{2}$$

  - Let $k\geq 1$, assume $P(k)$, and prove $P(k+1)$, i.e.

This is what you have to prove

$$\underbrace{1+2+\cdots+k+(k+1)=\frac{(k+1)[(k+1)+1]}{2}=\frac{(k+1)(k+2)}{2}}_{P(k+1)}$$

- ***Inductive step*** continues…

By inductive hypothesis P(k)

$$1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

$$= \frac{k^2 + 3k + 2}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

P(k+1)

- Therefore, by the principle of mathematical induction $P(n)$ is true for all integers n with $n \geq 1$

Discrete Structures and Graph Theory

Computer Engineering

Semester III (Structure for Regular Students)

Sr. No.	Course Type	Course Name	Teaching Scheme			Credits
			L	T	P	
1	BSC	Ordinary Differential Equations and Multivariate Calculus	2	1	0	3
2	MLC	Professional Laws, Ethics, Values and Harmony	1	0	0	0
3	HSMC	Innovation and Creativity	1	0	0	1
4	SBC	Development Tools Laboratory	1	0	2	2
5	IFC	Feedback Control Systems	1	1	0	2
6	PCC	Data Structures and Algorithms – I	2	0	0	2
7	LC	Data Structures and Algorithms -I Laboratory	0	0	2	1
8	PCC	Digital Logic Design	3	0	0	3
9	LC	Digital Logic Design Laboratory	0	0	2	1
10	PCC	Discrete Structures and Graph Theory	2	1	0	3
11	PCC	Principles of Programming Languages	3	0	0	3
12	LC	Principles of Programming Languages Laboratory	0	0	2	1
		Total	16	3	8	**22**
			27			

Teaching Scheme
Lectures: 2 Hrs / Week
Tutorials: 1 hr / week

Examination Scheme:
Assignment/Quizzes : 40 marks
End Semester Exam : 60 marks

Course Outcomes

Students will be able to:

1. Explain formal logic and different proof techniques.
2. Recognize relation between different entities using sets, functions, and relations.
3. Use Chinese Remainder Theorem & the Euclidean algorithm for modular arithmetic.
4. Solve problems based on graphs, trees and related algorithms.
5. Relate, interpret and apply the concepts to various areas of computer science.

Course Content

Set Theory, Logic and Proofs : Propositions, Conditional Propositions, Logical Connectivity, Propositional calculus, predicates and Quantifiers, First order logic, Proofs: Proof Techniques, Mathematical Induction, Set, Combination of sets, Finite and Infinite sets, countable and Uncountable sets, Principle of inclusion and exclusion,

[8 Hrs]

Relations, Functions, Recurrence Relations: Definitions, Properties of Binary Relations, Equivalence Relations and partitions, Partial ordering relations and lattices, Chains and Anti chains. Theorem on chain, Warshall's Algorithm & transitive closure, Recurrence relations. Functions: Definition, Domain, Range, Image, etc. Types of functions: Surjection, Injection, Bijection, Inverse, Identity, Composition of Functions, Generating Function

[8 Hrs]

Number Theory: Basics of Modulo Arithmetic, Basic Prime Number Theory, GCD, LCM, Divisibility, Euclid's algorithm, Factorization, Congruences, inverse , multiplicative inverse, Chinese Remainder Theorem

[4 Hrs]

Counting: Basic Counting Techniques (sum, product, subtraction, division, exponent), Pigeonhole and Generalized Pigeonhole Principle with many examples, Permutations and Combinations and numerical problems, Binomial Coefficients Pascal's, Identity and Triangle

[6 Hrs]

Graphs & Trees: Basic terminology, multi graphs and weighted graphs, paths and circuits, shortest path Problems, Euler and Hamiltonian paths and circuits, factors of a graph, planar graph and Kuratowskis graph and theorem, independent sets, connectivity graph coloring. Trees, rooted trees, path length in rooted trees, binary search trees, spanning trees and, theorems on spanning trees, cut sets , circuits, minimum spanning trees, Kruskal's and Prim's algorithms for minimum spanning tree.

[8 Hrs]

Algebraic Systems: Algebraic Systems, Groups, Semi Groups, Monoids, Subgroups, Permutation Groups, Codes and Group codes, Isomorphism and Automorphisms, Homomorphism and Normal Subgroups, Ring, Field.

[6 Hrs]

Text Books

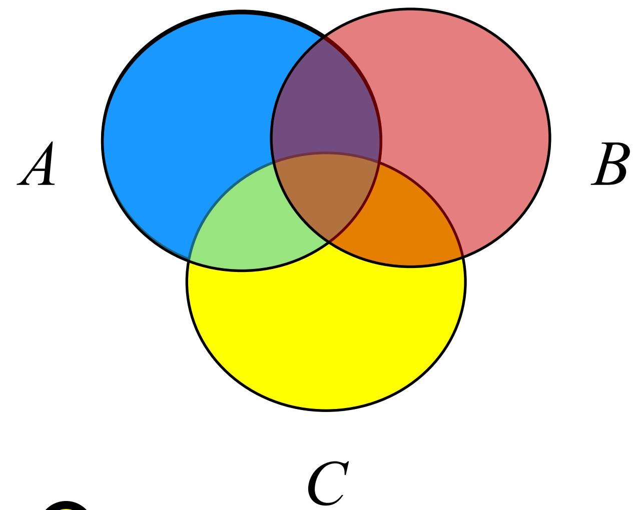
- “Discrete Mathematics and Its Applications”, Kenneth H. Rosen, 7th Edition, Tata McGraw-Hill, 2017, ISBN: 9780073383095.
- “Elements of Discrete Mathematics”, C. L. LIU, 4th Edition, Tata McGraw-Hill, 2017, ISBN-10: 1259006395 ISBN-13: 978125 9006395.

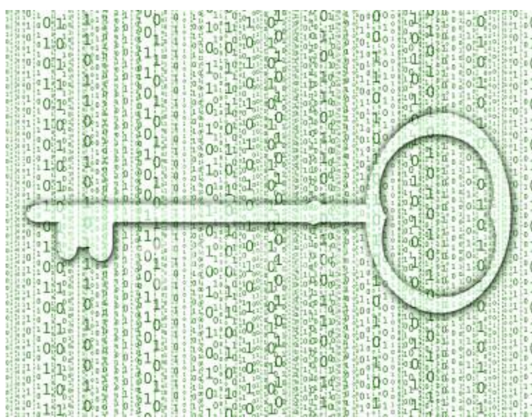
Reference Books

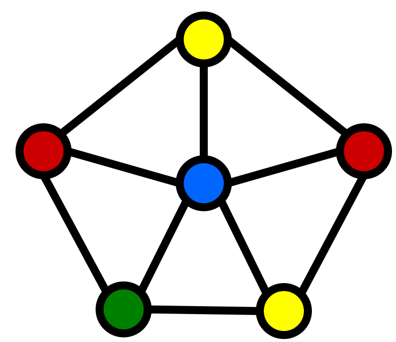
- ”Discrete Mathematical Structures”, G. Shanker Rao, 2 nd Edition2009, New Age International, ISBN-10: 8122426697, ISBN-13: 9788122426694
- “Discrete Mathematics”, Lipschutz, Lipson, 2nd Edition, 1999, Tata McGraw-Hill, ISBN: 007 463710X.
- “Graph Theory”, V. K. Balakrishnan, 1 st Edition, 2004, Tata McGraw-Hill , ISBN-10: 0- 07-058718-3, ISBN-13: 9780070587182.
- “Discrete Mathematical Structures”, B. Kolman, R. Busby and S. Ross, 4th Edition, Pearson Education, 2002, ISBN: 8178085569
- “Discrete Mathematical Structures with application to Computer Science”, J. Tremblay, R. Manohar, Tata McGraw-Hill, 2002, ISBN: 0070651426
- “Discrete Mathematics”, R. K. Bisht, H. S. Dhami, Oxford University Press, ISBN: 9780199452798

Introduction to Discrete Mathematics

$$\frac{x_1 + x_2 + \ldots + x_n}{n} \geq \sqrt[n]{x1 \cdot x_2 \cdots x_n}$$







a = qb+r $\Rightarrow$ gcd(a,b) = gcd(b,r)

Why is discrete mathematics?

Logic: artificial intelligence (AI), database, circuit design

Counting: probability, analysis of algorithm

Graph theory: computer network, data structures

Number theory: cryptography, coding theory

logic, sets, functions, relations, etc

Why is discrete mathematics?

GATE core subject

Competitive Exams

Learn Competitive Programming

It Improves:

- Mathematical thinking
- Problem solving ability
- Foundation of all subjects in computer Engineering

What are “discrete structures” ?

“***Discrete***” - Composed of distinct, separable parts. (Opposite of *continuous*.)
discrete:*continuous* :: *digital*:*analog*

“***Structures***” - Objects built up from simpler objects according to some definite pattern.

“***Discrete Mathematics***” - The study of discrete, mathematical objects and structures.

Lecture 1 Link

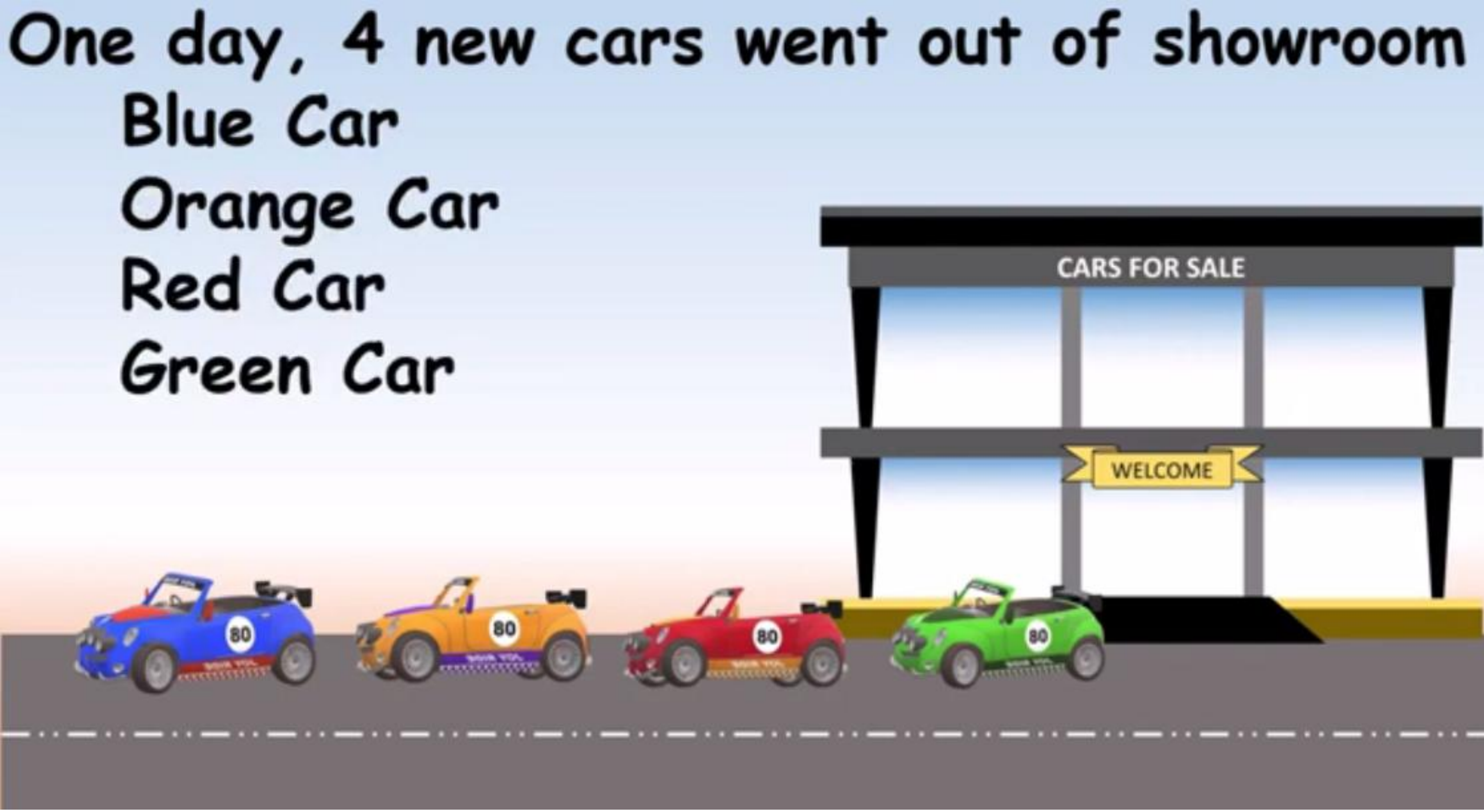
- https://web.microsoftstream.com/video/2c0044b2-bc32-4abe-bfa7-17ca741fa609

Logic, Proofs and Set Theory

https://www.youtube.com/watch?v=QmMnLxWVSGM

CAN YOU SOLVE THIS SIMPLE PUZZLE AND TELL,

WHICH CAR WAS STOLEN FROM THE SHOWROOM



3 out of the 4 cars which went out were driven by Showroom staff

But the 4th car was driven by a thief and was stolen

You have to Find out,

which car was stolen,

based on the clues,

which are:

1) Owner of the Showroom went home for Lunch in Blue car

2) Mechanic drove one car, but that was not the Green Car

3) Salesman took one car for Test Drive, but that was not Green or Orange Car

Based on these clues,

Can you tell which car

was stolen?

Lets see what is the Answer

Which car was stolen?

				
Owner	✓	✗	✗	✗
Mechanic	✗			
Salesman	✗			
Thief	✗			

1) Owner of the Showroom went home for Lunch in Blue car

This means, no one else took the Blue Car

Which car was stolen?

				
Owner	✓	✗	✗	✗
Mechanic	✗			✗
Salesman	✗			
Thief	✗			

1) Owner of the Showroom went home for Lunch in Blue car

2) Mechanic drove one car, but that was not the Green Car

This means, mechanic drove either Orange or Red Car

Which car was stolen?

				
Owner	✓	✗	✗	✗
Mechanic	✗			✗
Salesman	✗	✗	✓	✗
Thief	✗			

1) Owner of the Showroom went home for Lunch in Blue car

2) Mechanic drove one car, but that was not the Green Car

3) Salesman took one car for Test Drive, but that was not Green or Orange Car

This means, salesman drove the Red Car

Which car was stolen?

				
Owner	✓	✗	✗	✗
Mechanic	✗		✗	✗
Salesman	✗	✗	✓	✗
Thief	✗		✗	

1) Owner of the Showroom went home for Lunch in Blue car
2) Mechanic drove one car, but that was not the Green Car
3) Salesman took one car for Test Drive, but that was not Green or Orange Car

And no one else drove the red car

Which car was stolen?

				
Owner	✓	✗	✗	✗
Mechanic	✗	✓	✗	✗
Salesman	✗	✗	✓	✗
Thief	✗		✗	

1) Owner of the Showroom went home for Lunch in Blue car

2) Mechanic drove one car, but that was not the Green Car

3) Salesman took one car for Test Drive, but that was not Green or Orange Car

Which means, mechanic drove the Orange car

Which car was stolen?

				
Owner	✓	✗	✗	✗
Mechanic	✗	✓	✗	✗
Salesman	✗	✗	✓	✗
Thief	✗	✗	✗	

1) Owner of the Showroom went home for Lunch in Blue car

2) Mechanic drove one car, but that was not the Green Car

3) Salesman took one car for Test Drive, but that was not Green or Orange Car

And the Thief Stole the Green Car

Which car was stolen?

				
Owner	✓	✗	✗	✗
Mechanic	✗	✓	✗	✗
Salesman	✗	✗	✓	✗
Thief	✗	✗	✗	✓

GREEN CAR WAS STOLEN

Statements/ Proposition

- Proposition or Statement or An Assertion
- Primary (Primitive, atomic) statements
- Set of Declarative sentences which cannot be further broken down into simpler sentences.
- Those who have one and only one of two possible values called “Truth Values”.
- True and False or T and F or 1 and 0
- Two-valued logic
- Some statements can be assertion but not the propositions
    - Ex. “This statement is false”

Statement (Proposition)

A *Statement* is a sentence that is either **True** or **False**

Examples:

$2 + 2 = 4$ True

$3 \times 3 = 8$ False

787009911 is a prime

Non-examples:

$x+y>0$

$x^2+y^2=z^2$

They are true for some values of x and y
but are false for some other values of x and y.

The Statement/Proposition Game

- “Elephants are bigger than ant.”

Is this a proposition? **yes**

What is the truth value of the proposition? **true**

The Statement/Proposition Game

- “$520 < 111$”

Is this a proposition? **yes**

What is the truth value of the proposition? **false**

The Statement/Proposition Game

- “Please do not fall asleep.”

Is this a statement? no

It’s a request.

Is this a proposition? no

Only statements can be propositions.

Examples of statements/ Propositions

All the following declarative sentences are propositions.

1. Washington, D.C., is the capital of the United States of America.
2. Toronto is the capital of Canada.
3. $1 + 1 = 2$.
4. $2 + 2 = 3$.

Propositions 1 and 3 are true, whereas 2 and 4 are false.

Examples

- Consider the following sentences.

1. What time is it?
2. Read this carefully.
3. $x + 1 = 2$.
4. $x + y = z$.

Sentences 1 and 2 are not propositions because they are not declarative sentences.

Sentences 3 and 4 are not propositions because they are neither true nor false.

Note that each of sentences 3 and 4 can be turned into a proposition if we assign values to the variables

Class Assignement

- Which of these sentences are propositions? What are the truth values of those that are propositions?

a) Boston is the capital of Massachusetts.
b) Miami is the capital of Florida.
c) $2 + 3 = 5$.
d) $5 + 7 = 10$.
e) $x + 2 = 11$.
f) Answer this question.
g) Do not pass go.
h) What time is it?
i) There are no black flies in Maine.
j) $4 + x = 5$.
k) The moon is made of green cheese.
l) $2n \geq 100$

Class Assignement

- Which of these sentences are propositions? What are the truth values of those that are propositions?

a) Boston is the capital of Massachusetts. T
b) Miami is the capital of Florida.
c) $2 + 3 = 5$. T
d) $5 + 7 = 10$. F
e) $x + 2 = 11$.
f) Answer this question.
g) Do not pass go.
h) What time is it?
i) $4 + x = 5$.
j) The moon is made of green cheese. F
k) $2n \geq 100$

Lecture 2

- https://web.microsoftstream.com/video/2d775ffa-9508-4141-a3af-08b35c8d8073

Operators/ Connectives

- An *operator* or *connective* combines one or more *operand* expressions into a larger expression.
- Two types of declarative sentences
- First is Primitive or primary or atomic statement
- Denoted by letters A,B,C…..P,Q,R…or a,b,c,…p,q,r…

  P: London is capital of India.

  A:Ram is poor.
- Second types are obtained from primitives using **connectives and parenthesis,** Called molecular or compound statements
- Like statements connective also denoted by symbol

Examples

e.g.

1. India is country and Mumbai is capital of India.

    P:India is country

    Q:Mumbai is capital of India.

    P and Q P ^ Q

2. Ram is poor but he is clever.

    A: Ram is poor.

    B: Ram is clever.

    A and B

Connectives

1. Negation (Not)
2. Conjunction (and)
3. Disjunction (or)
4. Conditional (if…then) /implication
5. Bi-conditional (if and only if)

Connectives' Symbols

Formal Name	**Nickname**	**Property**	**Symbol**
Negation operator	NOT	Unary	$\neg$
Conjunction operator	AND	Binary	$\wedge$
Disjunction operator	OR	Binary	$\vee$
Exclusive-OR operator	XOR	Binary	$\oplus$
Implication operator	IMPLIES	Binary	$\rightarrow$
Biconditional operator	IFF	Binary	$\leftrightarrow$

Lecture 3

- https://web.microsoftstream.com/video/eaf401a0-8259-4d61-af55-87efd46b1b92
- https://web.microsoftstream.com/video/6ef37cbd-170d-440c-ac61-cfe331ac5816

Discrete Structures and Graph Theory

Connectives

1. Negation (Not)
2. Conjunction (and)
3. Disjunction (or)
4. Conditional (if…then) /implication
5. Bi-conditional (if and only if)

Negation (NOT)

- Statements Formed by introducing “not” word
- “P” is Statement then negation of p is written as “not p“ or It is not case that P.
- $\neg$ p
- Unary Connective
- If P is true then $\neg$ p is false and vice versa.

P	$\neg$ **P**
T	F
F	T

P:London is a city.

Then

┐p: London is not a city.

OR

┐p: It is not the case that London is a city.

Q: I went to my class yesterday

Then

┐Q:I did not go to my class yesterday

Conjunction (and)

- **Statements Formed by introducing "and" word**
- **Binary Connective**
- **Used to combine two or more statements.**
- **Denote by $\wedge$**
- If both the statements are true then $p \wedge Q$ is true otherwise false.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P:London is a capital of India.

Q: India is country.

London is a capital of India and India is country.

$P \wedge Q$

Disjunction (OR)

- Statements Formed by introducing “OR” word
- Binary Connective
- Denote by $\vee$
- If one statement is true then $p \vee Q$ is true otherwise false.

P	Q	P∨Q
T	T	T
T	F	T
F	T	T
F	F	F

P:London is a capital of India.

Q: India is country.

London is a capital of India or India is country.

$$P \vee Q$$

Conditional (if..then)

- Statements Formed by introducing "if…then" word
- Binary Connective
- Denote by $\rightarrow$
- If First statement is true and second statement is false then $p \rightarrow Q$ is false otherwise true.

P	Q	P→Q
T	T	T
T	F	F
F	T	T
F	F	T

“If elephants were red, then they could hide in cherry trees.”.

$$P \rightarrow Q$$

P is known as Antecedent

Q is known as consequent

For Q → P, vice versa

Implication

- If you study regularly you then you will get grade 'A'

Case 1 : You did regular study , you got A grade.

$(P \rightarrow Q)$: True

Case 2: You did regular study ,by chance you didn't get grade A. $(P \rightarrow Q)$: False

Case 3: You didn't study regularly, you may get grade A. $(P \rightarrow Q)$: True

Case 4: You didn't study regularly, you didn't get grade A. $(P \rightarrow Q)$: True

Some reading for P->Q

- "*p* implies *q*"
- "if *p*, then *q*"
- "if *p*, *q*"
- "when *p*, *q*"
- "whenever *p*, *q*"
- "*q* if *p*"
- "*q* when *p*"
- "*q* whenever *p*"
- "*p* only if *q*"
- "*p* is sufficient for *q*"
- "*q* is necessary for *p*"
- "*q* follows from *p*"
- "*q* is implied by *p*"

We will see some equivalent logic expressions later.

Bi-conditional (if and only if)

- **Statements Formed by introducing "if and only if " word**
- **Binary Connective**
- **Denote by $\leftrightarrow$**
- If both the statement has same truth value then $p \leftrightarrow Q$ is true otherwise false.

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

"$x < y$ if and only if $y > x$."

$$P \leftrightarrow Q$$

EX-OR (Either-Or)

- Statement formed by “Either Or” word.
- Exclusive Or
- P x Q proposition will be true, if exactly one of two propositions of both is true. Otherwise false

P	Q	$P \oplus Q$
T	T	F
T	F	T
F	T	T
F	F	F

Inclusive or OR Exclusive or

- In order to get a job in this multinational company , experience with C++ **or** Java is mandatory.

Inclusive or OR Exclusive or

- In order to get a job in this multinational company , experience with C++ **or** Java is mandatory.

Inclusive OR

Disjunction

Inclusive or OR Exclusive or

- “When you buy a mobile of xyz company, you get Rs.500 cashback or a mobile cover of worth Rs.500.”

Inclusive or OR Exclusive or

- “When you buy a mobile of xyz company, you get Rs.500 cashback **or** a mobile cover of worth Rs.500.”

Exclusive OR

Statement Formula and Truth Table

- Atomic statements/proposition
- Compound statements/proposition

$\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$, $\neg(P \wedge Q)$, $\neg(P \wedge Q)$

- Statement formula
- Truth Table
- 2^n where n is number of distinct statement variable
- $P \wedge \neg P$

  2 rows, n=1, 2^1

- $(P \wedge Q)$

  4 rows, n=2, 2^2

- Statements and operators (Connectives and parenthesis) can be combined in any way to form new statements.
- $(\neg P) \vee (\neg Q)$

P	Q			
T	T			
T	F			
F	T			
F	F			

- Statements and operators can be combined in any way to form new statements.
- $(\neg P) \vee (\neg Q)$

P	Q	$\neg P$		
T	T	F		
T	F	F		
F	T	T		
F	F	T		

- Statements and operators can be combined in any way to form new statements.
- $(\neg P)\vee(\neg Q)$

P	Q	$\neg P$	$\neg Q$	
T	T	F	F	
T	F	F	T	
F	T	T	F	
F	F	T	T	

- Statements and operators can be combined in any way to form new statements.
- $(\neg P)\vee(\neg Q)$

P	Q	$\neg P$	$\neg Q$	$(\neg P)\vee(\neg Q)$
T	T	F	F	
T	F	F	T	
F	T	T	F	
F	F	T	T	

- Statements and operators can be combined in any way to form new statements.
- $(\neg P)\vee(\neg Q)$

P	Q	$\neg P$	$\neg Q$	$(\neg P)\vee(\neg Q)$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

$$\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$$

P	Q		
T	T		
T	F		
F	T		
F	F		

$$\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$$

P	Q	$\neg P$	$\neg Q$
T	T	F	F
T	F	F	T
F	T	T	F
F	F	T	T

$$\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$$

P	Q	¬P	¬Q
T	T	F	F
T	F	F	T
F	T	T	F
F	F	T	T

(P∧Q)			
T			
T			
T			
F			

$$\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$$

P	Q	$\neg P$	$\neg Q$
T	T	F	F
T	F	F	T
F	T	T	F
F	F	T	T

$(P \wedge Q)$	$\neg(P \wedge Q)$		
T	F		
T	F		
T	F		
F	T		

$\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$

P	Q	$\neg P$	$\neg Q$
T	T	F	F
T	F	F	T
F	T	T	F
F	F	T	T

$(P \wedge Q)$	$\neg(P \wedge Q)$	$(\neg P) \vee (\neg Q)$	
T	F	F	
T	F	T	
T	F	T	
F	T	T	

$$\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$$

P	Q	$\neg P$	$\neg Q$
T	T	F	F
T	F	F	T
F	T	T	F
F	F	T	T

$(P \wedge Q)$	$\neg(P \wedge Q)$	$(\neg P) \vee (\neg Q)$	$\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$
T	F	F	T
F	T	T	T
F	T	T	T
F	T	T	T

$\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$

P	Q	$\neg P$	$\neg Q$
T	T	F	F
T	F	F	T
F	T	T	F
F	F	T	T

$(P \wedge Q)$	$\neg(P \wedge Q)$	$(\neg P) \vee (\neg Q)$	$\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$
T	F	F	T
F	T	T	T
F	T	T	T
F	T	T	T

Example

- Using the statements:

  R:Mark is Rich.

  H:Mark is happy

- Write the following statements in **symbolic form**:
- *(a) Mark is poor but happy.*

  $\neg R \wedge H$

- *(b) Mark is rich or unhappy;*

  $R \vee \neg H$

- *(c) Mark is neither rich nor happy.*

  $\neg R \wedge \neg H$

- *(d) Mark is poor or he is both rich and unhappy.*

  $\neg R \vee (R \wedge \neg H)$

Example

- Let *p be "It is cold"* and let *q be "It is raining". Give a simple* ***verbal sentence*** *which describes each of the following statements:*
- (a) $\neg p$; (b) $p \wedge q$; (c) $p \vee q$; (d) $q \vee \neg p$.
- (a) $\neg p$;

  It is not cold.
- (b) $p \wedge q$;

  It is cold and raining.
- (c) $p \vee q$;

  It is cold or it is raining
- (d) $q \vee \neg p$.

  It is raining or it is not cold.

Example 1.17 There are two restaurants next to each other. One has a sign that says, "Good food is not cheap," and the other has a sign that says, "Cheap food is not good." Are the signs saying the same thing?

Using the statements:

P:Food is good.

H:Food is cheap.

Good food is not cheap.

$P \rightarrow \neg H$

Cheap food is not good.

$H \rightarrow \neg P$

$H \rightarrow \neg P$

P	H	$\neg P$	$\neg H$	$P \to \neg H$	$H \to \neg P$
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

WFF (well formed formula)

- Now consider the proposition : $P \vee \sim Q \rightarrow P \wedge R$

Trying to construct a truth table for this is quite confusing. Which is to be assumed?

$$(P \vee \sim Q) \rightarrow (P \wedge R) \text{ or } P \vee (\sim Q \rightarrow P) \wedge R$$

Which part is calculated first?

for such cases we have order of precendence for these operators.

WFF (well formed formula)

- A statement formula is said to be WFF if it has :

1. Every Atomic statement is wff
2. If P is wff then ~ p is also wff
3. If P and Q are wff then $(P \wedge Q)$, $(P \vee Q)$, and $(P \rightarrow Q)$ are wff
4. Nothing else is wff

For example: $((P \wedge Q) \vee R)$ is wff w, where as $P \vee Q \wedge R$ is not a wff

Precedence of the operators

- $\sim$
- $\wedge$
- V, $\oplus$
- $\rightarrow$
- $\longleftrightarrow$

For example ,

$\sim P \wedge Q \rightarrow R \vee Q$ is not a wff,

can be converted to wff by using rules of precedence as $((\sim P) \wedge Q) \rightarrow (R \vee Q))$

Equivalent Statements

- If truth values of statement formula/proposition A is equal to the truth values of statement formula/proposition B for every possible truth values then A and B are logically equivalent to each other.

P	Q	$\neg P$	$\neg Q$	$\neg(P \wedge Q)$	$(\neg P) \vee (\neg Q)$
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

Denoted by symbol $\Leftrightarrow$

- *Let P be "Roses are red" and Q be "Violets are blue." Let S be the statement:*

"It is not true that roses are red and violets are blue."

- *Then S can be written in the form* $\neg(p \wedge q)$.
- *Accordingly, S* has the same meaning as the statement:

"Roses are not red, or violets are not blue."

Then S can be written in the form $\neg p \vee \neg q$.

However, as noted above, $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$.

Equivalent Statements

- The statements $\neg(P\wedge Q)$ and $(\neg P) \vee (\neg Q)$ are logically equivalent, since $\neg(P\wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$ is always true.

P	Q	$\neg P$	$\neg Q$	$\neg(P\wedge Q)$	$(\neg P)\vee(\neg Q)$	$\neg(P\wedge Q)\leftrightarrow(\neg P)\vee(\neg Q)$
T	T	F	F	F	F	T
T	F	F	T	T	T	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Convert the following English statements in symbolic form.

- You can access the internet from campus if you are computer science major or you are not a freshman.

Solution: P: You can access the internet from campus.

Q: you are computer major.

R: you are a freshman.

$P \rightarrow (Q \vee \neg R)$

- You can ride on roller coaster if you are under 4 feet tall unless you are older than 16 years old.

Solution :

P: You can ride on roller coaster

Q: You are under 4 feet

R: You are older than 16 years old.

$(Q \vee \neg R) \rightarrow P$

Logical Equivalence

The easiest way to check for logical equivalence is to see if the truth tables of both variants have *identical last columns*:

p	q	$p \rightarrow q$

p	q	$\neg q$	$\neg p$	$\neg q \rightarrow \neg p$

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T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

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T	F	**F**
F	T	**T**
F	F	**T**

p	q	$\neg q$	$\neg p$	$\neg q \rightarrow \neg p$
T	T			
T	F			
F	T			
F	F			

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F	T	**T**
F	F	**T**

p	q	$\neg q$	$\neg p$	$\neg q \rightarrow \neg p$
T	T	F		
T	F	T		
F	T	F		
F	F	T		

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F	T	**T**
F	F	**T**

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T	T	F	F	
T	F	T	F	
F	T	F	T	
F	F	T	T	

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T	F	**F**
F	T	**T**
F	F	**T**

p	q	$\neg q$	$\neg p$	$\neg q \rightarrow \neg p$
T	T	F	F	**T**
T	F	T	F	**F**
F	T	F	T	**T**
F	F	T	T	**T**

Exercises

- Prove that:

1) $(P \rightarrow Q) \Leftrightarrow \neg P \vee Q$

2) $P \rightarrow \mathbf{(}Q \rightarrow \mathbf{R)} \Leftrightarrow (P \wedge Q) \rightarrow \mathbf{R}$.

Tautologies and Contradictions

- Some propositions *P contain only T in the last column of their truth tables or, in other words, they* are true for any truth values of their variables. Such propositions are called *tautologies.* A tautology is a statement that is always true.

Examples:

- $R \vee (\neg R)$

$\forall$ $\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$

- If $S \rightarrow T$ is a tautology, we write $S \Rightarrow T$.
- If $S \leftrightarrow T$ is a tautology, we write $S \Leftrightarrow T$.

$$\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$$

P	Q	$\neg P$	$\neg Q$
T	T	F	F
T	F	F	T
F	T	T	F
F	F	T	T

$(P \wedge Q)$	$\neg(P \wedge Q)$	$(\neg P) \vee (\neg Q)$	$\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$
T	F	F	T
F	T	T	T
F	T	T	T
F	T	T	T

Tautology by truth table

p	q	$\neg p$	$p \vee q$	$\neg p \wedge (p \vee q)$	$[\neg p \wedge (p \vee q)] \rightarrow q$
T	T				
T	F				
F	T				
F	F				

Tautology by truth table

p	q	$\neg p$	$p \vee q$	$\neg p \wedge (p \vee q)$	$[\neg p \wedge (p \vee q)] \rightarrow q$
T	T	F			
T	F	F			
F	T	T			
F	F	T			

Tautology by truth table

p	q	$\neg p$	$p \vee q$	$\neg p \wedge (p \vee q)$	$[\neg p \wedge (p \vee q)] \rightarrow q$
T	T	F	T		
T	F	F	T		
F	T	T	T		
F	F	T	F		

Tautology by truth table

p	q	$\neg p$	$p \vee q$	$\neg p \wedge (p \vee q)$	$[\neg p \wedge (p \vee q)] \rightarrow q$
T	T	F	T	F	
T	F	F	T	F	
F	T	T	T	T	
F	F	T	F	F	

Tautology by truth table

p	q	$\neg p$	$p \vee q$	$\neg p \wedge (p \vee q)$	$[\neg p \wedge (p \vee q)] \rightarrow q$
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

Tautologies and Contradictions

- a proposition *P is called a contradiction if it contains only F in the last column of its truth table* or, in other words, if it is false for any truth values of its variables.
- *A* contradiction is a statement that is always false.

Examples:

- $R \wedge (\neg R)$

∀ $\neg(\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q))$

- The negation of any tautology is a contradiction, and the negation of any contradiction is a tautology.

- Two way to finding the Equivalences, Tautology and Contradiction
- Truth Table
- Without Truth Table Using Substitution (by formulas)

P	Q	$\neg P$	$\neg Q$	$\neg P \vee Q$	$P \rightarrow Q$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Logical Equivalences

- Identity Laws: $p \wedge \mathbf{T} \Leftrightarrow p$ and $p \vee \mathbf{F} \Leftrightarrow p$.
- Domination Laws: $p \vee \mathbf{T} \Leftrightarrow \mathbf{T}$ and $p \wedge \mathbf{F} \Leftrightarrow \mathbf{F}$.
- Idempotent Laws: $p \wedge p \Leftrightarrow p$ and $p \vee p \Leftrightarrow p$.
- Double Negation Law: $\neg(\neg p) \Leftrightarrow p$.
- Commutative Laws:
    - $(p \vee q) \Leftrightarrow (q \vee p)$ and $(p \wedge q) \Leftrightarrow (q \wedge p)$.
- Associative Laws: $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$
    - and $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$.

Logical Equivalences

- Distributive Laws:
  - $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ and
  - $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$.
- DeMorgan's Laws:
  - $\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$ and
  - $\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$.
- Absorption Laws:
  - $p \vee (p \wedge q) \Leftrightarrow p$ and $p \wedge (p \vee q) \Leftrightarrow p$.
- Negation Laws: $p \vee \neg p \Leftrightarrow \mathbf{T}$ and $p \wedge \neg p \Leftrightarrow \mathbf{F}$.

Logical Equivalences for Implication

$$p \to q \equiv \neg p \vee q$$

$$p \to q \equiv \neg q \to \neg p$$

$$p \vee q \equiv \neg p \to q$$

$$p \wedge q \equiv \neg(p \to \neg q)$$

$$\neg(p \to q) \equiv p \wedge \neg q$$

$$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$$

$$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$$

$$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$$

$$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$$

Logical Equivalences for Double Implication

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$

$$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$

$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

Substitution instance

- A formula A is called substitution instance of formula B if A can be obtained from B by substituting formulas for some variable of B.

Examples:

- B: $P \rightarrow (J \wedge P)$
- If P be $R \leftrightarrow S$
- A: $(R \leftrightarrow S) \rightarrow (J \wedge (R \leftrightarrow S))$
- **As like we can substitute the formula with another formula if both have same truth values**
- $(R \rightarrow S) \wedge (R \leftrightarrow S)$
- $(\neg R \vee S) \wedge (R \leftrightarrow S)$
- Equivalent formula can be substitute for each other.

- Prove that $P \rightarrow (Q \rightarrow \mathbf{R}) \Leftrightarrow (P \wedge Q) \rightarrow \mathbf{R}$.
- $P \rightarrow (Q \rightarrow \mathbf{R}) \Leftrightarrow P \rightarrow (\neg Q \vee R)$ **…..implication law**

$\Leftrightarrow \neg P \vee (\neg Q \vee R)$ **..implication law**

$\Leftrightarrow (\neg P \vee \neg Q) \vee R$ …Associative **law**

$\Leftrightarrow \neg (P \wedge Q) \vee R$ …Associative **law**

$\Leftrightarrow (P \wedge Q) \rightarrow \mathbf{R}$.

Prove: $(p \wedge \neg q) \vee q \Leftrightarrow p \vee q$

$(p \wedge \neg q) \vee q$	Left-Hand Statement
$\Leftrightarrow q \vee (p \wedge \neg q)$	Commutative
$\Leftrightarrow (q \vee p) \wedge (q \vee \neg q)$	Distributive
$\Leftrightarrow (q \vee p) \wedge T$	Or Tautology
$\Leftrightarrow q \vee p$	Identity
$\Leftrightarrow p \vee q$	Commutative

Prove: $(p \wedge \neg q) \vee q \Leftrightarrow p \vee q$

$(p \wedge \neg q) \vee q$ Left-Hand Statement

$\Leftrightarrow q \vee (p \wedge \neg q)$ Commutative

$\Leftrightarrow (q \vee p) \wedge (q \vee \neg q)$ Distributive

Why did we need this step?

Prove: $p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$

Contrapositive

$p \rightarrow q$

$\Leftrightarrow \neg p \vee q$ Implication Equivalence

$\Leftrightarrow q \vee \neg p$ Commutative

$\Leftrightarrow \neg(\neg q) \vee \neg p$ Double Negation

$\Leftrightarrow \neg q \rightarrow \neg p$ Implication Equivalence

If $p \rightarrow q$ is a statement then $q \rightarrow p$ is called converse.

$\neg p \rightarrow \neg q$ is inverse and

$\neg q \rightarrow \neg p$ is contrapositive.

Prove: $p \to p \vee q$ is a tautology

Must show that the statement is true for any value of p,q.

$p \to p \vee q$	
$\Leftrightarrow \neg p \vee (p \vee q)$	Implication Equivalence
$\Leftrightarrow (\neg p \vee p) \vee q$	Associative
$\Leftrightarrow (p \vee \neg p) \vee q$	Commutative
$\Leftrightarrow T \vee q$	Or Tautology
$\Leftrightarrow q \vee T$	Commutative
$\Leftrightarrow T$	Domination

This tautology is called the addition rule of inference.

Predicates & Quantifiers

Universal and Existential

Predicate Logic

- A predicate is an expression of one or more variables defined on some specific **domain**. A predicate with variables can be made a proposition by either assigning a value to the variable or by quantifying the variable.
- The following are some examples of predicates −
  - Let E(x, y) denote "x = y"
  - Let X(a, b, c) denote "a + b + c = 0"
  - Let M(x, y) denote "x is married to y“
  - Let P(x) denote “x is greater than 3”
    - In last statement first part variable x ,is the subject of the statement, the second part is predicate “is greater than 3”, P(x) is a propositional function P at x.

Example

- Let P(x) is x>3 what are the truth values for P(2) and P(4)? Unary
- Let Q(x,y) denote “x=y+3” what are the truth values for Q(1,2) and Q(3,0)? Binary
- Let R(x,y,x) denote “x+y=z” what are the truth values for R(1,2,3) & R(0,0,1)?
- Similarly for P(x1,x2,…xn) can be a value for n tuple, and P is also known as Predicate. N-ary predicate

Example

- Let P(x; y; z) denote that x + y = z and U (Universe of Discourse) be the integers for all three variables.
  - P(-4; 6; 2) is true.
  - P(5; 2; 10) is false.
  - P(5; x; 7) is not a proposition.

Quantifiers

- We need quantifiers to formally express the meaning of the words "all" and "some".
- The two most important quantifiers are:
  - Universal quantifier, "For all". Symbol: $\forall$
  - Existential quantifier, "There exists". Symbol: $\exists$
- $\forall x\, P(x)$ asserts that $P(x)$ is true for **every x in the domain.**
- $\exists x\, P(x)$ asserts that $P(x)$ is true for **some x in the domain.**
- The quantifiers are said to bind the variable x in these expressions.
- Variables in the scope of some quantifier are called **bound variables**. All other variables in the expression are called **free variables.**
- A propositional function that does not contain any free variables is a proposition and has a truth value.

Quantifiers

- The variable of predicates is quantified by quantifiers. There are two types of quantifier in predicate logic −
  - Universal Quantifier and
  - Existential Quantifier.

Universal Quantifier

- Universal quantifier states that the statements within its scope are true for every value of the specific variable. It is denoted by the symbol $\forall$
- $\forall x P(x)$ is read as for every value of x, P(x) is true.
- **Example** −
  - "Man is mortal" can be transformed into the propositional form $\forall x P(x)$
  - where P(x) is the predicate which denotes x is mortal and the universe of discourse is all men.

Existential Quantifier

- Existential quantifier states that the statements within its scope are true for some values of the specific variable. It is denoted by the symbol $\exists$
- $\exists x P(x)$ is read as for some values of x, P(x) is true.
  - **Example** – "Some people are dishonest" can be transformed into the propositional form $\exists x P(x)$
  - where P(x) is the predicate which denotes x is dishonest and the universe of discourse is some people.

Uniqueness Quantifier

- $\exists\, !\, x\, P(x)$ means that there exists one and only one x in the domain such that P(x) is true.
- $\exists_1 !\, x\, P(x)$ is an alternative notation for $\exists\, !x\, P(x)$.
- This is read as

    There is one and only one x such that P(x).

    There exists a unique x such that P(x).
- **Example: Let P(x) denote x + 1 = 0 and U are the integers.**
    - Then $\exists\, !x\, P(x)$ is true.
- **Example: Let P(x) denote x > 0 and U are the integers.**
    - **Then** $\exists !x\, P(x)$ is false.
- The uniqueness quantifier can be expressed by standard operations. $\exists !x\, P(x)$ is equivalent to

$$\exists x\, (P(x) \land \forall y\, (P(y) \rightarrow y = x)).$$

- Quantifiers ∀ and ∃ have higher precedence then all logical operators.
- $\forall x\, P(x) \wedge Q(x)$ means $(\forall x\, P(x)) \wedge Q(x)$ In particular, this expression contains a free variable.
- $\forall x\, (P(x) \wedge Q(x))$ means something different.

Example

- Translate the following sentence into predicate logic: "Every student in this class has taken a course in Java."
- Solution:
  - First decide on the domain U (Universe of discourse).
  - Solution 1: If U is all students in this class, define a propositional function J(x) denoting "x has taken a course in Java" and translate as $\forall x\, J(x)$.
  - Solution 2: But if U is all people, also define a propositional function S(x) denoting "x is a student in this class" and translate as $\forall x\, (S(x) \rightarrow J(x))$
- Note: $\forall x\, (S(x) \wedge J(x))$ is not correct. What does it mean?

- Some student in this class has visited Mexico
  - means that
  - "There is a student in this class with the property that the student has visited Mexico."
  - We can introduce a variable x, so that our statement becomes
  - "There is a student x in this class having the property that x has visited Mexico."
  - $M(x)$, which is the statement "x has visited Mexico
  - If the domain for x consists
  - of the students in this class, we can translate this first statement as $\exists x M(x)$.
  - if we are interested in people other than those in this class,
  - "There is a person x having the properties that x is a student in this class and x has visited Mexico."
  - $S(x)$ to represent "x is a student in this class."
  - Solution: $\exists x(S(x) \wedge M(x))$

- “Every student in this class has visited either Canada or Mexico”
  - $C(x)$ be “x has visited Canada.”
  - domain for x consists of
  - the students in this class, this second statement can be expressed as $\forall x(C(x) \vee M(x))$.
  - if the domain for x consists of all people
  - “For every person x, if x is a student in this class, then x has visited Mexico or x has visited Canada.”
  - $\forall x(S(x) \rightarrow (C(x) \vee M(x)))$.

Predicates and Quantifiers

Puzzle

Brown, Jones and Smith are suspected of income tax evasion. They testify under oath as follows:

Brown: Jones is guilty and Smith is innocent.

Jones: If Brown is guilty, then so is Smith.

Smith: I am innocent but at least one of the others is guilty.

Assume,

Brown

innocent   guilty

B	In	×
J	gui	
S	I	
	×	

B	G	G	G	G
J	I	I	G	G
S	I	G	I	G
	×	✓	×	×

$$\Downarrow$$

$$\neg(\neg J \wedge S)$$

$$J \vee \neg S$$

$B \rightarrow$ guilty

$J \rightarrow$ innocent

$S \rightarrow$ guilty.

Real use

- An important type of programming language is designed to reason using the rules of predicate logic. Prolog (from *Pro*gramming in *Log*ic), developed in the 1970s by computer scientists working in the area of artificial intelligence, is an example of such a language. Prolog programs include a set of declarations consisting of two types of statements, **Prolog facts** and **Prolog rules**.
- Prolog facts define predicates by specifying the elements that satisfy these predicates.
- Prolog rules are used to define new predicates using those already defined by Prolog facts.

Quantifiers as Conjunctions/Disjunctions

- If the domain is finite then universal/existential quantifiers can be expressed by conjunctions/ disjunctions.
- If U consists of the integers 1,2, and 3, then

$$\forall x\ P(x) \equiv P(1) \wedge P(2) \wedge P(3)$$
$$\exists x\ P(x) \equiv P(1) \vee P(2) \vee P(3)$$

- Even if the domains are infinite, you can still think of the quantifiers in this fashion, but the equivalent expressions without quantifiers will be infinitely long.

Negation for Quantifiers

- The rules for negating quantifiers are:
- We can say, De Morgan’s Law for Quantifiers

$$\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$$
$$\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$$

Negating Quantifiers

- Consider the quantified statement:
  - "Every student has at least one course where the lecturer is a teaching assistant."
  - Its negation is the statement:
  - "There is a student such that in every course the lecturer is not a teaching assistant."

Negate each of the following statements

(*a) All students live in the dormitories.*

(b) All mathematics majors are males.

(*c) Some students are 25 years old or older.*

solution

(*a) At least one student does not live in the dormitories. (Some students do not live in the dormitories.)*

(*b) At least one mathematics major is female. (Some mathematics majors are female.)*

(*c) None of the students is 25 years old or older. (All the students are under 25.)*

Negate each of the following statements:

(a) $\exists x \ \forall y, p(x, y);$

(b) $\exists x \ \forall y, p(x, y);$

(c) $\exists y \ \exists x \ \forall z, p(x, y, z).$

Use $\neg \forall x \, p(x) \equiv \exists x \neg p(x)$ *and* $\neg \exists x \, p(x) \equiv \forall x \neg p(x);$

Solution

(a) $\neg(\exists x \forall y, p(x, y)) \equiv \forall x \exists y \neg p(x, y)$

(b) $\neg(\forall x \forall y, p(x, y)) \equiv \exists x \exists y \neg p(x, y)$

(c) $\neg(\exists y \ \exists x \ \forall z, p(x, y, z)) \equiv \forall y \ \forall x \ \exists z \neg p(x, y, z)$

- Express the statement "Every student in this class has studied calculus" using predicates and quantifiers.
  - rewrite the statement
  - "For every student in this class, that student has studied calculus."
  - "For every student *x* in this class, *x* has studied calculus."

    C(x): "x has studied calculus."
  - domain for *x* consists of the students in the class
  - we can translate our statement as $\forall x C(x)$
  - If we change the domain to consist of all people
  - "For every person *x*, if person *x* is a student in this class then *x* has studied calculus."

    S(x): person *x* is in this class

    $\forall x(S(x) \rightarrow C(x))$.
  - Our statement cannot be expressed as $\forall x(S(x) \land C(x))$ because this statement says that all people are students in this class and have studied calculus!
  - *As this property,* $P \longrightarrow Q \equiv \sim P \vee Q$

- Express the statements “Some student in this class has visited Mexico” and “Every student in this class has visited either Canada or Mexico” using predicates and quantifiers
  - “There is a student in this class with the property that the student has visited Mexico.”
  - “There is a student *x* in this class having the property that *x* has visited Mexico.”
  - $M(x)$: *x* has visited Mexico
- domain for x consists of the students in this class, then $\exists x M(x)$.
  - Domain: all people.
  - “There is a person x having the properties that x is a student in this class and x has visited Mexico.”
  - $S(x)$:“*x* is a student in this class.”
  - Now, $\exists x(S(x) \land M(x))$

  Means: there is a person *x* who is a student in this class and who has visited Mexico.

- *Our statement cannot be expressed as $\exists x(S(x) \rightarrow M(x))$, which is true when there is someone not in the class because, in that case, for such a person x, $S(x) \rightarrow M(x)$ becomes either $F \rightarrow T$ or $F \rightarrow F$, both of which are true.*
- *Statement becomes,*
- *"For every x in this class, x has the property that x has visited Mexico or x has visited Canada."*

Example to transfer from English to Logical

- Consider these statements. The first two are premises and the third is the conclusion.
  - "All lions are fierce."
  - "Some lions do not drink coffee."
  - "Some fierce creatures do not drink coffee."
- Solution
  - Let P(x),Q(x) and R(x) be the statements "x is a lion", "x is fierce" and "x drinks coffee." respectively. Let the domain consists of all creatures. Now the statements are:
  - $\forall x (P(x) \rightarrow Q(x))$.
  - $\exists x (P(x) \land \neg R(x))$.
  - $\exists x (Q(x) \land \neg R(x))$.
- Not okay:
  - $\exists x (P(x) \rightarrow \neg R(x))$ here ,if creature is not lion then also they drink coffee.
  - $\exists x (Q(x) \rightarrow \neg R(x))$
- Not exact -- both are true even if P(x) and Q(x) both are not true!

- Consider these statements. The first three are premises and the fourth is a valid conclusion.
  - "All hummingbirds are richly colored."
  - "No large birds live on honey."
  - "Birds that do not live on honey are dull in color."
  - "Hummingbirds are small."
- Solution
  - Let P(x): "x is a hummingbird" ,
  - Q(x): "x is large",
  - R(x): "x lives on honey",
  - S(x) : "x is richly colored."
  - Let the domain consists of all birds. So the statements are:
  - $\forall x\, (P(x) \rightarrow S(x))$.
  - $\neg\exists x\, (Q(x) \wedge R(x))$.
  - $\forall x\, (\neg R(x) \rightarrow \neg S(x))$.
  - $\forall x\, (P(x) \rightarrow \neg Q(x))$.

Propositions for More than one variable

Let $B = \{1, 2, 3, \ldots, 9\}$ and let $p(x, y)$ denote "$x + y = 10$" Then $p(x, y)$ is a propositional function.

- The following is a statement since there is a quantifier for each variable:
  - $\forall x \exists y, p(x, y)$, *that is, "For every x, there exists a y such that x + y = 10"*
  - This statement is true. For example, if *x = 1, let y = 9; if x = 2, let y = 8, and so on.*
- The following is also a statement:
  - $\exists y \forall x, p(x, y)$, *that is, "There exists a y such that, for every x, we have x + y = 10"*
  - No such *y exists; hence this statement is false.*
- *Note: Change of order for different quantifiers can change the meaning.*

Quantifications of Two Variables **Statement**

Statement	When True?	When False
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	$P(x,y)$ is true for every pair x,y.	There is a pair x, y for which $P(x,y)$ is false.
$\forall x \exists y P(x,y)$	For every x there is a y for which $P(x,y)$ is true.	There is an x such that $P(x,y)$ is false for every y.
$\exists x \forall y P(x,y)$	There is an x for which $P(x,y)$ is true for every y.	For every x there is a y for which P(x,y) is false.
$\exists x \exists y P(x,y)$ $\exists y \exists x P(x,y)$	There is a pair x, y for which $P(x,y)$ is true.	P(x,y) is false for every pair x,y

Examples

- Determine the truth value of each of the following statements where **U = {1, *2, 3} is the universal set:***

(a) $\exists x \forall y,\ x2 < y + 1;$

(b) $\forall x \exists y,\ x2 + y2 < 12;$

Solution

(a) True. For if $x = 1$*, then 1, 2, and 3 are all solutions to* $1 < y + 1$*.*

(b) True. For each $x0$*, let* $y = 1$*; it is a true statement.*

If we change order meaning can get changed.

- Examples:
- $\forall x \exists y$ [x is married to y] is true,

  however, $\exists y\ \forall x$[x married to y] asserts that there is some person in the universe who married to everyone, this is false .
- $\forall x\ \exists y\ [x+y=0]$ (for all x, there exists a y such that x+y=0 is true, since for any value of s there is a value of y (i.e, -x) which makes it true.

However,

- $\exists y\ \forall x\ [x+y=0]$ (There exists a y such that for all x, x+y=0) asserts that value of y can be chosen independently of the value of x, since no y exists which yields zero when added to arbitrary integer x , this is false.

Examples in Mathematics Nested Quantifiers

- Translate the logical statement into Logical.

1. The sum of two integers is always positive.
   - To solve this, Read "For every two integers, if these integers are both positive, then the sum of these integers is positive".

$$\forall x \forall y((x > 0) \land (y > 0) \rightarrow (x + y > 0))$$

$$\forall x \forall y(x + y > 0)$$

2. “Every real number expect zero has a multiplicative inverse” (A multiplicative index of a real number x is a real number y such that xy=1.)

Solution:

We can rewrite as, “For every real number x expect 0, x has a multiplicative inverse.”

“For every real number x, if x $\neq$ 0”, then there exists a real number y such that xy=1”

$$\forall x((x \neq 0) \longrightarrow \exists y(xy = 1))$$

Valid, Satisfiable and unsatisfiable

- If $P(x_1,x_{2,}\dots x_n)$ is true for all values $C_1,C_{2,}\dots C_n$ from the universe U, then $P(x_1,x_{2,}\dots x_n)$ is **valid** in U.
- If $P(x_1,x_{2,}\dots x_n)$ is true for some values of $C_1,C_{2,}\dots C_n$ from the universe U, then $P(x_1,x_{2,}\dots x_n)$ is **Satisfiable** in U.
- If $P(x_1,x_{2,}\dots x_n)$ is not true for any values of $C_1,C_{2,}\dots C_n$ from the universe U, then $P(x_1,x_{2,}\dots x_n)$ is **Unsatisfiable** in U.

Nested Quantifiers

- Complex meanings require nested quantifiers.
  - "Every real number has an inverse w.r.t. addition."
    - Let the domain U be the real numbers. Then the property is expressed by

$$\forall x \; \exists y \; (x + y = 0)$$

  - "Every real number except zero has a multiplicative inverse."
    - Let the domain U be the real numbers. Then the property is expressed by

$$\forall x \; (x \neq 0 \rightarrow \exists y \; (x * y = 1))$$

Examples on Negation

- Negate the following :
- "There does not exist a woman who has taken a flight on every airline in the world "

  Solution:

  - "There is a woman who has taken a flight on every airline in the world " we can express,

$$\neg\exists w \forall a \exists f (P(w,f) \wedge Q(f,a))$$

Where, $P(w,f)$ is "w has taken f " $Q(f,a)$ is " f is a flight on a".

By applying Demorgon's law for quantifiers we can move negation inside successive quantifiers and by applying this in last step we will get the equation equivalent this.

$$\forall w \neg\forall a \exists f (P(w,f) \wedge Q(f,a))$$
$$\forall w \exists a \neg\exists f (P(w,f) \wedge Q(f,a))$$
$$\forall w \exists a \forall f \neg(P(w,f) \wedge Q(f,a))$$
$$\forall w \exists a \forall f (\neg P(w,f) \vee \neg Q(f,a))$$

Set Theory

A set is an unordered collection of objects

English alphabet vowels: $V = \{a, e, i, o, u\}$

$a \in V \qquad b \notin V$

Odd positive integers less than 10:

$$O = \{1,3,5,7,9\}$$

elements of set
members of set

Other set representations

Set of positive integers less than 100:

$$\{1,2,3,\ldots,99\}$$

omitted elements

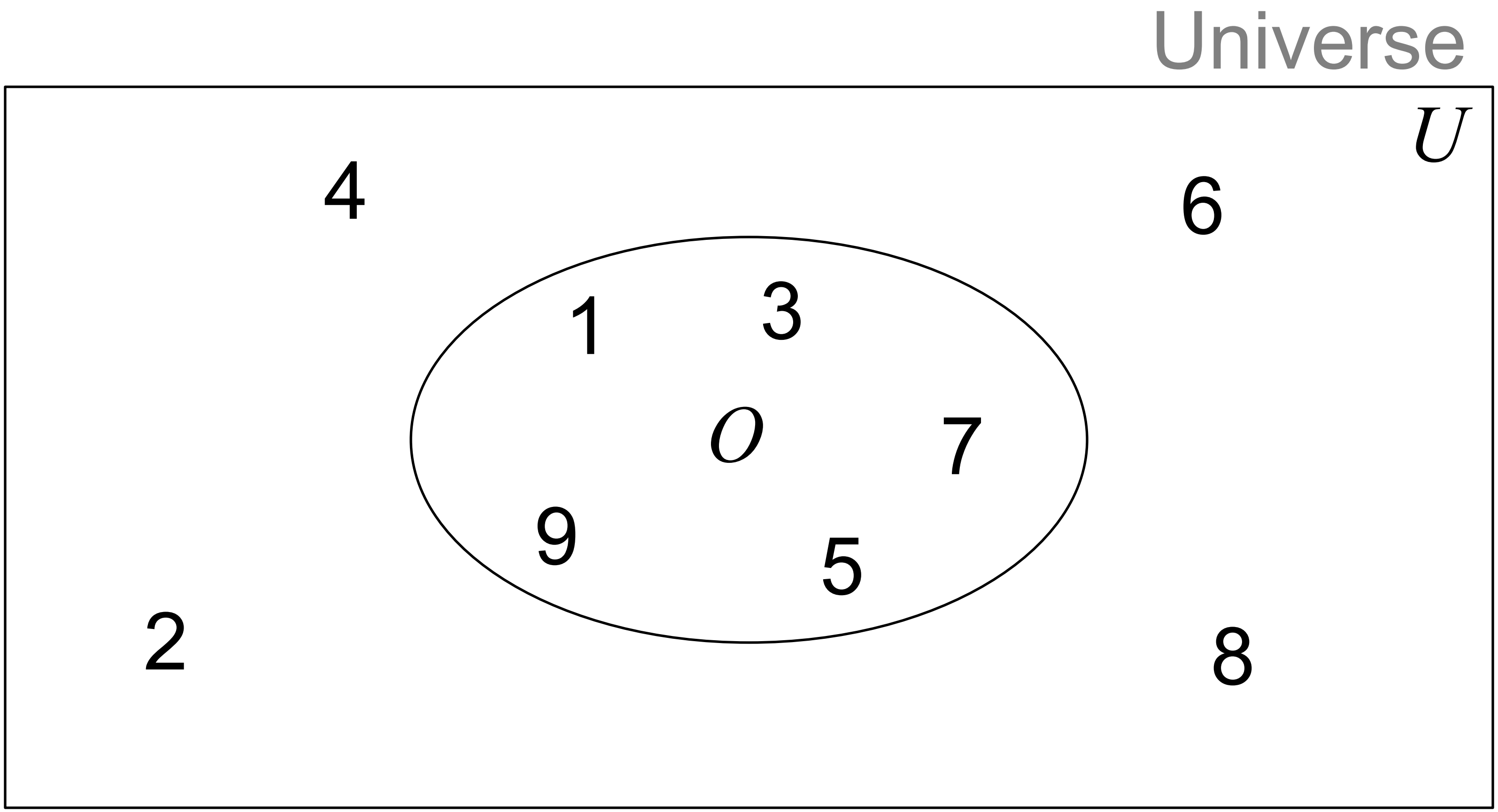
Odd positive integers less than 10:

$$O = \{1,3,5,7,9\}$$

$$O = \{x \mid x \text{ is an odd positive integer less than } 10\}$$

$$O = \{x \in Z^+ \mid x \text{ is odd and } x < 10\}$$

Venn Diagram



$$U = \{x \mid x \text{ is a positive integer less than } 10\}$$

$$O = \{x \mid x \text{ is an odd positive integer less than } 10\}$$

Useful sets

$N = \{0,1,2,3,\ldots\}$ Natural numbers

$Z = \{\ldots,-2,1,0,1,2,\ldots\}$ Integers

$Z^+ = \{1,2,3,\ldots\}$ Positive integers

$Q = \{p / q \mid p \in Z, q \in Z, q \neq 0\}$ Rational numbers

$R = \{\text{set of Real numbers}\}$ Real numbers

Empty set

$$\varnothing = \{\}$$

$$\varnothing \neq \{\varnothing\}$$

Cardinality (size) of set

Finite sets

Number of elements

$$S_1 = \{a, e, i, o, u\}$$

$$|S_1| = 5$$

$$S_2 = \{a, b, c, \ldots, z\}$$

$$|S_2| = 26$$

$$S_3 = \{1, 2, 3, \ldots, 99\}$$

$$|S_3| = 99$$

$$|\varnothing| = |\{\}| = 0$$

$$|\{\varnothing\}| = 1$$

Infinite set

$$N = \{0, 1, 2, 3, \ldots\}$$

infinite size

Equal sets

$$A = B$$

$$\forall x (x \in A \leftrightarrow x \in B)$$

Examples: $\{1,3,5\} = \{3,5,1\}$

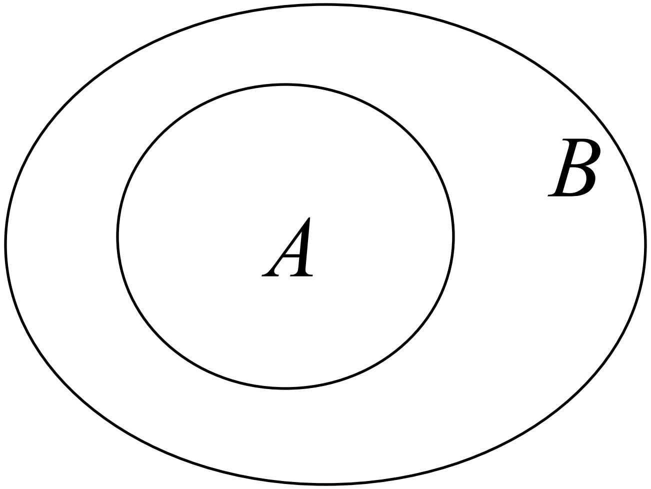
$$\{1,3,5\} = \{1,3,3,3,5,5,5,5\}$$

$$\{1,3,5,7,9\} = \{x \in Z^+ \mid x \text{ is odd and } x < 10\}$$

Subset

$$A \subseteq B$$

$$\forall x(x \in A \rightarrow x \in B)$$

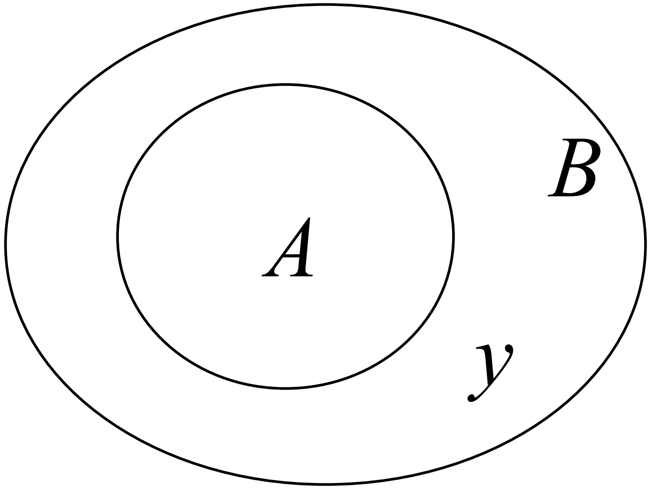


Examples: $\{1,3,5\} \subseteq \{0,1,3,5\}$ $N \subseteq Z$

For any set S: $S \subseteq S$ $\varnothing \subseteq S$

Proper Subset

$$A \subset B$$



$$A \subseteq B \wedge A \neq B$$

$$\forall x(x \in A \rightarrow x \in B \wedge \exists y(y \in B \wedge y \notin A))$$

Examples: $\{1,3,5\} \subset \{0,1,3,5\}$ $\qquad N \subset Z$

$$A = B$$

is equivalent to

$$A \subseteq B \quad \wedge \quad B \subseteq A$$

Notation

- $S=\{a, b, c\}$ refers to the set whose elements are a, b and c.
- $a \in S$ means "a is an element of set S".
- $d \notin S$ means "d is *not* an element of set S".
- $\{x \in S \mid P(x)\}$ is the set of all those x from S such that P(x) is true. *E.g.*, $T=\{x \in \mathbf{Z} \mid 0<x<10\}$.
- *Notes:*

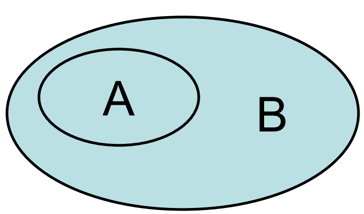
  1) $\{a,b,c\}$, $\{b,a,c\}$, $\{c,b,a,b,b,c\}$ all represent the same set.

  2) Sets can themselves be elements of other sets, *e.g.*, S={ {Mary, John}, {Tim, Ann}, …}

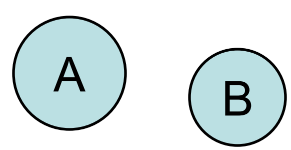
Relations between sets

- **Definition:** Suppose A and B are sets. Then A is called a **subset** of B: $\mathbf{A \subseteq B}$ iff every element of A is also an element of B. Symbolically,

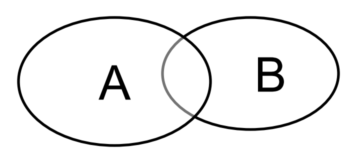
  $A \subseteq B \iff \forall x$, if $x \in A$ then $x \in B$.
- $A \not\subset B \iff \exists x$ such that $x \in A$ and $x \notin B$.



$A \subseteq B$



$A \not\subset B$



$A \not\subset B$

Relations between sets

- **Definition:** Suppose A and B are sets. Then

  A **equals** B: **A = B**

  iff every element of A is in B and

  every element of B is in A.

  Symbolically,

  $A=B \Leftrightarrow A \subseteq B$ and $B \subseteq A$.

- **Example:** Let $A = \{m \in \mathbf{Z} \mid m=2k+3$ for some integer $k\}$;

  B = the set of all odd integers.

  Then A=B.

Operations on Sets

Definition: Let A and B be subsets of a set U.

1. Union of A and B: $A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$
2. Intersection of A and B:

$$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$$

3. Difference of B minus A: $B - A = \{x \in U \mid x \in B \text{ and } x \notin A\}$
4. Complement of A: $A^c = \{x \in U \mid x \notin A\}$

Ex.: Let $U = \mathbf{R}$, $A = \{x \in \mathbf{R} \mid 3 < x < 5\}$, $B = \{x \in \mathbf{R} \mid 4 < x < 9\}$.

Then

1) $A \cup B = \{x \in \mathbf{R} \mid 3 < x < 9\}$.
2) $A \cap B = \{x \in \mathbf{R} \mid 4 < x < 5\}$.
3) $B - A = \{x \in \mathbf{R} \mid 5 \le x < 9\}$, $A - B = \{x \in \mathbf{R} \mid 3 < x \le 4\}$.
4) $A^c = \{x \in \mathbf{R} \mid x \le 3 \text{ or } x \ge 5\}$, $B^c = \{x \in \mathbf{R} \mid x \le 4 \text{ or } x \ge 9\}$

Properties of Sets

- **Theorem** 1 (*Some subset relations*):
  1) $A \cap B \subseteq A$
  2) $A \subseteq A \cup B$
  3) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

- To prove that $A \subseteq B$ use the "element argument":
  1. suppose that x is a particular but arbitrarily chosen element of A,
  2. show that x is an element of B.

Proving a Set Property

- **Theorem** 2 (*Distributive Law*):

  For any sets A,B and C:

  $$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) .$$

- **Proof:** We need to show that

  (I) $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ and

  (II) $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

Let's show (I).

Suppose $x \in A \cup (B \cap C)$ (1)

We want to show that $x \in (A \cup B) \cap (A \cup C)$ (2)

Proving a Set Property

- **Proof (cont.):**

 $x \in A \cup (B \cap C) \Rightarrow x \in A$ or $x \in B \cap C$.

(a) Let $x \in A$. Then

 $x \in A \cup B$ *and* $x \in A \cup C \Rightarrow x \in (A \cup B) \cap (A \cup C)$

(b) Let $x \in B \cap C$. Then $x \in B$ and $x \in C$.

$$\left.\begin{array}{l} x \in B \Rightarrow x \in A \cup B \\ x \in C \Rightarrow x \in A \cup C \end{array}\right\} \Rightarrow x \in (A \cup B) \cap (A \cup C)$$

Thus, (2) is true, and we have shown (I).

(II) is shown similarly (*left as exercise*). ■

Set Properties

- Commutative Laws:

$$(a)\ A \cap B = B \cap A$$

$$(b)\ A \cup B = B \cup A$$

- Associative Laws:

$$(a)\ (A \cap B) \cap C = A \cap (B \cap C)$$

$$(b)\ (A \cup B) \cup C = A \cup (B \cup C)$$

- Distributive Laws:

$$(a)\ A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$(b)\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Set Properties

- Double Complement Law:

$$(A^c)^c = A$$

- De Morgan's Laws:

$$(a)\ (A \cap B)^c = A^c \cup B^c$$

$$(b)\ (A \cup B)^c = A^c \cap B^c$$

- Absorption Laws:

$$(a)\ A \cup (A \cap B) = A$$

$$(b)\ A \cap (A \cup B) = A$$

Theorem: $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Proof: Show that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

Part 1: $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

$$x \in \overline{A \cap B}$$

$$\rightarrow x \notin A \cap B \rightarrow \neg(x \in A \cap B)$$

De Morgan's law from logic

$$\rightarrow \neg((x \in A) \wedge (x \in B)) \rightarrow \neg(x \in A) \vee \neg(x \in B)$$

$$\rightarrow (x \notin A) \vee (x \notin B) \rightarrow (x \in \overline{A}) \vee (x \in \overline{B})$$

$$\rightarrow x \in (\overline{A} \cup \overline{B})$$

Part 2: $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

$$x \in (\overline{A} \cup \overline{B})$$

$$\rightarrow (x \in \overline{A}) \vee (x \in \overline{B}) \rightarrow (x \notin A) \vee (x \notin B)$$

$$\rightarrow \neg(x \in A) \vee \neg(x \in B) \rightarrow \neg((x \in A) \wedge (x \in B))$$

$$\rightarrow \neg(x \in A \cap B)$$

De Morgan's law from logic

$$\rightarrow x \in \overline{A \cap B}$$

End of Proof

Showing that a set property is false

- **Statement:** For all sets A,B and C,

$$A-(B-C) = (A-B)-C \ .$$

The following counterexample shows that the statement is false.

- **Counterexample:**

Let $A=\{1,2,3,4\}$, $B=\{3,4,5,6\}$, $C=\{3\}$.

Then $B-C = \{4,5,6\}$ *and* $A-(B-C) = \{1,2,3\}$.

On the other hand,

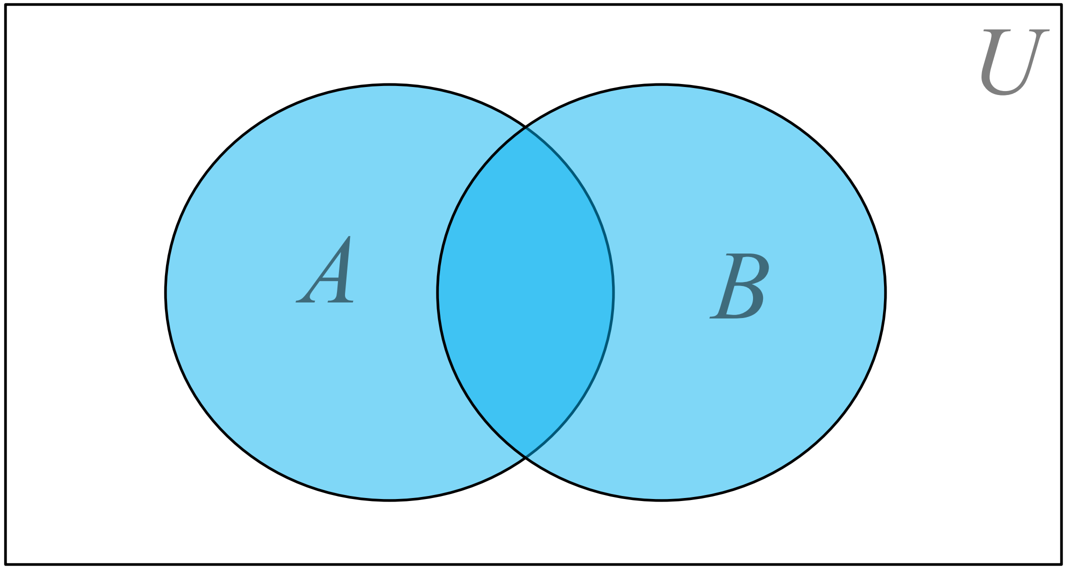
$$A-B = \{1,2\} \quad \textit{and} \quad (A-B)-C = \{1,2\} \ .$$

Thus, for this example

$$A-(B-C) \neq (A-B)-C \ .$$

Set operations

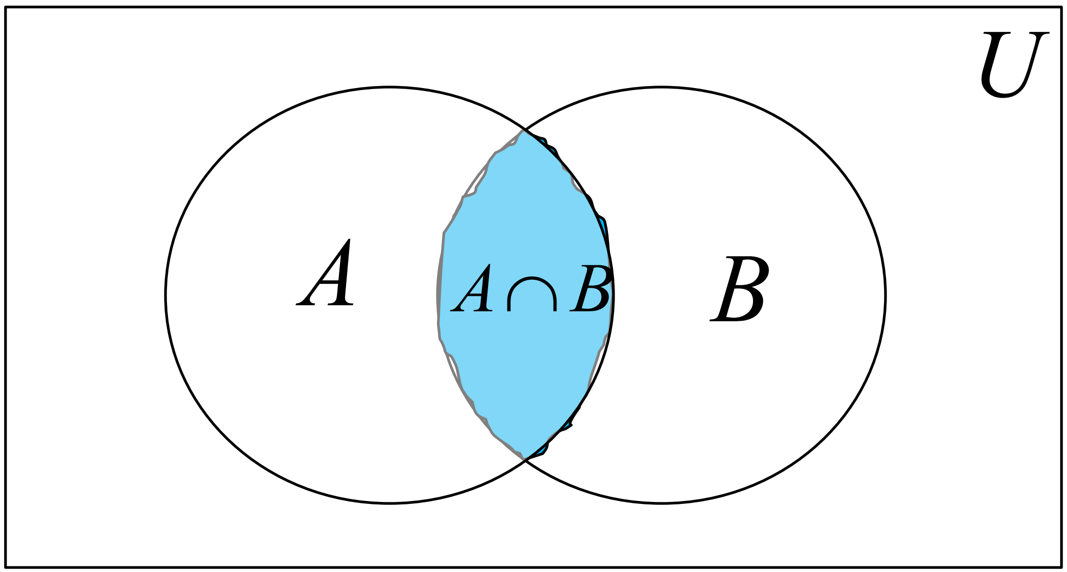
Union $A \cup B = \{x \mid x \in A \lor x \in B\}$



$A = \{1,3,5\}$ $B = \{1,2,3\}$ $A \cup B = \{1,2,3,5\}$

Intersection

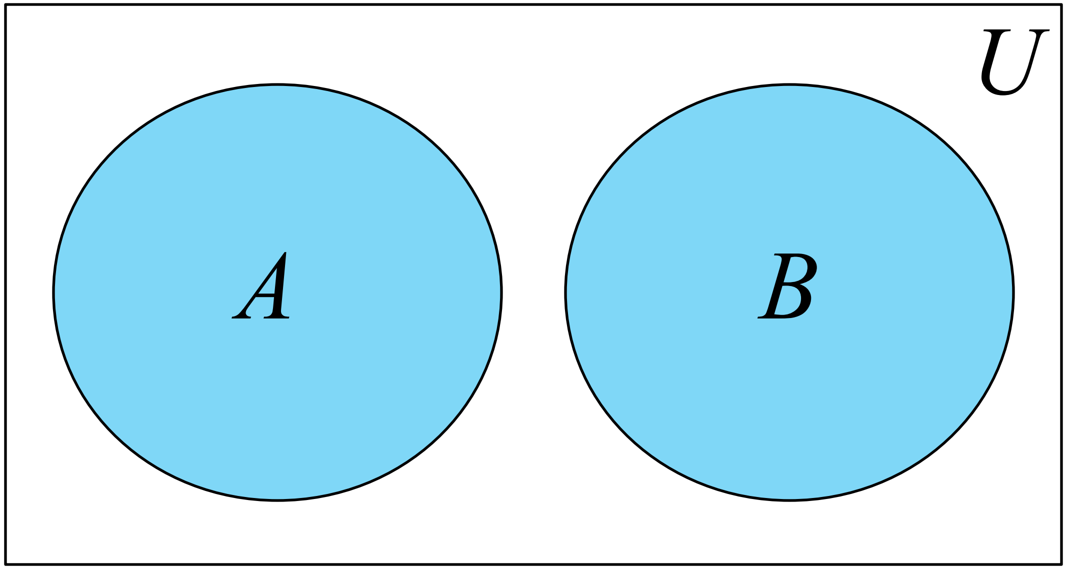
$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$



$A = \{1,3,5\}$ $\qquad B = \{1,2,3\}$ $\qquad A \cap B = \{1,3\}$

Disjoint sets A, B

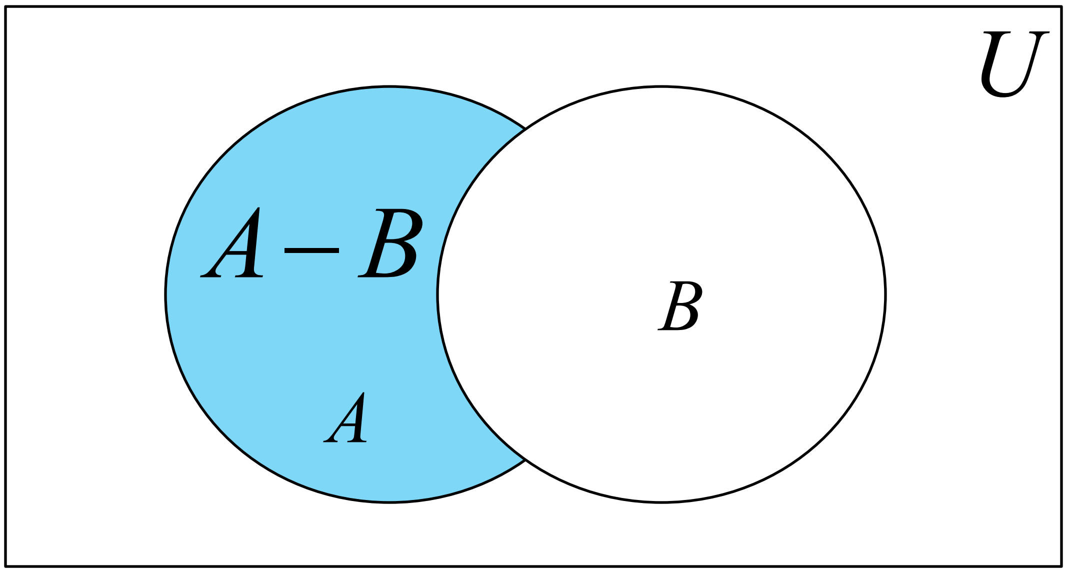
$$A \cap B = \varnothing$$



$A = \{1,3,5\}$ $B = \{2,9\}$ $A \cap B = \varnothing$

Set difference

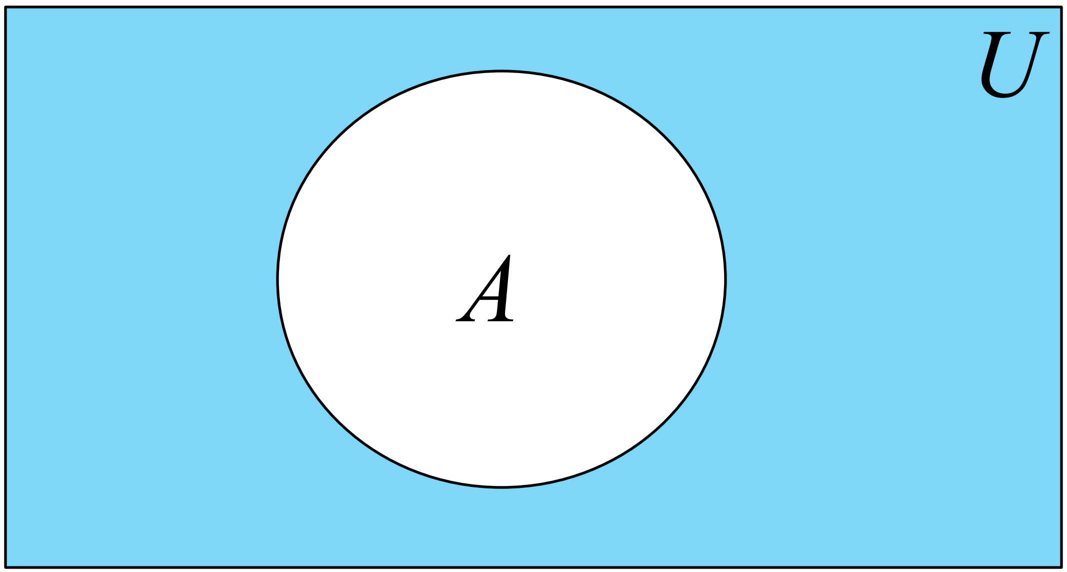
$$A - B = \{x \mid x \in A \wedge x \notin B\}$$



$A = \{1,3,5\}$ $B = \{1,2,3\}$ $A - B = \{5\}$

Complement

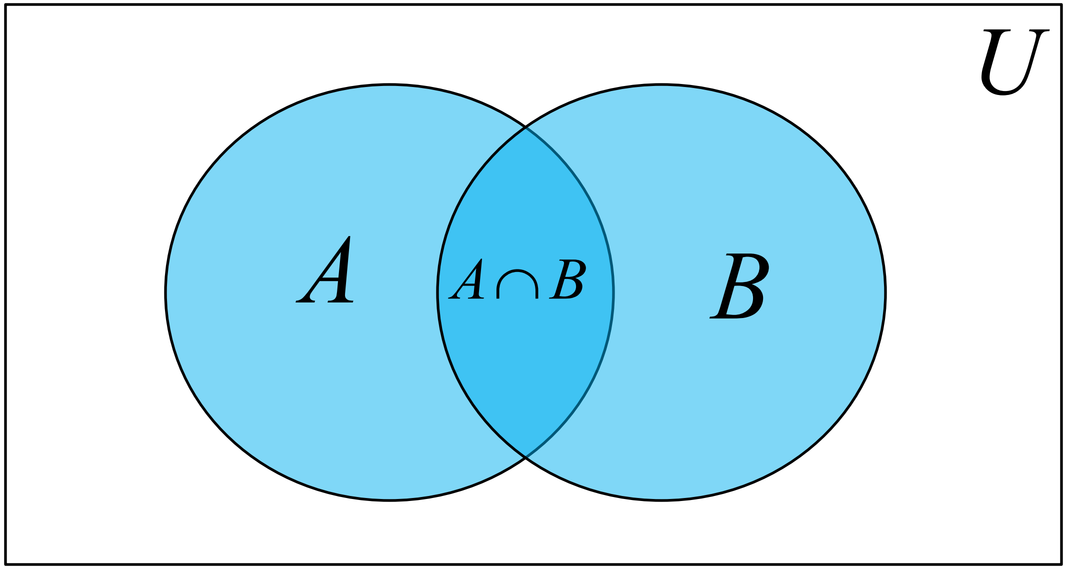
$$\overline{A} = \{x \mid x \notin A\}$$



$$A = \{1,3,5\} \qquad U = \{1,2,3,4,5\} \qquad \overline{A} = \{2,4\}$$

Size of union

$$|A \cup B| = |A| + |B| - |A \cap B|$$



$A = \{1,3,5\}$ $B = \{1,2,3\}$ $A \cup B = \{1,2,3,5\}$ $A \cap B = \{1,3\}$

$$|A \cup B| = |A| + |B| - |A \cap B| = 3 + 3 - 2 = 4$$

Empty Set

- The unique set with no elements
  is called **empty set** and denoted by $\varnothing$.
- Set Properties that involve $\varnothing$.

  For all sets A,

  1. $\varnothing \subseteq A$
  2. $A \cup \varnothing = A$
  3. $A \cap \varnothing = \varnothing$
  4. $A \cap A^c = \varnothing$

Disjoint Sets

- A and B are called **disjoint** iff $A \cap B = \varnothing$.
- Sets $A_1, A_2, \ldots, A_n$ are called **mutually disjoint**
  iff for all i,j = 1,2,…, n
  $A_i \cap A_j = \varnothing$ whenever $i \neq j$.
- *Examples:*
  1) A={1,2} and B={3,4} are disjoint.
  2) The sets of even and odd integers are disjoint.
  3) A={1,4}, B={2,5}, C={3} are mutually disjoint.
  4) $A-B$, $B-A$ and $A \cap B$ are mutually disjoint.

Partitions

- **Definition:** A collection of nonempty sets $\{A_1, A_2, \ldots, A_n\}$ is a **partition** of a set A *iff*
  1. $A = A_1 \cup A_2 \cup \ldots \cup A_n$
  2. $A_1, A_2, \ldots, A_n$ are mutually disjoint.
- *Examples*:
  1) $\{\mathbf{Z}^+, \mathbf{Z}^-, \{0\}\}$ is a partition of $\mathbf{Z}$.
  2) Let $S_0=\{n \in \mathbf{Z} \mid n=3k \text{ for some integer } k\}$
     $S_1=\{n \in \mathbf{Z} \mid n=3k+1 \text{ for some integer } k\}$
     $S_2=\{n \in \mathbf{Z} \mid n=3k+2 \text{ for some integer } k\}$
     Then $\{S_0, S_1, S_2\}$ is a partition of $\mathbf{Z}$.

Power Sets

- **Definition:** Given a set A,
  the **power set** of A, denoted $\mathcal{P}$**(A)** ,
  is the set of all subsets of A.
- *Example:* $\mathcal{P}(\{a,b\}) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$.
- **Properties:**
  1) *If* $A \subseteq B$ *then* $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.
  2) *If* a set A has n elements
     then $\mathcal{P}(A)$ has 2^n elements.

Power set

The power set of S contains all possible subsets of S (and the empty set)

$$S = \{1,2,3\}$$

Power set

$$P(S) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$$

$$\underbrace{|P(S)| = 2^{|S|}}_{\text{Size of power set}} = 2^3 = 8$$

Special cases

$$P(\varnothing) = \{\varnothing\}$$

$$P(\{\varnothing\}) = \{\varnothing, \{\varnothing\}\}$$

Cartesian product

Cartesian product of two sets A, B

$$A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$$

Example: $A = \{1,2\}$ $B = \{a,b,c\}$

$$A \times B = \{(1,a),(1,b),(1,c),(2,a),(2,b),(2,c)\}$$

$$B \times A = \{(a,1),(b,1),(c,1),(a,2),(b,2),(c,2)\}$$

For this case: $A \times B \neq B \times A$

Size: $|A \times B| = |A| \times |B| = 2 \times 3 = 6$

Cartesian product of sets $A_1, A_2, \ldots, A_n$

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i\}$$

Example: $A = \{1,2\} \quad B = \{a,b,c\} \quad C = \{x,y\}$

$$\begin{aligned} A \times B \times C = \{&(1,a,x),(1,b,x),(1,c,x),(2,a,x),(2,b,x),(2,c,x),\\ &(1,a,y),(1,b,y),(1,c,y),(2,a,y),(2,b,y),(2,c,y)\} \end{aligned}$$

Size: $|A \times B \times C| = |A| \times |B| \times |C| = 2 \times 3 \times 2 = 12$

$$|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \times |A_2| \times \cdots \times |A_n|$$

Theorem (Inclusion–Exclusion Principle)

- **Suppose A and B are finite sets. Then A ∪ B and A ∩ B are** finite and

  $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ or

  $|A \cup B| = |A| + |B| - |A \cap B|$
- That is, we find the number of elements in A or B (or both) by first adding n(A) and n(B) (inclusion) and then
- subtracting n(A ∩ B) (exclusion) since its elements were counted twice.

- Let A,B,C be the finite sets . Then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

Example For Inclusion Exclusion Formula

- A computer company wants to hire 25 programmers to habdle systems programming jobs and 40 programmers for applications programming. Of those hired , ten will be expected to perform jobs of both types. How many programmers must be hired.
- Solutions:
  - Let A be the set of systems programmers hired and B be the set of applications programmers hired.
  - The company must have $|A|=25$, $|B|=40$, and $|A \cap B|=10$.
  - The number of programmers that must be hired is $|A \cup B|$, but

    $$|A \cup B| = |A| + |B| - |A \cap B|$$
    $$=25+40-10$$
    $$= 55$$

Example 2

- Verify $|A \cup B \cup C| = |A| + |B| + |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$ where A={1,2,3,4,5}, B={2,3,4,6}, C={3,4,6,8}.

Solution:

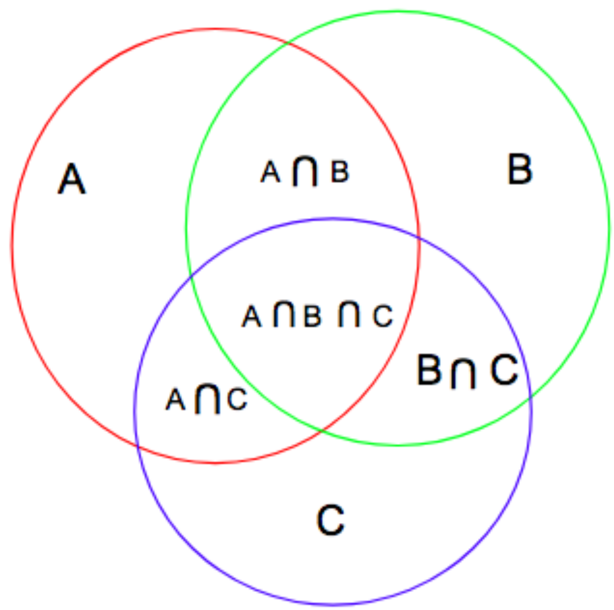
$A \cup B \cup C$ = {1,2,3,4,5,6,8}

$A \cap B = \{2,3,4\}, B \cap C = \{3,4,6\} and C \cap A = \{3,4\}$

$|A \cup B \cup C| = |A| + |B| + |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

= 5+4+4-3-3-2+2

= 7= $|A \cap B \cap C|$



$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Find the number of mathematics students at a college taking at least one of the languages French, German, and Russian given the following data:

65 study French	20 study French and German
45 study German	25 study French and Russian
42 study Russian	15 study German and Russian
	8 study all three languages

We want to find $n(F \cup G \cup R)$ where, F, G, and R denote the sets of students studying French, German, and Russian, respectively.

By the inclusion–exclusion principle,

$$n(F \cup G \cup R) = n(F) + n(G) + n(R) - n(F \cap G) - n(F \cap R) - n(G \cap R)$$
$$+ n(F \cap G \cap R)$$
$$= 65 + 45 + 42 - 20 - 25 - 15 + 8 = 100$$

Thus 100 students study at least one of the languages.

Now, suppose we have any finite number of finite sets, say, $A_1, A_2, \ldots, A_m$. Let s_k be the sum of the cardinalities

$$n(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k})$$

of all possible k-tuple intersections of the given m sets. Then we have the following general inclusion–exclusion principle.

How many binary strings of length 8 either start with a ‘1’ bit or end with two bits ’00’?

- **Solution:** If the string starts with one, there are 7 characters left which can be filled in 2^7=128 ways.
  If the string ends with ’00’ then 6 characters can be filled in 2^6=64ways.
  Now if we add the above sets of ways and conclude that it is the final answer, then it would be wrong.
- This is because there are strings with start with ‘1’ and end with ’00’ both, and since they satisfy both criteria they are counted twice.
  So we need to subtract such strings to get a correct count.
  Strings that start with ‘1’ and end with ’00’ have five characters that can be filled in 2^5=32 ways.
- So by the inclusion-exclusion principle we get-
  Total strings = 128 + 64 – 32 = 160

- In case of the usage of three toothpastes A,B,C, It is fount that 60 people like A, 55 like, 40 like C, 20 like A and B, 35 like B and C, 15 like A and C , and 10 like all three toothpastes. Find the following
  - Number of persons included in the survey.
  - Number of persons who like A only
  - Number of persons who like A and B but not C

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Solution A,B,C, Denote set of people who like toothpastes A, B and C resp.

given, |A|=60, |B|=55, |C|=40, |A∩B|=20, |B∩C|=35, |A∩C|=15, and |A∩B∩C|=10

Number of persons included in the survey.

$$|A \cup B \cup C| = |A| + |B| + |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

= 60+55+40-20-35-15+10=95

Number of persons who like A only

=|A|-(|A∩B|+|A∩C|- |A∩B∩C|)=60-(20+15-10)=35

Number of persons who like A and B but not C

=|A∩B|+ |A∩B∩C|=20-10=10

Countable Sets

Countable finite set:

Any finite set is countable by default

Countable infinite set:

An infinite set S is countable if there is a one-to-one correspondence from S to Z^+

Positive integers

Theorem: Even positive integers are countable

Proof:

$$\begin{array}{lccccc}
\text{Even positive integers:} & 2, & 4, & 6, & 8, & \ldots \\
\text{One-to-one Correspondence:} & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\
\text{Positive integers:} & 1, & 2, & 3, & 4, & \ldots
\end{array}$$

n corresponds to $2n$

End of Proof

Theorem: The set of rational numbers is countable

Proof:

We need to find a method to list

all rational numbers: $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots$

Naïve Approach

Start with nominator=1

Rational numbers: $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots$

One-to-one correspondence: $\updownarrow \quad \updownarrow \quad \updownarrow$

Positive integers: $1, \; 2, \; 3, \; \ldots$

Doesn't work:

we will never list numbers with nominator 2: $\frac{2}{1}, \frac{2}{2}, \frac{2}{3}, \ldots$

Better Approach: scan diagonals

Nomin.=1 $\frac{1}{1}$ $\frac{1}{2}$ $\frac{1}{3}$ $\frac{1}{4}$ $\cdots$

Nomin.=2 $\frac{2}{1}$ $\frac{2}{2}$ $\frac{2}{3}$ $\cdots$

Nomin.=3 $\frac{3}{1}$ $\frac{3}{2}$ $\cdots$

Nomin.=4 $\frac{4}{1}$ $\cdots$

first diagonal

$$\frac{1}{1} \qquad \frac{1}{2} \qquad \frac{1}{3} \qquad \frac{1}{4} \qquad \cdots$$

$$\frac{2}{1} \qquad \frac{2}{2} \qquad \frac{2}{3} \qquad \cdots$$

$$\frac{3}{1} \qquad \frac{3}{2} \qquad \cdots$$

$$\frac{4}{1} \qquad \cdots$$

second diagonal

$$\begin{array}{ccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\[2em]
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \cdots & \\[2em]
\frac{3}{1} & \frac{3}{2} & \cdots & & \\[2em]
\frac{4}{1} & \cdots & & &
\end{array}$$

third diagonal

$$
\begin{array}{ccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \cdots & \\
\frac{3}{1} & \frac{3}{2} & \cdots & & \\
\frac{4}{1} & \cdots & & &
\end{array}
$$

fourth diagonal…

$$
\begin{array}{ccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \cdots & \\
\frac{3}{1} & \frac{3}{2} & \cdots & & \\
\frac{4}{1} & \cdots & & &
\end{array}
$$

Every element will be eventually scanned

Diagonal listing

Rational Numbers: $\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{2}{2}, \ldots$

One-to-one correspondence: $\updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow$

Positive Integers: $1, \ 2, \ 3, \ 4, \ 5, \ \ldots$

End of Proof

We have proven: $(0,1) \subseteq R$ is uncountable

It can be proven: Every subset of a countable set is countable

$\Downarrow$

It follows that the set of real numbers R is uncountable

Multisets

- Sets:
  - An unordered collection of distinct objects.
- Multisets:
  - Sets in which some elements occur more than once
  - A={1,1,1,2,2,3}
- Notation to represent a multiset by:
  - $S=\{n_1.a_1,n_2.a_2,\ldots,n_i.a_i\}$
  - This denotes that a_1 occurs n_1 times
  - The number n_i=1,2,3,.. Are called multiplicities of the elements n_i.
  - A={3.1,2.2,1.3}

Union of Multisets

- The union of the multisets A and B is the multiset where the multiplicity of an element is the maximum of its multiplicities in A and B

$$A = \{1, 1, 1, 2, 2, 3\} \text{ and}$$
$$B = \{1, 1, 4, 3, 3\}$$
$$A \cup B = \{1, 1, 1, 4, 2, 2, 3, 3\}$$

Intersection Multisets

- The Intersection of A and B is the multiset where the multiplicity of an element the minimum of its multiplicities in A and B.

$A = \{1, 1, 1, 2, 2, 3\}$ and

$B = \{1, 1, 4, 3, 3\}$

$A \cap B = \{1, 1, 3\}$

Difference of Multisets

- The difference of A and B is the multiset where the multiplicity of an element is the multiplicity of element in A less its multiplicity in B unless this difference is negative, in which case the multiplicity is zero.

$$A = \{1, 1, 1, 2, 2, 3, 4, 4, 5\}$$
$$B = \{1, 1, 2, 2, 2, 3, 3, 4, 4, 6\}$$
$$A - B = \{1, 5\}$$

Sum of Multisets

- The Sum of A and B is the multiset where the multiplicity of an element is sum of multiplicities in set A and set B denoted by A+B.

$$A = \{1, 1, 2, 3, 3\} \quad \text{and} \quad B = \{1, 2, 2, 4\},$$

$$A + B = \{1, 1, 1, 2, 2, 2, 3, 3, 4\}$$

Multiset Examples

Let A and B be multisets as $A = \{3.a, 2.b, 1.c\}$ and $B = \{2.a, 3.b, 4.d\}$

Find

(a) $A \cup B = \{3.a, 3.b, 1.c, 4.d\}$

(b) $A \cap B = \{2.a, 2.b\}$

(c) $A - B = \{1.a, 1.c\}$

(d) $B - A = \{1.b, 4.d\}$

(e) $A + B = \{5.a, 5.b, 1.c, 4.d\}$