

Conditional Probability:

Suppose a new medical test is developed to diagnose a particular infection of the blood. The test is conducted on blood samples from 100 randomly selected patients and outcomes of the test are presented in following table.

Absolute frequencies of test results and infection status

		Infection		Total (row)
		Present	Absent	
Test	positive (+)	30	10	40
	Negative (-)	15	45	60
Total (column)		45	55	Total = 100

$\leftarrow T_1$

T_2 contains
the relative
frequencies of
 T_1

Relative frequencies of patients and tests

		Infection		Total row
		Present (IP)	Absent (IA)	
Test	positive (+)	0.30	0.10	0.40
	Negative (-)	0.15	0.45	0.60
Total column		0.45	0.55	Total = 1

$\leftarrow T_2$

There are the following four possible outcomes:

- The blood sample has an infection and the test diagnoses it; that is, the test is correctly diagnosis the infection.

- The blood sample does not have an infection and test does not diagnose it, that is, the test is correctly diagnosing that there is no infection.
- The blood sample has an infection and test does not diagnose it ; that is, the test is incorrect in stating that there is no infection.
- The blood sample does not have an infection but the test diagnoses it, that is, the test is incorrect in stating that there is an infection.

We interpret the relative frequencies as probabilities; i.e, we assume that the values in Table 2 would be observed if the number n of patients was much larger than 100.

It is easy to observe that the probability that a test is positive is $P(T+) = 0.30 + 0.10 = 0.4$ & the probability than an infection is present is $P(IP) = 0.30 + 0.15 = 0.45$

- If one already knows that the test is positive and wants to determine the probability that the infection is indeed present, then this can be achieved by the respective conditional probability $P(IP|T+)$ which is

$$P(IP|T+) = \frac{P(IP \cap T+)}{P(T+)} = \frac{0.3}{0.4} = 0.75$$

IP \cap T+: denotes the relative frequency of blood samples in which the disease is present and the test is positive which is 0.3.

We use the intersection operator \cap to describe events which occur for $A=a$ and $B=b$. This relates to the joint relative frequencies. The marginal relative frequencies (i.e. probabilities $P(A=a)$) can be observed from a column and row sums, respectively; and the conditional probabilities can be observed as the joint frequencies in relation to the marginal frequencies.

— Conditional Probability - Laplace experiment —
 e.g: For simplicity, assume that all simple events in $\Omega = \{\omega_1, \omega_2, \dots, \omega_k\}$ are equally probable, that is,
 $P(\omega_j) = \frac{1}{k}, j=1, 2, \dots, k$.

Let A and B be two events containing n_A & n_B number of simple events. Let further $A \cap B$ contain n_{AB} numbers of simple events. The Laplace probability is

$$P(A) = \frac{n_A}{k}, P(B) = \frac{n_B}{k}, P(A \cap B) = \frac{n_{AB}}{k}$$

Now, suppose we have prior information that A has already occurred. We want to find out how the probability of B is to be calculated.

Since A is already occurred, we know that the ~~sample~~ space is reduced by the number of simple events which are contained in A. There are n_A such simple events.

∴ The total sample space is reduced by the sample space of A. Therefore, the simple events in $A \cap B$ are those simple events which are realized when B is realized.

The Laplace probability of B under the prior information on A, or under the condition that A is known, is therefore

$$P(B|A) = \frac{n_{AB}/k}{n_A/k} = \frac{P(A \cap B)}{P(A)}.$$

This can be generalized to the case when the probabilities for simple events are unequal.

$P(A|B)$: conditional probability of A given B.

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Conditional Probability:

Let $P(A) > 0$. Then the conditional probability of event B occurring, given that event A has already occurred

is $P(B|A) = \frac{P(A \cap B)}{P(A)}$ —① $\leftarrow (B \text{ given } A\right)$

The role of A & B can be interchanged to define $P(A|B)$ as follows.

Let $P(B) > 0$. The conditional probability of A given B

is $P(A|B) = \frac{P(A \cap B)}{P(B)}$ —②

Thm: Multiplication Theorem of probability:

For two arbitrary events A and B , the following holds:

$$P(A \cap B) = P(A|B) P(B) = P(B|A) P(A). \quad \text{—③}$$

This follows from Defⁿ: ($P(A) > 0, P(B) > 0$ not required)

Thm Law of Total Probability:

Assume that A_1, A_2, \dots, A_m are events such that $\bigcup_{i=1}^m A_i = \Omega$ and $A_i \cap A_j = \emptyset$ for all $i \neq j$, $P(A_i) > 0$ for all i , that is, A_1, A_2, \dots, A_m form a complete decomposition of $\Omega = \bigcup_{i=1}^m A_i$ in pairwise disjoint events, then the probability of an event B can be calculated as

$$P(B) = \sum_{i=1}^m P(B|A_i) P(A_i). \quad \text{--- (4)}$$

★ Bayes' Theorem.

Bayes' Theorem gives a connection between $P(A|B)$ and $P(B|A)$. For events A and B with $P(A) > 0$ and $P(B) > 0$, then by (1), (2) & (3) we get

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B)}{P(A)} \cdot \frac{P(A)}{P(B)} = \frac{P(B|A) P(A)}{P(B)} \quad \text{--- (5)}$$

Let A_1, A_2, \dots, A_m be events such that $\bigcup_{i=1}^m A_i = \Omega$ and $A_i \cap A_j = \emptyset$ for all $i \neq j$, $P(A_i) > 0$ for all i , and B is another event than A_j , then using (4) and (5) we get

$$P(A_j|B) = \frac{P(B|A_j) P(A_j)}{\sum_{i=1}^k P(B|A_i) \cdot P(A_i)}$$

$P(A_i)$ are called prior probabilities,
 $P(B|A_i)$ are sometimes called model probabilities,
and $P(A_j|B)$ are called posterior probabilities.

Exa:

Suppose someone rents movies from two different DVD stores. Sometimes it happens that the DVD does not work because of scratches.

We consider the following events:

A_i ($i=1, 2$): the DVD is rented from store i .

Further, let B denote the event that the DVD is working without any problems.

Assume that $P(A_1) = 0.6$, $P(A_2) = 0.4$ (Note $A_2 = \overline{A_1}$), and $P(B|A_1) = 0.95$, $P(B|A_2) = 0.75$, we are interested in the probability that a rented DVD works fine.

Apply Law of Total Probability,

$$\begin{aligned} P(B) &= P(B|A_1) \cdot P(A_1) + P(B|A_2) \cdot P(A_2) \\ &= 0.6 \times 0.95 + 0.4 \cdot 0.75 \\ &= 0.87. \end{aligned}$$

The probability that movie was rented from store 1 is working is

$$P(B \cap A_1) = P(B|A_1) P(A_1) = 0.95 \times 0.6 = 0.57.$$

Now, suppose we have a properly working DVD. What is the probability that it is rented from store 1.

$$P(A_1|B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{0.57}{0.87} = 0.6552.$$

Assume we have a DVD which does not work, that is, \bar{B} occurs. The probability that a DVD is not working given that it is from store 1 is $P(\bar{B}|A_1) = 0.05$.

Similarly $P(\bar{B}|A_2) = 0.25$ for store 2.

The conditional probability that a DVD is from store 1 given that it is not working:

$$\begin{aligned} P(A_1|\bar{B}) &= \frac{P(\bar{B}|A_1) \cdot P(A_1)}{P(\bar{B}|A_1) P(A_1) + P(\bar{B}|A_2) P(A_2)} = P(\bar{B}) \\ &= \frac{0.05 \times 0.6}{0.05 \times 0.6 + 0.25 \times 0.4} = 0.13 \\ &= 0.2308 \end{aligned}$$

$$P(\bar{B}) = 1 - P(B) = 1 - 0.87 = 0.13$$

Stochastic independence:

Two events are independent if the occurrence or non-occurrence of one event does not affect the occurrence or non-occurrence of the other event.

- Two events A & B are independent if the probability of occurrence of B has no effect on the probability of occurrence of A.

We expect, $P(A|B) = P(A)$ & $P(A|\bar{B}) = P(A)$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \& \quad P(A|\bar{B}) = \frac{P(A \cap \bar{B})}{P(\bar{B})}$$

$$\therefore \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap \bar{B})}{P(\bar{B})}$$

$$\Rightarrow P(A \cap B) P(\bar{B}) = P(A \cap \bar{B}) \cdot P(B)$$

$$P(A \cap B) (1 - P(B)) = P(A \cap \bar{B}) \cdot P(B)$$

$$P(A \cap B) = (P(A \cap \bar{B}) + P(A \cap B)) \cdot P(B)$$

$$P(A \cap B) = P(A) \cdot P(B). \quad \left. \begin{array}{l} \text{Law of total} \\ \text{probability} \end{array} \right\}$$

This leads to the following definition of stochastic independence.

Stochastic independence:

Two random events A and B are called (stochastically) independent if

$$P(A \cap B) = P(A) \cdot P(B).$$

that is, the probability of simultaneous occurrence of events A and B is the product of the individual probabilities of occurrence of A and B.

Generalization

Defⁿ: The n events A_1, A_2, \dots, A_n are stochastically mutually independent, if for any subset of m events $A_{i_1}, A_{i_2}, \dots, A_{i_m}$ ($m \leq n$)

$$P(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_m}) = P(A_{i_1}) P(A_{i_2}) \dots P(A_{i_m}) \text{ holds.} \quad (A)$$

A weaker form of independence is pairwise independence.

If (A) is true for $m=2$, then the events are called pairwise independent.

- The difference between pairwise independence and general stochastic independence is explained in the following example.

exa: Consider an urn with four balls. The following combinations of zeroes and ones are printed on the balls : ~~001, 011, 101, 111~~ 001, 010, 100, 111.

One ball drawn from the urn. Define the following events:

A_1 : The first digit on the ball is zero

A_2 : The second digit on the ball is zero.

A_3 : The third digit on the ball is zero.

inform: there are two favourable simple events for each of
events $A_1, A_2 \nexists A_3$.

CS $\therefore P(A_1) = P(A_2) = P(A_3) = \frac{2}{4} = \frac{1}{2}$.

The probability that all the three events simultaneously occur is zero because there is no ball with 000 printed on it.

$\Rightarrow A_1, A_2 \& A_3$ are not stochastically independent because

$$P(A_1) \cdot P(A_2) \cdot P(A_3) = \frac{1}{8} \neq 0 = P(A_1 \cap A_2 \cap A_3)$$

However, $P(A_1 \cap A_2) = \frac{1}{4} = P(A_1) \cdot P(A_2)$

$$P(A_1 \cap A_3) = \frac{1}{4} = P(A_1) \cdot P(A_3)$$

$$P(A_2 \cap A_3) = \frac{1}{4} = P(A_2) \cdot P(A_3).$$

This means that the three events are pairwise independent.

exa: A driving licence examination consists of two parts which are based on a theoretical and practical examination. Suppose 30% of people fail the practical examination, 15% of people fail the theoretical examination, and 10% of people fail both the examination. If a person is randomly chosen, then what is the probability that this person

a) fails at least one of the examination

b) only fails the practical examination but not the theoretical examination?

c) successfully passes both the test.

d) fails any of ~~both~~ the two examination?

Given:

$$P(PE) = 0.3 \quad P(TE) = 0.15 \quad P(P \cap T) = 0.1$$

$$\begin{aligned} \text{a) } P(PE \cup TE) &= P(PE) + P(TE) - P(PE \cap TE) \\ &= 0.3 + 0.15 - 0.1 \\ &= 0.35. \end{aligned}$$

$$\text{b) } P(PE|TE) = P(PE) - P(PE \cap TE) = 0.3 - 0.1 = 0.2$$

$$\text{c) } P(\overline{PE \cup TE}) = 1 - P(PE \cup TE) = 1 - 0.35 = 0.65$$

d) We are interested in the probability of the person failing exactly in one exam. This corresponds to

$$\begin{aligned} P(\overline{PE} \cap TE \cup TE \cap \overline{PE}) &= P(PE|TE) + P(TE|PE) \\ &= P(PE) - P(P \cap T) + \\ &\quad P(TE) - P(P \cap T) \\ &= 0.3 - 0.1 + 0.15 - 0.1 \\ &= 0.25 \end{aligned}$$

1. Question: Ten numbered cards are there from 1 to 15, and two cards are chosen at random such that the sum of the numbers on both the cards is even. Find the probability that the chosen cards are odd-numbered.

Solution: Let, A \equiv event of selecting two odd-numbered cards, B \equiv event of selecting cards whose sum is even. Then, $n(B) =$ number of ways of choosing two numbers whose sum is even $= 8C2 + 7C2$. $n(A \cap B) =$ number of ways of choosing odd-numbered cards such that their sum is even. $= 8C2$. Now, $P(A|B) = P(A \cap B)/P(B) = n(A \cap B)/n(B) = 8C2 / (8C2 + 7C2) = 4/7$ 1.

2. Question: Let E and F be events of an experiment such that $P(E) = 3/10$, $P(F) = 1/2$, and $P(F|E) = 2/5$. Find the value of:

- o (i) $P(E \cap F)$
- o (ii) $P(E|F)$
- o (iii) $P(E \cup F)$

Solution: We know that $P(A|B) = P(A \cap B)/P(B)$. Therefore,

- o (i) $P(E \cap F) = P(F|E).P(E) = \frac{2}{5} \times \frac{3}{10} = 3/25$
- o (ii) $P(E|F) = P(E \cap F)/P(F) = (3/25)/(1/2) = 6/25$
- o (iii) $P(E \cup F) = P(E) + P(F) - P(E \cap F) = \frac{3}{10} + \frac{1}{2} - \frac{3}{25} = 17/25$ 1.

3. Question: The probability of a student passing in science is $4/5$ and the probability of the student passing in both science and maths is $1/2$. What is the probability of that student passing in maths knowing that he passed in science?

Solution: Let A \equiv event of passing in science, B \equiv event of passing in maths. Given, $P(B) = 4/5$ and $P(A \cap B) = 1/2$. Then, probability of passing maths after passing in science $= P(B|A) = P(A \cap B)/P(A) = (1/2)/(4/5) = 5/8$ 1.

4. Question: In a survey among a few people, 60% read Hindi newspaper, 40% read English newspaper, and 20% read both.

Solution: Let there be 100 people in the survey.

- o The number of people who read Hindi newspaper only is 40.
- o The number of people who read English newspaper only is 20.
- o The number of people who read both newspapers is 20.

Therefore, the number of people who read at least one newspaper is 80.

The probability that a person reads Hindi newspaper given that he/she reads English newspaper is given by:

$$\begin{aligned} P(\text{Hindi} | \text{English}) &= [P(\text{Hindi} \cap \text{English})] / [P(\text{English})] \\ &= [20/100] / [40/100] \\ &= **0.5** 1. \end{aligned}$$

5. Question: Two coins are tossed simultaneously. What is the probability that at least one head appears?

Solution: The sample space S consists of four equally likely outcomes: {HH, HT, TH, TT}. The event A consists of three outcomes: {HH, HT, TH}. Therefore,

$$\begin{aligned} P(\text{at least one head}) &= P(A) \\ &= [\text{number of outcomes in } A] / [\text{total number of outcomes}] \\ &= [3] / [4] \\ &= **0.75** 1. \end{aligned}$$

very hard level question with solution on bayes theorem

- ✓ Searching for: bayes theorem questions and solutions
- ✓ Generating answers for you...

Here are five hard level questions with solutions on Bayes' theorem:

1. Question: A company produces two types of products, A and B. The probability of a product being faulty is 0.02 for product A and 0.05 for product B. If a product is selected at random and is found to be faulty, what is the probability that it is product A?

Solution: Let $E_1 \equiv$ event of selecting product A, $E_2 \equiv$ event of selecting product B, $F \equiv$ event of selecting a faulty product. Then, $P(E_1) = 1/2$, $P(E_2) = 1/2$, $P(F|E_1) = 0.02$, $P(F|E_2) = 0.05$.

We need to find $P(E_1|F)$. By Bayes' theorem,

$$P(E_1|F) = [P(F|E_1).P(E_1)] / [P(F|E_1).P(E_1) + P(F|E_2).P(E_2)] = (0.02 \times 0.5) / (0.02 \times 0.5 + 0.05 \times 0.5) = 0.286^1.$$

2. Question: In a certain population, the probability of having a certain disease is 0.01. A test for the disease has been developed that has a false positive rate of 5% and a false negative rate of 10%. If a person tests positive for the disease, what is the probability that they actually have the disease?

Solution: Let $D \equiv$ event of having the disease, $D' \equiv$ event of not having the disease, $T \equiv$ event of testing positive for the disease, $T' \equiv$ event of testing negative for the disease.

Then, $P(D) = 0.01$, $P(D') = 0.99$, $P(T|D) = 0.05$ (false positive rate), $P(T'|D) = 0.10$ (false negative rate). We need to find $P(D|T)$. By Bayes' theorem,

$$P(D|T) = [P(T|D).P(D)] / [P(T|D).P(D) + P(T|D').P(D')] = (0.99 \times 0.01) / (0.99 \times 0.01 + 0.05 \times 0.99) = 0.166^2.$$

3. Question: A box contains two coins: one is fair and the other has heads on both sides. One coin is chosen at random and flipped twice; both times it shows heads. What is the probability that the coin chosen was fair?

Solution: Let $F \equiv$ event of choosing the fair coin, $U \equiv$ event of choosing the unfair coin, $H \equiv$ event of getting heads twice in a row when flipping a coin twice. Then, $P(F) = P(U) = 1/2$, $P(H|F) = (1/2)^2 = 1/4$ (probability of getting heads twice in a row with a fair coin), $P(H|U) = 1$ (probability of getting heads twice in a row with an unfair coin). We need to find $P(F|H)$.

By Bayes' theorem,

$$P(F|H) = [P(H|F).P(F)] / [P(H|F).P(F) + P(H|U).P(U)] = (1/4 \times 1/2) / (1/4 \times 1/2 + 1 \times 1/2) = 0.2^1.$$

1 8 2 5 7 4 8 2 4 3 3 8 0 1 8 1 4 8 2 6 6 3 7 = 5 5 4 2 6 5 5

6 4 7 2 7 9 4 5 0 2 1 9 4 1 8 4 0 2 4 5 6 2 9 7 1 1 7 6 8

A bank considers changing its credit card policy. Currently 5% of credit card owners are not able to pay their bill in any month, i.e., they never pay their bills. Among those who are generally able to pay their bills, there is still a 20% probability that the bill is paid too late in particular month.

- (a) What is the probability that someone is not paying his bill in a particular month?
- (b) A credit card owner did not pay his bill in a particular month. What is the probability that he never pays back the money?
- (c) Should the bank consider blocking the credit card if a customer does not pay his bill on time?

Solⁿ: Define

$$A: \text{Bill never paid} \quad P(A) = 0.05 \quad P(\bar{A}) = 0.95$$

$$M: \text{Bill paid too late} \quad P(M) = ?$$

$$P(M|\bar{A}) = 0.2$$

$P(M|A) = 1$ because someone who never pays will always pay too late.

- (a) We are interested in $P(M)$, the probability that someone does not pay his bill in a particular month, either because he is not able to or he pays

too late. We can use the law of total probability to obtain the results:

$$\begin{aligned} P(M) &= P(M|A)P(A) + P(M|\bar{A}) \cdot P(\bar{A}) \\ &= 0.05 \times 1 + 0.2 \times 0.95 \\ &= 0.24 \end{aligned}$$

(b) We can use Bayes' Theorem to obtain the results:

$$P(A|M) = \frac{P(A)P(M|A)}{P(M)} = \frac{0.05}{0.24} = 0.208.$$

(c) If the bill was not paid in particular month, the probability is 20.8% that it will never^{be} paid and 78.2% that it will be paid.

One could argue that a preventive measure that affects almost 79% of trustworthy customers are not ideal and the bank should therefore not block a credit card if a bill is not paid on time.

(2) A football practice target is a portable wall with two holes (which are the target) in it for training shots. Suppose there are two players A and B. The probabilities of hitting the target by A and B are 0.6 and 0.4, respectively.

(a) What is the probability that at least one of the

jersey succeeds with his shot?

What is the probability that exactly one of the players hits the target?

(c) What is the probability that only B scores?

Solⁿ: (a) The two shots are independent of each other

and thus $P(A \cap B) = \cancel{0.6} \times \cancel{0.4} = P(A) \cdot P(B)$
 $= 0.6 \times 0.4 = 0.24$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.6 + 0.4 - 0.24 \\ = \underline{\underline{0.76}}$$

(b) $P(A|B \cup B|A) = P(A|B) + P(B|A) = \cancel{P(A|A)} P(A) - P(A \cap B)$
 $+ P(B) - P(A \cap B)$
 $= 0.6 - 0.24 + 0.4 - 0.24 = \underline{\underline{0.52}}$

(c) $P(B|A) = 0.6 - 0.24 = \underline{\underline{0.16}}$

(3) Mr. Patil asks his neighbour to take care of his lemon plant while he is away on leave. He assumes that his neighbour does not take care of the lemon plant with probability of $\frac{4}{10}$. The lemon dies with probability $\frac{1}{2}$ when ~~when~~ someone takes care of it and with probability $\frac{3}{4}$ if no one takes care of it.

- (a) Calculate the probability of the ~~lemon~~ plant surviving after its owner's leave.
- (b) It turns out that the lemon^{plant}, eventually died. What is the probability that Mr. Patil's neighbours did not take care of the plant?

Soln: Define:

A : Lemon^{plant} is treated well \bar{A} : Lemon^{plant} is not treated well

B : Lemon^{plant} survives \bar{B} : Lemon^{plant} dies.

$$P(\bar{A}) = \frac{4}{10} \Rightarrow P(A) = \frac{6}{10}; P(\bar{B})$$

$$P(B|A) = \frac{1}{2} \quad P(B|\bar{A}) = \frac{3}{4}.$$

Using law of Total probability, we get

$$P(B) = P(B|A) \cdot P(A) + P(B|\bar{A}) \cdot P(\bar{A}).$$

$$= \frac{1}{2} \cdot \frac{6}{10} + \frac{3}{4} \times \frac{4}{10} = \frac{6}{10} = 0.6$$

$$(b) P(\bar{A}|\bar{B}) = \frac{P(\bar{B}|\bar{A}) \cdot P(\bar{A})}{P(\bar{B})} = \frac{\frac{1}{4} \times \frac{4}{10}}{\frac{4}{10}} = \frac{1}{4}$$

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There are epidemics which affect animals such as cows, pigs and others. Suppose 200 cows are tested to see whether they are infected with virus or not. Let event A describe whether a cow has been transported by a truck recently or not and B denote the event that cow has been tested positive with a virus. The data are summarized in the following table.

	B	\bar{B}	Total
A	40	60	100
\bar{A}	20	80	100
Total	60	140	200

- ① What is the probability that a cow is infected and has been transported by a truck recently?
- ② What is the probability of having an infected cow given that it has been transported by the truck?
- ③ Determine and interpret $P(B)$?

Soln: The "and" operator refers to the joint distribution of two variables. In our example, we want to know the probability of being infected and having been transported by the truck. This probability can be directly obtained from the respective entry in the contingency table:

40 out of 200 cows fulfill both criteria and thus

$$P(B \cap A) = \frac{40}{200} = \underline{\underline{\frac{1}{5}}}$$

(b) We can use $P(A) = \frac{100}{200} = P(\bar{A})$ to obtain:

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{40/200}{100/200} = \frac{40}{100}$$

(c) Using these results and $P(B) = \frac{60}{200}$ and

$$P(\bar{B}) = \frac{140}{200} = 1 - P(B) \text{ we obtain}$$

$$P(B \cap \bar{A}) = \frac{P(B \cap \bar{A})}{P(\bar{A})} = \frac{20/200}{100/200} = \frac{20}{100}$$

by using the law of total probability.

$$\begin{aligned} P(B) &= P(B|A) \cdot P(A) + P(B|\bar{A}) \cdot P(\bar{A}) \\ &= 0.40 \times 0.50 + 0.20 \times 0.50 \\ &= 0.30 \end{aligned}$$

This means that the probability of a cow being infected is 30 %. Alternatively, we could have simply looked at the marginal distribution of

$$\text{the contingency table to get } P(B) = \frac{60}{200} = \underline{\underline{0.3}}$$