College of Engineering Pune

Linear Algebra and Univariate Calculus(D.S.Y)

Tutorial 2

Vector Space, Subspace, Linear combination, Linearly dependence and Independence

1. Show that $\mathbb{R}^n$ forms a vector space over R.

2. Show that set of all nxn matrices over $\mathbb{R}$ i.e., $M_{n\times n}(\mathbb{R})$ forms a vector space over $\mathbb{R}$.

3. Show that set of all continuous functions from set of real numbers to set of real numbers i.e., $C(\mathbb{R}, \mathbb{R})$ forms a vector space over $\mathbb{R}$.

4. Which of the following forms subspaces?

   (a) $S_1 = \{(x, y) \in \mathbb{R}^2 | x = y\}$

   (b) $S_2 = \{(x, y) \in \mathbb{R}^2 | x = 2y\}$

   (c) $S_3 = \{(x, y) \in \mathbb{R}^2 | x = cy, c \in \mathbb{R}\backslash 0\}$

   (d) $S_4 = \{(x, y) \in \mathbb{R}^2 | x = y + 1\}$

   (e) $S_5 = \{(x, y) \in \mathbb{R}^2 | x = y + c, c \in \mathbb{R}\backslash 0\}$

   (f) $S_6 = \{(x, y, z) \in \mathbb{R}^3 | x + y + z = 0\}$

   (g) $S_7 = \{(x, y, z) \in \mathbb{R}^3 | x = y \text{ and } 2y = z\}$

   (h) $S_8 = \{(x, y, z) \in \mathbb{R}^3 | x + y = 3z\}$

   (i) $S_9 = \{(x, y, z) \in \mathbb{R}^3 | x = 0\}$

5. Which of the following forms a subspace for $M_{n\times n}(\mathbb{R})$?

   (a) Set of upper triangular matrices.

   (b) Set of lower triangular matrices.

   (c) Set of diagonal matrices.

   (d) Set of scalar matrices.

   (e) Set of matrices whose determinant is non-zero.

   (f) Set of matrices whose determinant is zero.

(g) Set of matrices whose trace (Sum of diagonal entries) is zero.

(h) Set of matrices whose trace (Sum of diagonal entries) is non-zero.

(i) Set of symmetric matrices.

(j) Set of skew-symmetric matrices.

6. Which of the following forms subspaces for $C(\mathbb{R}, \mathbb{R})$.

    (a) $S_9 = \{f \in C(\mathbb{R}, \mathbb{R}) | f(0) = 0\}$

    (b) $S_9 = \{f \in C(\mathbb{R}, \mathbb{R}) | f(0) = 1\}$

    (c) $S_9 = \{f \in C(\mathbb{R}, \mathbb{R}) | f(x) = f(-x), \forall x \in \mathbb{R}\}$

    (d) $S_9 = \{f \in C(\mathbb{R}, \mathbb{R}) | f(x) = -f(-x), \forall x \in \mathbb{R}\}$

    (e) $S_9 = \{f \in C(\mathbb{R}, \mathbb{R}) | f(x+1) = f(x), \forall x \in \mathbb{R}\}$

7. Which of the following are subspaces of $\mathbb{R}^\infty$.

    (a) All sequence like $(1, 0, 1, 0, 1, 0, ...)$ i.e., zero at even positions.

    (b) All sequences $(x_1, x_2, x_3, ...)$ with $x_j = 0$ from some point onwards.

    (c) All decreasing sequences: $x_{j+1} \le x_j$ for each j.

8. If U and W are subspaces of a vactor space V then show that $U \cap W$ and $U + W$ are also subspaces of V. What can you say about $U \cup W$, does it forms a subspace in general?

9. Construct a subset of the $x - y$ plane in $\mathbb{R}^2$ that is:

    (a) closed under vector addition and subtraction but not under scalar multiplication.

    (b) closed under scalar multiplication but not under vector addition.

10. Express the given vector X as a linear combination of the given vectors A,B, and find the coordinates of X with respect to A,B.

    (a) $X = {}^t(1, 0)$, $A = {}^t(1, 1)$, $B = {}^t(0, 1)$ 1,-1

    (b) $X = {}^t(2, 1)$, $A = {}^t(1, -1)$, $B = {}^t(1, 1)$ 3/2 , 1/2

    (c) $X = {}^t(1, 0, 0)$, $A = {}^t(1, 1, 1)$, $B = {}^t(-1, 1, 0)$, $C = {}^t(1, 0, -1)$ -1,1,1

    (d) $X = {}^t(1, 1, 1)$, $A = {}^t(0, 1, -1)$, $B = {}^t(1, 1, 0)$, $C = {}^t(1, 0, 2)$ 1,0,1

11. Check linear independence and dependence of following vectors.

   (a) ${}^t(1,2,3), {}^t(0,0,0), {}^t(1,0,0)$. Dependent
   (b) ${}^t(1,1,0), {}^t(1,1,1), {}^t(0,1,-1)$. ID
   (c) ${}^t(0,1,1), {}^t(0,2,1), {}^t(1,5,3)$. ID
   (d) ${}^t(1,1,2), {}^t(1,2,3), {}^t(2,2,4)$. Dependent
   (e) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ ID
   (f) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix}$ ID

12. Show that v_1, v_2, v_3 are independent but v_1, v_2, v_3, v_4 are dependent:
$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_4 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

13. If w_1, w_2, w_3 are independent vectors, show that the differences $v_1 = w_2 - w_3, v_2 = w_1 - w_3$, and $v_3 = w_1 - w_2$ are dependent. (Hint: Find a combination of the $v's$ that gives 0.)

14. If w_1, w_2, w_3 are independent vectors, show that the sum $v_1 = w_2 + w_3, v_2 = w_1 + w_3$, and $v_3 = w_1 + w_2$ are linearly independent.

15. Suppose v_1, v_2, v_3, v_4 are vectors in $\mathbb{R}^3$.

   (a) These four vectors are dependent because ...
   (b) The two vectors v_1 and v_2 will be dependent if ...
   (c) The vectors v_1 and $(0,0,0)$ are dependent because...

16. True or false. Justify

   T (a) Subset of linearly independent set is linearly independent.
   F (b) Subset of linearly dependent set is linearly dependent. $\{(2,4,6),(1,2,3)\}$
   F (c) Superset of linearly independent set is linearly independent. ← same
   T (d) Superset of linearly dependent set is linearly dependent.