College of Engineering Pune

Linear Algebra and Univariate Calculus(D.S.Y)

Tutorial on Rank Nullity, Eigenvalues, and Eigenvectors.

1. What is the dimension of space of solutions for the following systems of linear equations? In each case, find a basis for the space of solutions.

   (a)
   $$\begin{aligned} 2x + y - z &= 0 \\ 2x + y + z &= 0 \end{aligned}$$

   (b)
   $$\begin{aligned} x + y + z &= 0 \\ x - y &= 0 \\ y + z &= 0 \end{aligned}$$

   (c)
   $$\begin{aligned} 2x - 3y + z &= 0 \\ x + y - z &= 0 \\ 3x + 4y &= 0 \\ 5x + y + z &= 0 \end{aligned}$$

   (d)
   $$\begin{aligned} 4x + 7y - \pi z &= 0 \\ 2x - y + z &= 0 \end{aligned}$$

2. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 7 \end{bmatrix}$. Find Nullity(T) and Rank(T). Further verify Rank-Nullity theorem.

3. Take a 3 x 4 matrix of your choice and do the above things. (Don't take a null matrix :) . Also try to take distinct entries!)

4. Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$. Compute eigenvalues and eigenvectors of $A - 7I$. How are they related to those of A.

5. Verify that sum of eigenvalues of A(above) is equal to trace of A and product of eigenvalues of A is equal to determinant of A. Is this true in general?

6. Prove that eigenvalues of a matrix and its transpose are always same.

7. Prove that similar matrices have same eigenvalues. What can you say about eigenvectors?

8. If a matrix M has λ as an eigenvalue then what can say about eigenvalue of M^{-1}. What about eigenvectors of M and M^{-1}?

9. If a matrix M has λ as an eigenvalue then what can say about eigenvalue of kM where k is some real number. What about eigenvectors of M and kM?

10. Consider a 2×2 matrix whose trace is 5 and determinant is 6. Find its eigenvalues.

11. For the following matrices:

    (a) Compute real eigenvalues and eigenvectors.

    (b) Write down algebraic and geometric multiplities for each eigenvalues.

    (c) Are the matrices diagonalizable? Justify. Further write down the diagonal matrix D and the invertible matrix P.

(a) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$

(f) $\begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix}$

(g) $\begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$

(h) $\begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

(i) $\begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$