Discrete Structures

Mathematical Logic- Rules of Inference

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Rules of Inference

- ***An argument:*** a sequence of statements that end with a conclusion.
  - **argument ("valid") :** never lead from correct statements to an incorrect conclusion**.**
  - **argument ("fallacies")** : can lead from true statements to an incorrect conclusion.

- ***A logical argument*** consists of premises/hypotheses and a single proposition called the conclusion.

- ***Logical rules of inference:*** Templates for constructing valid arguments.

Valid Arguments

- Example: A logical argument

  If I dance all night, then I get tired.

  I danced all night.

  Therefore I got tired.

- Logical representation of underlying variables:

  p: I dance all night. q: I get tired.

- Logical analysis of argument:

$$\begin{array}{ll} p \rightarrow q & \text{premise 1} \\ \underline{p} & \underline{\text{premise 2}} \\ \therefore q & \text{conclusion} \end{array}$$

Inference Rules: General Form

- An *Inference Rule is*

  A pattern establishing that if we know that a set of *premise statements of certain* forms are all true, then we can validly deduce that a certain related *conclusion* statement is true.

$$\boxed{\begin{array}{l} \quad \textit{premise 1} \\ \quad \textit{premise 2} \\ \quad \dots \\ \hline \therefore\ \textit{conclusion} \end{array}}$$

“∴” means “therefore”

Valid Arguments

- A form of logical argument is ***valid*** if whenever every premise is true, the conclusion is also true.

- A form of argument that is not valid is called a ***fallacy.***

Inference Rules Summary

- Each valid logical inference rule corresponds to an implication that is a tautology.

$$\begin{array}{ll} & \textit{premise 1} \\ & \textit{premise 2} \\ & \ldots \\ \hline \therefore & \textit{conclusion} \end{array}$$

Inference rule

- Corresponding tautology:

$$((\textit{premise 1}) \land (\textit{premise 2}) \land \ldots.) \rightarrow \textit{conclusion}$$

Rules of inference

- Modus ponens
- Modus tollens
- Hypothetical syllogism
- Disjunctive syllogism
- Resolution
- Addition
- Simplification
- Conjunction

Modus Ponens

$$\begin{array}{l} \quad p \\ \quad p \rightarrow q \\ \hline \therefore\ q \end{array}$$

Rule of ***Modus ponens***
(a.k.a. *law of detachment*)

"the mode of affirming"

$(p \land (p \rightarrow q)) \rightarrow q$ is a tautology

p	q	$p \rightarrow q$	$p \land (p \rightarrow q)$	$(p \land (p \rightarrow q)) \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

- Notice that the first row is the only one where premises are all true

Modus Ponens: Example

$$\text{If}\ \left\{\begin{array}{ll} p \to q & : \text{“If it snows today} \\ & \quad \text{then we will go skiing”} \\ p & : \text{“It is snowing today”} \end{array}\right\}\ \text{assumed TRUE}$$

$$\text{Then}\ \therefore q \quad : \text{“We will go skiing”} \qquad \text{is TRUE}$$

$$\text{If}\ \left\{\begin{array}{ll} p \to q & : \text{“If } n \text{ is divisible by 3} \\ & \quad \text{then } n^2 \text{ is divisible by 3”} \\ p & : \text{“} n \text{ is divisible by 3”} \end{array}\right\}\ \text{assumed TRUE}$$

$$\text{Then}\ \therefore q \quad : \text{“} n^2 \text{ is divisible by 3”} \qquad \text{is TRUE}$$

Modus Tollens

$$\begin{array}{r} \neg q \\ p \rightarrow q \\ \hline \therefore \neg p \end{array}$$

Rule of ***Modus tollens***

"the mode of denying"

$(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$ is a tautology

Modus Tollens: Example

- Example

$$\text{If} \left\{ \begin{array}{ll} p \to q : & \text{“If this jewel is really a diamond then it will scratch glass”} \\ \neg q & : \text{“The jewel doesn't scratch glass”} \end{array} \right\} \text{assumed TRUE}$$

$$\text{Then} \quad \therefore \neg p \quad : \text{“The jewel is not a diamond”} \quad \text{is TRUE}$$

More Inference Rules

- $$\begin{array}{l} p \\ \hline \therefore\ p \vee q \end{array}$$
  Rule of ***Addition***
  Tautology: $p \rightarrow (p \vee q)$
- $$\begin{array}{l} p \wedge q \\ \hline \therefore\ p \end{array}$$
  Rule of ***Simplification***
  Tautology: $(p \wedge q) \rightarrow p$
- $$\begin{array}{l} p \\ q \\ \hline \therefore\ p \wedge q \end{array}$$
  Rule of ***Conjunction***
  Tautology: $[(p) \wedge (q)] \rightarrow p \wedge q$

Examples

- State which rule of inference is the basis of the following arguments:
  - It is below freezing and raining now. Therefore, it is below freezing now.

Examples

- State which rule of inference is the basis of the following arguments:
  - It is below freezing and raining now. Therefore, it is below freezing now.

- *p: It is below freezing now.*
- *q: It is raining now.*
  - ***$(p \wedge q) \rightarrow p$ (rule of simplification)***

Quick Quiz 4.1:

- State which rule of inference is the basis of the following argument:

  "It is below freezing now. Therefore, it is below freezing or raining now."

  Visit moodle to submit answer

Quick Quiz 4.1:

- State which rule of inference is the basis of the following argument:

  "It is below freezing now. Therefore, it is below freezing or raining now."

Answer:

$$\frac{p}{\therefore p \vee q}$$

This is an argument that uses **the addition rule.**

Examples

- State which rule of inference is the basis of the following arguments:
  - It is below freezing, It is raining now. Therefore, it is below freezing and raining now.

- *p: It is below freezing now.*
- *q: It is raining now.*
  - $(p) \land (q) \rightarrow (p \land q)$ ***(rule Conjunction)***

Hypothetical Syllogism

$$\boxed{\begin{array}{r} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}}$$

Rule of ***Hypothetical syllogism***

Tautology:

$$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$$

Example:

- State the rule of inference used in the argument:

“If it rains today, then we will not have a barbecue today. If we do not have a barbecue today, then we will have a barbecue tomorrow. Therefore, if it rains today, then we will have a barbecue tomorrow.”

"If $\underbrace{\text{it rains today}}_{p}$, then $\underbrace{\text{we will not have a barbecue today}}_{q}$. If $\underbrace{\text{we do not have a barbecue today}}_{q}$, then $\underbrace{\text{we will have a barbecue tomorrow}}_{r}$. Therefore, if $\underbrace{\text{it rains today}}_{p}$, then $\underbrace{\text{we will have a barbecue tomorrow}}_{r}$."

$$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$$

Disjunctive Syllogism

$$\begin{array}{l} \quad p \vee q \\ \quad \neg p \\ \hline \therefore q \end{array}$$

Rule of ***Disjunctive syllogism***

Tautology: $[(p \vee q) \wedge (\neg p)] \rightarrow q$

Example

Ed’s wallet is in his back pocket or it is on his desk. Ed’s wallet is not in his back pocket. Therefore, Ed’s wallet is on his desk.

Example

- Ed's wallet is in his back pocket or it is on his desk. ($p \vee q$)
  p q
- Ed's wallet is not in his back pocket. ($\neg p$)
- Therefore, Ed's wallet is on his desk. (q)

Resolution

$$\begin{array}{r} p \lor q \\ \neg p \lor r \\ \hline \therefore\ q \lor r \end{array}$$

Rule of ***Resolution***

Tautology:

$$[(p \lor q) \land (\neg p \lor r)] \rightarrow (q \lor r)$$

- When $q = r$:

$$[(p \lor q) \land (\neg p \lor q)] \rightarrow q$$

- When r = **F**:

$$[(p \lor q) \land (\neg p)] \rightarrow q \quad \text{(Disjunctive syllogism)}$$

Resolution: Example

- Use resolution to show that the hypotheses

"Jasmine is skiing or it is not snowing" and "It is snowing or Bart is playing hockey" imply that "Jasmine is skiing or Bart is playing hockey"

"$\underbrace{\text{Jasmine is skiing}}_{r}$ or $\underbrace{\text{it is not snowing}}_{\neg p}$" and "$\underbrace{\text{It is snowing}}_{p}$ or $\underbrace{\text{Bart is playing hockey}}_{q}$" imply that "$\underbrace{\text{Jasmine is skiing}}_{r}$ or $\underbrace{\text{Bart is playing hockey}}_{q}$"

$$(p \vee q) \wedge (\neg p \vee r) \rightarrow (q \vee r)$$

Formal Proofs

- A formal proof of a conclusion *C, given premises*

  p1, p2,…, pn

  consists of a sequence of steps, each of which applies some inference rule to premises or previously-proven statements to yield a new true statement (the *conclusion).*

- A proof demonstrates that *if the premises are* true, *then the conclusion is true.*

Formal Proof Example

- Suppose we have the following premises:
  "It is not sunny and it is cold."
  "We will swim only if it is sunny."
  "If we do not swim, then we will take a trip."
  "If we take a trip, then we will be home by sunset."

- Given these premises, prove the conclusion **"We will be home by sunset" using** inference rules.

- **Step 1:** Identify the propositions (Let us adopt the following abbreviations)

  sunny = ***"It is sunny";***

  cold = ***"It is cold";***

  swim = ***"We will swim";***

  canoe = ***"We will take a trip"*;**

  sunset = ***"We will be home by* sunset".**

- **Step 2:** Identify the argument. (Build the argument form)

It is not sunny and it is cold.

We will swim only if it is sunny.

If we do not swim, then we will take a trip.

If we take a trip, then we will be home by sunset.

We will be home by sunset.

- **Step 2:** Identify the argument. (Build the argument form)

It is not sunny and it is cold.

We will swim only if it is sunny.

If we do not swim, then we will canoe.

If we take a trip, then we will be home by sunset.

We will be home by sunset.

$$\begin{array}{l} \neg sunny \land cold \\ swim \rightarrow sunny \\ \neg swim \rightarrow canoe \\ canoe \rightarrow sunset \\ \hline \therefore\ sunset \end{array}$$

- **Step 3:** Verify the reasoning using the rules of inference

$$\begin{array}{l} \neg sunny \land cold \\ swim \rightarrow sunny \\ \neg swim \rightarrow canoe \\ canoe \rightarrow sunset \\ \hline \therefore\ sunset \end{array}$$

Step	Proved by
1. $\neg sunny \land cold$	Premise #1.
2. $\neg sunny$	Simplification of 1.
3. $swim \rightarrow sunny$	Premise #2.
4. $\neg swim$	Modus tollens on 2 and 3.
5. $\neg swim \rightarrow canoe$	Premise #3.
6. $canoe$	Modus ponens on 4 and 5.
7. $canoe \rightarrow sunset$	Premise #4.
8. $sunset$	Modus ponens on 6 and 7.

Quick Quiz 4.2:

- **True or False?**
- The premises $(p \land q) \lor r$ *and* $r \rightarrow s$ *imply the conclusion* $p \lor s$.

- *Visit Moodle to Submit Answer.*

Quick Quiz 4.2:

True or False?

- The premises $(p \land q) \lor r$ *and* $r \rightarrow s$ *imply the conclusion* $p \lor s$.

Solution: True

- rewrite the premises $(p \land q) \lor r$ as two clauses, $p \lor r$ and $q \lor r$.
- replace $r \rightarrow s$ by the equivalent clause $\neg r \lor s$.
- $(\neg r \lor s) \land (p \lor r) \land (q \lor r)$
- $(p \lor s) \land (q \lor r)$
- Using the two clauses $p \lor r$ and $\neg r \lor s$, we can use resolution to conclude $p \lor s$.

Common Fallacies

- A ***fallacy*** is an inference rule or other proof method that is not logically valid.
- **A fallacy may yield a false conclusion!**

- **Ex.** $((p \vee q) \wedge p) \rightarrow \sim q$ is not a tautology.
- **Fallacy of Disjunction**

- ***Fallacy of affirming the conclusion:***
- “*$p \rightarrow q$ is true, and q is true, so p must be* true.”

  (No, because **F → T is true.)**

- **Example**
- If David Cameron (DC) is president of the US, then he is at least 40 years old. *($p \rightarrow q$)*
- DC is at least 40 years old. *(q)*
- Therefore, DC is president of the US. *(p)*

- ***Fallacy of denying the hypothesis:***
- "*$p \rightarrow q$ is true, and p is false, so q must be* false."

  (No, again because **F → T is true.)**

  Or
- The proposition $((p \rightarrow q) \land \neg p) \rightarrow \neg q$ is not a tautology

- **Example**
- If a person does arithmetic well then his/her checkbook will balance. ($p \rightarrow q$)
- I cannot do arithmetic well. ($\neg p$)
- Therefore my checkbook does not balance. ($\neg q$)

Quick Quiz 5.1

- Is the following argument valid?

  "If you do every problem in this book, then you will learn discrete mathematics." "You learned discrete mathematics.“ Therefore, "you did every problem in this book."

- **Visit Moodle : To Submit Your Answer.**

Quick Quiz 5.1

- Is the following argument valid?

  "If you do every problem in this book, then you will learn discrete mathematics." "You learned discrete mathematics.“ Therefore, "you did every problem in this book."

- **Solution:**

  Let

  p : “You did every problem in this book.”

  q : “You learned discrete mathematics.”

  Argument is of the form: if $p \to q$ and q, then p.

  This is an example of an incorrect argument using the **fallacy of affirming the conclusion.**

Inference Rules for Quantifiers

- $$\frac{\forall x\, P(x)}{\therefore P(c)}$$ **Universal instantiation**

  (substitute *any* specific member *c* in the domain)

- $$\frac{P(c)}{\therefore \forall x\, P(x)}$$ (for an arbitrary element *c* of the domain)

  Universal generalization

- $$\frac{\exists x\, P(x)}{\therefore P(c)}$$ **Existential instantiation**

  (substitute an element *c* for which $P(c)$ is true)

- $$\frac{P(c)}{\therefore \exists x\, P(x)}$$ (for some element *c* in the domain)

  Existential generalization

Example

Show that the premises imply the conclusion

- *“Every animal has brain.” and “Human is a animal.” Therefore, “Human has brain.”*

Example

- *Every animal has brain. Human is a animal. Therefore, Human has brain.*
- **Proof**
- Define the predicates:

  M(x): x is a animal

  L(x): x has brain

  J: Human, a member of the universe
- The argument becomes

$$
\begin{array}{ll}
1. & \forall x\,[M(x) \rightarrow L(x)] \\
2. & M(J) \\
\hline
\therefore & L(J)
\end{array}
$$

- The proof is

$$\begin{array}{l} \forall x\,(M(x) \longrightarrow L(x)) \\ M(J) \\ \hline \therefore L(J) \end{array}$$

- **Note:** Using the rules of inference requires lots of practice.
  - Try example problems in the textbook.

$$\begin{array}{l} \forall x\,(M(x) \longrightarrow L(x)) \\ M(J) \\ \hline \therefore L(J) \end{array}$$

- The proof is

1. $\forall x\,[M(x) \rightarrow L(x)]$ — *Premise 1*
2. $M(J) \rightarrow L(J)$ — *U. I. from (1)*
3. $M(J)$ — *Premise 2*
4. $L(J)$ — *Modus Ponens from (2) and (3)*

- **Note:** Using the rules of inference requires lots of practice.
  - Try example problems in the textbook.

Combining Rules of Inference for Propositions and Quantified Statements

Universal Modus Ponens:

- If $\forall x(P(x) \rightarrow Q(x))$ is true, and if $P(a)$ is true for a particular element a in the domain of the universal quantifier, then $Q(a)$ must also be true.

$$\begin{array}{l} \forall x(P(x) \rightarrow Q(x)) \\ \underline{P(a), \text{ where } a \text{ is a particular element in the domain}} \\ \therefore Q(a) \end{array}$$

EXAMPLE

- Assume that

“For all positive integers *n, if n is greater than 4, then* n^2 *is less than* 2^n”

is true. Use universal modus ponens to show that $100^2 < 2^{100}$.

EXAMPLE

Solution:

P(n): "n > 4"

Q(n): "$n^2 < 2$"

"For all positive integers n, if n is greater than 4, then n^2 is less than 2^n"

Represented by $\forall n(P(n) \rightarrow Q(n))$, Assumed True

- *P(100) is true because 100 > 4.*
- *It follows by universal modus ponens that Q(100)* is true, namely, that $100^2 < 2^{100}$.

Universal Modus tollens

$$\forall x(P(x) \rightarrow Q(x))$$

$$\neg Q(a), \text{ where } a \text{ is a particular element in the domain}$$

$$\therefore \neg P(a)$$

Universal Modus tollens

$$\begin{array}{l} \forall x(P(x) \to Q(x)) \\ \neg Q(a), \text{ where } a \text{ is a particular element in the domain} \\ \hline \therefore \neg P(a) \end{array}$$