Mathaholic ODE 21 / 03 / 2003

Order → Highest derivative
Degree → power of highest derivate (Eliminate all radicals first, i.e. power wale fractions).

Solution → A function y is a solution over an interval, if it satisfies the ODE
- → General Solution involves an arbitary constant $x^2+y^2=C$
- → Particular solution doesn't involve arbitary constant. $x^2+y^2=3$

Variable Separable Method:

→ Separate dependent and independent variables then itegrate them.

Reduction to Variable Separable Method

$F(x,y,y') \longleftarrow$ Notations $\longrightarrow \frac{dy}{dx}=f(x,y)$

$\frac{dy}{dx}=F(x,y)$ → For 1^{st} order linear ODE put everything other than $\frac{dy}{dx}$ to right side

Case I:

$\frac{dy}{dx}=f(\underline{ax+by}+c)$ ↪ replace this with u

1) $y'=\sin(x+y)$

Case II : Homogeneous Differential Equations

$\frac{dy}{dx}=f(\underbrace{y/x}_{u})$

e.g ⇒ $\frac{dy}{dx}=\frac{4x^2+y^2}{xy}$

Total Derivative:

$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$

$f(x,y)=x^2y^3+e^{xy}$

⇓

$df=\frac{\partial(x^2y^3+e^{xy})}{\partial x}dx+\frac{\partial(x^2y^3+e^{xy})}{\partial y}dy$

$=\left[2xy^3+e^{xy}(y)\right]dx+\left[x^2\,3y^2+xe^{xy}\right]dy$

2) Exact differential Equation

$$\frac{\partial y}{\partial x} = \frac{x^3 y}{x^2+y^{10}}$$

$$x^2+y^{10}\,\partial y = x^3 y\,\partial x$$

$$x^3 y\,dx - (x^2+y^{10})\,dy = 0$$

$$x^3 y\,dx + (-x^2-y^{10})\,dy = 0$$

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad \cdots (1) \longleftarrow \text{Required form.}$$

is exact ODE if there exists a differentiable function $u \equiv u(x,y)$ such that

$$\partial u = M\,dx + N\,dy \quad \cdots (2)$$

$$du = 0$$

$$u(x,y) = C$$

Existence Theorem:

$M\,dx + N\,dy = 0$ is exact if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$$\frac{\partial u}{\partial x} = M \qquad \frac{\partial u}{\partial y} = N$$

$$\frac{\partial u}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial M}{\partial y} \quad \& \quad \frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial N}{\partial x}$$

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial M}{\partial y} \quad \& \quad \frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ by Mixed derivative theorem,

$$\underline{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.}$$

4) Integrating Factor Method

Integrating factor:

function f is a function such that whenever you multiply a non-exact differential equation by f the equation becomes exact.

Method 1:

$R(x) = \frac{1}{N}(M_y - N_x)$

.. if the term is independent of y

Integrating factor $= e^{\int R(x)\,dx}$

else:

Method 2: $\frac{1}{M}(N_x - M_y)$

if it is independent of x

Integrating Factor $= e^{\int R(y)\,dy}$

* Linear differential equations of first order:

$\frac{\partial y}{\partial x} + p(x) \cdot y = q(x)$

1) $\left(\frac{dy}{dx}\right)^1 \longleftarrow$ this should be one.

2) degree of dependent variable should be 1. (y's degree is 1)

3) No product of dependent variable & derivative.

Types:

Homogeneous $\longrightarrow q(x) = 0$ else non-homogeneous $q(x) \,!= 0$

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① $x + y\frac{dy}{dx} = 0$, $x^2+y^2=c$ ...verify

$y\frac{dy}{dx} = -x$

$y\,dy = -x\,dx$

$\int y\,dy = -\int x\,dx$

$\frac{y^2}{2} = -\frac{x^2}{2} + C$

$\frac{y^2}{2}+\frac{x^2}{2} = C$

$y^2+x^2 = 2c$

$x^2+y^2=2c$

$x^2+y^2 = c$

② $x - y\frac{dy}{dx} = 0$, $y^2-x^2=c$

$\Rightarrow x - y\frac{dy}{dx} = 0$

$x = y\frac{dy}{dx}$

$x\,dx = y\,dy$

$\frac{x^2}{2} = \frac{y^2}{2} + c$

$\frac{x^2-y^2}{2} = c$

$x^2-y^2 = 2c$

$x^2-y^2 = c$

$-(y^2-x^2) = -c$

$y^2-x^2 = c$

Verify: $y''-y=0$. check following are ODE's

1) $y_1(x) = \sin(x)$

$y = \sin(x)$

$y' = \cos(x)$

$y'' = -\sin x$

$\therefore y''-y = -\sin x - \sin x$

$= -2\sin x \neq 0$

$y''+y = -\sin x + \sin x \Rightarrow 0$ ✓

2) $y_2(x) = \cos x$

$y' = -\sin x$

$y'' = -\cos x$

$y''-y = -\cos x - \cos x$

$= -2\cos x \neq 0$

$y''+y = -\cos x + \cos x$

$= 0$ ✓

3) $y_3(x) = \sin(x)+5$

$y' = \cos x$

$y'' = -\sin x$

$y''-y = -\sin x - (\sin x + 5)$

$\neq 0$

$y''+y = -\sin x + \sin x + 5$

$= 5$ ✗

4) $3\sin(x) + 7\cos(x)$

$y' = 3\cos x - 7\sin x$

$y'' = -3\sin x - 7\cos x$

$y''-y = -6\sin x - 14\cos x$

$\neq 0$

$y''+y = -3\sin x - 7\cos x + 3\sin x + 7\cos x$

$= 0$

Variable Separable Method

① $\frac{dy}{dx} + 4y = 1.4$

$\frac{dy}{dx} = 1.4 - 4y$

$\frac{dy}{1.4-4y} = dx$

$\int \frac{dy}{1.4-4y} = \int dx$

$\frac{\ln(1.4-4y)}{-4} = x + C_1$

$\ln(1.4-4y) = -4x - 4C_1$

$1.4-4y = e^{-4x-4C_1}$

$1.4-4y = e^{-4x}\cdot e^{-4C_1}$

$4y = 1.4 - e^{-4x}\cdot e^{-4C_1}$

$y = \frac{1.4 - Ce^{-4x}}{4}$

Particular solution for $y(0)=10$

$x_0 = 0$

$y_0 = 10$.

$y = \frac{1.4 - Ce^{-4x}}{4}$

$10\times 4 = 1.4 - Ce^{-4(0)}$

$40 = 1.4 - Ce^0$

$40 - 1.4 = -C(1)$

$38.6 = -C$

$C = -38.6$

Solution:

$y = \frac{1.4 + 38.6e^{-4x}}{4}$

Particular solution (as no arbitrary const exist)

② $\sqrt{(1+4x^2)}\,dy - xy^3\,dx = 0$

$\sqrt{(1+4x^2)}\,dy = xy^3\,dx$

$\frac{1}{y^3}dy = \frac{x}{\sqrt{(1+4x^2)}}dx$

$\int \frac{1}{y^3} = \int \frac{x}{\sqrt{1+4x^2}}dx$

Ⓘ $\int \frac{1}{y^3} = \int y^{-3} = \frac{y^{-2}}{-2} = \frac{1}{-2y^2} = -\frac{1}{2y^2}$

Ⓘⓘ $\int \frac{x}{\sqrt{1+4x^2}}dx$, put $1+4x^2 = u$

$8x = \frac{du}{dx}$

$dx = du/8x$

$\int \frac{x}{\sqrt{u}} \times \frac{du}{8x}$

$1/8 \int \frac{du}{\sqrt{u}} \rightarrow 1/8 \int u^{-1/2}$

$= 1/8 \; \frac{u^{1/2}}{1/2}$

$= 1/8 \; 2\sqrt{u}$

$= \frac{\sqrt{u}}{4}$

$= \frac{\sqrt{1+4x^2}}{4}$

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$$\frac{-1}{2y^2} = \frac{\sqrt{1+4x^2}}{4} + \frac{4c}{4}$$

$$\frac{-1}{y^2} = \frac{\sqrt{1+4x^2}}{2} + \frac{4c}{2}$$

$$y^2 = \frac{-2}{\sqrt{1+4x^2}+4C}$$

HW

① $y^3 y' + x^3 = 0$

$$y^3 \frac{dy}{dx} = -x^3$$

$$y^3 dy = -x^3 dx$$

$$\int y^3 dy = -\int x^3 dx$$

$$\frac{y^4}{4} = -\frac{x^4}{4} + C_1$$

$$y^4 + x^4 = 4C_1$$

$$x^4 + y^4 = C_2$$

$$\therefore C_2 = 4C_1$$

② $y' = \sin(x+y)$

$$\frac{dy}{dx} = \sin(x+y)\ dx$$

let $u = x + y$

$$\frac{du}{dx} = 1 + y'$$

$$u' = 1 + \sin(u)$$

$$\frac{du}{1+\sin u} = dx$$

$$\frac{1-\sin u\ du}{1-\sin^2 u} = dx$$

$$\frac{1-\sin u\ du}{\cos^2 u} = dx$$

$$\frac{1}{\cos^2 u} - \frac{\sin u}{\cos^2 u} = dx$$

$$\sec^2 u - \tan u \cdot \sec u\ du = dx$$

$$\int \sec^2 u\, du - \int \tan u \cdot \sec u\ du = \int dx$$

$$\tan u - \sec u = x + C$$

$$\tan(x+y) - \sec(x+y) = x + C$$

③ $y' \sin(2\pi x) = \pi y \cos(2\pi x)$

put $u = 2\pi x$

$$\frac{du}{dx} = 2\pi$$

$$\frac{du}{2\pi} = dx$$

$$dy\ \sin(u) = \cancel{\pi} y \cos(u)\ \frac{du}{2\cancel{\pi}}$$

$$dy\ \sin(u) = y\ \cos(u)\ \frac{du}{2}$$

$$dy\ \frac{\sin u}{y} = \cos(u)\frac{du}{2}$$

$$\int dy\ \frac{1}{y} = \frac{1}{2}\int \frac{\cos u}{\sin u}\ du$$

$$\ln(y) = \frac{1}{2}\int \cot u\ du$$

$$2\ln(y) = \ln(\sin u) + \ln(c)$$

$$2e^{\ln(y)} = e^{\ln(\sin u)+\ln(c)}$$

$$2y = e^{\ln(\sin u)} \cdot e^{\ln(c)}$$

$$2y = \sin u \cdot C$$

$$y = \frac{\sin(2\pi x) + C}{2}$$

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Reducing to Variable Separable Method

① $y' = (9x + y + 17)^2 \quad \cdots ①$

let $u = 9x + y + 17$

$$\therefore \frac{du}{dx} = 9 + y'$$

$$u' = 9 + y'$$

$$y' = u' - 9$$

in Ⓘ

$$u' - 9 = u^2$$

$$u' = u^2 + 9$$

$$\frac{du}{dx} = u^2 + 9$$

$$\frac{du}{u^2+9} = dx$$

$$\int \frac{du}{u^2+3^2} = \int dx$$

$$\frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right) = x + C \qquad \cdots \frac{1}{x^2+a^2} = \frac{1}{a}\tan^{-1}(x/a)$$

$$\frac{u}{3} = \tan(3x + C_2) \qquad \cdots C_2 = 3C$$

$$u = 3\tan(3x + C_2)$$

$$9x + y + 17 = 3\tan(3x + C_2)$$

$$y = 3\tan(3x + C_2) - 9x - 17$$

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② $y' = e^{7x+4y+13}$

let $u = 7x+4y+13$

$\frac{du}{dx} = 7+4y'$

$\frac{u'-7}{4} = y'$

$\frac{u'-7}{4} = e^{u}$

$u'-7 = 4e^{u}$

$u' = 4e^{u}-7$

$\frac{du}{dx} = 4e^{u}-7$

$\frac{du}{4e^{u}-7} = dx$

$\int \frac{du}{4e^{u}-7} = \int dx$

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$y' = e^{7x+4y+13}$

$\frac{dy}{dx} = e^{7x+13} \cdot e^{4y}$

$\frac{dy}{e^{4y}} = e^{7x+13}\,dx$

(I) (II)

(I) $u = 4y$

$\frac{du}{dy} = 4$

$du = 4\,dy$

$\frac{dy}{e^{4y}} \Rightarrow \frac{du}{4} \cdot \frac{1}{e^{u}}$

$\frac{1}{4} \int \frac{du}{e^{u}}$

$\frac{1}{4} \int e^{-u}\,du$

$\left(\frac{1}{4}\right)\left(-e^{-u}\right) + C_1$ ①

(II) $u = 7x+13$

$\frac{du}{dx} = 7$

$\frac{du}{7} = dx$

$e^{u} \times \frac{du}{7}$

$\frac{1}{7} \int e^{u} \cdot du$

$\frac{1}{7} e^{u} + C_2$ ②

$\frac{-e^{-u}}{4} = \frac{e^{u}}{7} + C$ ... $C = C_2 - C_1$

$\frac{-e^{-4y}}{4} = \frac{e^{7x+13}}{7} + C$

$-e^{-4y} = \frac{4e^{7x+13}}{7} + 4C$

logarithim on both sides,

$-4y = \ln\left(\frac{4e^{7x+13}}{7} \times 4C\right)$

$y = -\frac{1}{4} \ln\left(\frac{4e^{7x+13} \times 4C}{7}\right)$

Case II:

① $\frac{dy}{dx} = \frac{4x^2+y^2}{xy}$

$= \frac{4x^2+y^2 \div x^2}{xy \div x^2}$

$= \frac{4+(y/x)^2}{y/x}$

put $u = y/x$

$y = ux$

$y' = u + u'x$

$u + u'x = \frac{4+u^2}{u}$

$u'x = \frac{4+u^2-u^2}{u}$

$u'x = \frac{4}{u}$

$\frac{du}{dx} x = \frac{4}{u}$

$u\,du = \frac{4\,dx}{x}$

$\int u\,du = 4 \int \frac{dx}{x}$

$\frac{u^2}{2} = 4\ln(x) + C$

$u^2 = 8\ln(x) + C$

$\frac{y^2}{x^2} = 8\ln(x) + C(x)$

$y^2 = 8\ln(x)x^2 + C(x)$

HW: $2xyy' = 3y^2 + x^2$

$2y' = \frac{3y^2+x^2}{xy}$

$2y' = \frac{3y}{x} + \frac{x}{y}$

let $u = y/x$

$y = xu$

$y' = u + u'x$

$2(u+u'x) = 3u + \frac{1}{u}$

$2u + 2u'x = 3u + \frac{1}{u}$

$2u'x = 3u - 2u + 1/u$

$2u'x = u + \frac{1}{u}$

$2u'x = \frac{u^2+1}{u}$

$2\frac{du}{dx}x = \frac{u^2+1}{u}$

$2 \cdot \frac{u}{u^2+1} du = \frac{dx}{x}$

(I)

(I) $\frac{u}{u^2+1} du$

let $u^2+1 = t$

$\frac{dt}{du} = 2u$

$\frac{dt}{2u} = du$

$\frac{\cancel{u}}{t} \frac{dt}{2\cancel{u}}$

$\frac{1}{2} \int \frac{1}{t} dt$

$\frac{1}{2} \ln(t)$

$\frac{1}{2} \ln(u^2+1)$

$\therefore 2 \cdot \frac{u\,du}{u^2+1} = \frac{dx}{x}$

$\cancel{2} \cdot \frac{1}{\cancel{2}} \ln(u^2+1) = \ln(x) + \ln(C)$

$\ln(u^2+1) = \ln(x) + \ln(C)$

$u^2+1 = xC$

$\frac{y^2+x^2}{x^2} = xC$

$y^2 = x^3C - x^2$

$y^2 = x^3C - x^2$

$y^2 = x^2(xC-1)$

$y(1) = 2$

$x_0 = 1$; $y_0 = 2$

$2^2 = 1^2(1 \times C - 1)$

$4 = 1(C-1)$

$4 = C - 1$

$C = 5$

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Find total derivative of:

① $f(x,y,z) = x^2y^3 + e^{xy} + z^2xy$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$= \left[2xy^3 + ye^{xy} + z^2y\right]dx + \left[x^2 3y^2 + xe^{xy} + z^2x\right]dy + \left[2zxy\right]dz$$

ex) $e^y dx + (2y + xe^y)\,dy = 0$

$M(x,y) = e^y$
$N(x,y) = 2y + xe^y$

$$\frac{\partial M}{\partial y} = e^y$$

$$\frac{\partial N}{\partial x} = 0 + 1\cdot e^y$$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ i.e. $M_y = N_x$ $\therefore$ Exact ODE

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$\frac{\partial u}{\partial x} = e^{x}$$

$$\frac{\partial u}{\partial y} = 2y + xe^y$$

$$\int \frac{\partial u}{\partial x} = \int e^y$$

$$\int du = \int e^y dx$$

$u = xe^y + c(y)$ ···①

$$\frac{\partial u}{\partial y} = xe^y + \frac{\partial c(y)}{\partial y}$$

$$2y + xe^y = xe^y + \frac{\partial c(y)}{\partial y}$$

$$\frac{\partial c(y)}{\partial y} = 2y$$

$$\int \frac{\partial c(y)}{\partial y} = \int 2y$$

$$c(y) = \cancel{2}\cdot\frac{y^2}{\cancel{2}}$$

$$c(y) = y^2$$

$$u = xe^y + y^2$$

example, again

$$e^y dx + (2y + xe^y)\, dy = 0 \quad \cdots (1)$$

$M(x,y) = e^y$

$N(x,y) = 2y + xe^y$

$\frac{\partial M}{\partial y} = e^y$

$\frac{\partial N}{\partial x} = 0 + e^y$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \therefore$ (1) is Exact ODE

$du = e^y dx + (2y + xe^y)\, dy$

$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = e^y dx + (2y + xe^y)\, \partial y$

$\frac{\partial u}{\partial x} = e^y \quad \cdots (1)$

$\frac{\partial u}{\partial y} = 2y + xe^y \quad \cdots (2)$

Integrate (1) w.r.t. x

$\int \frac{\partial u}{\partial x} = \int e^y$

$\int du = \int e^y\, dx$

$u = xe^y + c(y) \quad \cdots (3)$

differentiate (3) w.r.t. y

$\frac{\partial u}{\partial y} = xe^y + \frac{d\, c(y)}{\partial y}$

$2y + \cancel{xe^y} = \cancel{xe^y} + \frac{\partial c(y)}{\partial y}$

$\int \frac{\partial c(y)}{\partial y} = \int 2y$

$c(y) = y^2 + C_1$

Put values in (3)

$u = xe^y + y^2 + C_1$

Solution:

General $\Rightarrow u(x,y) = C$

$\therefore \quad xe^y + y^2 = C \qquad C = C - C_1$

HW. (1) $(6x^2 - y + 3)\, dx + (3y^2 - x - 2)\, dy = 0$

$M = 6x^2 - y + 3$

$N = 3y^2 - x - 2$

$\frac{\partial M}{\partial y} = 0 - 1$

$\frac{\partial N}{\partial x} = 0 - 1$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ Equation is exact ODE

$du = (6x^2 - y + 3)\, dx + (3y^2 - x - 2)\, dy$

$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$

$\therefore \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = (6x^2 - y + 3)\, dx + (3y^2 - x - 2)\, dy$

Comparing coefficients

$\therefore \frac{\partial u}{\partial x} = 6x^2 - y + 3 \quad \cdots (1) \qquad \frac{\partial u}{\partial y} = 3y^2 - x - 2 \quad \cdots (2)$

Integrating (1) w.r.t. x

$\int du = \int 6x^2 - y + 3\, dx$

$u = \int 6x^2 dx + \int -y + 3\, dx$

$= 6 \int x^2 dx + (-y+3) \int 1\, dx$

$= 6 \times x^3/3 + (-y+3)x \quad + C$

$= 2x^3 - yx + 3x + C(y) \cdots (3)$

diff. w.r.t. y,

$\frac{\partial u}{\partial y} = 0 - x + 0 + \frac{\partial c(y)}{\partial y}$

$3y^2 - \cancel{x} - 2 = \cancel{x} + \frac{\partial\, c(y)}{\partial y}$

$\int \frac{\partial c(y)}{\partial y} = \int 3y^2 \partial y - 2 \int 1\, \partial y$

$c(y) = y^3 - 2y$

$u = 2x^3 - yx + 3x + y^3 - 2y$

$\therefore$ Solution,

$2x^3 - yx + 3x + y^3 - 2y = C$

② HW ⇒ $(2xy - \sin x)\,dx + (x^2 - \cos y)\,dy$

$$M = 2xy - \sin x$$
$$N = x^2 - \cos y$$
$$\frac{\partial M}{\partial y} = 2x$$
$$\frac{\partial N}{\partial x} = 2x$$
$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ Exact ODE

$$du = (2xy - \sin x)\,dx + (x^2 - \cos y)\,dy$$
$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$
$$\therefore \frac{\partial u}{\partial x} = 2xy - \sin x \quad \cdots ①$$
$$\frac{\partial u}{\partial y} = x^2 - \cos y \quad \cdots ②$$

Integrate ① w.r.t x

$$\int du = \int 2xy - \sin x \; dx$$
$$u = \int 2xy\, dx - \int \sin x\, dx$$
$$u = 2y\int x\, dx - \int \sin x\, dx$$
$$u = \cancel{2}y\frac{x^2}{\cancel{2}} - (-\cos x) + C(y)$$
$$u = x^2y + \cos x + C(y) \quad \cdots ③$$

Diff ③ w.r.t. y,

$$\frac{\partial u}{\partial y} = x^2 + \frac{\partial C(y)}{\partial y}$$
$$\cancel{x^2} - \cos y = \cancel{x^2} + \frac{\partial C(y)}{\partial y}$$
$$\int \partial C(y) = \int -\cos y \; dy + C_2$$
$$C(y) = -\sin y + C_2$$

$$u = x^2y + \cos x - \sin y$$

General solution ⇒ $u(x,y) = C$

$$\therefore x^2y + \cos x - \sin y = C$$

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Constructive proof :

Consider $Mdx + Ndy = 0$ & $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

To prove : $\exists\ u(x,y)$ such that

$$du = Mdx + Ndy$$

$$\frac{\partial u}{\partial x}\partial x + \frac{\partial u}{\partial y}dy = Mdx + Ndy$$

i.e. $\frac{\partial u}{\partial x} = \partial x \qquad \frac{\partial u}{\partial y} = N$

$$\int M dx = G(x,y)$$

$$\frac{\partial}{\partial x}\int M \partial x = \frac{\partial}{\partial x} G(x,y)$$

$$M = \frac{\partial}{\partial x} G(x,y)$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial G(x,y)}{\partial x}\right)$$

$$\frac{\partial M}{\partial y} = \frac{\partial^2 G(x,y)}{\partial y \partial x}$$

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} = \frac{\partial^2 G(x,y)}{\partial y\, \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial G(y)}{\partial y}\right)$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial G(y)}{\partial y}\right)$$

$$\frac{\partial N}{\partial x} - \frac{\partial}{\partial x}\left(\frac{\partial G(y)}{\partial y}\right)$$

$$\frac{\partial}{\partial x}\left(N - \frac{\partial G(y)}{\partial y}\right) = 0$$

$$N - \frac{\partial G}{\partial x} = \phi y$$

$$u(x,y) = G(x,y) + \int_y \phi y \, dy$$

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Integrating factor:

1) $2y\,dx + 3x\,dy = 0$

$M = 2y$
$N = 3x$

$\frac{\partial M}{\partial y} = 2$

$\frac{\partial N}{\partial x} = 3$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

i.e. $M_y \neq N_x$

$F = \frac{1}{xy}$

$FM\,dx = FN\,dy$

$\frac{1}{x\cancel{y}} 2\cancel{y} = \frac{1}{\cancel{x}y} 3\cancel{x}$

$\frac{\partial}{\partial y}\frac{2}{x} = \frac{\partial\, 3/y}{\partial x}$

$0 = 0$

2) $(x+y)\sin y\,dx + (x\sin y + \cos y)\,dy = 0$

$\Rightarrow$

$M = (x+y)\sin y\,dx$
$N = (x\sin y + \cos y)\,dy$

M_y:

...by Integration by parts.
$\int uv' = uv - \int v\,du$

$u = (x+y)$
$v' = \sin y\,dx$
$u' = 1$
$v = -\cos y$

$M_y = -(x+y)\cos y - (-\cos y)1$

$M_y = -(x+y)\cos y + \int \cos y$
$= -(x+y)\cos y + \sin y$

$N_y = \sin y$

$R(x) = \frac{1}{N}\left(M_y - N_x\right)$

$= \frac{1}{x\sin y + \cos y} \cdot -(x+y)\cos y \cancel{+\sin y} - \cancel{\sin y}$

$= \frac{-x+y}{x\sin y + \cos y}$

..but since y is present we can't use this,

$R(y) = \frac{1}{M}(N_x - M_y)$

$= \frac{1}{\cancel{(x+y)}\sin y} \cdot -\cancel{(x+y)}\cos y$

$= \frac{-\cos y}{\sin y}$

$R(y) = -\cot y$

$IF = e^{-\int \cot y}$

$IF = e^{-\ln(\sin y)}$

$IF = e^{\ln(\sin y^{-1})}$

$IF = \sin y^{-1}$

Mul both sides by Integrating factor

$(x+y)\cancel{\sin y}\,\cancel{\sin y^{-1}}\,dx + (x\cancel{\sin y}\,\cancel{\sin^{-1} y} + \cos y \sin^{-1} y)\,dy = 0$

$x+y\,dx + (x + \cot y)\,dy = 0$

$M = x+y$ $\quad$ $N = x + \cot y$
$M_y = 1$ $\quad$ $N_x = 1$

$$\frac{\partial u}{\partial x} = x + y$$

$$\frac{\partial u}{\partial y} = x + \cot y$$

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$$\int du = \int x + y \, dx$$

$$u = \int x \, dx + \int y \, dx$$

$$u = \frac{x^2}{2} + xy + c(y) \cdots 1$$

$$\frac{du}{dy} = \frac{\partial x^2/2}{\partial y} + \frac{\partial xy}{\partial y}$$

$$\cancel{x} + \cot y = 0 + \cancel{x} + \frac{\partial c(y)}{\partial y}$$

$$\frac{\partial c(y)}{\partial y} = \cot y$$

$$\int \partial c(y) = \int \cot y \, dy$$

$$c(y) = \ln|\sin y| + C$$

$$u = \frac{x^2}{2} + xy + \ln|\sin y| + C$$

$$\frac{x^2}{2} + xy + \ln|\sin y| = C$$

2) $(3xy - y^2)\,dx + x(x-y)\,dy = 0 \quad \cdots 1$

$M = 3xy - y^2$

$M_y = 3x - 2y$

$N = x^2 - yx$

$N_x = 2x - y$

$N_x \neq M_y$

$$R(x) = \frac{1}{N}\left(M_y - N_x\right)$$

$$= \frac{1}{x^2 - yx}\left[3x - 2y - 2x + y\right)$$

$$= \frac{1}{x(x-y)}\left[x - y\right]$$

$$= \frac{1}{x}$$

$IF = e^{\int R(x)}$

$IF = e^{\int 1/x}$

$= e^{\ln(x)}$

$\boxed{IF = x}$

Multiplying both sides in 1 by x

$x(3xy - y^2)\,dx + x \cdot x(x-y)\,dy = 0$

$M = 3x^2y - xy^2$

$N = x^3 - x^2y$

$M_y = 3x^2 - 2xy$

$N_x = 3x^2 - 2xy$

$M_y = N_x$

$\therefore$ Exact ODE :

$$\frac{\partial u}{\partial x} = 3x^2y - xy^2$$

$$\frac{du}{dy} = x^3 - x^2y$$

$$\int du = \int 3x^2y \, dx - \int xy^2 \, dx$$

$$= 3y \int x^2 \, dx - y^2 \int x \, dx$$

$$= \cancel{3}y\, x^3/\cancel{3} \, dx - y^2 \cdot x^2/2$$

$$u = x^3y - \frac{x^2y^2}{2} + c(y) \cdots 2$$

$$\frac{du}{dy} = x^3 - \frac{2x^2y}{2} + \frac{\partial c(y)}{\partial y}$$

$$x^3 - x^2y = x^3 - x^2y + \frac{\partial c(y)}{\partial (y)}$$

$$\int \partial c(y) = \int 0 \, dy$$

$$c(y) = 0 + C$$

$$u = x^3y - \frac{x^2y^2}{2} + C$$

$$x^3y - \frac{x^2y^2}{2} = C$$

/ /

HW $(x-\cos y)dx - \sin y\, dy = 0 \quad \cdots ①$

$M = x - \cos y$

$N = -\sin y$

$M_y = -(-\sin y) = \sin y$

$N_x = 0$

$M_y \neq N_x$

$R(x) = \frac{1}{N}\left[M_y - N_x\right]$

$= \frac{1}{-\sin y} \cdot \sin y$

$= -1$

$IF = e^{\int R(x)\,dx} \Rightarrow e^{\int -1\,dx}$

$= e^{-\int dx} \Rightarrow e^{-x}$

Mul ① by e^{-x}

$M = e^{-x}x - e^{-x}\cos y$

$N = -e^{-x}\sin y$

$M_y = 0 - e^{-x}(-\sin y)$

$M_y = e^{-x}\sin y$

$N_x = -(-e^{-x}\sin y)$

$= e^{-x}\sin y$

$\frac{\partial u}{\partial x} = e^{-x}x - e^{-x}\cos y$

$\frac{\partial u}{\partial y} = -e^{-x}\sin y$

$\int du = \int e^{-x}x - e^{-x}\cos y\, dx$

$u = \underbrace{\int e^{-x}x\,dx}_{Ⅰ} - \cos y\int e^{-x}dx \quad \cdots ②$

Ⅰ $\int e^{-x}x\,dx \Rightarrow \int u\,dv = uv - \int v\,du$

$u = x \qquad du = 1$

$dv = e^{-x} \qquad v = -e^{-x}$

$\therefore \int e^{-x}x\,dx \Rightarrow -xe^{-x} - \int 1\cdot(-e^{-x})$

$= -xe^{-x} + \int e^{-x}$

$= -xe^{-x} - e^{-x}$

$= -1(xe^{-x} + e^{-x})$

$\therefore u = -xe^{-x} - e^{-x} + \cos y\, e^{-x} + c(y) \quad \cdots ③$

$\frac{\partial u}{\partial y} = 0 - 0 - \sin y\, e^{-x} + \frac{\partial c(y)}{\partial y}$

$-e^{-x}\cancel{\sin y} = -e^{-x}\cancel{\sin y} + \frac{\partial c(y)}{\partial y}$

$\int \partial c(y) = \int 0\, dy$

$c(y) = 0 + C$

$u = -xe^{-x} - e^{-x} + \cos y\, e^{-x} + C$

$\therefore -xe^{-x} - e^{-x} + \cos y\, e^{-x} = C$

$e^{-x}(-x - 1 + \cos y) = C$

$$\boxed{\frac{\cos y - x - 1}{e^{x}} = C}$$

2) $(xy^2 - 2y^3)dx + (3 - 2xy^2)\,dy = 0 \quad \ldots (1)$

$M = xy^2 - 2y^3$

$N = 3 - 2xy^2$

$M_y = 2xy - 6y^2$

$N_x = -2y^2$

$M_y \neq N_x$

$R(x) = \frac{1}{N}\left[M_y - N_x\right]$

$= \frac{1}{3-2xy^2}\left[2xy - 6y^2 + 2y^2\right]$

$= \frac{1}{3-2xy^2}\left[2xy - 4y^2\right]$

$R(y) = \frac{1}{M}\left[N_x - M_y\right]$

$= \frac{1}{xy^2 - 2y^3}\left[-2y^2 - 2xy + 6y^2\right]$

$= \frac{4y^2 - 2xy}{xy^2 - 2y^3}$

$= \frac{4y^2 - 2xy}{y(xy - 2y^2)}$

$= \frac{2\cancel{(2y^2 - xy)}}{-y\cancel{(-xy + 2y^2)}}$

$R(y) = \frac{-2}{y}$

$IF = e^{\int R(y)\,dy}$

$= e^{\int -2/y\,dy} = e^{-2\int 1/y\,dy}$

$= e^{-2\ln(y)}$

$= e^{\ln(y^{-2})}$

$= y^{-2} \Rightarrow IF = 1/y^2$

Mul both sides in (1) by $\frac{1}{y^2}$

$M = x - 2y$

$N = \frac{1}{y^2}(3 - 2xy^2)$

$N = \frac{3}{y^2} - 2x$

$M_y = -2$

$N_x = -2$

$\therefore M_y = N_x$

$\frac{\partial u}{\partial x} = x - 2y$

$\frac{\partial u}{\partial y} = \frac{3}{y^2} - 2x$

$\int du = \int x - 2y\,dx$

$u = \int x - \int 2y\,dx$

$u = \frac{x^2}{2} - 2xy + c(y)$

$\frac{\partial u}{\partial y} = 0 - 2x + \frac{\partial c(y)}{\partial y}$

$\frac{3}{y^2} \cancel{-2x} = \cancel{-2x} + \frac{\partial c(y)}{\partial y}$

$\frac{\partial c(y)}{\partial y} = \frac{3}{y^2}$

$\int \partial c(y) = \int \frac{3}{y^2}\partial y$

$c(y) = 3\int \frac{1}{y^2}dy$

$= 3\int y^{-2}dy$

$= 3 \cdot \frac{y^{-1}}{-1} + C$

$= -\frac{3}{y} + C$

$u = x - 2xy - \frac{3}{y} + C$

$\therefore \frac{x^2}{2} - 2xy - 3y = C$

* Linear differential equation

(I) Homogeneous:

(1) $\frac{dy}{dx} + p(x)\cdot y = 0$

$\frac{dy}{dx} = -p(x)y$

$\frac{dy}{y} = -p(x)\,dx$

$\int \frac{dy}{y} = -\int p(x)\,dx$

$\ln|y| = -\int p(x)\,dx + C$

$y = e^{-\int p(x)} \cdot e^C$

$y = ke^{-\int p(x)dx} \quad \ldots k = e^C$

$$\boxed{\therefore y = e^{-\int p(x)dx}\int e^{\int p(x)dx}\cdot q(x)\,dx + C}$$

Case (II) Non-Homogeneous

$\frac{dy}{dx} + p(x)y = q \quad \ldots q \neq 0$

$\frac{dy}{dx} = -p(x)y + q$

$dy = (-p(x)y + q)\,dx$

$0 = q - p(x)y\,dx - dy$

$M = q - p(x)y$

$N = -1$

$M_y = -p(x)$

$N_x = 0$

$\therefore M_y \neq N_x$

$R(x) = \frac{1}{N}(M_y - N_x)$

$= \frac{1}{-1}(-p(x))$

$= p(x)$

$IF = e^{\int p(x)\,dx}$

$e^{\int p} q - e^{\int p} p(x)y\,dx - e^{\int p}dy = 0$

$e^{\int p(x)dx} q = \left(e^{\int p(x)dx}\cdot y\right)'$

$y \cdot e^{\int p(x)dx} = \int e^{\int p(x)dx}\cdot q(x)\,dx + C$

$\therefore$

dx

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1) $\cos x\, y' + \sin x \cdot y = 2\cos^3 x \sin x - 1$

div by $\cos x$

$$y' + \tan x \cdot y = 2\cos^2 x \sin x - \sec x$$

$$y \cdot e^{\int \tan(x)\,dx} = \int e^{\int \tan x\,dx} \cdot (2\cos^2 x \sin x - \sec x)$$

$$y \cdot e^{\ln(\sec x)} = \int \sec x \cdot (2\cos^2 x \sin x - \sec x)\,dx$$

$$y \cdot \sec x = \int 2\cos x \sin x \,dx - \int \sec^2 x\,dx + C$$

$$y \cdot \sec x = \int \sin 2x - \tan x + C$$

$$= \frac{\cos 2x}{2} - \tan x$$

$$y = \frac{1}{\sec x}\left(\frac{\cos 2x}{2} - \tan x + C\right)$$

2) $xy' - 2y = x^5 \sin(2x) - x^3 + 2x^4 \quad | \; y(\pi) = \frac{3}{2}\pi^4$

$$y' - \frac{2}{x}y = x^4 \sin 2x - x^2 + 2x^3$$

$$y \cdot e^{\int -\frac{2}{x}dx} = \int e^{\int -\frac{2}{x}dx}(x^4 \sin 2x - x^2 + 2x^3)$$

$$y \cdot x^{-2} = \int x^{-2} x^4 \sin 2x - x^{-2} \cdot x^2 + 2x^3 \cdot x^{-2}$$

$$y \cdot x^{-2} = \underset{\text{I}}{\int x^2 \sin 2x} - \int 1\,dx + \int 2x\,dx$$

$x^2 \sin 2x$: $u = x^2$, $du = 2x$
$dv = \sin 2x$, $v = -\frac{\cos 2x}{2}$

$$\int x^2 \sin 2x = -x^2 \cdot \frac{\cos 2x}{2} - \int -\frac{\cos 2x}{\cancel{2}} \cdot \frac{\cancel{2}x}{1}$$

$$= \frac{-x^2 \cos 2x}{2} + \underset{\text{II}}{\int \cos 2x \cdot x}$$

$\int \cos 2x \cdot x \Rightarrow u = x$, $du = 1$
$dv = \cos 2x$, $v = \frac{\sin 2x}{2}$

$$\int \cos 2x \cdot x \Rightarrow x\frac{\sin 2x}{2} - \int \frac{\sin 2x}{2}$$

$$\Rightarrow x\frac{\sin 2x}{2} - \left(-\frac{\cos 2x}{4}\right)$$

$$\Rightarrow \frac{x \sin 2x}{2} + \frac{\cos 2x}{4}$$

$$= -\frac{x^2 \cos 2x}{2} + \frac{x \sin 2x}{2} + \frac{\cos 2x}{4}$$

$$y \cdot x^{-2} = -\frac{x^2 \cos 2x}{2} + \frac{x \sin 2x}{2} + \frac{\cos 2x}{4} - x + x^2$$

$$y = x^2\left(-\frac{x^2 \cos 2x}{2} + \frac{x \sin 2x}{2} + \frac{\cos 2x}{4} - x + x^2\right)$$

$$= -\frac{x^4 \cos 2x}{2} + \frac{x^3 \sin 2x}{2} + \frac{x^2 \cos 2x}{4} - x^3 + x^4$$

$$y = x^4\left(1 - \frac{\cos 2x}{2}\right) + x^3\left(\frac{\sin 2x}{2} - 1\right) + x^2\left(\frac{\cos 2x}{4}\right) + C$$

$\underline{y(\pi) = \frac{3}{2}\pi^4}$

$x_0 = \pi$

$y_0 = \frac{3}{2}\pi^4$

$$\frac{3}{2}\pi^4 = \pi^4\left(1 - \frac{\cos 2\pi}{2}\right) + \pi^3\left(\frac{\sin 2\pi}{2} - 1\right) + \pi^2\left(\frac{\cos 2\pi}{4}\right) + C$$

$$\frac{3}{2}\pi^4 - 22.14 = C$$

$$C = 123.97$$

$$\boxed{\int AB = A\int B - \int \frac{d}{dx}A \times \int B \, . \, dx}$$

Bernoulli's theorem

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e.g. ① $y' = 5y + e^{-2x} y^{-2}$

$$\frac{dy}{dx} - 5y = e^{-2x} y^{-2}$$

$$y^2 \frac{dy}{dx} - 5y \cdot y^2 = e^{-2x} \cancel{y^{-2}} \cancel{y^2}$$

$$y^2 \frac{dy}{dx} - 5y^3 = e^{-2x}$$

$$u = y^3$$

$$\frac{du}{dx} = 3y^2 \cdot \frac{dy}{dx} \quad \text{...chain rule}$$

$$\frac{du}{3y^2\,dx} = \frac{dy}{dx}$$

$$\frac{\cancel{y^2}}{3\cancel{y^2}} \frac{du}{dx} - 5u = e^{-2x}$$

$$u' - \underbrace{15u}_{p(x)} = \underbrace{3e^{-2x}}_{q(x)}$$

$$IF = e^{\int -15\,dx}$$

$$IF = e^{-15x}$$

$$y = \frac{1}{e^{\int p(x)dx}} \times \int e^{\int p(x)dx} \cdot q(x)$$

$$u \cdot e^{-15x} = \int \left(e^{-15x} \cdot 3e^{-2x}\right) dx$$

$$u \cdot e^{-15x} = \int e^{-15x} \cdot (3e^{-2x})\,dx + C$$

$$ue^{-15x} = \int 3e^{-17x}\,dx + C$$

$$u = \frac{3e^{-17x}}{-17} + C$$

$$u = \frac{3e^{-17x} \cdot e^{15x}}{-17} + Ce^{15x}$$

$$u = \frac{3e^{-2x}}{-17} + Ce^{15x}$$

$$y^3 = \frac{3e^{-2x}}{-17} + Ce^{15x}$$

$$\int e^{-17x}\,dx$$

$$u = -17x$$

$$\frac{du}{dx} = -17$$

$$\frac{du}{-17}\ dx$$

$$e^u \frac{du}{-17}$$

$$\frac{1}{-17} \int e^u\,du$$

$$\frac{e^u}{-17} \Rightarrow \frac{e^{-17x}}{-17}$$

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1) $y' + \frac{y}{x} - \sqrt{y} = 0$ $\quad y(1) = 0$

$\frac{xy' + y - x\sqrt{y}}{x} = 0$

$xy' + y - x\sqrt{y} = 0$

$xy' + y = xy^{1/2}$

$x\frac{dy}{dx} + y = xy^{1/2}$

Mul both sides by y^{1-n} i.e. $y^{1-1/2} \Rightarrow y^{1/2}$

$xy^{1/2}\frac{dy}{dx} + y\sqrt{y} = x$

$y' + y/x = \sqrt{y}$

$p(x) = 1/x \quad q(x) = 1 \quad a = 1/2$

$u = y^{1-1/2} = y^{1/2}$

$u' = 1/2 \cdot \frac{1}{y^{1/2}} \cdot y'$

$u' = \frac{1}{2} \times \frac{1}{\sqrt{y}} \times \left[\sqrt{y} + \frac{y}{x}\right]$

$u' = \frac{1}{2}\left[1 + y^{1/2} \times \frac{1}{x}\right]$

$y' + p(x)y = q(x)y^{a}$

$u = y^{1-a} = y^{-2}$

$u = y^{-2} = 1/y^2$

$\frac{du}{dx} = -\frac{2}{y^3} \cdot \frac{dy}{dx}$

$u' = -\frac{2}{y^3}\left[- // - \right]$

$u' + p(x)u = q(x)$

$y' \cdot y^{-1/2}$

$y' + p(x)y = q(x)$

$u' + p(x)u = q(x)$

$y' + y/x - \sqrt{y} = 0$

$y' + y/x = \sqrt{y}$

$p(x) = 1/x \quad q(x) = 1 \quad a = 1/2$

$u = y^{1-a} = y^{1/2}$

$u' = \frac{1}{2} \times \frac{1}{y^{1/2}}\left[y'\right]$

$= \frac{1}{2\sqrt{y}}\left[\sqrt{y} - y/x\right]$

$u' = \frac{1}{2} - \frac{1}{2}\sqrt{y} \times \frac{1}{x}$

$u' + 1/2\sqrt{y} \times \frac{1}{x} = 1/2$

$u' + \frac{1}{2x}u = 1/2$

$p(x) = 1/2x \quad q(x) = 1/2$

$e^{\int 1/2x\,dx} = \sqrt{x}$

$$u \cdot \sqrt{x} = \int \sqrt{x} \cdot 1/2 \; dx$$

$$u \cdot \sqrt{x} = \frac{1}{\cancel{2}} \cdot \frac{x^{1/2}}{3/\cancel{2}} + C$$

$$u \cdot \sqrt{x} = \frac{x^{3/2}}{3} + C$$

$$\sqrt{y} \cdot \sqrt{x} = \frac{x^{3/2}}{3} + C \longrightarrow 0 \cdot 1 = \frac{1^{3/2}}{3} + C$$

$$\boxed{\frac{-1}{3} = C}$$

$$y \cdot x = \frac{x^3}{9} + c^2$$

$$y = \frac{x^2}{9} + \frac{c^2}{x}$$

$$y = \frac{x^3 + 9c^2}{9x}$$

Q.

$$y' + \frac{y}{x} - \sqrt{y} = 0$$

$$y' + \frac{y}{x} = \sqrt{y} \Rightarrow y' = \sqrt{y} - \frac{y}{x}$$

$p(x) = \frac{1}{x}$; $q(x) = 1$; $n = 1/2$

$$u = y^{1/2}$$

$$\frac{\partial u}{\partial x} = \frac{1}{2\sqrt{y}} \frac{\partial y}{\partial x}$$

$$= \frac{1}{2\sqrt{y}} \left[\sqrt{y} - \frac{y}{x} \right)$$

$$= \frac{1}{2} \left[1 - \frac{\sqrt{y}}{x} \right]$$

$$\frac{du}{\partial x} = \frac{1}{2} - \frac{\sqrt{y}}{2x}$$

$$\frac{du}{\partial x} = \frac{1}{2} - \frac{u}{2x}$$

$$u' + \frac{u}{2x} = \frac{1}{2}$$

$p(x) = \frac{1}{2x}$ $\quad y(x) = \frac{1}{2}$

$$IF = e^{\int 1/2x \, dx}$$

$$= e^{1/2 \ln x}$$

$$= e^{\ln \sqrt{x}}$$

$$= \sqrt{x}$$

$$u = \frac{1}{e^{\int 1/2x\, dx}} \int \left[e^{\int 1/2x \, dx} \; y(x) \right]$$

$$u\sqrt{x} = \int \sqrt{x} \cdot \frac{1}{2} \, dx$$

$$u\sqrt{x} = \frac{1}{\cancel{2}} \frac{x^{1/2+1}}{3/\cancel{2}}$$

$$= \frac{x^{3/2}}{3} + C$$

$$u = \frac{x^{3/2} \cdot x^{-1/2}}{3} + \frac{C}{\sqrt{x}}$$

$$= \frac{x}{3} + \frac{C}{\sqrt{x}}$$

$$\sqrt{y} = \frac{x}{3} + \frac{C}{\sqrt{x}}$$

$$\sqrt{y} = \frac{x\sqrt{x} + 3C}{3\sqrt{x}}$$

$$y(1) = 0$$

$$0 = \frac{1\sqrt{1} + 3C}{3\sqrt{1}}$$

$$= \frac{1 + 3C}{3}$$

$$\frac{1}{3} + \frac{\cancel{3}C}{\cancel{3}} \Rightarrow -1/3 = C$$

/ /

$$6y' - 2y = xy^4$$

$$y' - \frac{1}{3}y = \frac{1}{6}xy^4 \Rightarrow \frac{dy}{dx} = \frac{1}{6}xy^4 + \frac{1}{3}y$$

$$p(x) = -\frac{1}{3} \;;\; q(x) = \frac{1}{6}x \;;\; n = 4$$

$$u = y^{1-n} = y^{1-4} = y^{-3}$$

$$\frac{du}{dx} = \frac{-3}{y^4}\frac{dy}{dx}$$

$$= \frac{-3}{y^4}\left[\frac{1}{6}xy^4 + \frac{1}{3}y\right]$$

$$= -3\left[\frac{1}{6}\frac{x\cancel{y^4}}{\cancel{y^4}} + \frac{1}{3}\frac{y}{y^{\cancel{4}3}}\right]$$

$$= -3\left[\frac{1}{6}x + \frac{1}{3y^3}\right]$$

$$= \frac{-3x}{6} - \frac{3}{3y^3}$$

$$\frac{du}{dx} = -\frac{1}{2}x - \frac{1}{y^3}$$

$$u' = -\frac{1}{2}x - \frac{1}{y^3}$$

$$u' = -\frac{1}{2}x - y^{-3}$$

$$u' = -\frac{1}{2}x - u$$

$$u' + u = -\frac{1}{2}x$$

$$p(x) = 1 \qquad q(x) = -\frac{1}{2}x$$

$$IF = e^{\int p(x)dx}$$
$$= e^{1\int dx}$$
$$= e^x$$

$$u \cdot e^x = \int e^x\left(-\frac{1}{2}x\right)dx$$

$$u \cdot e^x = -\frac{1}{2}\int e^x x \, dx$$

$$u = x \qquad du = 1$$
$$dv = e^x \qquad v = e^x$$

$$\int u\,dv = uv - \int v\,du$$
$$= xe^x - \int e^x$$
$$= xe^x - e^x$$

$$u \cdot e^x = -\frac{1}{2}e^x(x-1) + C$$

$$u = -\frac{1}{2}(x-1)\cancel{e^x} \cdot \cancel{e^{-x}} + Ce^{-x}$$

$$u = \frac{-x+1}{2} + Ce^{-x}$$

$$2y^{-3} = 1 - x + Ce^{-x}$$

$$\frac{2}{y^3} = (1-x) + Ce^{-x}$$

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$$y'' + y = 0$$
$$\lambda^2 + 1 = 0$$
$$\lambda^2 = -1$$
$$\lambda = \pm\sqrt{-1}$$
$$\lambda = \pm i$$

Q. $xy'' + 2y' + xy = 0$; $y(x) = x^{-1}\cos x$

$y_2 = uy_1$

$u = \int U\,dx$ ... $U = \frac{1}{y_1^2} \cdot e^{-\int p(x)\,dx}$

$U = \frac{1}{y_1^2} e^{-\int p(x)dx}$ coefficient of y' Provided y'''s coeff. is 1.

$y'' + \frac{2}{x}y' + y = 0$

$y_1(x) = x^{-1}\cos x$

$y_2 = uy_1$; $U = \frac{1}{(x^{-1}\cos x)^2} \cdot e^{-\int \frac{2}{x}dx}$

$u = \int U\,dx$

$U = e^{-2\int 1/x\,dx} \cdot \frac{1}{x^{-2}\cos^2 x}$

$= e^{\ln(x^{-2})} \cdot \frac{1}{x^{-2}}$ – ıı –

$= \frac{\cancel{x^{-2}}\ 1}{\cancel{x^{-2}}\cos^2 x}$

$U = \sec^2 x$

$u = \int \sec^2 x\,dx$

$u = \tan x + C_1$

$y_2 = uy_1$

$= (\tan x + C_1)\,x^{-1}\cos x$

$= x^{-1}\frac{\sin x}{\cancel{\cos x}} \cdot \cancel{\cos x} + C_1' x^{-1}\cos x$

$y_2 = x^{-1}\sin x + C_1' x^{-1}\cos x$

$y(x) = C_1 x^{-1}\cos x + C_2 x^{-1}\sin x + C_2 x^{-1}\cos x\, C_1'$

$= x^{-1}\cos x\,(\underbrace{C_1 + C_1' C_2}_{C_3}) + C_2 x^{-1}\sin x$

$y(x) = C_3 x^{-1} \cos x + C_2 x^{-1} \sin x$

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$(1-x^2) y'' - 2xy' + 2y = 0 \quad ; \quad y_1(x) = x$

$$y'' - \frac{2xy}{1-x^2} + \frac{2y}{1-x^2} = 0$$

$$U = \frac{1}{y_1^2} e^{\int \frac{2x}{1-x^2} dx}$$

$$= \frac{1}{x^2} e^{-\ln(1-x^2)}$$

$$= \frac{1}{x^2} e^{\ln(1-x^2)^{-1}}$$

$$= \frac{1}{x^2} \cdot \frac{1}{1-x^2}$$

$$U = \frac{1}{x^2 - x^4}$$

$$u = \int U dx$$

$$u = \frac{1}{x^2 - x^4}$$

$$\int \frac{2x}{1-x^2}$$

$1 - x^2 = u$

$-2x = \frac{du}{dx}$

$$\int \frac{\cancel{2x}}{u} \frac{du}{\cancel{-2x}}$$

$dx = \frac{du}{-2x}$

$$- \int \frac{1}{u} du$$

$\ln(1-x^2)$

/ /

$y'' + \sin x \, y' = 0$

$y'' = -\sin x \, y'$ $\qquad y' = z \, ; \; y'' = z'$

$z' = -\sin x \, z$

$\frac{dz}{dx} = -\sin x \, z$

$\int \frac{dz}{z} = -\int \sin x \, dx$

$\ln(z) = -(-\cos x) + C$

$\ln(z) = \cos x + C$

$z = k e^{\cos x}$ $\quad \ldots \quad k = e^{C}$

$y' = k e^{\cos x}$

$\int dy = \int k e^{\cos x} \, dx$

$y = k \int e^{\cos x} \, dx$

$y = k \; e^{\cos x}$

2) $y'' = k y'$

$z = y'$; $\qquad \frac{dy}{dx} = z$;

$z' = kz$

$= \int k \, dx$

$\ln(z) = kx$

$z = e^{kx}$

$y' = e^{kx}$

$\frac{dy}{dx} = e^{kx}$

$\int dy = \int e^{kx} \, dx$

$y = \frac{e^{kx}}{k} + C$

$y = e^{kx} + kC$

/ /

• If y_1, y_2 are linearly dependent, then $W(y_1, y_2) = 0$

$y_1 = \sin x$ $\quad y_2 = \cos x$

$$W = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} \qquad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= -\sin^2 x - \cos^2 x$$
$$= -(\sin^2 x + \cos^2 x)$$
$$= -1$$

$$y'' + p(x)y' + q(x)y = 0$$

1) if y_1 & y_2 are LD $\Rightarrow$ $W(y_1, y_2) = 0$

2) if $W(y_1, y_2) \neq 0$ over I $\Rightarrow$ y_1 & y_2 are L.I

3) If y_1 & y_2 are solutions and $W(y_1, y_2) = 0 \Rightarrow$ L.D

4) If y_1 & y_2 are solutions and L.I $\Rightarrow$ $W \neq 0$ on I

Q. $y_1 = x^2$ $\quad y_2 = 1 - x^2$, prove that y_1, y_2 are sol_n of Second order Homo. Linear DE.

Suppose y_1 & y_2 are solutions of

$$y'' + p(x)y' + q(x)y = 0$$

$$W = \begin{vmatrix} x^2 & 1-x^2 \\ 2x & -2x \end{vmatrix} = -\cancel{2x^4} - 2x \cancel{+2x^4}$$
$$= -2x \neq 0 \text{ on } I$$

∴ the two can never be solutions.

① $y_1 = e^{2x}$ $\quad y_2 = e^{3x}$

$$W \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = 3e^{3x} \cdot e^{2x} - 2e^{2x} \cdot e^{3x}$$
$$= 3e^{3x \times 2x} - 2e^{2x \cdot 3x}$$
$$= 3e^{6x^2} - 2e^{6x^2}$$
$$= e^{6x^2} \neq 0$$

∴ y_1, y_2 are linearly independent

② $1, x, x^2$

$$\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 1(2-0) - x(0) + x^2(0-0)$$
$$= 1(2) - 0 + 0$$
$$= 2 \neq 0$$

$\therefore 1, x, x^2$ is linearly independent.

③ $\sin x, \cos x$ on $\mathbb{R}$

$$\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x$$
$$= -(1)$$
$$= -1 \neq 0$$

$\therefore$ LI.

④ $x^2, x|x|$ on $[7, 100$

$$\begin{vmatrix} x^2 & x|x| \\ 2x & 2|x| \end{vmatrix} \Rightarrow x^2 \cdot 2|x| - 2x^2|x|$$
$$= 2x^2|x| - 2x^2|x|$$
$$= 0$$

$\therefore$ LD

⑤ $$\begin{vmatrix} \sinh x & e^x & e^{-x} \\ \cosh x & e^x & -e^{-x} \\ \sinh x & e^x & e^{-x} \end{vmatrix} \Rightarrow \sinh x\left(e^x \cdot e^{-x} - e^x \cdot -e^{-x}\right) - e^x\left(\cosh x\, e^{-x} + \sinh x\, e^{-x}\right)$$
$$- e^{-x}\left(-\cosh x\, e^x + \sinh x\, e^x\right)$$
$$\Rightarrow \sinh x\left(e^0 + e^0\right) -$$

Abel's Identity:

· 2nd Order linear Homo DE.

$$y'' + p(x)y' + q(x)y = 0 \quad \cdots ①$$

y_1, y_2 are Solutions.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 \cdot y_2' - y_1' y_2$$

$$W' = \cancel{y_1' y_2'} + y_1 y_2'' - y_1'' y_2 - \cancel{y_1' y_2'}$$

$$= -y_1 P y_2' - y_1 q y_2 + y_2 P y_1' + y_2 q y_1$$

$$= -P(y_1 y_2' - y_2 y_1') - q(y_1 y_2 - y_2 y_1)$$

$$= -P(W) - q(0)$$

$$y_1'' + p(x)y_1' + q(x)y = 0$$

$$\therefore y_1'' = -p(x)y' - q(x)y$$

$$W' = -P(W)$$

$$\int \frac{W'}{W} = \int -P(x)$$

$$\ln|W| = -P(x)\,dx + C$$

$$W = e^{-\int p(x)dx + C}$$

$$W = e^{-\int p(x)dx} \cdot e^{C}$$

$$W = Ce^{-\int p(x)\,dx}$$

* Find Wronskein when $W(x_0)$ and $p(x)$ given.

$$W(x) = W(x_0)\, e^{-\int_{x_0}^{x} p(x)\,dx}$$

Q. e.g. $y'' - 3y' + 4y = 0$

$$p(x) = -3$$

$$W_x = C \cdot e^{-\int -3\,dx}$$

$$= C \cdot e^{3x}$$

e.g. 2) $x^2 y'' + xy' + y = 0$

$$y'' + \frac{\cancel{x}}{x \times \cancel{x}} y' + \frac{y}{x^2} = 0$$

$$p(x) = 1/x$$

$$W(x) = C \cdot e^{-\int p(x)dx}$$

$$= C \cdot e^{-\int 1/x\,dx}$$

$$= C \cdot e^{-\log x}$$

$$= C \cdot e^{\log(x^{-1})}$$

$$= \frac{C}{x}$$

/ /

$y'' + y = 0$

$y'' + 0y + y = 0$

$p(x) = 0$

$W(x) = C \cdot e^{\int p(x)\,dx}$

$= C \cdot e^{\int 0\,dx}$

$= C \cdot e^{0}$

$W(x) = C$

/ /

Second Order Linear Homogeneous DE with constant coefficients.

$y'' + ay' + by = 0$

$y(x) = e^{\lambda x}$

$y(x)' = \lambda e^{\lambda x}$

$y(x)'' = \lambda^2 e^{\lambda x}$

$\lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + b e^{\lambda x} = 0$

$e^{\lambda x}(\lambda^2 + a\lambda + b) = 0$

Now, $e^{\lambda x}$ is never zero,

$\therefore \lambda^2 + a\lambda + b = 0$, $\leftarrow$ Auxillary equation

λ_1 and λ_2 are two roots

Cases:

1) λ_1 & λ_2 are real & independent

$\therefore y_1 = e^{\lambda_1 x}$

$y_2 = e^{\lambda_2 x}$

2) $\lambda_1 = \lambda_2$ real & equal

$y_1 = e^{\frac{-ax}{2}}$

$y_2 = x \cdot y_1 \Rightarrow y_2 = x e^{\frac{-ax}{2}}$

3) λ_1 & λ_2 are complex

$y = e^{-x} [A \cos \omega x + B \sin \omega x)$

$y(x) = e^{(\text{real part})x} (A \cos(\text{imaginary})x + B \sin(\text{imaginary})x)$

$e^{i\theta} = \cos\theta + i\sin\theta$

Case I: real & distinct

$y'' + 7y' + 10y = 0$
$\lambda^2 + 7\lambda + 10 = 0$
$\lambda^2 + 5\lambda + 2\lambda + 10 = 0$
$\lambda(\lambda+5) + 2(\lambda+5) = 0$
$(\lambda+2)(\lambda+5)$
$\lambda = -2$ or $\lambda = -5$
$\therefore y_1 = e^{-2x}$
$y_2 = e^{-5x}$
$y = c_1 y_1 + c_2 y_2$
$= c_1 e^{-2x} + c_2 e^{-5x}$

Case II Real & equal

$y'' + 10y' + 25y = 0$
$\lambda^2 + 10\lambda + 25 = 0$
$\lambda^2 + 5\lambda + 5\lambda + 25 = 0$
$\lambda(\lambda+5) + 5(\lambda+5) = 0$
$\lambda = -5$ or $\lambda = -5$
$\therefore y_1 = e^{-5x}$
$y_2 = xe^{-5x}$

Case 3 Complex roots

$y'' + 4y' + 9y = 0$
$\lambda^2 + 4\lambda + 9 = 0$
$\lambda = \dfrac{-4 \pm \sqrt{4^2 - 4\times 9}}{2\cdot 1}$
$\lambda = -4 \pm \sqrt{16-36}$
$\lambda = -4 \pm \sqrt{-20}$
$= -4 \pm \sqrt{2\times 2\times -5}$
$= -4 \pm 2\sqrt{-5}$
$= \dfrac{\cancel{2}(-2 \pm 1\sqrt{-5})}{\cancel{2}}$
$= -2 \pm \sqrt{-5}$
$= -2 \pm \sqrt{5}i$

e.g.) 1) $y'' + 7y' + 12y = 0$
$\lambda^2 + 7\lambda + 12 = 0$
$\lambda^2 + 3\lambda + 4\lambda + 12 = 0$
$\lambda(\lambda+3) + 4(\lambda+3) = 0$
$(\lambda+4)(\lambda+3) = 0$
$\lambda = -4$ OR $\lambda = -3$
$y_1 = e^{-4x}$
$y_2 = e^{-3x}$
$y(x) = c_1 e^{-4x} + c_2 e^{-3x}$

2) $y'' + 14y' + 49y = 0$
$\lambda^2 + 14\lambda + 49 = 0$
$\lambda^2 + 7\lambda + 7\lambda + 49 = 0$
$\lambda(\lambda+7) + 7(\lambda+7) = 0$
$(\lambda+7)(\lambda+7) = 0$
$\lambda_1 = \lambda_2 = -7$
$\therefore y_1 = e^{-7x}$
$y_2 = xe^{-7x}$
$y = c_1 e^{-7x} + c_2 x e^{-7x}$

3) $y'' + 6y' + 13y = 0$
$\lambda^2 + 6\lambda + 13 = 0$
$= \dfrac{-6 \pm \sqrt{36 - 4\times 13}}{2} \Rightarrow \dfrac{-6 \pm \sqrt{36-52}}{2}$
$\Rightarrow \dfrac{-6 \pm \sqrt{-16}}{2}$
$\Rightarrow \dfrac{-6 \pm 4\sqrt{-1}}{2}$
$\Rightarrow \dfrac{2(-3 \pm 2i)}{2}$
$\Rightarrow -3 \pm 2i$
$\therefore y = e^{-3x}[A\cos 2x + B\sin 2x]$

where
$A = c_1 + c_2$
$B = i(c_1 - c_2)$

/ /

$x^2y'' - 2xy' + 2y = 0$; $y(1) = 1.5$, $y'(1) = 1$

, $a = -2$, $b = 2$

$y_0 = 1.5$
$x_0 = 1$

put $y = x^m$

$y' = m x^{m-1}$

$y'' = m(m-1)x^{(m-2)}$

$m^2 + (a-1)m + b$

$m^2 + (-2-1)m + 2$

$m^2 - 3m + 2$

$m^2 - 2m - m + 2 = 0$

$m(m-2) - 1(m-2)$

$m = 1, 2$

$y_1 = x^1$

$y_2 = x^2$

$y = x^m$

$y = x^m$

$y' = m x^{(m-1)}$

$y'' = m(m-1)x^{m-2}$

$y_0 = c_1 x_0^1 + c_2 x_0^2$

$y' = c_1 + 2c_2 x$

$1.5 = c_1 + c_2$

$1 = c_1 + 2c_2$

$c_1 + c_2 = 1.5$

$c_1 + 2c_2 = 1$

$-1c_2 = 0.5$

$\boxed{c_1 = 2}$

$\boxed{c_2 = -0.5}$

$\cancel{x^2}(m(m-1)x^{m-2} - 2x(mx^{m-1}) + 2x^m = 0$

$x^m(m^2 - m) - 2mx^m + 2x^m$

$x^m(m^2 - m) - 2mx^m + x^m$

$x^m(m^2 - m - 2m + 1)$

$x^m(m^2 - 3m + 1) = 0$

$x^m \neq 0$,

$\therefore m^2 - 3m + 1 = 0$

/ /

③ $\underline{(x^2D^2 + 3xD + 1)}\ \underline{y} = 0$; $y(1) = 3$; $y'(1) = -4$

$y = x^m$

$y' = mx^{m-1}$

$y'' = m(m-1)\, x^{m-2}$

$x^2y'' + 3xy' + y = 0$

$x^2\,(m(m-1)x^{m-2}) + 3x\, m\, x^{m-1} + x^m = 0$

$x^m\,(\ m^2 - m + 3m + 1) = 0$

$m^2 - 2m + 1 = 0$

$m^2 - m - m + 1 = 0$

$m^2 - m - m + 1 = 0$

$m(m-1) - 1(m-1)$

$m = 1$

$m = 1$

$\boxed{x}$ $\boxed{x\ln x}$

$m = 1, 1$

$y_1 = x$

$y_2 = x\ln x$

$y = C_1x + C_2\, x\ln x \longrightarrow$ $3 = C_1$

$y' = C_1 + C_2 \frac{dy}{dx}\, x\ln x$

$\cancel{x}\cdot\frac{1}{\cancel{x}} + \ln x$

$1 + \ln x$

$y' = C_1 + C_2 + C_2\ln(x)$

$y'(1) = -4$

$x_1 = 1$

$y_1 = -4$

$-4 = C_1 + C_2 + 0$

$-4 = 3 + C_2$

$\boxed{C_2 = -7}$

/ /

② $x^2y'' + xy' + 9y = 0$, $y(1) = 2$, $y'(1) = 0$

$y = x^m$

$y' = mx^{m-1}$

$y'' = m(m-1)x^{m-2}$

$\cancel{x^2}\cancel{x^{m-2}}(m^2 - m) + x\,m x^{m-1} + 9x^m$

$x^m(m^2 - m) + mx^m + 9x^m = 0$

$x^m(m^2 - m + m + 9) = 0$

$m^2 + 9 = 0$

$m^2 = \sqrt{-9}$

$m = \pm 3\sqrt{-1}$

$m = \pm 3i$

$y_0 = e^{0x}\cos 3x$

$y_1 = e^{0x}\sin 3x$

$y = C_1\cos 3x + C_2\sin 3x \rightarrow 2 = C_1\cos(3\cdot 1) + C_2\sin(3\cdot 1) \Rightarrow 0.95C_1 + 0.05C_2$

$y' = -C_1\dfrac{\sin 3x}{3} + C_2\dfrac{\cos 3x}{3} \rightarrow 0 = -0.05C_1 + 0.95C_2$

$0.95C_1 + 0.05C_2$

$\div\ 0.05 \quad +0.95C_2$

$2 = 0.9C_1 + 1C_2 \longrightarrow$ ①

$0.95C_1 + 0.05C_2$

$+0.05C_1 - 0.95C_2$

$2 = 1C_1 - 0.9C_2$ — ②

$C_2 + 0.9C_1 = 2$

$C_2 + 0.9\times 20 = 2$

$C_2 + 18 = 2$

$C_2 = 2 - 18$

$\boxed{C_2 = -16}$

$= 0.9(20) - 16$

$= 18 - 16$

$= 2$

$C_2 = 2 - 0.9C_1$ — ③

$2 = 1C_1 - 0.9(2 - 0.9C_1)$

$2 = C_1 - 1.8 - 0.81C_1$

$3.8 = 0.19C_1$

$380 = 19C_1$

$C_1 = \dfrac{380}{19} \Rightarrow \boxed{C_1 = 20}$

/ /

(a) $(D^4 + 4D^3 + 8D^2 + 8D + 4)\, y = 0$ $\quad n \quad (a+b)^2$

$\rightarrow (D^2 + (2D+2))^2 = 0$

$(D^2 + (2D+2))^2 = 0$

$D^2 + (2D+2) = 0$

$D^2 + 2D + 2 = 0$

$\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $\quad 2 < \frac{2}{2}$

$u = \int \frac{1}{y_1^2} \times e^{-\int p(x)\,dx}$

$= \int \frac{1}{\cancel{e^{-2x}}} \times \cancel{e^{-2x}}$

$= x$

$-1+i$
$-1+i$
$-1-i$
$-1-i$

$y_1 = e^{-x} \cos x$ $\quad y_2 = u y_1$

$y_2 = e^{-x} \sin x$

$y_3 = x e^{-x} \cos x$

$y_4 = x e^{-x} \sin x$

$y = e^{-x}(A\cos x + B\sin x) + x e^{-x}(C\cos x + D\sin x)$

$\rightarrow y'' + 9y = 0$

$y_1 = \cos 3x \quad y \rightarrow LI$ $\quad y'(0) = 6$, $y(0) = 4$

$y_2 = \sin 3x$

$y_1 \rightarrow \cos 3x$

$y_2 \rightarrow$

$\frac{\cancel{\cos 3x}}{2\cancel{\cos 3x}} \boxed{\frac{1}{2}}$

$y_1 \nrightarrow y_2$

$\frac{y_1}{y_2} = \frac{\cos 3x}{\sin 3x} = \cot 3x$

which is not scalar

$y = c_1 \cos 3x + c_2 \sin 3x$ $\quad 4 = c_1$ $\quad u = \int$

$\boxed{c_1 = 4}$

$y' = -c_1 \frac{\sin 3x}{3} + c_2 \frac{\cos 3x}{3}$

$\boxed{6 = c_2}$ $\quad \boxed{\begin{matrix} c_2 = 6 \\ c_1 = 4 \end{matrix}}$

$\boxed{y = 4\cos 3x + 6\sin 3x}$

/ /

$4x^2y'' - 3y = 0 \qquad y(1) = 3, \quad y'(1) = 2.5, \quad y_1 = x^{-1/2}$

$y_2 = x^{3/2}$

$\frac{y_1}{y_2} = \frac{x^{-1/2}}{x^{3/2}} = x^2$ which is not scaler.

$= \frac{1}{x^{3/2} + x^{1/2}} = \frac{1}{x^2} = x^{-2}$

$u = \frac{1}{(y_1)^2} \int e^{-\int p(x)dx} \cdot q(x)$

$y = C_1 x^{-1/2} + C_2 x^{3/2}$

$3 = C_1 1^{-1/2} + C_2 1^{3/2}$

$3 = C_1 + C_2 \quad \cdots 1$

$y' = \frac{-1}{2} C_1 x^{-3/2} + \frac{3}{2} C_2 x^{1/2}$

$2.5 = -\frac{1}{2} C_1 + \frac{3}{2} C_2$

$5 = -C_1 + 3C_2$

$5+3 = C_1 - C_1 + C_2 + 3C_2$

$8 = 4C_2$

$\boxed{C_2 = 2}$

$\boxed{C_1 = 1}$

$6y^2 dx - (2x^4 + xy)\, dy = 0$

$dx - \frac{2x^4 + xy}{6y^2} dy = 0$

$\frac{dx}{dy} = \frac{2x^4 + xy}{6y^2}$

$\frac{dx}{dy} - \frac{x^4}{3y^2} = \frac{x}{6y}$

$\frac{dx}{dy} - \frac{x}{6y} = \frac{x^4}{3y^2}$

$x' - \frac{x}{6y} = \frac{x^4}{3y^2}$

$p(x) = -1/6y \quad ; \quad$ put $x^{1-4} \Rightarrow x^{-3} = u$

$\frac{du}{dy} = -3x^{-4} \frac{dx}{dy}$

$$x' = -\frac{u' x^4}{3}$$

/ /

$$\frac{-u'}{3} x^4 - \frac{x}{6y} = \frac{x^4}{3y^2}$$

$$\frac{-u'}{3} - \frac{1}{6yx^3} = \frac{1}{3y^2}$$

mul by -3

$$u' + \frac{1}{2yx^3} = -\frac{1}{y^2}$$

$$u' + \frac{u}{2y} = -\frac{1}{y^2}$$

$$P = \frac{1}{2y} \qquad q = -\frac{1}{y^2}$$

$$IF = e^{\int p(y) dy}$$

$$= e^{1/2 \int y}$$

$$= e^{1/2 \ln(y)}$$

$$= e^{\ln \sqrt{y}}$$

$$= \sqrt{y}$$

Solution: $u \cdot \sqrt{y} = \int \sqrt{y} \cdot -\frac{1}{y^2}$

$$= u \cdot \sqrt{y} = -1 \int y^{1/2} \cdot y^{-2}$$

$$= u \cdot \sqrt{y} = -1 \int y^{-3/2} dy$$

$$= u \cdot \sqrt{y} = \left(\frac{2}{y^{1/2}} \right)$$

$$u = \frac{2}{\sqrt{y}} \cdot \frac{1}{\sqrt{y}}$$

$$\boxed{u = 2/y + c}$$

$$u = \frac{1}{x^3}$$

$$\frac{1}{x^3} = 2/y + C$$

$$\frac{1}{x^3} = \underline{\frac{2}{} + Cy}$$

$y = 2x^3 + cx^3y$

/ /

Second Order Homogeneous Equations with constant coefficients:

$y'' + ay' + by = 0$

$y(x) = C_1y_1 + C_2y_2$

$y_1 = e^{\lambda_1 x}$

$y_2 = e^{\lambda_2 x}$

Case I: λ_1, λ_2 are real and distinct

$y_1 = e^{\lambda x}$

e.g. $y'' + 7y' + 10y = 0$

$\lambda^2 + 7\lambda + 10 = 0$

$\lambda = -5$ or $\lambda = -2$

$y_1 = e^{-5x}$ $\quad y_2 = e^{-2x}$

$y(x) = C_1e^{-5x} + C_2e^{-2x}$

Case II: $\lambda_1 = \lambda_2$

$y_1 : xe^{\lambda}x$

$y_2 : xe^{\lambda}x$

$y'' + 10y' + 25y = 0$

$\lambda^2 + 10\lambda + 25 = 0$

$\lambda = -5, -5$

$\therefore y_1 = e^{-5\lambda}$

$y_2 = xe^{-5\lambda}$

$y(x) = e^{-5\lambda}(C_1 + xC_2)$

Case 3: Roots are complex

$y'' + 4y' + 9y = 0$

$\lambda^2 + 4\lambda + 9 = 0$

$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\Rightarrow -4 \pm \sqrt{-20}/2$

$-2(2 \pm \sqrt{-5})/2$

$= -2 \pm \sqrt{5}\, i$

$\Rightarrow$

$= e^{-2x}[C_1 \cos(\sqrt{5}x) + iC_2 \sin\sqrt{5}x$

$C_1 \cos(\sqrt{5}x) - iC_1 \sin(\sqrt{5}x)]$

$= e^{-2x}[A\cos\sqrt{5}\,x + B\sin(\sqrt{5})x]$

$\ldots e^{i\theta} = \cos\theta + i\sin\theta$

e.g. 1) $\lambda = 0, 0, 0, 0, \pm 2i$

$y = C_1 + C_2x + C_3x^2 + C_4x^3 + A\cos 2x + B\sin 2x$

→ Euler Cauchy:

$x^n y^{(n)} + P_1 x^{n-1} y^{(n-1)} \ldots + P_{n-1}x^2 y' + P_n y = 0$

$y(x) = x^m$

$y' = mx^{m-1}$

$y'' = (m-1)mx^{m-2}$

$y''' = (m-2)(m-1)mx^{m-3}$

e.g $x^2y'' + axy' + by = 0$

$x^2 x^{m-2}(m^2 - m) + axmx^{m-1} + bx^m$

$x^m(m^2 - m) + x^m(am) + x^m(b)$

$x^m(m^2 - m + am + b) = 0$

Case 1:

$m_1 \neq m_2$

$y_1(x) = x^{m_1}$

$y_2(x) = x^{m_2}$

$y(x) = C_1x^{m_1} + C_2x^{m_2}$

Case 2:

$m_1 = m_2 = m \in \mathbb{R}$

x^m

$\ln(x)\, x^m$

$\ln(x)^2\, x^m$

$y(x) = x^m(C_1 + C_2\ln(x) \ldots)$

Case III:

m_1, m_2 are complex,

$y(x) = x^{real}[A\cos(i\text{-part}\ \ln|x|) + B\sin(i\text{-part}\ \ln|x|)]$

i-part
↓
imaginary part

Non-Homogeneous higher Order ODE:

$$y'' + p(x)y' + q(x)y = r(x) \quad \dots ①$$

Let $y(x)$ be any General sol$_n$ to ①

Let $y_p(x)$ be a particular solution to ①.

① Undetermined Coefftients:

form : $y'' + ay' + by = r(x) \quad \dots ①$

$$y(x) = y_h(x) + y_p(x)$$

$$y'' + ay' + by = 0$$

$r(x)$	Choice of $y_p(x)$
$e^{\alpha x}$	$Ke^{\alpha x}$
Polynomial	$K_n x^n + K_{n-1}x^{n-1} + K_{n-2}x^{n-2}$
$\left.\begin{matrix}\cos(\omega x)\\ \sin(\omega x)\end{matrix}\right\}$	$K\cos(\omega x) + M\sin(\omega x)$
$\left\{\begin{matrix}e^{\alpha x}\cos(\omega x)\\ e^{\alpha x}\sin(\omega x)\end{matrix}\right.$	$e^{\alpha x}(K\cos(\omega x) + M\sin(\omega x))$

e.g. $y'' + 5y' + 6y = 17e^{4x} \quad \dots ①$ → Step 1 find y_h ⟹

$$y'' + 5y' + 6y = 0$$
$$\lambda^2 + 5\lambda + 6 = 0$$
$$(\lambda+3)(\lambda+2) = 0$$
$$\lambda = -3, -2$$
$$y(x) = C_1 e^{-3x} + C_2 e^{-2x}$$

Step 2 find y_p

Choose $y_p(x) = ke^{4x}$,

$$y_p' = 4ke^{4x}$$

$$y_p'' = 16ke^{4x}$$

① becomes,

$$16ke^{4x} + 20ke^{4x} + 6ke^{4x} = 17e^{4x}$$

$$e^{4x}(16k + 20k + 6k) = 17e^{4x}$$

$$42k = 17$$
$$k = 17/42$$

$$y_p = \frac{17}{42}e^{4x} \quad \dots ①$$

$$y = y_h + y_p \Rightarrow C_1 e^{-3x} + C_2 e^{-2x} + \frac{17}{42}e^{4x}.$$

/ /

② $y'' + 5y' + 6y = 17e^{-2x}$

$\lambda^2 + 5\lambda + 6 = 0$

$\lambda = -2, -3$

$y_1 = e^{-2x}$

$y_2 = e^{-3x}$

for y_p ;

$y(x) = ke^{-2x}$, but e^{-2x} is already a solution,

thus we take $e^{-2x} \cdot x$

$\Rightarrow y(x) = kxe^{-2x}$

$y' = k(-2xe^{-2x} + e^{-2x}) \Rightarrow y' = ke^{-2x}(-2x+1)$

$y'' = k(-2(-2xe^{-2x} + e^{-2x}) - 2e^{-2x})$

$= k(+4xe^{-2x} - 2e^{-2x} - 2e^{-2x})$

$y'' = ke^{-2x}(4x-4)$

$y'' = 4ke^{-2x}(x-1)$

$= 4ke^{-2x}(x-1) + 5ke^{-2x}(-2x+5) + 6ke^{-2x}(x)$

$= ke^{-2x}(4x-4+(-10x+25)+6x)$

$= ke^{-2x}(4x-4-10x+25+6x)$

$= ke^{-2x}\quad(0x+21)$

$17e^{-2x} = 21ke^{-2x}$

$21k = 17$

$k = \frac{17}{21}$ ... (iii)

$\therefore y_p = \frac{17}{21}e^{-2x}$

$\therefore$ Sol^n $\Rightarrow e^{-2x} + e^{-3x} + \frac{17}{21}e^{-2x}$

/ /

1) $y'' + 5y' + 4y = x^2 + x$

$y_h(x) \Rightarrow \lambda^2 + 5\lambda + 4 = 0$

$\lambda^2 + 4\lambda + \lambda + 4 = 0$

$\lambda(\lambda+4) + 1(\lambda+4) = 0$

$(\lambda+1)(\lambda+4) = 0$

$\lambda = -1, \lambda = -4$

$y = C_1 e^{-1x} + C_2 e^{-4x}$

$y_p(x) = K_2 x^2 + K_1 x + K_0$

$y' = 2K_2 x + K_1$

$y'' = 2K_2$

$\rightarrow eq_n = 2K_2 + 10K_2 x + 5K_1 + 4K_2 x^2 + 4K_1 x + 4K_0$

$= x^2(4K_2) + x(10K_2 + 4K_1) + (2K_2 + 5K_1 + 4K_0)$

Comparing coefficients:

$4K_2 = 1 \quad \cdots ①$

$K_2 = 1/4$

$10K_2 + 4K_1 = 1 \quad \cdots ②$

$\frac{5}{2} + 4K_1 = 1$

$4K_1 = 1 - \frac{5}{2}$

$4K_1 = -\frac{3}{2}$

$K_1 = \frac{-3}{8}$

$2K_2 + 5K_1 + 4K_0 = 0$

$\frac{2}{4} - \frac{15}{8} + 4K_0 = 0$

$\frac{-15+4}{8} + 4K_0 = 0$

$-\frac{11}{8} + 4K_0 = 0$

$4K_0 = -\frac{11}{8}$

$K_0 = \frac{-11}{36}$

$\therefore y_1 = \frac{1}{4}x^2$

$y_2 = \frac{-3}{8}x$

$y_3 = \frac{-11}{36}$

$y = \frac{C_1 x^2}{4} - \frac{3C_2 x}{8} \quad \frac{-11 C_3}{36}$

HW

2) $y_h = C_1 e^{x} + C_2 e^{-x} + C_3 \cos(7x) + C_4 \sin(7x)$

$r(x) = 11e^{x} + 14\sin(7x)$

$y_p = Kxe^{x} + \quad x(K\cos\omega x + M\sin\omega x)$

Q2) [Session 26: 17:27]

$y_p = (K_3 x^3 + K_2 x^2 + K_1 x + K_0)x^4 + xe^{-2x}(K\cos\omega x + M\sin\omega x)$

Method:

Variation of Parameters:

$y'' + p(x)y' + q(x)y = r(x)$ ① $y(x) = y_h + y_p$

$y'' + ay' + by = r(x)$

$y_p = u_1 y_1 + u_2 y_2 + u_3 y_3$

where,

$u_1 = \int \frac{W_1 r}{W} dx \qquad u_2 = \int \frac{W_2 r}{W} dx \qquad u_3 = \int \frac{W_3 r}{W} dx$

$$y_p = \sum_{i=1}^{n} u_i y_i$$

$$u_i = \int \frac{W_i r}{W} dx$$

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} \quad W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ 1 & y_2'' & y_3'' \end{vmatrix} \quad W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & 1 & y_3'' \end{vmatrix} \quad W_3 = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & 1 \end{vmatrix}$$

e.g. $y'' + y = \operatorname{cosec} x$

$y(h) \Rightarrow \quad y'' + y = 0$

$\lambda^2 + 1 = 0$

$\lambda^2 = -1$

$\lambda = \pm i$

$y_1 = \cos x$

$y_2 = \sin x$

$y_h(x) = C_1 \cos x + C_2 \sin x$

$y_p(x) = u_1 y_1 + u_2 y_2$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$W_1 = \begin{vmatrix} 0 & \sin x \\ 1 & \cos x \end{vmatrix} = 0 - \sin x = -\sin x$$

$$W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & 1 \end{vmatrix} = \cos x$$

$$u_1 = \int \frac{W_1 r}{W} dx = \int \frac{-\sin x \operatorname{cosec} x}{1} dx$$

$$= \int -1 \, dx$$

$$= -x$$

$$u_2 = \int \frac{\cos x \cdot \operatorname{cosec} x}{1} dx$$

$$u_2 = \int \cot x \, dx$$

$$u_2 = \ln|\sin x|$$

$$y_p = -x\cos x + \ln|\sin x| \sin x$$

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$$Q \Rightarrow x^2y'' + xy' - 4y = \frac{1}{x^2}$$

$$y(h) \Rightarrow x^2y'' + xy' - 4y = 0$$

$$y = x^m,$$

$$y' = mx^{m-1}$$

$$y'' = m(m-1)\ x^{m-2}$$

$$x^2x^{m-2}(m^2-m) + xmx^{m-1} - 4x^m = 0$$

$$x^m(m^2-m+m-4) = 0$$

$$m^2-4=0$$

$$m^2=4$$

$$m = \pm 2$$

$$\therefore y_1 = x^2 \quad \text{or} \quad y_2 = x^{-2}$$

$$y_h(x) = C_1x^2 + C_2x^{-2}$$

$$y_p(x) \Rightarrow v_1y_1 + v_2y_2,$$

$$W = \begin{vmatrix} x^2 & x^{-2} \\ 2x & -2x^{-3} \end{vmatrix} = -2x^{-3}x^2 - 2xx^{-2} = -2x^{-1} - 2x^{-1} = -4x^{-1}$$

$$W_1 = \begin{vmatrix} 0 & x^{-2} \\ 1 & -2x^{-3} \end{vmatrix} = -x^{-2}$$

$$W_2 = \begin{vmatrix} x^2 & 0 \\ 2x & 1 \end{vmatrix} = x^2$$

$$v_1 = \int \frac{-x^{-2}\cdot x^{-4}}{-4x^{-1}}\,dx$$

$$= \frac{-1}{4}\int \frac{x^{-6}}{x^{-1}}\,dx$$

$$= -\frac{1}{4}\int x^{-5}\,dx$$

$$= -\frac{1}{4}\left(\frac{x^{-4}}{-4}\right)$$

$$v_1 = \frac{x^{-4}}{16}$$

$$v_2 = \int \frac{x^2\cdot x^{-4}}{-4x^{-1}}\,dx$$

$$= \int \frac{x^{-2}}{-4x^{-1}}$$

$$= \frac{-1}{4}\int (x^{-1})$$

$$= -\frac{1}{4}\int\left(\frac{1}{x}\right)$$

$$= -\frac{1}{4}\ln(x)$$

$$= \ln(x^{-1/4})$$

$$v_2 = \ln(1/\sqrt[4]{x})$$

$$y_p = v_1y_1 + v_2y_2$$

$$= \frac{x^{-4}x^2}{16} + \ln\left(\frac{1}{\sqrt[4]{x}}\right)x^{-2}$$

$$y'' + y = \tan x$$

$$y_h(x) \Rightarrow y'' + y = 0$$

$$\lambda^2 + 1 = 0$$

$$\lambda = \pm i$$

$$y_1 = \cos x$$

$$y_2 = \sin x$$

$$y = c_1 \cos x + c_2 \sin x$$

$$y_p(x) \Rightarrow$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1 \qquad W_1 = \begin{vmatrix} 0 & \sin x \\ 1 & \cos x \end{vmatrix} = -\sin x \qquad W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & 1 \end{vmatrix} = \cos x$$

$$u_1 = \int \frac{-\sin x \cdot \tan x}{1} = -\int \sin^2 x \cdot \sec x = -\int (1 - \cos^2 x) \sec x = -\int \sec x - \cos x$$

$$= -\left(\int \sec x - \int \cos x \right) = -\ln|\sec x + \tan x| + \sin x$$

$$\therefore u_1 = \sin x - \ln|\sec x + \tan x|$$

$$u_2 = \int \cos x \cdot \tan x \, dx = \int \sin x = -\cos x$$

$$y_p = \cos x \sin x - \cos x \ln|\sec x + \tan x| - \cos x \sin x$$