Introduction

- An equivalence relation is a relation that is reflexive, symmetric, and transitive
- A partial ordering (or partial order) is a relation that is reflexive, *anti*symmetric, and transitive
  - Recall that antisymmetric means that if $(a,b) \in R$, then $(b,a) \notin R$ unless $b = a$
  - Thus, (a,a) is allowed to be in R
  - But since it's reflexive, all possible (a,a) must be in R

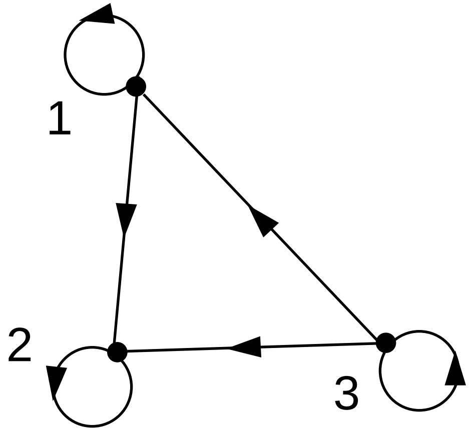
Partially Ordered Set (POSET)

A relation R on a set S is called a *partial ordering* or *partial order* if it is *reflexive*, *antisymmetric*, and *transitive*. A set S together with a partial ordering R is called a *partially ordered set*, or *poset*, and is denoted by (S, R)

Example (1)

Let S = {1, 2, 3} and

let R = {(1,1), (2,2), (3,3), (1, 2), (3,1), (3,2)}



$\preccurlyeq$

In a poset the notation $a \preccurlyeq b$ denotes that

- **(a,b) belong to R**

This notation is used because the **“*less than or equal to*”** relation is a paradigm for a partial ordering. (Note that the symbol $\preccurlyeq$ is used to denote the relation in *any* poset, not just the “less than or equals” relation.) The notation $a \prec b$ denotes that $a \preccurlyeq b$, but **“a not equal to b”**

Example

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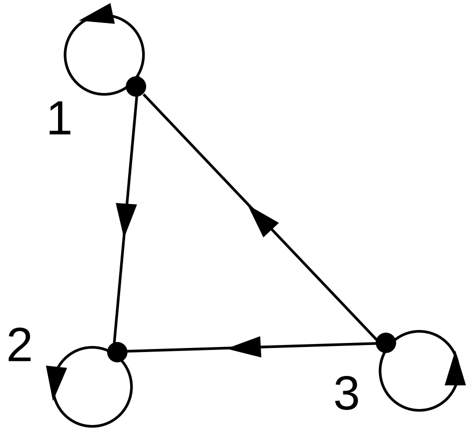


$2 \preccurlyeq 2 \qquad 3 \prec 2$

Example (2)

- Show that ≥ is a partial order on the set of integers
  - It is reflexive: $a \geq a$ for all $a \in \mathbf{Z}$
  - It is antisymmetric: if $a \geq b$ then the only way that $b \geq a$ is when $b = a$
  - It is transitive: if $a \geq b$ and $b \geq c$, then $a \geq c$
- Note that ≥ is the partial ordering on the set of integers
- ($\mathbf{Z}$, ≥) is the partially ordered set, or poset

Example (3)

Consider the power set of $\{a, b, c\}$ and the subset relation.

Comparable / Incomparable

The elements a and b of a poset $(S, \preccurlyeq)$ are called *comparable* if either $a \preccurlyeq b$ or $b \preccurlyeq a$. When a and b are elements of S such that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, a and b are called *incomparable*.

Totally Ordered, Chains

If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, S is called *totally ordered* or *linearly ordered* set, and $\preccurlyeq$ is called a *total order* or a *linear order*. A totally ordered set is also called a *chain*.

- In the poset ($\mathbf{Z}^+, \leq$), are the integers 3 and 9 comparable?
  - Yes, as $3 \leq 9$
- Are 7 and 5 comparable?
  - Yes, as $5 \leq 7$
- As all pairs of elements in $\mathbf{Z}^+$ are comparable, the poset ($\mathbf{Z}^+, \leq$) is a total order
  - totally ordered poset, linear order, or chain

- In the poset $(\mathbf{Z}^+, |)$ with "divides" operator $|$, are the integers 3 and 9 comparable?
  - Yes, as $3 \mid 9$
- Are 7 and 5 comparable?
  - No, as $7 \nmid 5$ and $5 \nmid 7$
- Thus, as there are pairs of elements in $\mathbf{Z}^+$ that are not comparable, the poset $(\mathbf{Z}^+, |)$ is a partial order. It is not a chain.

Hasse Diagrams

Given any partial order relation defined on a finite set, it is possible to draw the directed graph so that all of these properties are satisfied.

This makes it possible to associate a somewhat simpler graph, called a *Hasse diagram*, with a partial order relation defined on a finite set.

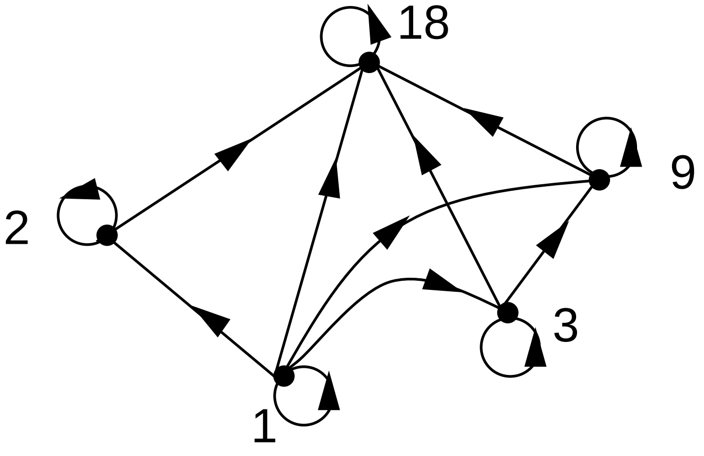
Hasse Diagrams (continued)

Start with a directed graph of the relation in which all arrows point upward. Then eliminate:

1. the loops at all the vertices,
2. all arrows whose existence is implied by the transitive property,
3. the direction indicators on the arrows.

Example

Let A = {1, 2, 3, 9, 18} and consider the “divides” relation on A:

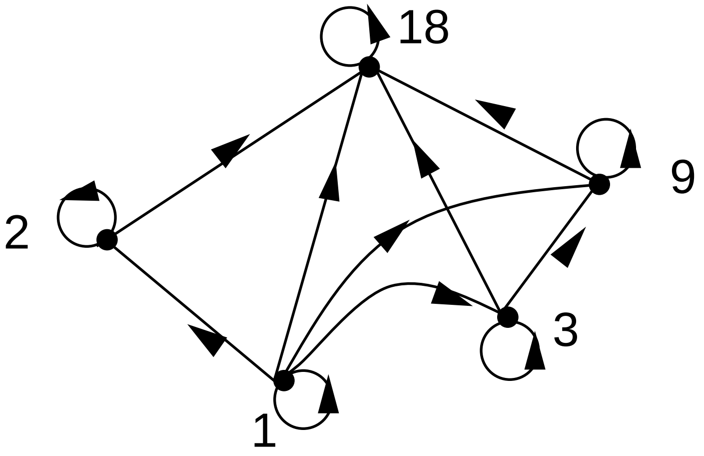


Example

Eliminate the loops at all the vertices.

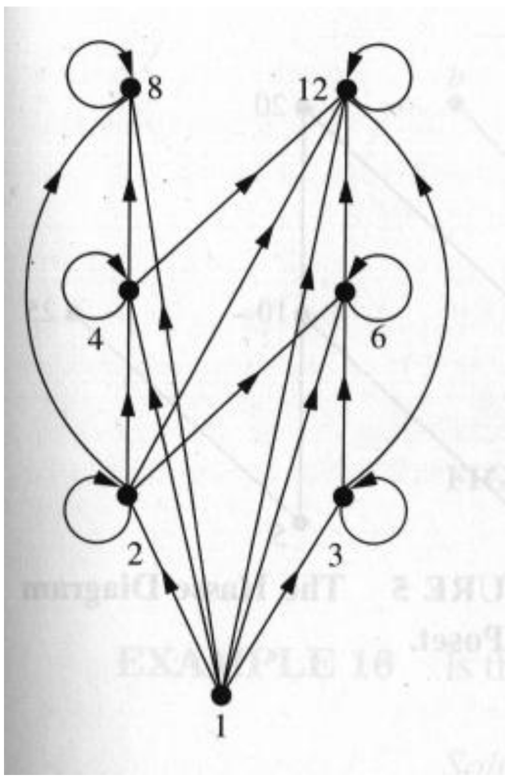
Eliminate all arrows whose existence is implied by the transitive property.

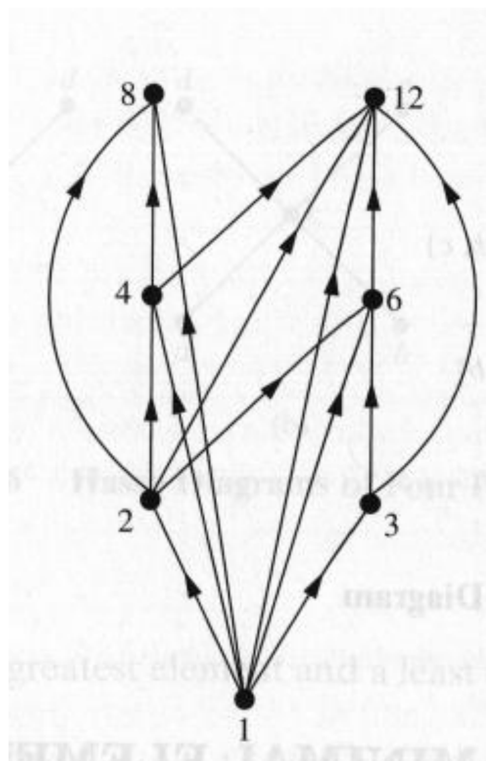
Eliminate the direction indicators on the arrows.

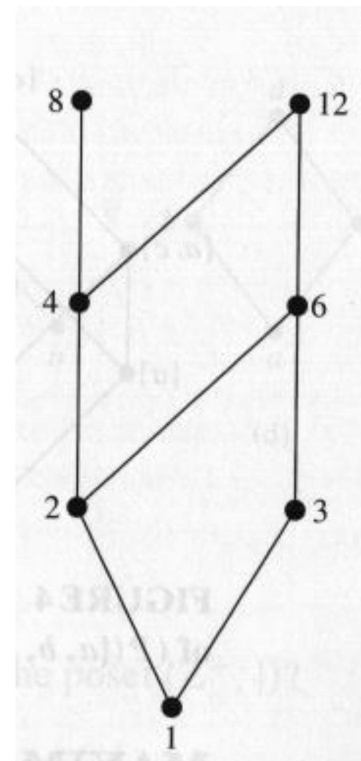


Hasse Diagram

- For the poset ({1,2,3,4,6,8,12}, |)







Construct the Hasse diagram of $(P(\{a, b, c\}), \subseteq)$.

The elements of $P(\{a, b, c\})$ are

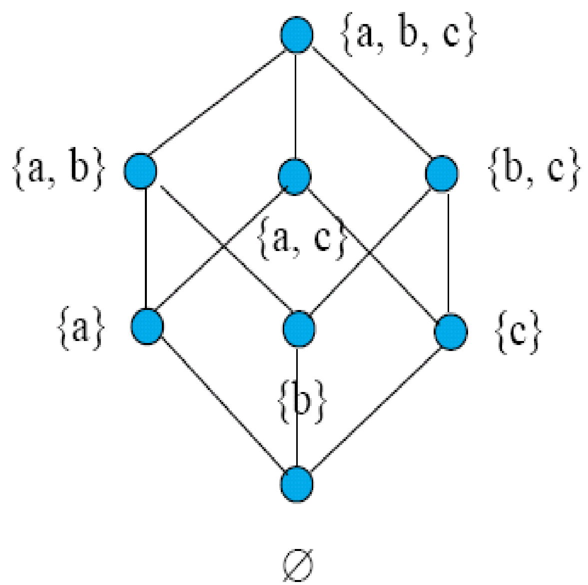
$\varnothing$

{a}, {b}, {c}

{a, b}, {a, c}, {b, c}

{a, b, c}

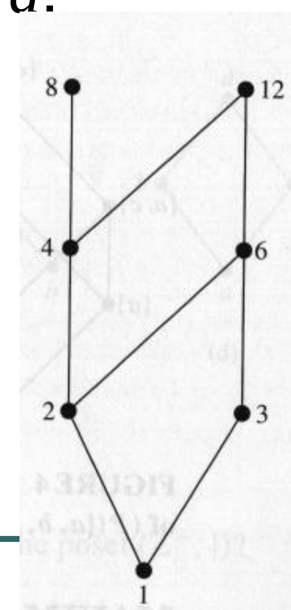
The digraph is



Maximal and Minimal Elements

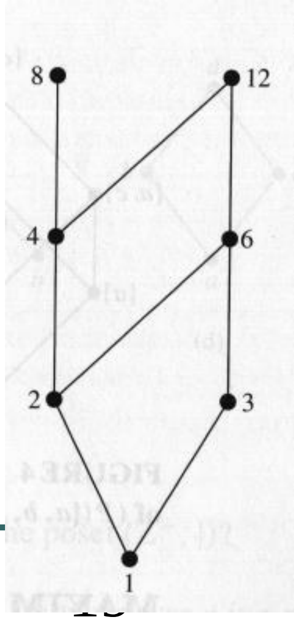
a is a *maximal* in the poset $(S, \preccurlyeq)$ if there is no b belonging to S such that $a \prec b$. Similarly, an element of a poset is called *minimal* if it is not greater than any element of the poset. That is, a is *minimal* if there is no element b belonging to S such that $b \prec a$.

It is possible to have multiple minimals and maximals.



Greatest Element
Least Element

a is the *greatest element* in the poset $(S, \preccurlyeq)$ if $b \preccurlyeq a$ for all b belonging to S. Similarly, an element of a poset is called the *least element* if it is less or equal than all other elements in the poset. That is, a is the *least element* if $a \preccurlyeq b$ for all b belonging to S.



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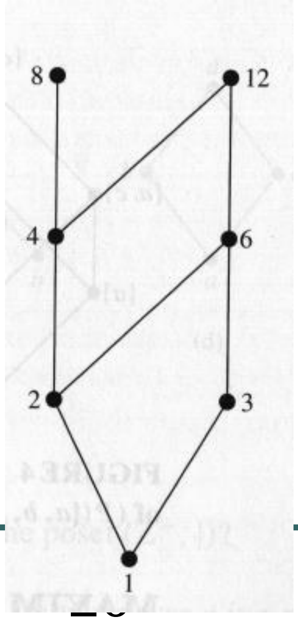


Upper bound, Lower bound

Sometimes it is possible to find an element that is greater than or = all the elements in a subset A of a poset $(S, \preccurlyeq)$. If u is an element of S such that $a \preccurlyeq u$ for all elements a belongs to A , then u is called an *upper bound* of A. Likewise, there may be an element less than all the elements in A. If l is an element of S such that $l \preccurlyeq a$ for all elements a belongs to A , then l is called a *lower bound* of A.

Least Upper Bound, Greatest Lower Bound

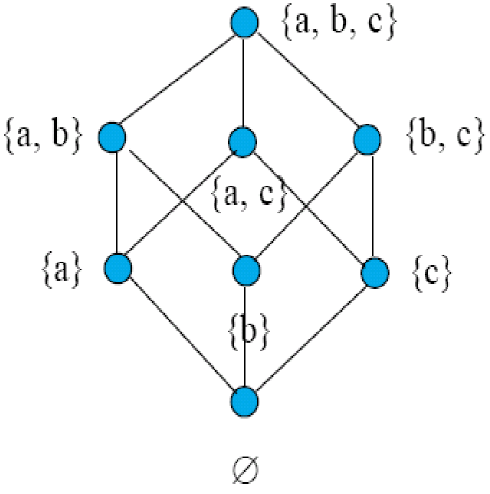
The element x is called the *least upper bound* (lub) of the subset A if x is an upper bound that is less than every other upper bound of A.

The element y is called the *greatest lower bound* (glb) of A if y is a lower bound of A and $z \preccurlyeq y$ whenever z is a lower bound of A.

Lattices

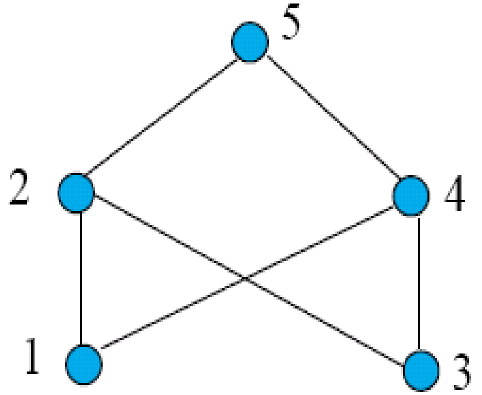
A partially ordered set in which *every pair* of elements has both a least upper bound and a greatest lower bound is called a *lattice*.

$(P(\{a, b, c\}), \subseteq)$



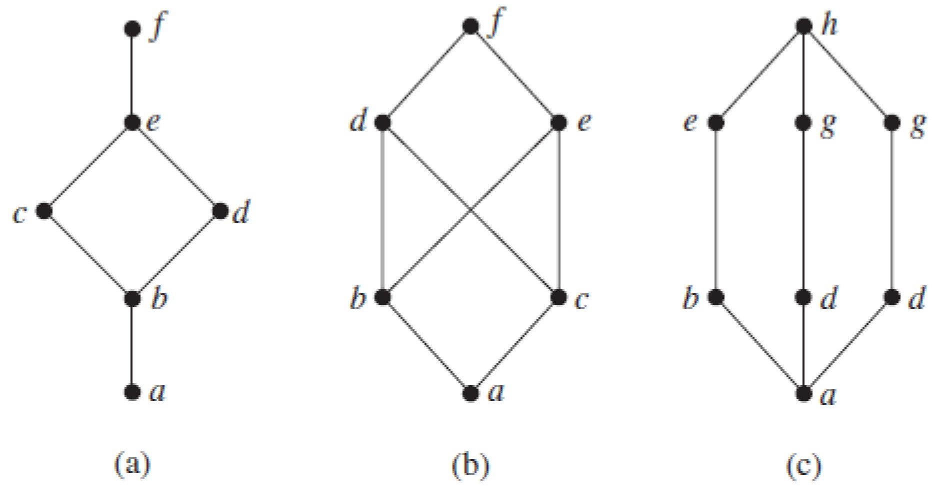
Consider the elements 1 and 3.

- Upper bounds of 1 are 1, 2, 4 and 5.
- Upper bounds of 3 are 3, 2, 4 and 5.
- 2, 4 and 5 are upper bounds for the pair 1 and 3.
- There is no lub since
  - 2 is not related to 4
  - 4 is not related to 2
  - 2 and 4 are both related to 5.
- There is no glb either.

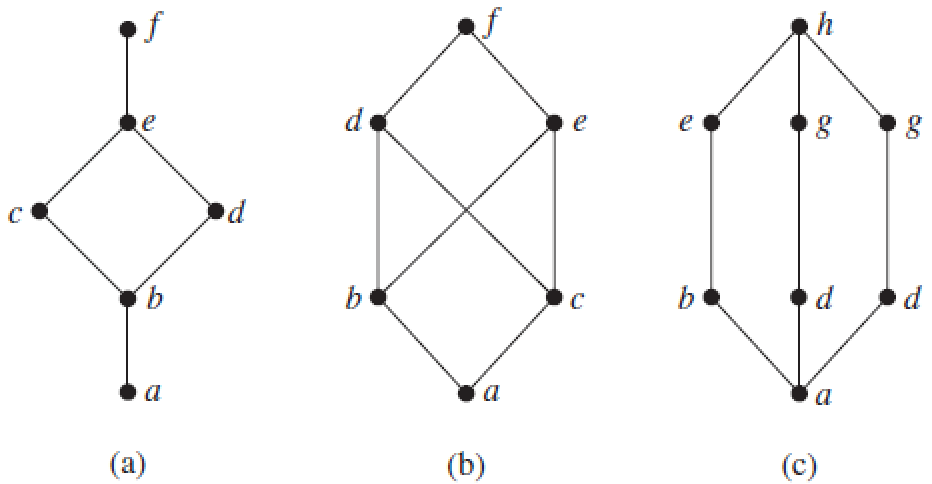


The poset is not a lattice.

Which of the following are Lattices



Which of the following are Lattices



- The posets represented by the Hasse diagrams in (a) and (c) are both lattices because in each poset every pair of elements has both a least upper bound and a greatest lower bound, as the reader should verify. On the other hand, the poset with the Hasse diagram shown in (b) is not a lattice, because the elements *b and c have no least upper bound. To see this, note that* each of the elements *d, e, and f is an upper bound, but none of these three elements precedes* the other two with respect to the ordering of this poset.

- Determine whether the posets *({1, 2, 3, 4, 5}, |) and ({1, 2, 4, 8, 16}, |) are lattices.*

- Determine whether the posets *({1, 2, 3, 4, 5}, |) and ({1, 2, 4, 8, 16}, |) are lattices.*

Solution: Because 2 and 3 have no upper bounds in ({1, 2, 3, 4, 5}, |), they certainly do not have a least upper bound. Hence, the first poset is not a lattice.

- Every two elements of the second poset have both a least upper bound and a greatest lower bound. The least upper bound of two elements in this poset is the larger of the elements and the greatest lower bound of two elements is the smaller of the elements, as the reader should verify.
- Hence, this seconsd poset is a lattice.

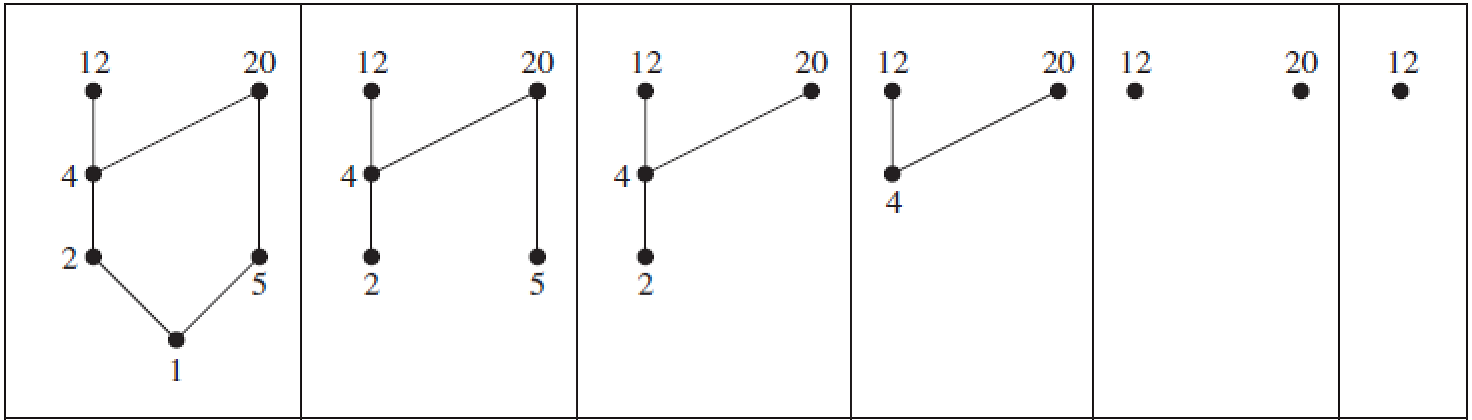
Topological Sorting

- a project is made up of 20 different tasks
- Some tasks can be completed only after others have been finished.
- we set up a partial order on the set of tasks so that $a \prec b$ if and only if a and b are tasks where b cannot be started until a has been completed.
- To produce a schedule for the project, we need to produce an order for all 20 tasks that is compatible with this partial order

- We begin with a definition. A total ordering is said to be **compatible** with the partial
- ordering *R* if *a* *b* whenever *aRb*. Constructing a compatible total ordering from a partial
- ordering is called **topological sorting**.
- Topological sorting has an application to the scheduling of projects.
- Lemma:Every finite nonempty poset *(S,)* has at least one minimal element.

Example

- Find a compatible total ordering for the poset ({1, 2, 4, 5, 12, 20}, |).

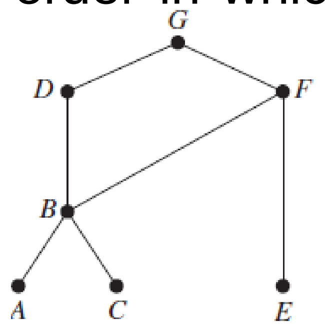


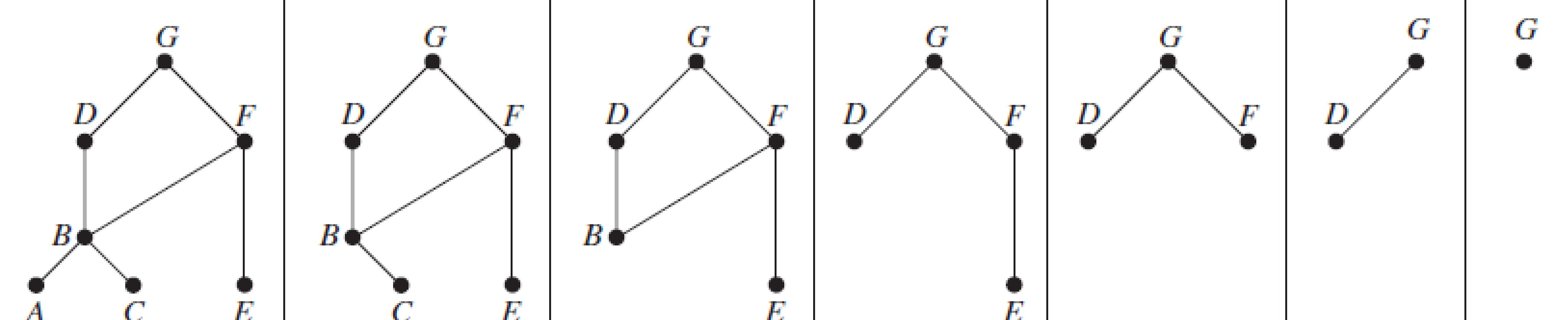
Minimal element chosen	1	5	2	4	20	12

$$1 \prec 5 \prec 2 \prec 4 \prec 20 \prec 12.$$

- A development project at a computer company requires the completion of seven tasks. Some of these tasks can be started only after other tasks are finished.A partial ordering on tasks is set up by considering task $X \prec$ task Y if task Y cannot be started until task X has been completed.

- The Hasse diagram for the seven tasks, with respect to this partial ordering, is shown in Figure Find an order in which these tasks can be carried out to complete the project.





Minimal element chosen	A	C	B	E	F	D	G