College of Engineering Pune
Ordinary Differential Equations and Multivariate Calculus
Tutorial-2 (2020-2021)

1. Define Linear independence and dependence of functions.

2. Apply the given operator to the given function (show all the steps in detail).

   a) $8D^2 + 2D - I, \qquad \cosh\frac{x}{2}\ ,\ \sinh\frac{x}{2}\ ,\ e^{\frac{x}{2}}$

   b) $(D+5I)(D-I), \qquad e^{-3x}\sin x\ ,\ e^{3x}\ ,\ x^2$

   c) $(D-4I)(D+3I), \qquad x^3 - x^2\ ,\ \sin 4x\ ,\ e^{-3x}$

3. Check whether the following functions are linearly independent or dependent on the given interval ?

   a) $\sin^2 x\ ,\ \sin(x^2), \quad (0 < x < \sqrt{\pi})$

   b) $x^2\ ,\ x|x|, \quad [-1, 1]$

   c) $0\ ,\ \tan x, \quad (|x| < \frac{\pi}{4})$

   d) $e^x\cos x,\ e^x\sin x,\ e^x, \quad (x > 0)$

4. Find Linear ODE for which the following functions are linearly independent solutions:

   a) $1\ ,\ e^{-2x}$

   b) $e^{-(s+it)x}\ ,\ e^{-(s-it)x}$

   c) $1\ ,\ x\ ,\ \cos 2x\ ,\ \sin 2x$

   d) $e^x\ ,\ xe^x\ ,\ \cos x\ ,\ \sin x\ ,\ x\cos x\ ,\ x\sin x$

   e) $x^2\ ,\ x^3$

   f) $x\ ,\ x\ln x$

5. State and prove the Fundamental theorem for the homogeneous linear ODE,
   $y'' + P(x)y' + Q(x)y = 0$

6. Obtain the general solution of following homogeneous linear ODEs.

   a) $100y'' + 20y' - 99y = 0$

   b) $y'' - y' + 2.5y = 0$

   c) $9y'' + 18y' - 16y = 0$

   d) $y^{iv} + 5y''' + 5y'' - 5y' - 6y = 0$

   e) $y''' + y = 0$

   f) $y^{iv} - 18y'' + 18y = 0$

   g) $y''' - y'' - y' - y = 0$

   h) $y^{iv} + 3y'' - 4y = 0$

   i) $x^2y'' + 3xy' + y = 0$

   j) $x^2y'' - xy' + 2y = 0$

7. Solve the following homogeneous linear ODEs:

a) $(D+2I)^2y=0$
b) $(D^4+k^4)y=0$
c) $(D^3-3D^2+9D-27I)y=0$
d) $(D-I)^2(D^2+I)y=0$
e) $(D^4+8D^2+16I)y=0$
f) $\left(\frac{d}{dx}+\frac{1}{x}\right)^2 \; y=0$
g) $(10x^2D^2-20xD+22.4\ I)y=0$
h) $x^2D^2+5xD+3I)y=0$

8. Solve the following IVP:

a) $y^{''}+\pi y^{'}=0,$ $\quad y(0)=3 \;,\; y^{'}(0)=-\pi$
b) $y^{''}+18y^{'}+5.6y=0,$ $\quad y(0)=4 \;,\; y^{'}(0)=-3.8$
c) $y^{''}-2y^{'}-24y=0,$ $\quad y(0)=0 \;,\; y^{'}(0)=-24$
d) $y^{'''}+3.2y^{''}+4.81y^{'}=0,$ $\quad y(0)=3.4 \;,\; y^{'}(0)=-4.6 \;,\; y^{''}(0)=9.91$
e) $y^{iv}-9y^{''}-400y=0,$ $\quad y(0)=3.4 \;,\; y^{'}(0)=0 \;,\; y^{''}(0)2.5 \;,\; y^{'''}(0)=3.5$

9. If the roots of the auxillary equation of 2^{nd} order homogeneous linear ODE $y^{''}+by^{'}+cy=0$ are real and equal then find the first solution, and the second solution using the method of reduction of order, and hence write the basis.

10. Using the method of undetermined coefficients, obtain a real general solution of following non-homogeneous differential equations:

a) $y^{''}-y^{'}-2y=3e^x$
b) $3y^{''}+10y^{'}+3y=9x+5\cos x$
c) $y^{''}+6y^{'}+9y=50e^{-x}\cos x$
d) $y^{''}+2y^{'}+10y=25x^2+3$
e) $y^{''}+4y^{'}+4y=18\cosh x$
f) $y^{''}+y^{'}=2+2x+x^2$
g) $y^{''}+y^{'}-6y=6x^3-3x^2+12x$
h) $y^{''}+10y^{'}+25y=100\sinh x$
i) $y^{''}-2y^{'}=12e^{2x}-8e^{-2x}$
j) $y^{''}-9y^{'}=x^3+e^{2x}-\sin 3x$
k) $y^{'''}+y^{'}=3x^2+4\sin x-2\cos x$

11. Using the method of variation of parameters, obtain a real general solution of following non-homogeneous differential equations:

a) $y'' - 4y' + 4y = \dfrac{e^{2x}}{x}$

b) $y'' + 9y = \sec 3x$

c) $y'' - 4y' + 5y = e^{2x}\ cosec\ x$

d) $(D^2 + 6D + 9)y = \dfrac{16e^{-3x}}{x^2 + 1}$

e) $y'' + 4y' + 4y = \dfrac{e^{-2x}}{x^2}$

f) $(x^2 D^2 - 2xD + 2)y = x^3 \cos x$

g) $y'' - y' = (3 + x)x^2 e^x$

h) $x^2 y'' - xy' + y = x \ln x$

i) $(D^2 + I)y = \cot x$

j) $(D^3 + D)y = cosec\ x$

12. For the following non-homogeneous equation, a solution y_1 of the corresponding homogeneous equation is given. Find a second solution y_2 of the corresponding homogeneous equation and the general solution of the non-homogeneous equation using the method of variation of parameters.
$(1 + x^2)y'' - 2xy' + 2y = x^3 + x, \qquad y_1(x) = x$

13. Solve the differential equations / IVP:

a) $(D^4 + 4D^3 + 8D^2 + 8D + 4)y = 0$

b) $(D^4 + 10D^2 + 9)y = 0, \qquad y(0) = 0\ ,\ y'(0) = 0\ ,\ y''(0) = 32\ ,\ y'''(0) = 0$

c) $(D^5 - 3D^4 + 3D^3 - D^2)y = 0$

d) $y''' - y' = 2x^2 e^x$

e) $4x^3 y''' + 3xy' - 3y = 4x^{11/2}$

f) $(D^3 + 3D^2 + 3D + 1)y = e^{-x} \sin x, \ \ y(0) = 2\ ,\ y'(0) = 0\ ,\ y''(0) = -1$

g) $(x^3 D^3 - 3x^2 D^2 + 6xD - 6)y = 12/x, \qquad y(1) = 5\ ,\ y'(1) = 13\ ,\ y''(1) = 10$

14. A capacitor $C = 0.2\ farad$ in series with a resistor $R = 20\ ohms$ is charged from a source $E_0 = 24\ volts$. Find the voltage $v(t)$ on the capacitor, assuming that at $t = 0$ the capacitor is completely uncharged.

15. Consider the RC circuit equation $R\,\dfrac{dQ}{dt}+\dfrac{Q}{C}=E(t)$ Determine the charge and current at time $t>0$ if $R=10\ ohms,\ \ C=2\times10^{-4}\ farads$, and $E(t)=100\ volts$. Given that $Q(t=0)=0$.

16. The charge Q on the plate of a condenser of capacity C charged through a resistance R by a steady voltage V satisfies the differential equation $R\,\dfrac{dQ}{dt}+\dfrac{Q}{C}=V$. If $Q=0$ at $t=0$, show that $Q=CV\left(1-e^{-\dfrac{t}{RC}}\right)$. Find the current flowing into the plate at any time t. (Ans: $i(t)=\frac{V}{R}e^{-\frac{t}{RC}}$)

17. A decaying e.m.f. $E=200\ e^{-5t}$ is connected in series with a 20 ohm resistor and 0.01 farad capacitor. Find the charge and current at any time assuming $Q=0$ at $t=0$. Show that the charge reaches a maximum, calculate it and find the time when it is reached. (Ans: $t=\frac{1}{5}$, Max. of $Q=0.74$)

18. In a circuit containing inductance L, resistance R and voltage E, the current I is given by $E=RI+L\,\dfrac{dI}{dt}$. Given $L=640\ H,\ R=250$ ohm and $E=500$ volts. I being zero when $t=0$. Find the time that elapses, before it reaches 90 % of its maximum value. (Ans: $\frac{64}{25}\ ln\ 10$)

19. Show that the current in RL circuit when a constant e.m.f. E_0 is applied reaches 63 % of its final value in $\dfrac{L}{R}$ seconds. Further if $L=10$ henries, determine the value of R so that the current will reach 99 % of its final value at $t=1$ seconds ? (Ans: $R=46.06$)

20. Find the current $I(t)$ in the RC circuit with $E=100$ volts, $C=0.25$ farads, R is variable according to

$$\begin{aligned} R &= (200-t)\text{ ohms}, && 0\le t\le 200\text{ sec}\\ &= 0 && t\ >200\text{ sec}\end{aligned}$$

and $I(0)=1$ amp. (Ans: $I=(200)^{-3}(200-t)^3$ and 0)

21. Find the time when the capacitor in an RC circuit with no external e.m.f. has lost 99 % of its initial charge of Q_0 Coulomb. (Ans: $t=4.605\ RC$)

22. Find the steady state solution for $Q(t)$ in an RC circuit when $R=50$ ohm, $C=0.04$ farad, and $E(t)=100\ cos2t+25\ sin2t+200\ cos4t+25\ sin4t$.

23. Find the frequency of vibration of a ball of mass $m=3\ kgs$ on a spring of modules (i) $k_1=27\ nt/m$, (ii) $k_2=75\ nt/m$, (iii) on those springs in parallel, (iv) in series, i.e the ball hangs on one spring, which in turn hangs on another spring.

24. What is the frequency of a harmonic oscillation if the the static equilibrium position of the ball is 10 cm lower than the lower end of the spring before the ball attached?

25. Consider the under-damped motion of a body of mass $m = 2\ kg$. If the time between two consecutive maxima is 2 *sec* and the maimum amplitude decreases to 1/4 of its initial value after 15 *cycles*, what is the damping constant of the system?

26. Find the overdamped motion that starts from y_0 with initial velocity v_0.

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