Feedback Control System
Unit 1
Introduction

Introduction

Control systems are an integral part of modern society.

Numerous applications are all around us:

The rockets fire, and the space shuttle lifts off to earth orbit; a self-guided vehicle delivering material to workstations in an aerospace assembly plant glides along the floor seeking its destination. These are just a few examples of the automatically controlled systems

The domestic applications are

an air conditioner, a refrigerator, a bathroom toilet tank, an automatic iron, elevators, washing machines and many processes within a car like car braking system, fuel injection system, and son

Introduction

Bio control systems

Within our own bodies we have numerous control systems, such as

the **pancreas**, which regulates our blood sugar.

In time of “fight or flight,” our adrenaline increases along with our heart rate, causing more oxygen to be delivered to our cells.

Our eyes follow a moving object to keep it in view;

our hands grasp the object and place it precisely at a predetermined location.

Definition

A control system consists of subsystems and processes (or plants) assembled for the purpose of obtaining a desired output with desired performance, given a specified input.

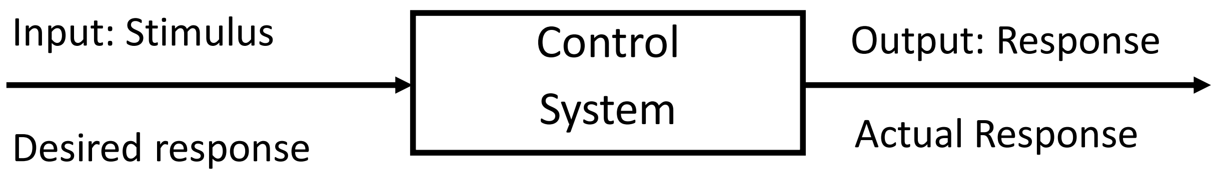
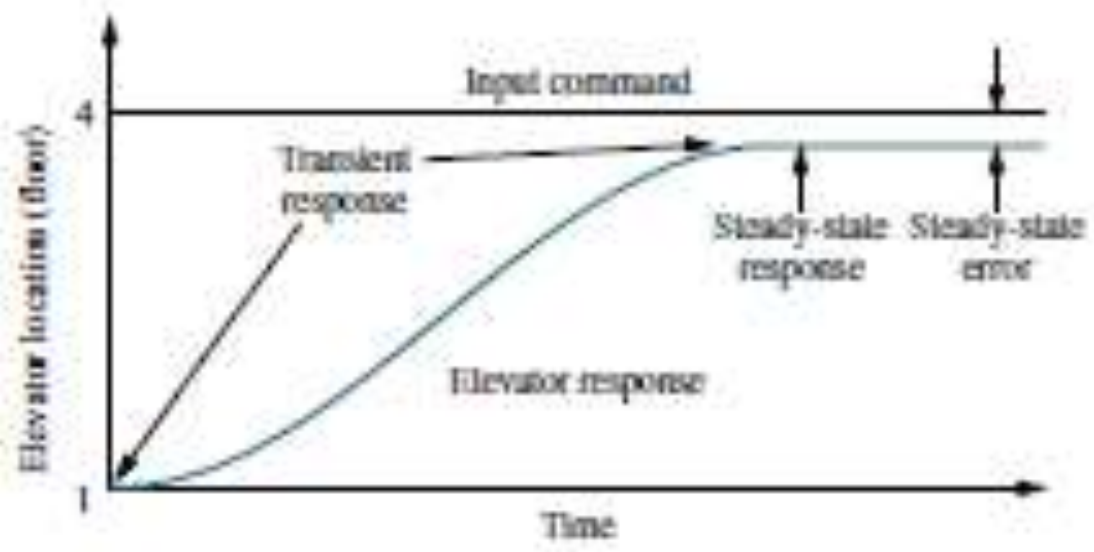


Figure shows a control system in its simplest form, where the input represents a desired output.

Definition

- For example, consider an elevator.
- When the fourth-floor button is pressed on the first floor, the elevator rises to the fourth floor with a speed and floor-leveling accuracy designed for passenger comfort.
- The push of the fourth-floor button is an input that represents our desired output, shown as a step function in Figure



Definition

The performance of the elevator can be seen from the elevator response curve in the figure.

Two major measures of performance are apparent:

(1) the transient response and

(2) the steady-state error.

Definition

In our example, passenger comfort and passenger patience are dependent upon the transient response.

If this response is too fast, passenger comfort is sacrificed;

if too slow, passenger patience is sacrificed.

The steady-state error is another important performance specification since passenger safety and convenience would be sacrificed if the elevator did not properly level.

Advantages of Control Systems

With control systems we can move large equipment with precision that would otherwise be impossible.

We can point huge antennas toward the farthest reaches of the universe to pick up faint radio signals; controlling these antennas by hand would be impossible.

Because of control systems, elevators carry us quickly to our destination, automatically stopping at the right floor.

We alone could not provide the power required for the load and the speed; motors provide the power, and control systems regulate the position and speed.

Advantages of Control Systems

We build control systems for four primary reasons:

1. Power amplification
2. Remote control
3. Convenience of input form
4. Compensation for disturbances

For example, a radar antenna, positioned by the low-power rotation of a knob at the input, requires a large amount of power for its output rotation. A control system can produce the needed power amplification, or power gain.

Robots designed by control system principles can compensate for human disabilities.

Control systems are also useful in remote or dangerous locations. For example, a remote-controlled robot arm can be used to pick up material in a radioactive environment.

Advantages of Control Systems

For example, in a temperature control system, the input is a position on a thermostat. The output is heat. Thus, a convenient position input yields a desired thermal output.

Another advantage of a control system is the ability to compensate for disturbances.

When the rope of elevator is cut, or the speed of elevator increases over the rated speed safety brake applied.

The system must be able to yield the correct output even with a disturbance.

For example, consider an antenna system that points in a commanded direction.

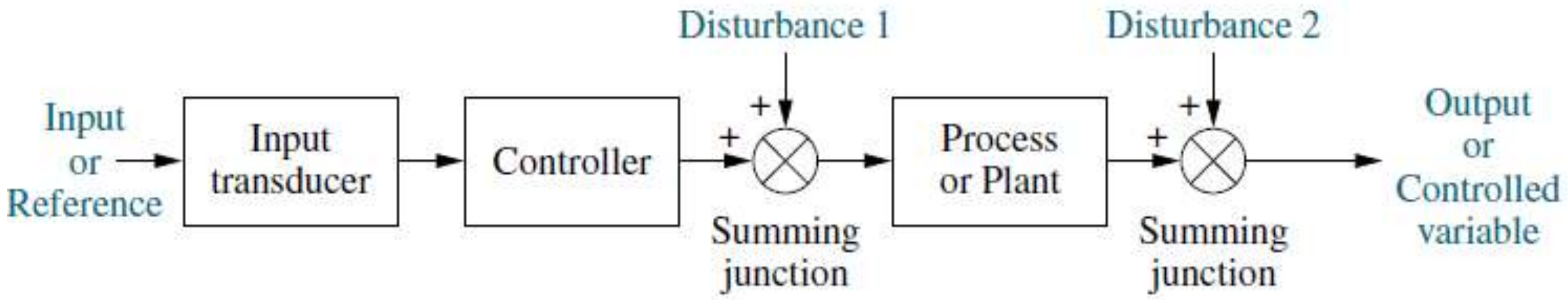
If wind forces the antenna from its commanded position, or if noise enters internally, the system must be able to detect the disturbance and correct the antenna's position.

Advantages of Control Systems

Obviously, the system’s input will not change to make the correction. Consequently, the system itself must measure the amount that the disturbance has repositioned the antenna and then return the antenna to the position commanded by the input.

Open Loop Control Systems

A generic open-loop system is shown in Figure.



(*a*)

It starts with a subsystem called an input transducer, which converts the form of the input to that used by the controller.

The controller drives a process or a plant.

The input is sometimes called the reference, while the output can be called the controlled variable.

Open Loop Control Systems

Other signals, such as disturbances, are shown added to the controller and process outputs via summing junctions, which yield the algebraic sum of their input signals using associated signs.

For example, the plant can be a furnace or air conditioning system, where the output variable is temperature.

The controller in a heating system consists of fuel valves and the electrical system that operates the valves.

The distinguishing characteristic of an open-loop system is that **it cannot compensate for any disturbances** that add to the controller's driving signal (Disturbance 1 in Figure).

Open Loop Control Systems

For example, if the controller is an electronic amplifier and Disturbance 1 is noise, then any additive amplifier noise at the first summing junction will also drive the process, corrupting the output with the effect of the noise. The output of an open-loop system is corrupted not only by signals that add to the controller's commands but also by disturbances at the output (Disturbance 2 in Figure). The system cannot correct for these disturbances, either.

Open Loop Control Systems

Open-loop systems, then, do not correct for disturbances and are simply commanded by the input.

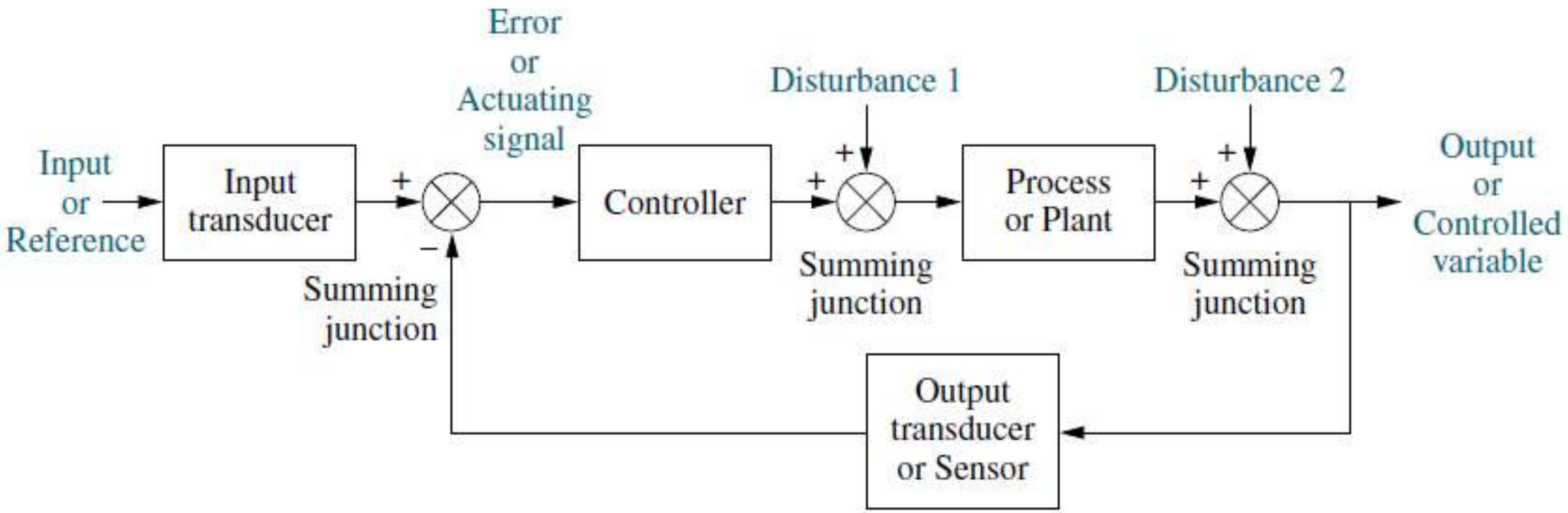
For example, toasters are open-loop systems, as anyone with burnt toast can attest.

The controlled variable (output) of a toaster is the color of the toast. The device is designed with the assumption that the toast will be darker the longer it is subjected to heat.

The toaster does not measure the color of the toast; it does not correct for the fact that the toast is rye, white, or sourdough, nor does it correct for the fact that toast comes in different thicknesses.

Closed Loop Control Systems

The disadvantages of open-loop systems, namely sensitivity to disturbances and inability to correct for these disturbances, may be overcome in closed-loop systems.



The generic architecture of a closed-loop system is shown in Figure.

Closed Loop Control Systems

The input transducer converts the form of the input to the form used by the controller.

An output transducer, or sensor, measures the output response and converts it into the form used by the controller.

For example, if the controller uses electrical signals to operate the valves of a temperature control system, the input position and the output temperature are converted to electrical signals.

The input position can be converted to a voltage by a potentiometer, a variable resistor, and the output temperature can be converted to a voltage by a thermistor, a device whose electrical resistance changes with temperature.

Closed Loop Control Systems

The first summing junction algebraically adds the signal from the input to the signal from the output, which arrives via the feedback path, the return path from the output to the summing junction.

In Figure, the output signal is subtracted from the input signal. The result is generally called the actuating signal.

However, in systems where both the input and output transducers have unity gain (that is, the transducer amplifies its input by 1), the actuating signal's value is equal to the actual difference between the input and the output. Under this condition, the actuating signal is called the error.

Closed Loop Control Systems

The closed-loop system compensates for disturbances by measuring the output response, feeding that measurement back through a feedback path, and comparing that response to the input at the summing junction.

If there is any difference between the two responses, the system drives the plant, via the actuating signal, to make a correction.

If there is no difference, the system does not drive the plant, since the plant's response is already the desired response.

Closed Loop Control Systems

Closed-loop systems have the obvious advantage of greater accuracy than open-loop systems.

They are less sensitive to noise, disturbances, and changes in the environment.

Transient response and steady-state error can be controlled more conveniently and with greater flexibility in closed-loop systems, often by a simple adjustment of gain (amplification) in the loop and sometimes by redesigning the controller.

We refer to the redesign as compensating the system and to the resulting hardware as a compensator.

Closed Loop Control Systems

On the other hand, closed-loop systems are more complex and expensive than open-loop systems.

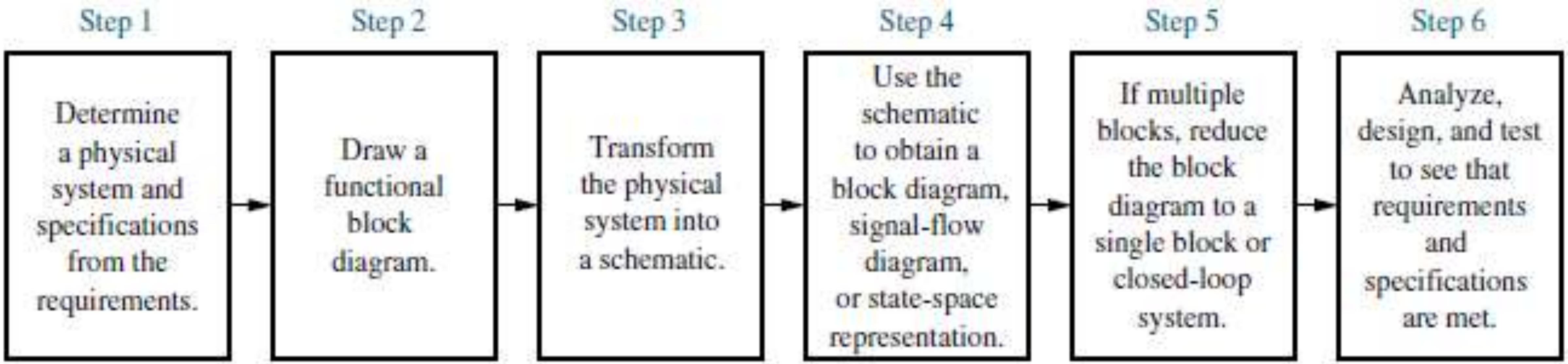
A standard, open-loop toaster serves as an example: It is simple and inexpensive. A closed-loop toaster oven is more complex and more expensive since it has to measure both color (through light reflectivity) and humidity inside the toaster oven.

Thus, the control systems engineer must consider the trade-off between the simplicity and low cost of an open-loop system and the accuracy and higher cost of a closed-loop system.

In summary, systems that perform the previously described measurement and correction are called closed-loop, or feedback control, systems. Systems that do not have this property of measurement and correction are called open-loop systems

Designed Procedure

Orderly sequence for the design of feedback control systems



Designed Procedure: Step 1

Step 1: Transform Requirements Into a Physical System

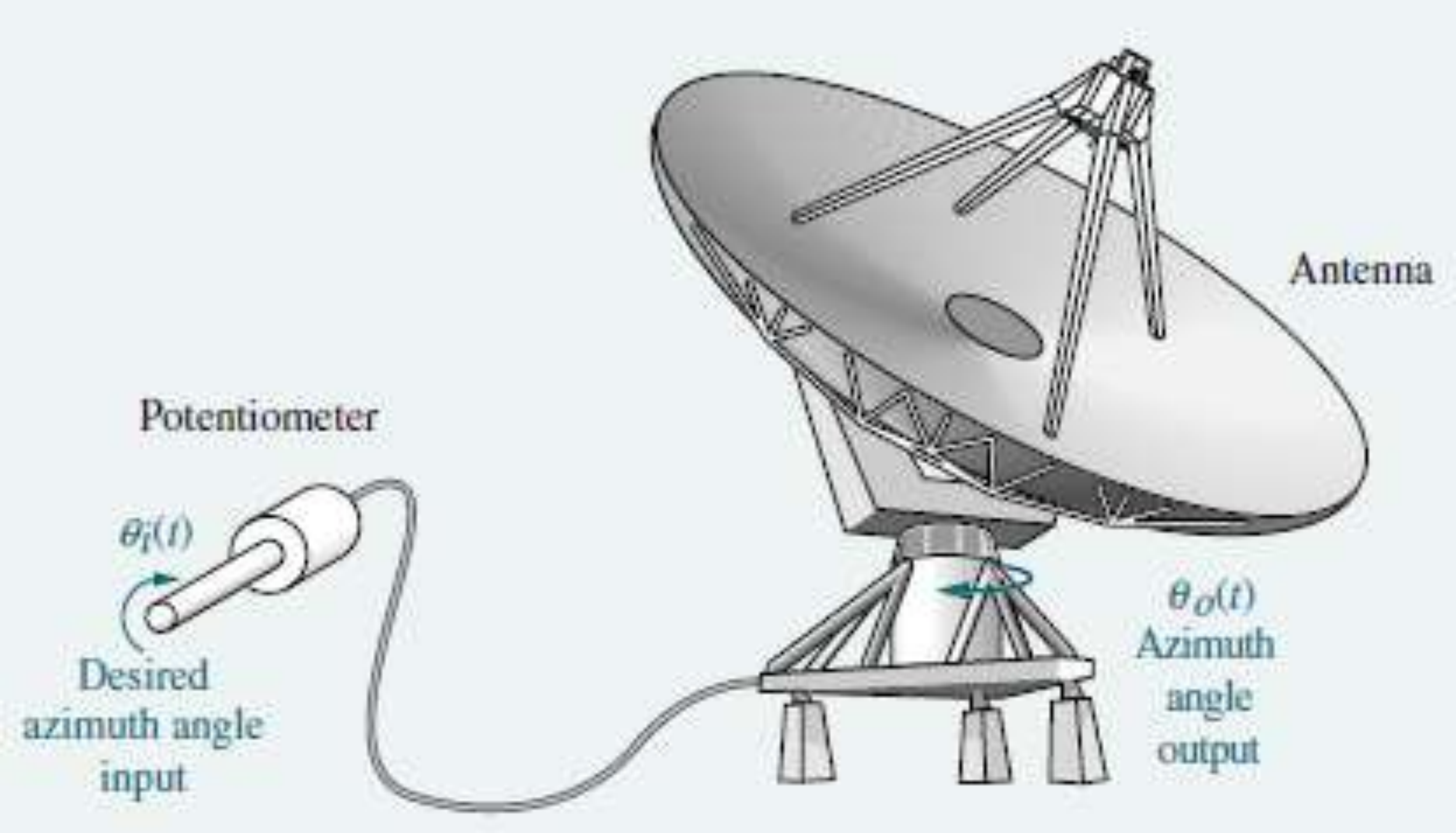
We begin by transforming the requirements into a physical system.

For example, in the antenna azimuth position control system, the requirements would state the desire to position the antenna from a remote location and describe such features as weight and physical dimensions.

Using the requirements, design specifications, such as desired transient response and steady-state accuracy, are determined.

Designed Procedure: Step 1

Step 1



Designed Procedure: Step 2

Step 2: Draw a Functional Block Diagram

The designer now translates a qualitative description of the system into a functional block diagram that describes the component parts of the system (that is, function and/or hardware) and shows their interconnection. Figure 1.9(d) is an example of a functional block diagram for the antenna azimuth position control system. It indicates functions such as input transducer and controller, as well as possible hardware descriptions such as amplifiers and motors. At this point the designer may produce a detailed layout of the system, such as that shown in Figure 1.9(b), from which the next phase of the analysis and design sequence, developing a schematic diagram, can be launched

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Following Figure is an example of a functional block diagram for the antenna azimuth position control system.

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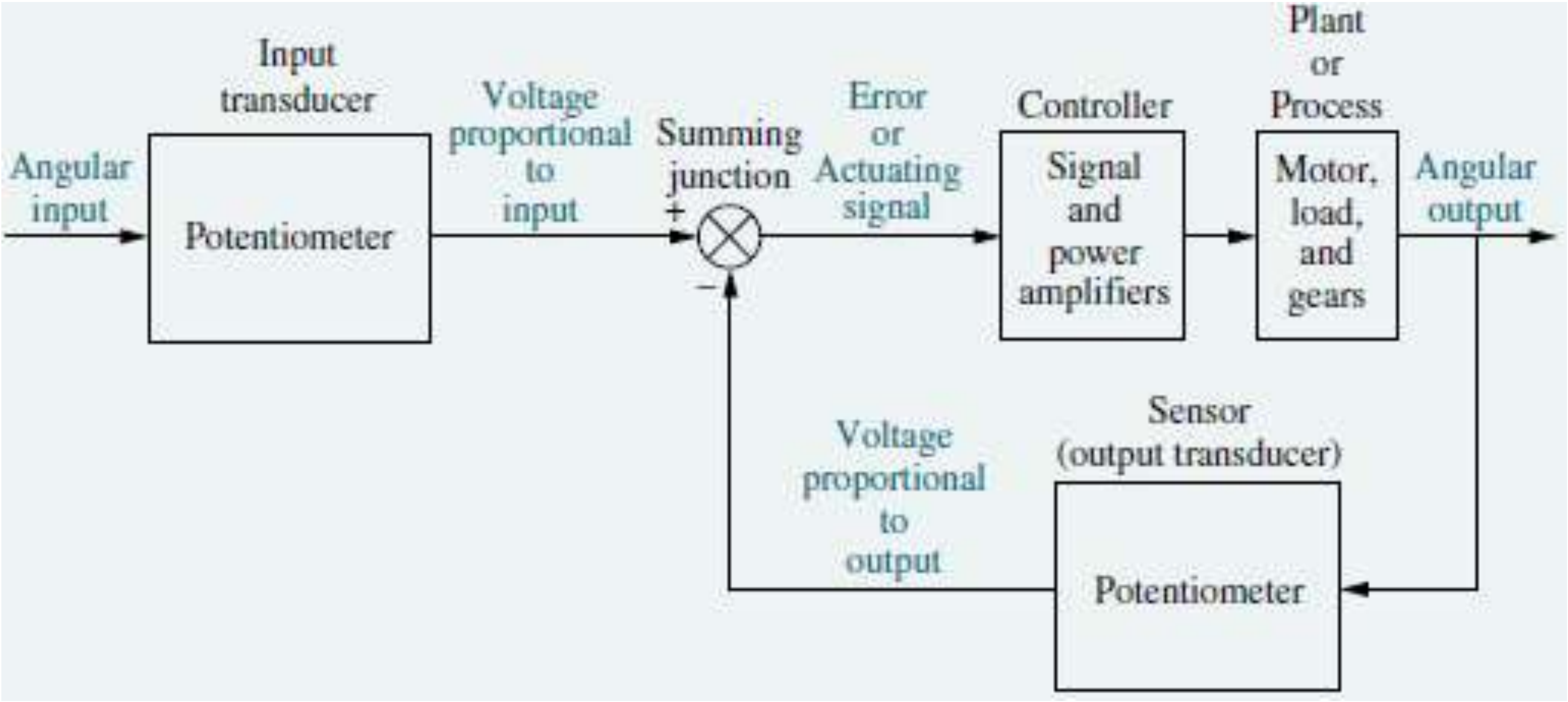

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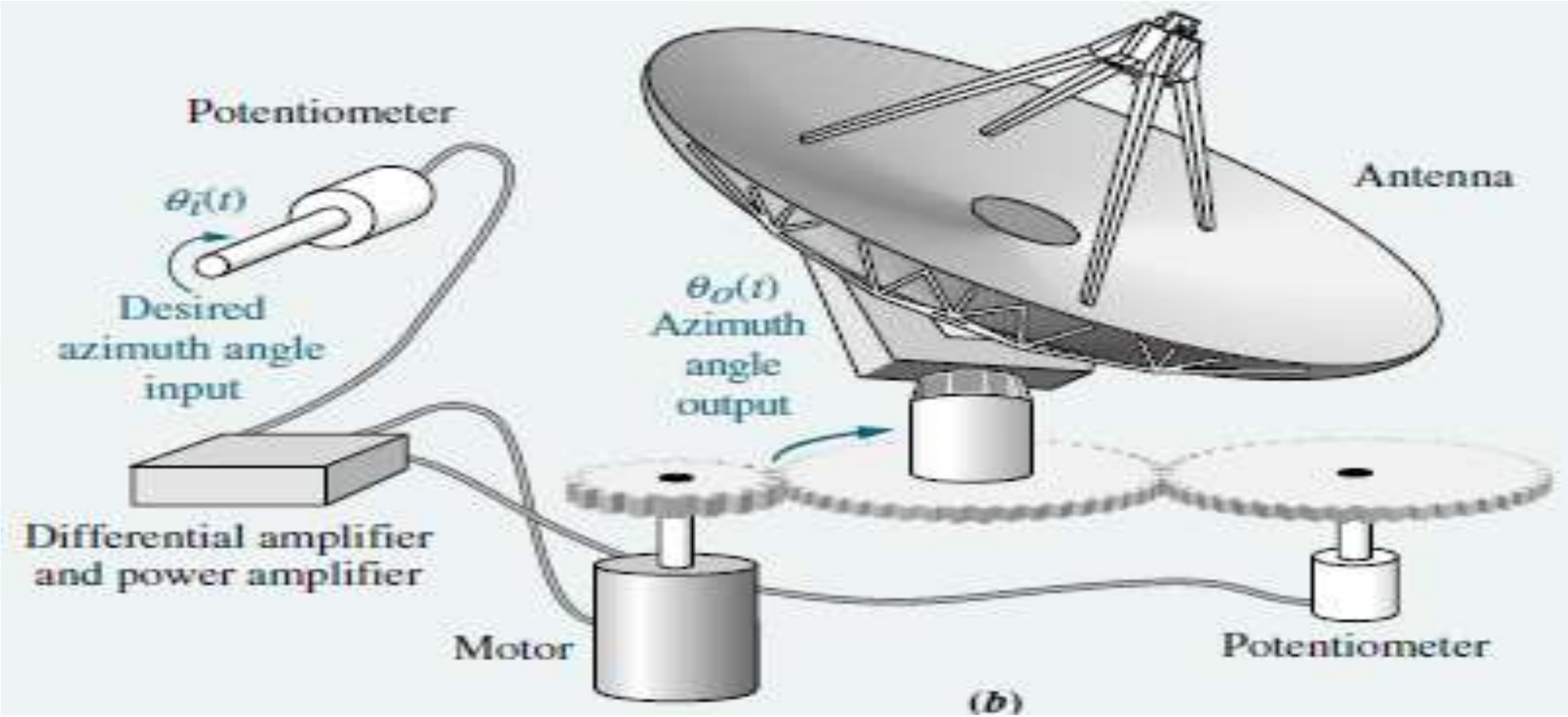


Designed Procedure: Step 3

Step 3: Create a Schematic

As we have seen, position control systems consist of electrical, mechanical, and electromechanical components.

After producing the description of a physical system, the control systems engineer transforms the physical system into a schematic diagram. The control system designer can begin with the physical description, to derive a schematic.

The engineer must make approximations about the system and neglect certain phenomena, or else the schematic will be unwieldy, making it difficult to extract a useful mathematical model during the next phase of the analysis and design sequence.

Designed Procedure: Step 3

Step 3: Create a Schematic

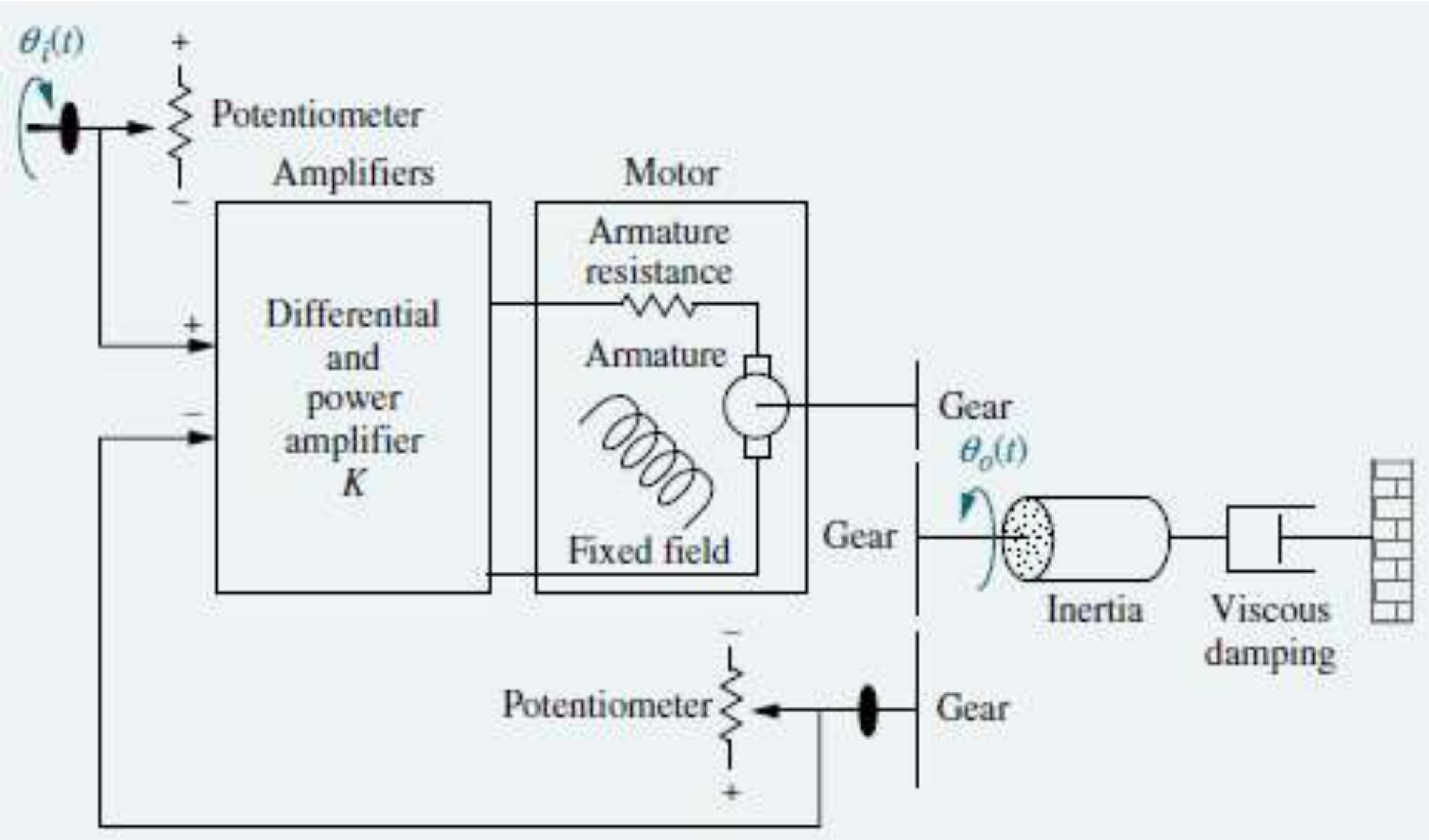
The designer starts with a simple schematic representation and, at subsequent phases of the analysis and design sequence, checks the assumptions made about the physical system through analysis and computer simulation.

If the schematic is too simple and does not adequately account for observed behavior, the control systems engineer adds phenomena to the schematic that were previously assumed negligible.

A schematic diagram for the antenna azimuth position control system is shown in following figure.

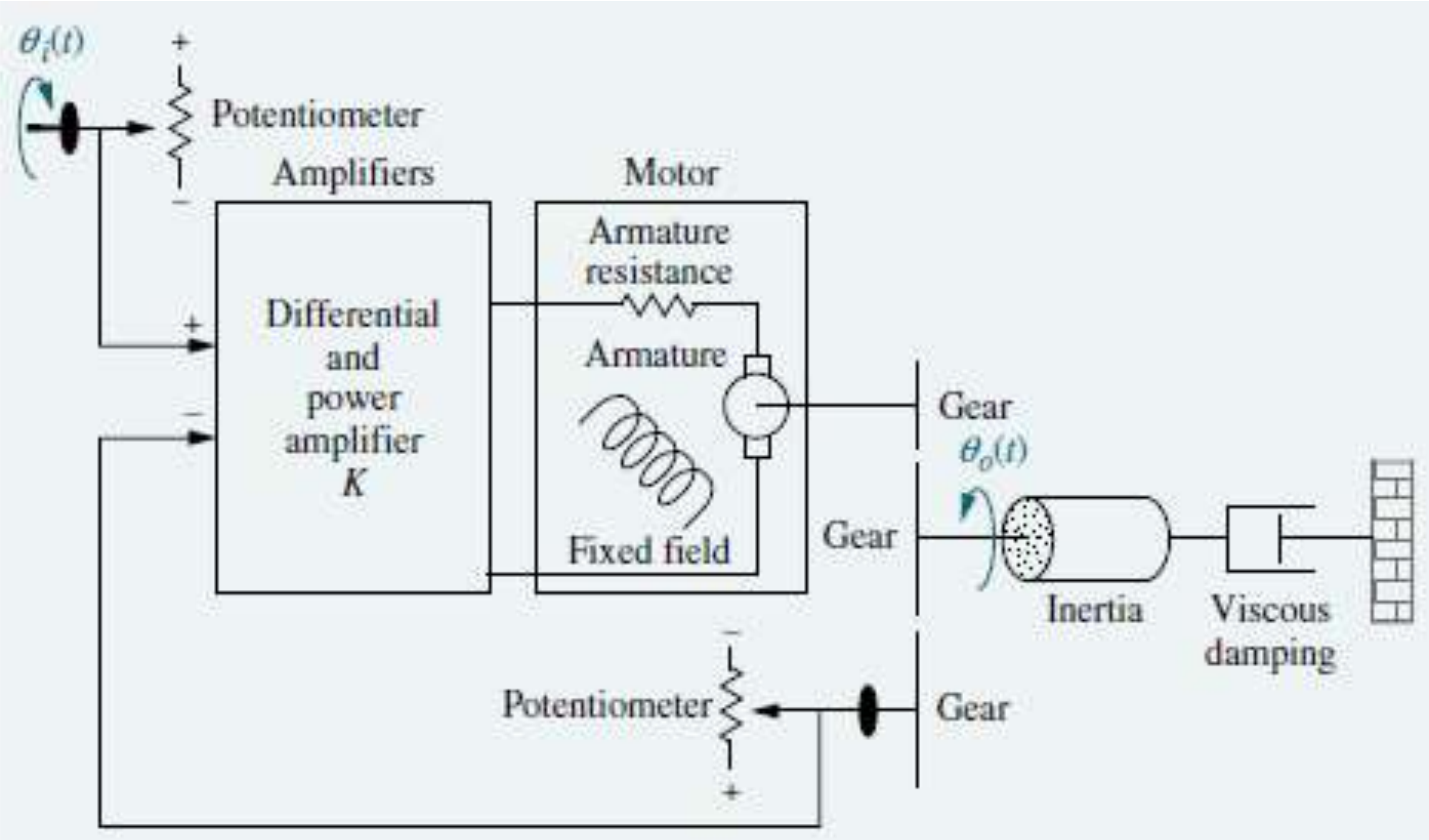
Designed Procedure: Step 3

Step 3: Create a Schematic



Designed Procedure: Step 3

Step 3: Create a Schematic



Designed Procedure: Step 3

Step 3: Create a Schematic

When we draw the potentiometers, we make our first simplifying assumption by neglecting their friction or inertia.

These mechanical characteristics yield a dynamic, rather than an instantaneous, response in the output voltage.

We assume that these mechanical effects are negligible and that the voltage across a potentiometer changes instantaneously as the potentiometer shaft turns.

A differential amplifier and a power amplifier are used as the controller to yield gain and power amplification, respectively, to drive the motor.

Designed Procedure: Step 3

Step 3: Create a Schematic

Again, we assume that the dynamics of the amplifiers are rapid compared to the response time of the motor; thus, we model them as a pure gain, K .

A dc motor and equivalent load produce the output angular displacement. The speed of the motor is proportional to the voltage applied to the motor's armature circuit. Both inductance and resistance are part of the armature circuit.

We assume the effect of the armature inductance is negligible for a dc motor. The designer makes further assumptions about the load.

The load consists of a rotating mass and bearing friction. Thus, the model consists of inertia and viscous damping whose resistive torque increases with speed, as in an automobile's shock absorber or a screen door damper.

Designed Procedure: Step 3

Step 3: Create a Schematic

The decisions made in developing the schematic stem from knowledge of the physical system, the physical laws governing the system's behavior, and practical experience.

Designed Procedure: Step 4

Step 4: Develop a Mathematical Model (Block Diagram)

Once the schematic is drawn, the designer uses physical laws, such as Kirchhoff's laws for electrical networks and Newton's law for mechanical systems, along with simplifying assumptions, to model the system mathematically. These laws are

Kirchhoff's voltage law: The sum of voltages around a closed path equals zero.

Kirchhoff's current law: The sum of electric currents flowing from a node equals zero.

Newton's laws: The sum of forces on a body equals zero; the sum of moments on a body equals zero

Kirchhoff's and Newton's laws lead to mathematical models that describe the relationship between the input and output of dynamic systems. One such model is the linear, time-invariant differential equation,

Designed Procedure: Step 4

Step 4: Develop a Mathematical Model (Block Diagram)

$$\frac{d^m c(t)}{dt^n} + d_{n-1}\frac{d^{m-1} c(t)}{dt^{n-1}} + \cdots + d_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1}\frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots + b_0 r(t)$$

Many systems can be approximately described by this equation, which relates the output, c(t), to the input, r(t), by way of the system parameters, a_i and b_j.

In addition to the differential equation, the transfer function is another way of mathematically modeling a system.

The model is derived from the linear, time-invariant differential equation using what we call the Laplace transform. Although the transfer function can be used only for linear systems.

Designed Procedure: Step 4

Step 4: Develop a Mathematical Model (Block Diagram)

We will be able to change system parameters and rapidly sense the effect of these changes on the system response. The transfer function is also useful in modeling the interconnection of subsystems by forming a block diagram with a mathematical function inside each block.

Designed Procedure: Step 4

Step 4: Develop a Mathematical Model (Block Diagram)

Another model is the **state-space representation**.

One advantage of state space methods is that they can also be used for systems that cannot be described by linear differential equations.

Further, state-space methods are used to model systems for simulation on the digital computer.

Basically, this representation turns an n^{th} order differential equation into n simultaneous first-order differential equations.

Designed Procedure: Step 4

Step 4: Develop a Mathematical Model (Block Diagram)

Finally, we should mention that to produce the mathematical model for a system, we require knowledge of the parameter values, such as equivalent resistance, inductance, mass, and damping, which is often not easy to obtain. Analysis, measurements, or specifications from vendors are sources that the control systems engineer may use to obtain the parameters.

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Designed Procedure: Step 5

Step 5: Reduce the Block Diagram

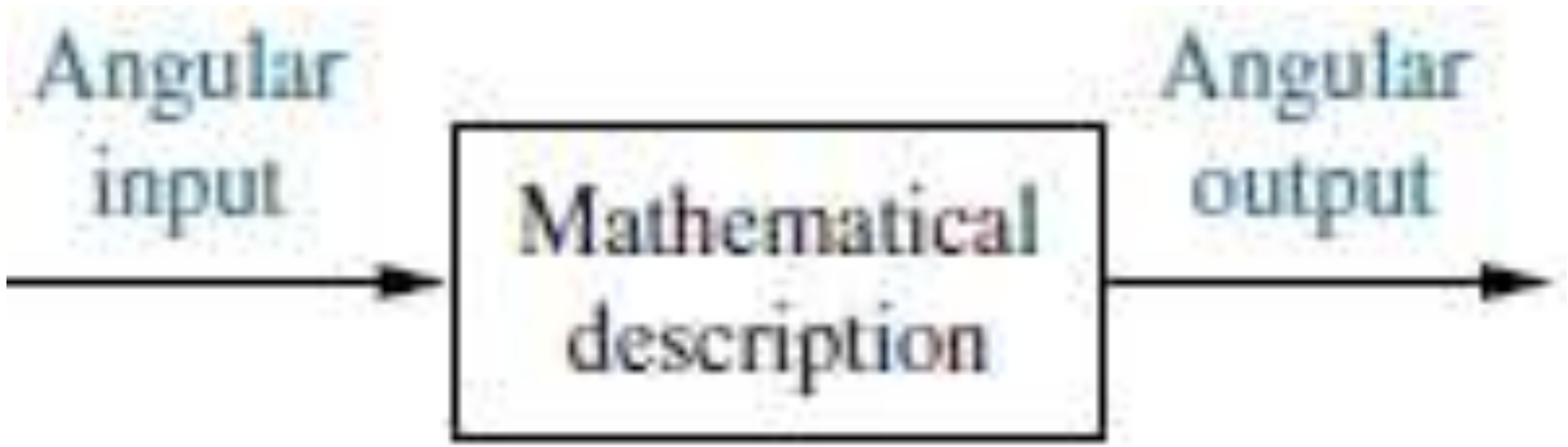
Subsystem models are interconnected to form block diagrams of larger systems where each block has a mathematical description.

Notice that many signals, such as proportional voltages and error, are internal to the system. There are also two signals – angular input and angular output – that are external to the system.

In order to evaluate system response in this example, we need to reduce this large system's block diagram to a single block with a mathematical description that represents the system from its input to its output, as shown in following figure. Once the block diagram is reduced, we are ready to analyze and design the system.

Designed Procedure: Step 5

Step 5: Reduce the Block Diagram



Designed Procedure: Step 6

Step 6: Analyze and Design

The next phase of the process, following block diagram reduction, is analysis and design.

If you are interested only in the performance of an individual subsystem, you can skip the block diagram reduction and move immediately into analysis and design.

In this phase, the engineer analyzes the system to see if the response specifications and performance requirements can be met by simple adjustments of system parameters.

If specifications cannot be met, the designer then designs additional hardware in order to effect a desired performance.

Designed Procedure: Step 6

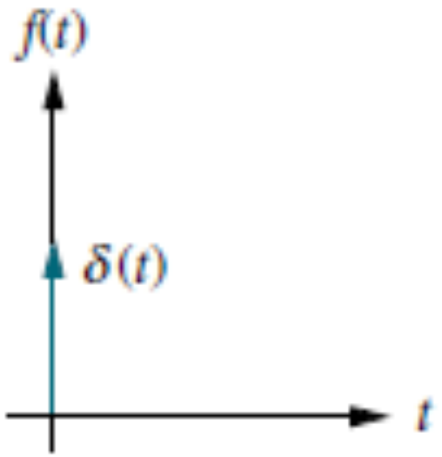
Step 6: Analyze and Design

Test input signals are used, both analytically and during testing, to verify the design.

It is neither necessarily practical nor illuminating to choose complicated input signals to analyze a system's performance.

Thus, the engineer usually selects standard test inputs.

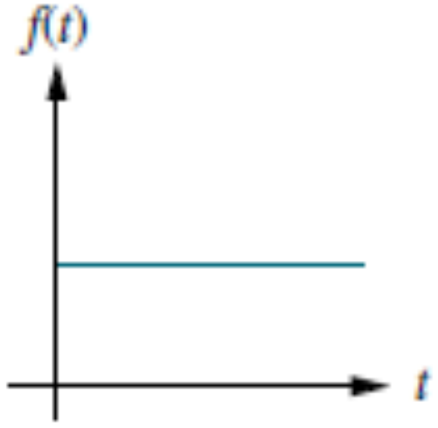
Designed Procedure: Step 6

I/P	Function	Description	Sketch	Use
Impulse	$\delta(t)$	$\delta(t) = \infty$ for $-0 < t < +0$ $= 0$ elsewhere $\int_{-0}^{+0} \delta(t)dt = 1$		Transient response Modelling

An impulse is infinite at $t = 0$ and zero elsewhere.

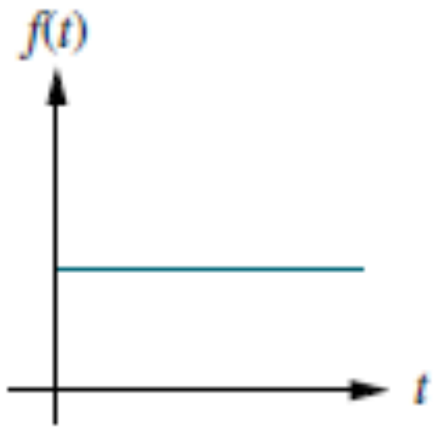
The area under the unit impulse is 1. An approximation of this type of waveform is used to place initial energy into a system so that the response due to that initial energy is only the transient response of a system. From this response the designer can derive a mathematical model of the system

Designed Procedure: Step 6

I/P	Function	Description	Sketch	Use
Step	u(t)	$u(t) = 1 \quad \text{for } t > 0$ $\quad\;\; = 0 \quad \text{for } t < 0$		Transient response Steady-state error

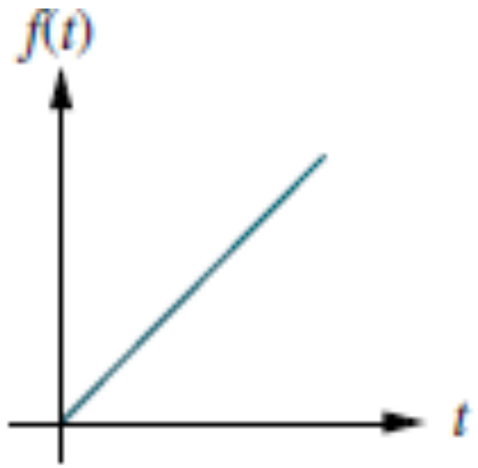
A step input represents a constant command, such as position, velocity, or acceleration. Typically, the step input command is of the same form as the output. For example, if the system's output is position, as it is for the antenna azimuth position control system, the step input represents a desired position, and the output represents the actual position.

Designed Procedure: Step 6

I/P	Function	Description	Sketch	Use
Step	u(t)	$u(t) = 1$ for $t > 0$ $= 0$ for $t < 0$	 	Transient response Steady-state error

If the system's output is velocity, as is the spindle speed for a video disc player, the step input represents a constant desired speed, and the output represents the actual speed. The designer uses step inputs because both the transient response and the steady-state response are clearly visible and can be evaluated

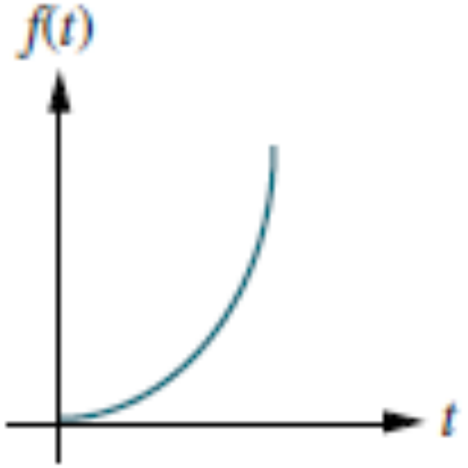
Designed Procedure: Step 6

I/P	Function	Description	Sketch	Use
Ramp	t u(t)	t u(t) = 1 for $t \geq 0$ = 0 otherwise		Steady-state error

The ramp input represents a linearly increasing command.

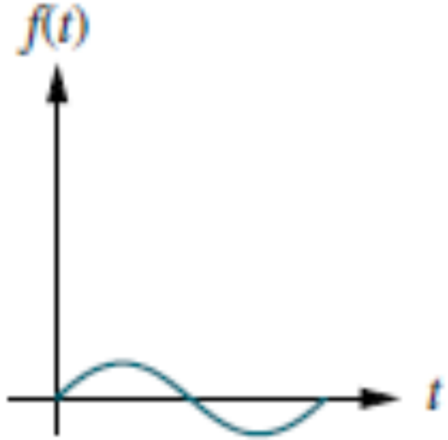
For example, if the system's output is position, the input ramp represents a linearly increasing position, such as that found when tracking a satellite moving across the sky at constant speed.

Designed Procedure: Step 6

I/P	Function	Description	Sketch	Use
Parabola	$\frac{1}{2} t^2 u(t)$	$\frac{1}{2} t^2 u(t) = \frac{1}{2} t^2$ for $t \geq 0$ $= 0$ otherwise		Steady-state error

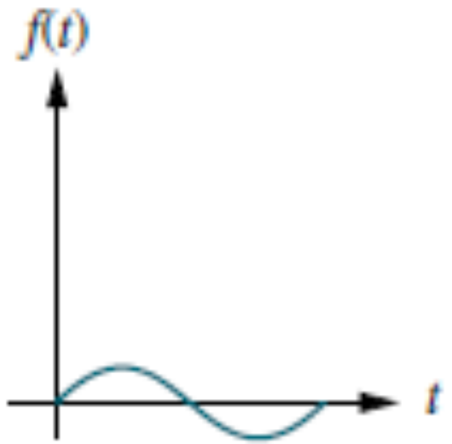
If the system's output is velocity, the input ramp represents a linearly increasing velocity. The response to an input ramp test signal yields additional information about the steady-state error. The previous discussion can be extended to parabolic inputs, which are also used to evaluate a system's steady-state error.

Designed Procedure: Step 6

I/P	Function	Description	Sketch	Use
Sinusoid	sin ωt	sin ωt		Transient response Modeling Steady-state error

Sinusoidal inputs can also be used to test a physical system to arrive at a mathematical model.

Designed Procedure: Step 6

I/P	Function	Description	Sketch	Use
Sinusoid	sin ωt	sin ωt		Transient response Modeling Steady-state error

Sinusoidal inputs can also be used to test a physical system to arrive at a mathematical model.

Laplace Transform

A system represented by a differential equation is difficult to model as a block diagram. By using the Laplace transform, with which we can represent the input, output, and system as separate entities.

Further, their interrelationship will be simply algebraic.

Let us first define the Laplace transform and then show how it simplifies the representation of physical systems

The Laplace transform is defined a

$$L[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$$

Where $s = \sigma + j\omega$ is a complex variable.

Laplace Transform

The notation for the lower limit means that even if f(t) is discontinuous at t = 0, we can start the integration prior to the discontinuity as long as the integral converges.

Thus, we can find the Laplace transform of impulse functions.

This property has distinct advantages when applying the Laplace transform to the solution of differential equations where the initial conditions are discontinuous at t = 0.

Using differential equations, we have to solve for the initial conditions after the discontinuity knowing the initial conditions before the discontinuity

Laplace Transform

The inverse Laplace transform is defined a

$$L^{-1}[F(s)] = f(t) = \frac{1}{2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds = f(t)u(t)$$

Where u(t) = 1; t > 0

= 0; t < 0

is the unit step function. Multiplication of f(t) by u(t) yields a time function that is zero for t < 0.

Laplace Transform Table

SN	f(t)	F(s)
1	$\delta(t)$	1
2	$u(t)$	$1/s$
3	$t\ u(t)$	$1/s^2$
4	$t^n\ u(t)$	$n!\ /(s^n +1)$
5	$e^{-at}\ u(t)$	$1/(s+a)$
6	$\sin \omega t\ u(t)$	$\omega/(s^2 + \omega^2)$
7	$\cos \omega t\ u(t)$	$s/(s^2 + \omega^2)$

Laplace Transform Theorems

SN	Theorem	Name
1	$L[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2	$L[kf(t)] = kF(s)$	Linearity
3	$L[f_1(t) + f_1(t)] = F_1(s) + F_2(s)$	Linearity
4	$L[e^{-at}f(t)] = F(s+a)$	Frequency shift theorem
5	$L[f(t-T)] = e^{-sT}F(s)$	Time shift theorem
6	$L[f(at)] = \left(\frac{1}{a}\right)F(s/a)$	Scaling theorem
7	$L\left[\frac{df(t)}{dt}\right] = sF(s) - f(0-)$	Differentiation theorems

Laplace Transform Theorems

SN	Theorem	Name
8	$L\left[\frac{d^2f(t)}{dt^2}\right]=s^2F(s)-sf(0-)-f'(0-)$	Differentiation theorems
9	$L\left[\frac{d^nf(t)}{dt^n}\right]=s^nF(s)-\sum_{k=1}^{n}s^{n-k}f^{k-1}(0-)$	Differentiation theorems
10	$L\left[\int_{0-}^{t}f(\tau)d\tau\right]=F(s)/s$	Integration theorem
11	$f(\infty)=\lim_{s\to 0}sF(s)$	Final value theorem
12	$f(0+)=\lim_{s\to\infty}sF(s)$	Initial value theorem

Laplace Transform Theorems

SN	Theorem	Name
8	$L\left[\frac{d^2f(t)}{dt^2}\right] = s^2F(s) - sf(0-) - f'(0-)$	Differentiation theorems
9	$L\left[\frac{d^nf(t)}{dt^n}\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorems
10	$L\left[\int_{0-}^{t} f(\tau)d\tau\right] = F(s)/s$	Integration theorem
11	$f(\infty) = \lim_{s\to 0} sF(s)$	Final value theorem
12	$f(0+) = \lim_{s\to\infty} sF(s)$	Initial value theorem

Laplace Transform

Find the Laplace transform of $f(t) = Ae^{-at}u(t)$.

Laplace Transform

Find the Laplace transform of $f(t) = Ae^{-at}u(t)$.

Since the time function does not contain an impulse function, we can replace the lower limit of 0- with 0.

Hence,

$$F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty Ae^{-at}e^{-st}dt$$

$$= A\int_0^\infty e^{-(s+a)t}dt = -\frac{A}{s+a}e^{-(s+a)t}\Big|_0^\infty = \frac{A}{s+a}$$

Laplace Transform

Find the inverse Laplace transform of $F_1(s) = 1/(s+3)^2$.

Laplace Transform

Find the inverse Laplace transform of $F_1(s) = 1/(s+3)^2$.

We know the frequency shift theorem

$$L[e^{-at} f(t)] = F(s + a)$$

the Laplace transform of $f(t) = tu(t)$ is $1/s^2$,

If the inverse transform of $F(s) = 1/s^2$ is $tu(t)$,

the inverse transform of $F(s + a) = 1/(s + a)^2$ is $e^{-at}\, t\, u(t)$.

Hence, $f_1(t) = e^{-3t}\, t\, u(t)$

Laplace Transform: Partial-Fraction Expansion

To find the inverse Laplace transform of a complicated function,

Convert the function to a sum of simpler terms

The result is called a partial-fraction expansion.

If $F1(s) = N(s)/D(s)$,

If the order of $N(s)$ is less than the order of $D(s)$, then a partial-fraction expansion can be made.

If the order of $N(s)$ is greater than or equal to the order of $D(s)$, then $N(s)$ must be divided by $D(s)$ successively until the result has a remainder whose numerator is of order less than its denominator. For example, if

Laplace Transform: Partial-Fraction Expansion

For example, if $F_1(s) = \frac{s^3+2s^2+6s+7}{s^2+s+5}$

we must perform the indicated division until we obtain a remainder whose numerator is of order less than its denominator. Hence $F_1(s) = s + 1 + \frac{2}{s^2+s+5}$

Taking the inverse Laplace transform, using the above Tables we obtain $f_1(t) = \frac{d\delta(t)}{dt} + \delta(t) + L^{-1}\left[\frac{2}{s^2+s+5}\right]$ Using partial-fraction expansion, we will be able to expand functions like $F(s) = \frac{2}{s^2+s+5}$ into a sum of terms and then find the inverse Laplace transform for each term

Case 1. Roots of the Denominator of F(s) Are Real and Distinct

Let $F(s) = \frac{2}{(s+1)(s+2)}$

The roots of the denominator are distinct, since each factor is raised only to unity power. We can write the partial-fraction expansion as a sum of terms where each factor of the original denominator forms the denominator of each term, and constants, called residues, form the numerators. Hence, $F(s) = \frac{2}{(s+1)(s+2)} = \frac{K_1}{(S+1)} + \frac{K_2}{(S+2)}$

To find K_1, we first multiply the equation by (s + 1), which isolates K_1. Thus,

$\frac{2}{(s+1)(s+2)} = K_1 + \frac{K_2(s+1)}{(S+2)}$

Let s → -1 eliminates the last term and gives K1 = 2. Similarly, K2 can be found by multiplying the equation by (s + 2) and then letting s → -2; hence, K2 = -2.

Case 1. Roots of the Denominator of F(s) Are Real and Distinct

Let s → -1 eliminates the last term and gives $K_1 = 2$. Similarly,

K_2 can be found by multiplying the equation by (s + 2) and then letting s → -2; hence,

$K_2 = -2$.

Therefore, $F(s) = \frac{2}{S+1} - \frac{2}{s+2}$

By referring the tables above

$f(t) = (2e^{-t} - 2e^{-2t})\, u(t)$

Case 1. Roots of the Denominator of F(s) Are Real and Distinct

Given the following differential equation, solve for y(t) if all initial conditions are zero. Use the Laplace transform. $\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 32y = 32u(t)$

Substitute the corresponding Laplace transform for each term in above equation, using Table, and the initial conditions of $y(t)$ and $\frac{dy(t)}{dt}$ given by $y(0-) = 0$ and $\frac{dy(0-)}{dt} = 0$, respectively. Hence, the Laplace transform of equation is

$$s^2Y(s) + 12sY(s) + 32Y(s) = \frac{32}{s}$$

Solving for the response Y(s) we have $Y(s) = \frac{32}{s(s^2+12s+32)} = \frac{32}{s(s+4)(s+8)}$

Now y(t) is obtained by taking the inverse Laplace transform, for that we need to partial fraction method. Therefore

Case 1. Roots of the Denominator of F(s) Are Real and Distinct

Therefore, $Y(s) = \frac{32}{s(s+4)(s+8)} = \frac{K_1}{s} + \frac{K_2}{s+4} + \frac{K_3}{s+8}$

$$K_1 = \left.\frac{32}{(s+4)(s+8)}\right|_{s \to 0} = 1$$

$$K_2 = \left.\frac{32}{s(s+8)}\right|_{s \to -4} = -2$$

$$K_3 = \left.\frac{32}{s(s+4)}\right|_{s \to -8} = 1$$

Case 1. Roots of the Denominator of F(s) Are Real and Distinct

Hence $Y(s) = \frac{1}{s} - \frac{2}{s+4} + \frac{1}{s+8}$

This is the simplest form, and we can easily find the LT of each term

$$y(t) = (1 - 2e^{-4t} + e^{-8t})u(t)$$

The u(t) shows that the response is zero until t = 0.

Thus, output responses will also be zero until t = 0. For convenience, we will leave off the u(t) notation. Accordingly, we write the output response as $y(t) = (1 - 2e^{-4t} +$

Case 2. Roots of the Denominator of F(s) Are Real and Repeated

Let $F(s) = \frac{2}{(s+1)(s+2)^2}$

The roots of $(s + 2)^2$ in the denominator are repeated, since the factor is raised to an integer power higher than 1. In this case, the denominator root at 2 is a multiple root of multiplicity 2

We can write the partial-fraction expansion as a sum of terms, where each factor of the denominator forms the denominator of each term. In addition, each multiple root generates additional terms consisting of denominator factors of reduced multiplicity. For example, if $F(s) = \frac{2}{(s+1)(s+2)^2} = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)^2} + \frac{K_3}{(s+2)}$

Case 2. Roots of the Denominator of F(s) Are Real and Repeated

For K_1

$$\frac{2}{(s+2)^2} = K_1 + \frac{(s+1)K_2}{(s+2)^2} + \frac{(s+1)K_3}{(s+2)}, \ \therefore K_1{=}2$$

K_2 can be isolated by multiplying by $(s + 2)^2$, yielding

$$\frac{2}{s+1} = (s+2)^2\frac{K_1}{(s+1)} + K_2 + (S+2)K_3$$. Letting s → -2 ; K_2 = - 2.

To find K_3 we differentiate above equation with respect to s

$$\frac{-2}{(s+1)^2} = \frac{(s+2)K_1}{(s+1)^2} + K_3$$, K_3 is isolated and can be found by s → -2 Hence, K3 = -2

$$\therefore F(s) = \frac{2}{(s+1)(s+2)^2} = \frac{2}{(s+1)} - \frac{2}{(s+2)^2} - \frac{2}{(s+2)}$$

$$\therefore f(t) = 2e^{-t} - 2te^{-2t} - 2e^{-2t}$$

Case 3. Roots of the Denominator of F(s) Are Complex or Imaginary

Let $F(s) = \frac{3}{s(s^2+2s+5)}$ *this*

This function can be expanded in the following form

$$\frac{3}{s(s^2+2s+5)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{(s^2+2s+5)}$$

K_1 is found in the usual way to be 3/5. K_2 and K_3 can be found by first multiplying the above equation by the lowest common denominator, $s(s^2 + 2s + 5)$, and clearing the fractions. After simplification with $K_1 = 3/5$, we obtain

$$3 = \left(K_2 + \frac{3}{5}\right)s^2 + \left(K_3 + \frac{6}{5}\right)s + 3$$

Case 3. Roots of the Denominator of F(s) Are Complex or Imaginary

Balancing coefficients, $(K_2 + 3/5) = 0$ and $(K_3 + 6/5) = 0$. Hence $K_2 = -3/5$ and $K_3 = -6/5$. Thus,

Let $F(s) = \frac{3}{s(s^2+2s+5)} = \frac{3/5}{s} - \frac{3}{5}\frac{s+2}{(s^2+2s+5)}$

The last term can be shown to be the sum of the Laplace transforms of an exponentially damped sine and cosine. Using $L[\sin(\omega t)\, u(t)] = \frac{\omega}{(s^2+\omega^2)}$; $L[\cos \omega t\, u(t)] = \frac{s}{(s^2+\omega^2)}$, $L[kf(t)] = kF(s)$; and $L[e^{-at}f(t)] = F(s+a)$

$L[Ae^{-at}\cos \omega t] = \frac{A(s+a)}{(s+a)^2+\omega^2}$ and $L[Be^{-at}\sin \omega t] = \frac{B\omega}{(s+a)^2+\omega^2}$

If we add these two equations

Case 3. Roots of the Denominator of F(s) Are Complex or Imaginary

If we add these two equations

$$L[Ae^{-at}\cos\omega t + Be^{-at}\sin\omega t] = \frac{A(s+a)+B\omega}{(s+a)^2+\omega^2}$$

Now rearrange $F(s) = \frac{3}{s(s^2+2s+5)} = \frac{3/5}{s} - \frac{3}{5}\frac{s+2}{(s^2+2s+5)}$ by completing the squares in the denominator and adjusting terms in the numerator without changing its value

$$F(s) = \frac{3/5}{s} - \frac{3}{5}\frac{s+1+(1/2)(2)}{(s+1)^2+2^2}$$

$$\therefore f(t) = \frac{3}{5} - \frac{3}{5}e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right)$$

Case 3. Roots of the Denominator of F(s) Are Complex or Imaginary

The alternative method is

$$F(s) = \frac{3}{s(s^2+2s+5)} = \frac{3}{s(s+1+j2)(s+1-j2)}$$

$$= \frac{k_1}{s} + \frac{k_2}{s+1+j2} + \frac{k_3}{s+1-j2}$$

$$K_2 = \left.\frac{3}{s(s+1-j2)}\right|_{s\to -1-j2} = -\frac{3}{20}(2+j1)$$

We know k1 = 3/5; and K3 is complex conjugate of K2.

Case 3. Roots of the Denominator of F(s) Are Complex or Imaginary

Hence

$$F(s) = \frac{3/5}{s} - \frac{3}{20}\left(\frac{2+j1}{S+1+j2} + \frac{2-j1}{S+1-j2}\right)$$

$$\therefore f(t) = \frac{3}{5} - \frac{3}{20}e^{-t}\left((2+j1)e^{-(1+j2)t} + (2-j1)e^{-(1-j2)t}\right)$$

$$= \frac{3}{5} - \frac{3}{20}e^{-t}\left[4\left(\frac{e^{j2t}+e^{-j2t}}{2}\right) + 2\left(\frac{e^{j2t}+e^{-j2t}}{2j}\right)\right]$$

$$= \frac{3}{5} - \frac{3}{5}e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right)$$

Feedback Control System
Unit 1
Introduction

Introduction

Control systems are an integral part of modern society.

Numerous applications are all around us:

The rockets fire, and the space shuttle lifts off to earth orbit; a self-guided vehicle delivering material to workstations in an aerospace assembly plant glides along the floor seeking its destination. These are just a few examples of the automatically controlled systems

The domestic applications are

an air conditioner, a refrigerator, a bathroom toilet tank, an automatic iron, elevators, washing machines and many processes within a car like car braking system, fuel injection system, and son

Introduction

Bio control systems

Within our own bodies we have numerous control systems, such as

the **pancreas**, which regulates our blood sugar.

In time of "fight or flight," our adrenaline increases along with our heart rate, causing more oxygen to be delivered to our cells.

Our eyes follow a moving object to keep it in view;

our hands grasp the object and place it precisely at a predetermined location.

Definition

A control system consists of subsystems and processes (or plants) assembled for the purpose of obtaining a desired output with desired performance, given a specified input.

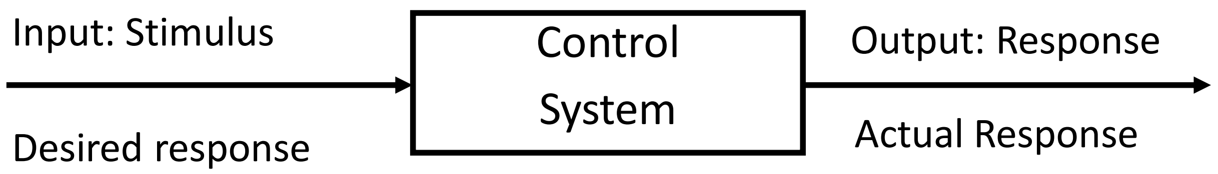
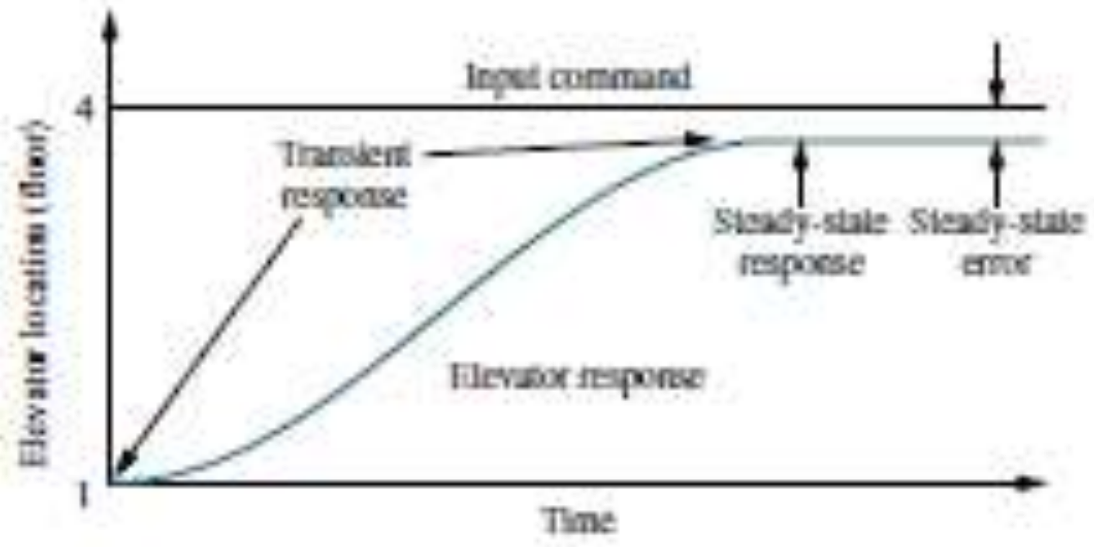


Figure shows a control system in its simplest form, where the input represents a desired output.

Definition

- For example, consider an elevator.
- When the fourth-floor button is pressed on the first floor, the elevator rises to the fourth floor with a speed and floor-leveling accuracy designed for passenger comfort.
- The push of the fourth-floor button is an input that represents our desired output, shown as a step function in Figure



Definition

The performance of the elevator can be seen from the elevator response curve in the figure.

Two major measures of performance are apparent:

(1) the transient response and

(2) the steady-state error.

Definition

In our example, passenger comfort and passenger patience are dependent upon the transient response.

If this response is too fast, passenger comfort is sacrificed;

if too slow, passenger patience is sacrificed.

The steady-state error is another important performance specification since passenger safety and convenience would be sacrificed if the elevator did not properly level.

Advantages of Control Systems

With control systems we can move large equipment with precision that would otherwise be impossible.

We can point huge antennas toward the farthest reaches of the universe to pick up faint radio signals; controlling these antennas by hand would be impossible.

Because of control systems, elevators carry us quickly to our destination, automatically stopping at the right floor.

We alone could not provide the power required for the load and the speed; motors provide the power, and control systems regulate the position and speed.

Advantages of Control Systems

We build control systems for four primary reasons:

1. Power amplification
2. Remote control
3. Convenience of input form
4. Compensation for disturbances

For example, a radar antenna, positioned by the low-power rotation of a knob at the input, requires a large amount of power for its output rotation. A control system can produce the needed power amplification, or power gain.

Robots designed by control system principles can compensate for human disabilities.

Control systems are also useful in remote or dangerous locations. For example, a remote-controlled robot arm can be used to pick up material in a radioactive environment.

Advantages of Control Systems

For example, in a temperature control system, the input is a position on a thermostat. The output is heat. Thus, a convenient position input yields a desired thermal output.

Another advantage of a control system is the ability to compensate for disturbances.

When the rope of elevator is cut, or the speed of elevator increases over the rated speed safety brake applied.

The system must be able to yield the correct output even with a disturbance.

For example, consider an antenna system that points in a commanded direction.

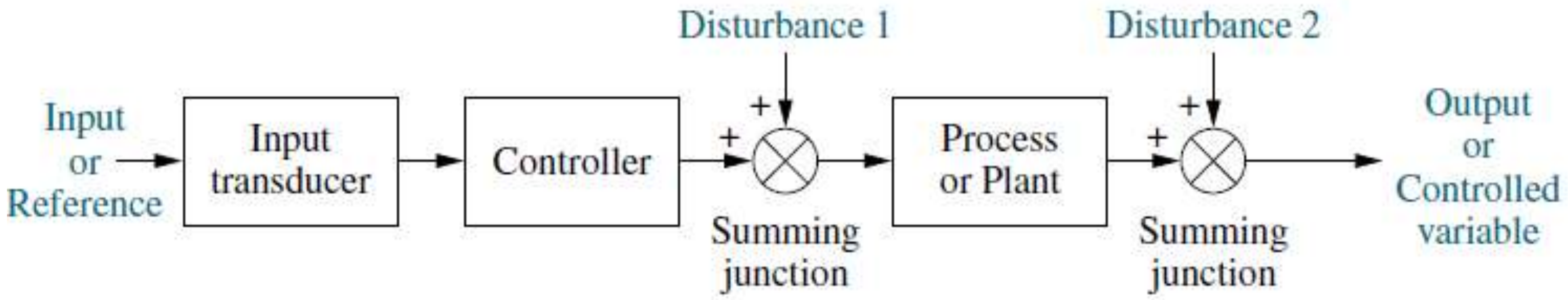
If wind forces the antenna from its commanded position, or if noise enters internally, the system must be able to detect the disturbance and correct the antenna's position.

Advantages of Control Systems

Obviously, the system’s input will not change to make the correction. Consequently, the system itself must measure the amount that the disturbance has repositioned the antenna and then return the antenna to the position commanded by the input.

Open Loop Control Systems

A generic open-loop system is shown in Figure.



(a)

It starts with a subsystem called an input transducer, which converts the form of the input to that used by the controller.

The controller drives a process or a plant.

The input is sometimes called the reference, while the output can be called the controlled variable.

Open Loop Control Systems

Other signals, such as disturbances, are shown added to the controller and process outputs via summing junctions, which yield the algebraic sum of their input signals using associated signs.

For example, the plant can be a furnace or air conditioning system, where the output variable is temperature.

The controller in a heating system consists of fuel valves and the electrical system that operates the valves.

The distinguishing characteristic of an open-loop system is that **it cannot compensate for any disturbances** that add to the controller's driving signal (Disturbance 1 in Figure).

Open Loop Control Systems

For example, if the controller is an electronic amplifier and Disturbance 1 is noise, then any additive amplifier noise at the first summing junction will also drive the process, corrupting the output with the effect of the noise. The output of an open-loop system is corrupted not only by signals that add to the controller's commands but also by disturbances at the output (Disturbance 2 in Figure). The system cannot correct for these disturbances, either.

Open Loop Control Systems

Open-loop systems, then, do not correct for disturbances and are simply commanded by the input.

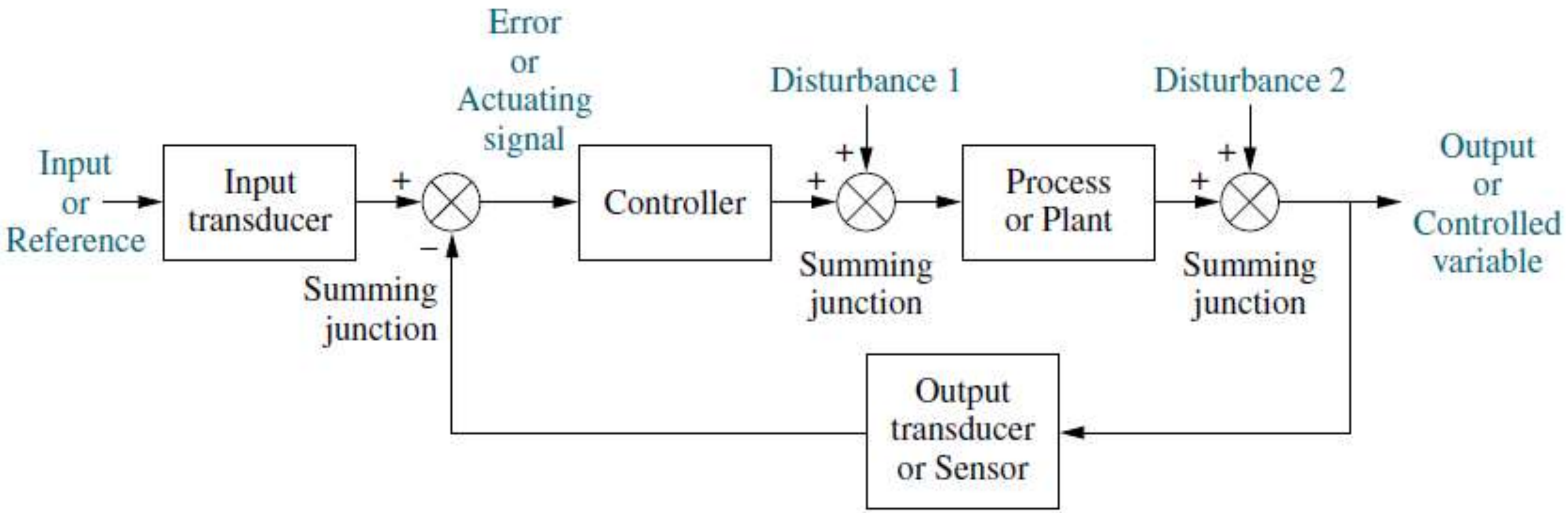
For example, toasters are open-loop systems, as anyone with burnt toast can attest.

The controlled variable (output) of a toaster is the color of the toast. The device is designed with the assumption that the toast will be darker the longer it is subjected to heat.

The toaster does not measure the color of the toast; it does not correct for the fact that the toast is rye, white, or sourdough, nor does it correct for the fact that toast comes in different thicknesses.

Closed Loop Control Systems

The disadvantages of open-loop systems, namely sensitivity to disturbances and inability to correct for these disturbances, may be overcome in closed-loop systems.



The generic architecture of a closed-loop system is shown in Figure.

Closed Loop Control Systems

The input transducer converts the form of the input to the form used by the controller.

An output transducer, or sensor, measures the output response and converts it into the form used by the controller.

For example, if the controller uses electrical signals to operate the valves of a temperature control system, the input position and the output temperature are converted to electrical signals.

The input position can be converted to a voltage by a potentiometer, a variable resistor, and the output temperature can be converted to a voltage by a thermistor, a device whose electrical resistance changes with temperature.

Closed Loop Control Systems

The first summing junction algebraically adds the signal from the input to the signal from the output, which arrives via the feedback path, the return path from the output to the summing junction.

In Figure, the output signal is subtracted from the input signal. The result is generally called the actuating signal.

However, in systems where both the input and output transducers have unity gain (that is, the transducer amplifies its input by 1), the actuating signal's value is equal to the actual difference between the input and the output. Under this condition, the actuating signal is called the error.

Closed Loop Control Systems

The closed-loop system compensates for disturbances by measuring the output response, feeding that measurement back through a feedback path, and comparing that response to the input at the summing junction.

If there is any difference between the two responses, the system drives the plant, via the actuating signal, to make a correction.

If there is no difference, the system does not drive the plant, since the plant's response is already the desired response.

Closed Loop Control Systems

Closed-loop systems have the obvious advantage of greater accuracy than open-loop systems.

They are less sensitive to noise, disturbances, and changes in the environment.

Transient response and steady-state error can be controlled more conveniently and with greater flexibility in closed-loop systems, often by a simple adjustment of gain (amplification) in the loop and sometimes by redesigning the controller.

We refer to the redesign as compensating the system and to the resulting hardware as a compensator.

Closed Loop Control Systems

On the other hand, closed-loop systems are more complex and expensive than open-loop systems.

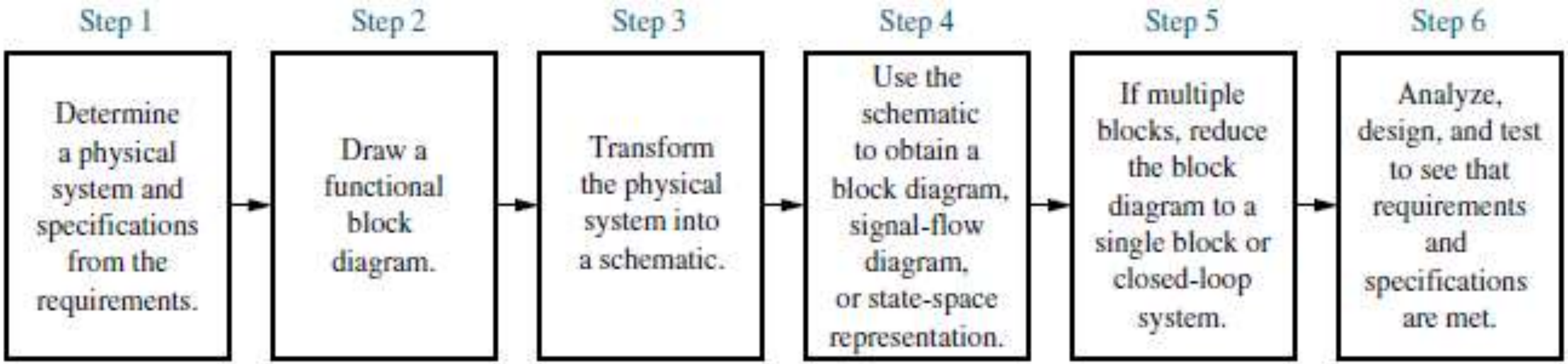
A standard, open-loop toaster serves as an example: It is simple and inexpensive. A closed-loop toaster oven is more complex and more expensive since it has to measure both color (through light reflectivity) and humidity inside the toaster oven.

Thus, the control systems engineer must consider the trade-off between the simplicity and low cost of an open-loop system and the accuracy and higher cost of a closed-loop system.

In summary, systems that perform the previously described measurement and correction are called closed-loop, or feedback control, systems. Systems that do not have this property of measurement and correction are called open-loop systems

Designed Procedure

Orderly sequence for the design of feedback control systems



Designed Procedure: Step 1

Step 1: Transform Requirements Into a Physical System

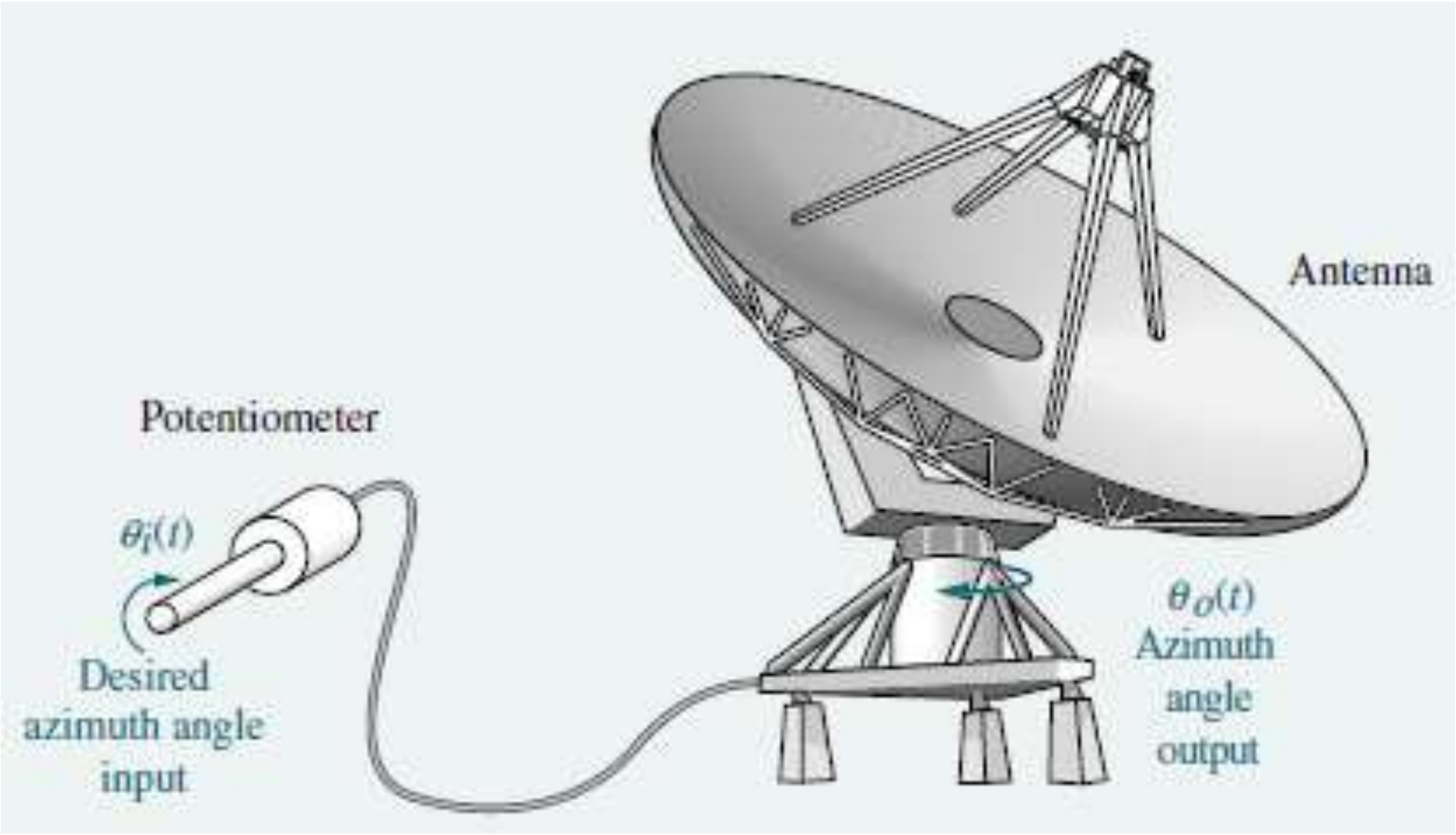
We begin by transforming the requirements into a physical system.

For example, in the antenna azimuth position control system, the requirements would state the desire to position the antenna from a remote location and describe such features as weight and physical dimensions.

Using the requirements, design specifications, such as desired transient response and steady-state accuracy, are determined.

Designed Procedure: Step 1

Step 1



Designed Procedure: Step 2

Step 2: Draw a Functional Block Diagram

The designer now translates a qualitative description of the system into a functional block diagram that describes the component parts of the system (that is, function and/or hardware) and shows their interconnection. Figure 1.9(d) is an example of a functional block diagram for the antenna azimuth position control system. It indicates functions such as input transducer and controller, as well as possible hardware descriptions such as amplifiers and motors. At this point the designer may produce a detailed layout of the system, such as that shown in Figure 1.9(b), from which the next phase of the analysis and design sequence, developing a schematic diagram, can be launched

Designed Procedure: Step 2

Step 2: Draw a Functional Block Diagram

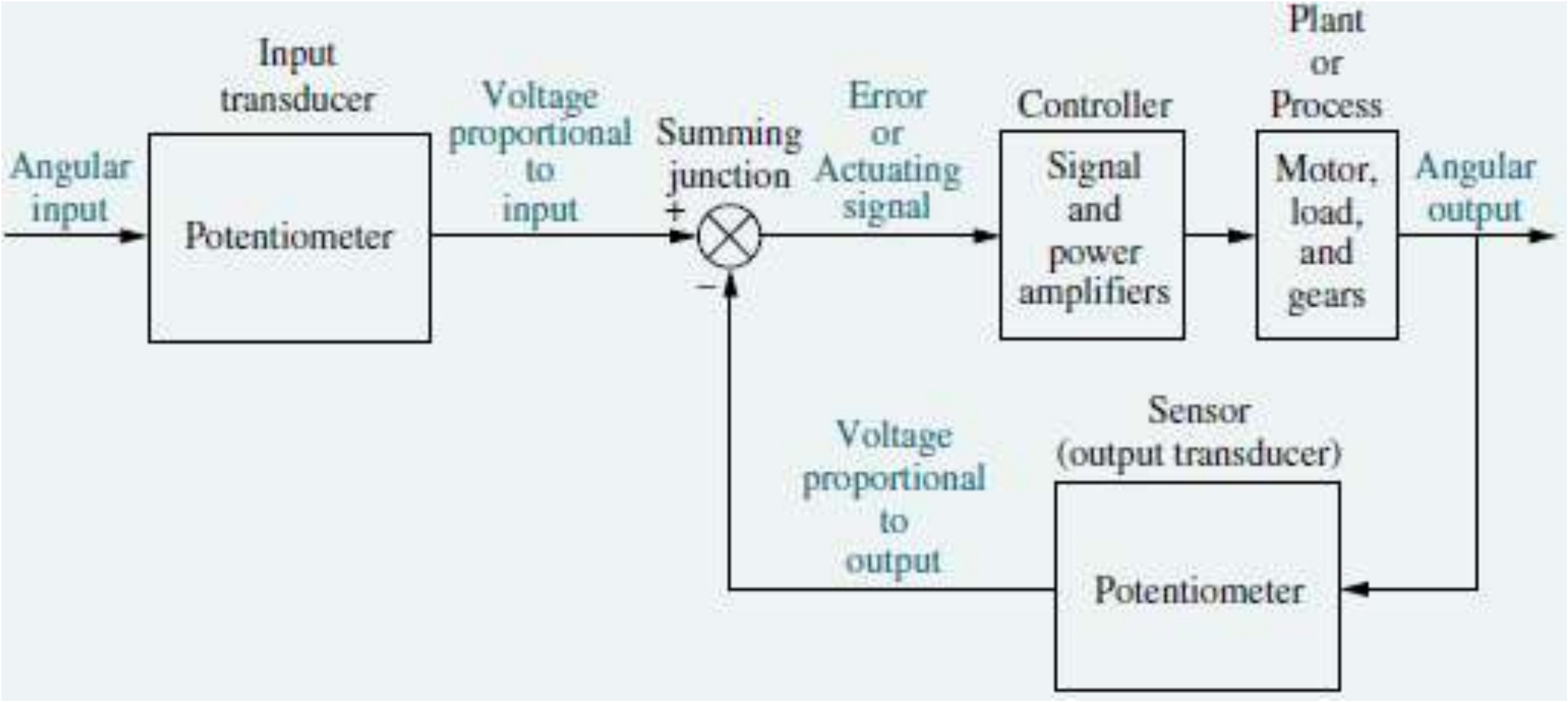
The designer now translates a qualitative description of the system into a functional block diagram that describes the component parts of the system (that is, function and/or hardware) and shows their interconnection.

Following Figure is an example of a functional block diagram for the antenna azimuth position control system.

It indicates functions such as input transducer and controller, as well as possible hardware descriptions such as amplifiers and motors.

Designed Procedure: Step 2

Step 2: Draw a Functional Block Diagram



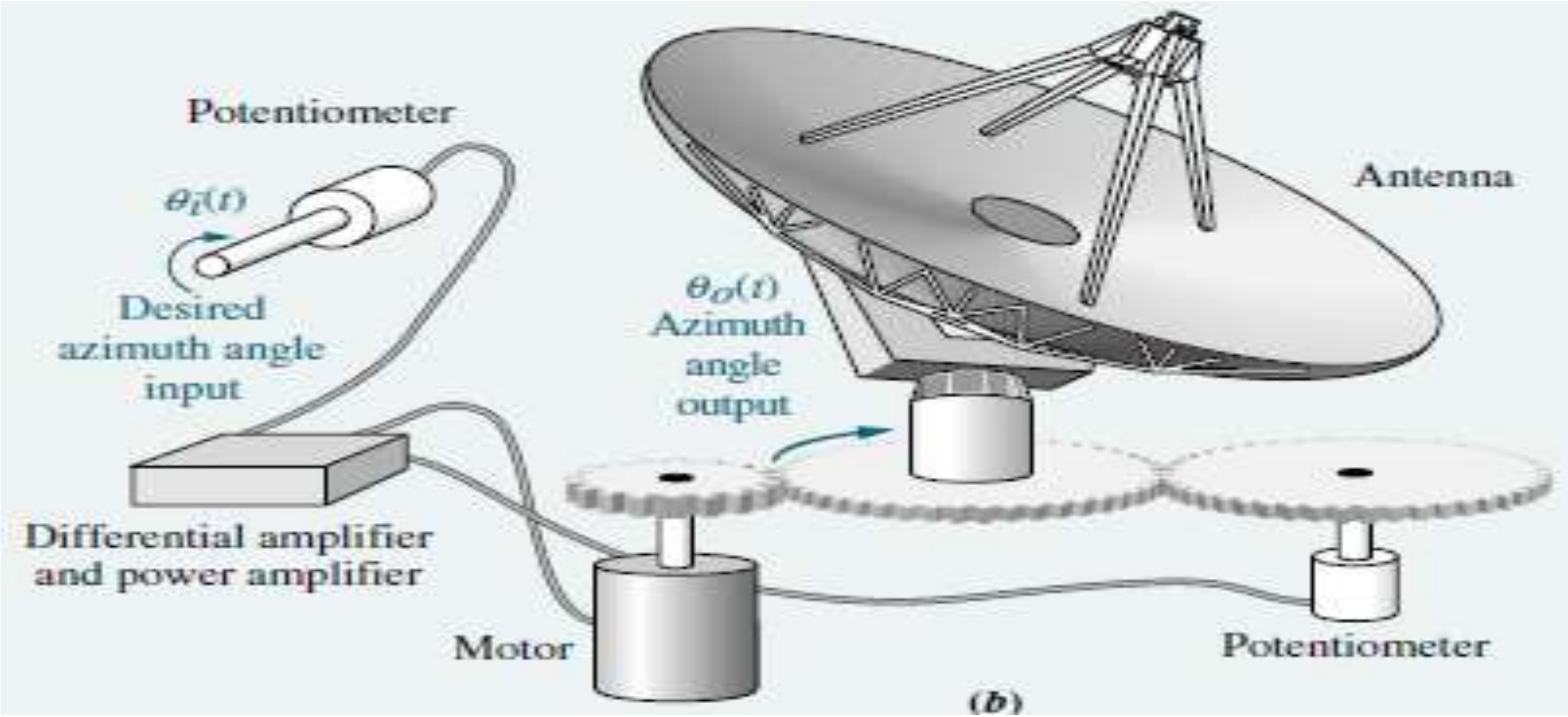
Designed Procedure: Step 2

Step 2: Draw a Functional Block Diagram

At this point the designer may produce a detailed layout of the system, such as that shown in following figure, from which the next phase of the analysis and design sequence, developing a schematic diagram, can be launched

Designed Procedure: Step 2

Step 2: Draw a Functional Block Diagram



Designed Procedure: Step 3

Step 3: Create a Schematic

As we have seen, position control systems consist of electrical, mechanical, and electromechanical components.

After producing the description of a physical system, the control systems engineer transforms the physical system into a schematic diagram. The control system designer can begin with the physical description, to derive a schematic.

The engineer must make approximations about the system and neglect certain phenomena, or else the schematic will be unwieldy, making it difficult to extract a useful mathematical model during the next phase of the analysis and design sequence.

Designed Procedure: Step 3

Step 3: Create a Schematic

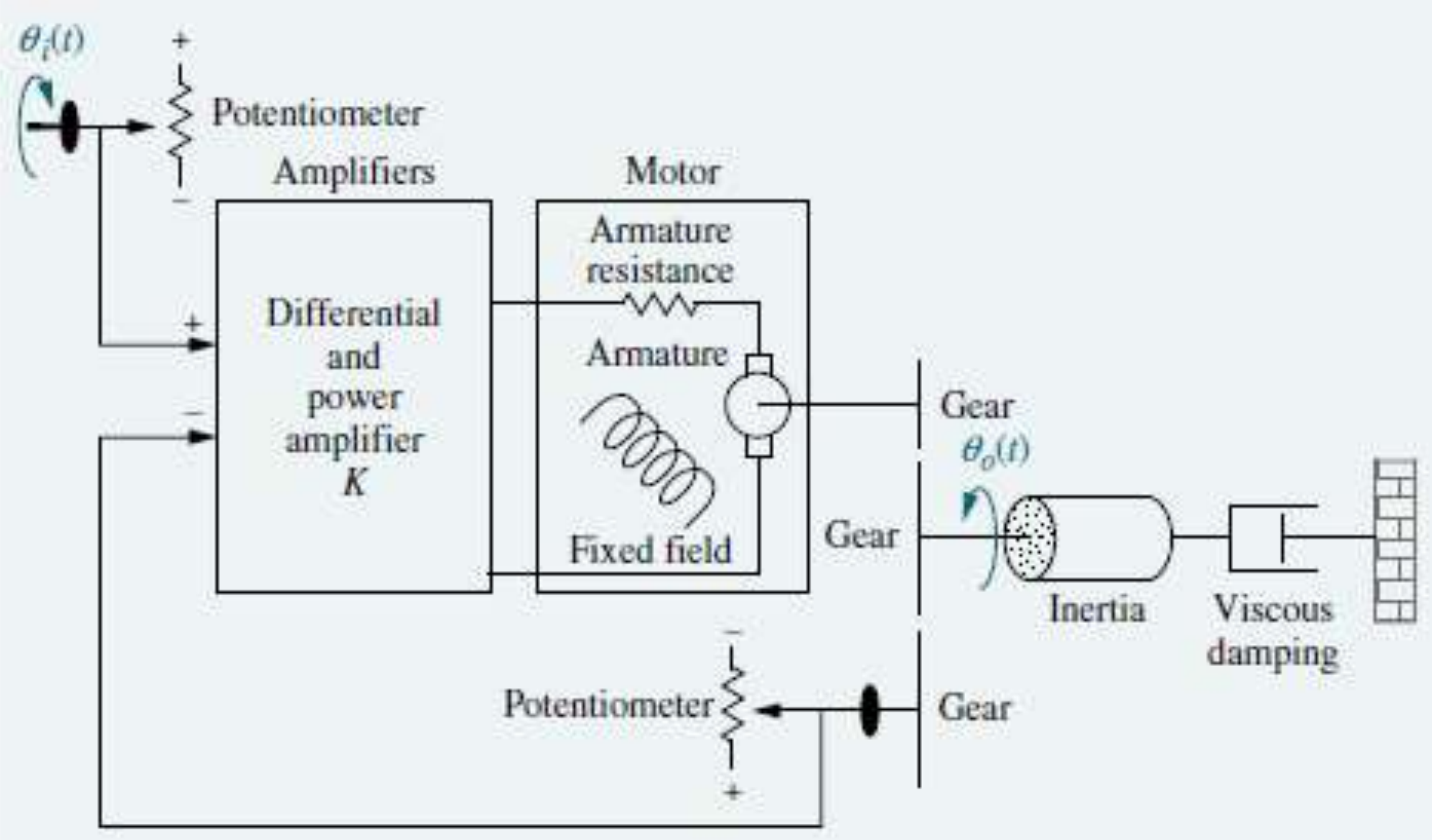
The designer starts with a simple schematic representation and, at subsequent phases of the analysis and design sequence, checks the assumptions made about the physical system through analysis and computer simulation.

If the schematic is too simple and does not adequately account for observed behavior, the control systems engineer adds phenomena to the schematic that were previously assumed negligible.

A schematic diagram for the antenna azimuth position control system is shown in following figure.

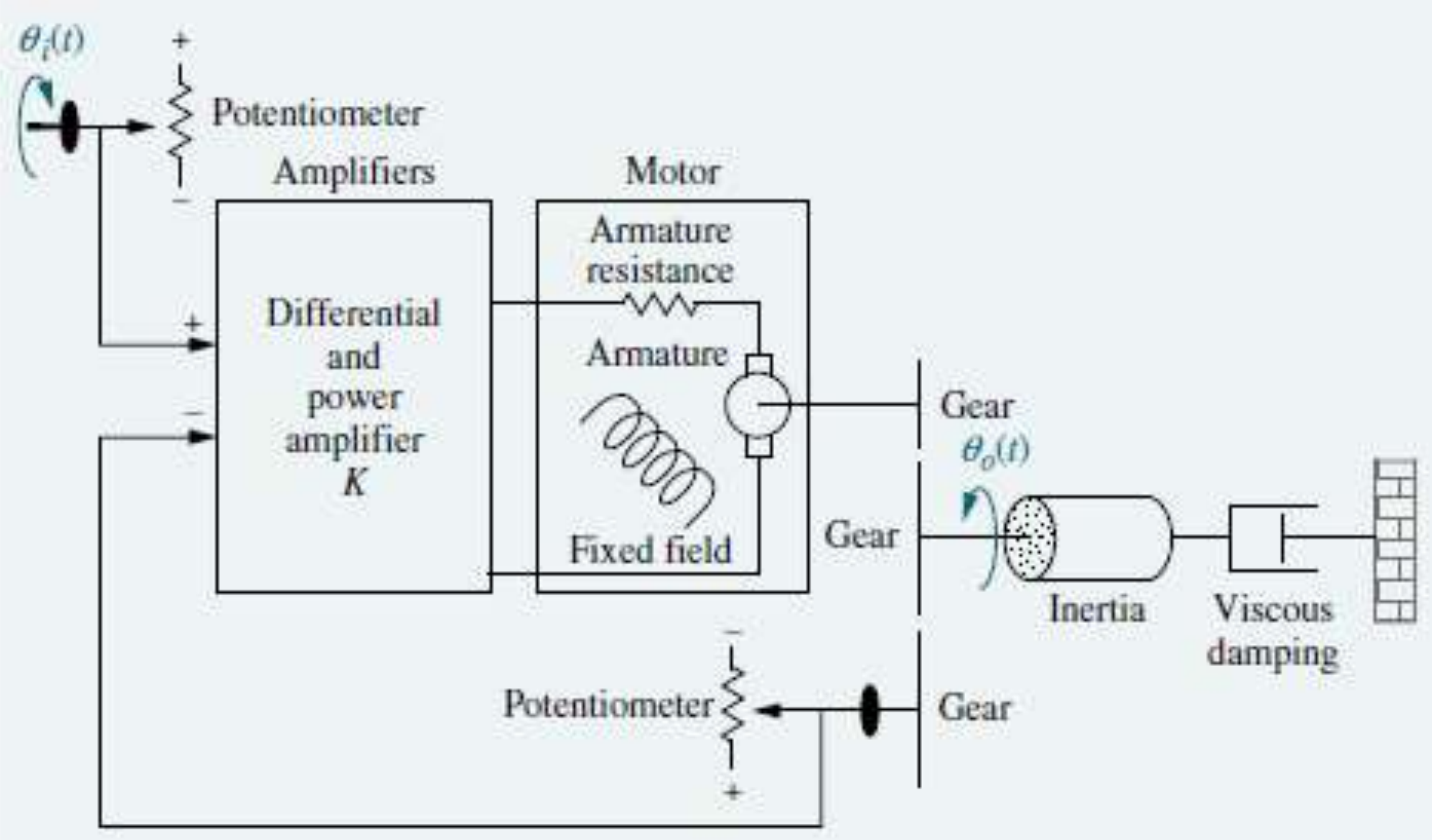
Designed Procedure: Step 3

Step 3: Create a Schematic



Designed Procedure: Step 3

Step 3: Create a Schematic



Designed Procedure: Step 3

Step 3: Create a Schematic

When we draw the potentiometers, we make our first simplifying assumption by neglecting their friction or inertia.

These mechanical characteristics yield a dynamic, rather than an instantaneous, response in the output voltage.

We assume that these mechanical effects are negligible and that the voltage across a potentiometer changes instantaneously as the potentiometer shaft turns.

A differential amplifier and a power amplifier are used as the controller to yield gain and power amplification, respectively, to drive the motor.

Designed Procedure: Step 3

Step 3: Create a Schematic

Again, we assume that the dynamics of the amplifiers are rapid compared to the response time of the motor; thus, we model them as a pure gain, K .

A dc motor and equivalent load produce the output angular displacement. The speed of the motor is proportional to the voltage applied to the motor's armature circuit. Both inductance and resistance are part of the armature circuit.

We assume the effect of the armature inductance is negligible for a dc motor. The designer makes further assumptions about the load.

The load consists of a rotating mass and bearing friction. Thus, the model consists of inertia and viscous damping whose resistive torque increases with speed, as in an automobile's shock absorber or a screen door damper.

Designed Procedure: Step 3

Step 3: Create a Schematic

The decisions made in developing the schematic stem from knowledge of the physical system, the physical laws governing the system's behavior, and practical experience.

Designed Procedure: Step 4

Step 4: Develop a Mathematical Model (Block Diagram)

Once the schematic is drawn, the designer uses physical laws, such as Kirchhoff's laws for electrical networks and Newton's law for mechanical systems, along with simplifying assumptions, to model the system mathematically. These laws are

Kirchhoff's voltage law: The sum of voltages around a closed path equals zero.

Kirchhoff's current law: The sum of electric currents flowing from a node equals zero.

Newton's laws: The sum of forces on a body equals zero; the sum of moments on a body equals zero

Kirchhoff's and Newton's laws lead to mathematical models that describe the relationship between the input and output of dynamic systems. One such model is the linear, time-invariant differential equation,

Designed Procedure: Step 4

Step 4: Develop a Mathematical Model (Block Diagram)

$$\frac{d^m c(t)}{dt^n} + d_{n-1}\frac{d^{m-1} c(t)}{dt^{n-1}} + \cdots + d_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1}\frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots + b_0 r(t)$$

Many systems can be approximately described by this equation, which relates the output, c(t), to the input, r(t), by way of the system parameters, a_i and b_j.

In addition to the differential equation, the transfer function is another way of mathematically modeling a system.

The model is derived from the linear, time-invariant differential equation using what we call the Laplace transform. Although the transfer function can be used only for linear systems.

Designed Procedure: Step 4

Step 4: Develop a Mathematical Model (Block Diagram)

We will be able to change system parameters and rapidly sense the effect of these changes on the system response. The transfer function is also useful in modeling the interconnection of subsystems by forming a block diagram with a mathematical function inside each block.

Designed Procedure: Step 4

Step 4: Develop a Mathematical Model (Block Diagram)

Another model is the **state-space representation**.

One advantage of state space methods is that they can also be used for systems that cannot be described by linear differential equations.

Further, state-space methods are used to model systems for simulation on the digital computer.

Basically, this representation turns an n^{th} order differential equation into n simultaneous first-order differential equations.

Designed Procedure: Step 4

Step 4: Develop a Mathematical Model (Block Diagram)

Finally, we should mention that to produce the mathematical model for a system, we require knowledge of the parameter values, such as equivalent resistance, inductance, mass, and damping, which is often not easy to obtain. Analysis, measurements, or specifications from vendors are sources that the control systems engineer may use to obtain the parameters.

Designed Procedure: Step 4

Step 4: Develop a Mathematical Model (Block Diagram)

Finally, we should mention that to produce the mathematical model for a system, we require knowledge of the parameter values, such as equivalent resistance, inductance, mass, and damping, which is often not easy to obtain. Analysis, measurements, or specifications from vendors are sources that the control systems engineer may use to obtain the parameters.

Designed Procedure: Step 5

Step 5: Reduce the Block Diagram

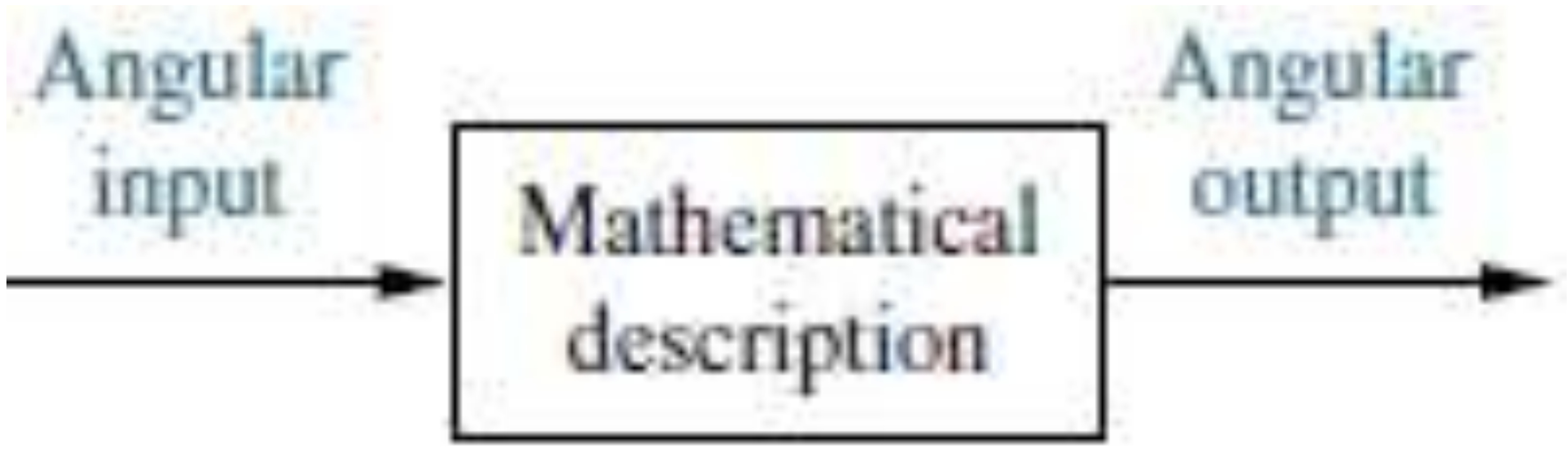
Subsystem models are interconnected to form block diagrams of larger systems where each block has a mathematical description.

Notice that many signals, such as proportional voltages and error, are internal to the system. There are also two signals – angular input and angular output – that are external to the system.

In order to evaluate system response in this example, we need to reduce this large system's block diagram to a single block with a mathematical description that represents the system from its input to its output, as shown in following figure. Once the block diagram is reduced, we are ready to analyze and design the system.

Designed Procedure: Step 5

Step 5: Reduce the Block Diagram



Designed Procedure: Step 6

Step 6: Analyze and Design

The next phase of the process, following block diagram reduction, is analysis and design.

If you are interested only in the performance of an individual subsystem, you can skip the block diagram reduction and move immediately into analysis and design.

In this phase, the engineer analyzes the system to see if the response specifications and performance requirements can be met by simple adjustments of system parameters.

If specifications cannot be met, the designer then designs additional hardware in order to effect a desired performance.

Designed Procedure: Step 6

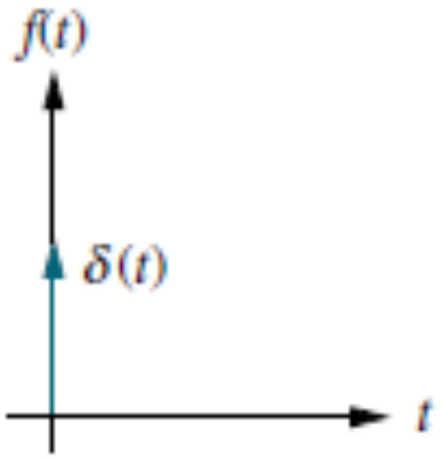
Step 6: Analyze and Design

Test input signals are used, both analytically and during testing, to verify the design.

It is neither necessarily practical nor illuminating to choose complicated input signals to analyze a system's performance.

Thus, the engineer usually selects standard test inputs.

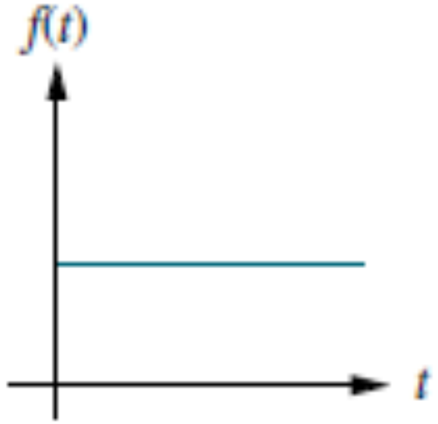
Designed Procedure: Step 6

I/P	**Function**	**Description**	**Sketch**	**Use**
Impulse	$\delta(t)$	$\delta(t) = \infty$ for $-0 < t < +0$ $= 0$ elsewhere $\int_{-0}^{+0} \delta(t)dt = 1$		Transient response Modelling

An impulse is infinite at $t = 0$ and zero elsewhere.

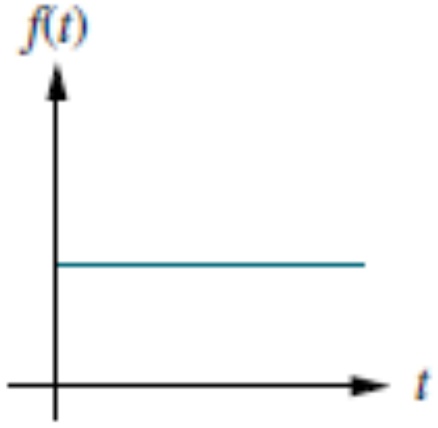
The area under the unit impulse is 1. An approximation of this type of waveform is used to place initial energy into a system so that the response due to that initial energy is only the transient response of a system. From this response the designer can derive a mathematical model of the system

Designed Procedure: Step 6

I/P	Function	Description	Sketch	Use
Step	u(t)	$u(t) = 1 \quad \text{for } t > 0$ $\quad\;\; = 0 \quad \text{for } t < 0$		Transient response Steady-state error

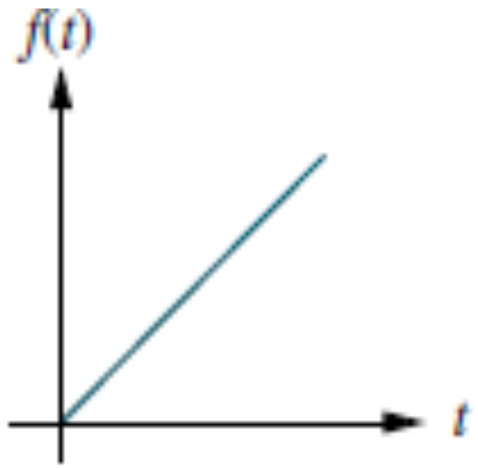
A step input represents a constant command, such as position, velocity, or acceleration. Typically, the step input command is of the same form as the output. For example, if the system's output is position, as it is for the antenna azimuth position control system, the step input represents a desired position, and the output represents the actual position.

Designed Procedure: Step 6

I/P	Function	Description	Sketch	Use
Step	u(t)	$u(t) = 1$ for $t > 0$ $= 0$ for $t < 0$		Transient response Steady-state error

If the system's output is velocity, as is the spindle speed for a video disc player, the step input represents a constant desired speed, and the output represents the actual speed. The designer uses step inputs because both the transient response and the steady-state response are clearly visible and can be evaluated

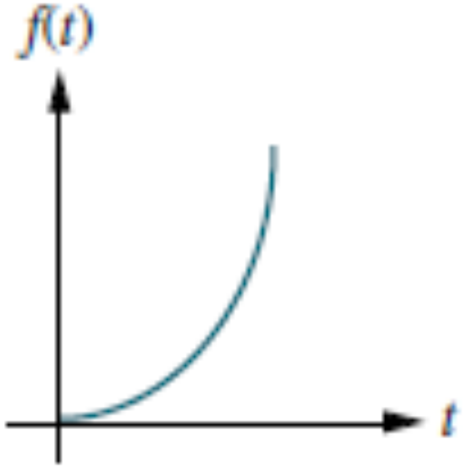
Designed Procedure: Step 6

I/P	Function	Description	Sketch	Use
Ramp	$t\,u(t)$	$t\,u(t) = 1$ for $t \geq 0$ $= 0$ otherwise		Steady-state error

The ramp input represents a linearly increasing command.

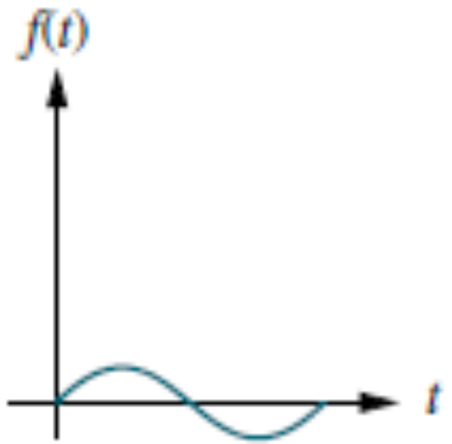
For example, if the system's output is position, the input ramp represents a linearly increasing position, such as that found when tracking a satellite moving across the sky at constant speed.

Designed Procedure: Step 6

I/P	Function	Description	Sketch	Use
Parabola	$\frac{1}{2} t^2 u(t)$	$\frac{1}{2} t^2 u(t) = \frac{1}{2} t^2$ for $t \geq 0$ $= 0$ otherwise		Steady-state error

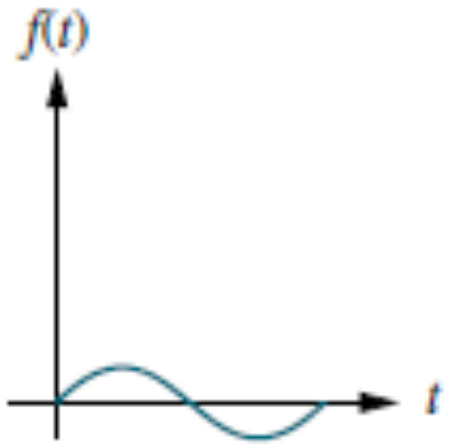
If the system's output is velocity, the input ramp represents a linearly increasing velocity. The response to an input ramp test signal yields additional information about the steady-state error. The previous discussion can be extended to parabolic inputs, which are also used to evaluate a system's steady-state error.

Designed Procedure: Step 6

I/P	Function	Description	Sketch	Use
Sinusoid	sin ωt	sin ωt		Transient response Modeling Steady-state error

Sinusoidal inputs can also be used to test a physical system to arrive at a mathematical model.

Designed Procedure: Step 6

I/P	**Function**	**Description**	**Sketch**	**Use**
Sinusoid	$\sin \omega t$	$\sin \omega t$		Transient response Modeling Steady-state error

Sinusoidal inputs can also be used to test a physical system to arrive at a mathematical model.

Laplace Transform

A system represented by a differential equation is difficult to model as a block diagram. By using the Laplace transform, with which we can represent the input, output, and system as separate entities.

Further, their interrelationship will be simply algebraic.

Let us first define the Laplace transform and then show how it simplifies the representation of physical systems

The Laplace transform is defined a

$$L[f(t)] = F(s) = \int_{0-}^{\infty} f(t) e^{-st} dt$$

Where $s = \sigma + j\omega$ is a complex variable.

Laplace Transform

The notation for the lower limit means that even if f(t) is discontinuous at $t = 0$, we can start the integration prior to the discontinuity as long as the integral converges.

Thus, we can find the Laplace transform of impulse functions.

This property has distinct advantages when applying the Laplace transform to the solution of differential equations where the initial conditions are discontinuous at $t = 0$.

Using differential equations, we have to solve for the initial conditions after the discontinuity knowing the initial conditions before the discontinuity

Laplace Transform

The inverse Laplace transform is defined a

$$L^{-1}[F(s)] = f(t) = \frac{1}{2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds = f(t)u(t)$$

Where u(t) = 1; t > 0

= 0; t < 0

is the unit step function. Multiplication of f(t) by u(t) yields a time function that is zero for t < 0.

Laplace Transform Table

SN	f(t)	F(s)
1	$\delta(t)$	1
2	$u(t)$	$1/s$
3	$t\ u(t)$	$1/s^2$
4	$t^n\ u(t)$	$n!\ /(s^n +1)$
5	$e^{-at}\ u(t)$	$1/(s+a)$
6	$\sin \omega t\ u(t)$	$\omega/(s^2 + \omega^2)$
7	$\cos \omega t\ u(t)$	$s/(s^2 + \omega^2)$

Laplace Transform Theorems

SN	Theorem	Name
1	$L[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2	$L[kf(t)] = kF(s)$	Linearity
3	$L[f_1(t) + f_1(t)] = F_1(s) + F_2(s)$	Linearity
4	$L[e^{-at}f(t)] = F(s+a)$	Frequency shift theorem
5	$L[f(t-T)] = e^{-sT}F(s)$	Time shift theorem
6	$L[f(at)] = \left(\frac{1}{a}\right)F(s/a)$	Scaling theorem
7	$L\left[\frac{df(t)}{dt}\right] = sF(s) - f(0-)$	Differentiation theorems

Laplace Transform Theorems

SN	Theorem	Name
8	$L\left[\dfrac{d^2f(t)}{dt^2}\right] = s^2F(s) - sf(0-) - f'(0-)$	Differentiation theorems
9	$L\left[\dfrac{d^nf(t)}{dt^n}\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorems
10	$L\left[\int_{0-}^{t} f(\tau)d\tau\right] = F(s)/s$	Integration theorem
11	$f(\infty) = \lim_{s\to 0} sF(s)$	Final value theorem
12	$f(0+) = \lim_{s\to\infty} sF(s)$	Initial value theorem

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Laplace Transform

Find the Laplace transform of $f(t) = Ae^{-at} u(t)$.

Laplace Transform

Find the Laplace transform of $f(t) = Ae^{-at}u(t)$.

Since the time function does not contain an impulse function, we can replace the lower limit of 0- with 0.

Hence,

$$F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty Ae^{-at}e^{-st}dt$$

$$= A\int_0^\infty e^{-(s+a)t}dt = -\frac{A}{s+a}e^{-(s+a)t}\Big|_0^\infty = \frac{A}{s+a}$$

Laplace Transform

Find the inverse Laplace transform of $F_1(s) = 1/(s+3)^2$.

Laplace Transform

Find the inverse Laplace transform of $F_1(s) = 1/(s+3)^2$.

We know the frequency shift theorem

$$L[e^{-at} f(t)] = F(s + a)$$

the Laplace transform of $f(t) = tu(t)$ is $1/s^2$,

If the inverse transform of $F(s) = 1/s^2$ is $tu(t)$,

the inverse transform of $F(s + a) = 1/(s + a)^2$ is $e^{-at}\, t\, u(t)$.

Hence, $f_1(t) = e^{-3t}\, t\, u(t)$

Laplace Transform: Partial-Fraction Expansion

To find the inverse Laplace transform of a complicated function,

Convert the function to a sum of simpler terms

The result is called a partial-fraction expansion.

If $F1(s) = N(s)/D(s)$,

If the order of $N(s)$ is less than the order of $D(s)$, then a partial-fraction expansion can be made.

If the order of $N(s)$ is greater than or equal to the order of $D(s)$, then $N(s)$ must be divided by $D(s)$ successively until the result has a remainder whose numerator is of order less than its denominator. For example, if

Laplace Transform: Partial-Fraction Expansion

For example, if $F_1(s) = \frac{s^3+2s^2+6s+7}{s^2+s+5}$

we must perform the indicated division until we obtain a remainder whose numerator is of order less than its denominator. Hence $F_1(s) = s + 1 + \frac{2}{s^2+s+5}$

Taking the inverse Laplace transform, using the above Tables we obtain $f_1(t) = \frac{d\delta(t)}{dt} + \delta(t) + L^{-1}\left[\frac{2}{s^2+s+5}\right]$ Using partial-fraction expansion, we will be able to expand functions like $F(s) = \frac{2}{s^2+s+5}$ into a sum of terms and then find the inverse Laplace transform for each term

Case 1. Roots of the Denominator of F(s) Are Real and Distinct

Let $F(s) = \frac{2}{(s+1)(s+2)}$

The roots of the denominator are distinct, since each factor is raised only to unity power. We can write the partial-fraction expansion as a sum of terms where each factor of the original denominator forms the denominator of each term, and constants, called residues, form the numerators. Hence, $F(s) = \frac{2}{(s+1)(s+2)} = \frac{K_1}{(S+1)} + \frac{K_2}{(S+2)}$

To find K_1, we first multiply the equation by (s + 1), which isolates K_1. Thus,

$$\frac{2}{(s+1)(s+2)} = K_1 + \frac{K_2(s+1)}{(S+2)}$$

Let s → -1 eliminates the last term and gives K1 = 2. Similarly, K2 can be found by multiplying the equation by (s + 2) and then letting s → -2; hence, K2 = -2.

Case 1. Roots of the Denominator of F(s) Are Real and Distinct

Let $s \rightarrow -1$ eliminates the last term and gives $K_1 = 2$. Similarly,

K_2 can be found by multiplying the equation by $(s + 2)$ and then letting $s \rightarrow -2$; hence,

$K_2 = -2$.

Therefore, $F(s) = \frac{2}{S+1} - \frac{2}{s+2}$

By referring the tables above

$f(t) = (2e^{-t} - 2e^{-2t})\, u(t)$

Case 1. Roots of the Denominator of F(s) Are Real and Distinct

Given the following differential equation, solve for y(t) if all initial conditions are zero. Use the Laplace transform. $\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 32y = 32u(t)$

Substitute the corresponding Laplace transform for each term in above equation, using Table, and the initial conditions of $y(t)$ and $\frac{dy(t)}{dt}$ given by $y(0-) = 0$ and $\frac{dy(0-)}{dt} = 0$, respectively. Hence, the Laplace transform of equation is

$$s^2Y(s) + 12sY(s) + 32Y(s) = \frac{32}{s}$$

Solving for the response Y(s) we have $Y(s) = \frac{32}{s(s^2+12s+32)} = \frac{32}{s(s+4)(s+8)}$

Now y(t) is obtained by taking the inverse Laplace transform, for that we need to partial fraction method. Therefore

Case 1. Roots of the Denominator of F(s) Are Real and Distinct

Therefore, $Y(s) = \frac{32}{s(s+4)(s+8)} = \frac{K_1}{s} + \frac{K_2}{s+4} + \frac{K_3}{s+8}$

$$K_1 = \left.\frac{32}{(s+4)(s+8)}\right|_{s\to 0} = 1$$

$$K_2 = \left.\frac{32}{s(s+8)}\right|_{s\to -4} = -2$$

$$K_3 = \left.\frac{32}{s(s+4)}\right|_{s\to -8} = 1$$

Case 1. Roots of the Denominator of F(s) Are Real and Distinct

Hence $Y(s) = \frac{1}{s} - \frac{2}{s+4} + \frac{1}{s+8}$

This is the simplest form, and we can easily find the LT of each term

$$y(t) = (1 - 2e^{-4t} + e^{-8t})u(t)$$

The u(t) shows that the response is zero until t = 0.

Thus, output responses will also be zero until t = 0. For convenience, we will leave off the u(t) notation. Accordingly, we write the output response as $y(t) = (1 - 2e^{-4t} +$

Case 2. Roots of the Denominator of F(s) Are Real and Repeated

Let $F(s) = \frac{2}{(s+1)(s+2)^2}$

The roots of $(s + 2)^2$ in the denominator are repeated, since the factor is raised to an integer power higher than 1. In this case, the denominator root at 2 is a multiple root of multiplicity 2

We can write the partial-fraction expansion as a sum of terms, where each factor of the denominator forms the denominator of each term. In addition, each multiple root generates additional terms consisting of denominator factors of reduced multiplicity. For example, if $F(s) = \frac{2}{(s+1)(s+2)^2} = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)^2} + \frac{K_3}{(s+2)}$

Case 2. Roots of the Denominator of F(s) Are Real and Repeated

For K_1

$$\frac{2}{(s+2)^2} = K_1 + \frac{(s+1)K_2}{(s+2)^2} + \frac{(s+1)K_3}{(s+2)}, \;\; \therefore K_1=2$$

K_2 can be isolated by multiplying by $(s + 2)^2$, yielding

$$\frac{2}{s+1} = (s+2)^2\frac{K_1}{(s+1)} + K_2 + (S+2)K_3$$. Letting s → -2 ; K_2 = - 2.

To find K_3 we differentiate above equation with respect to s

$$\frac{-2}{(s+1)^2} = \frac{(s+2)K_1}{(s+1)^2} + K_3$$, K_3 is isolated and can be found by s → -2 Hence, K3 = -2

$$\therefore F(s) = \frac{2}{(s+1)(s+2)^2} = \frac{2}{(s+1)} - \frac{2}{(s+2)^2} - \frac{2}{(s+2)}$$

$$\therefore f(t) = 2e^{-t} - 2te^{-2t} - 2e^{-2t}$$

Case 3. Roots of the Denominator of F(s) Are Complex or Imaginary

Let $F(s) = \frac{3}{s(s^2+2s+5)}$ *this*

This function can be expanded in the following form

$$\frac{3}{s(s^2+2s+5)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{(s^2+2s+5)}$$

K_1 is found in the usual way to be 3/5. K_2 and K_3 can be found by first multiplying the above equation by the lowest common denominator, $s(s^2 + 2s + 5)$, and clearing the fractions. After simplification with $K_1 = 3/5$, we obtain

$$3 = \left(K_2 + \frac{3}{5}\right)s^2 + \left(K_3 + \frac{6}{5}\right)s + 3$$

Case 3. Roots of the Denominator of F(s) Are Complex or Imaginary

Balancing coefficients, $(K_2 + 3/5) = 0$ and $(K_3 + 6/5) = 0$. Hence $K_2 = - 3/5$ and $K_3 = - 6/5$. Thus,

Let $F(s) = \frac{3}{s(s^2+2s+5)} = \frac{3/5}{s} - \frac{3}{5}\frac{s+2}{(s^2+2s+5)}$

The last term can be shown to be the sum of the Laplace transforms of an exponentially damped sine and cosine. Using $L[\sin(\omega t)\,u(t)] = \frac{\omega}{(s^2+\omega^2)}$; $L[\cos\omega t\,u(t)] = \frac{s}{(s^2+\omega^2)}$, $L[kf(t)] = kF(s)$; and $L[e^{-at}f(t)] = F(s+a)$

$L[Ae^{-at}\cos\omega t] = \frac{A(s+a)}{(s+a)^2+\omega^2}$ and $L[Be^{-at}\sin\omega t] = \frac{B\omega}{(s+a)^2+\omega^2}$

If we add these two equations

Case 3. Roots of the Denominator of F(s) Are Complex or Imaginary

If we add these two equations

$$L[Ae^{-at}\cos \omega t + Be^{-at}\sin \omega t] = \frac{A(s+a)+B\omega}{(s+a)^2+\omega^2}$$

Now rearrange $F(s) = \frac{3}{s(s^2+2s+5)} = \frac{3/5}{s} - \frac{3}{5}\frac{s+2}{(s^2+2s+5)}$ by completing the squares in the denominator and adjusting terms in the numerator without changing its value

$$F(s) = \frac{3/5}{s} - \frac{3}{5}\frac{s+1+(1/2)(2)}{(s+1)^2+2^2}$$

$$\therefore f(t) = \frac{3}{5} - \frac{3}{5}e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right)$$

Case 3. Roots of the Denominator of F(s) Are Complex or Imaginary

The alternative method is

$$F(s) = \frac{3}{s(s^2 + 2s + 5)} = \frac{3}{s(s + 1 + j2)(s + 1 - j2)}$$

$$= \frac{k_1}{s} + \frac{k_2}{s + 1 + j2} + \frac{k_3}{s + 1 - j2}$$

$$K_2 = \left. \frac{3}{s(s + 1 - j2)} \right|_{s \to -1 - j2} = -\frac{3}{20}(2 + j1)$$

We know k1 = 3/5; and K3 is complex conjugate of K2.

Case 3. Roots of the Denominator of F(s) Are Complex or Imaginary

Hence

$$F(s) = \frac{3/5}{s} - \frac{3}{20}\left(\frac{2+j1}{S+1+j2} + \frac{2-j1}{S+1-j2}\right)$$

$$\therefore f(t) = \frac{3}{5} - \frac{3}{20}e^{-t}\left((2+j1)e^{-(1+j2)t} + (2-j1)e^{-(1-j2)t}\right)$$

$$= \frac{3}{5} - \frac{3}{20}e^{-t}\left[4\left(\frac{e^{j2t}+e^{-j2t}}{2}\right) + 2\left(\frac{e^{j2t}+e^{-j2t}}{2j}\right)\right]$$

$$= \frac{3}{5} - \frac{3}{5}e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right)$$

Feedback Control System Unit 1_2
Transfer function of physical systems

The Transfer Function

Consider a n[th]-order, linear, time-invariant continuous time system describe by the differential equation

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \cdots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots + b_0 r(t)$$

where c(t) is the output, r(t) is the input, and the a_n, a_{n-1},…, a_0 and b_m, b_{m-1}, …, b_0 are the coefficients. Taking the Laplace transform of both sides,

$$a_n s^n C(s) + a_{n-1} s^{n-1} C(s) + \cdots + a_0 C(s) + \text{initial condition terms involving } c(t) =$$
$$b_m s^m R(s) + b_{m-1} s^{m-1} R(s) + \cdots + b_0 R(s) + \text{initial condition terms involving } r(t)$$

If we assume that all initial conditions are zero, then

$$(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0) C(s) = (b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0) R(s)$$

The Transfer Function

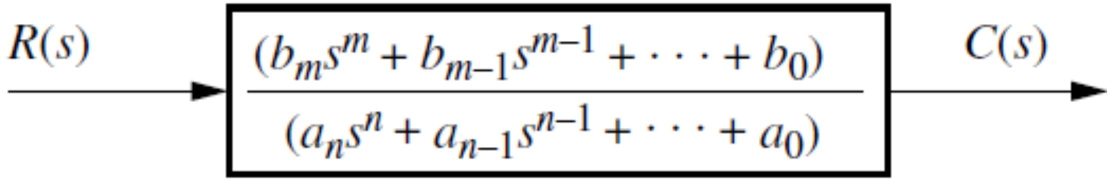
The ratio of the output transform, C(s), divided by the input transform, R(s) is:

$$\frac{C(s)}{R(s)} = G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0}$$

The transfer function G(s) is defined as the ratio of Laplace Transform of output C(s) to the Laplace Transform of input R(s) with zero initial conditions

The transfer function can be represented as a block diagram, as shown in Figure, with the input on the left, the output on the right, and the system transfer function inside the block.

Also, we can find the output, C(s) by using C(s) = R(s)G(s)



The Transfer Function

Find the transfer function of a system represented by

$$\frac{dc(t)}{dt} + 2c(t) = r(t)$$

Taking the Laplace transform of both sides, assuming zero initial

$$sC(s) + 2C(s) = R(s)$$

$$(s+2)C(s) = R(s)$$

The transfer function is

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{s+2}$$

The Transfer Function

Find step response of the linear time invariant system described by the differential equation

$$\frac{dc(t)}{dt} + 2c(t) = r(t)$$

Taking the Laplace transform of both sides, assuming zero initial

$$sC(s) + 2C(s) = R(s)$$

$$(s+2)C(s) = R(s)$$

The transfer function is

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{s+2}$$

Now the we know C(s) = R(s) G(s)

The Transfer Function

Now the we know C(s) = R(s) G(s)

r(t) = u(t) therefore R(s) = 1/s

$$C(s) = \frac{1}{s}\frac{1}{s+2}$$

Using partial fraction

$$C(s) = \frac{1/2}{s} - \frac{1/2}{s+2}$$

Finally, taking the inverse Laplace transform of each term yields

$$c(t) = \frac{1}{2} - \frac{1}{2}e^{-2t}$$

Electrical Network Transfer Functions

- the mathematical modeling of electric circuits
- three passive linear components: resistors, capacitors, inductors, and OPAMP.
- combine electrical components into circuits, decide on the input and output, and find the transfer function.
- Our guiding principles are Kirchhoff's laws.
- Sum voltages around loops or sum currents at nodes, depending on which technique involves the least effort in algebraic manipulation, and then equate the
- result to zero.
- From these relationships write the differential equations for the circuit.
- Then find the Laplace transforms of the differential equations and finally solve for the transfer function

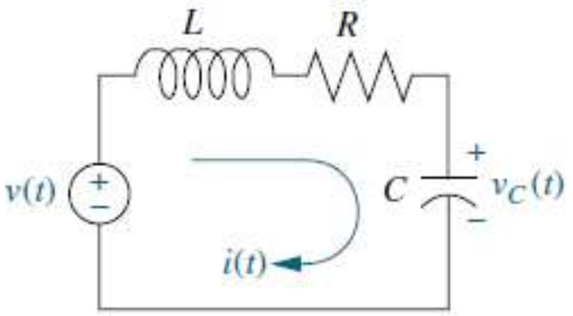
Electrical Network Transfer Functions

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance Z(s)=V(s)/I(s)	Admittance Y(s)=I(s)/V(s)
Capacitor	$v(t) = \frac{1}{C}\int_0^1 i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^1 v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Electrical Network : RLC Circuit (Mesh Analysis)

Transfer functions can be obtained using **Kirchhoff's voltage law** and summing voltages around loops or meshes

Find the transfer function relating the capacitor voltage VC(s) to the input voltage V(s).



In any problem, the designer must first decide what the input and output should be.

In this network, several variables could have been chosen to be the output—for example, the inductor voltage, the capacitor voltage, the resistor voltage, or the current.

In this problem it is stated as statement the capacitor voltage is the output the applied voltage as the input.

Electrical Network : RLC Circuit (Mesh Analysis)

Summing the voltages around the loop, assuming zero initial conditions, we get

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v(t) \dots 1$$

Changing variables from current to charge using $i(t) = dq(t)/dt$

$$\therefore L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{1}{C}q(t) = v(t) \dots 2$$

From the voltage-charge relationship for a capacitor from the Table $q(t) = Cv_C(t)$

$$\therefore LC\frac{d^2v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t) = v(t) \dots 3$$

Electrical Network : RLC Circuit (Mesh Analysis)

Taking the Laplace transform assuming zero initial conditions, rearranging terms, and simplifying

$$(LCs^2 + RCs + 1)V_C(s) = V(s) \dots 4$$

Hence the transfer function is

$$\frac{V_C(s)}{V(s)} = \frac{1}{LCs^2 + RCs + 1} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \dots 5$$

Electrical Network : RLC Circuit (Mesh Analysis)

Alternative:

First, take the Laplace transform of the equations in the voltage-current assuming zero initial conditions.

Component	**Voltage current**	**LT of voltage current**
Capacitor	$v(t) = \frac{1}{C}\int_0^1 i(\tau)d\tau$	$V(s) = \frac{1}{Cs}I(s)$
Resistor	$v(t) = Ri(t)$	$V(s) = RI(s)$
Inductor	$v(t) = L\frac{di(t)}{dt}$	$V(s) = LsI(s)$

Now define the transfer function $\frac{V(s)}{I(s)} = Z(s)$

Electrical Network : RLC Circuit (Mesh Analysis)

Alternative:

Notice that this function is similar to the definition of resistance, that is, the ratio of voltage to current.

But this function is applicable to capacitors and inductors and carries information on the dynamic behavior of the component, since it represents an equivalent differential equation. We call this particular transfer function impedance. The impedance for each of the electrical elements is

Component	**Impedance Z(s)=V(s)/I(s)**
Capacitor	$\frac{1}{Cs}$
Resistor	R
Inductor	Ls

Electrical Network : RLC Circuit (Mesh Analysis)

Let us use the concept of impedance for simplified solution

for the transfer function. The Laplace transform of $L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v(t)$, assuming zero initial conditions, is

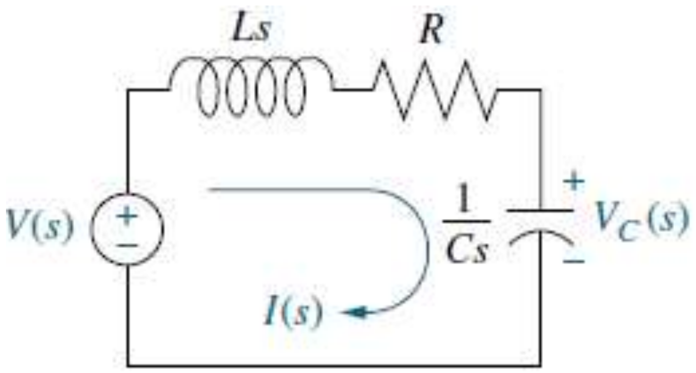
$$\left(Ls + R + \frac{1}{Cs}\right)I(s) = V(s) \ldots 6$$

This equation is in the form

(Sum of impedances) I(s) = (Sum of applied voltages)

Electrical Network : RLC Circuit (Mesh Analysis)

From equation 6 we can have the series circuit as shown. This circuit could have been obtained immediately from the original RLC circuit simply by replacing each element with its impedance. Let us call this altered circuit the transformed circuit.



From the transformed circuit we can write Eq. (6) immediately, if we add impedances in series as we add resistors in series.

Electrical Network : RLC Circuit (Mesh Analysis)

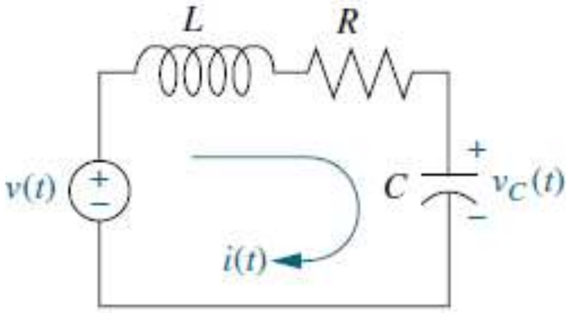
So instead of writing the differential equation first and then taking the Laplace transform, we can draw the transformed circuit and obtain the Laplace transform of the differential equation simply by applying Kirchhoff's voltage law to the transformed circuit.

We conclude this discussion as follows

1. Redraw the original network showing all time variables, such as $v(t)$, $i(t)$, and $v_C(t)$, as Laplace transforms $V(s)$, $I(s)$, and $V_C(s)$, respectively.
2. Replace the component values with their impedance values. This replacement is similar to the case of dc circuits, where we represent resistors with their resistance values.

Electrical Network : RLC Circuit (Mesh Analysis)

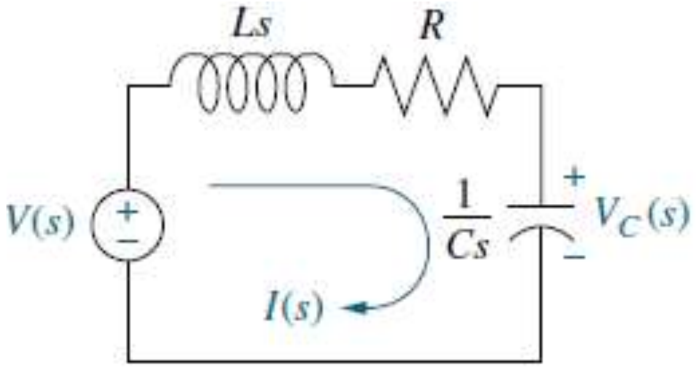
Find the transfer function relating the capacitor voltage VC(s) to the input voltage V(s)



by transform method.

Steps

1. Draw the transform network



Electrical Network : RLC Circuit (Mesh Analysis)

2. Write the equation

$\left(Ls + R + \frac{1}{Cs}\right) I(s) = V(s)$ solving for I(s)/V(s)

$$\frac{I(s)}{V(s)} = \frac{1}{\left(Ls + R + \frac{1}{Cs}\right)}$$

But the voltage across the capacitor, $V_C(s)$, is the product of the current and the impedance of the capacitor.

$$V_{C(s)} = I(s)\frac{1}{Cs} = \frac{1}{Cs}\frac{V(s)}{\left(Ls + R + \frac{1}{Cs}\right)} = \frac{V(s)}{(LCs^2 + RCs + 1)} = \frac{(1/LC)V(s)}{\left(s^2 + \left(\frac{R}{L}\right)Cs + (1/LC)\right)}$$

Electrical Network : RLC Circuit (Mesh Analysis)

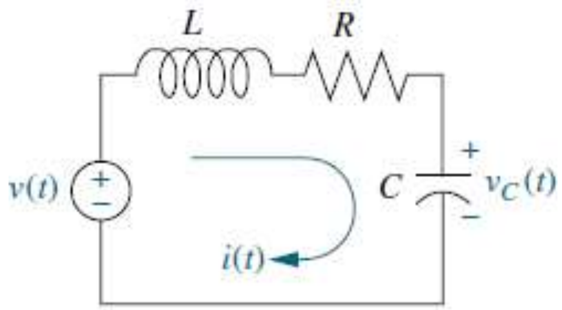
$$\frac{V_{C(s)}}{V(s)} = \frac{(1/LC)}{\left(s^2 + \left(\frac{R}{L}\right)Cs + (1/LC)\right)}$$

Same as equation 5.

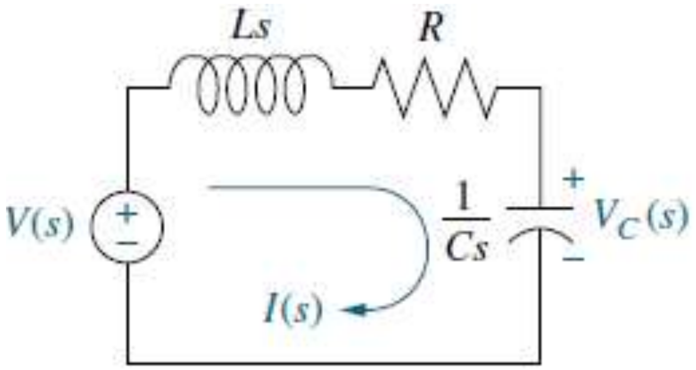
Electrical Network : RLC Circuit (Nodal Analysis)

Transfer functions also can be obtained using **Kirchhoff's current law** and summing currents flowing from nodes. This is nodal analysis method.

Find the transfer function relating the capacitor voltage VC(s) to the input voltage V(s) by nodal analysis method and without writing a differential equation i.e. using transform method



transform to



Electrical Network : RLC Circuit (Nodal Analysis)

The transfer function is obtained by summing currents flowing out of the node whose voltage is VC(s) transformed figure with assumption that currents leaving the node are positive and currents entering the node are negative.

The currents consist of the current through the capacitor and the current flowing through the series resistor and inductor. Therefore for each I(s) = V(s)/Z(s). Hence,

$$\frac{V_C(s)}{1/Cs} + \frac{V_C(s) - V(s)}{R + Ls} = 0 \ldots 1$$

where $V_C(s)\,/(1/Cs)$ is the current flowing out of the node through the capacitor, and $[V_C(s) - V(s)]/(R + Ls$ is the current flowing out of the node through the series resistor and inductor. Solve eq. (1) for the transfer function, $V_C(s)\,/V(s)$

Electrical Network : RLC Circuit (Nodal Analysis)

$$\frac{V_C(s)}{1/Cs}+\frac{V_C(s)-V(s)}{R+Ls}=\frac{V_C(s)}{1/Cs}+\frac{V_C(s)}{R+Ls}-\frac{V(s)}{R+Ls}=0$$

$$\frac{V_C(s)}{1/Cs}+\frac{V_C(s)}{R+Ls}=\frac{V(s)}{R+Ls};\rightarrow\frac{(R+Ls)V_C(s)}{1/Cs}+VC(s)=V(s);$$

$$V_C(s)\left(1+\frac{(R+Ls)}{1/Cs}\right)=V(s);\rightarrow V_C(s)\left(1+\frac{(R+Ls)}{1/Cs}\right)=V(s)$$

$$V_C(s)\left(\frac{\frac{1}{Cs}+R+Ls}{1/Cs}\right)=V(s);\rightarrow\frac{V_C(s)}{V(s)}=\frac{1/Cs}{\frac{1}{Cs}+R+Ls}$$

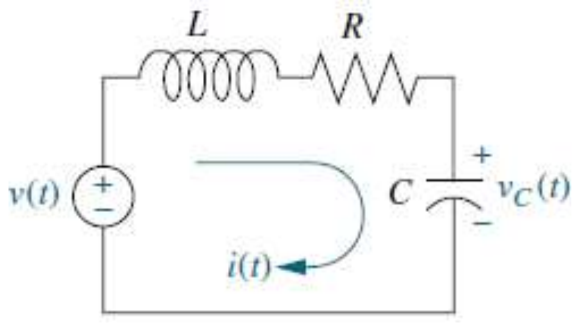
Electrical Network : RLC Circuit (Nodal Analysis)

$$V_C(s)\left(\frac{\frac{1}{Cs}+R+Ls}{1/Cs}\right)=V(s);\rightarrow\frac{V_C(s)}{V(s)}=\frac{1/Cs}{\frac{1}{Cs}+R+Ls}=\frac{1}{LCs^2+RCs+1}$$

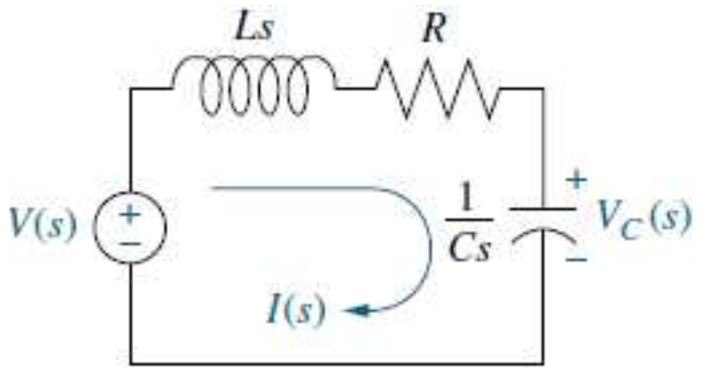
$$=\frac{1/LC}{s^2+\frac{R}{L}s+\frac{1}{LC}}$$

Electrical Network : RLC Circuit (Voltage Division)

Find the transfer function relating the capacitor voltage VC(s) to the input voltage V(s) by votage division method



transformed to



The voltage across the capacitor is some proportion of the input voltage, namely the impedance of the capacitor divided by the sum of the impedances. Thus

$$V_C(s) = \frac{\frac{1}{Cs}}{Ls + R + \frac{1}{Cs}} V(s)$$

Solve this equation for the transfer function, $V_C(s) / V(s)$

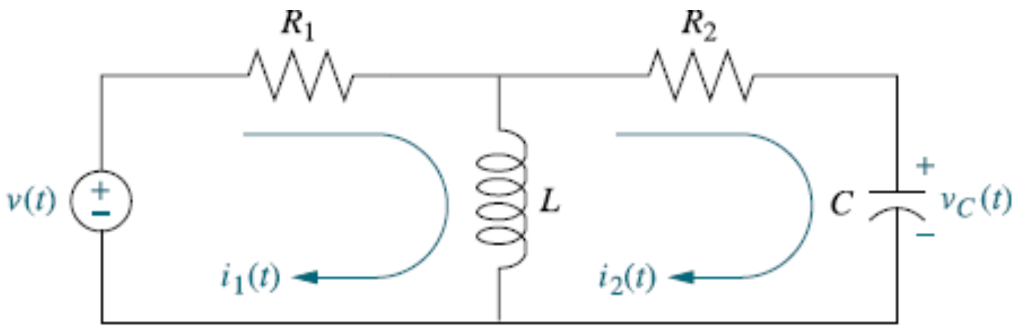
Complex Circuits via Mesh Analysis

To solve complex electrical networks—those with multiple loops and nodes—using mesh analysis, we can perform the following steps:

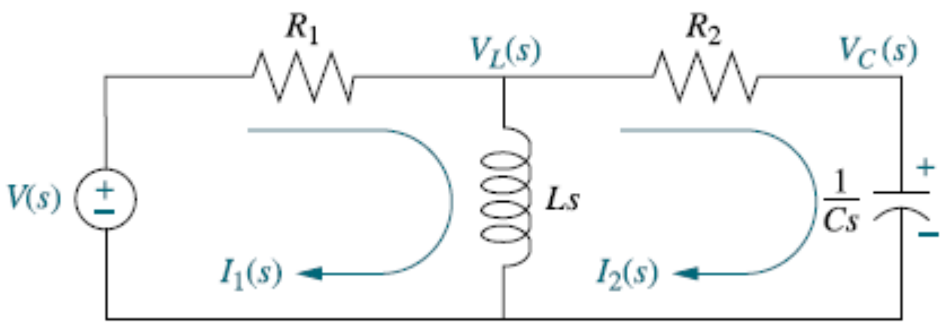
1. Replace passive element values with their impedances.
2. Replace all sources and time variables with their Laplace transform.
3. Assume a transform current and a current direction in each mesh
4. Write Kirchhoff's voltage law around each mesh.
5. Solve the simultaneous equations for the output.
6. Form the transfer function

Complex Circuits via Mesh Analysis

Find the transfer function $I_2(s)/V(s)$ for the circuit given below.



The first step in the solution is to convert the network into Laplace transforms for impedances and circuit variables, assuming zero initial conditions.



Complex Circuits via Mesh Analysis

Find the two simultaneous equations for the transfer function by summing voltages around each mesh.

For mesh 1

$$R_1 I_1(s) + LsI_1(s) - LsI_2(s) = V(s) \ldots 1$$

For mesh 2

$$LsI_2(s) + R_2 I_2(s) + \frac{1}{Cs} I_2(s) - LsI_1(s) = 0 \ldots 2$$

The first step in the solution is to convert the network into Laplace transforms for impedances and circuit variables, assuming zero initial conditions.

Complex Circuits via Mesh Analysis

Combining terms, equation 1 & 2 become simultaneous equat

$$(R_1+Ls)I_1(s) \qquad\qquad -LsI_2(s) = V(s) \ldots 3$$

$$-LsI_1(s) + \left(Ls + R_2 + \frac{1}{Cs}\right)I_2(s) = 0 \ldots 4$$

$$I_2(s) = \frac{\begin{vmatrix} (R_1+Ls) & V(s) \\ -Ls & 0 \end{vmatrix}}{\Delta} = \frac{LsV(s)}{\Delta}$$

$$\Delta = \begin{vmatrix} (R_1+Ls) & -Ls \\ -Ls & \left(Ls+R_2+\frac{1}{Cs}\right) \end{vmatrix}$$

We can use Cramer's rule (or any other method for solving simultaneous equations) to solve equation (3 & 4) for $I_2(s)$. Hence,

$$I_2(s) = \frac{\begin{vmatrix} (R_1+Ls) & V(s) \\ -Ls & 0 \end{vmatrix}}{\Delta} = \frac{LsV(s)}{\Delta}$$

Complex Circuits via Mesh Analysis

Where

$$\Delta = \begin{vmatrix} (R_1+Ls) & - \\ -Ls & \left(Ls + \ldots\, Cs\right) \end{vmatrix}$$

$$G(s) = \frac{I_2(s)}{V(s)} = \frac{Ls}{\Delta} = \frac{LCs^2}{(R_1+R_2)LCs^2+(R_1R_2C+L)s+R_1}$$

Therefore, the transfer function $I_2(s)/V(s)$ is

$$\mathrm{G}(s) = \frac{Ls}{\Delta} = \frac{LCs^2}{(R_1+R_2)LCs^2+(R_1R_2C+L)s+R_1}$$

Translational Mechanical System: Transfer Function

Mechanical systems are analogous to electrical networks. They mechanical systems also have passive components, energy storage components etc.

The symbols and units are:

f(t): force, N (newtons),

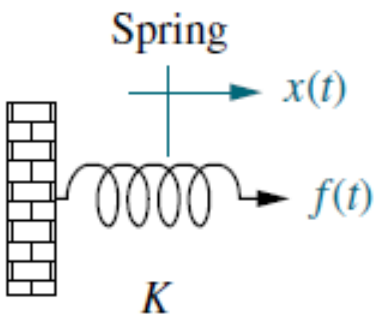
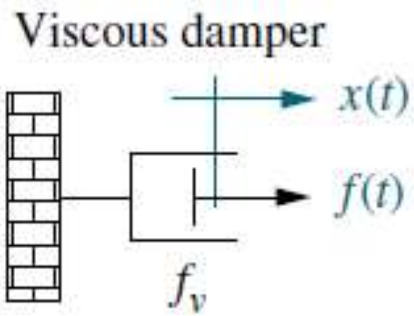
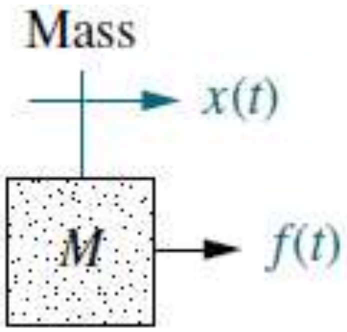
x(t): displacement, m (meters), v(t): velacity m/s (meters/second),

K: spring constant, N/m (newtons/meter),

f_v: coefficient of viscous friction, N-s/m (newton-seconds/meter),

M: Mass, kg (kilograms = newton-seconds^2/meter).

Translational Mechanical System: Transfer Function

Component	Force-velocity	Force-displacement	Impedance ZM(s) =F(s)/X(s)
	$f(t) = K\int_0^t v(\tau)d\tau$	$f(t) = K\mathrm{x}(\mathrm{t})$	K
	$f(t) = fv\, v(t)$	$f(t) = fv\,\frac{dx(t)}{dt}$	$f_v s$
	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2

Translational Mechanical System: Transfer Function

Component	**Voltage Current**	**Component**	**Force velocity**
Capacitor	$v(t) = \frac{1}{C}\int_0^1 i(\tau)d\tau$	Spring	$f(t) = K\int_0^t v(\tau)d\tau$
Resistor	$v(t) = Ri(t)$	Viscous Damper	$f(t) = f v\, v(t)$
Inductor	$v(t) = L\frac{di(t)}{dt}$	Mass	$f(t) = M\frac{dv(t)}{dt}$

The mechanical **force** is analogous to electrical **voltage** and mechanical **velocity** is analogous to electrical **current**.

The **spring** is analogous to the **capacitor**, the **viscous damper** is analogous to the **resistor**, and the **mass** is analogous to the **inductor**.

Translational Mechanical System: Transfer Function

Thus, summing forces written in terms of velocity is analogous to summing voltages written in terms of current, and the resulting mechanical differential equations are analogous to mesh equations.

If the forces are written in terms of displacement, the resulting mechanical equations resemble, but are not analogous to, the mesh equations.

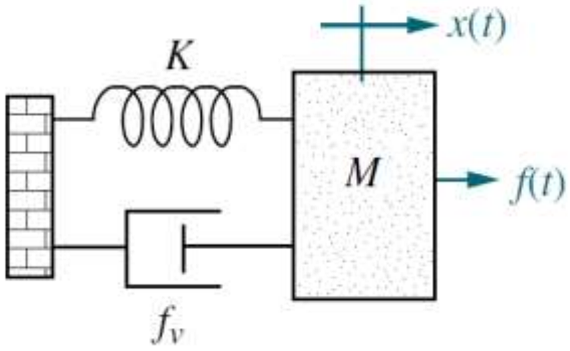
Translational Mechanical System: Transfer Function

Component	**Current voltage**	**Component**	**Force velocity**
Capacitor	$i(t) = C\dfrac{dv(t)}{dt}$	Mass	$f(t) = M\dfrac{dv(t)}{dt}$
Resistor	$i(t) = \dfrac{1}{R}v(t)$	Viscous Damper	$f(t) = f_v\, v(t)$
Inductor	$i(t) = \dfrac{1}{L}\int_0^1 v(\tau)d\tau$	Spring	$f(t) = K\int_0^t v(\tau)d\tau$

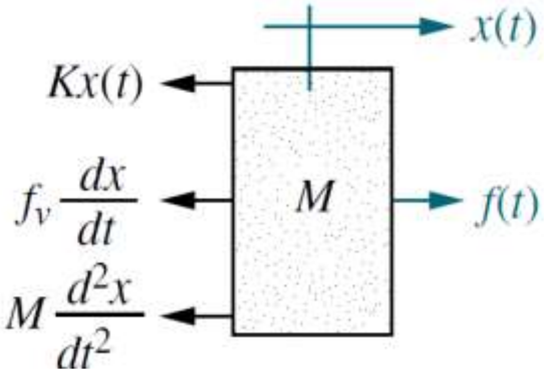
Here the **analogy** is between **force** and **current** and between **velocity** and **voltage**. The **spring** is analogous to the **inductor**, the **viscous damper** is analogous to the **resistor**, and the **mass** is analogous to the **capacitor**. Thus, summing forces written in terms of velocity is analogous to summing currents written in terms of voltage and the resulting mechanical **differential equations** are analogous to **nodal** equations.

Translational Mechanical System: Transfer Function

Find the transfer function, X(s)/F(s) for the system as shown



→ draw the free body diagram



Place all forces felt by the mass. We assume the mass is traveling toward the right. Thus, only the applied force points to the right; all other forces impede the motion and act to oppose it. Hence, the spring, viscous damper, and the force due to acceleration point to the left.

Write the differential equation of motion using Newton's law to sum of all the forces on the mass is zero

Translational Mechanical System: Transfer Function

$$M\frac{d^2x(t)}{dt^2} + f v\frac{dx(t)}{dt} + Kx(t) = f(t)$$

Taking the Laplace transform, assuming zero initial conditions,

$$Ms^2X(s) + f v\, sX(s) + KX(s) = F(s)$$

$$(Ms^2 + f v\, s + K)X(s) = F(s)$$

Hence the transfer function is

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{(Ms^2 + f v\, s + K)}$$

Translational Mechanical System: Transfer Function

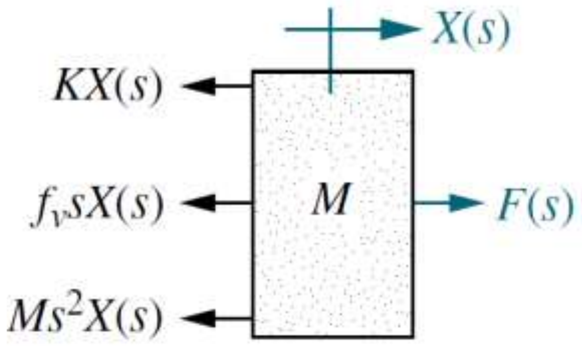
Component	**Force-displacement**	**LT of Force-displacement**	**Impedance** $Z_M(s) = F(s)/X(s)$
Spring	$f(t) = Kx(t)$	$F(s) = KX(s)$	K
Viscous Damper	$f(t) = fv\dfrac{dx(t)}{dt}$	$F(s) = fv\, sX(s)$	$f_v s$
Mass	$f(t) = M\dfrac{d^2x(t)}{dt^2}$	$F(s) = Ms^2X(s)$	Ms^2

Last column is for the impedance of mechanical components.

Replacing each force in free body diagram by its Laplace transform, which is in the format

$F(s) = Z_M(s)X(s)$

Translational Mechanical System: Transfer Function



we obtain the figure as shown, from which we could have obtained the equation immediately without writing the differential equation.

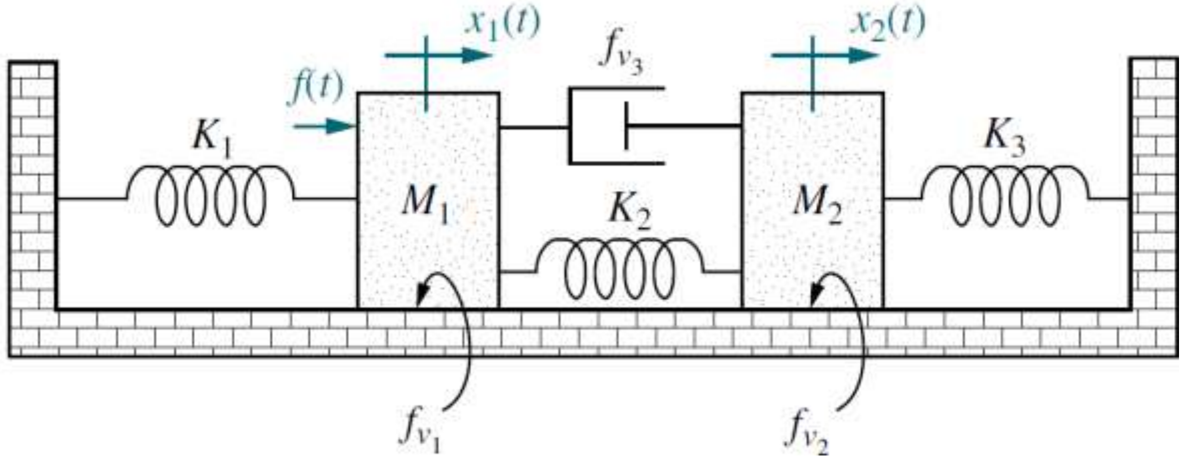
$$Ms^2X(s) + fv\,sX(s) + KX(s) = F(s)$$

And the equation $(Ms^2 + fv\,s + K)X(s) = F(s)$ is of the form of

{Sum of impedances} X(s) = {Sum of applied force}

Translational Mechanical System: Transfer Function

Find the transfer function, X2(s)/F(s), for the system as shown Figure 1.



The system has two degrees of freedom, since each mass can be moved in the horizontal direction while the other is held still.

Thus, two simultaneous equations of motion will be required to describe the system.

The two equations come from free-body diagrams of each mass.

Translational Mechanical System: Transfer Function

Forces acting on mass M_1

only due to motion of M_1

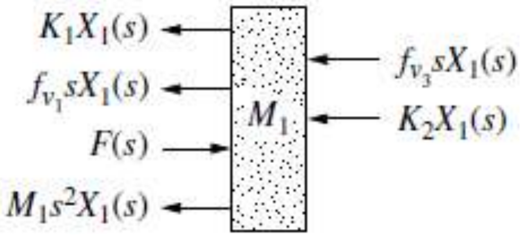


figure 2 a

Forces acting on mass M_1

only due to motion of M_2

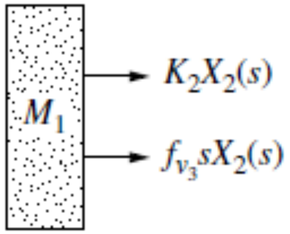


figure 2 b

All forces acting on mass M_1

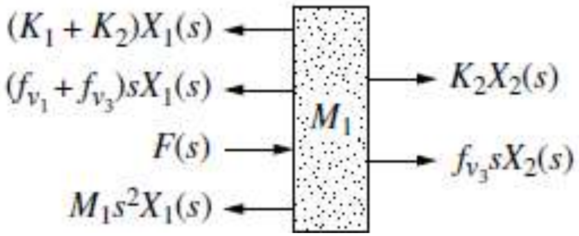


figure 2 c

Translational Mechanical System: Transfer Function

Forces acting on mass M_2 only due to motion of M_2

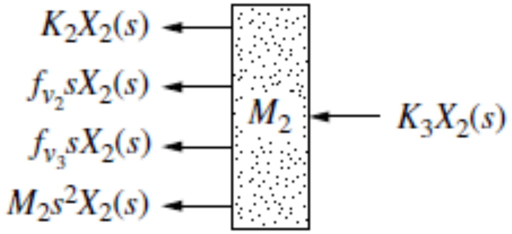


figure 3 a

Forces acting on mass M_2 only due to motion of M_1

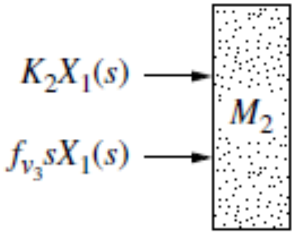


figure 3 b

All forces acting on mass M_2

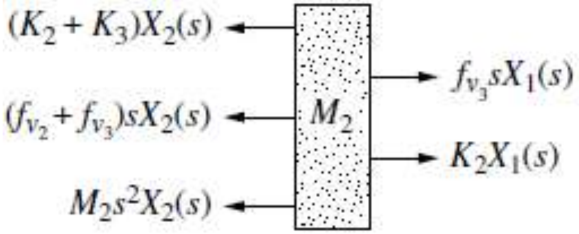


figure 3 c

Translational Mechanical System: Transfer Function

The Laplace transform of the equations of motion can now be written from figures 2 (c) and 3(c)

$$[M_1s^2(f_{v1}+f_{v3})s+(K_1+K_2)]X_1(s) \qquad - (f_{v3}s+K_2)X_2(s) = F(s)$$

$$-(f_{v3}s+K_2)X_{1(s)} + [M_2s^2(f_{v2}+f_{v3})s+(K_2+K_3)]X_2(s) = 0$$

From this, $X_2(s)$is

$$X_2(s) = \frac{\begin{vmatrix} [M_1s^2(f_{v1}+f_{v3})s+(K_1+K_2) & F(s) \\ -(f_{v3}s+K_2) & 0 \end{vmatrix}}{\Delta}$$

Translational Mechanical System: Transfer Function

and the transfer function, $X_2(s)/F(s)$, is

$$\frac{X_2(s)}{F(s)} = \frac{(f_{v3}s + K_2)}{\Delta}$$

Where,

$$\Delta = \begin{vmatrix} M_1 s^2 (f_{v1} + f_{v3})s + (K_1 + K_2) & -f_{v3}s + K_2 \\ -f_{v3}s + K_2 & M_2 s^2 (f_{v2} + f_{v3})s + (K_2 + K_3) \end{vmatrix}$$

Feedback Control System Unit 1_2
Transfer function of physical systems

The Transfer Function

Consider a n[th]-order, linear, time-invariant continuous time system describe by the differential equation

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \cdots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots + b_0 r(t)$$

where c(t) is the output, r(t) is the input, and the a_n, a_{n-1},…, a_0 and b_m, b_{m-1}, …, b_0 are the coefficients. Taking the Laplace transform of both sides,

$$a_n s^n C(s) + a_{n-1} s^{n-1} C(s) + \cdots + a_0 C(s) + \text{initial condition terms involving } c(t) =$$
$$b_m s^m R(s) + b_{m-1} s^{m-1} R(s) + \cdots + b_0 R(s) + \text{initial condition terms involving } r(t)$$

If we assume that all initial conditions are zero, then

$$(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0) C(s) = (b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0) R(s)$$

The Transfer Function

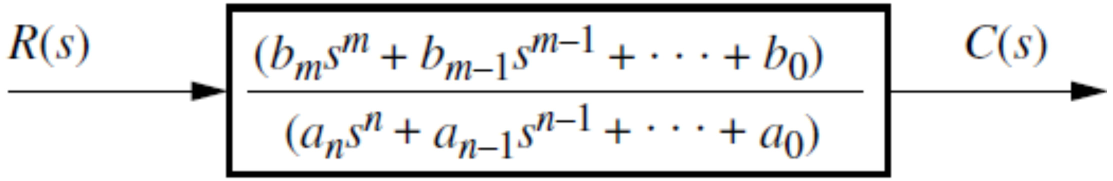
The ratio of the output transform, C(s), divided by the input transform, R(s) is:

$$\frac{C(s)}{R(s)} = G(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

The transfer function G(s) is defined as the ratio of Laplace Transform of output C(s) to the Laplace Transform of input R(s) with zero initial conditions

The transfer function can be represented as a block diagram, as shown in Figure, with the input on the left, the output on the right, and the system transfer function inside the block.

Also, we can find the output, C(s) by using C(s) = R(s)G(s)



The Transfer Function

Find the transfer function of a system represented by

$$\frac{dc(t)}{dt} + 2c(t) = r(t)$$

Taking the Laplace transform of both sides, assuming zero initial

$$sC(s) + 2C(s) = R(s)$$

$$(s + 2)C(s) = R(s)$$

The transfer function is

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{s + 2}$$

The Transfer Function

Find step response of the linear time invariant system described by the differential equation

$$\frac{dc(t)}{dt} + 2c(t) = r(t)$$

Taking the Laplace transform of both sides, assuming zero initial

$$sC(s) + 2C(s) = R(s)$$

$$(s+2)C(s) = R(s)$$

The transfer function is

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{s+2}$$

Now the we know C(s) = R(s) G(s)

The Transfer Function

Now the we know C(s) = R(s) G(s)

r(t) = u(t) therefore R(s) = 1/s

$$C(s) = \frac{1}{s}\frac{1}{s+2}$$

Using partial fraction

$$C(s) = \frac{1/2}{s} - \frac{1/2}{s+2}$$

Finally, taking the inverse Laplace transform of each term yields

$$c(t) = \frac{1}{2} - \frac{1}{2}e^{-2t}$$

Electrical Network Transfer Functions

- the mathematical modeling of electric circuits
- three passive linear components: resistors, capacitors, inductors, and OPAMP.
- combine electrical components into circuits, decide on the input and output, and find the transfer function.
- Our guiding principles are Kirchhoff's laws.
- Sum voltages around loops or sum currents at nodes, depending on which technique involves the least effort in algebraic manipulation, and then equate the
- result to zero.
- From these relationships write the differential equations for the circuit.
- Then find the Laplace transforms of the differential equations and finally solve for the transfer function

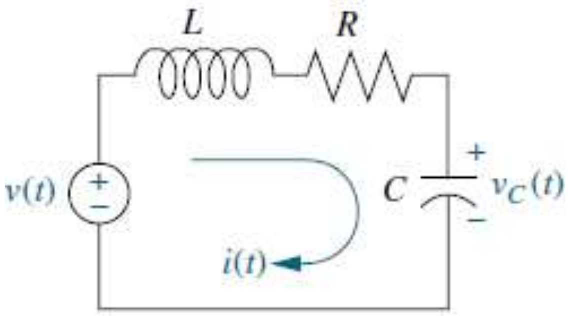
Electrical Network Transfer Functions

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s)=V(s)/I(s)$	Admittance $Y(s)=I(s)/V(s)$
Capacitor	$v(t) = \frac{1}{C}\int_0^1 i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^1 v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Electrical Network : RLC Circuit (Mesh Analysis)

Transfer functions can be obtained using **Kirchhoff's voltage law** and summing voltages around loops or meshes

Find the transfer function relating the capacitor voltage VC(s) to the input voltage V(s).



In any problem, the designer must first decide what the input and output should be.

In this network, several variables could have been chosen to be the output—for example, the inductor voltage, the capacitor voltage, the resistor voltage, or the current.

In this problem it is stated as statement the capacitor voltage is the output the applied voltage as the input.

Electrical Network : RLC Circuit (Mesh Analysis)

Summing the voltages around the loop, assuming zero initial conditions, we get

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v(t) \ldots 1$$

Changing variables from current to charge using $i(t) = dq(t)/dt$

$$\therefore L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{1}{C}q(t) = v(t) \ldots 2$$

From the voltage-charge relationship for a capacitor from the Table $q(t) = Cv_C(t)$

$$\therefore LC\frac{d^2v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t) = v(t) \ldots 3$$

Electrical Network : RLC Circuit (Mesh Analysis)

Taking the Laplace transform assuming zero initial conditions, rearranging terms, and simplifying

$$(LCs^2 + RCs + 1)V_C(s) = V(s) \ldots 4$$

Hence the transfer function is

$$\frac{V_C(s)}{V(s)} = \frac{1}{LCs^2 + RCs + 1} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \ldots 5$$

Electrical Network : RLC Circuit (Mesh Analysis)

Alternative:

First, take the Laplace transform of the equations in the voltage-current assuming zero initial conditions.

Component	**Voltage current**	**LT of voltage current**
Capacitor	$v(t) = \frac{1}{C}\int_0^1 i(\tau)d\tau$	$V(s) = \frac{1}{Cs}I(s)$
Resistor	$v(t) = Ri(t)$	$V(s) = RI(s)$
Inductor	$v(t) = L\frac{di(t)}{dt}$	$V(s) = LsI(s)$

Now define the transfer function $\frac{V(s)}{I(s)} = Z(s)$

Electrical Network : RLC Circuit (Mesh Analysis)

Alternative:

Notice that this function is similar to the definition of resistance, that is, the ratio of voltage to current.

But this function is applicable to capacitors and inductors and carries information on the dynamic behavior of the component, since it represents an equivalent differential equation. We call this particular transfer function impedance. The impedance for each of the electrical elements is

Component	**Impedance Z(s)=V(s)/I(s)**
Capacitor	$\frac{1}{Cs}$
Resistor	R
Inductor	Ls

Electrical Network : RLC Circuit (Mesh Analysis)

Let us use the concept of impedance for simplified solution

for the transfer function. The Laplace transform of $L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v(t)$,

assuming zero initial conditions, is

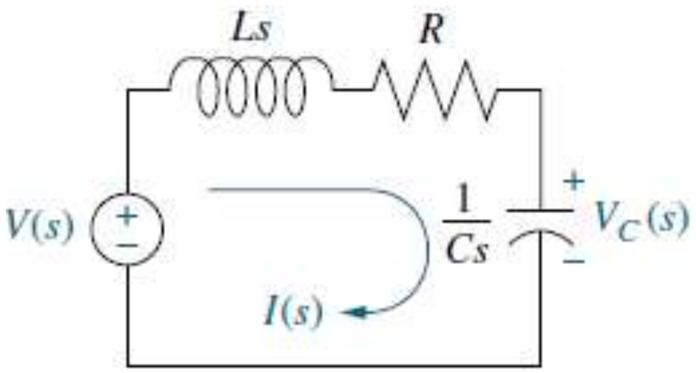
$$\left(Ls + R + \frac{1}{Cs}\right)I(s) = V(s) \ldots 6$$

This equation is in the form

(Sum of impedances) I(s) = (Sum of applied voltages)

Electrical Network : RLC Circuit (Mesh Analysis)

From equation 6 we can have the series circuit as shown. This circuit could have been obtained immediately from the original RLC circuit simply by replacing each element with its impedance. Let us call this altered circuit the transformed circuit.



From the transformed circuit we can write Eq. (6) immediately, if we add impedances in series as we add resistors in series.

Electrical Network : RLC Circuit (Mesh Analysis)

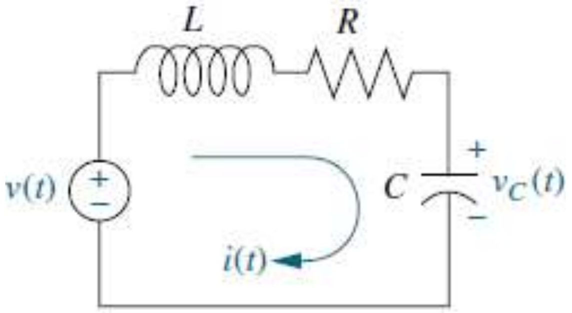
So instead of writing the differential equation first and then taking the Laplace transform, we can draw the transformed circuit and obtain the Laplace transform of the differential equation simply by applying Kirchhoff's voltage law to the transformed circuit.

We conclude this discussion as follows

1. Redraw the original network showing all time variables, such as $v(t)$, $i(t)$, and $v_C(t)$, as Laplace transforms $V(s)$, $I(s)$, and $V_C(s)$, respectively.
2. Replace the component values with their impedance values. This replacement is similar to the case of dc circuits, where we represent resistors with their resistance values.

Electrical Network : RLC Circuit (Mesh Analysis)

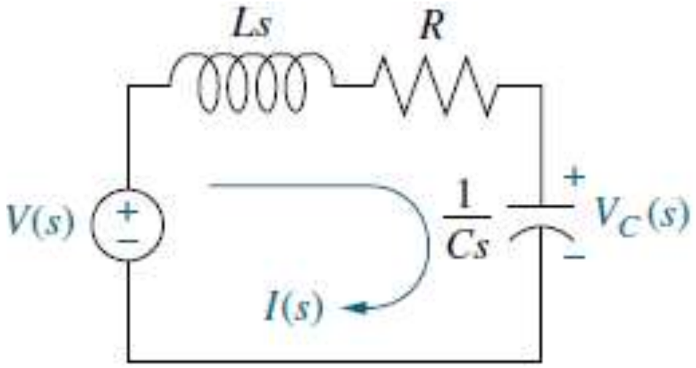
Find the transfer function relating the capacitor voltage VC(s) to the input voltage V(s)



by transform method.

Steps

1. Draw the transform network



Electrical Network : RLC Circuit (Mesh Analysis)

2. Write the equation

$\left(Ls + R + \frac{1}{Cs}\right) I(s) = V(s)$ solving for I(s)/V(s)

$$\frac{I(s)}{V(s)} = \frac{1}{\left(Ls + R + \frac{1}{Cs}\right)}$$

But the voltage across the capacitor, $V_C(s)$, is the product of the current and the impedance of the capacitor.

$$V_{C(s)} = I(s)\frac{1}{Cs} = \frac{1}{Cs}\frac{V(s)}{\left(Ls + R + \frac{1}{Cs}\right)} = \frac{V(s)}{(LCs^2 + RCs + 1)} = \frac{(1/LC)V(s)}{\left(s^2 + \left(\frac{R}{L}\right)Cs + (1/LC)\right)}$$

Electrical Network : RLC Circuit (Mesh Analysis)

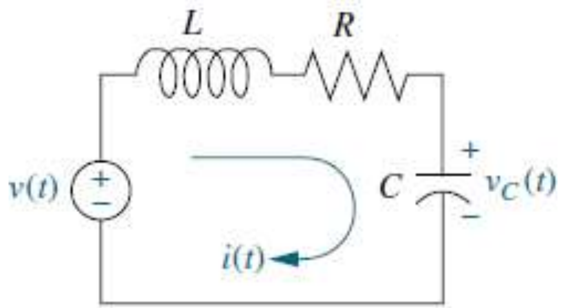
$$\frac{V_{C(s)}}{V(s)} = \frac{(1/LC)}{\left(s^2 + \left(\frac{R}{L}\right)Cs + (1/LC)\right)}$$

Same as equation 5.

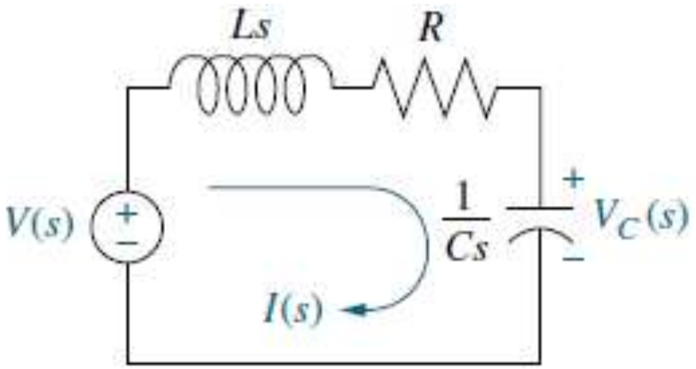
Electrical Network : RLC Circuit (Nodal Analysis)

Transfer functions also can be obtained using **Kirchhoff's current law** and summing currents flowing from nodes. This is nodal analysis method.

Find the transfer function relating the capacitor voltage VC(s) to the input voltage V(s) by nodal analysis method and without writing a differential equation i.e. using transform method



transform to



Electrical Network : RLC Circuit (Nodal Analysis)

The transfer function is obtained by summing currents flowing out of the node whose voltage is VC(s) transformed figure with assumption that currents leaving the node are positive and currents entering the node are negative.

The currents consist of the current through the capacitor and the current flowing through the series resistor and inductor. Therefore for each I(s) = V(s)/Z(s). Hence,

$$\frac{V_C(s)}{1/Cs} + \frac{V_C(s) - V(s)}{R + Ls} = 0 \ldots 1$$

where $V_C(s)\,/(1/Cs)$ is the current flowing out of the node through the capacitor, and $[V_C(s) - V(s)]/(R + Ls$ is the current flowing out of the node through the series resistor and inductor. Solve eq. (1) for the transfer function, $V_C(s)\,/V(s)$

Electrical Network : RLC Circuit (Nodal Analysis)

$$\frac{V_C(s)}{1/Cs}+\frac{V_C(s)-V(s)}{R+Ls}=\frac{V_C(s)}{1/Cs}+\frac{V_C(s)}{R+Ls}-\frac{V(s)}{R+Ls}=0$$

$$\frac{V_C(s)}{1/Cs}+\frac{V_C(s)}{R+Ls}=\frac{V(s)}{R+Ls};\rightarrow\frac{(R+Ls)V_C(s)}{1/Cs}+VC(s)=V(s);$$

$$V_C(s)\left(1+\frac{(R+Ls)}{1/Cs}\right)=V(s);\rightarrow V_C(s)\left(1+\frac{(R+Ls)}{1/Cs}\right)=V(s)$$

$$V_C(s)\left(\frac{\frac{1}{Cs}+R+Ls}{1/Cs}\right)=V(s);\rightarrow\frac{V_C(s)}{V(s)}=\frac{1/Cs}{\frac{1}{Cs}+R+Ls}$$

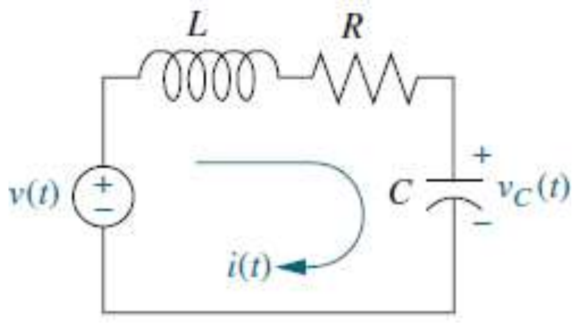
Electrical Network : RLC Circuit (Nodal Analysis)

$$V_C(s)\left(\frac{\frac{1}{Cs}+R+Ls}{1/Cs}\right)=V(s);\rightarrow\frac{V_C(s)}{V(s)}=\frac{1/Cs}{\frac{1}{Cs}+R+Ls}=\frac{1}{LCs^2+RCs+1}$$

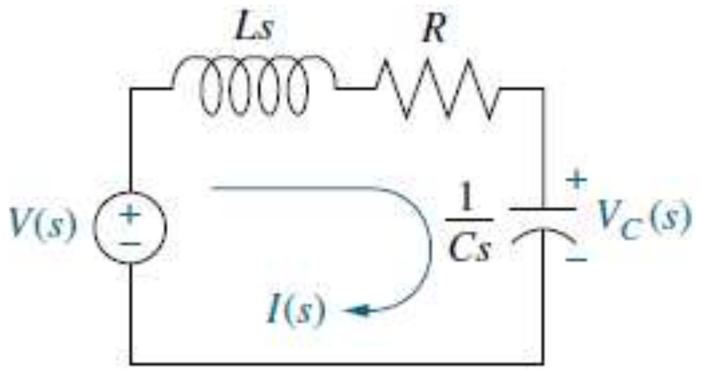
$$=\frac{1/LC}{s^2+\frac{R}{L}s+\frac{1}{LC}}$$

Electrical Network : RLC Circuit (Voltage Division)

Find the transfer function relating the capacitor voltage VC(s) to the input voltage V(s) by votage division method



transformed to



The voltage across the capacitor is some proportion of the input voltage, namely the impedance of the capacitor divided by the sum of the impedances. Thus

$$V_C(s) = \frac{\frac{1}{Cs}}{Ls + R + \frac{1}{Cs}} V(s)$$

Solve this equation for the transfer function, $V_C(s)\,/V(s)$

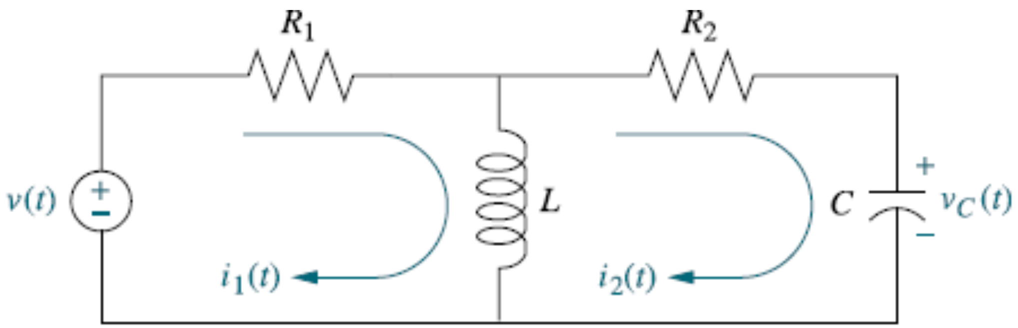
Complex Circuits via Mesh Analysis

To solve complex electrical networks—those with multiple loops and nodes—using mesh analysis, we can perform the following steps:

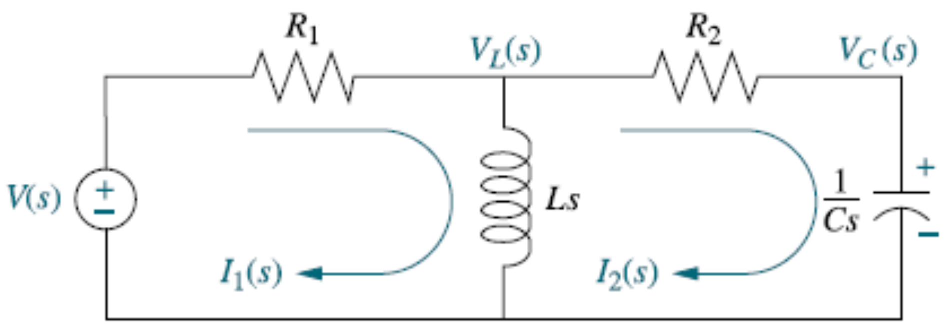
1. Replace passive element values with their impedances.
2. Replace all sources and time variables with their Laplace transform.
3. Assume a transform current and a current direction in each mesh
4. Write Kirchhoff's voltage law around each mesh.
5. Solve the simultaneous equations for the output.
6. Form the transfer function

Complex Circuits via Mesh Analysis

Find the transfer function $I_2(s)/V(s)$ for the circuit given below.



The first step in the solution is to convert the network into Laplace transforms for impedances and circuit variables, assuming zero initial conditions.



Complex Circuits via Mesh Analysis

Find the two simultaneous equations for the transfer function by summing voltages around each mesh.

For mesh 1

$$R_1 I_1(s) + LsI_1(s) - LsI_2(s) = V(s) \dots 1$$

For mesh 2

$$LsI_2(s) + R_2 I_2(s) + \frac{1}{Cs} I_2(s) - LsI_1(s) = 0 \dots 2$$

The first step in the solution is to convert the network into Laplace transforms for impedances and circuit variables, assuming zero initial conditions.

Complex Circuits via Mesh Analysis

Combining terms, equation 1 & 2 become simultaneous equat

$$(R_1+Ls)I_1(s) \qquad - LsI_2(s) = V(s) \ldots 3$$

$$-LsI_1(s) + \left(Ls + R_2 + \frac{1}{Cs}\right)I_2(s) = 0 \ldots 4$$

$$I_2(s) = \frac{\begin{vmatrix} (R_1 + Ls) & V(s) \\ -Ls & 0 \end{vmatrix}}{\Delta} = \frac{LsV(s)}{\Delta}$$

$$\Delta = \begin{vmatrix} (R_1 + Ls) & -Ls \\ -Ls & \left(Ls + R_2 + \frac{1}{Cs}\right) \end{vmatrix}$$

We can use Cramer's rule (or any other method for solving simultaneous equations) to solve equation (3 & 4) for $I_2(s)$. Hence,

$$I_2(s) = \frac{\begin{vmatrix} (R_1+Ls) & V(s) \\ -Ls & 0 \end{vmatrix}}{\Delta} = \frac{LsV(s)}{\Delta}$$

Complex Circuits via Mesh Analysis

Where

$$\Delta = \begin{vmatrix} (R_1 + Ls) & -Ls \\ -Ls & \left(Ls + R_2 + \frac{1}{Cs}\right) \end{vmatrix}$$

$$G(s) = \frac{I_2(s)}{V(s)} = \frac{Ls}{\Delta} = \frac{LCs^2}{(R_1 + R_2)LCs^2 + (R_1 R_2 C + L)s + R_1}$$

Therefore, the transfer function $I_2(s)/V(s)$ is

$$G(s) = \frac{Ls}{\Delta} = \frac{LCs^2}{(R_1 + R_2)LCs^2 + (R_1 R_2 C + L)s + R_1}$$

Translational Mechanical System: Transfer Function

Mechanical systems are analogous to electrical networks. They mechanical systems also have passive components, energy storage components etc.

The symbols and units are:

$f(t)$: force, N (newtons),

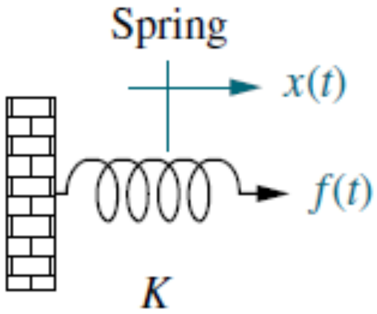
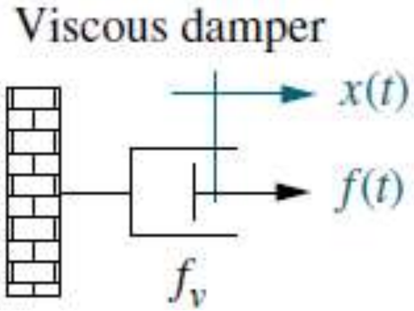
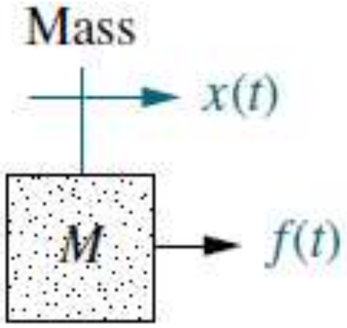
$x(t)$: displacement, m (meters), $v(t)$: velacity m/s (meters/second),

K: spring constant, N/m (newtons/meter),

f_v: coefficient of viscous friction, N-s/m (newton-seconds/meter),

M: Mass, kg (kilograms = newton-seconds^2/meter).

Translational Mechanical System: Transfer Function

Component	Force-velocity	Force-displacement	Impedance ZM(s) =F(s)/X(s)
	$f(t) = K\int_0^t v(\tau)d\tau$	$f(t) = K\mathrm{x}(\mathrm{t})$	K
	$f(t) = fv\, v(t)$	$f(t) = fv\dfrac{dx(t)}{dt}$	$f_v s$
	$f(t) = M\dfrac{dv(t)}{dt}$	$f(t) = M\dfrac{d^2x(t)}{dt^2}$	Ms^2

Translational Mechanical System: Transfer Function

Component	**Voltage Current**	**Component**	**Force velocity**
Capacitor	$v(t) = \frac{1}{C}\int_0^1 i(\tau)d\tau$	Spring	$f(t) = K\int_0^t v(\tau)d\tau$
Resistor	$v(t) = Ri(t)$	Viscous Damper	$f(t) = fv\, v(t)$
Inductor	$v(t) = L\frac{di(t)}{dt}$	Mass	$f(t) = M\frac{dv(t)}{dt}$

The mechanical **force** is analogous to electrical **voltage** and mechanical **velocity** is analogous to electrical **current**.

The **spring** is analogous to the **capacitor**, the **viscous damper** is analogous to the **resistor**, and the **mass** is analogous to the **inductor**.

Translational Mechanical System: Transfer Function

Thus, summing forces written in terms of velocity is analogous to summing voltages written in terms of current, and the resulting mechanical differential equations are analogous to mesh equations.

If the forces are written in terms of displacement, the resulting mechanical equations resemble, but are not analogous to, the mesh equations.

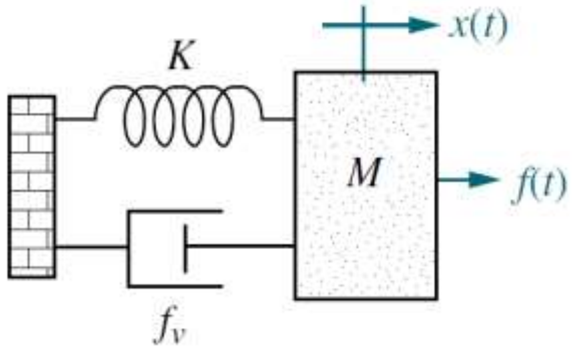
Translational Mechanical System: Transfer Function

Component	**Current voltage**	**Component**	**Force velocity**
Capacitor	$i(t) = C\dfrac{dv(t)}{dt}$	Mass	$f(t) = M\dfrac{dv(t)}{dt}$
Resistor	$i(t) = \dfrac{1}{R}v(t)$	Viscous Damper	$f(t) = fv\,v(t)$
Inductor	$i(t) = \dfrac{1}{L}\displaystyle\int_0^1 v(\tau)d\tau$	Spring	$f(t) = K\displaystyle\int_0^t v(\tau)d\tau$

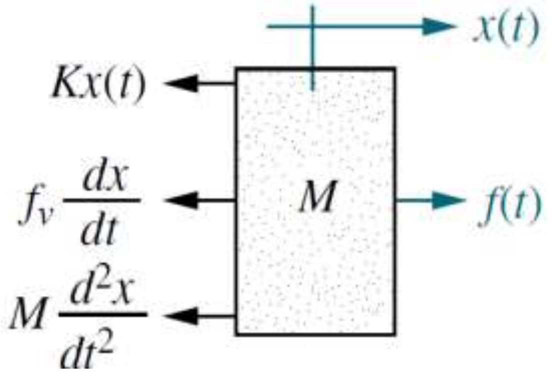
Here the **analogy** is between **force** and **current** and between **velocity** and **voltage**. The **spring** is analogous to the **inductor**, the **viscous damper** is analogous to the **resistor**, and the **mass** is analogous to the **capacitor**. Thus, summing forces written in terms of velocity is analogous to summing currents written in terms of voltage and the resulting mechanical **differential equations** are analogous to **nodal** equations.

Translational Mechanical System: Transfer Function

Find the transfer function, X(s)/F(s) for the system as shown



→ draw the free body diagram



Place all forces felt by the mass. We assume the mass is traveling toward the right. Thus, only the applied force points to the right; all other forces impede the motion and act to oppose it. Hence, the spring, viscous damper, and the force due to acceleration point to the left.

Write the differential equation of motion using Newton's law to sum of all the forces on the mass is zero

Translational Mechanical System: Transfer Function

$$M\frac{d^2x(t)}{dt^2} + f_v\frac{dx(t)}{dt} + Kx(t) = f(t)$$

Taking the Laplace transform, assuming zero initial conditions,

$$Ms^2X(s) + f_v\,sX(s) + KX(s) = F(s)$$

$$(Ms^2 + f_v\,s + K)X(s) = F(s)$$

Hence the transfer function is

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{(Ms^2 + f_v\,s + K)}$$

Translational Mechanical System: Transfer Function

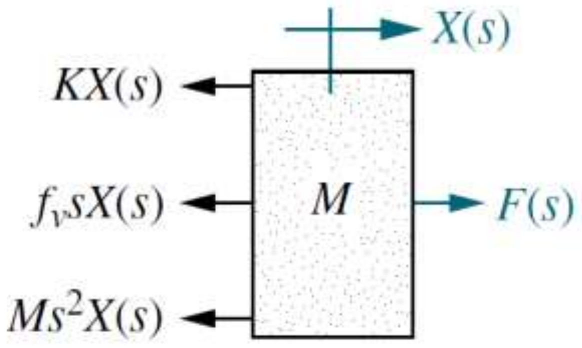
Component	**Force-displacement**	**LT of Force-displacement**	**Impedance $Z_M(s) = F(s)/X(s)$**
Spring	$f(t) = Kx(t)$	$F(s) = KX(s)$	K
Viscous Damper	$f(t) = fv \frac{dx(t)}{dt}$	$F(s) = fv\, sX(s)$	$f_v s$
Mass	$f(t) = M \frac{d^2x(t)}{dt^2}$	$F(s) = Ms^2X(s)$	Ms^2

Last column is for the impedance of mechanical components.

Replacing each force in free body diagram by its Laplace transform, which is in the format

$F(s) = Z_M(s)X(s)$

Translational Mechanical System: Transfer Function



we obtain the figure as shown, from which we could have obtained the equation immediately without writing the differential equation.

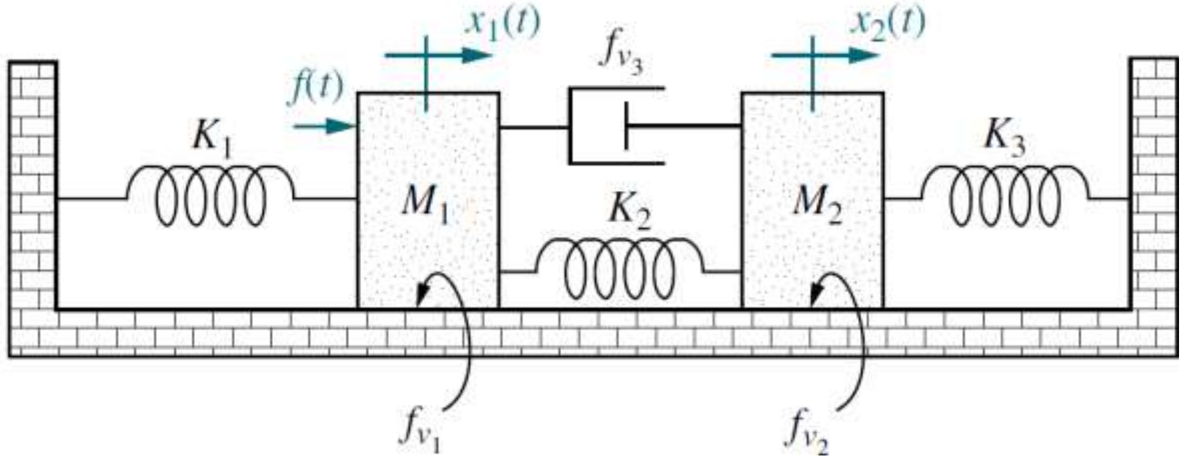
$$Ms^2X(s) + fv\, sX(s) + KX(s) = F(s)$$

And the equation $(Ms^2 + fv\, s + K)X(s) = F(s)$ is of the form of

{Sum of impedances} X(s) = {Sum of applied force}

Translational Mechanical System: Transfer Function

Find the transfer function, X2(s)/F(s), for the system as shown Figure 1.



The system has two degrees of freedom, since each mass can be moved in the horizontal direction while the other is held still.

Thus, two simultaneous equations of motion will be required to describe the system.

The two equations come from free-body diagrams of each mass.

Translational Mechanical System: Transfer Function

Forces acting on mass M_1

only due to motion of M_1

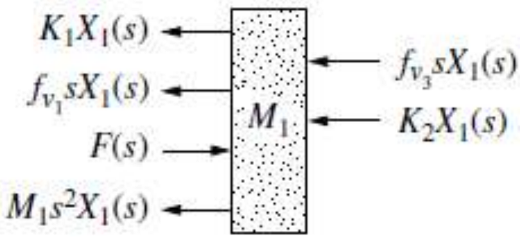


figure 2 a

Forces acting on mass M_1

only due to motion of M_2

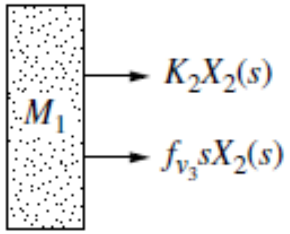


figure 2 b

All forces acting on mass M_1

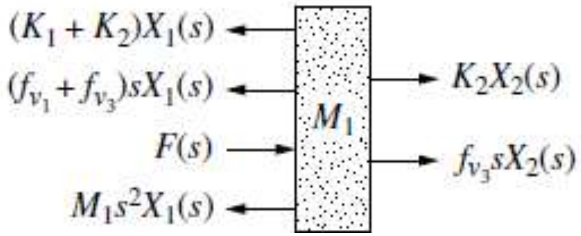


figure 2 c

Translational Mechanical System: Transfer Function

Forces acting on mass M_2

only due to motion of M_2

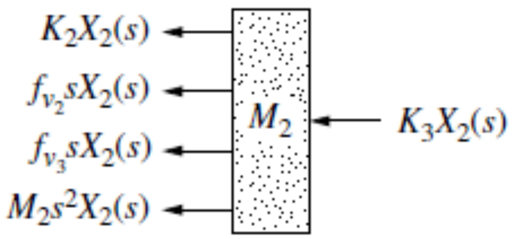


figure 3 a

Forces acting on mass M_2

only due to motion of M_1

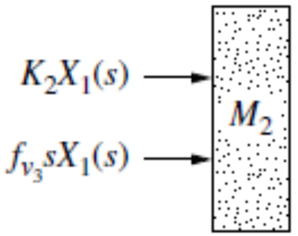


figure 3 b

All forces acting on mass M_2

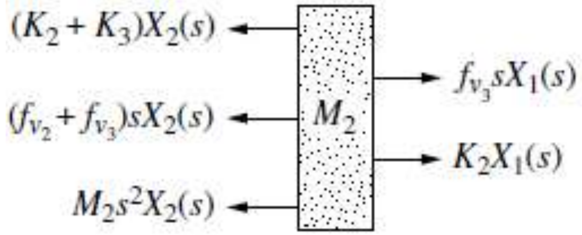


figure 3 c

Translational Mechanical System: Transfer Function

The Laplace transform of the equations of motion can now be written from figures 2 (c) and 3(c)

$$[M_1s^2(f_{v1}+f_{v3})s+(K_1+K_2)]X_1(s) \qquad -(f_{v3}s+K_2)X_2(s)=F(s)$$

$$-(f_{v3}s+K_2)X_{1(s)}+[M_2s^2(f_{v2}+f_{v3})s+(K_2+K_3)]X_2(s)=0$$

From this, $X_2(s)$is

$$X_2(s)=\frac{\begin{vmatrix}[M_1s^2(f_{v1}+f_{v3})s+(K_1+K_2) & F(s)\\ -(f_{v3}s+K_2) & 0\end{vmatrix}}{\Delta}$$

Translational Mechanical System: Transfer Function

and the transfer function, $X_2(s)/F(s)$, is

$$\frac{X_2(s)}{F(s)} = \frac{(f_{v3}s + K_2)}{\Delta}$$

Where,

$$\Delta = \begin{vmatrix} M_1 s^2 (f_{v1} + f_{v3})s + (K_1 + K_2) & -f_{v3}s + K_2 \\ -f_{v3}s + K_2 & M_2 s^2 (f_{v2} + f_{v3})s + (K_2 + K_3) \end{vmatrix}$$

Feedback Control System Unit 1_3
Time Analysis

Poles of a Transfer Function

The poles of a transfer function are

(1) the values of the Laplace transform variable, **s**, that cause the transfer function to become infinite or

(2) any roots of the denominator of the transfer function that are common to roots of the numerator.

Poles of a Transfer Function

- The first part completely satisfy definition of the poles of a transfer function i.e. the roots of the characteristic polynomial in the denominator are values of ‘**s**’ that make the transfer function infinite, so they are thus poles.
- However, if a factor of the denominator can be canceled by the same factor in the numerator, the root of this factor no longer causes the transfer function to become infinite.
- In control systems, we often refer to the root of the canceled factor in the denominator as a pole even though the transfer function will not be infinite at this value. Hence, the later part is included in the definition.

Zeros of a Transfer Function

The zeros of a transfer function are

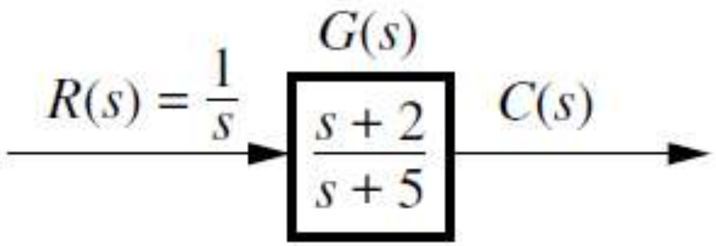
(1) the values of the Laplace transform variable, **s**, that cause the transfer function to become zero, or

(2) any roots of the numerator of the transfer function that are common to roots of the denominator.

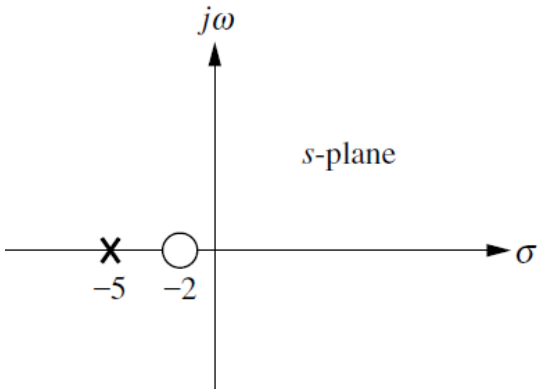
Zeros of a Transfer Function

- The first part completely satisfy definition of the poles of a transfer function i.e. the roots of the characteristic polynomial in the denominator are values of '**s**' that make the transfer function zero, so they are thus zeros.
- However, if a factor of the numerator can be canceled by the same factor in the denominator, the root of this factor no longer causes the transfer function to become zero.
- In control systems, we often refer to the root of the canceled factor in the numerator as a zero even though the transfer function will not be zero at this value. Hence, the later part is included in the definition.

Poles and Zeros of a First-Order System



a



b

Given the transfer function G(s) = (s+2)/(s+5) in figure (a), a pole exists at s = - 5, and a zero exists at - 2. These pole and zero are are plotted on the complex s-plane in figure (b).

To show the properties of the poles and zeros, let us find the unit step response of the system.

Poles and Zeros of a First-Order System & System Response

$\frac{C(s)}{R(s)} = G(s) = \frac{s+2}{s+5}$ the step response means input $r(t) = u(t)$ i.e. $R(s) = \frac{1}{s}$

$$\therefore C(s) = G(s)R(s) = \frac{1}{s}\frac{s+2}{s+5} \quad \dots 1$$

$$\therefore C(s) = \frac{A}{s} + \frac{B}{s+5} = \frac{2/5}{s} + \frac{3/5}{s+5} \quad \dots 2$$

Where $A = \left.\frac{s+2}{s+5}\right|_{s\to 0} = \frac{2}{5}$ $and\ B = \left.\frac{s+2}{s}\right|_{s\to -5} = \frac{3}{5}$

Thus the output response of the first order system c(t) is

$$c(t) = \frac{2}{5} + \frac{3}{5}e^{-5t} \quad \dots 3$$

Poles and Zeros of a First-Order System & System Response

The output response of a system is the sum of two responses: the **forced response** and the **natural response**.

The **forced response** is also called the **steady-state response** or **particular solution**. The **natural response** is also called the **homogeneous solution**.

$= G(s) = \frac{s+2}{s+5}$ the step response means input $r(t) = u(t)$ i.e. $R(s) = \frac{1}{s}$

$$\therefore C(s) = G(s)R(s) = \frac{1}{s}\frac{s+2}{s+5} \ldots 1$$

$$\therefore C(s) = \frac{A}{s} + \frac{B}{s+5} = \frac{2/5}{s} + \frac{3/5}{s+5} \ldots 2$$

Where $A = \left.\frac{s+2}{s+5}\right|_{s\to 0} = \frac{2}{5}$ and $B = \left.\frac{s+2}{s}\right|_{s\to -5} = \frac{3}{5}$

Poles and Zeros of a First-Order System & System Response

Thus the output response of the first order system c(t) is

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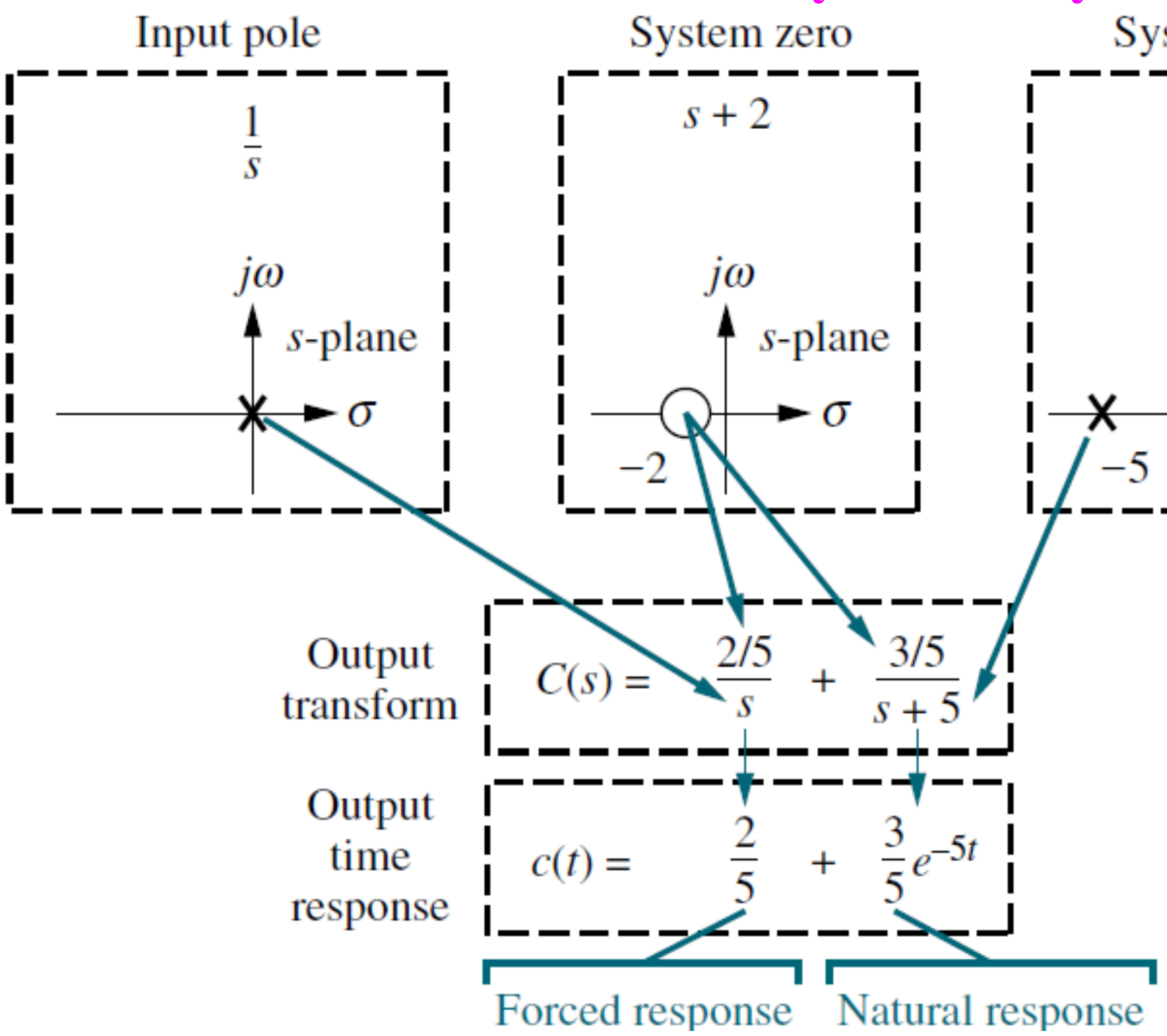
Poles and Zeros of a First-Order System & System Response



Evolution of a system response. The blue arrows show the evolution of the response component generated by the pole or zero.

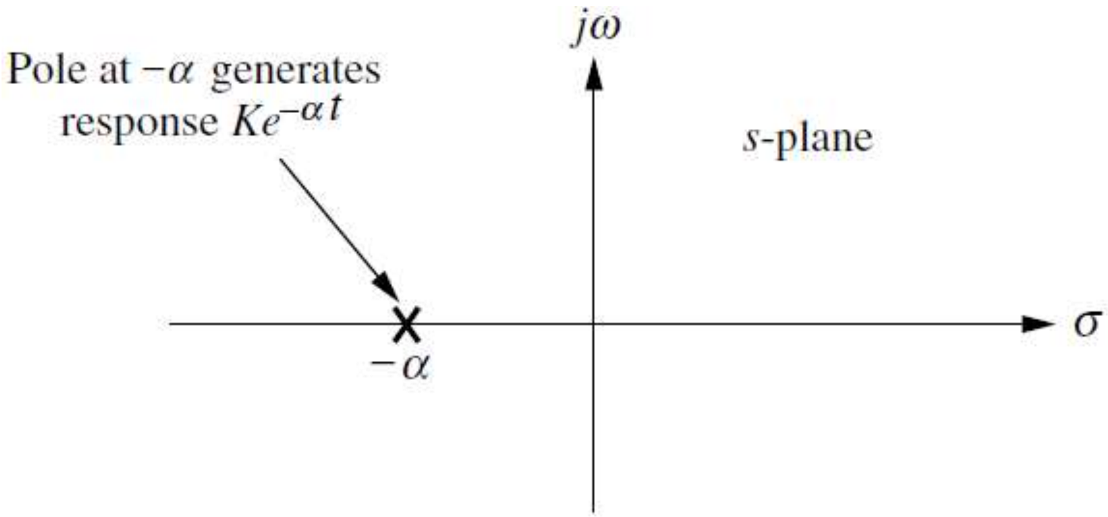
Poles and Zeros of a First-Order System & System Response

The conclusions are

1. A **pole** of the **input function** generates the form of the **forced response** (that is, the **pole at the origin generated a step function at the output**).
2. A **pole** of the **transfer function** generates the form of the **natural response** (that is, **the pole at -5 generated e^{-5t}**).
3. A pole on the real axis generates an exponential response of the form $e^{-\alpha t}$, where $-\alpha$ is the pole location on the real axis. Thus, the farther to the left a pole is on the negative real axis, the faster the exponential transient response will decay to zero.
4. The zeros and poles generate the amplitudes for both the forced and natural responses

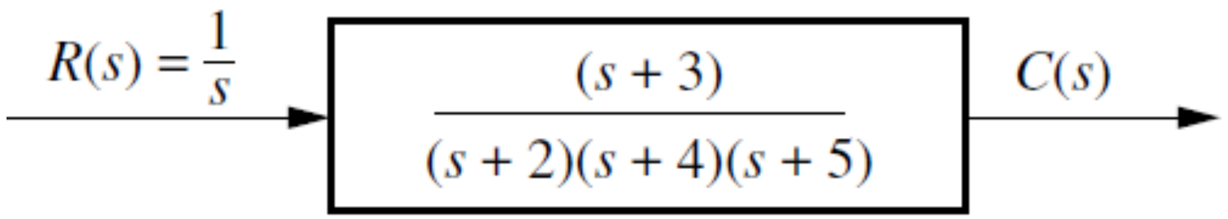
Poles and Zeros of a First-Order System & System Response

Effect of a real-axis pole upon transient response



Poles and Zeros of a First-Order System & System Response

Find the output c(t) Specify the forced and natural parts of the solution.



Poles and Zeros of a First-Order System & System Response

Find the output c(t) and specify the forced and natural parts of the solution if input is unit step.

$$G(s) = \frac{10(s+4)(s+6)}{(s+1)(s+7)(s+8)(s+10)}$$

First-Order Systems without zero

A first-order system without zeros can be described by the transfer function $G(s) = \frac{a}{(s+a)}$. If the input is a unit step, $R(s) = \frac{1}{s}$

the Laplace transform of the step response is C(s), where

$$C(s) = R(s)G(s) = \frac{a}{s(s+a)}$$

Taking the inverse transform, the step response is given by

$$c(t) = c_f(t) + c_n(t) = 1 - e^{-at} \quad \ldots 1$$

where the input pole at the origin generated the forced response $\mathbf{c_f(t) = 1}$, and the system pole at $-\mathbf{a}$ generated the natural response $\mathbf{c_n(t) = e^{-at}}$. Let us examine the significance of parameter a, the only parameter needed to describe the transient response.

First-Order Systems without zero

The parameter **a** is the only parameter needed to describe the transient response. When t = 1/a

$$\left. e^{-at} \right|_{t=1/a} = e^{-1} = 0.37 \ldots 2$$

$$\left. c(t) \right|_{t=1/a} = 1 - \left. e^{-at} \right|_{t=1/a} = 1 - 0.37 = 0.63 \ldots 3$$

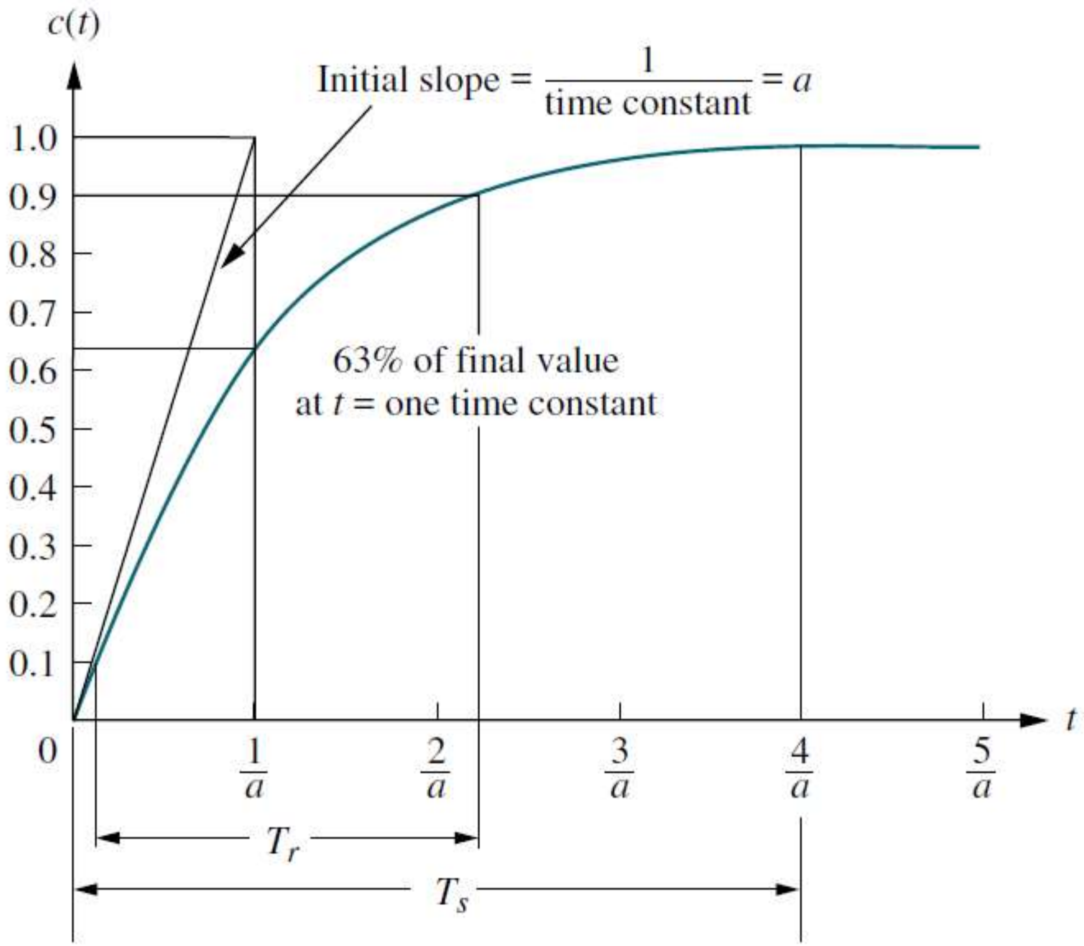
Time constant

The **1/a** is time constant of the response.

From equation $e^{-at}|_{t=1/a} = e^{-a} = 0.37$, the time constant can be described as the time for e^{-at} to decay to 37% of its initial value.

Alternately, from equation $c(t)|_{t=1/a} = 1 - e^{-at}|_{t=1/a} = 1 - 0.37 = 0.63$ the time constant is the time it takes for the step response to rise to 63% of its final value.

Time constant



First-order system response to a unit step

Time constant

The reciprocal of the time constant has the units (1/seconds), or frequency.

Thus, the parameter **'a'** the exponential frequency.

The derivative of $\mathbf{e^{-at}}$ is **'-a'** when t = 0, a is the initial rate of change of the exponential at t = 0.

Thus, the time constant can be considered a transient response specification for a first order system, since it is related to the speed at which the system responds to a step input.

The time constant can also be evaluated from the pole plot. Since the pole of the transfer function is at **'-a'**, we can say the pole is located at the reciprocal of the time constant, and the farther the pole from the imaginary axis, the faster the transient response.

Rise Time, T_r

Rise time is defined as the time for the waveform to go from 0.1 to 0.9 of its final value. Rise time is found by solving equation $c(t) = c_f(t) + c_n(t) = 1 - e^{-at}$ for the difference in time at c(t) = 0:9 and c(t) = 0:1. Hence

$$t_r = \frac{2.31}{a} - \frac{0.11}{a} = \frac{2.2}{a}$$

Settling Time, T_s

Settling time is defined as the time for the response to reach, and stay within, 2% of its final value.

Letting **c(t) = 0.98** in equation $c(t) = c_f(t) + c_n(t) = 1 - e^{-at}$ and solving for time, t, we find the settling time to be

$$T_s = \frac{4}{a}$$

Transfer function from laboratory test

It is not always possible or practical to obtain a system’s transfer function analytically. Because the system is closed, and the component parts are not easily identifiable.

Since the transfer function is a representation of the system from input to output, the system’s step response can help to find out the transfer function even though the inner construction is not known.

With a step input, we can measure the time constant and the steady-state value, from which the transfer function can be calculated

Transfer function from laboratory test

Consider a simple first-order system, $G(s) = \frac{K}{s+a}$, whose step response is

$$C(s) = \frac{K}{s(s+a)} = \frac{K/a}{s} - \frac{K/a}{(s+a)}$$

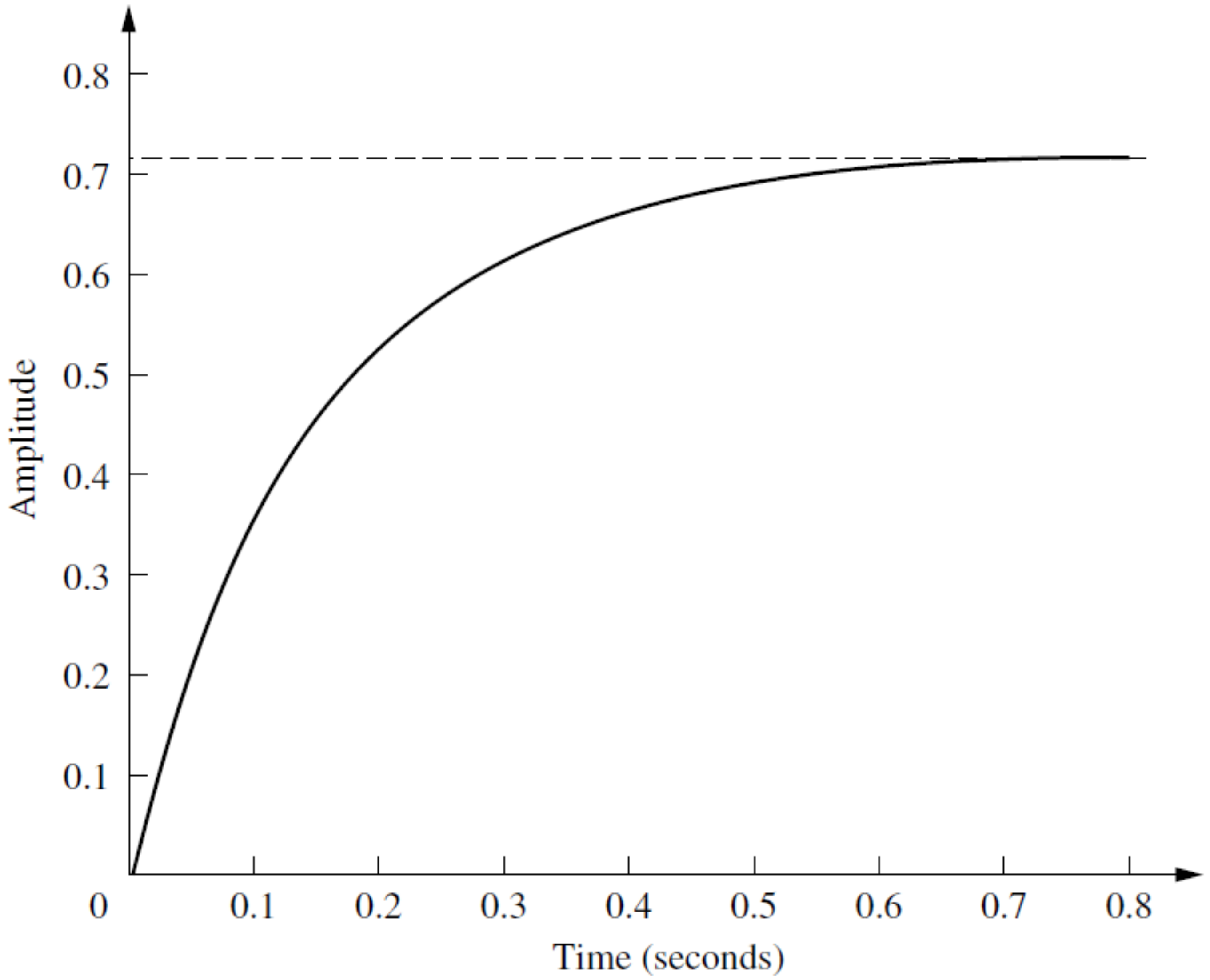
Now find out the values of identify **K** and **a** from laboratory testing, we can obtain the transfer function of the system.

Let the unit step response is given by the figure as shown. It is the first-order characteristics as no overshoot and nonzero initial slope.

From the response, we measure the time constant, that is, the time for the amplitude to reach 63% of its final value. Since the final value is about 0.72, the time constant is evaluated where the curve reaches 0.63 x 0.72 = 0.45, or about 0.13 second.

Hence, a = 1/0.13 = 7.7

Transfer function from laboratory test



Laboratory results of a system step response test

Transfer function from laboratory test

To find K, we realize from $C(s) = \frac{K}{s(s+a)} = \frac{K/a}{s} - \frac{K/a}{(s+a)}$ that the forced response reaches a steady state value of **K/a = 0.72**.

Substituting the value of **a**, we find **K = 5.54**.

Thus, the transfer function for the system is $G(s) = \frac{5.54}{(s+7.7)}$.

It is interesting to note that the response shown in the figure was generated using the transfer function $G(s) = \frac{5}{(s+7)}$

Transfer function from laboratory test

A system has a transfer function, $G(s) = \frac{50}{(s+50)}$. Find the time constant Tc, settling time Ts, and rise time Tr.

Second-Order Systems

Let us now extend the concepts of poles and zeros and transient response to second order systems.

Compared to the simplicity of a first-order system, a second-order system exhibits a wide range of responses

Whereas varying a first-order system's parameter simply changes the speed of the response, changes in the parameters of a second-order system can change the form of the response.

Second-Order Systems

Consider a general second order system of the form of $G(s) = \frac{b}{s^2+as+b}$

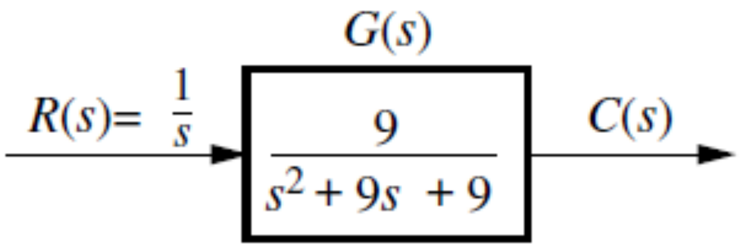
The term in the numerator is simply a scale or input multiplying factor that can take on any value without affecting the form of the derived results.

By assigning appropriate values to parameters **a** and **b**, we can show all possible second-order transient responses.

The unit step response then can be found using C(s) = R(s)G(s), where R(s) = 1/s.

The possible step response are based on the values of the parameters **a** and **b** are Overdamped, Underdamped, Undamped, and Critically damped.

Overdamped Response



For this second order system the transfer function $G(s) = \frac{9}{s^2+9s+9}$ and the response is

$$C(s) = \frac{1}{s}\frac{9}{s^2+9s+9} = \frac{9}{s(s+7.854)(s+1.146)}$$

The output response C(s) has a pole at the origin that comes from the unit step input and two real poles that come from the system.

The input pole at the origin **generates the constant forced response**;

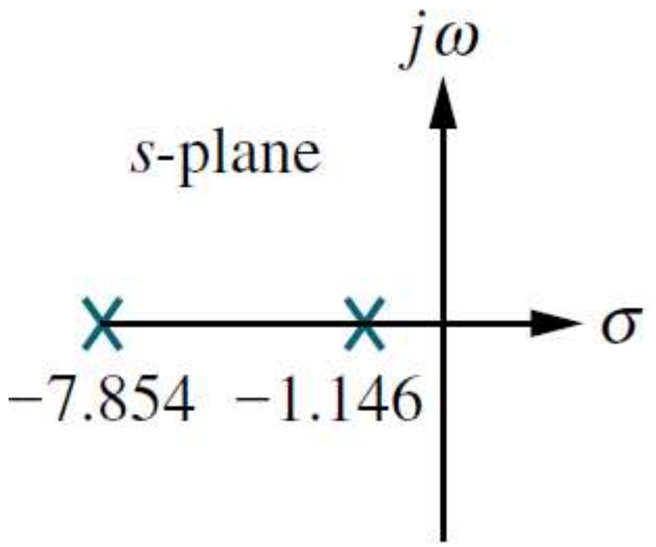
and the two system poles on the real axis **generates an exponential natural response** whose exponential frequency is equal to the pole location

Overdamped Response

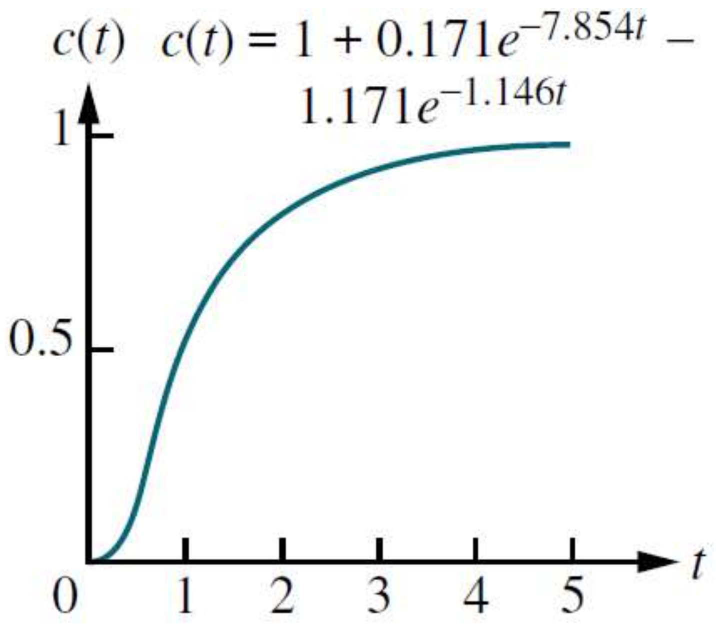
The output initially could have been written as $c(t) = K_1 + K_2 e^{-7.854t} + K_3 e^{-7.854t}$

This response, shown in following figure is called overdamped.

From the poles zero plot we can easily find out or visualized the nature of the response without the tedious calculation of the inverse Laplace transform.

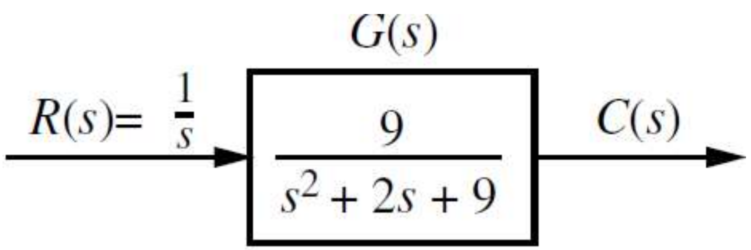


Pole zero plot



overdamped response

Underdamped Response



For this second order system the transfer function $G(s) = \frac{9}{s^2+2s+9}$ and the response is

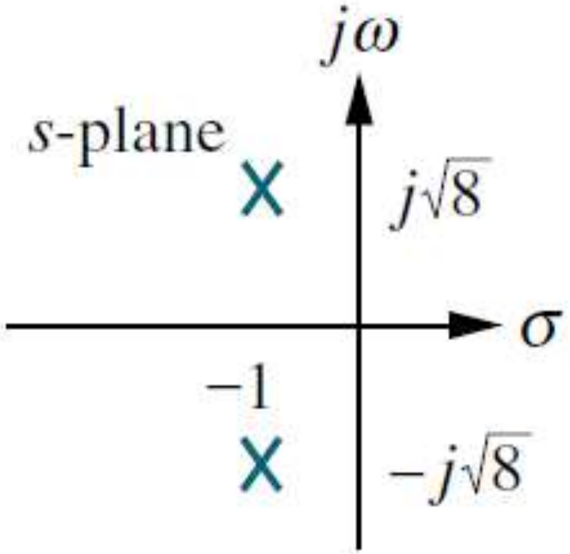
$$C(s) = \frac{1}{s}\frac{9}{s^2 + 2s + 9}$$

This function has a pole at the origin that comes from the unit step input and two complex poles that come from the system.

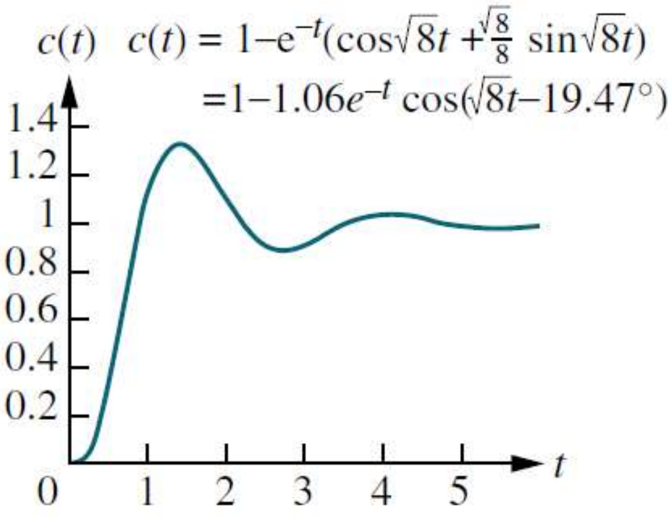
The input pole at the origin **generates the constant forced response**;

and the two system poles on the real axis **generates an exponential natural response** whose exponential frequency is equal to the pole location

Underdamped Response



Pole zero plot

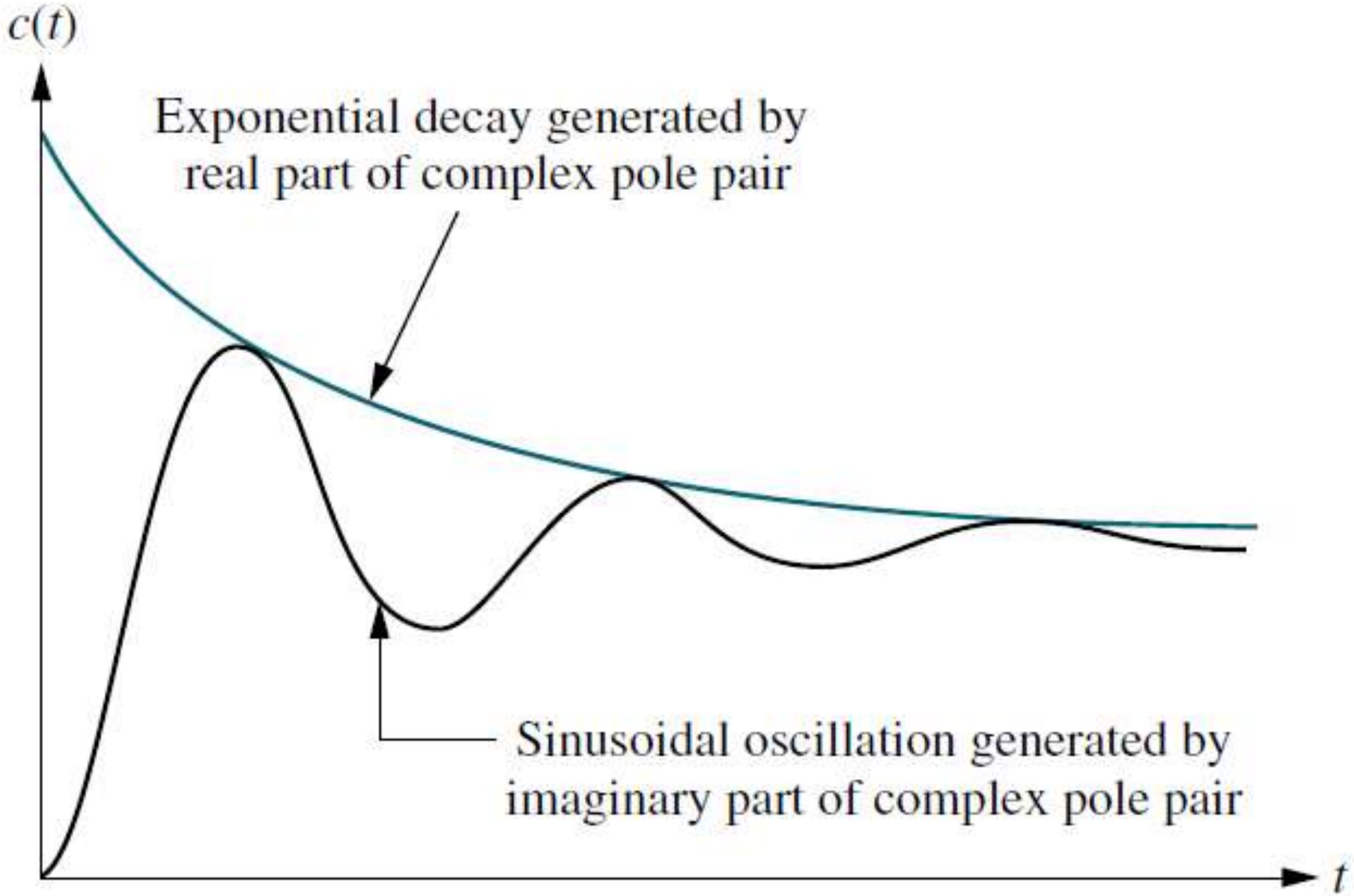


Underdamped response

The poles that generate the natural response are at $s = -1 \pm j\sqrt{8}$.

The real part of the pole matches the exponential decay frequency of the sinusoid's amplitude, while the imaginary part of the pole matches the frequency of the sinusoidal oscillation.

Underdamped Response



Second-order step response components generated by complex pole

Underdamped Response

Figure shows a general, damped sinusoidal response for a second order system.

The transient response consists of an **exponentially decaying amplitude generated by the real part** of the **system pole** times a sinusoidal waveform generated by the imaginary part of the system pole.

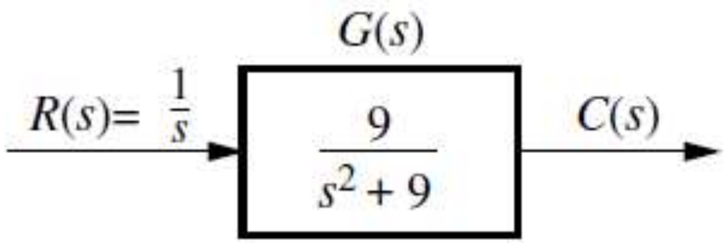
The time constant of the exponential decay is equal to the reciprocal of the real part of the system pole.

The value of the imaginary part is the actual frequency of the sinusoid.

This sinusoidal frequency is given the name damped frequency of oscillation, ω_d.

Finally, the steady-state response (unit step) was generated by the input pole located at the origin. This type of response is called as underdamped response, which approaches a steady-state value via a transient response that is a damped oscillation.

Undamped Response



For this second order system the transfer function $G(s) = \frac{9}{s^2+9}$ and the response is

$$C(s) = \frac{1}{s}\frac{9}{s^2+9}$$

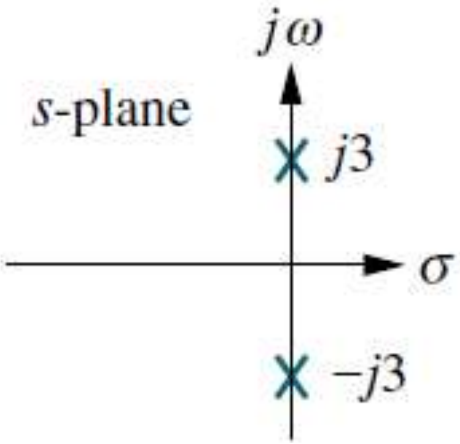
This function has a pole at the origin that comes from the unit step input and two imaginary poles that come from the system.

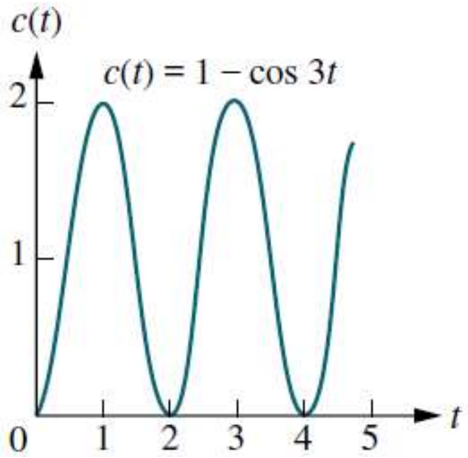
The input pole at the origin generates the constant forced response,

and the two system poles on the imaginary axis at $\pm j3$ generate a sinusoidal natural response whose frequency is equal to the location of the imaginary poles.

Undamped Response

Hence, the output can be estimated as $c(t) = K_1 + K_2\cos(3t - \varphi)$.





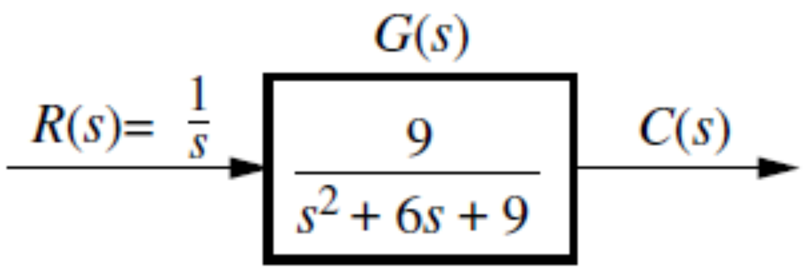
Pole zero plot

undamped response

This type of response is called undamped.

Note that the absence of a real part in the pole pair corresponds to an exponential that does not decay. Mathematically, the exponential is $e^{-0t} = 1$.

Critically damped Response



For this second order system the transfer function $G(s) = \frac{9}{s^2+6s+9}$ and the response is

$$C(s) = \frac{1}{s}\frac{9}{s^2 + 6s + 9}$$

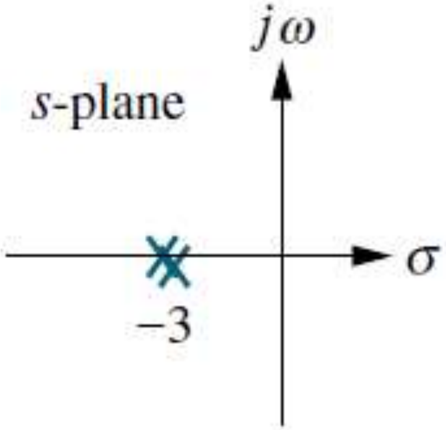
This function has a pole at the origin that comes from the unit step input and

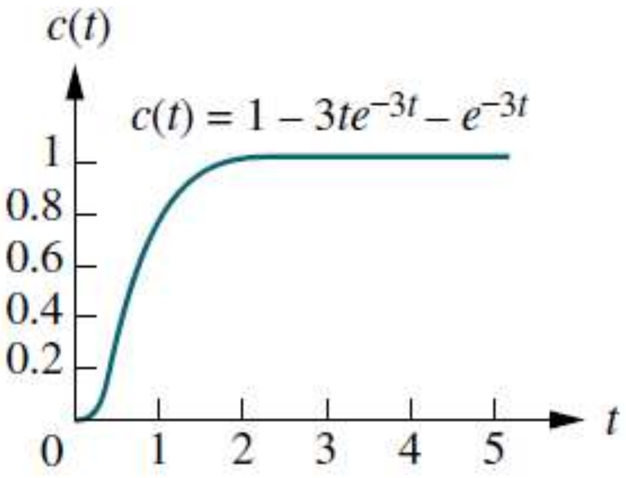
Two multiple real poles that come from the system.

The input pole at the origin generates the constant forced response, and the two poles on the real axis at 3 generate a natural response consisting of an exponential and an exponential multiplied by time, where the exponential frequency is equal to the location of the real poles.

Critically damped Response

Hence, the output can be estimated as $c(t) = K_1 + K_2 e^{-\sigma_1 t} + K_3 t e^{-\sigma_1 t}$





Pole zero plot critically damped response

The output can be estimated as $c(t) = K_1 + K_2 e^{-3t} + K_3 t e^{-3t}$.

This type of response is called critically damped.

Critically damped responses are the fastest possible without the overshoot that is characteristic of the underdamped response.

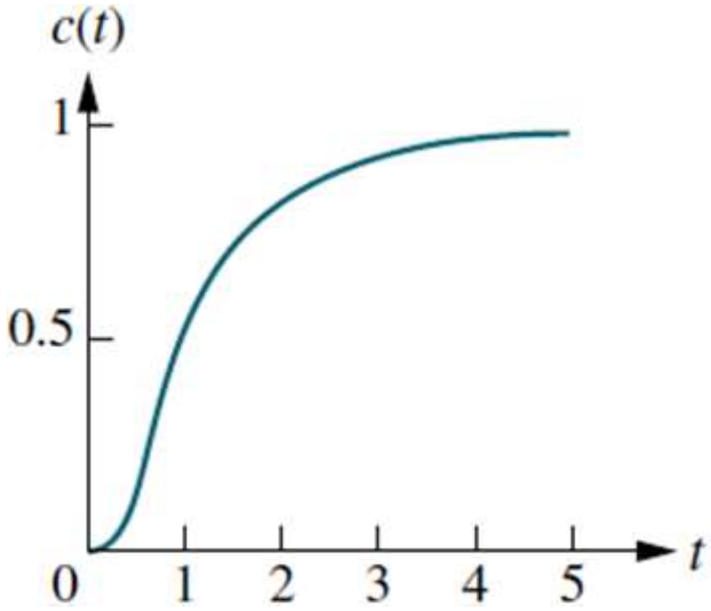
Step responses for second-order system damping cases: Conclusion

1. Overdamped responses

Poles: Two real at $-\sigma_1, -\sigma_2$

Natural response: Two exponentials with time constants equal to the reciprocal of the pole locations, or

$$c(t) = K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t}$$



Step responses for second-order system damping cases: Conclusion

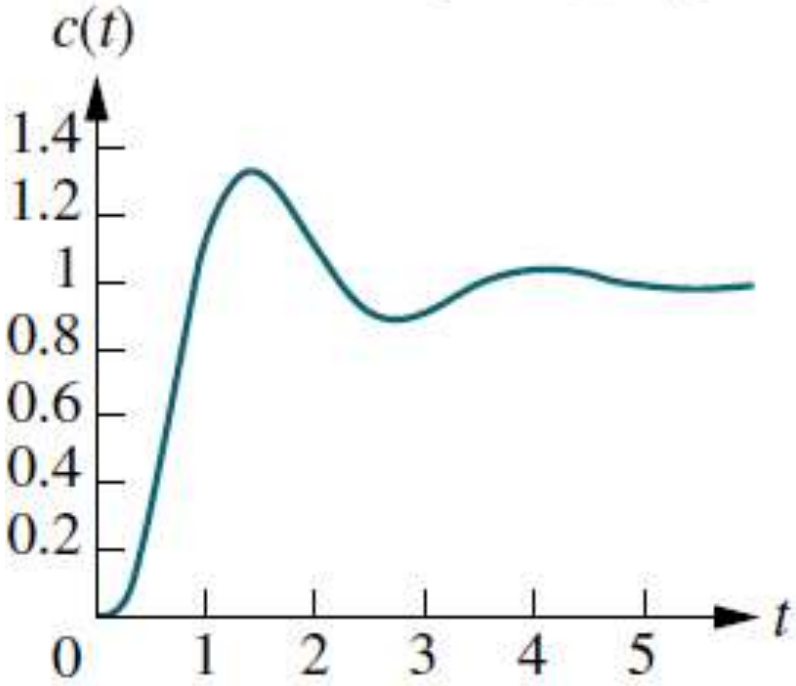
2. Underdamped responses

Poles: Two complex at $-\sigma_d \pm j\omega_d$

Natural response: Damped sinusoid with an exponential envelope whose time constant is equal to the reciprocal of the pole's real part.

The radian frequency of the sinusoid, the damped frequency of oscillation, is equal to the imaginary part of the poles, or

$$c(t) = Ae^{-\sigma_d t}\cos(\omega_d t - \phi)$$



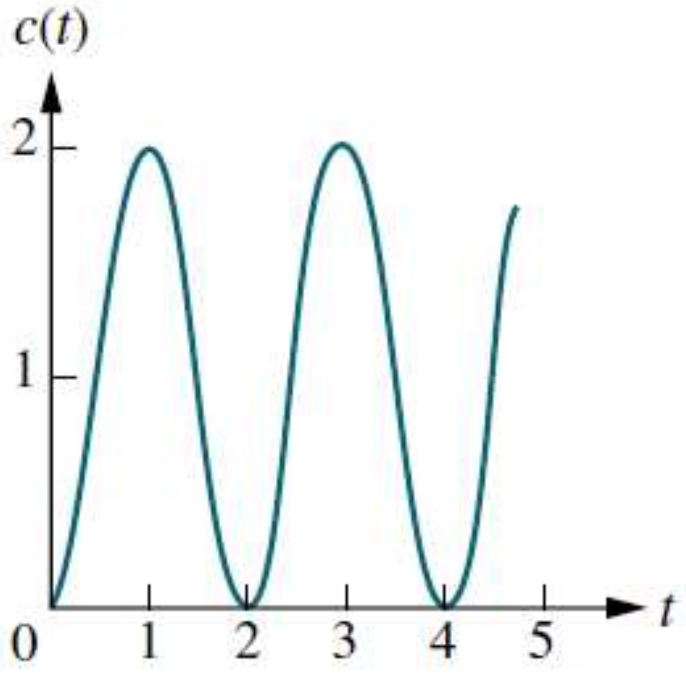
Step responses for second-order system damping cases: Conclusion

3. Undamped responses

Poles: Two imaginary at $\pm j\omega_1$

Natural response: Undamped sinusoid with radian frequency equal to the imaginary part of the poles, or

$$c(t) = A\cos(\omega_1 t - \phi)$$



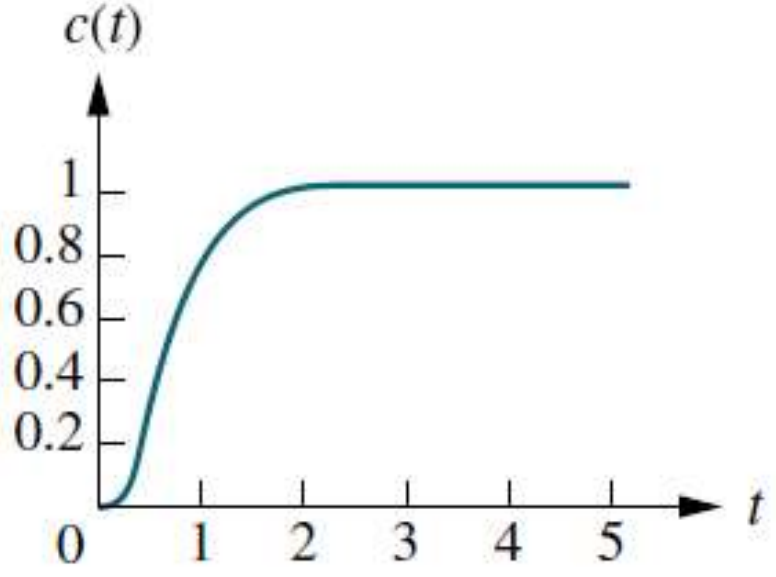
Step responses for second-order system damping cases: Conclusion

4. Critically damped responses

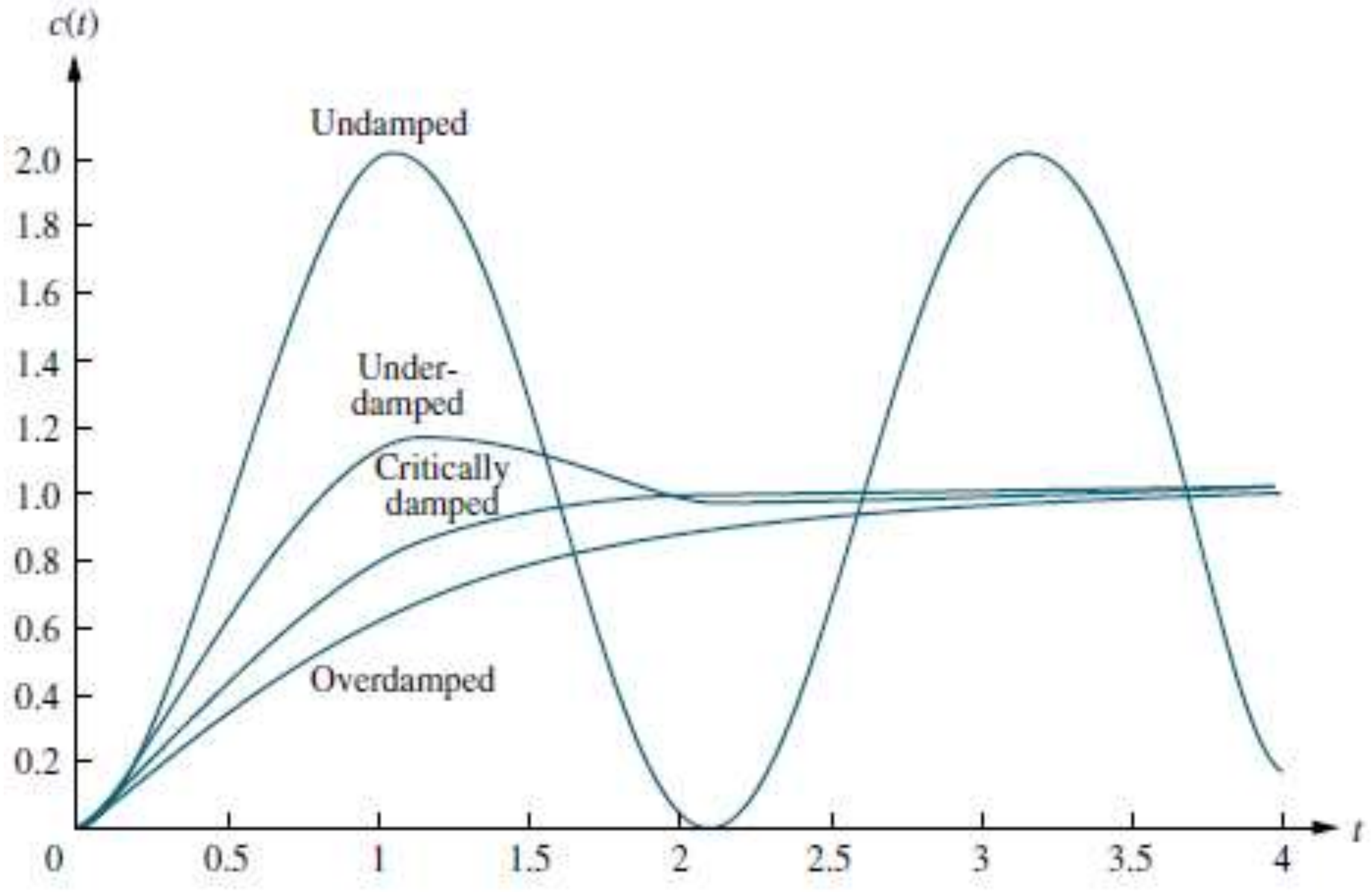
Poles: Two real at $-\sigma_1$

Natural response: One term is an exponential whose time constant is equal to the reciprocal of the pole location. Another term is the product of time, t, and an exponential with time constant equal to the reciprocal of the pole location, or

$$c(t) = K_1 e^{-\sigma_1 t} + K_2 t e^{-\sigma_1 t}$$



Step responses for second-order system damping cases: Conclusion



The step responses for the four cases of damping

Natural frequency and Damping ratio of second order system

The natural frequency and damping ratio can be used to describe the characteristics of the second-order transient response just as time constants describe the first-order system response.

Natural Frequency, ω_n

The natural frequency of a second-order system is the frequency of oscillation of the system without damping.

For example, the frequency of oscillation of a series RLC circuit with the resistance shorted would be the natural frequency.

Natural frequency and Damping ratio of second order system

Damping Ratio, ζ

This is the ratio of exponential decay frequency to Natural frequency (rad/second).

$$\zeta = \frac{\text{Exponential decay frequency}}{\text{Natual frequency (rad/second)}} = \frac{1}{2\pi}\frac{\text{Natural Period}}{\text{Exponential Period}}$$

Natural frequency and Damping ratio of second order system

Consider the general second order system

$$G(s) = \frac{b}{s^2 + as + b} \quad \dots 1$$

Without damping, the poles would be on the jω-axis, and the response would be an undamped sinusoid. For the poles to be purely imaginary, $a = 0$. Hence

$$G(s) = \frac{b}{s^2 + b} \dots 2$$

By definition, the natural frequency, ω_n, is the frequency of oscillation of this system. Since the poles of this system are on the jω -axis at $\pm j\sqrt{b}$

$$\omega_n = \sqrt{b} \dots 3$$

Natural frequency and Damping ratio of second order system

$$b = \omega_n^2 \quad \dots 4$$

Assuming an underdamped system, where the complex poles have a real part, σ, equal to - a/2.

The magnitude of this value is then the exponential decay frequency. Hence,

$$\zeta = \frac{\text{Exponential decay frequency}}{\text{Natual frequency (rad/second)}} = \frac{|\sigma|}{\omega_n} = \frac{a/2}{\omega_n} \quad \dots 5$$

From which

$$a = 2\zeta\omega_n \quad \dots 6$$

Therefore, the transfer function of general second order system becomes

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad \dots 7$$

Natural frequency and Damping ratio of second order system

Find ω_n and ζ for the system described by the transfer function

$$G(s) = \frac{36}{s^2 + 4.2s + 36}$$

Use

$b = \omega_n^2$ and $a = 2\zeta\omega_n$

Natural frequency, Damping ratio and pole locations of second order system

The poles can be obtained by solving the following transfer function

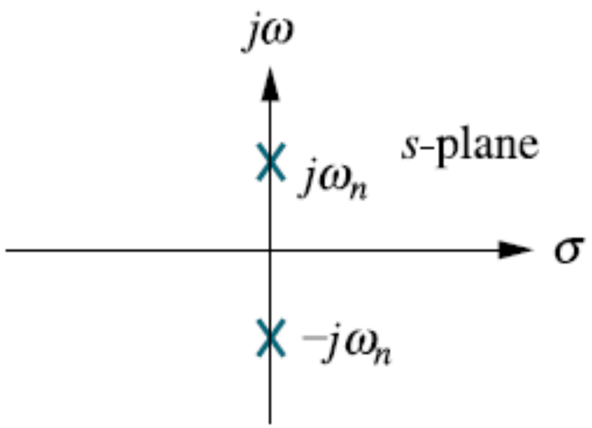
$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

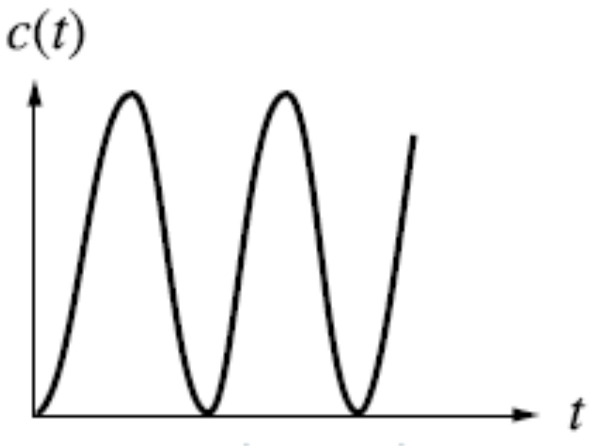
We get

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Natural frequency, Damping ratio and pole locations of second order system

For $\zeta = 0$



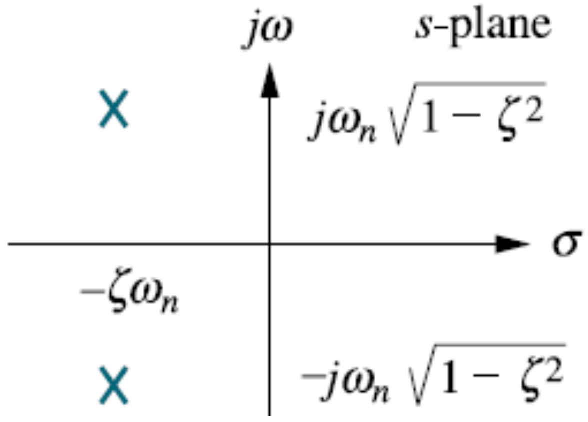


Poles

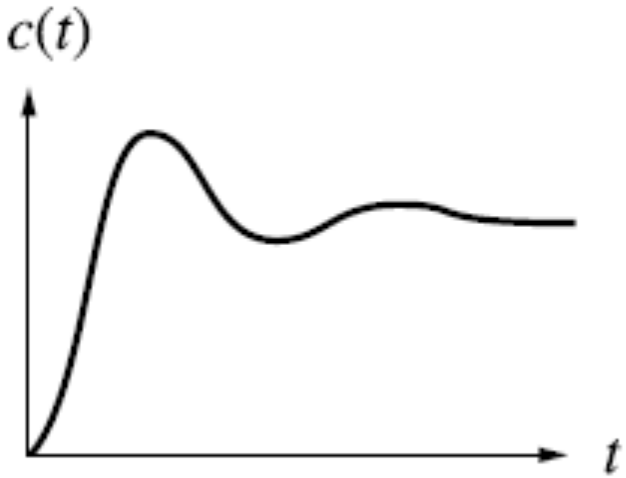
Step response: undamped

Natural frequency, Damping ratio and pole locations of second order system

For $0 < \zeta < 1$



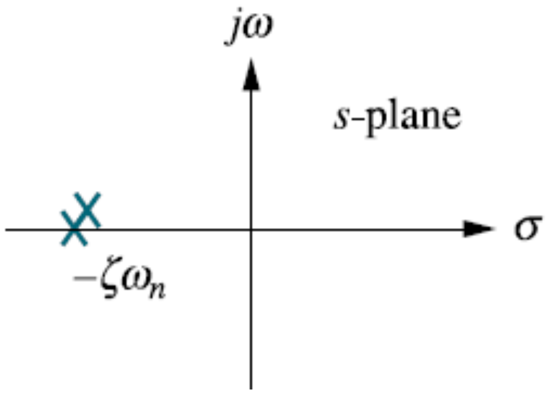
Poles



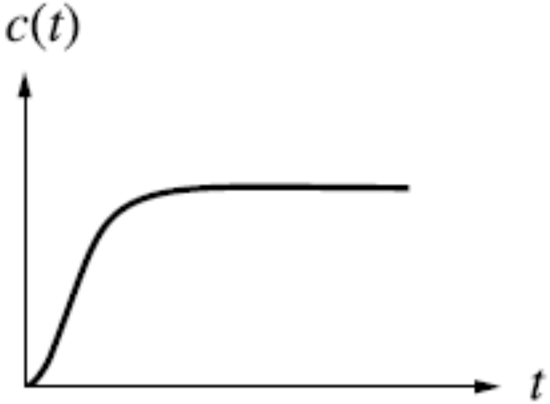
Step response: underdamped

Natural frequency, Damping ratio and pole locations of second order system

For $\zeta = 1$



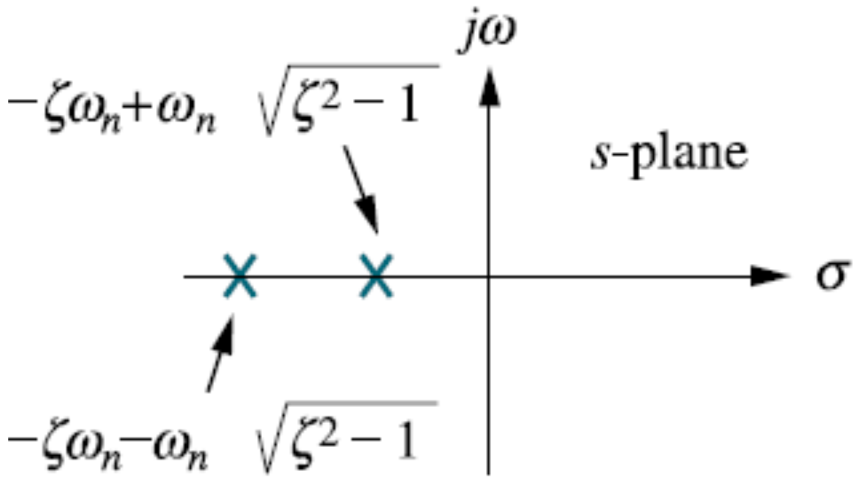
Poles



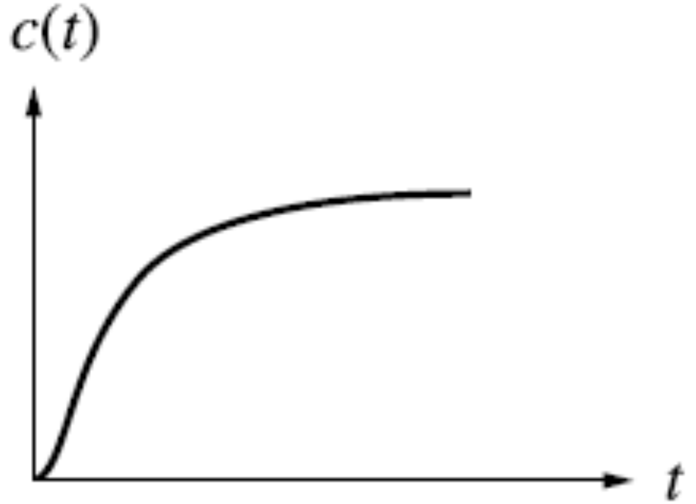
Step response: critically damped

Natural frequency, Damping ratio and pole locations of second order system

For $\zeta > 1$



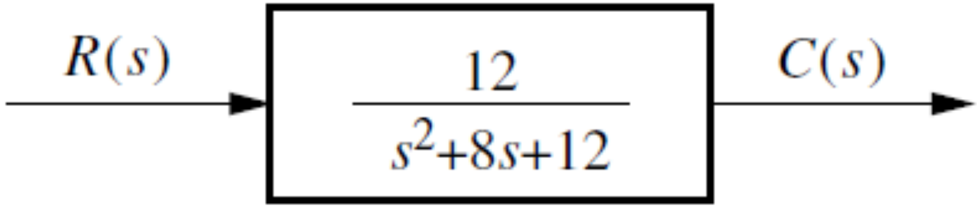
Poles

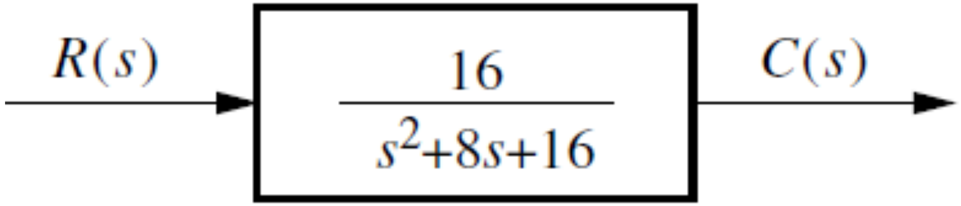


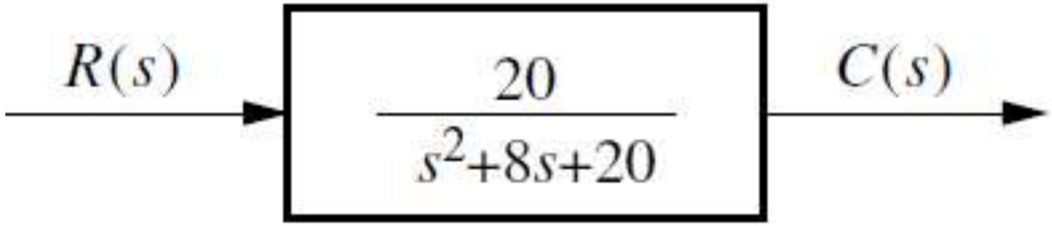
Step response: overdamped

Natural frequency, Damping ratio and pole locations of second order system

For each of the systems shown, find the value of ζ and report the kind of response expected.

A) 

B) 

C) 

Use $a = 2\,\zeta\omega_n$ and $\omega_n = \sqrt{b}$ therefore $\zeta = \frac{a}{2\sqrt{b}}$

Natural frequency, Damping ratio and pole locations of second order system

For each of the transfer functions

(1) Find the values of ζ and ω_n;

(2) characterize the nature of the response.

a. $G(s) = \dfrac{400}{s^2 + 12s + 400}$

b. $G(s) = \dfrac{900}{s^2 + 90s + 900}$

c. $G(s) = \dfrac{225}{s^2 + 30s + 225}$

d. $G(s) = \dfrac{625}{s^2 + 625}$

Natural frequency, Damping ratio and pole locations of second order system

For each of the transfer functions

(1) Find the values of ζ and ω_n;

(2) characterize the nature of the response.

a. $G(s) = \dfrac{400}{s^2 + 12s + 400}$ $\zeta = 0.3\ and \omega_n = 20;$ underdamped

b. $G(s) = \dfrac{900}{s^2 + 90s + 900}$ $\zeta = 1.5\ and \omega_n = 30;$ overdamped

c. $G(s) = \dfrac{225}{s^2 + 30s + 225}$ $\zeta = 1\ and \omega_n = 15;$ critically damped

d. $G(s) = \dfrac{625}{s^2 + 625}$ $\zeta = 0\ and \omega_n = 25;$ undamped

Underdamped Second-Order Systems

The underdamped second order system is a common model for physical problems.

A detailed description of the underdamped response is necessary for both analysis and design. The objectives of this study are

1. To define transient specifications associated with underdamped responses.
2. To relate these specifications to the pole location, drawing an association between pole location and the form of the underdamped second-order response.
3. to tie the pole location to system parameters.

Underdamped Second-Order Systems

Let us begin by finding the step response for the general second-order system.

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad \dots 1$$

The transform of the response, C(s), is the transform of the input times the transfer function,

$$C(s) = \frac{1}{s}\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad \dots 2$$

where it is assumed that $\zeta < 1$ (as it is the underdamped case). Expanding by partial fractions,

$$C(s) = \frac{1}{s} - \frac{(s + \zeta\omega_n) + \frac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s + \zeta\omega_n)^2 + \omega_n^2\left(\sqrt{1-\zeta^2}\right)} \quad \dots 3$$

Underdamped Second-Order Systems

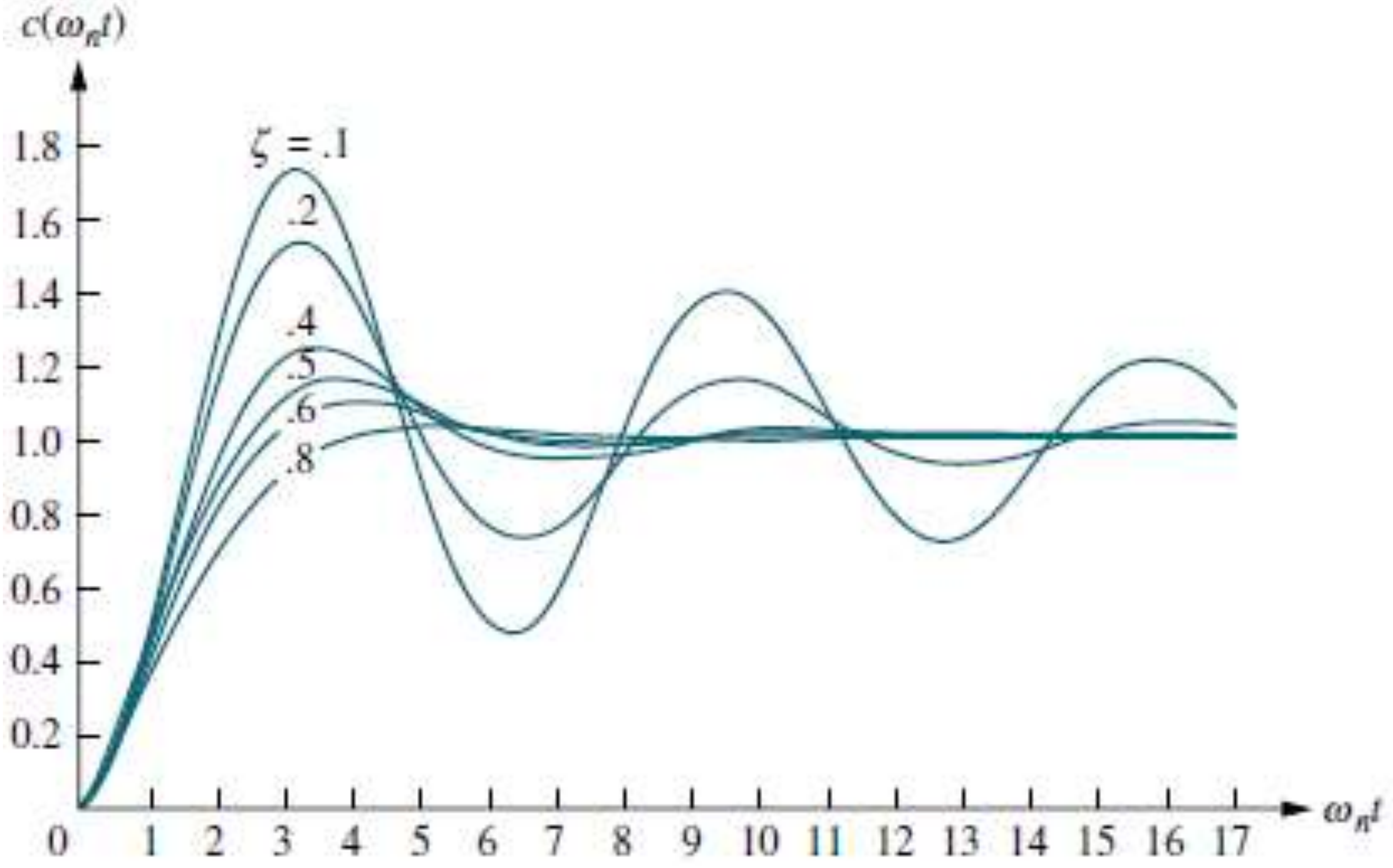
Taking the inverse Laplace transform

$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_n\sqrt{1-\zeta^2}\,t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_n\sqrt{1-\zeta^2}\,t\right) \dots 4$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_n\sqrt{1-\zeta^2}t + \phi\right) \dots 5$$

where $\phi = tan^{-1}\left(\sqrt{1-\zeta^2}/\zeta\right) \dots 6$

Underdamped Second-Order Systems



Second-order underdamped responses for damping ratio (ζ) values

Underdamped Second-Order Systems

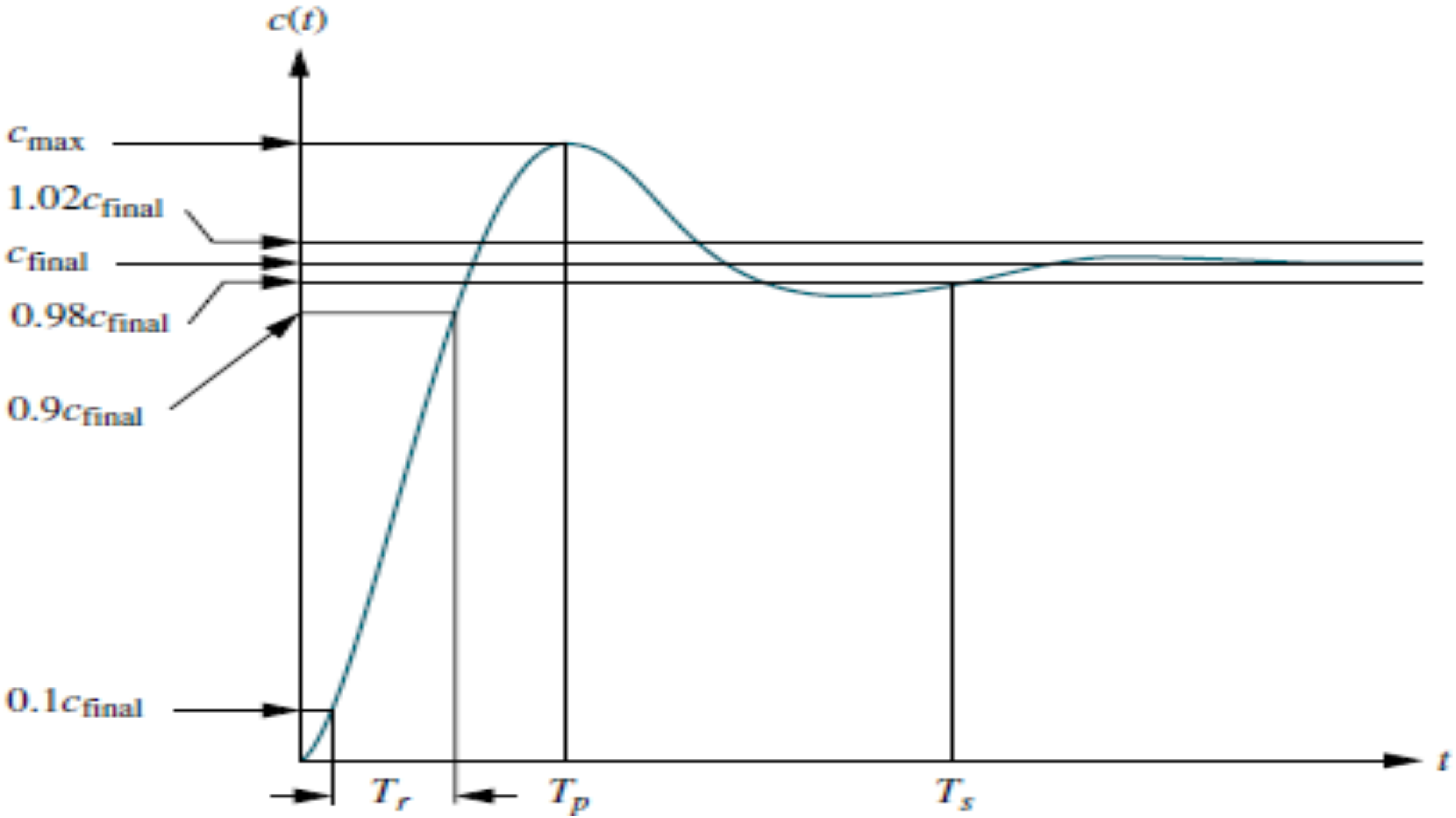
The relationship between the value of damping ratio (ζ) and the type of response obtained

The lower the value of damping ratio (ζ), the response is the more oscillatory.

The natural frequency (ω_n) is a time-axis scale factor and does not affect the nature of the response other than to scale it in time.

Other parameters associated with the underdamped response are **rise time**, **peak time**, **percent overshoot**, and **settling time**.

Underdamped Second-Order Systems



Second-order underdamped response specifications

Underdamped Second-Order Systems

1. **Rise time** (T_r) **:** The time required for the waveform to go from 0.1 of the final value to 0.9 of the final value.
2. **Peak time,** (T_p)**:** The time required to reach the first, or maximum, peak.
3. **Percent overshoot** $(\%OS)$ **:** The amount that the waveform overshoots the steady state, or final, value at the peak time, expressed as a percentage of the steady-state value.
4. **Settling time,** (T_s)**:** The time required for the transient's damped oscillations to reach and stay within $\pm 2\%$ of the steady-state value.

Underdamped Second-Order Systems

The definitions for settling time and rise time are basically the same as the definitions for the first-order response.

All definitions are also valid for systems of order higher than 2, although analytical expressions for these parameters cannot be found unless the response of the higher-order system can be approximated as a second-order system.

Underdamped Second-Order Systems

Rise time, peak time, and settling time yield information about the speed of the transient response. This information can help a designer determine if the speed and the nature of the response do or do not degrade the performance of the system.

For example, the speed of an entire computer system depends on the time it takes for a hard drive head to reach steady state and read data;

passenger comfort depends in part on the suspension system of a car and the number of oscillations it goes through after hitting a bump.

Underdamped Second-Order Systems: Evaluation of T_p

T_p : is found by differentiating c(t) and finding the first zero crossing after t = 0. This task is simplified by "differentiating" in the frequency domain by differentiation property of Laplace Transform assuming zero initial conditions, we get

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Underdamped Second-Order Systems: Evaluation of %OS

the percent overshoot, %OS, is given by

$$\%OS = \frac{c_{max} - c_{final}}{c_{final}} \times 100 \ldots 1$$

The term c_{max} is found by evaluating $c(t)$ at the peak time, $c(T_p)$.

$$\therefore \%OS = e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)} \times 100 \ldots 2$$

The percent overshoot is a function only of the damping ratio (ζ).

by equation 2 allows we can find %OS for given the damping ratio (ζ). Also we can find the damping ratio (ζ) for given %OS by following equation.

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}} \ldots 3$$

Underdamped Second-Order Systems: Evaluation of T_s

T_s, the settling time, is the time for which c(t) reaches and stays within 2% of the steady-state value, c_{final}.

The settling time is the time it takes for the amplitude of the decaying sinusoid in to reach 0.02,

$$T_s = \frac{4}{\zeta\omega_n}$$

Underdamped Second-Order Systems: Evaluation of T_r

T_r, the rise time, is the time for which c(t) reaches from 0.1 of c_{final} to o.9 of c_{final} for overdamped systems and 0 to 100 % is for underdamped system.

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_n\sqrt{1-\zeta^2}t + \phi\right)$$

where $\phi = tan^{-1}\left(\sqrt{1-\zeta^2}/\zeta\right)$

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_n\sqrt{1-\zeta^2}t + \phi\right) = 1$$

$$\therefore \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_n\sqrt{1-\zeta^2}t + \phi\right) = 0$$

Underdamped Second-Order Systems: Evaluation of T_r

$$\therefore \sin\left(\omega_n\sqrt{1-\zeta^2}t+\phi\right)=0$$

$$\therefore \omega_n\sqrt{1-\zeta^2}t_r+\phi=\pi$$

$$\therefore t_r=\frac{\pi-\phi}{\omega_n\sqrt{1-\zeta^2}}$$

where $\phi = tan^{-1}\left(\sqrt{1-\zeta^2}/\zeta\right)$

Underdamped Second-Order Systems:

Find the find T_p, %OS, T_s, and T_r For the system described by the transfer function

$$G(s) = \frac{100}{s^2 + 15s + 100}$$

Underdamped Second-Order Systems:

Find the find T_p, %OS, T_s, and T_r For the system described by the transfer function

$$G(s) = \frac{100}{s^2 + 15s + 100}$$

$\zeta = 0.75, \omega_n = 10$

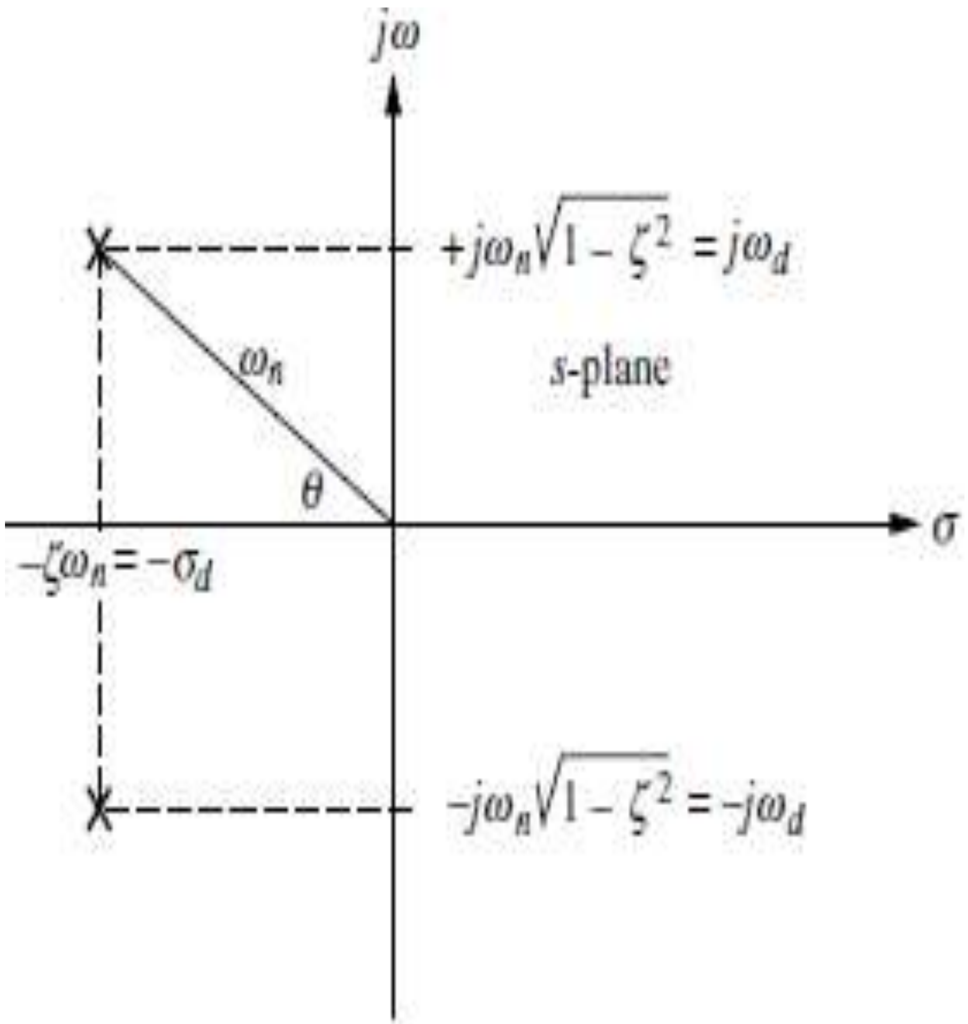
$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.475\ second$$

$$\%OS = e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)} \times 100 = 2.838$$

$$T_s = \frac{4}{\zeta\omega_n} = 0.533\ second$$

$$T_r = \frac{\pi-\phi}{\omega_n\sqrt{1-\zeta^2}} = 0.23\ second$$

Underdamped Second-Order Systems:



Pole plot for general underdamped second order system

The pole plot for a general, underdamped second-order system, shown in Figure.

We see from the Pythagorean theorem that the radial distance from the origin to the pole is the natural frequency ω_n and the $\cos\theta = \zeta$.

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d} \quad \ldots 1$$

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma_d} \quad \ldots 2$$

Underdamped Second-Order Systems:

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d} \quad \ldots 1$$

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma_d} \quad \ldots 2$$

where ω_d is the imaginary part of the pole and is called the damped frequency of oscillation, and

σ_d is the magnitude of the real part of the pole and is the exponential damping frequency.

Underdamped Second-Order Systems:

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d} \dots 1$$

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma_d} \dots 2$$

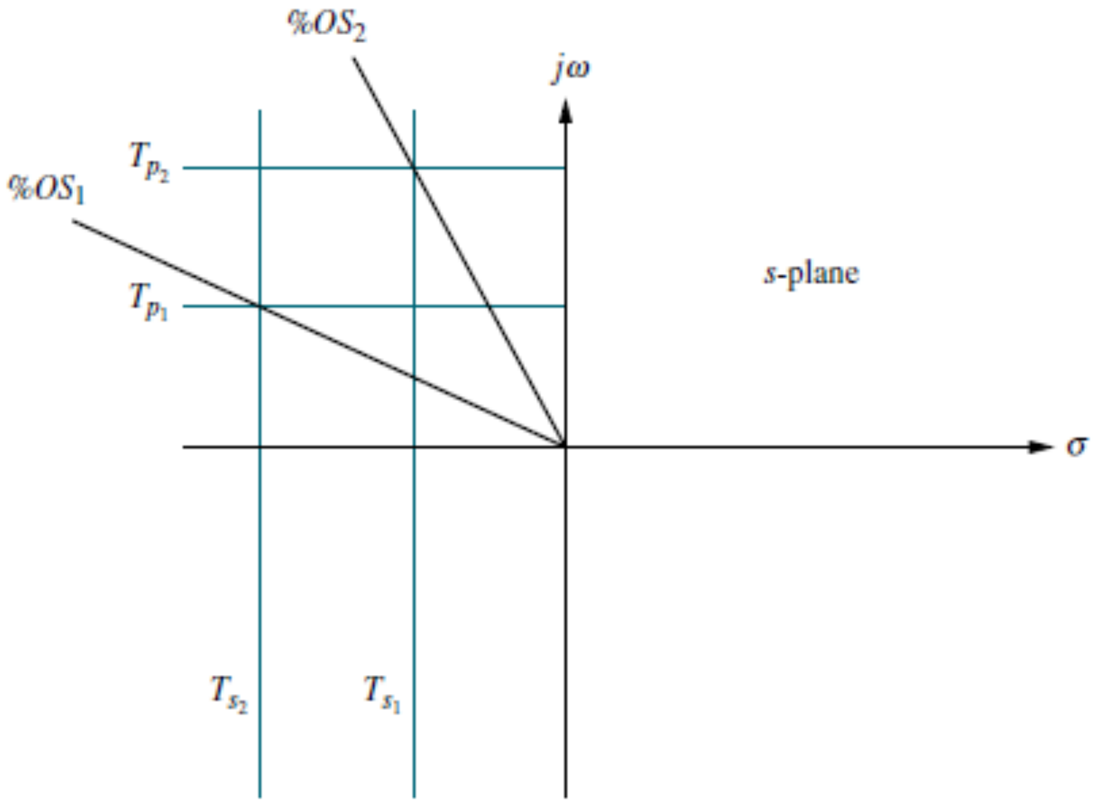
Equation (1) shows that T_p is inversely proportional to the imaginary part of the pole.

Since horizontal lines on the s-plane are lines of constant imaginary value, they are also lines of constant peak time.

Similarly, Eq. (2) tells us that T_s settling time is inversely proportional to the real part of the pole.

Since vertical lines on the s-plane are lines of constant real value, they are also lines of constant settling time.

Underdamped Second-Order Systems:

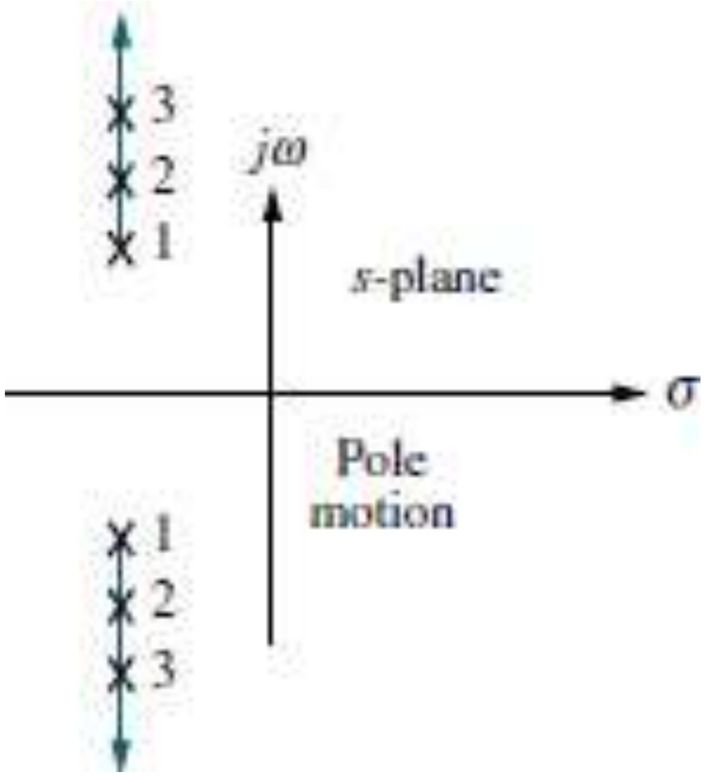


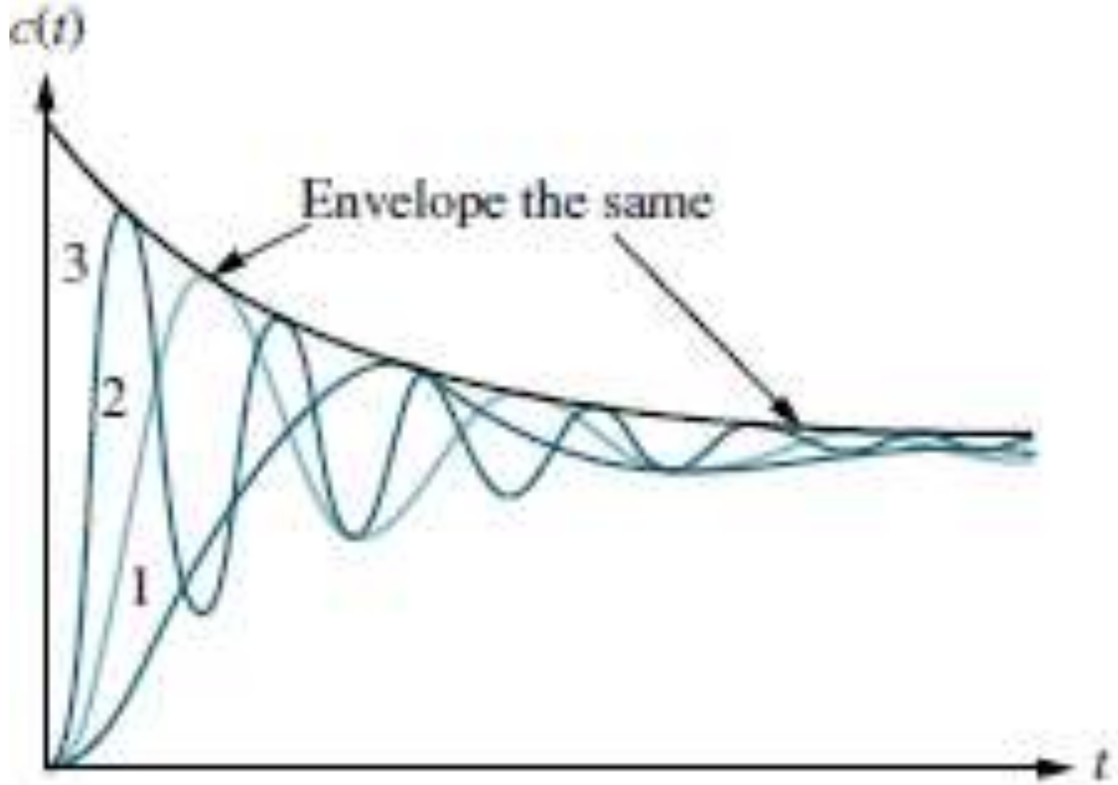
Lines of constant peak time, T_p, settling time, T_s, and percent overshoot, %OS.
Note: $T_{s2} < T_{s1}$;
$T_{p2} < T_{p1}$;
%OS1 < %OS2

Finally, since $\zeta = \cos\theta$, radial lines are lines of constant ζ. Since percent overshoot is only a function of ζ, radial lines are thus lines of constant percent overshoot, %OS.

The lines of constant Tp, Ts, and %OS are labeled on the s-plane.

Underdamped Second-Order Systems:





the poles are moved in a vertical direction, keeping the real part the same.

As the poles move in a vertical direction, the frequency increases, but the envelope remains the same since the real part of the pole is not changing.

Underdamped Second-Order Systems:

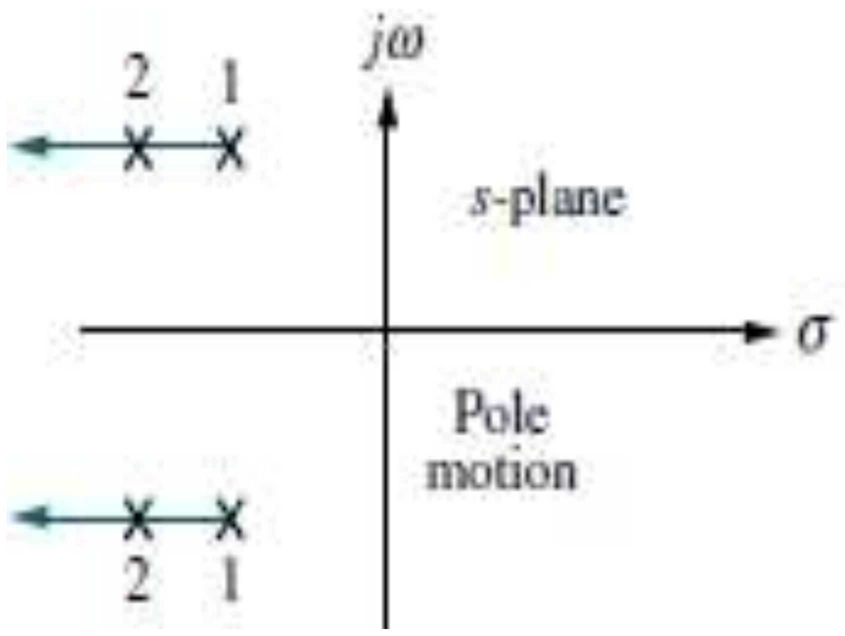
Figure shows the step responses as the poles are moved in a vertical direction, keeping the real part the same.

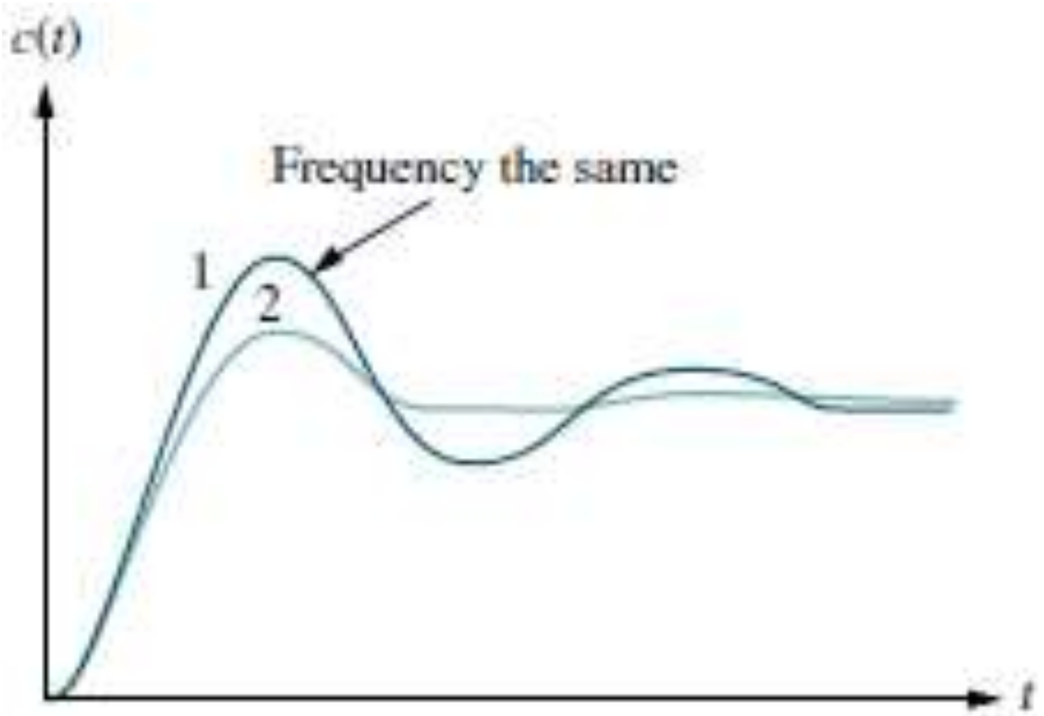
As the poles move in a vertical direction, the frequency increases, but the envelope remains the same since the real part of the pole is not changing.

The figure shows a constant exponential envelope, even though the sinusoidal response is changing frequency.

Since all curves fit under the same exponential decay curve, the settling time is virtually the same for all waveforms. Note that as overshoot increases, the rise time decreases

Underdamped Second-Order Systems:





the poles are moved in a horizontal direction, keeping the imaginary part the same.

As the poles move to the left, the response damps out more rapidly, while the frequency remains the same.

the peak time is the same for all waveforms because the imaginary part remains the same.

Underdamped Second-Order Systems:

Now move the poles to the right or left.

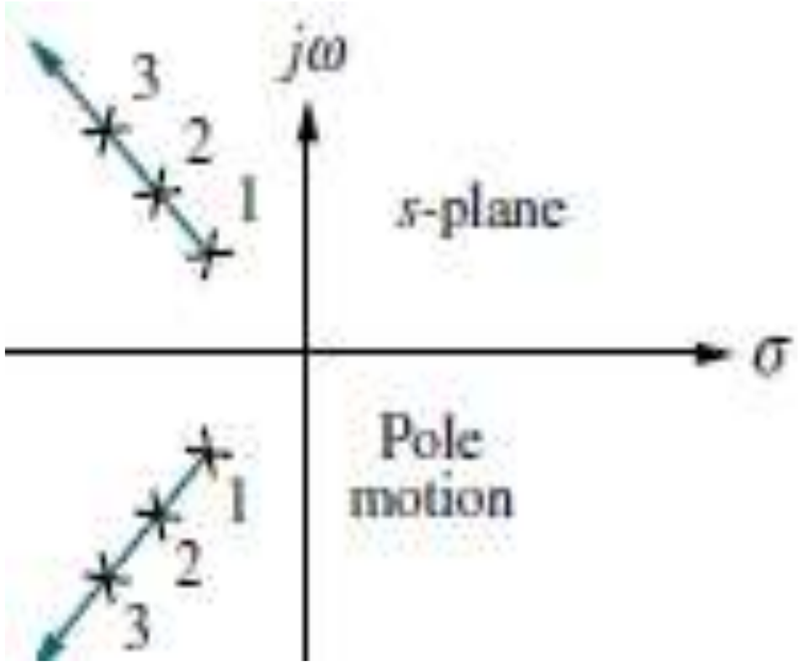
Since the imaginary part is now constant, movement of the poles yields the responses of as shown in figure.

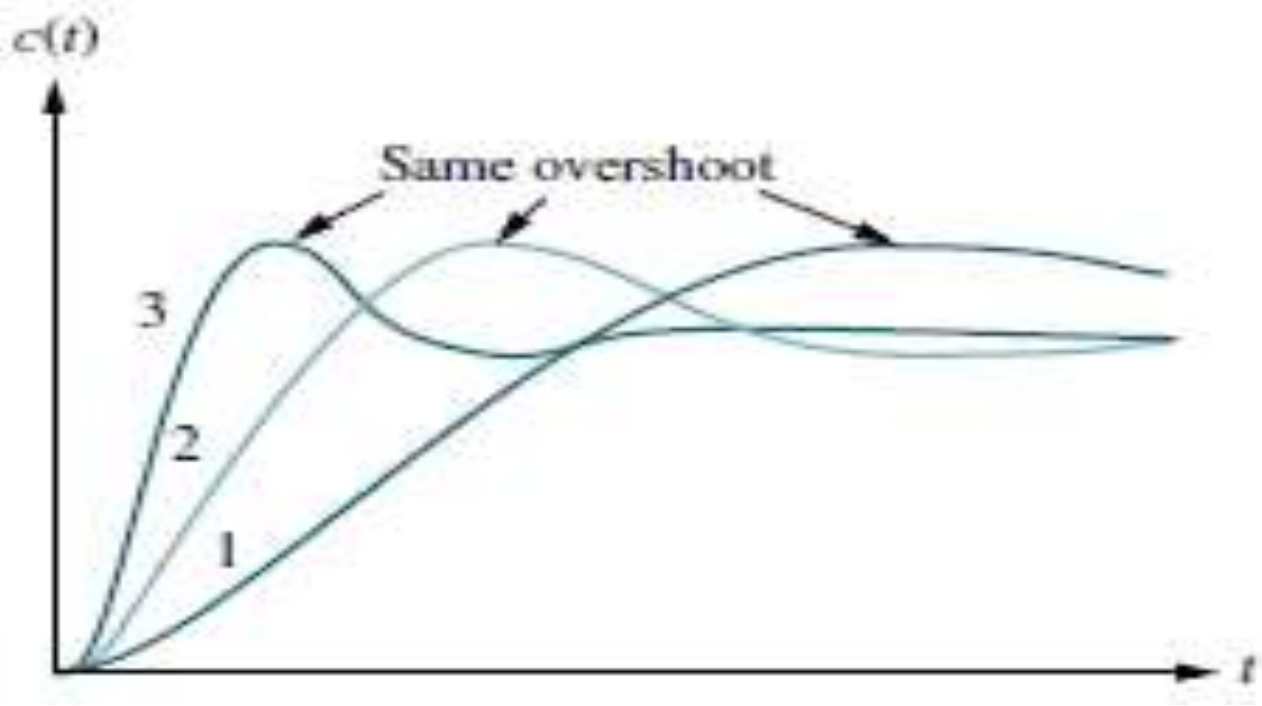
Here the frequency is constant over the range of variation of the real part.

As the poles move to the left, the response damps out more rapidly, while the frequency remains the same.

Notice that the peak time is the same for all waveforms because the imaginary part remains the same.

Underdamped Second-Order Systems:





the poles are moved in a along a constant radial line.

The responses look exactly alike, except for their speed.

The farther the poles are from the origin, the more rapid the response.

Underdamped Second-Order Systems:

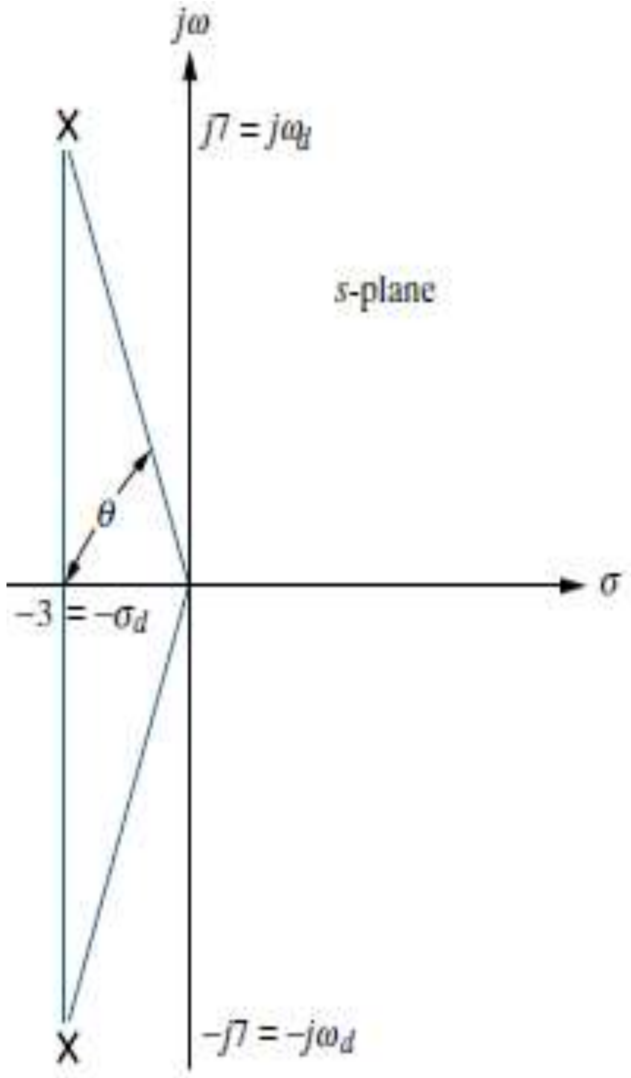
Now move the poles along a constant radial line yields the responses shown in figure.

Here the percent overshoot remains the same.

Notice also that the responses look exactly alike, except for their speed.

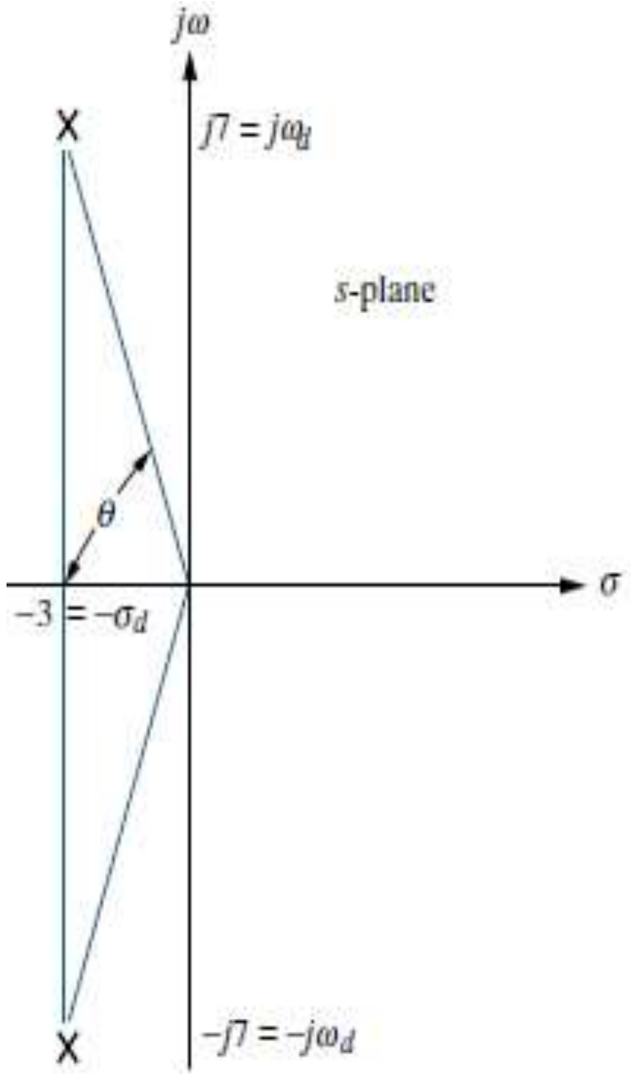
The farther the poles are from the origin, the more rapid the response.

Underdamped Second-Order Systems:



For the given the pole plot, find ζ, ω_n, Tp, %OS and Ts.

Underdamped Second-Order Systems:



The damping ratio is given by $\zeta = \cos\theta = 0.394$.

The natural frequency, ω_n, is the radial distance from the origin to the pole, or $\omega_n = \sqrt{7^2 + 3^2} = 7.616$

The peak time is

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{7} = 0.449\ seconds$$

The percent overshoot is

$$\%OS = e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)} \times 100 = 26\%$$

The approximate settling time is

$$T_s = \frac{4}{\sigma_d} = \frac{4}{3} = 1.333\ seconds$$

Underdamped Second-Order Systems:

Find $\zeta, \omega_n, Ts, Tp, Tr\ and\ \%Os$ for a system whose transfer function is

$$G(s) = \frac{361}{s^2 + 16s + 361}$$

Underdamped Second-Order Systems:

Find $\zeta, \omega_n, Ts, Tp, Tr\ and\ \%Os$ for a system whose transfer function is

$$G(s) = \frac{361}{s^2 + 16s + 361}$$

$\zeta = 0.421, \omega_n = 19, Ts = 0.5\ s, Tp = 0.182\ s, Tr = 0.079\ s,$ and $\%OS = 23.3\%$

Underdamped Second-Order Systems:

Find $\zeta, \omega_n, Ts, Tp, Tr\ and\ \%Os$ for a system whose transfer function is

$$G(s) = \frac{361}{s^2 + 16s + 361}$$

$\zeta = 0.421, \omega_n = 19, Ts = 0.5\ s, Tp = 0.182\ s, Tr = 0.079\ s,$ and $\%OS = 23.3\%$

System Response with Additional Poles

Consider a three-pole system with complex poles and a third pole on the real axis.

Assuming that the complex poles are at $s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}$ the real pole is at α_r, the step response of the system can be determined from a partial-fraction expansion. Thus, the output transform is

$$C(s) = \frac{A}{s} + \frac{B(s+\zeta\omega_n) + C\omega_d}{(s+\zeta\omega_n)^2 + \omega_d^2} + \frac{D}{s+\alpha_r} \quad \ldots 1$$

And the output response is

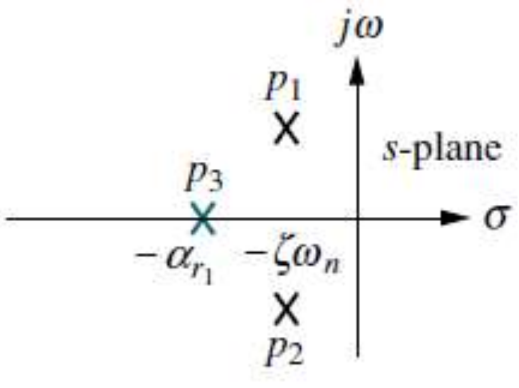
$$c(t) = Au(t) + e^{-\zeta\omega_n t}(B\cos\omega_d t + C\sin\omega_d t) + De^{-\alpha_r t} \quad \ldots 2$$

Consider the three cases of α_r.

Case I, $\alpha_r = \alpha_{r1}$ and is not much larger than $\zeta\omega_n$; for Case II, $\alpha_r = \alpha_{r2}$ and is much larger than $\zeta\omega_n$; and for Case III, $\alpha_r = \infty$.

System Response with Additional Poles

Case I, $\alpha_r = \alpha_{r1}$ and is not much larger than $\zeta\omega_n$



Case I

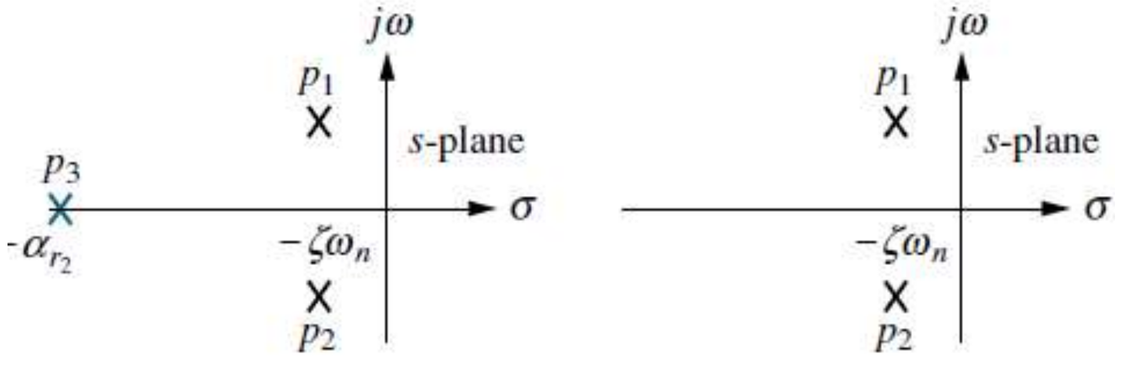
The real pole's transient response will not decay to insignificance at the peak time or settling time generated by the second-order pair.

In this case, the exponential decay is significant, and the system cannot be represented as a second-order system.

System Response with Additional Poles

Case II, $\alpha_r = \alpha_{r2}$ and is much larger than $\zeta\omega_n$ and

Case III, $\alpha_r = \infty$



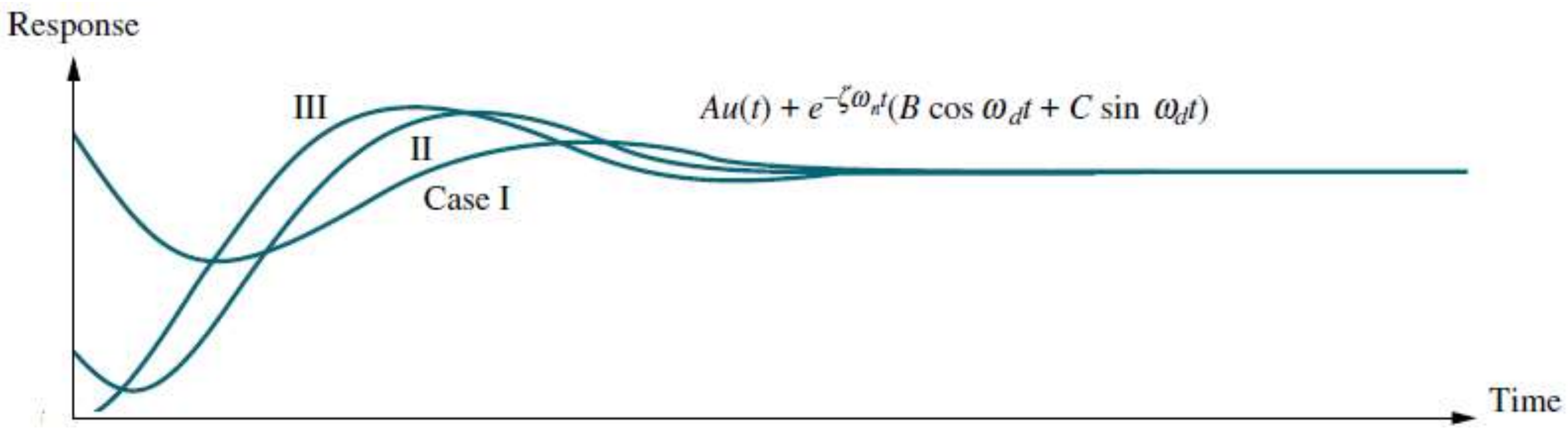
Case II Case III

The pure exponential will die out much more rapidly than the second-order underdamped step response.

If the pure exponential term decays to an insignificant value at the time of the first overshoot, such parameters as percent overshoot, settling time, and peak time will be generated by the second-order underdamped step response component.

Thus, the total response will approach that of a pure second-order system.

System Response with Additional Poles



component responses: Nondominant pole is near dominant second-order pair (Case I), Far from the pair (Case II), and at infinity (Case III).

System Response with Additional Poles

In a three-pole system with dominant second-order poles and no zeros, will actually decrease in magnitude as the third pole is moved farther into the left half-plane.

Assume a step response, C(s), of a three-pole system

$$C(s) = \frac{bc}{s(s^2+as+b)(s+c)} = \frac{A}{s} + \frac{Bs+C}{(s^2+as+b)} + \frac{D}{s+c}$$

here we assume that the nondominant pole is located at c on the real axis and that the steady-state response approaches unity. Evaluating the constants in the numerator of each term

$$A = 1; \quad B = \frac{ca - c^2}{c^2 + b - ca}; \quad C = \frac{ca^2 - c^2a - bc}{c^2 + b - ca}; \quad \text{and } D = \frac{-b}{c^2 + b - ca}$$

System Response with Additional Poles

In a three-pole system with dominant second-order poles and no zeros, will actually decrease in magnitude as the third pole is moved farther into the left half-plane.

Assume a step response, C(s), of a three-pole system

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here we assume that the nondominant pole is located at c on the real axis and that the steady-state response approaches unity. Evaluating the constants in the numerator of each term

$A = 1;\quad B = \dfrac{ca - c^2}{c^2 + b - ca};\quad C = \dfrac{ca^2 - c^2a - bc}{c^2 + b - ca};\quad$ and $D = \dfrac{-b}{c^2 + b - ca}$

System Response with Additional Poles

$$A = 1; \quad B = \frac{ca - c^2}{c^2 + b - ca}; \quad C = \frac{ca^2 - c^2 a - bc}{c^2 + b - ca}; \quad \text{and } D = \frac{-b}{c^2 + b - ca}$$

As the nondominant pole approaches ∞; or $c \rightarrow \infty$,

$A = 1; B = -1; C = -a;$ and $D = 0.$

Thus, for this example, D, the residue of the nondominant pole and its response, becomes zero as the nondominant pole approaches infinity.

System Response with Zero

We know that the zeros of a response affect the residue, or amplitude, of a response component but do not affect the nature of the response - exponential, damped sinusoid.

The zero will be added first in the left half-plane

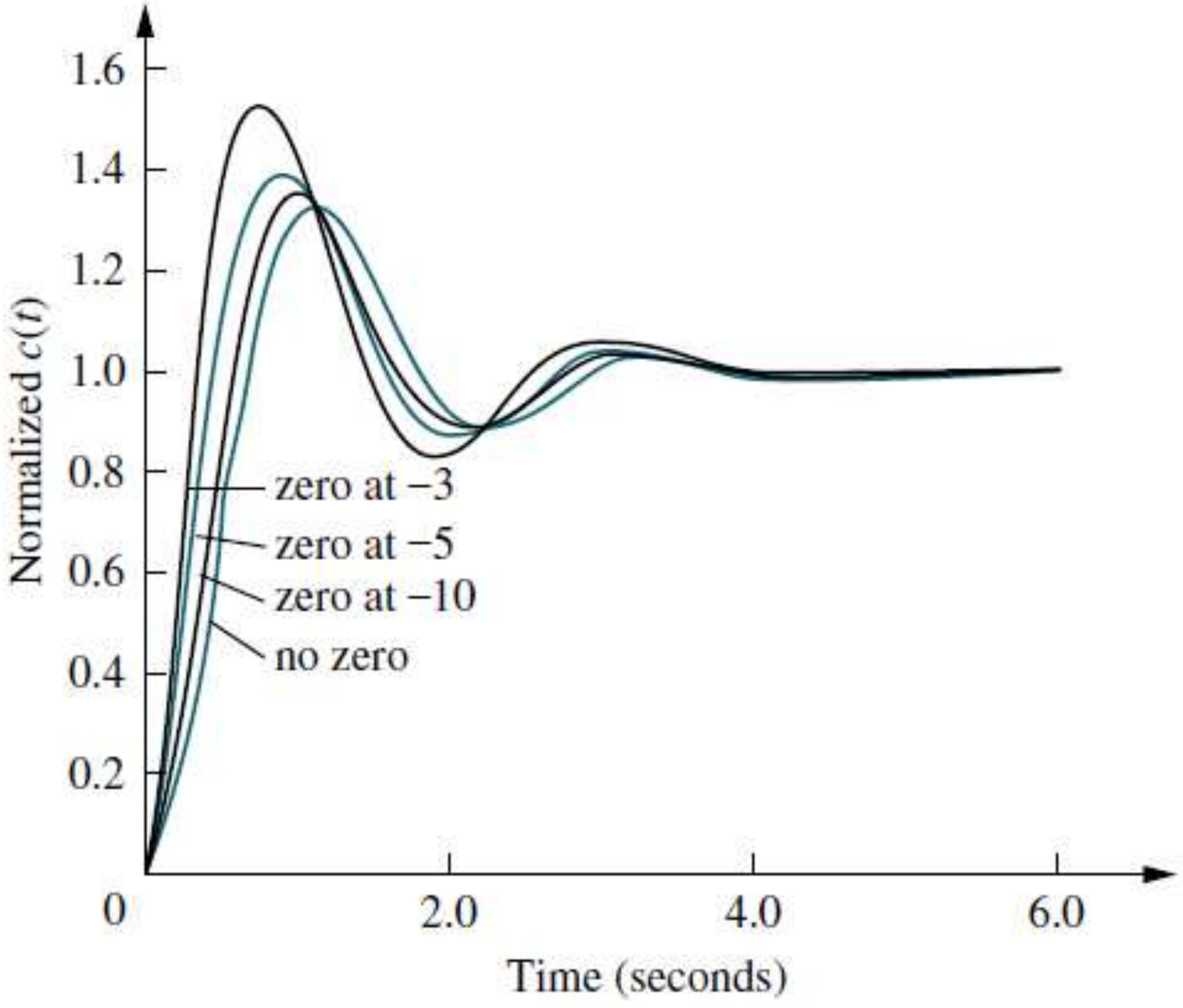
Starting with a two-pole system with poles at $-1 \pm j2.828$, we consecutively add zeros at $-3, -5$ and -10.

The results, normalized to the steady-state value, are plotted in Figure.

the closer the zero is to the dominant poles, the greater its effect on the transient response.

As the zero moves away from the dominant poles, the response approaches that of the two-pole system.

System Response with Zero



Effect of adding a zero to a two-pole system

System Response with Zero

Let's assume a system describes by the transfer function

$$G(s) = \frac{s+a}{(s+b)(s+c)} = \frac{A}{s+b} + \frac{B}{s+c}$$

$$= \frac{(-b+a)/(-b+c)}{s+b} + \frac{(-c+a)/(-c+b)}{s+c}$$

If the zero is far from the poles, then a is large compared to b and c.

$$G(s) \approx a\left[\frac{\frac{1}{-b+c}}{s+b} + \frac{\frac{1}{-c+b}}{s+c}\right] = \frac{a}{(s+b)(s+c)}$$

Hence, the zero looks like a simple gain factor and does not change the relative amplitudes of the components of the response.

System Response with Zero

Let $C(s)$ be the response of a system with unity in the numerator and $G(s)$be the transfer function of the system

$$\therefore C(s) = G(s)R(s) = \frac{1}{D(s)}$$

If we add a zero to the transfer function i.e. now $\mathrm{G}(s) = (s+a)G(s)$

Now the Laplace transform response is $(\mathrm{s}+\mathrm{a})C(s)$

$$\therefore (\mathrm{s}+\mathrm{a})C(s) = sC(s) + aC(s)$$

Thus, the response of a system with a zero consists of two parts:

1. the derivative of the original response $sC(s)$ and

2. a scaled version of the original response $aC(s)$.

System Response with Zero

$(s + a)C(s) = sC(s) + aC(s)$

If a is very large, the Laplace transform of the response is approximately $aC(s)$, or a scaled version of the original response.

$$\therefore (s + a)C(s) \approx aC(s)$$

If a is not very large, the response has an additional component consisting of the derivative of the original response.

As a becomes smaller, the derivative term contributes more to the response and has a greater effect. For step responses, the derivative is typically positive at the start of a step response. Thus, for small values of a, we can expect more overshoot in second order systems because the derivative term will be additive around the first overshoot.

System Response with Zero

An interesting phenomenon occurs if a is negative, i.e. placing the zero in the right half-plane.

In the equation $(s + a)C(s) = sC(s) + aC(s)$ the derivative term, which is typically positive initially, will be of opposite sign from the scaled response term.

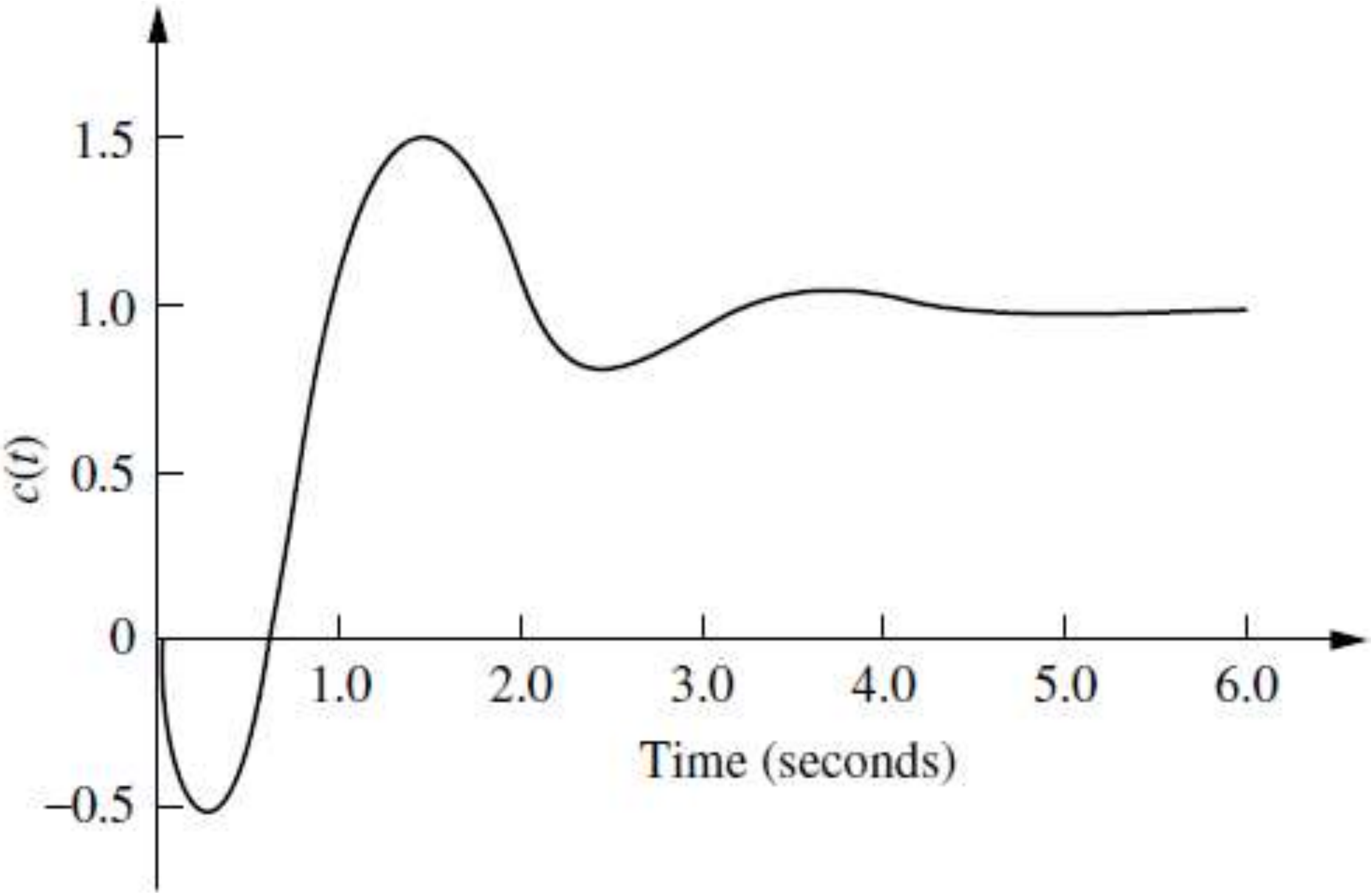
Thus, if the derivative term, $sC(s)$, is larger than the scaled response, $aC(s)$, the response will initially follow the derivative in the opposite direction from the scaled response. The result for a second-order system is shown in Figure.

where the sign of the input was reversed to yield a positive steady-state value.

Notice that the response begins to turn toward the negative direction even though the final value is positive.

A system that exhibits this phenomenon is known as a **nonminimum-phase** system.

System Response with Zero



Step response of a nonminimum-phase system

pole-zero cancellation

Assume a three pole system with a zero as represented by following equation.

$$G(s) = \frac{K(s+z)}{(s+p_3)(s^2+as+b)} \ldots 1$$

If the pole term, $(\mathrm{s}+\mathrm{p}_3)$, and the zero term, $(\mathrm{s}+\mathrm{z})$, cancel out, we get

$$G(s) = \frac{K}{(s^2+as+b)}$$

as a second-order transfer function.

From another perspective, if the zero at z is very close to the pole at p_3, then a partial-fraction expansion of Eq. 1 will show that the residue of the exponential decay is much smaller than the amplitude of the second-order response.

pole-zero cancellation

Determine whether there is cancellation between the zero and the pole closest to the zero. For any function for which pole-zero cancellation is valid, find the approximate response

$$C_1(s) = \frac{26.25(s+4)}{s(s+4.01)\,(s+5)(s+6)} \quad \ldots 1$$

If the partial fraction expansion for $C_1(s)$ is

$$C_1(s) = \frac{0.87}{s} - \frac{5.3}{s+5} + \frac{4.4}{s+6} + \frac{0.033}{s+4.01}$$

Here the pole at $s = -4.01$ is closer to zero at $s = -4$. The residue of the pole at $s = -4.01$ is equal to 0.033. It is too low as compared to other residues. Hence, we make a second-order approximation by neglecting the response generated by the pole at $s = -4.01$.

pole-zero cancellation

$$C_1(s) \approx \frac{0.87}{s} - \frac{5.3}{s+5} + \frac{4.4}{s+6}$$

and the output response is

$$c_1(t) \approx 0.87 - 5.3e^{-5t} + 4.4e^{-6t}$$

pole-zero cancellation

Determine whether there is cancellation between the zero and the pole closest to the zero. For any function for which pole-zero cancellation is valid, find the approximate response

$$C_2(s) = \frac{26.25(s+4)}{s(s+3.5)\,(s+5)(s+6)} \quad \dots 1$$

If the partial fraction expansion for $C_1(s)$ is

$$C_2(s) = \frac{1}{s} - \frac{3.5}{s+5} + \frac{3.5}{s+6} - \frac{1}{s+3.5}$$

Here the pole at $s = -3.5$ is closer to zero at $s = -4$. The residue of the pole at $s = -4.01$ is equal to 1. It is not negligible as compared to other residues. Hence, we can not make a second-order approximation by neglecting the response generated by the pole at $s = 3.5$.

pole-zero cancellation

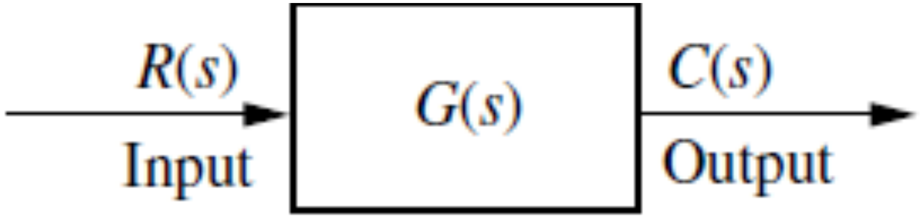
and the output response is

$$c_1(t) = 1 - 3.5\,e^{-5t} + 3.5\,e^{-6t} - e^{-3.5t}$$

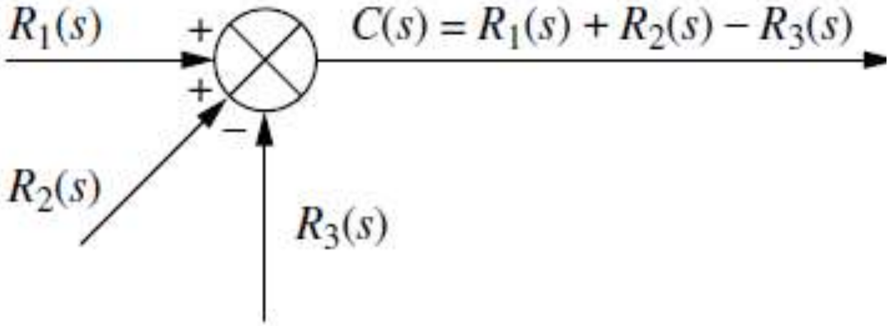
Feedback Control System
Unit 2
Block Diagrams and Signal flow graphs

Components of a block diagram for a LTI system

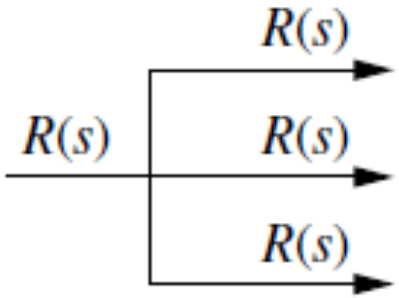
Signals



system



Summing junction



pickoff point

Cascade Form

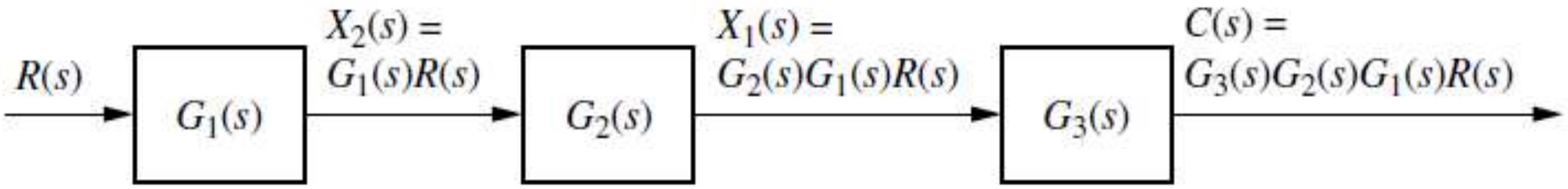
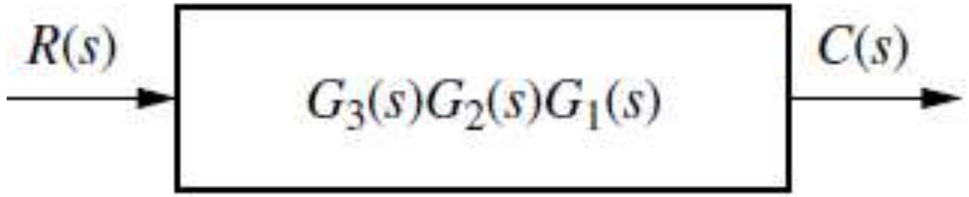


Figure shows a cascaded subsystems. Intermediate signal values are shown at the output of each subsystem. Each signal is derived from the product of the input times the transfer function.



The equivalent transfer function, $G_e(s) = G_1(s)G_2(s)G_3(s)$ ... 1,

is the output Laplace transform divided by the input Laplace which is the product of the subsystems' transfer functions.

Cascade Form

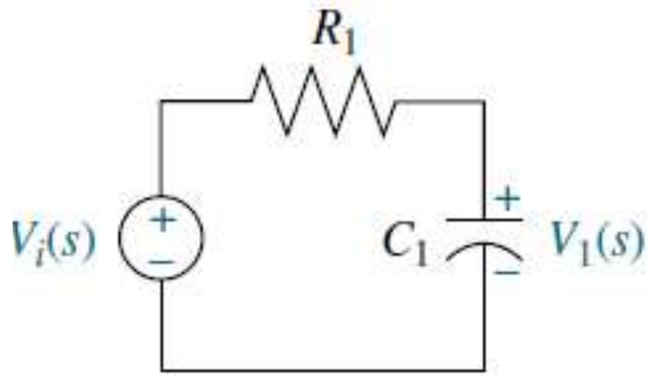
Equation (1) was derived under the assumption that interconnected sub systems do not load adjacent subsystems. That is,

a subsystem's output remains the same whether the subsequent subsystem is connected or not.

If there is a change in the output the subsequent subsystem loads the previous subsystem, and the equivalent transfer function is not the product of the individual transfer functions.

Cascade Form

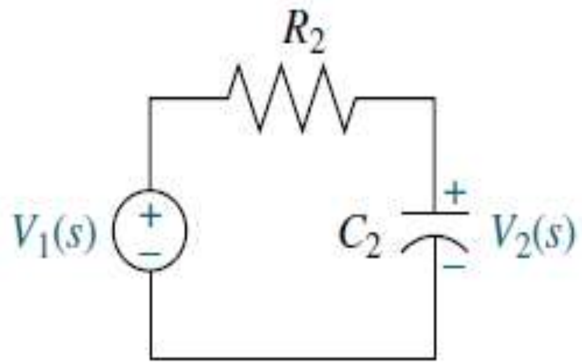
1



the transfer function of the first RC circuit is

$$G_1(s) = \frac{V_1(s)}{V_i(s)} = \frac{1/(R_1C_1)}{s + (1/(R_1C_1)}$$

2



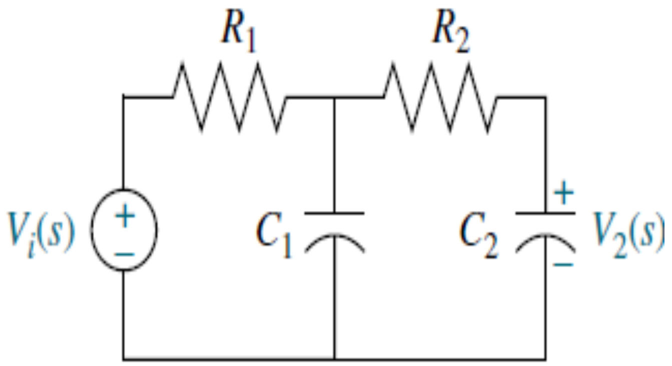
the transfer function of the second RC circuit is

$$G_2(s) = \frac{V_2(s)}{V_1(s)} = \frac{1/(R_2C_2)}{s + (1/(R_2C_2)}$$

If we couple both the circuits to have cascaded combination, then we get

Cascade Form

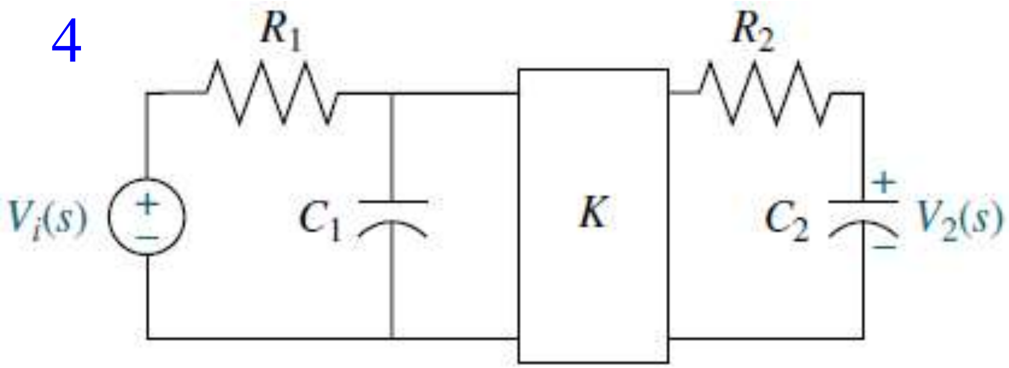
3



the transfer function of the cascaded combination of first and second RC circuits is

$G_3(s) \neq G_2(s)G_1(s)$

4



the transfer function of the cascaded combination of first and second RC circuits with an amplifier as a coupler

$G_4(s) = KG_2(s)G_1(s)$

Cascade Form

To prevent loading is to use an amplifier between the two networks.

The amplifier has a high-impedance input, so that it does not load the previous network.

At the same time, it has a low-impedance output, so that it looks like a pure voltage source to the subsequent network.

With the amplifier included, the equivalent transfer function is the product of the transfer functions and the gain, K, of the amplifier.

Parallel Form

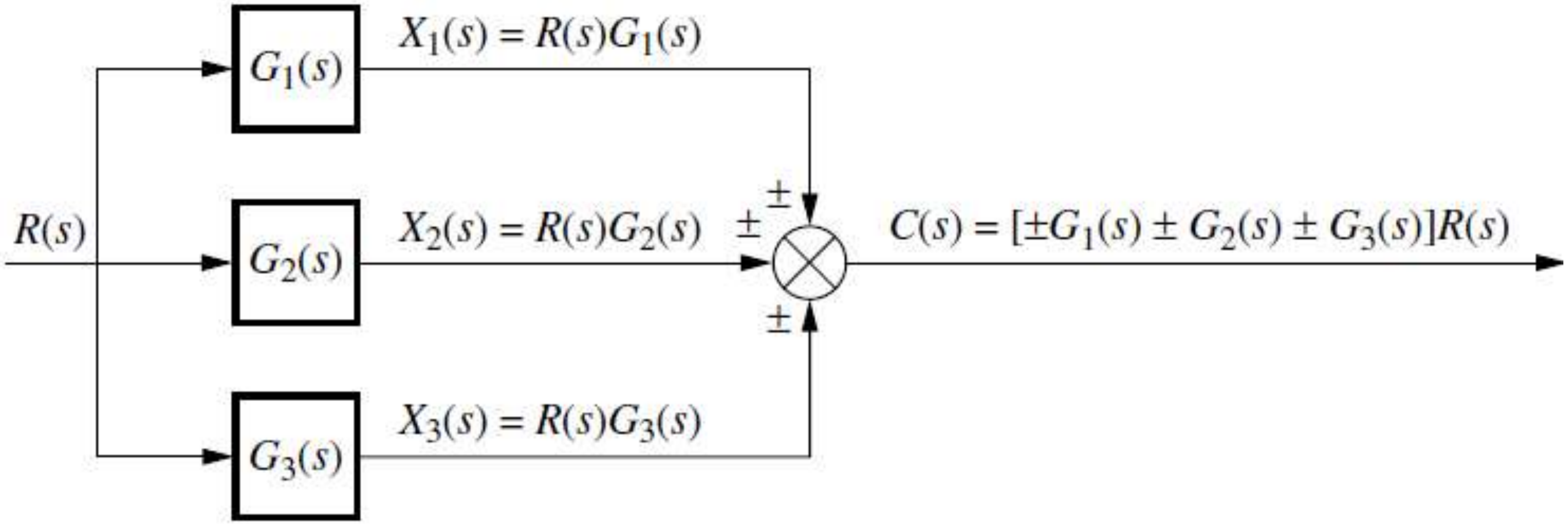
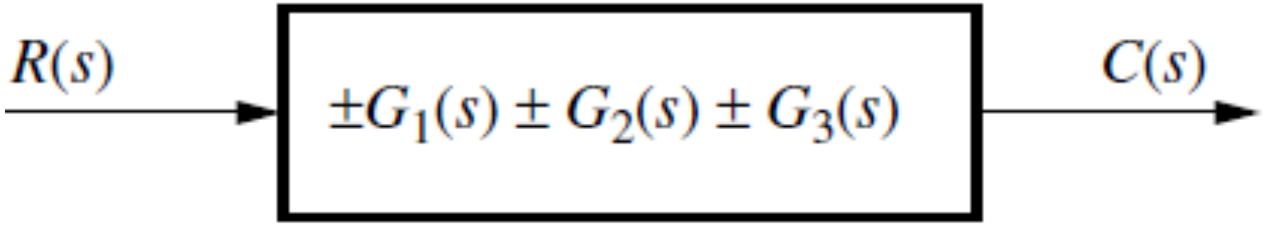


Figure shows an example of parallel subsystems. Again, by writing the output of each subsystem, we can find the equivalent transfer function.

Parallel subsystems have a common input and an output formed by the algebraic sum of the outputs from all of the subsystems.

Parallel Form

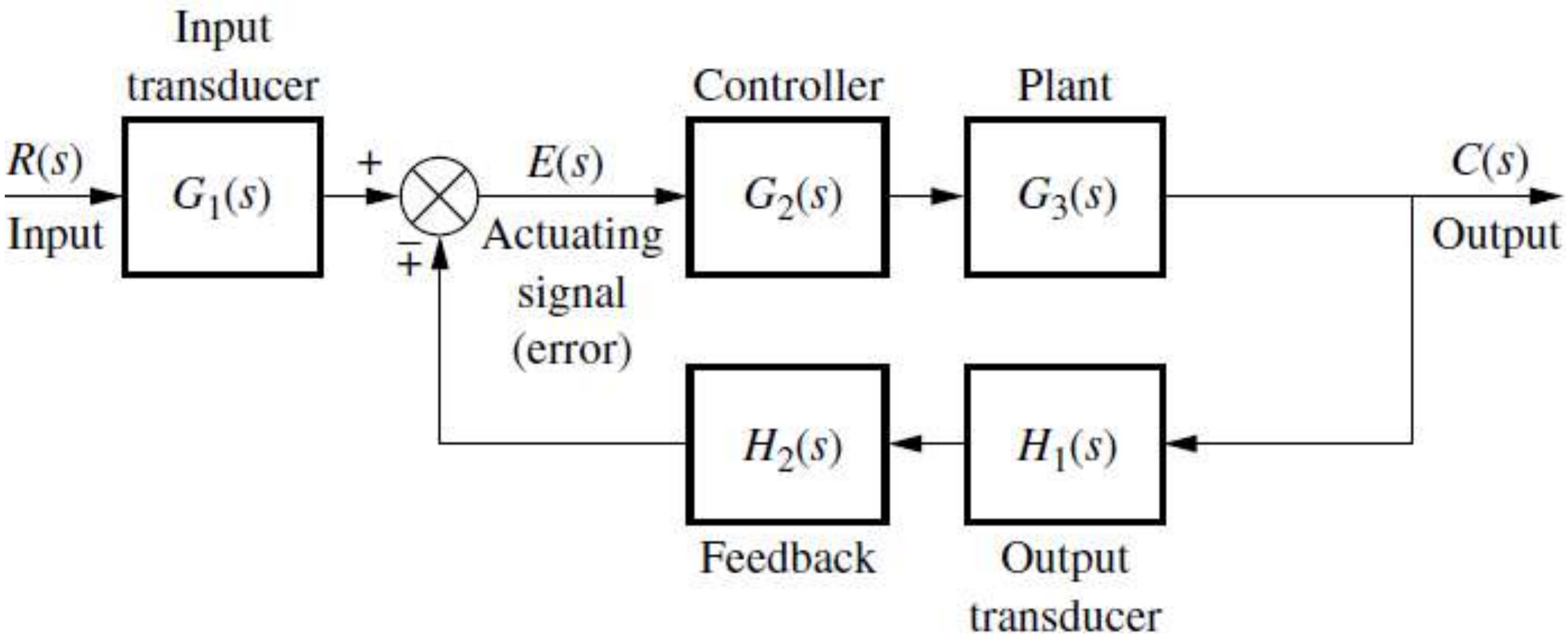


The equivalent transfer function, Ge(s), is the output transform divided by the input transform from Figure

$$G_e(s) = \pm G_1(s) \pm G_2(s) \pm G_3(s)$$

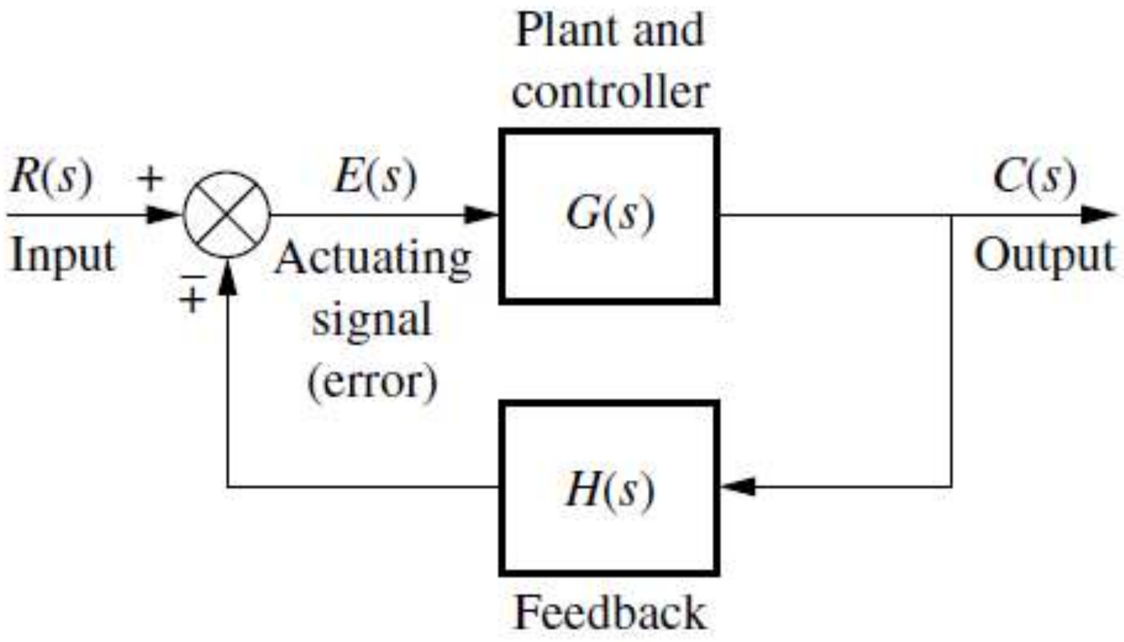
which is the algebraic sum of the subsystems' transfer functions.

Feedback Form

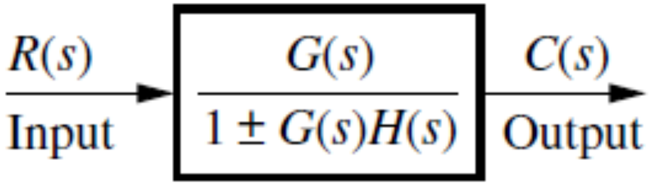


The typical feedback system.

Feedback Form



The simplified model of feedback system.



$E(s) = R(s) \mp C(s)H(s) \ldots 1$

But since $C(s) = E(s)G(s)$

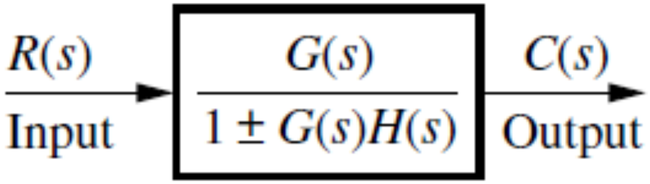
$$E(s) = \frac{C(s)}{G(s)} \ldots 2$$

Substituting Eq. (2) into Eq. (1) and solving

for the transfer function, $\frac{C(s)}{R(s)} = \text{Ge}(s)$

we obtain the equivalent, or closed-loop, transfer function shown in Figure

Feedback Form

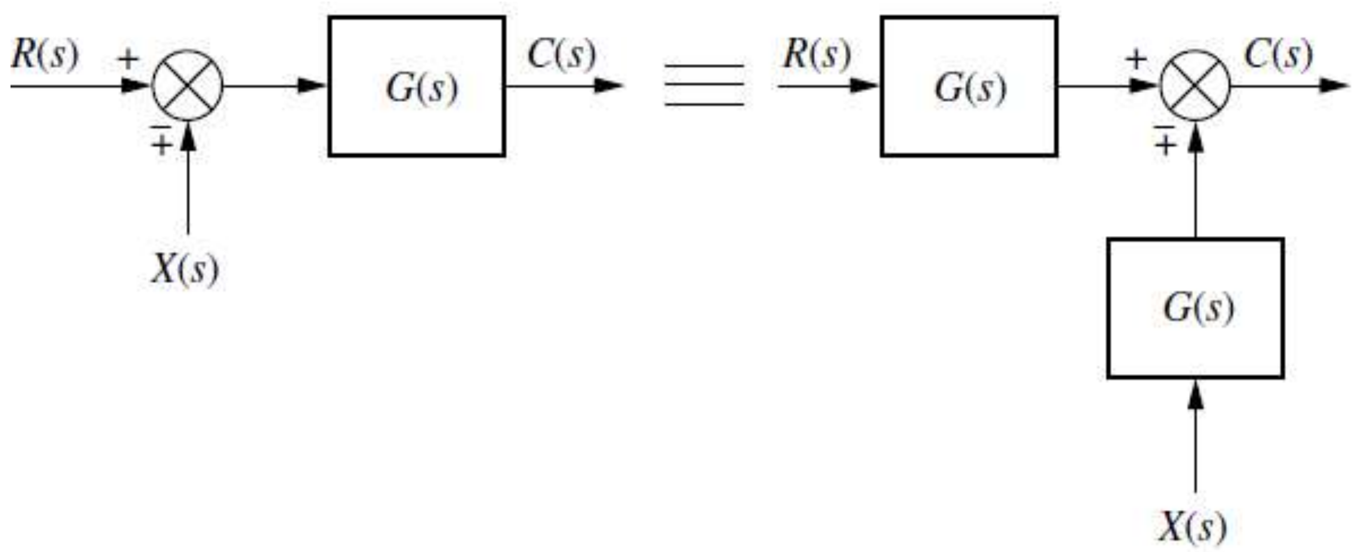


The closed loop transfer function is

$$G_e(s) = \frac{G(s)}{1 \pm G(s)H(s)} \quad \dots 3$$

The product, G(s)H(s) in Eq. (3) is called the open-loop transfer function, or loop gain.

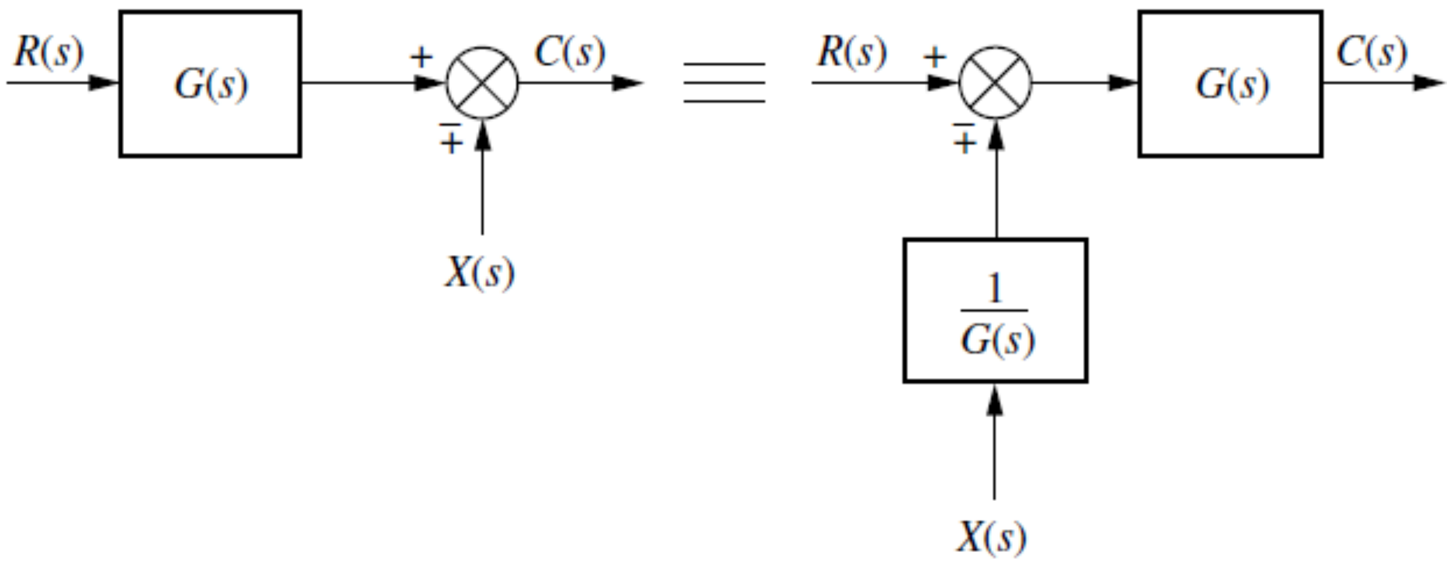
Feedback Form: Block diagram algebra for summing junctions



Block diagram algebra for summing junctions - equivalent forms for moving a block to the left past a summing junction

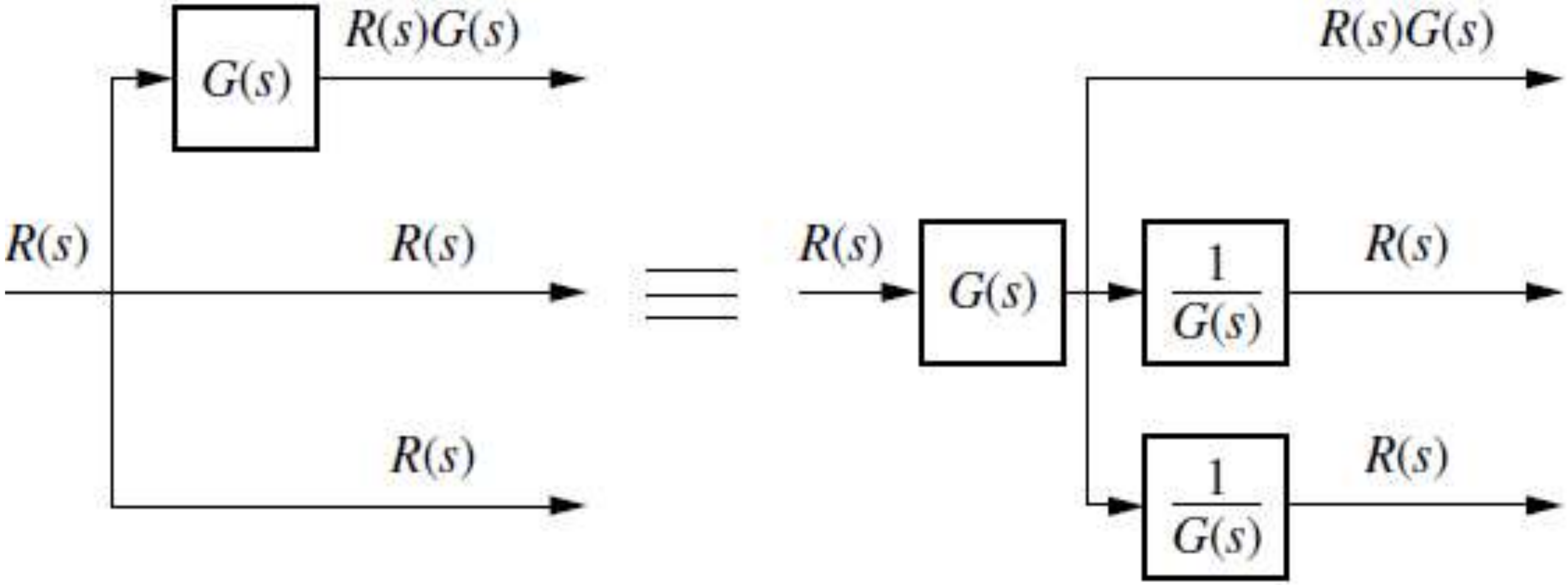
The signals $R(s)$ and $X(s)$ are multiplied by $G(s)$ before reaching the output. Hence, both block diagrams are equivalent, with $C(s) = R(s)G(s) \mp X(s)G(s)$

Feedback Form: Block diagram algebra for summing junctions



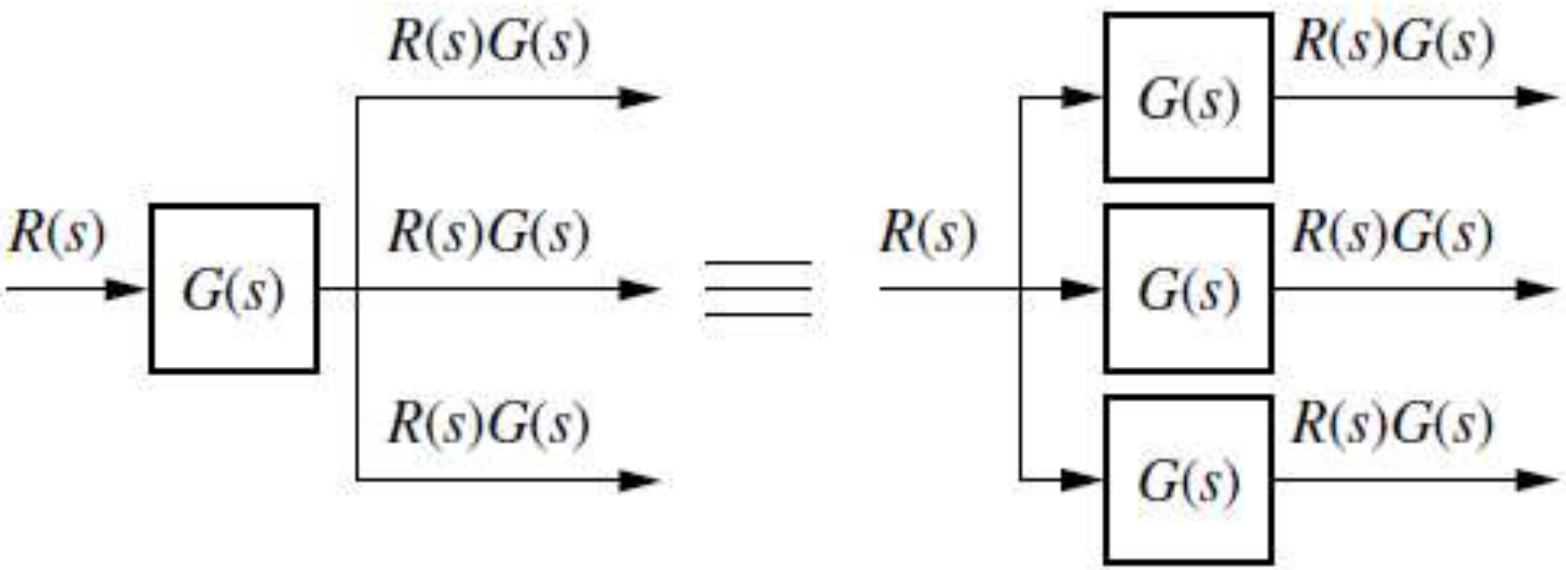
Block diagram algebra for summing junctions - equivalent forms for moving a block to the right past a summing junction. $R(s)$ is multiplied by $G(s)$ before reaching the output, but $X(s)$ is not. Hence, both block diagrams in Figure are equivalent, with $C(s) = R(s)G(s) \mp X(s)$

Feedback Form: Block diagram algebra for pickoff points



Block diagram algebra for pickoff points - equivalent forms for moving a block to the left past a pickoff point

Feedback Form: Block diagram algebra for pickoff points



Block diagram algebra for pickoff points - equivalent forms for moving a block to the right past a pickoff point

Block Diagram Reduction Rules

Follow these rules for simplifying (reducing) the block diagram, which is having many blocks, summing points and take-off points.

Rule 1 – Check for the blocks connected in series and simplify.

Rule 2 – Check for the blocks connected in parallel and simplify.

Rule 3 – Check for the blocks connected in feedback loop and simplify.

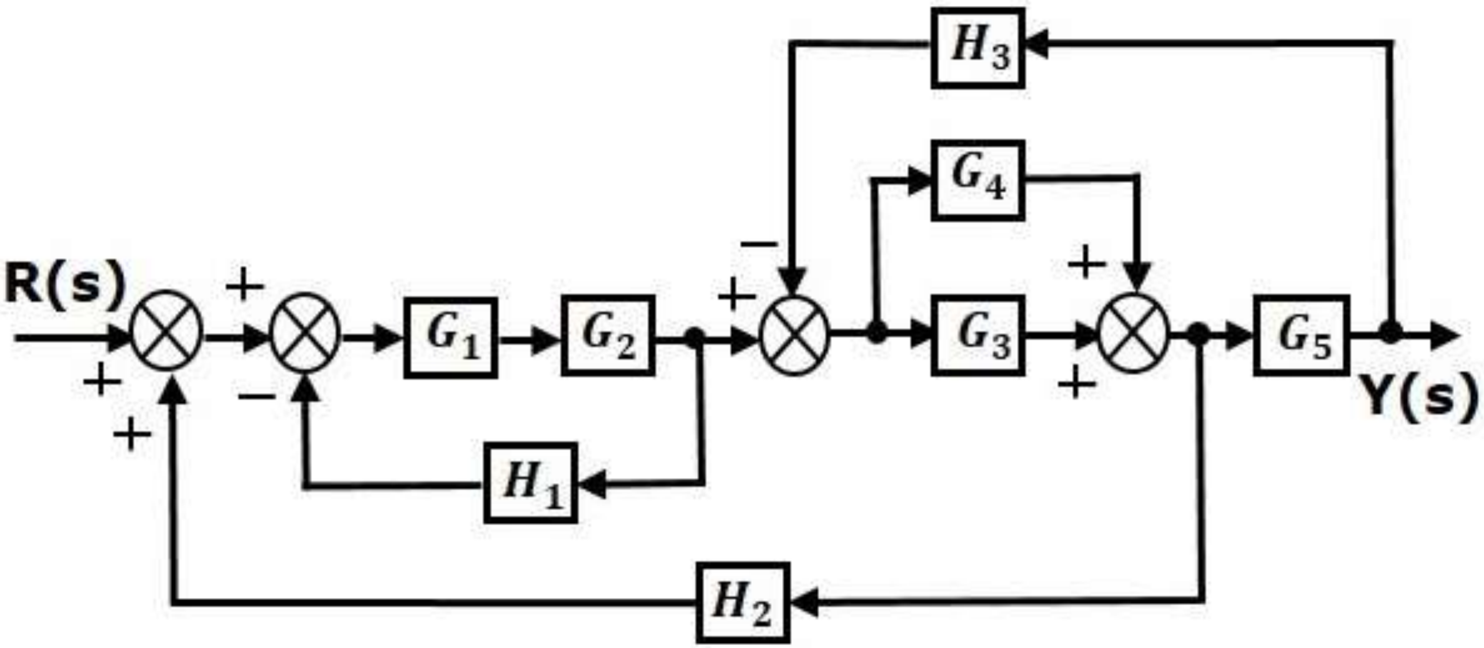
Rule 4 – If there is difficulty with take-off point while simplifying, shift it towards right.

Rule 5 – If there is difficulty with summing point while simplifying, shift it towards left.

Rule 6 – Repeat the above steps till you get the simplified form, i.e., single block.

Block Diagram Reduction Rules

Numerical: Consider the block diagram shown in the following figure. Let us simplify (reduce) this block diagram using the block diagram reduction rules.

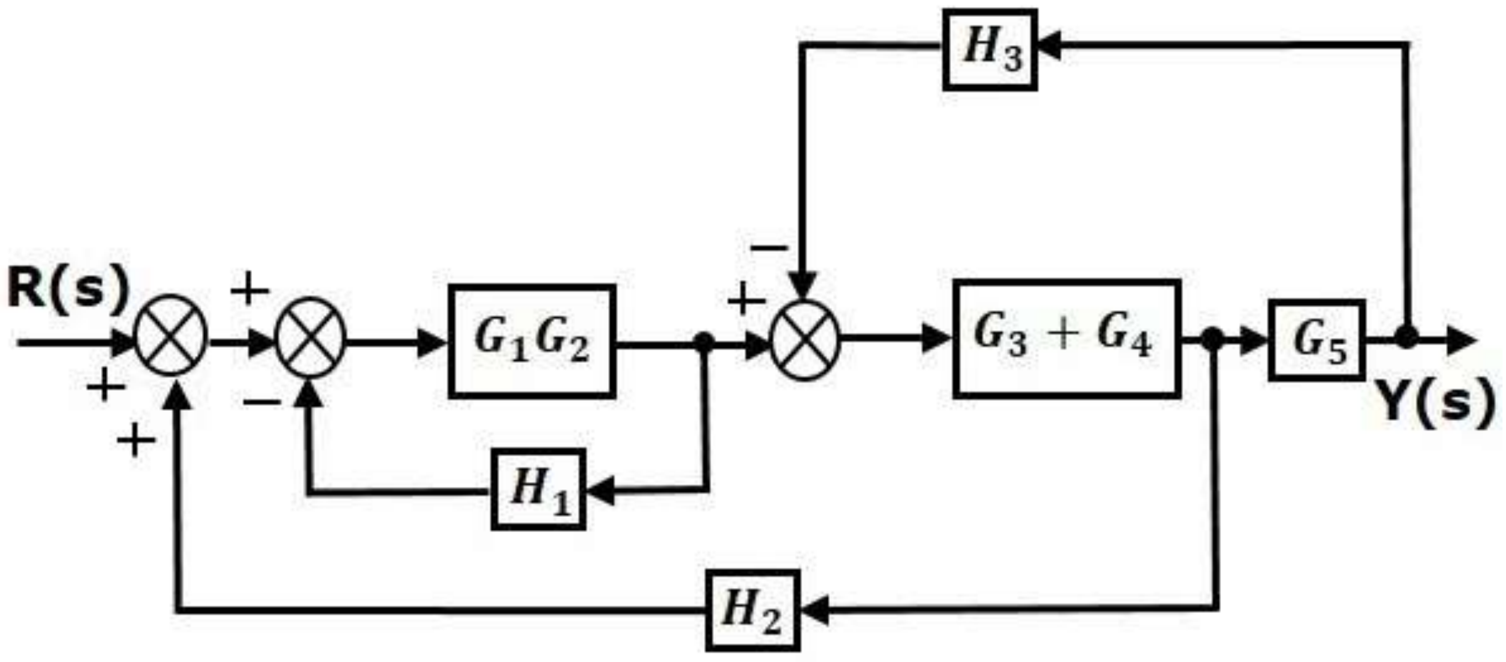


Block Diagram Reduction Rules

Step 1 − Use Rule 1 for blocks G_1 and G_1.

Use Rule 2 for blocks G_3 and G_4.

The modified block diagram is shown in the following figure.

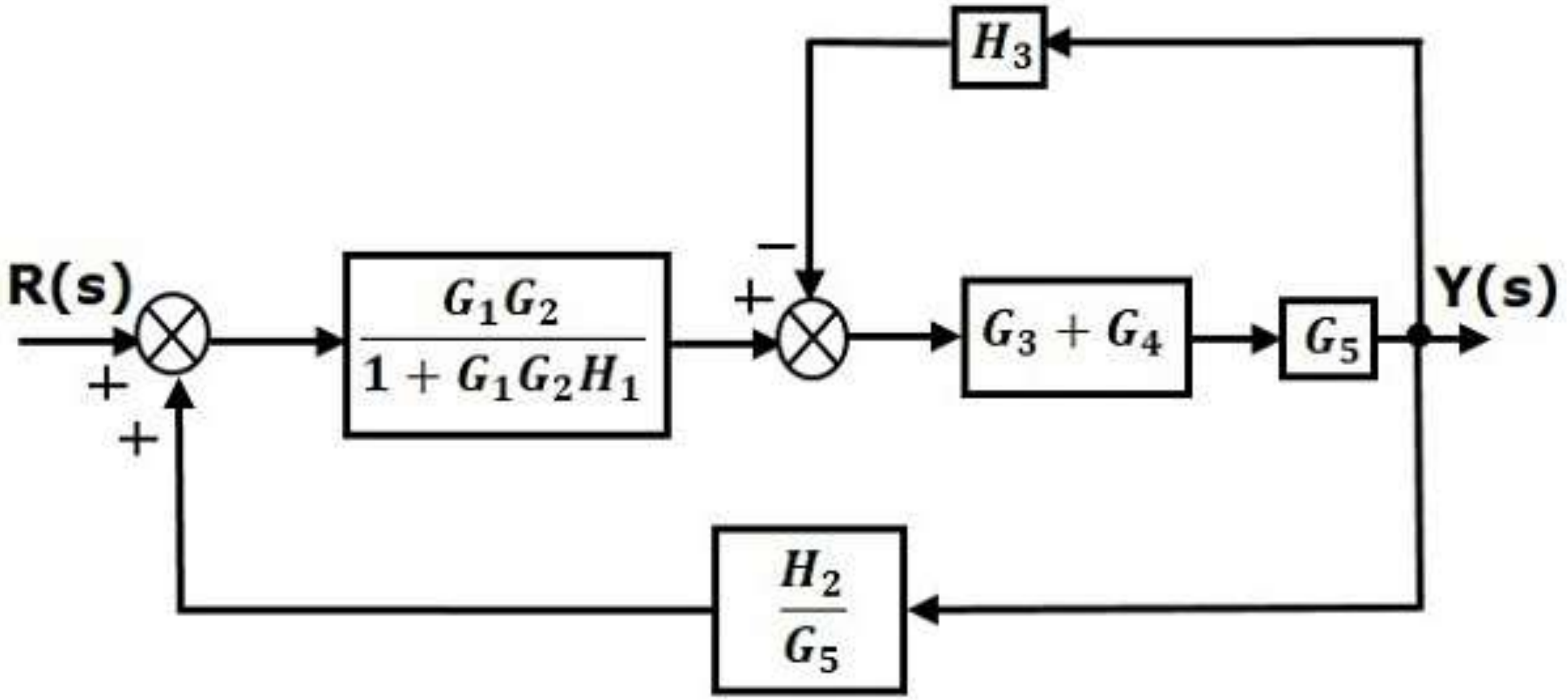


Block Diagram Reduction Rules

Step 2 − Use Rule 3 for blocks G_1G_2 and H_1.

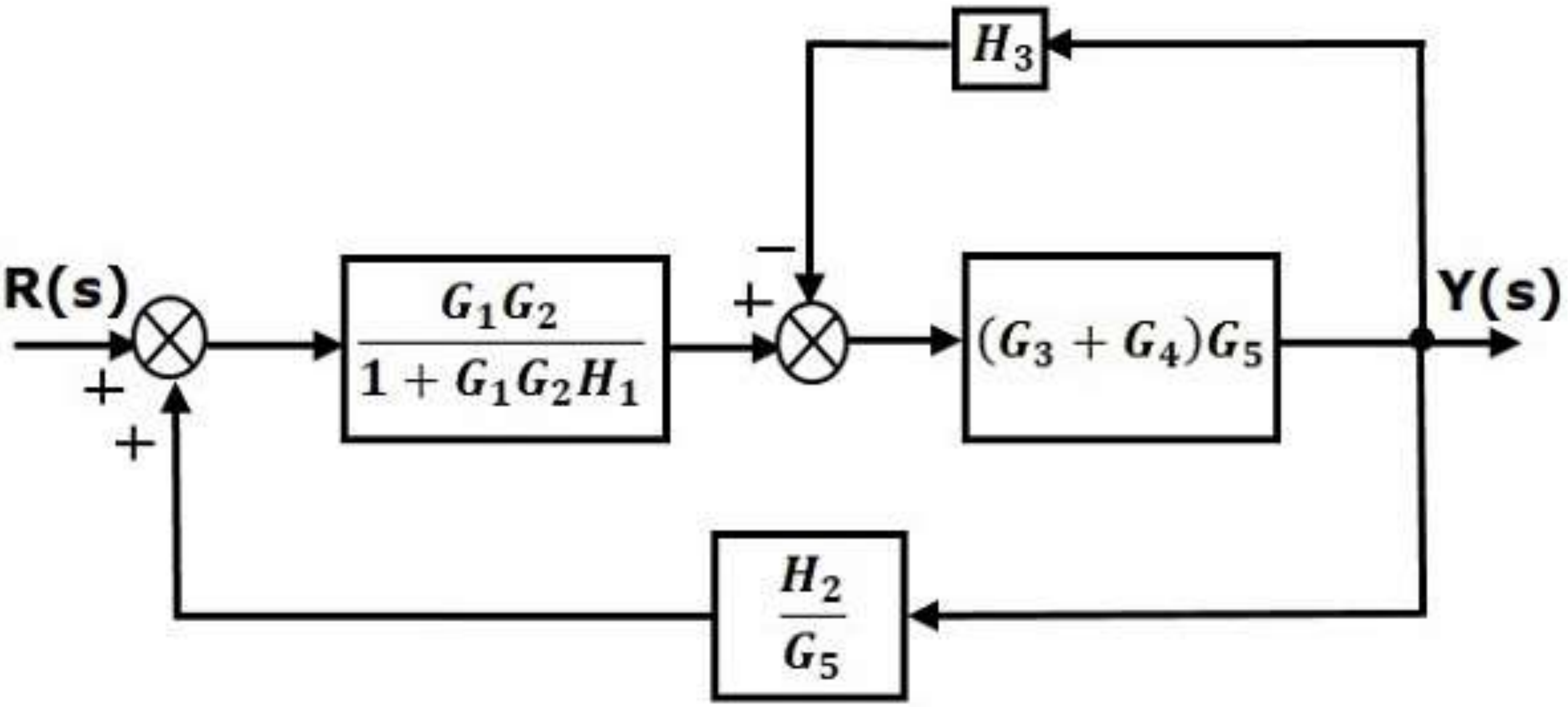
Use Rule 4 for shifting take-off point after the block G_5.

The modified block diagram is shown in the following figure.



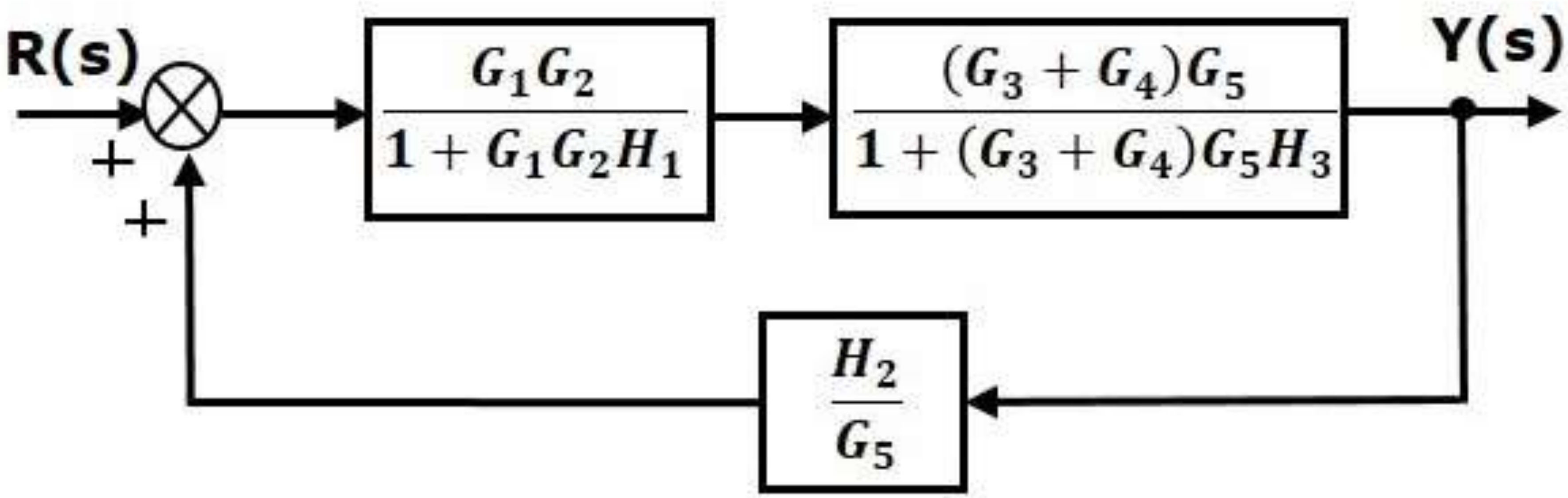
Block Diagram Reduction Rules

Step 3 – Use Rule 1 for blocks $(G3 + G4)$ and $G5$.
The modified block diagram is shown in the following figure.



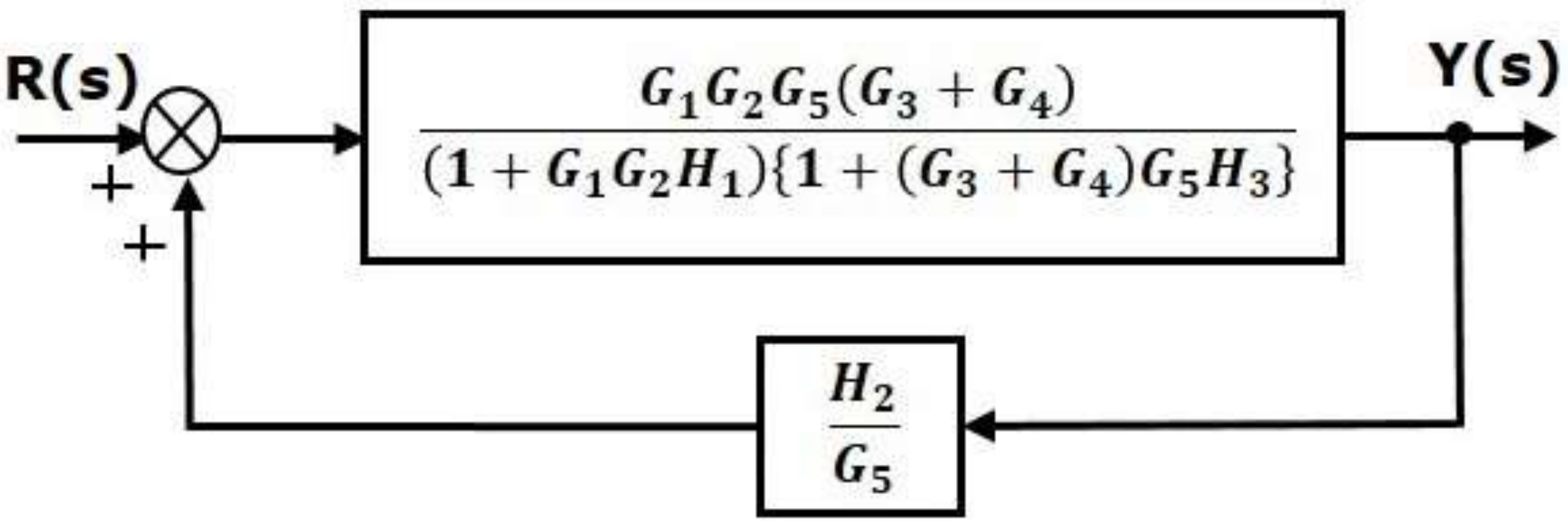
Block Diagram Reduction Rules

Step 4 − Use Rule 3 for blocks $(G3 + G4)G5$ and $(H3)$.
The modified block diagram is shown in the following figure.



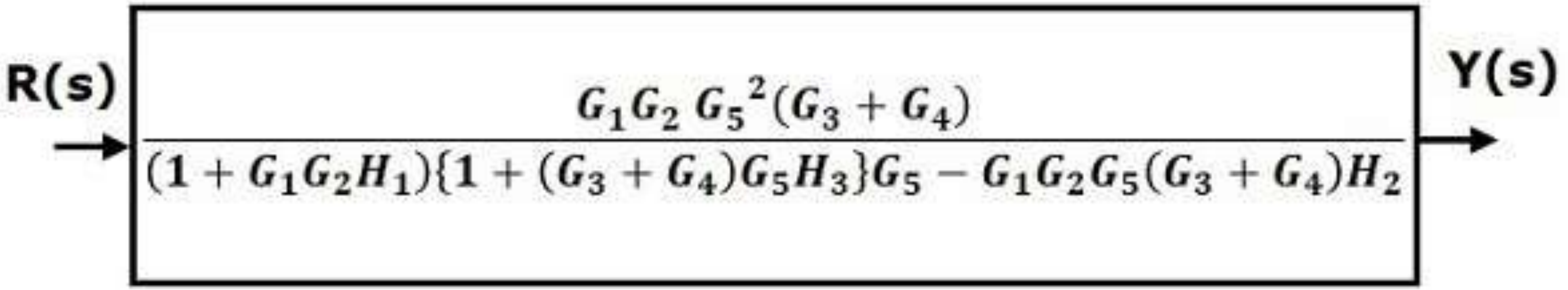
Block Diagram Reduction Rules

Step 5 – Use Rule 1 for blocks connected in series.
The modified block diagram is shown in the following figure.



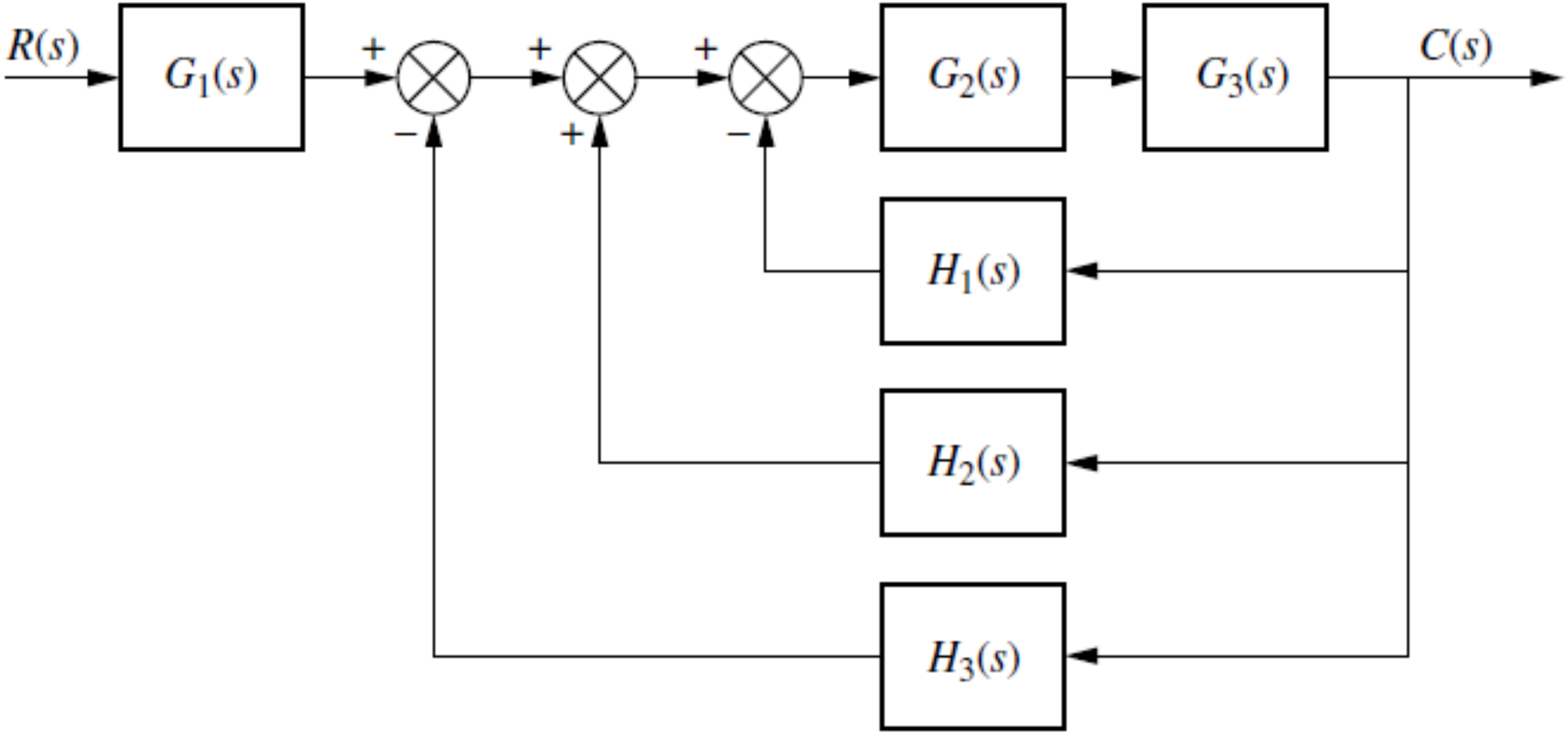
Block Diagram Reduction Rules

Step 6 − Use Rule 3 for blocks connected in feedback loop.
The modified block diagram is shown in the following figure.
This is the simplified block diagram.



Block diagram

Numerical: Reduce the block diagram to a single transfer function

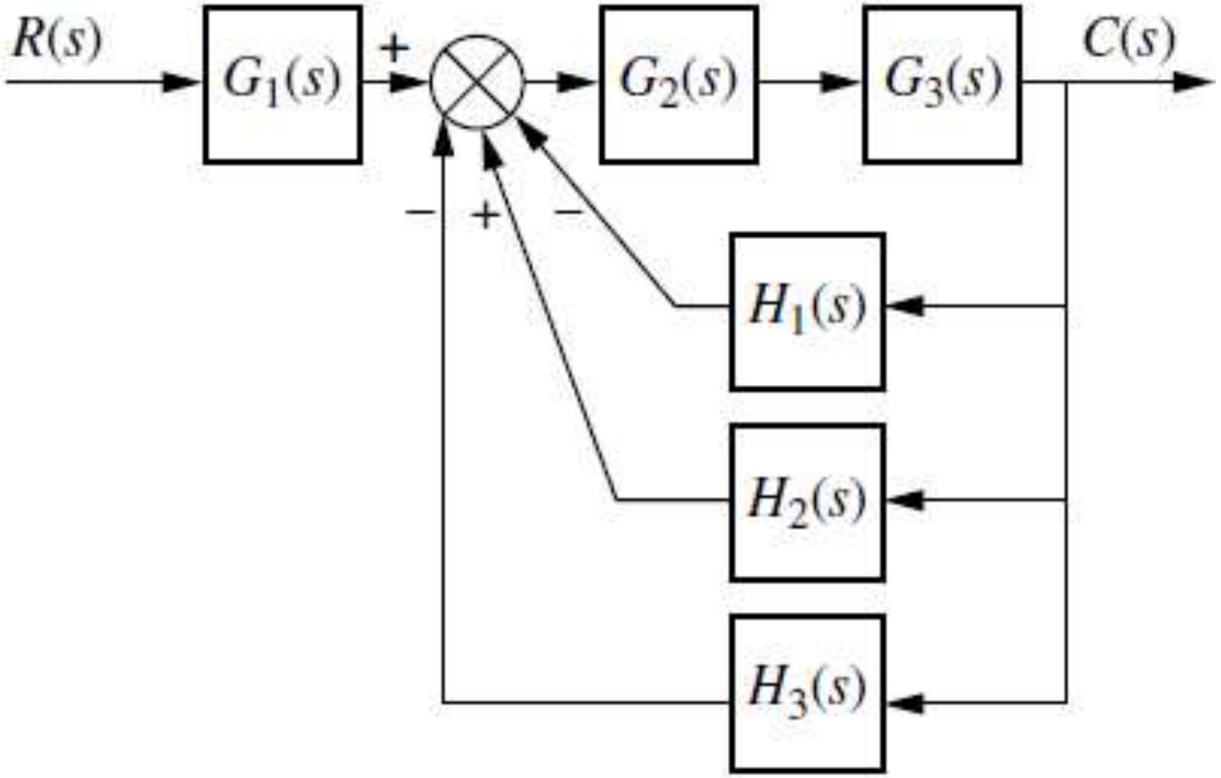


Block diagram

Step1: The three summing junctions can be combined into a single summing junction

Block diagram

The three summing junctions can be combined into a single summing junction



Block diagram

Step 2: The three feedback functions, $H_1(s), H_2(s)$ and $H_3(s)$ are in parallel. The equivalent function is $H_1(s) - H_2(s) + H_3(s)$. Also, $G_2(s)$ and $G_3(s)$ are connected in cascade. Thus, the equivalent transfer function is the product, $G_2(s)G_3(s)$.

Block diagram

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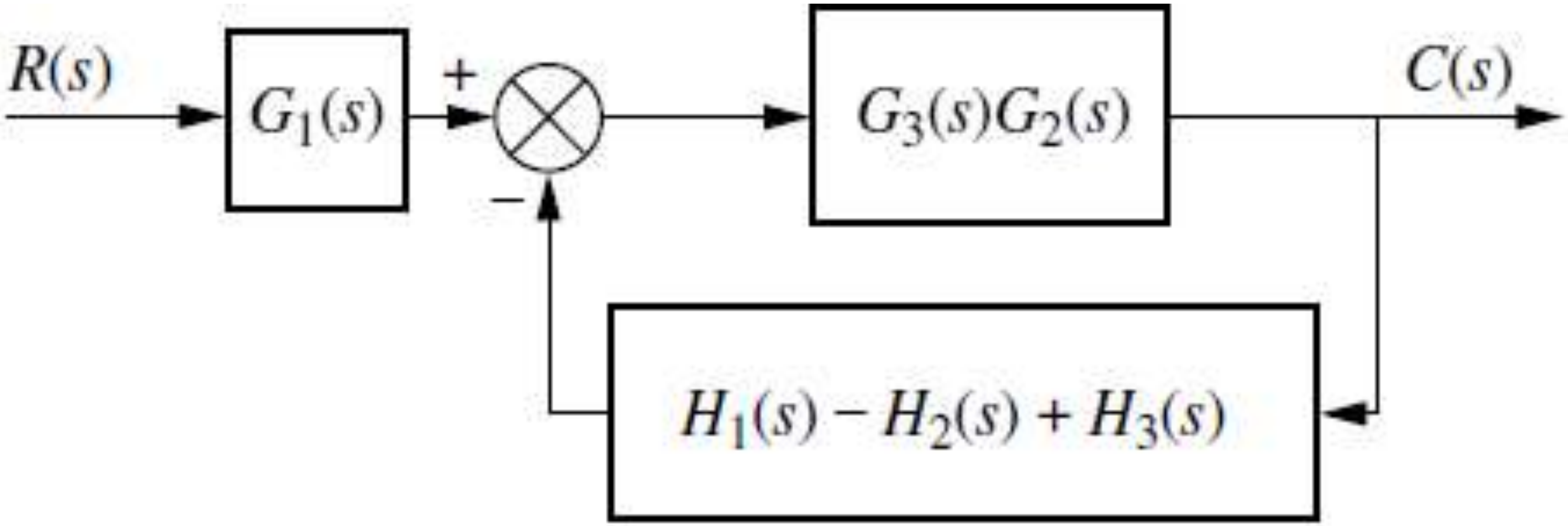


Block diagram

Step 3: the feedback system is reduced and multiplied by $G_1(s)$ to yield the equivalent transfer function

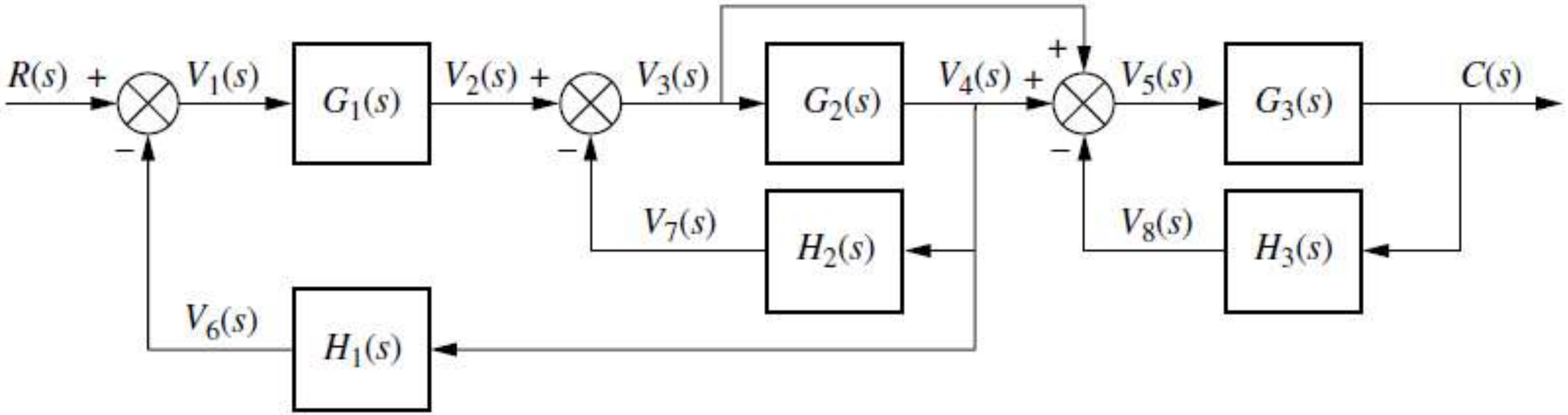
Block diagram

the feedback system is reduced and multiplied by $G_1(s)$ to yield the equivalent transfer function as shown

$$R(s) \longrightarrow \boxed{\dfrac{G_3(s)G_2(s)G_1(s)}{1 + G_3(s)G_2(s)[H_1(s) - H_2(s) + H_3(s)]}} \longrightarrow C(s)$$

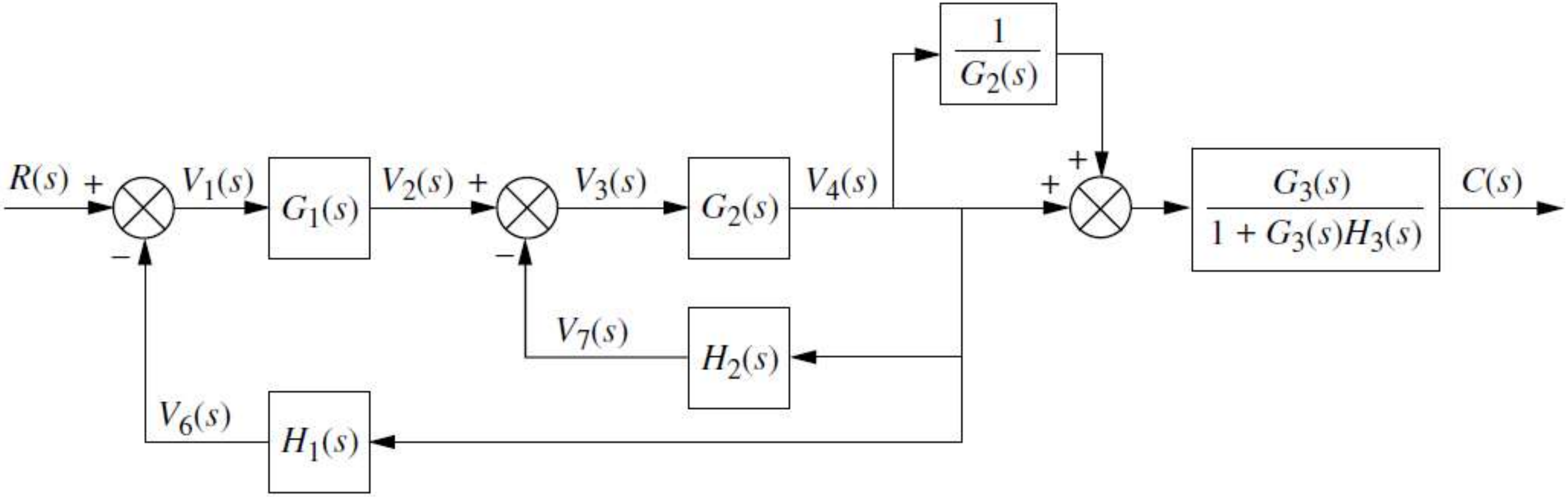
Block diagram

Numerical: Reduce the block diagram shown to a single transfer function



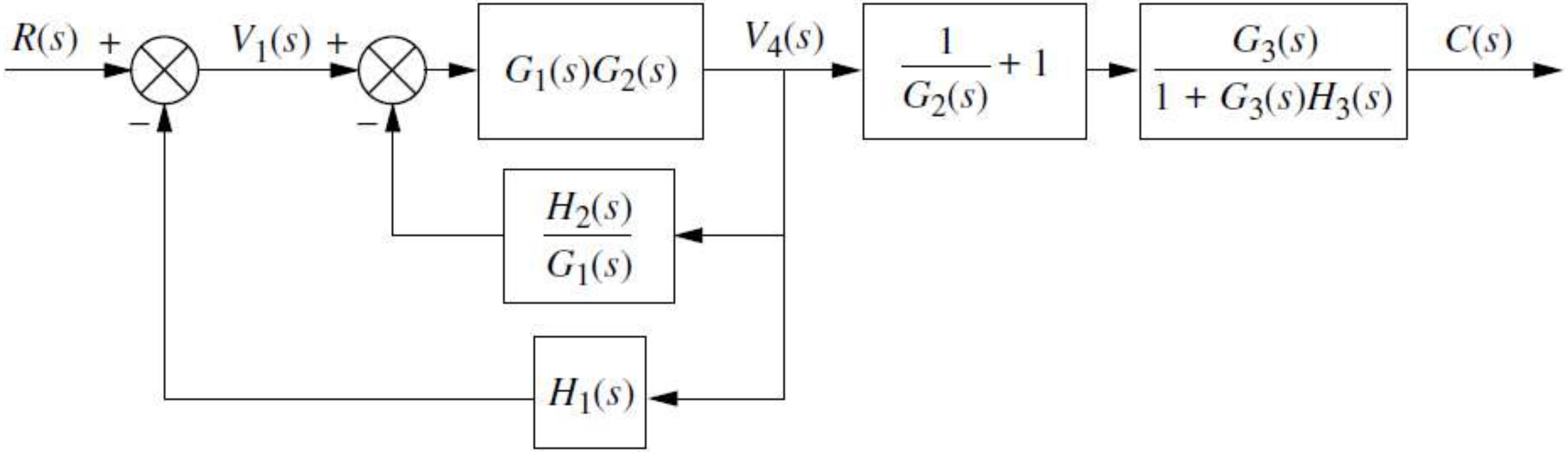
Block diagram

Step 1: move $G_2(s)$ to the left past the pickoff point to create parallel subsystems, and reduce the feedback system consisting of $G_3(s)$ and $H_3(s)$



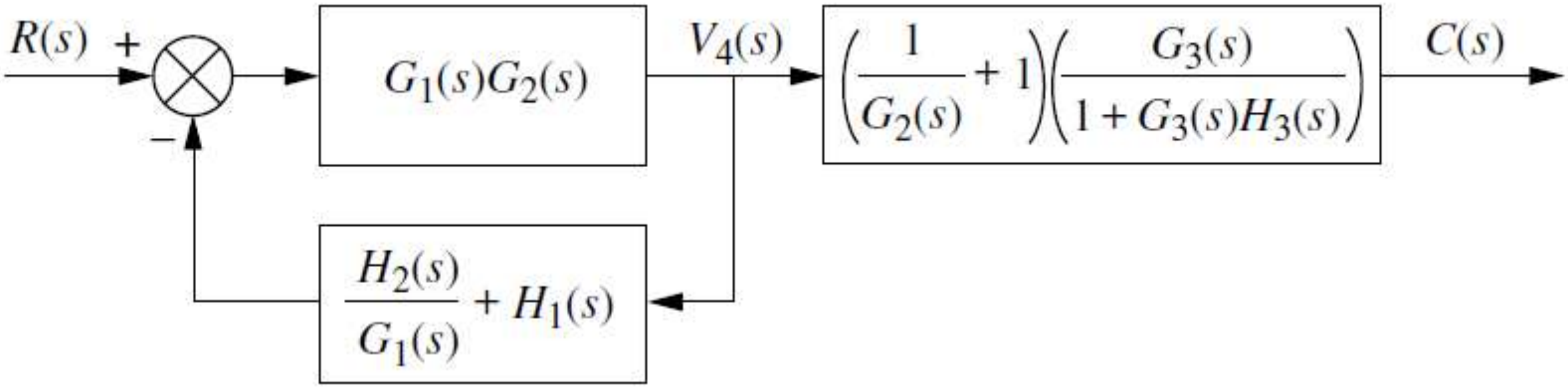
Block diagram

Step2: Second, reduce the parallel pair consisting of $1/G_2(s)$ and unity, and push $G_1(s)$ to the right past the summing junction, creating parallel subsystems in the feedback.



Block diagram

Step3: combine the summing junctions, add the two feedback elements together, and combine the last two cascaded blocks.



Block diagram

Step4: use the feedback formula.

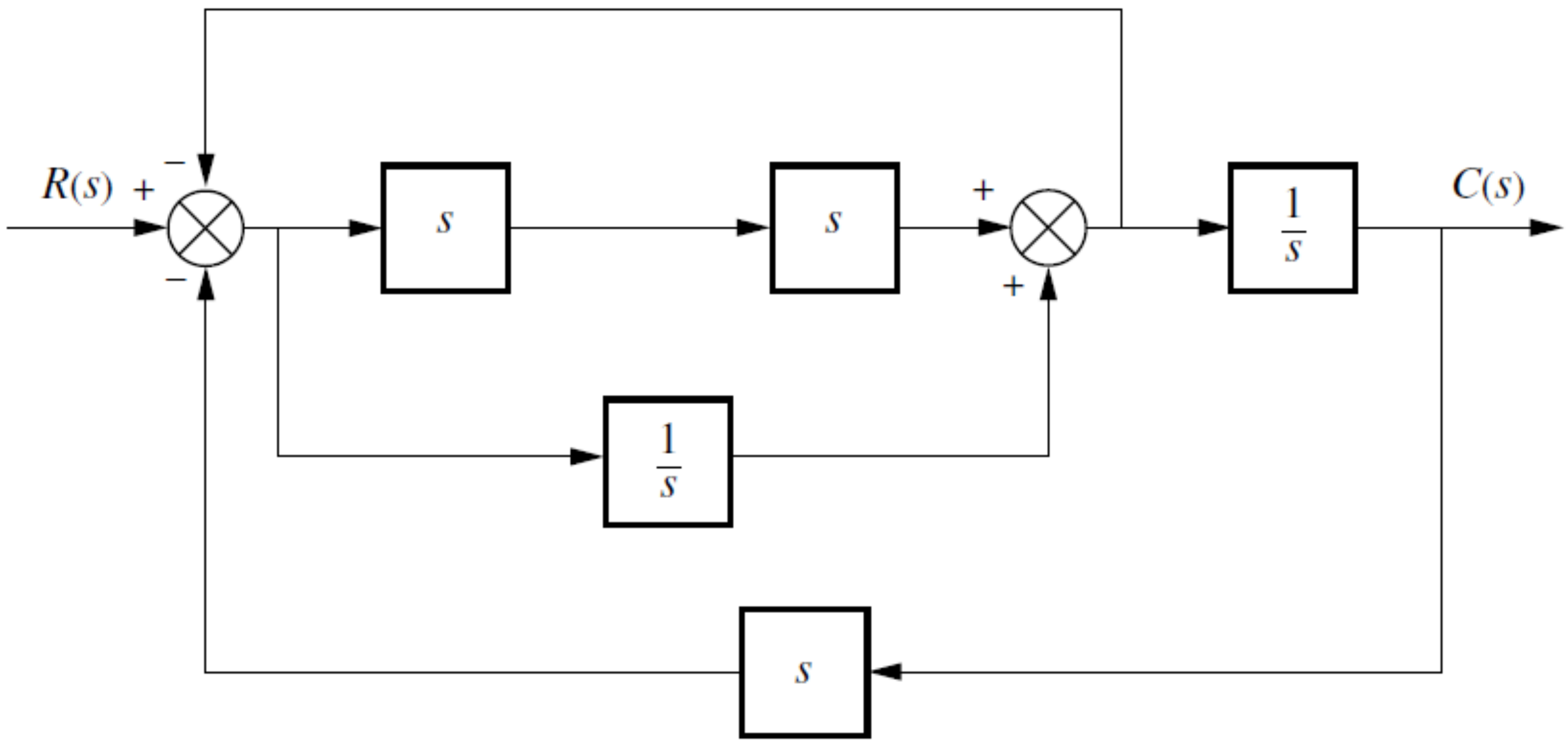
$$R(s) \longrightarrow \boxed{\dfrac{G_1(s)G_2(s)}{1 + G_2(s)H_2(s) + G_1(s)G_2(s)H_1(s)}} \xrightarrow{V_4(s)} \boxed{\left(\dfrac{1}{G_2(s)} + 1\right)\left(\dfrac{G_3(s)}{1 + G_3(s)H_3(s)}\right)} \longrightarrow C(s)$$

Step5: Finally, multiply the two cascaded blocks and obtain the final result

$$R(s) \longrightarrow \boxed{\dfrac{G_1(s)G_3(s)[1 + G_2(s)]}{[1 + G_2(s)H_2(s) + G_1(s)G_2(s)H_1(s)][1 + G_3(s)H_3(s)]}} \longrightarrow C(s)$$

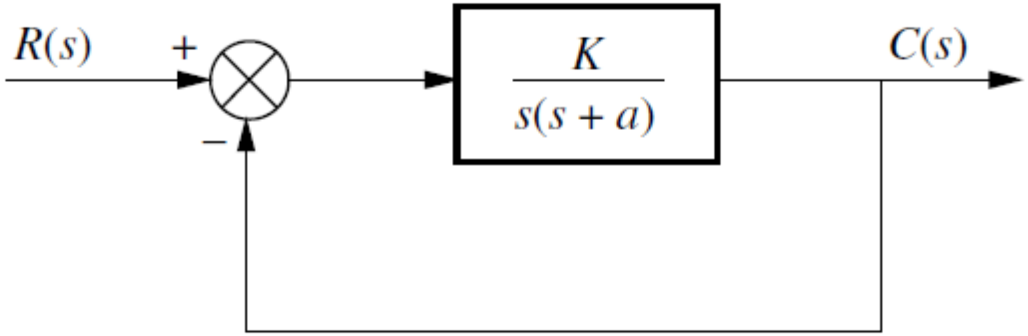
Block diagram

Numerical: Reduce the block diagram and find the transfer function



$$\frac{s^3+1}{2s^4+s^2+2s}$$

Analysis and Design of Feedback Systems



Consider the system shown, the open loop transfer function, $K/s(s+a)$

The closed-loop transfer function for this system is

$$T(s) = \frac{K}{s^2 + as + K}$$

As K varies, the poles move through the three ranges of operation of a second-order system: overdamped, critically damped, and underdamped. For example, for K between 0 and $a^2/4$, the poles of the system are real and are locate

For example, for K between 0 and $a/4$, the poles of the system are real and are locate

$$s_{1,2} = -\frac{a}{2} \pm \frac{\sqrt{a^2 - 4K}}{2}$$

As K increases, the poles move along the real axis, and the system remains overdamped until $K = a^2/4$. At that gain, or amplification, both poles are real and equal, and the system is critically damped.

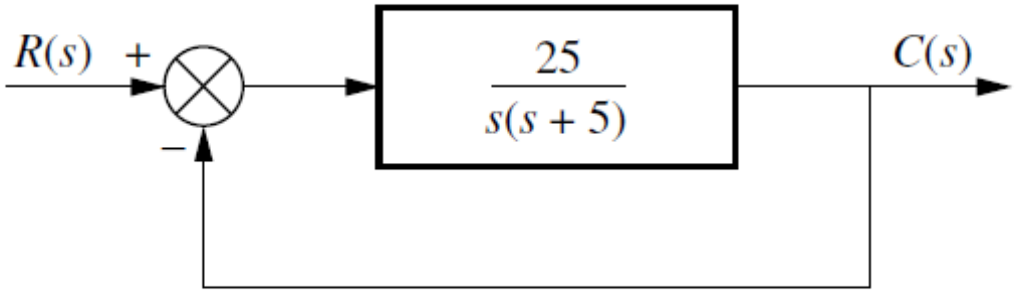
For gains above $a^2/4$, the system is underdamped, with complex poles located at

$$s_{1,2} = -\frac{a}{2} \pm j\frac{\sqrt{4K - a^2}}{2}$$

Now as K increases, the real part remains constant and the imaginary part increases. Thus, the peak time decreases and the percent overshoot increases, while the settling time constant.

Analysis and Design of Feedback Systems

For the system shown, find the peak time,



The closed-loop transfer function found from is

$$T(s) = \frac{25}{s^2 + 5s + 25}$$

$$\omega_n = \sqrt{25} = 5; 2\zeta\omega_n = 5; \therefore \zeta = 0.5$$

Analysis and Design of Feedback Systems

Using the values for ζ and ω_n we find respectively

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.726\ second$$

$$\%OS = \frac{e^{\zeta\pi}}{\sqrt{1-\zeta^2}} \times 100 = 16.303$$

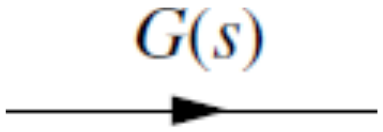
$$T_s = \frac{4}{\zeta\omega_n} = 1.6\ second$$

Signal-Flow Graphs

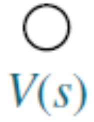
Signal-flow graphs are an alternative to block diagrams. The signal-flow graph consists of

branches which represent systems, and

nodes, which represent signals.



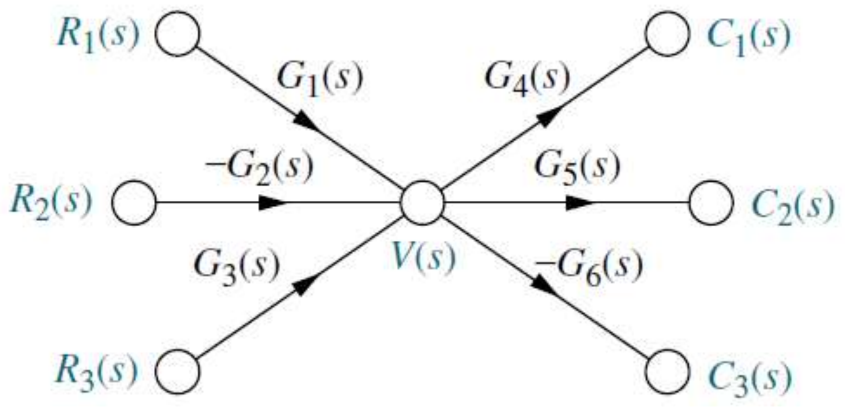
node (signal)



A system is represented by a line with an arrow showing the direction of signal flow through the system. Adjacent to the line we write the transfer function.

A signal is a node with the signal's name written adjacent to the node.

Signal-Flow Graphs



interconnection of systems and signals

Each signal is the sum of signals flowing into it. For example, the signals

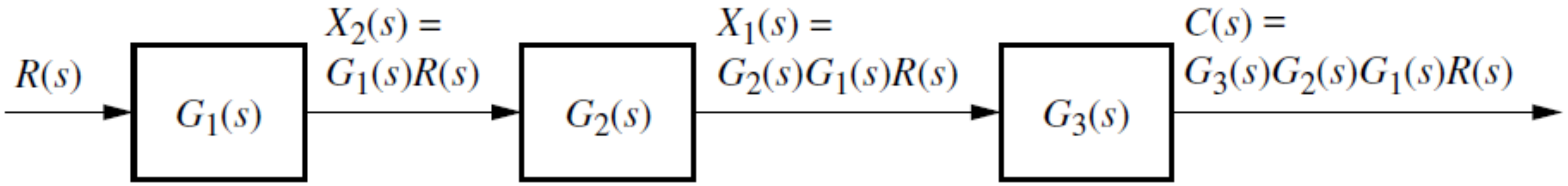
$$V(s) = R_1(s)G_1(s) - R_2(s)G_2(s) + R_3(s)G_3(s)$$

$$C_2(s) = V(s)G_5(s) = R_1(s)G_1(s)G_5(s) - R_2(s)G_2(s)G_5(s) + R_3(s)G_3(s)G_5(s)$$

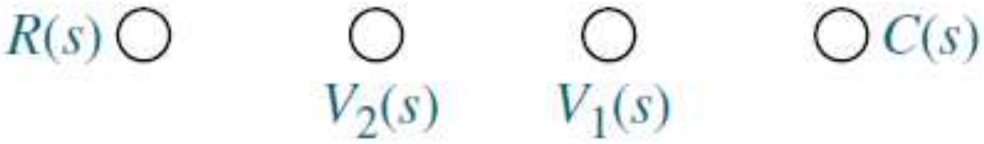
$$C_3(s) = -V(s)G_{6(s)} = -R_1(s)G_1(s)G_{6(s)} + R_2(s)G_2(s)G_{6(s)} - R_3(s)G_3(s)G_6(s)$$

Signal-Flow Graphs

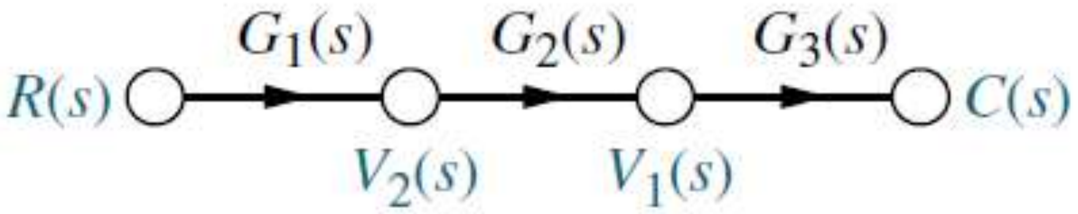
Convert the block diagram into signal-flow graphs



Step1: Draw the signal nodes for that system.



Step2: Interconnect the signal nodes with system branches.



Signal-Flow Graphs

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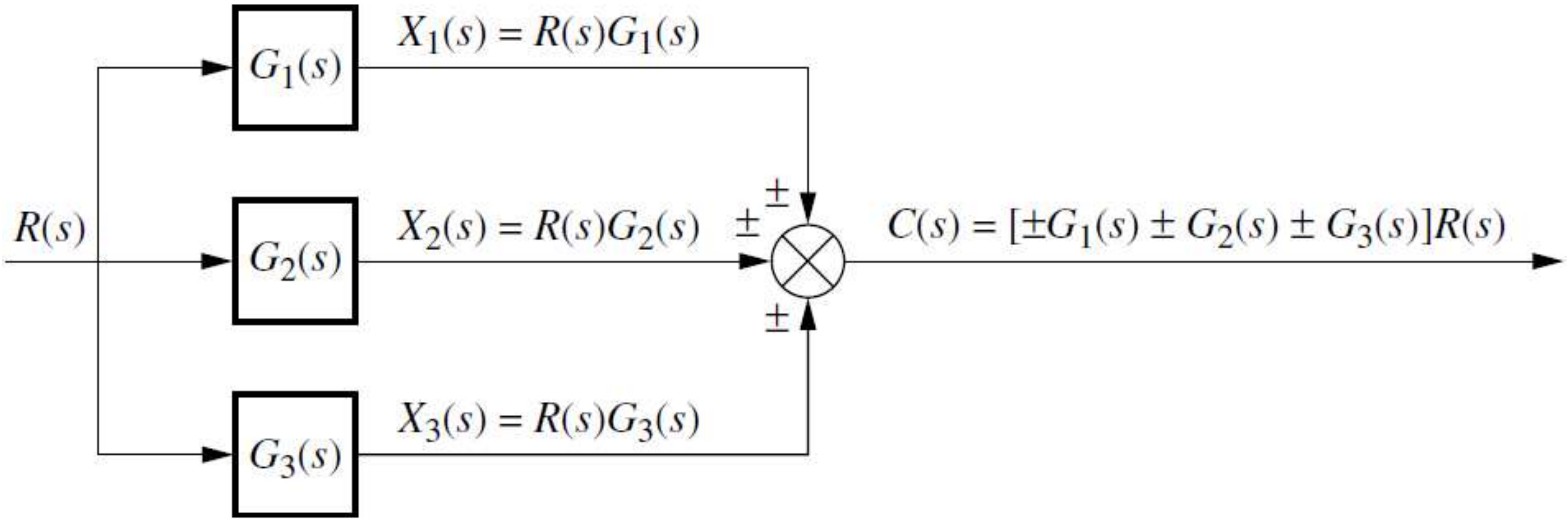

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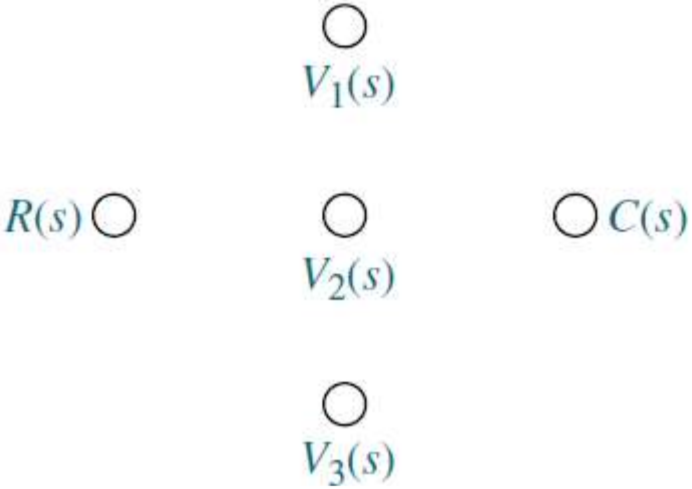

Signal-Flow Graphs

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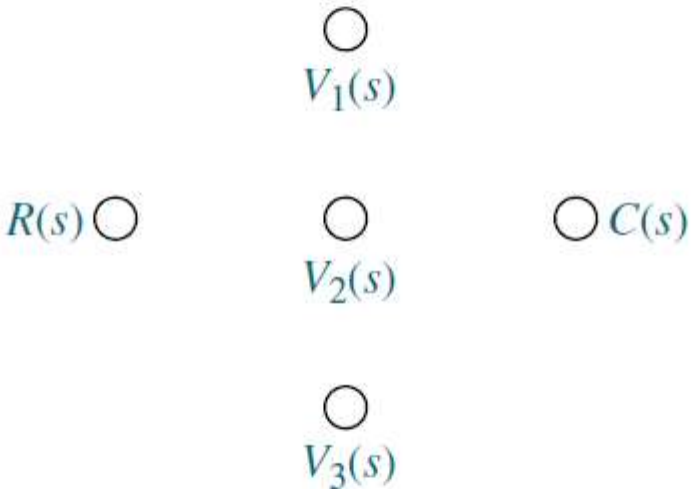
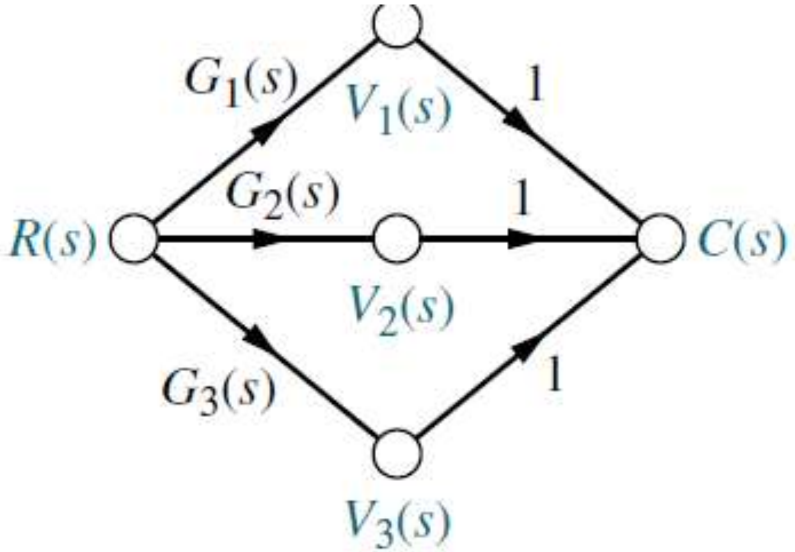


Step2: Interconnect the signal nodes with system branches.



Signal-Flow Graphs

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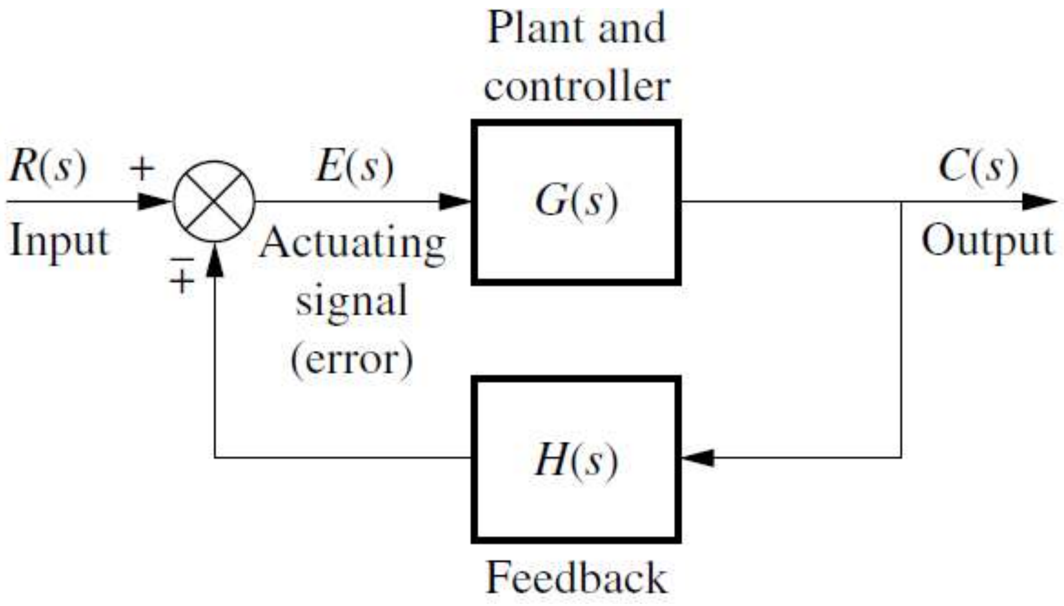

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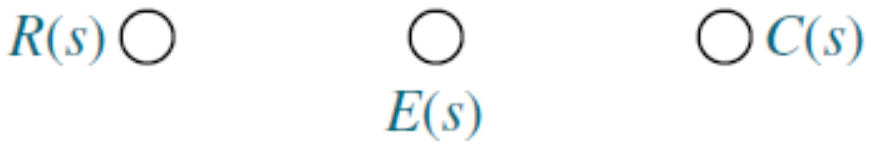

Signal-Flow Graphs

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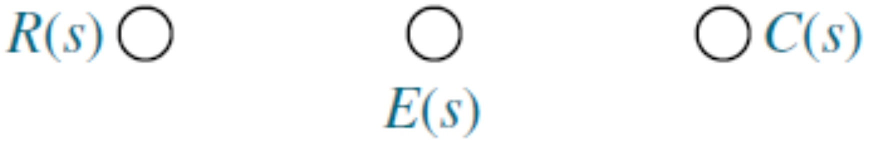
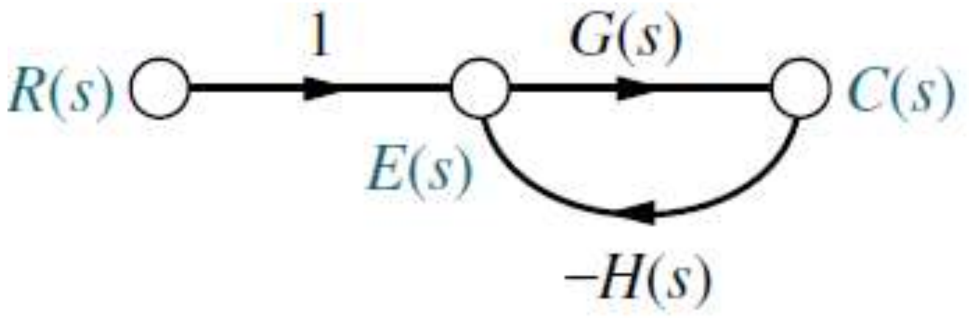
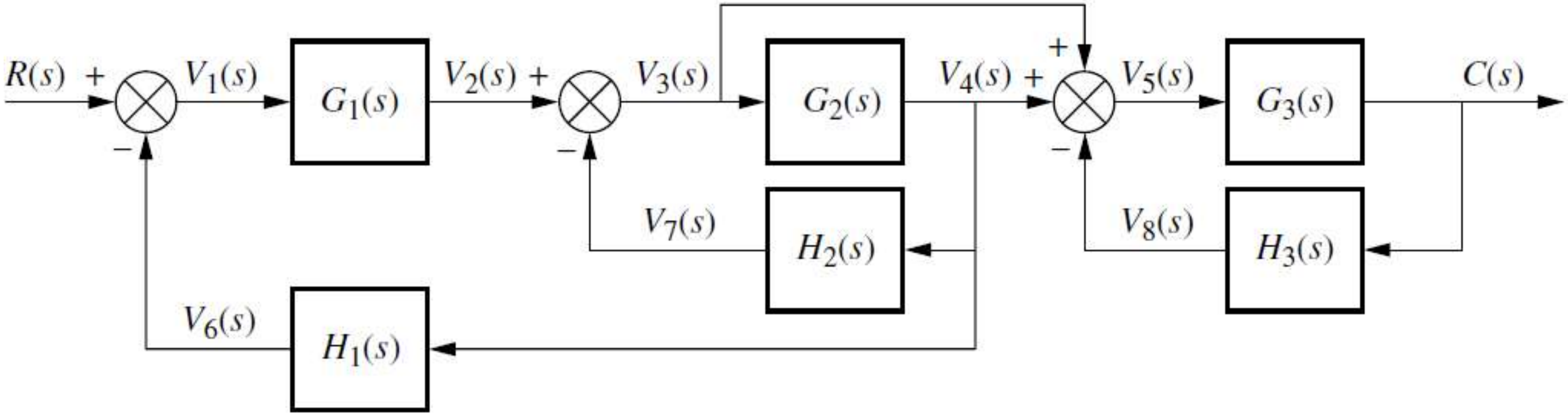


Step2: Interconnect the signal nodes with system branches.



Signal-Flow Graphs

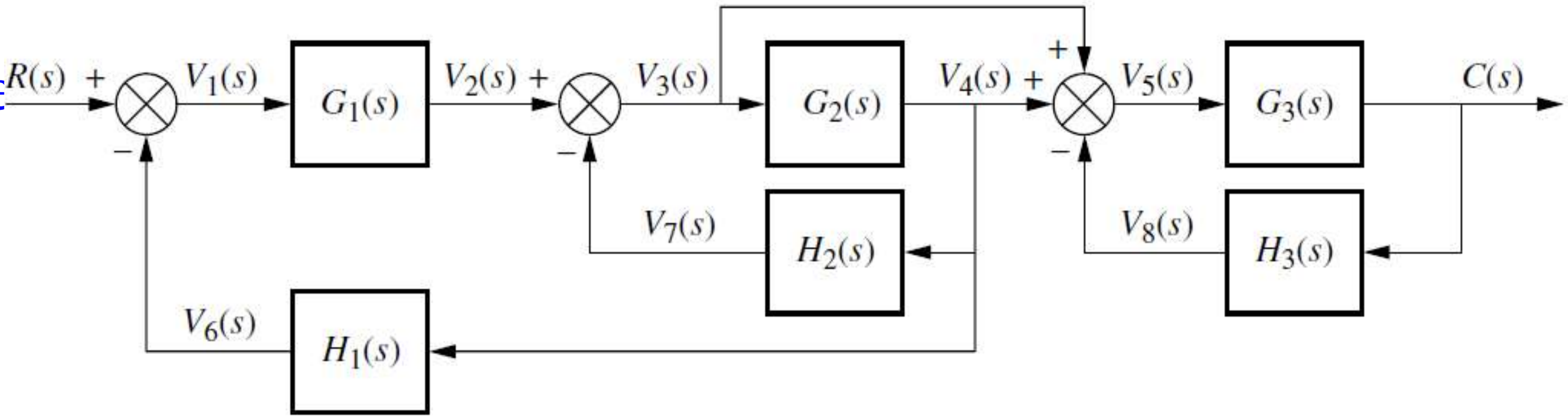
Convert the block diagram into signal-flow graphs

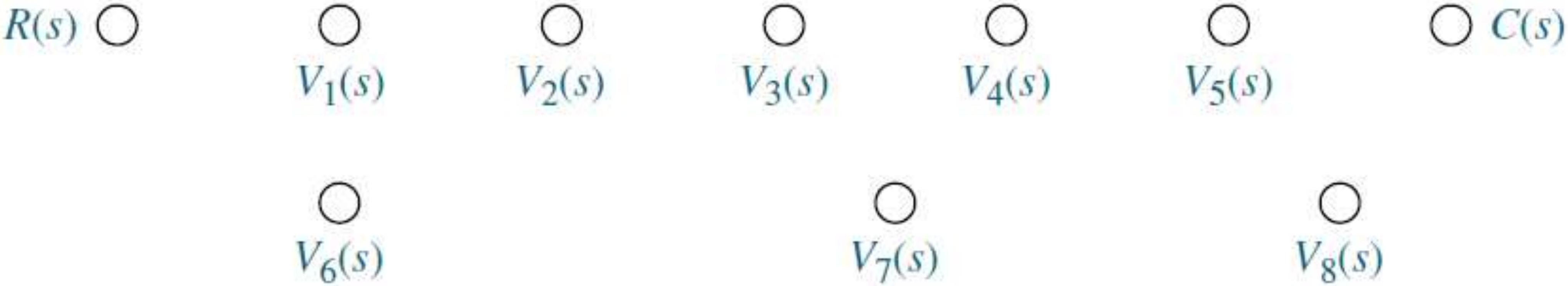


Step1: Draw the signal nodes for that system.

Signal-Flow Graphs

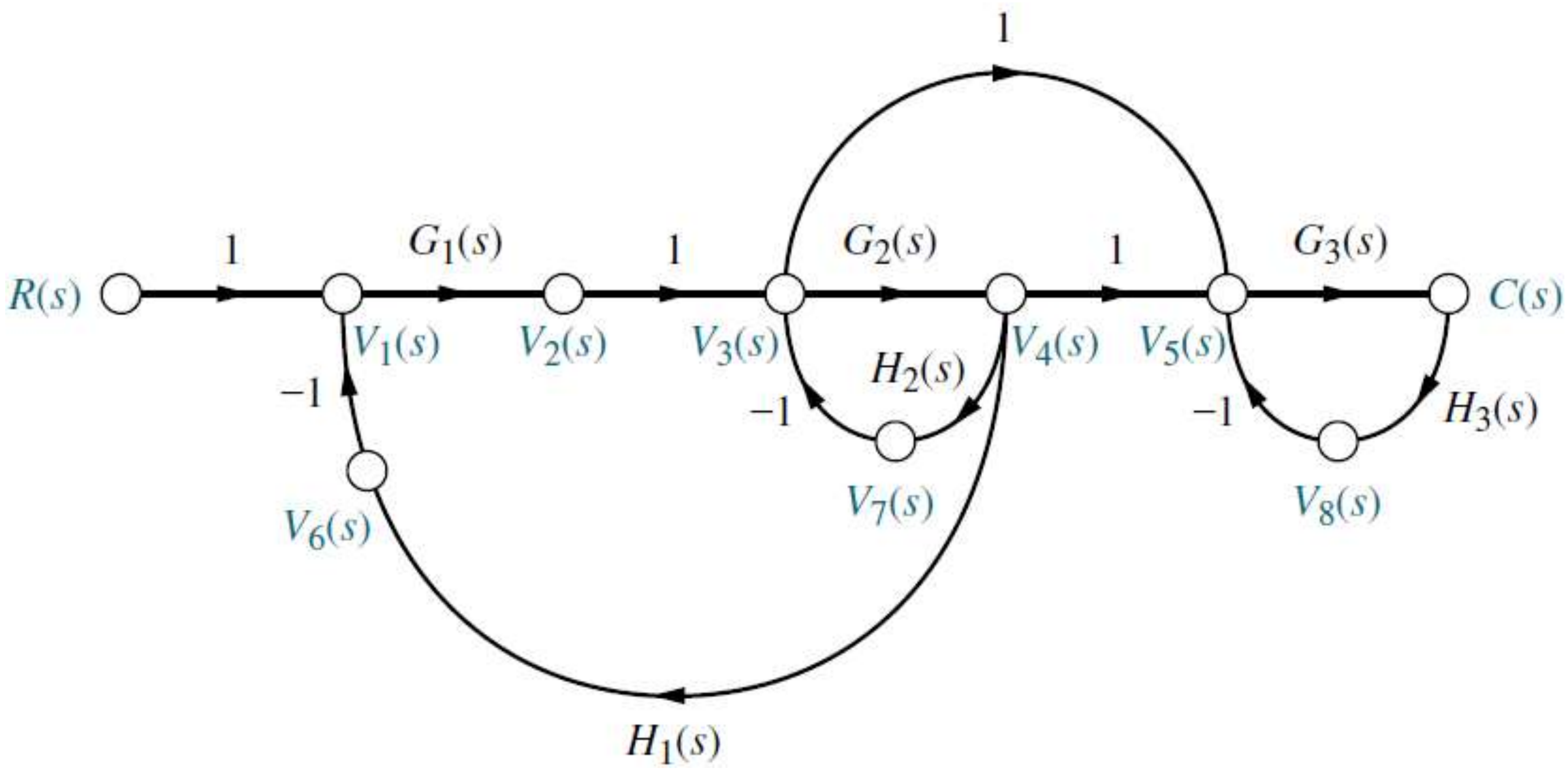
Step1: Draw the





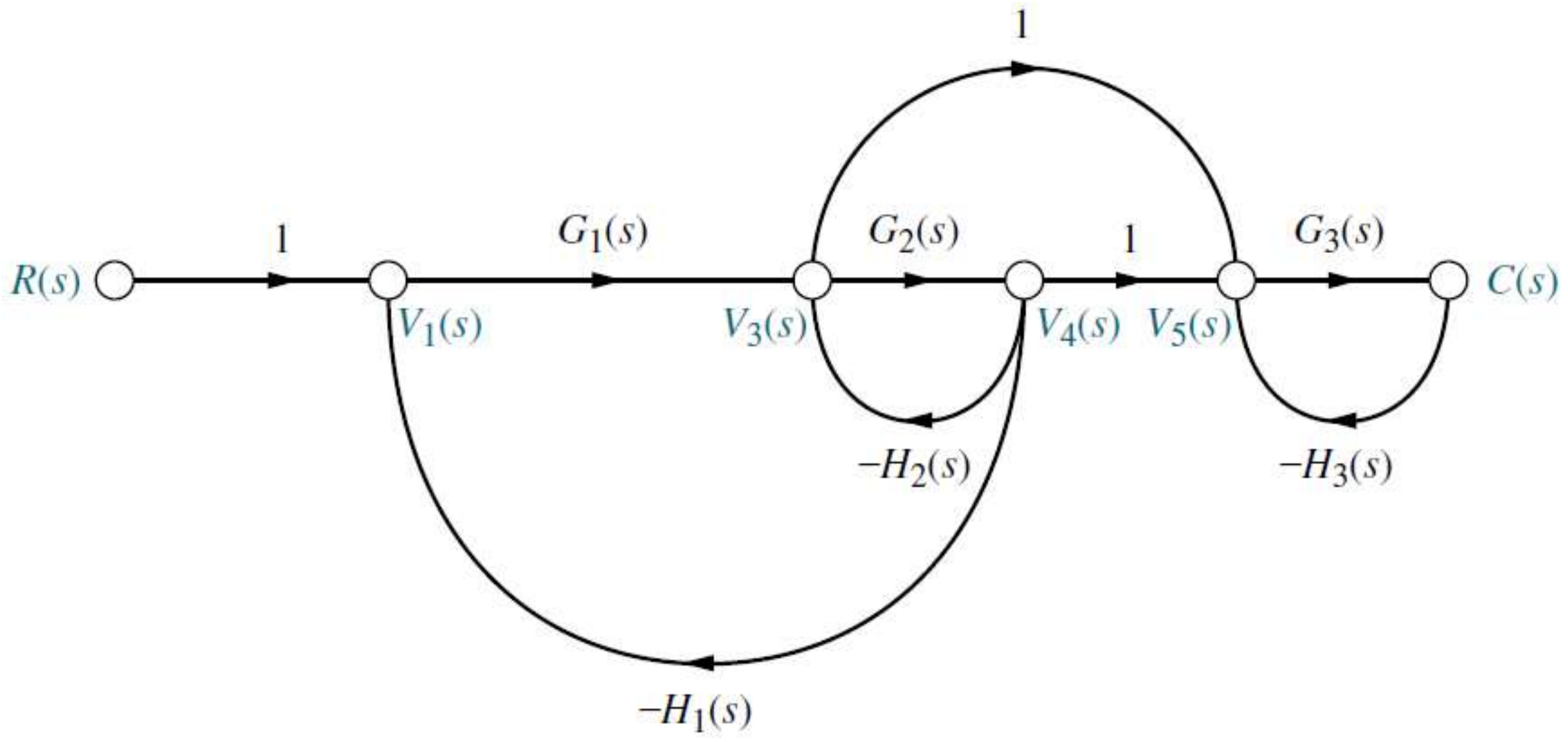
Signal-Flow Graphs

Step2: Interconnect the signal nodes with system branches.



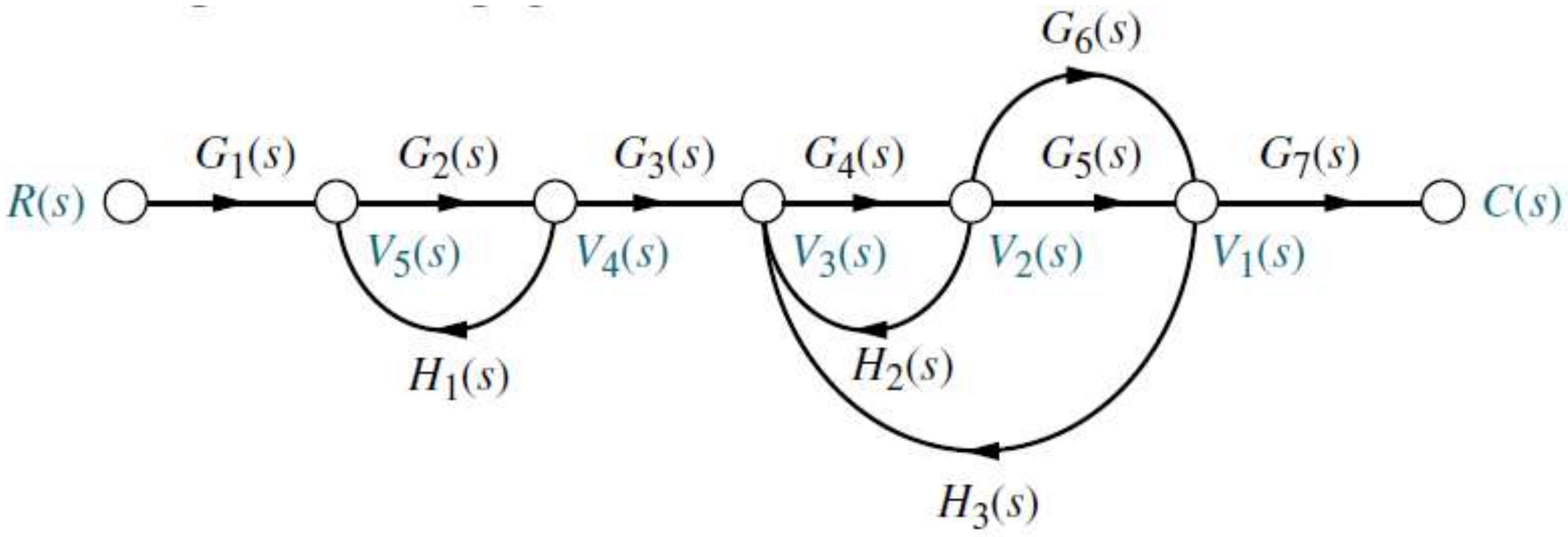
Signal-Flow Graphs

Step3: Simplified signal flow graph



Mason's Rule

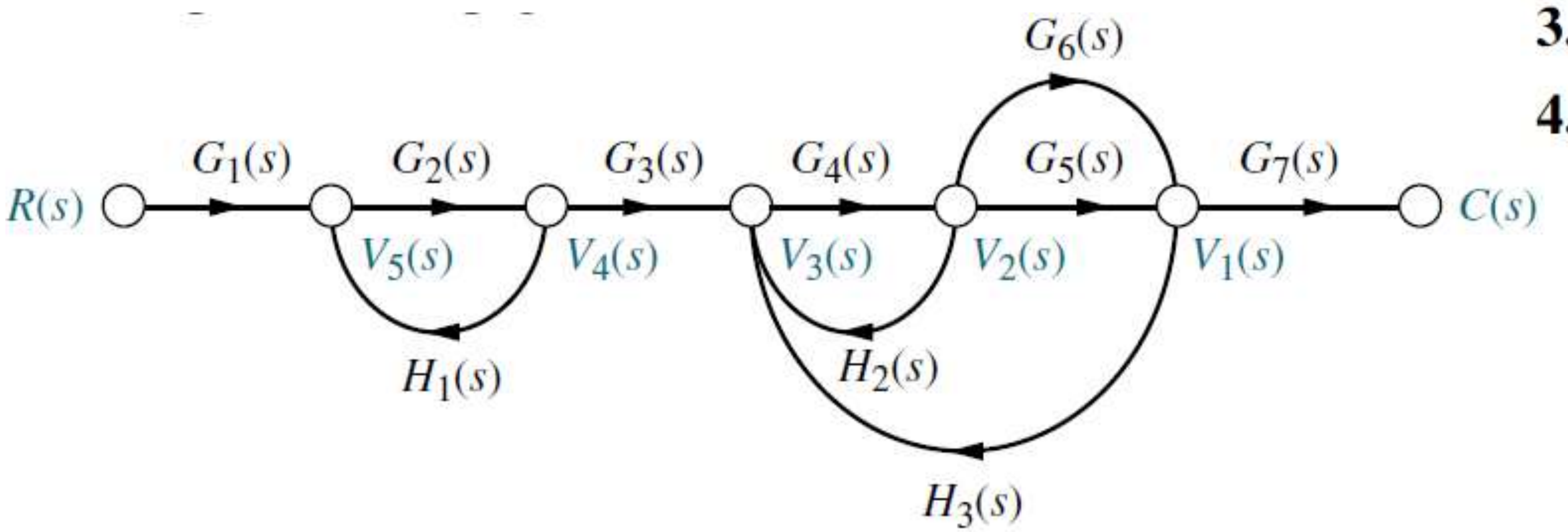
Mason's rule for reducing a signal-flow graph to a single transfer function requires the application of one formula. It has several components.



Mason's Rule

Loop gain: The product of branch gains found by traversing a path that starts at a node and ends at the same node, following the direction of the signal flow, without passing through any other node more than once. Here we have four loops

1. $G_2(s)H_1(s)$
2. $G_4(s)H_2(s)$
3. $G_4(s)G_5(s)H_3(s)$
4. $G_4(s)G_6(s)H_3(s)$

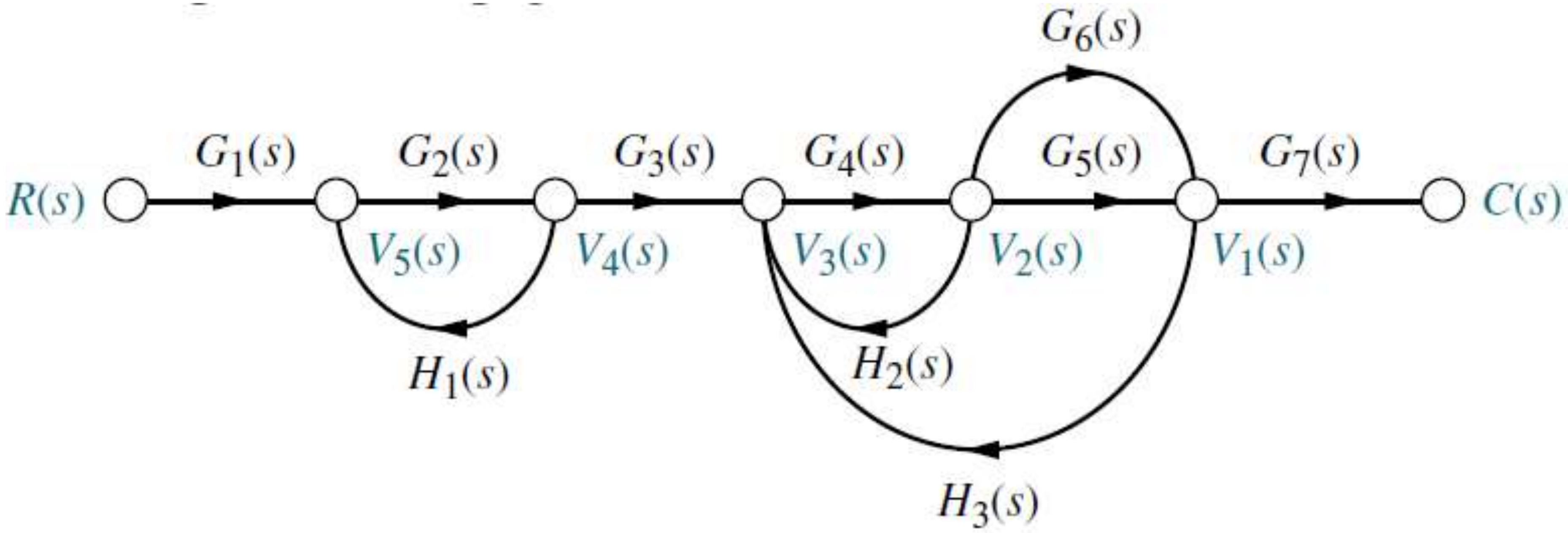


Mason's Rule

Forward-path gain: The product of gains found by traversing a path from the input node to the output node of the signal-flow graph in the direction of signal flow.

Here we have two forward-path gains:

1. $G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)G_7(s)$
2. $G_1(s)G_2(s)G_3(s)G_4(s)G_6(s)G_7(s)$

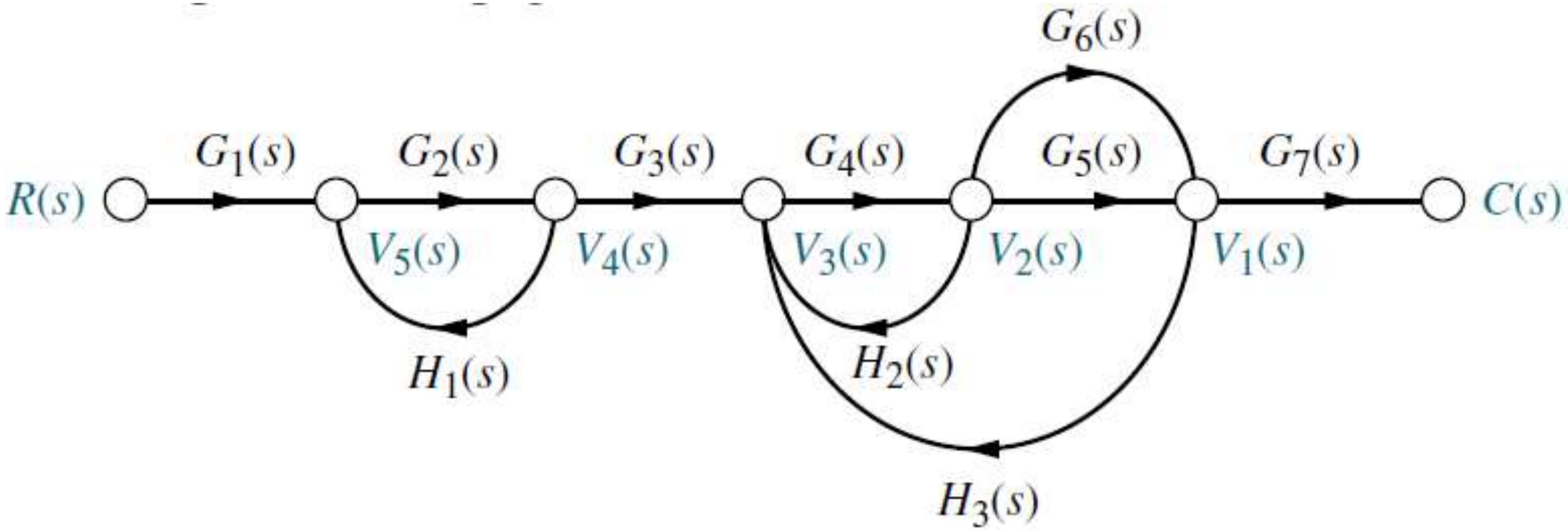


Mason's Rule

Forward-path gain: The product of gains found by traversing a path from the input node to the output node of the signal-flow graph in the direction of signal flow.

Here we have two forward-path gains:

1. $G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)G_7(s)$
2. $G_1(s)G_2(s)G_3(s)G_4(s)G_6(s)G_7(s)$

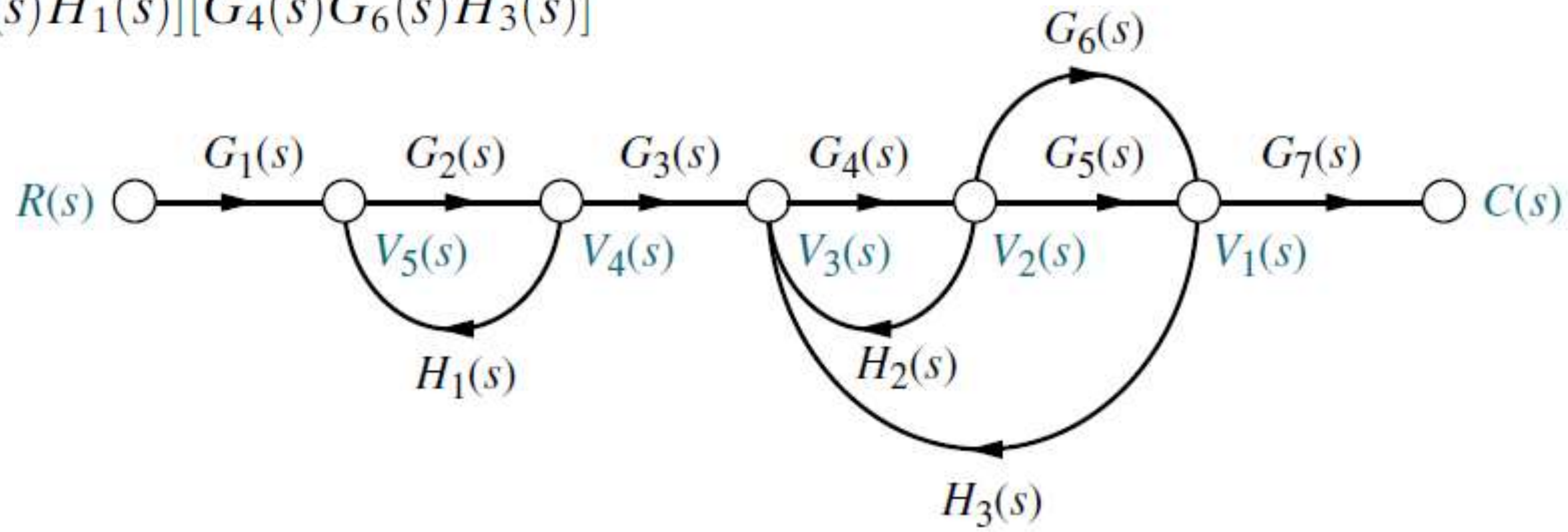


Mason's Rule

Nontouching loops:. Loops that do not have any nodes in common

Nontouching-loop gain: The product of loop gains from nontouching loops taken two, three, four, or more at a time. Here we have three sets of nontouching loops.

1. $[G_2(s)H_1(s)][G_4(s)H_2(s)]$
2. $[G_2(s)H_1(s)][G_4(s)G_5(s)H_3(s)]$
3. $[G_2(s)H_1(s)][G_4(s)G_6(s)H_3(s)]$



Mason's Rule

The transfer function of a system represented by a signal-flow graph is

$$G(s) = \frac{C(s)}{R(s)} = \frac{\sum_k T_k \Delta_k}{\Delta}$$

Where k = number of forward paths

T_k = the kth forward-path gain

$\Delta = 1 - \sum$ loop gains

$+\sum$ nontouching–loop gains taken two at a time

$-\sum$ nontouching–loop gains taken three at a time

$+\sum$ nontouching–loop gains taken four at a time

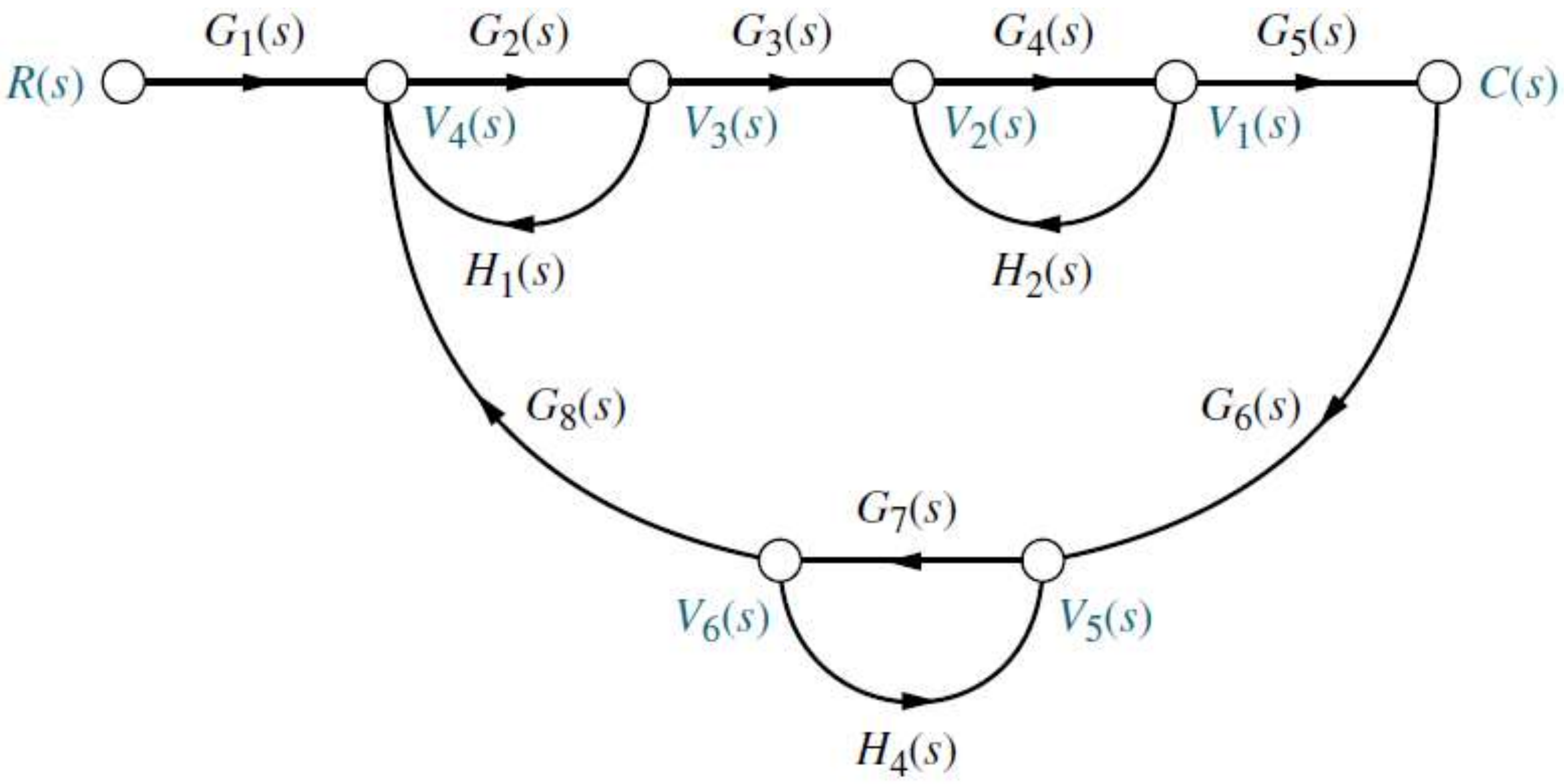
Mason's Rule

$\Delta_k = \Delta - \sum$ loop gains in Δ that touch the kth forward path:

In other words; Δ_k is formed by eliminating from Δ those loop gains that touch the kth forward path

Mason's Rule

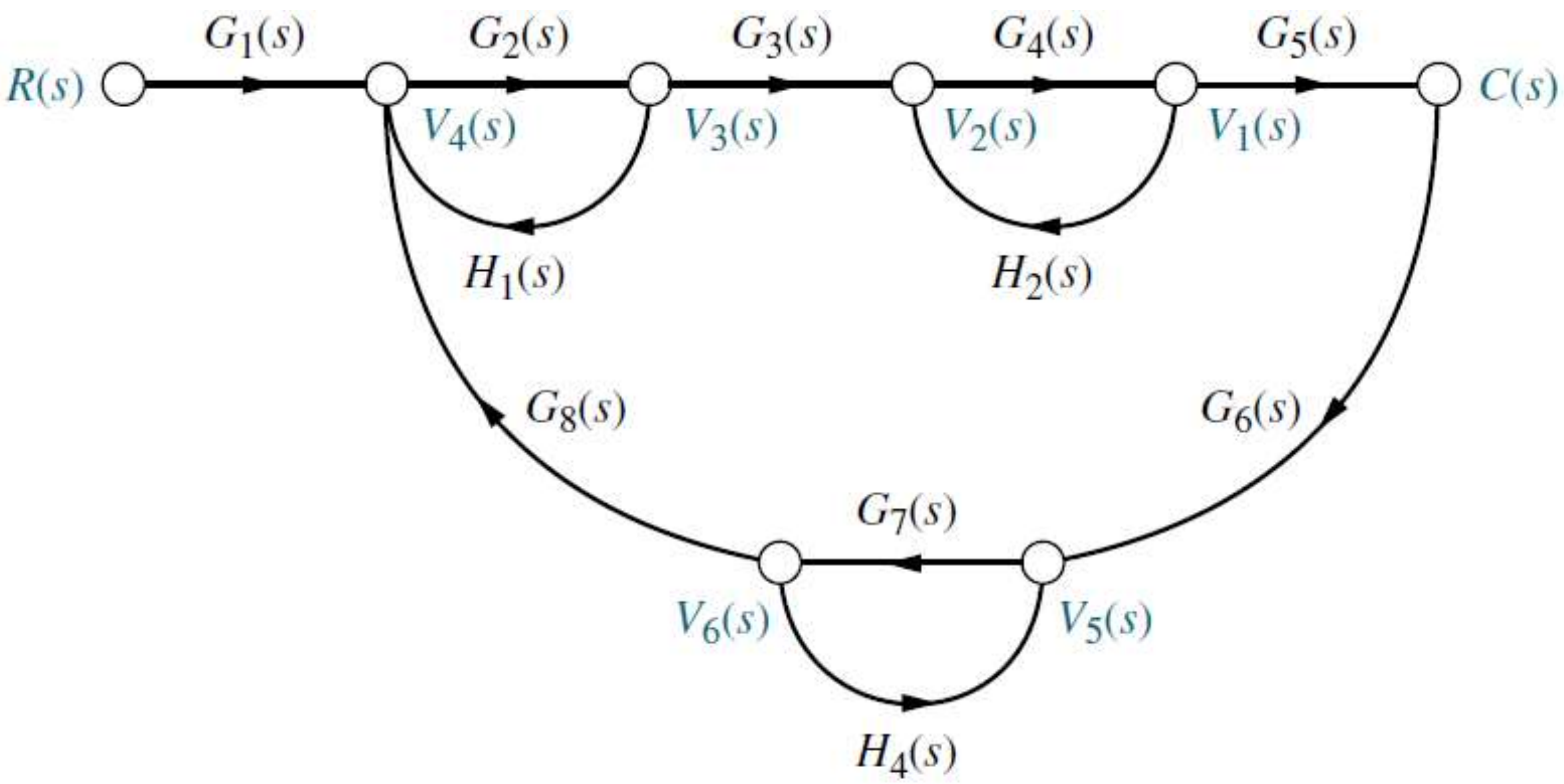
Find the transfer function, C(s)/R(s), for the signal-flow graph



Mason's Rule

Step1: Identify the forward-path gains. Here we have only one forward path

$$G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)$$



Mason's Rule

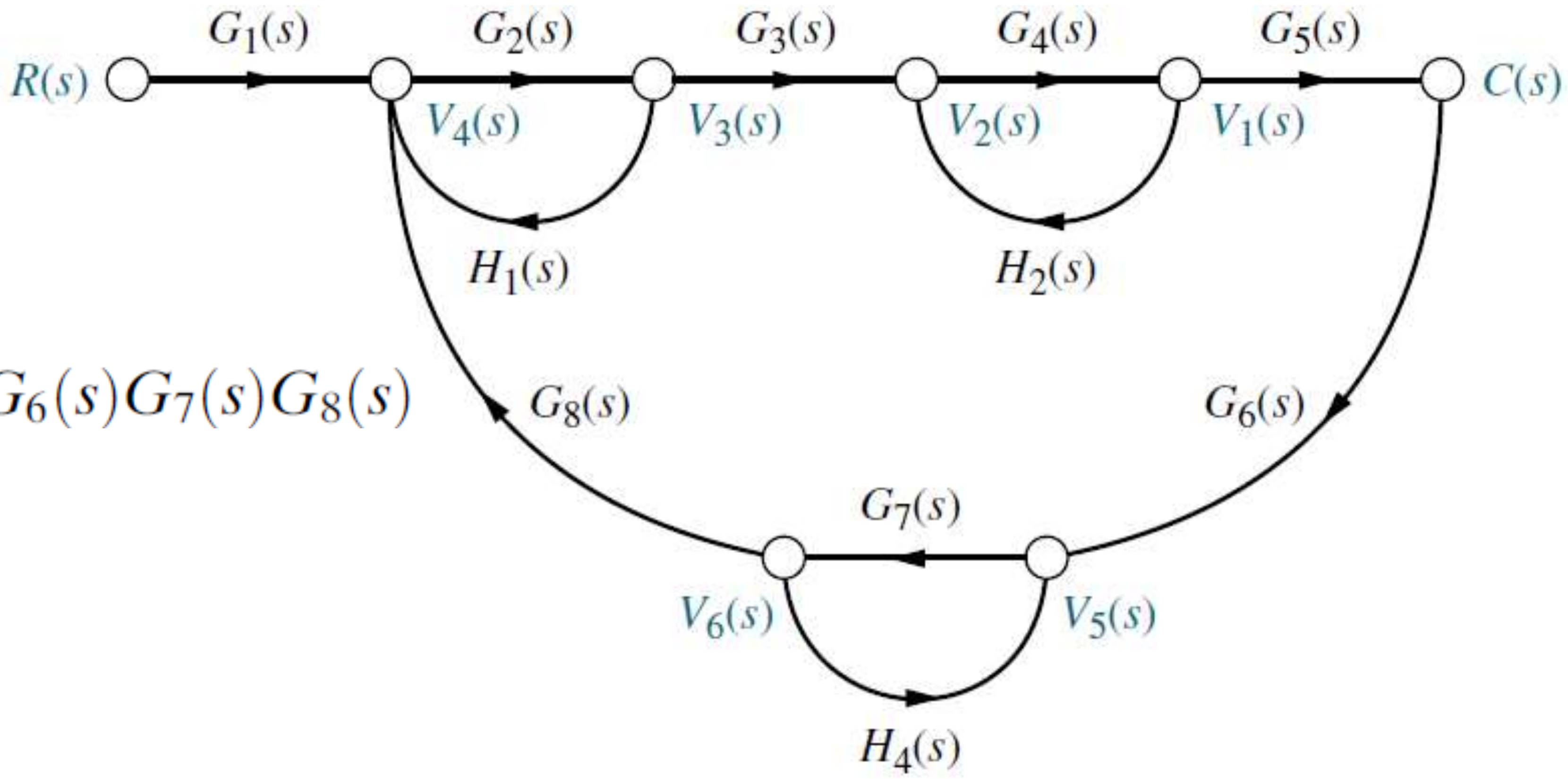
Step2: Identify the loop gains. Here we have four, as follows:

1. $G_2(s)H_1(s)$

2. $G_4(s)H_2(s)$

3. $G_7(s)H_4(s)$

4. $G_2(s)G_3(s)G_4(s)G_5(s)G_6(s)G_7(s)G_8(s)$



Mason's Rule

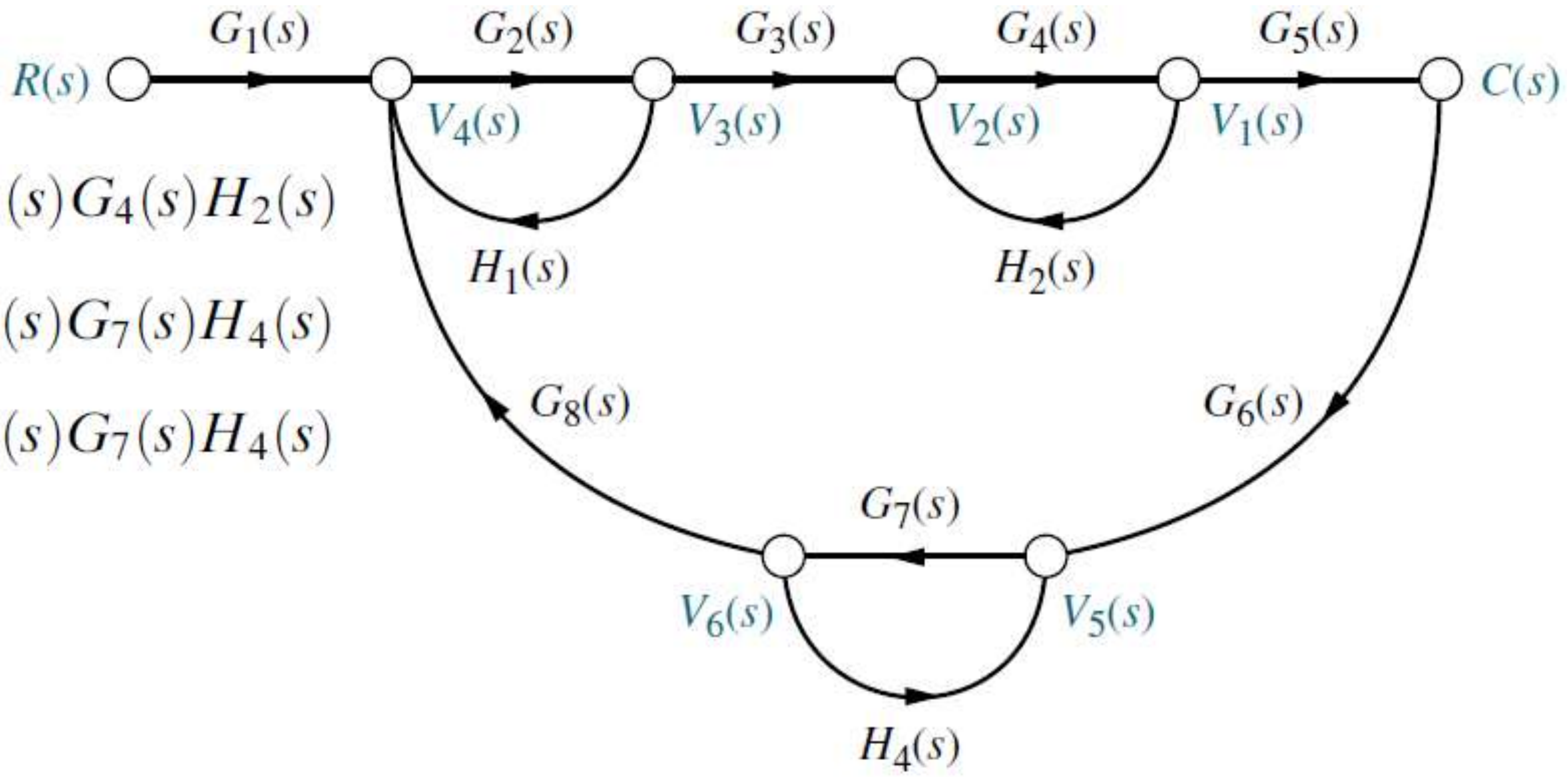
Step3: Identify the nontouching loops taken two at a time.

we can see that loop 1 does not touch loop 2, loop 1 does not touch loop 3, and loop 2 does not touch loop 3. Notice that loops 1, 2, and 3 all touch loop 4.

Loop 1 and loop 2 : $G_2(s)H_1(s)G_4(s)H_2(s)$

Loop 1 and loop 3 : $G_2(s)H_1(s)G_7(s)H_4(s)$

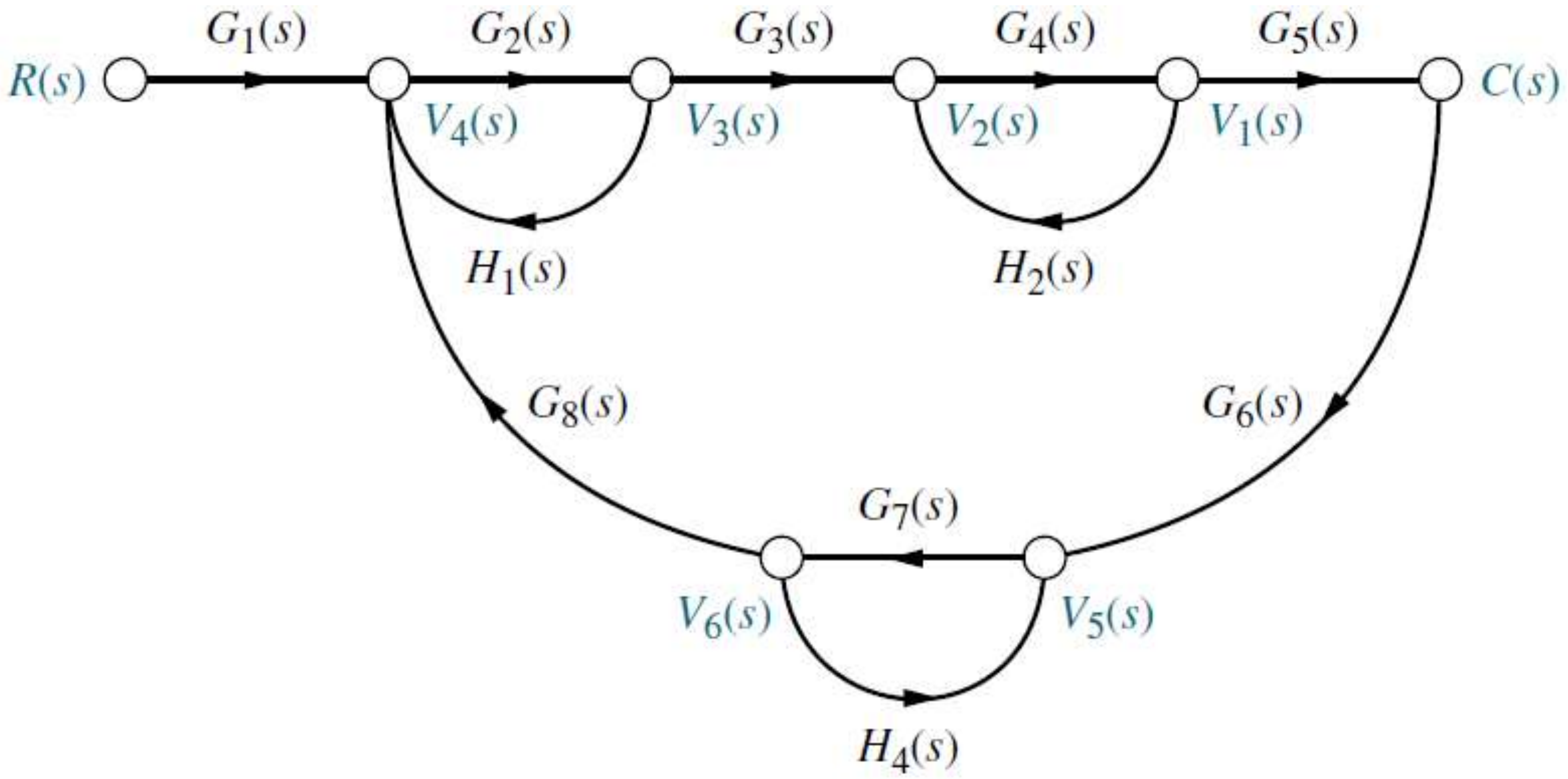
Loop 2 and loop 3 : $G_4(s)H_2(s)G_7(s)H_4(s)$



Mason's Rule

Step4: Finally, the nontouching loops taken three at a time are as follows

Loops 1, 2, and 3 : $G_2(s)H_1(s)G_4(s)H_2(s)G_7(s)H_4(s)$



Mason's Rule

Now find Δ *and* Δ_k

$$\Delta = 1 - [G_2(s)H_1(s) + G_4(s)H_2(s) + G_7(s)H_4(s) + G_2(s)G_3(s)G_4(s)G_5(s)G_6(s)G_7(s)G_8(s)]$$
$$+ [G_2(s)H_1(s)G_4(s)H_2(s) + G_2(s)H_1(s)G_7(s)H_4(s) + G_4(s)H_2(s)G_7(s)H_4(s)]$$
$$- [G_2(s)H_1(s)G_4(s)H_2(s)G_7(s)H_4(s)]$$

$\Delta = 1 - \sum$ loop gains

$+ \sum$ nontouching–loop gains taken two at a time

$- \sum$ nontouching–loop gains taken three at a time

$+ \sum$ nontouching–loop gains taken four at a time

Mason's Rule

We form Δ_k by eliminating from Δ the loop gains that touch the kth forward path

$$\Delta_1 = 1 - G_7(s)H_4(s)$$

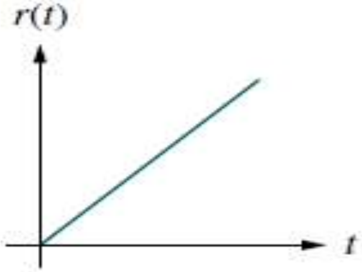
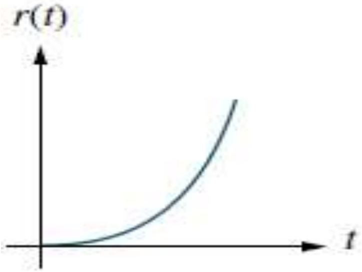
Therefore, the transfer function

$$G(s) = \frac{T_1\Delta_1}{\Delta} = \frac{[G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)][1 - G_7(s)H_4(s)]}{\Delta}$$

Feedback Control System
Unit 3
Steady state Error

Definition and Test Inputs

Steady-state error is the difference between the input and the output for a prescribed test input as $t \to \infty$.

Waveform	Name	Physical interpretation	Time function	Laplace transform
	Step	Constant position	1	$\frac{1}{s}$
	Ramp	Constant velocity	t	$\frac{1}{s^2}$
	Parabola	Constant acceleration	$\frac{1}{2}t^2$	$\frac{1}{s^3}$

Definition and Test Inputs: Position

Consider a position control system, where the output position follows the input commanded position.

Step inputs represent **constant position** and thus are useful in determining the ability of the control system to **position itself with respect to a stationary target**, such as a satellite in geostationary orbit.

An antenna position control is an example of a system that can be tested for accuracy using step input.

Definition and Test Inputs: Constant Velocity

Ramp inputs represent **constant-velocity inputs** to a position control system by their linearly increasing amplitude.

These waveforms can be used to test a system's ability to follow a linearly increasing input or, equivalently, to track a constant velocity target.

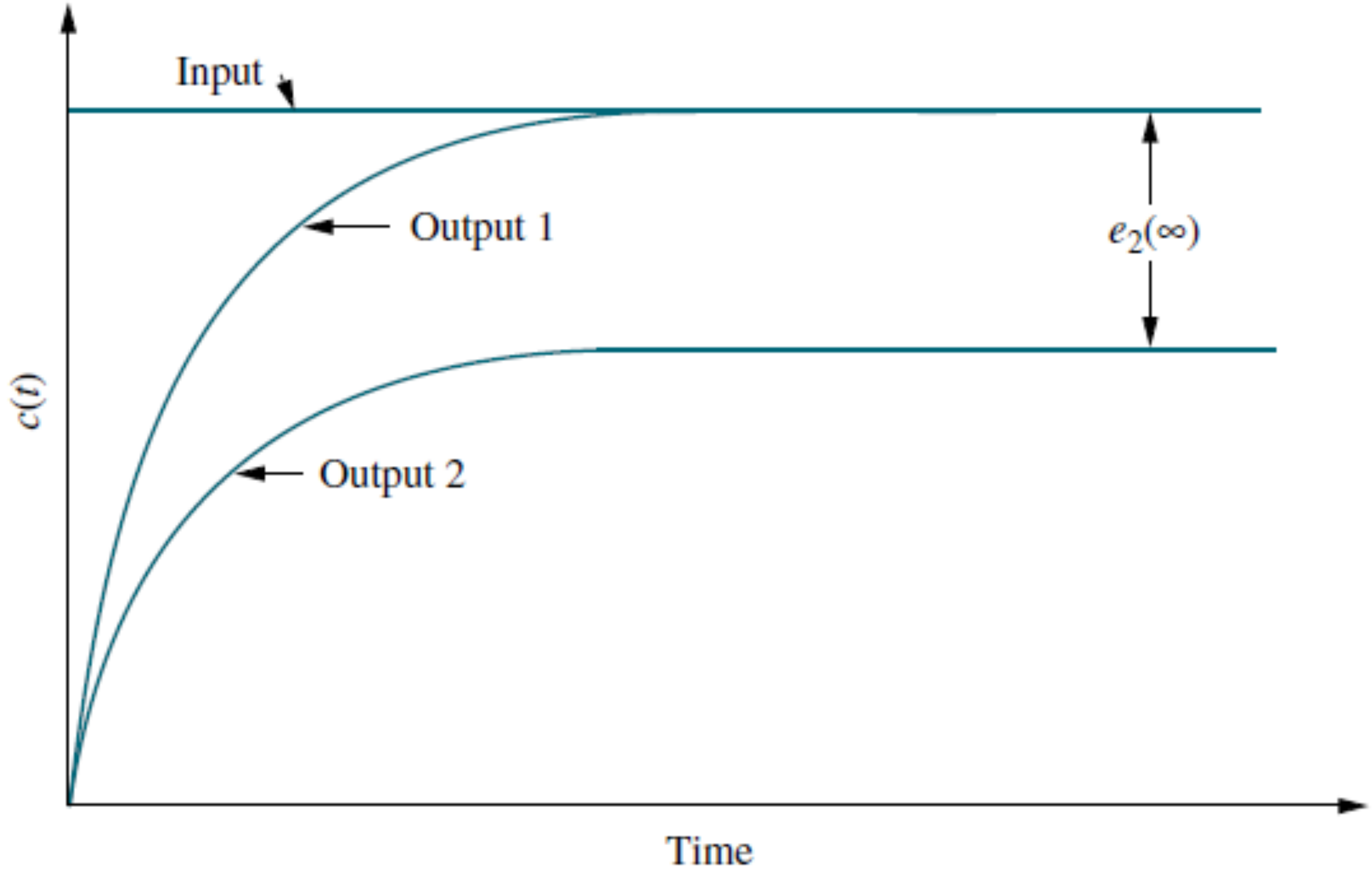
For example, a position control system that tracks a satellite that moves across the sky at a constant angular velocity would be tested with a ramp input to evaluate the steady-state error between the satellite's angular position and that of the control system.

Definition and Test Inputs: Constant Velocity

Parabola inputs represent **constant acceleration inputs** to position control systems.

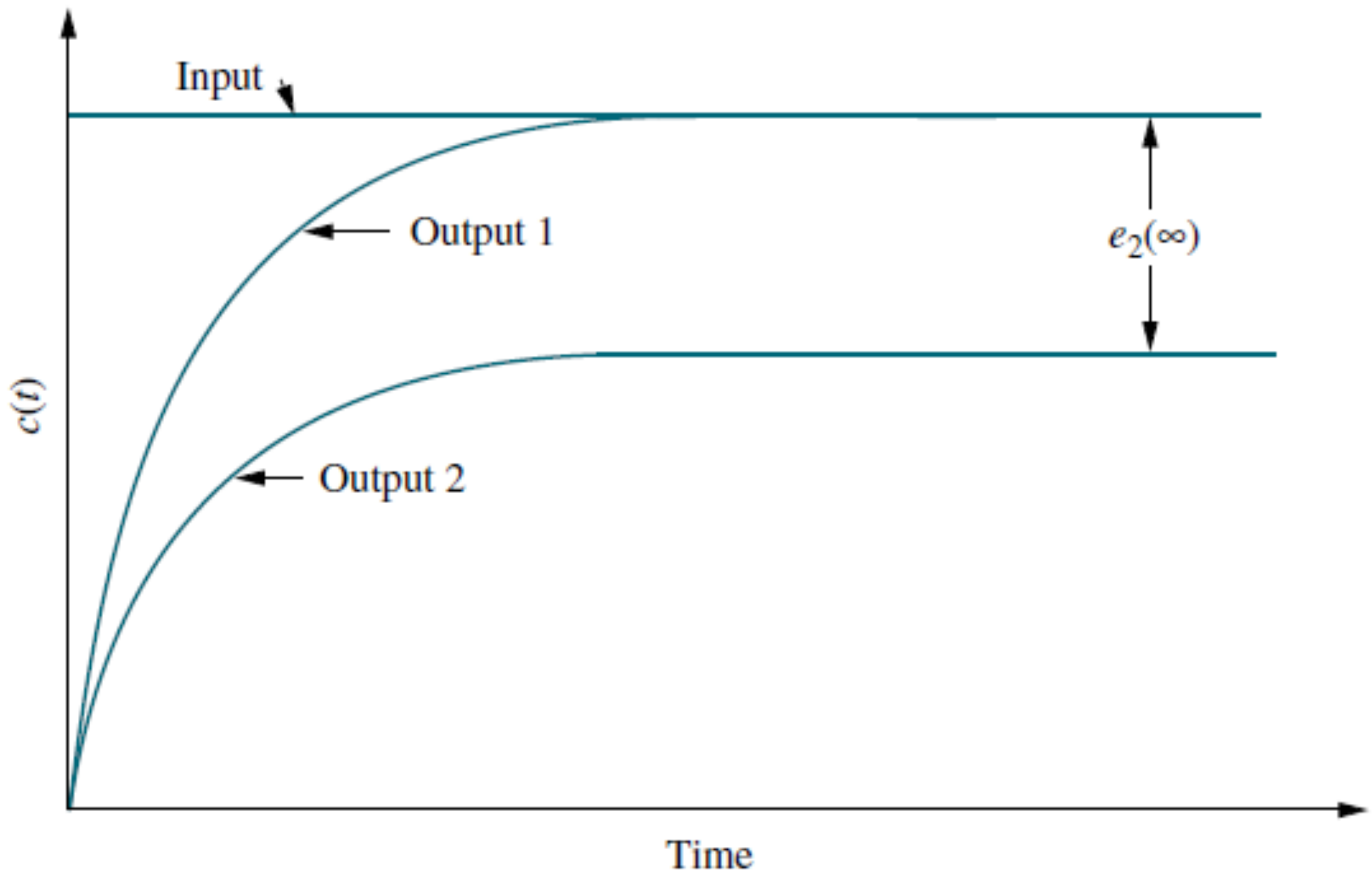
These can be used to represent accelerating targets to determine the steady-state error performance.

Evaluating Steady-State Errors



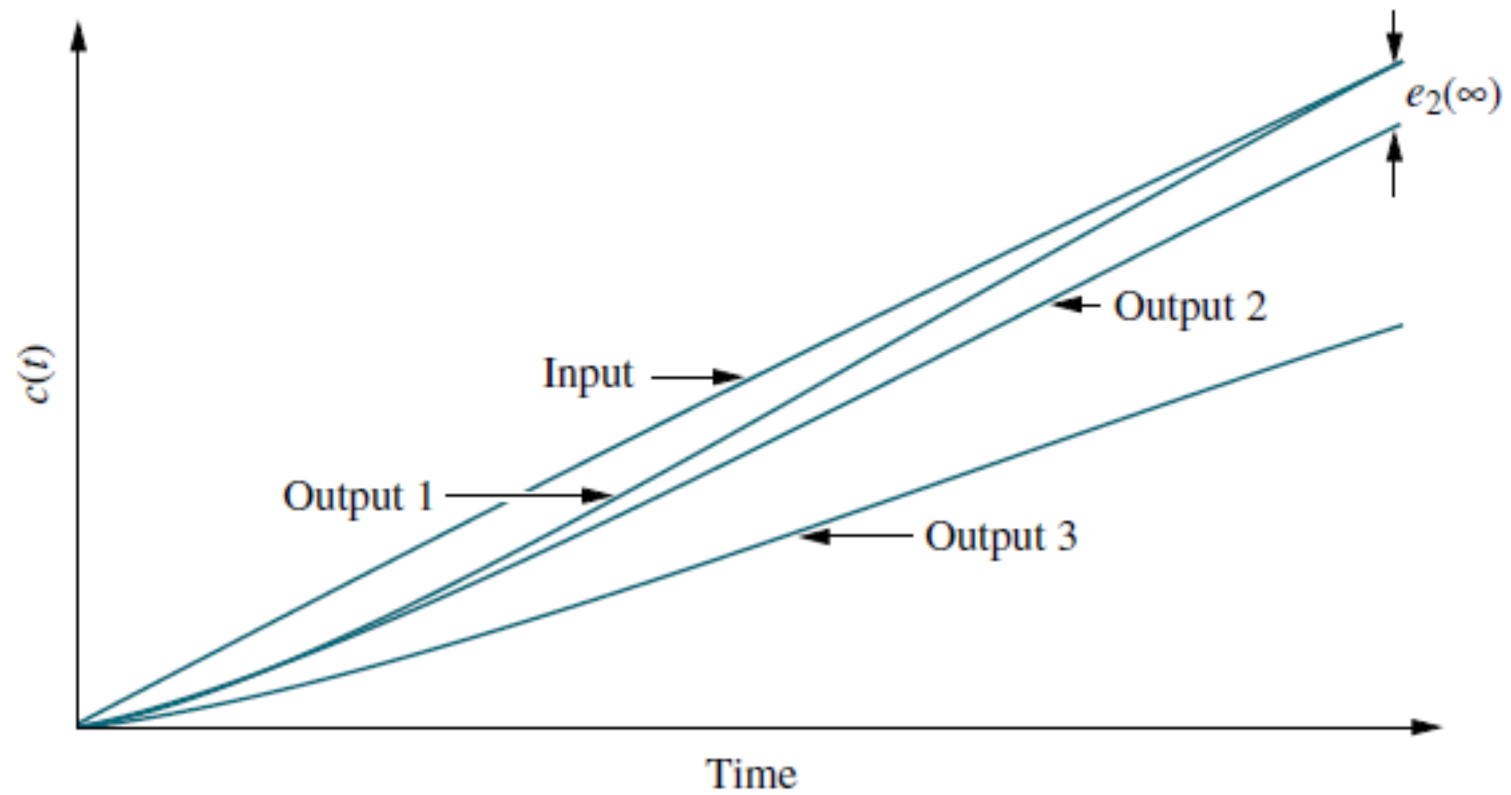
Consider a system with a step input and two possible outputs. Output 1 has **zero** steady-state error, and output 2 has a finite steady-state error, $e_2(\infty)$.

Evaluating Steady-State Errors



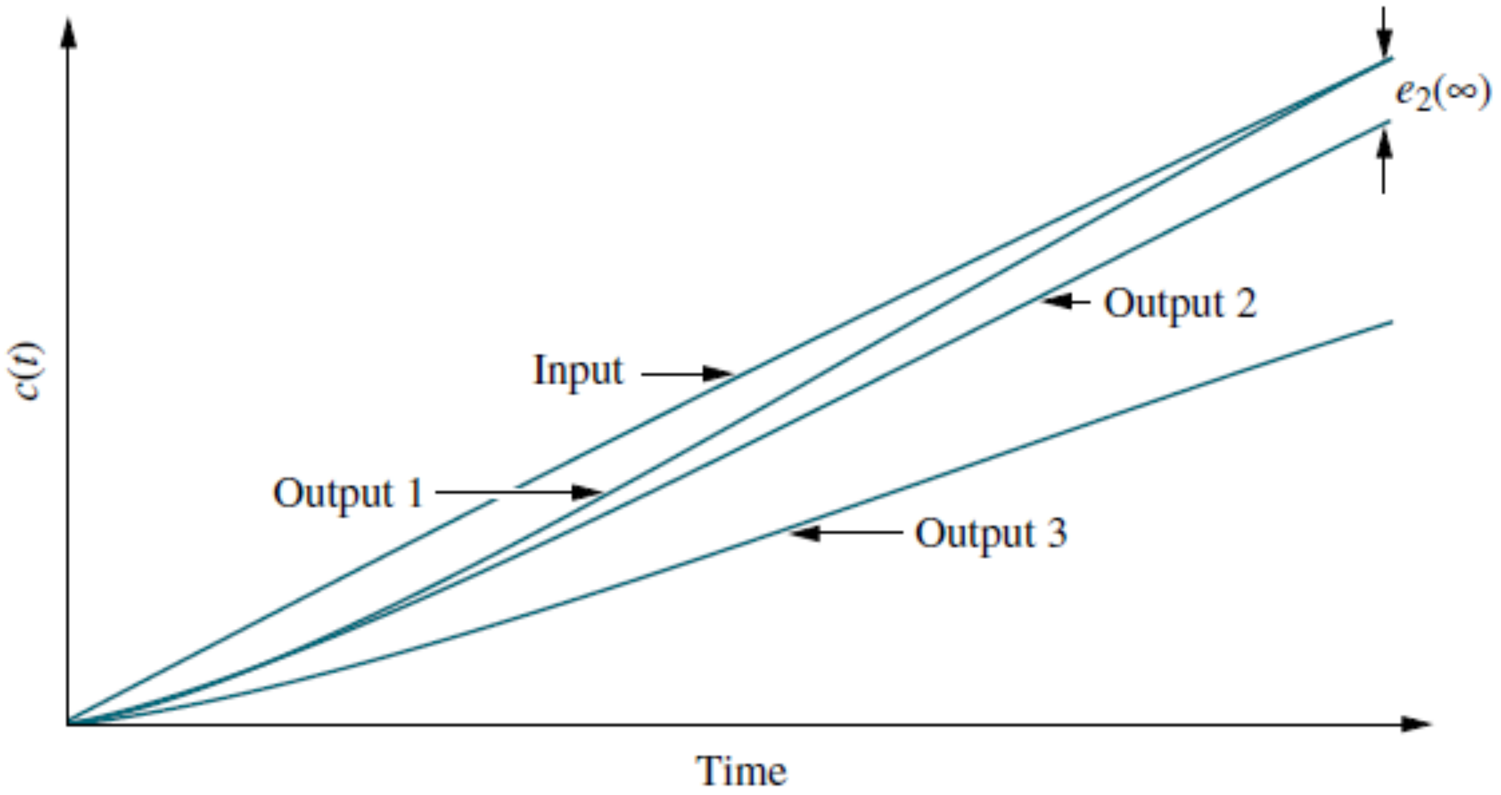
Step input is compared with output 1, which has zero steady-state error, and output 2, which has a finite steady-state error, $e_2(\infty)$, as measured vertically between the input and output 2 after the transients.

Evaluating Steady-State Errors



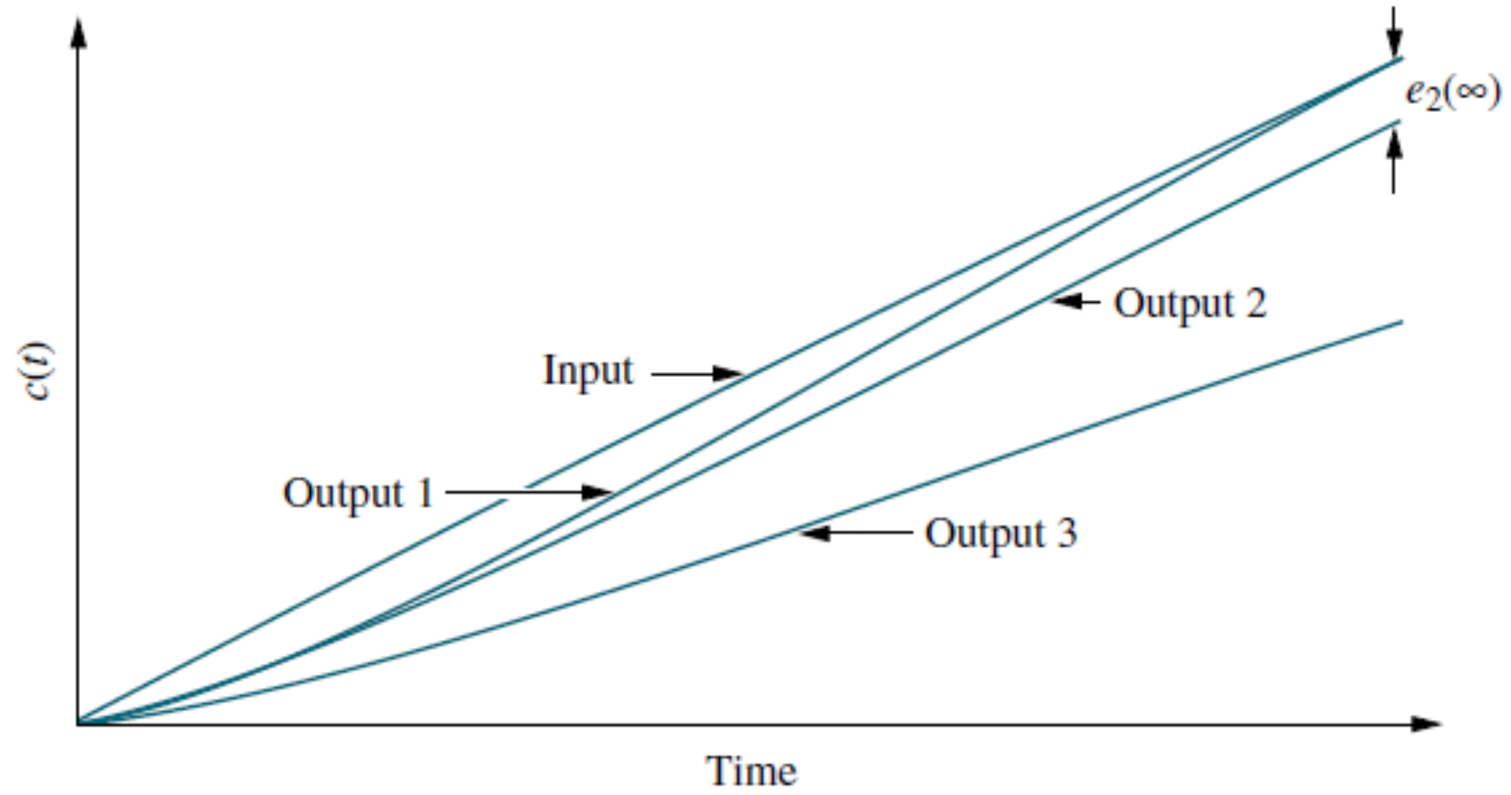
Consider a system with a ramp input which is compared with output 1, which has zero steady-state error, and output 2, which has a finite steady-state error, $e_2(\infty)$, as measured vertically between the input and output 2 after the transients.

Evaluating Steady-State Errors



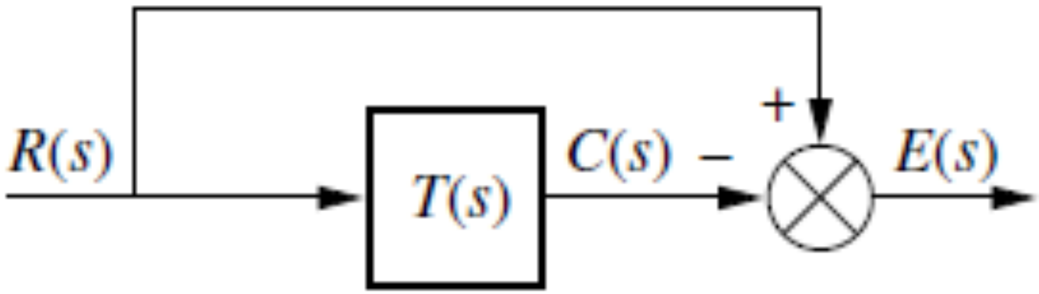
For the ramp input another possibility exists. If the output's slope is different from that of the input, then output 3 results. Here the steady-state error is infinite as measured vertically between the input and output 3 after the transients have died down, and t approaches infinity.

Evaluating Steady-State Errors



For the ramp input another possibility exists. If the output's slope is different from that of the input, then output 3 results. Here the steady-state error is infinite as measured vertically between the input and output 3 after the transients have died down, and t approaches infinity.

Evaluating Steady-State Errors



Since the error is the difference between the input and the output of a system, we assume a closed-loop transfer function, $T(s)$, and form the error, $E(s)$, by taking the difference between the input and the output. Here we are interested in the steady-state, or final, value of $e(t)$.

Evaluating Steady-State Errors

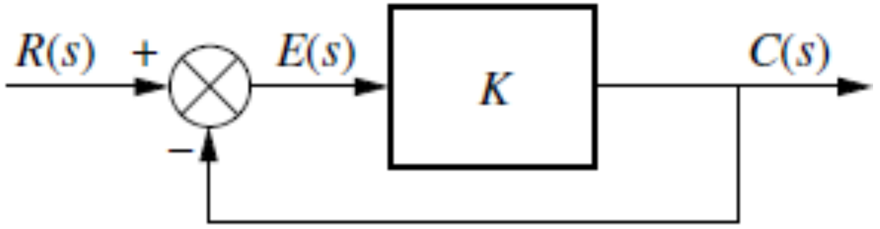
For unity feedback systems, $E(s)$, can be obtained as shown.

Sources of Steady-State Error

There are many sources for steady state errors such as backlash in gears or a motor that will not move unless the input voltage exceeds a threshold. These are the nonlinearities which causes the steady state error.

We study here are errors that arise from the configuration of the system itself and the type of applied input.

Sources of Steady-State Error



Consider the system as shown, where $R(s)$ is the input, $C(s)$ is the output, and $E(s) = R(s) - C(s)$ is the error. Consider a step input.

In the steady state, if $c(t)$ equals $r(t)$, $e(t)$ will be zero. But with a pure gain, K, the error, $e(t)$, cannot be zero if $c(t)$ is to be finite and nonzero.

Thus, by virtue of the configuration of the system (a pure gain of K in the forward path), an error must exist.

Sources of Steady-State Error

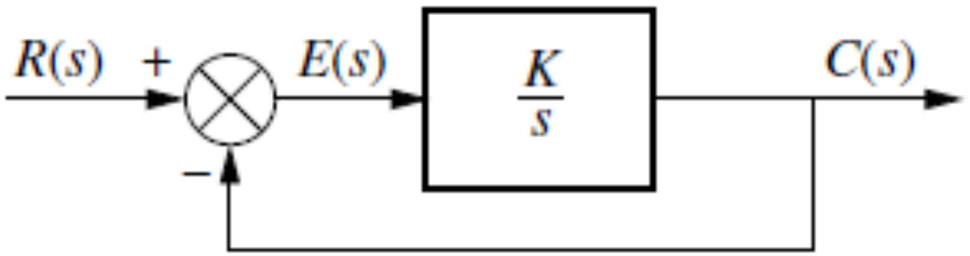
If we call $c_{steady-state}$ the steady state value of the output and $e_{steady-state}$ the steady-state value of the error, then $c_{steady-state} = Ke_{steady-state}$, or

$$e_{steady-state} = \frac{1}{K} c_{steady-state} \quad \dots 1$$

Thus, the larger the value of K, the smaller the value of $e_{steady-state}$ would have to be to yield a similar value of $c_{steady-state}$.

So, we conclude that "with a pure gain in the forward path, there will always be a steady-state error for a step input". This error reduces as the value of K increases.

Sources of Steady-State Error



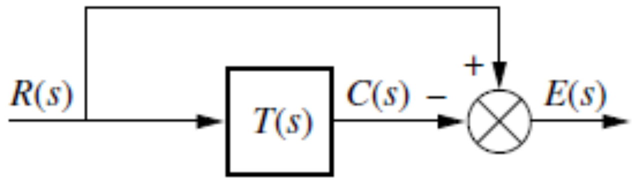
If the forward-path gain is replaced by an integrator, as shown, there will be zero error in the steady state for a step input.

The reasoning is as follows: As $c(t)$ increases, $e(t)$ will decrease, since $e(t) = r(t) - c(t)$. This decrease will continue until there is zero error, but there will still be a value for $c(t)$ since an integrator can have a constant output without any input.

Steady-State Error for Unity Feedback Systems

Steady-state error can be calculated from a system's closed-loop transfer function, $T(s)$, or the open-loop transfer function, $G(s)$, for unity feedback systems.

Steady-State Error in Terms of T(s)



Consider the system shown by Figure. To find $E(s)$, the error between the input, $R(s)$, and

Let the $E(s)$ be the error, $R(s)$ be the input, and $C(s)$ be the output

Therefore, $E(s) = R(s) - C(s)$ ... 1

But $C(s) = R(s)T(s)$ ... 2

Therefore from equation 1 and 2

$$E(s) = R(s)[1 - T(s)] \qquad \ldots 3$$

Steady-State Error for Unity Feedback Systems

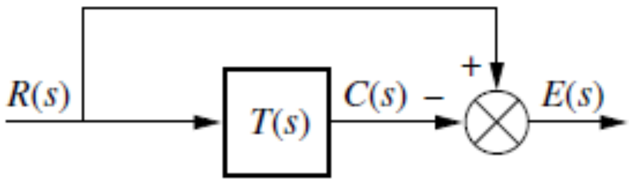
The final value of the error, $e(\infty)$ can be obtained by applying the final value theorem, which allows us to use the final value of $e(t)$ without taking the inverse Laplace transform of $E(s)$, andt hen letting t approach infinity, we obtain

$$e(\infty) = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) \qquad \ldots 4$$

$$\therefore e(\infty) = \lim_{s\to 0} sR(s)[1 - T(s)] \qquad \ldots 5$$

Steady-State Error for Unity Feedback Systems

Find the steady-state error for the system of Figure as shown if $\mathrm{T}(s) = 5/(s^2 + 7s + 10)$ and the input is a unit step



From the problem statement, $\mathrm{R}(s) = 1/s$ and $\mathrm{T}(s) = 5/(s^2 + 7s + 10)$

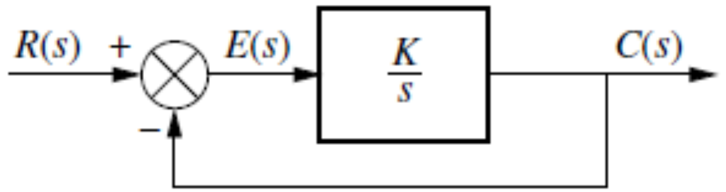
$$E(s) = \frac{s^2 + 7s + 5}{s(s^2 + 7s + 10)}$$

Applying the final value theorem

$$\therefore\ e(\infty) = \lim_{s \to 0} sR(s)[1 - T(s)] = \frac{1}{2}$$

Steady-State Error in Terms of G(s)

Consider the feedback control system shown in Figure



Since the feedback, $H(s) = 1$, the system has unity feedback.

The implication is that $E(s)$ is actually the error between the input, $R(s)$, and the output, $C(s)$.

Thus, if we solve for $E(s)$, we will have an expression for the error.

We will then apply the final value theorem to evaluate the steady-state error.

Steady-State Error in Terms of G(s)

Therefore, $E(s) = R(s) - C(s)$ ... 1

But $C(s) = R(s)G(s)$ ... 2

$$E(s) = \frac{R(s)}{1+G(s)} \qquad \dots 3$$

We now apply the final value theorem, before that we must check to see whether the closed-loop system is stable. Assume that the closed-loop system is stable and

$$\therefore\ e(\infty) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)} \qquad \dots 4$$

Equation 4 gives us the steady-state error, $e(\infty)$, for the given the input, $e(\infty)$, and the system, $G(s)$.

Steady-State Error in Terms of G(s): Step input

Now substitute several inputs for $R(s)$ and then draw conclusions about the relationships that exist between the open-loop system,

Step Input: Using Eq. (4) with $R(s) = 1/s$, we find

$$e(\infty) = e_{step}(\infty) = \lim_{s\to 0} \frac{s(1/s)}{1 + G(s)} = \frac{1}{1 + \lim_{s\to 0} G(s)} \qquad \ldots 5$$

The term $\lim_{s\to 0} G(s)$ is the dc gain of the forward transfer function, since s the frequency variable is approaching zero. In order to have zero steady-state error,

$$\lim_{s\to 0} G(s) = \infty \qquad \ldots 6$$

Hence, to satisfy Eq. (6), $G(s)$ must take on the following form

Steady-State Error in Terms of G(s) : Step input

Hence, to satisfy Eq. (6), $G(s)$ must take on the following form

$$G(s) = \frac{(s+z_1)(s+z_2)\ldots}{S^n(s+p_1)(s+p_2)\ldots} \qquad \ldots 7$$

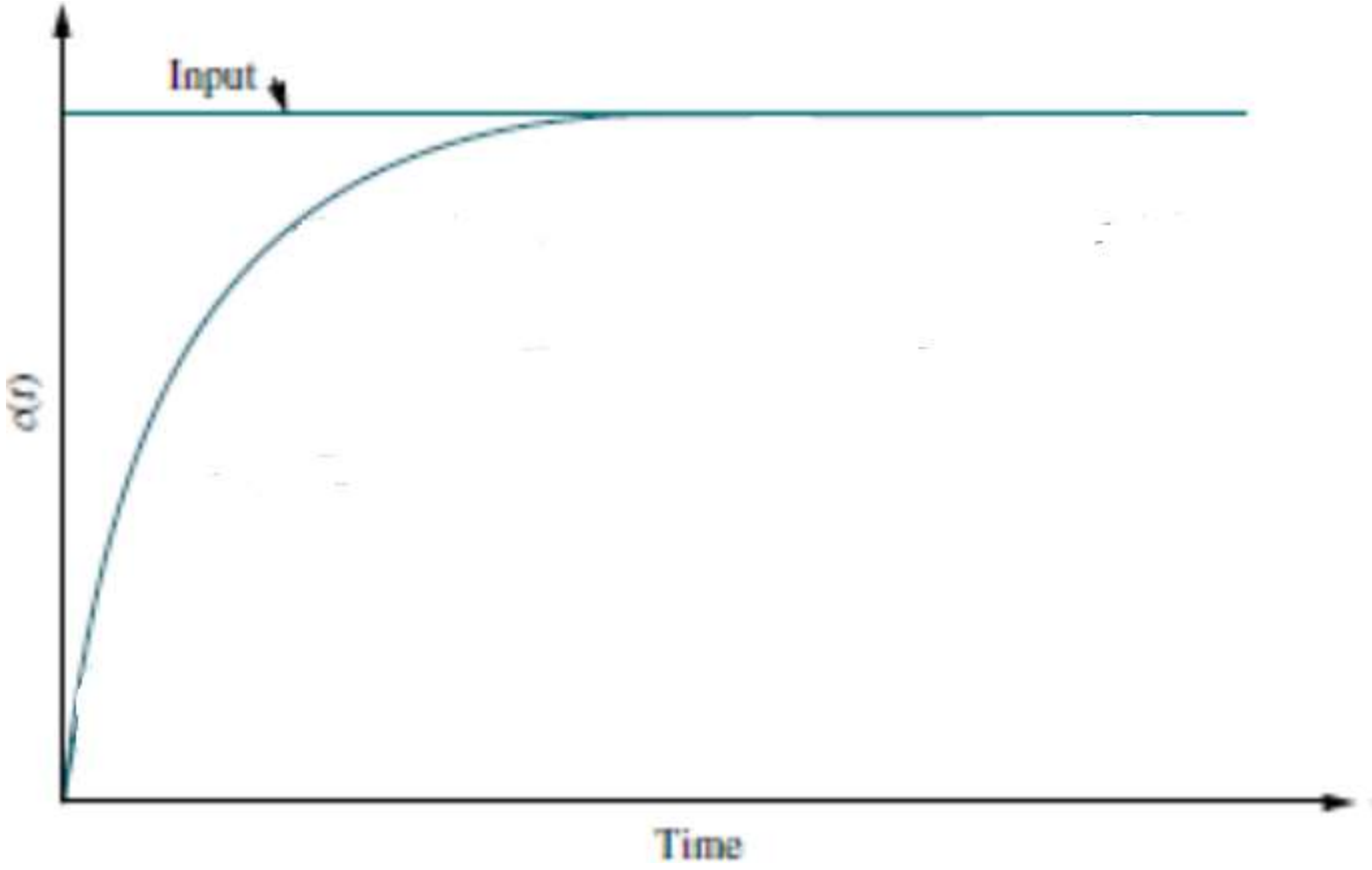
and for the limit to be infinite, the denominator must be equal to zero as s goes to zero. Thus, n ≥ 1; that is, at least one pole must be at the origin.

Since division by s in the frequency domain is integration in the time domain that means at least one pure integration must be present in the forward path.

The steady-state response for this case of zero steady-state error is similar to that shown in Figure output 1.

Steady-State Error in Terms of G(s) : Step input

The steady-state response for this case of zero steady-state error is shown in Figure.



Steady-State Error in Terms of G(s) : Step input

If there are no integrations, then n = 0. Using Eq. (7), we have

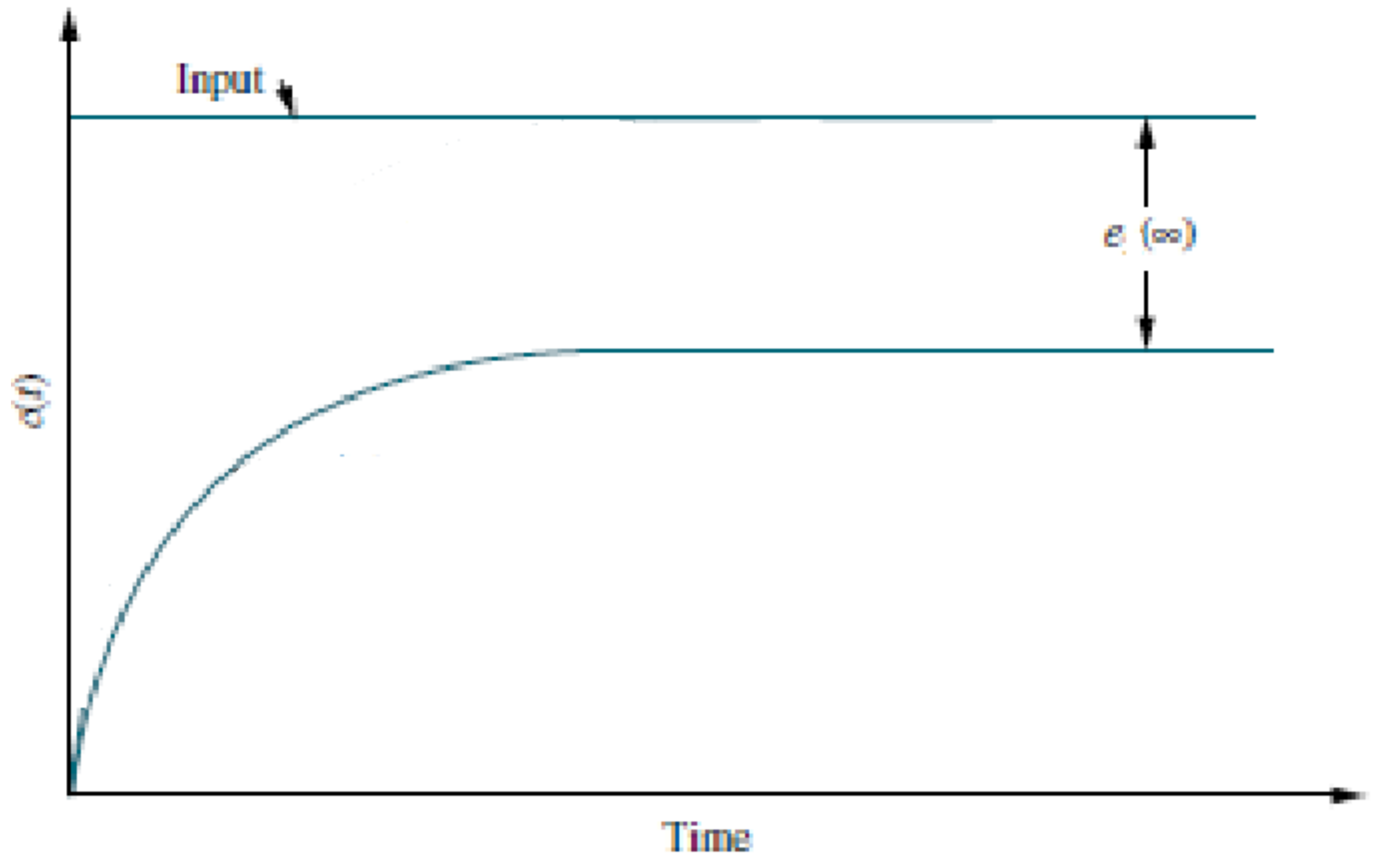
$$\lim_{s \to \infty} G(s) = \frac{z_1 z_2 z_3 \ldots}{p_1 p_2 p_3 \ldots} \qquad \ldots 8$$

which is finite and yields a finite error from Eq. (5).

Figure shows the steady-state response for this case of finite steady-state error.

Steady-State Error in Terms of G(s) : Step input

Figure shows the steady-state response for this case of finite steady-state error.



Steady-State Error in Terms of G(s) : Step input

Therefore, for a step input to a unity feedback system,

- the steady-state error will be zero if there is at least one pure integration in the forward path.
- If there are no integrations, then there will be a nonzero finite error.

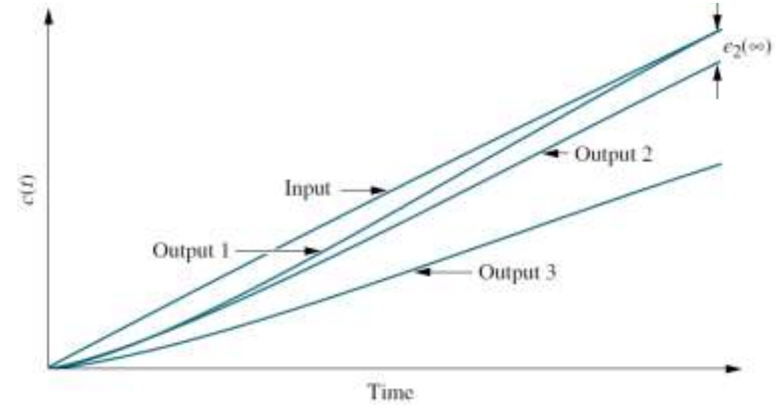
Steady-State Error in Terms of G(s): Ramp Input

Using Eq. (4) with $R(s) = 1/s^2$, we find

$$e(\infty) = e_{ramp}(\infty) = \lim_{s\to 0} \frac{s(1/s^2)}{1 + G(s)} = \lim_{s\to 0} \frac{1}{s + G(s)} = \frac{1}{\lim_{s\to 0} sG(s)} \qquad \dots 9$$

In order to have zero steady-state error for ramp input

$$\lim_{s\to 0} sG(s) = \infty \qquad \dots 10$$



Hence, to satisfy Eq. (10), $G(s)$ must take on the following form

$$G(s) = \frac{(s + z_1)(s + z_2)\dots}{S^n(s + p_1)(s + p_2)\dots}$$

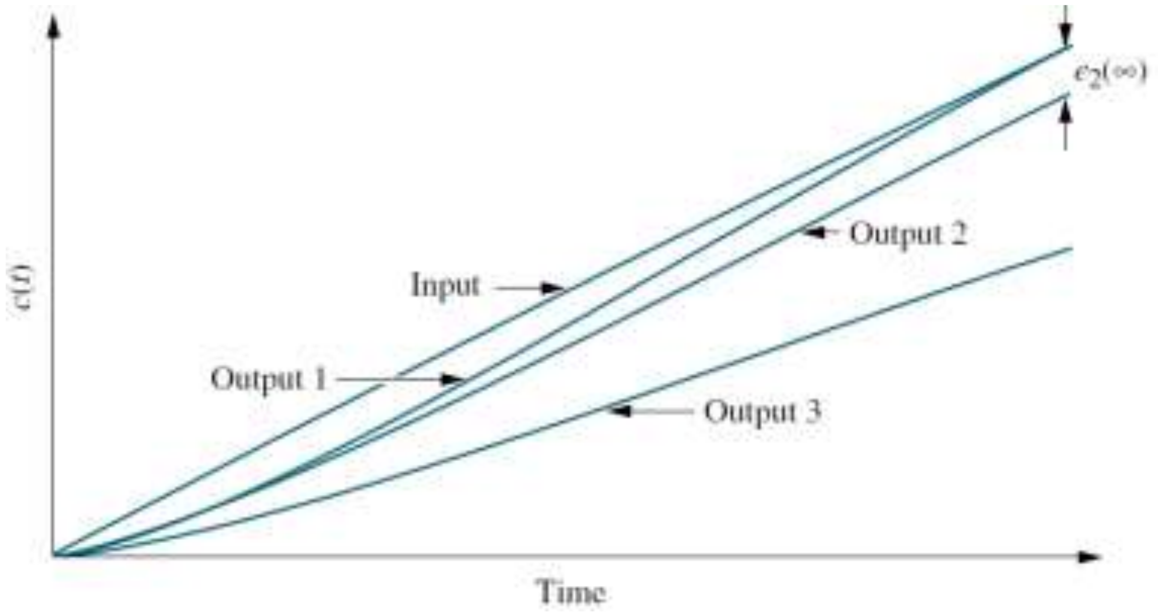
Here the s must be grater than 2. i.e. there must be at least two integrations in the forward path. (Output 1)

Steady-State Error in Terms of G(s): Ramp Input

If only one integration exists in the forward path, then we have

$$\lim_{s\to\infty} sG(s) = \frac{z_1 z_2 z_3 \ldots}{p_1 p_2 p_3 \ldots}$$

which is finite rather hence this configuration leads to a constant error.

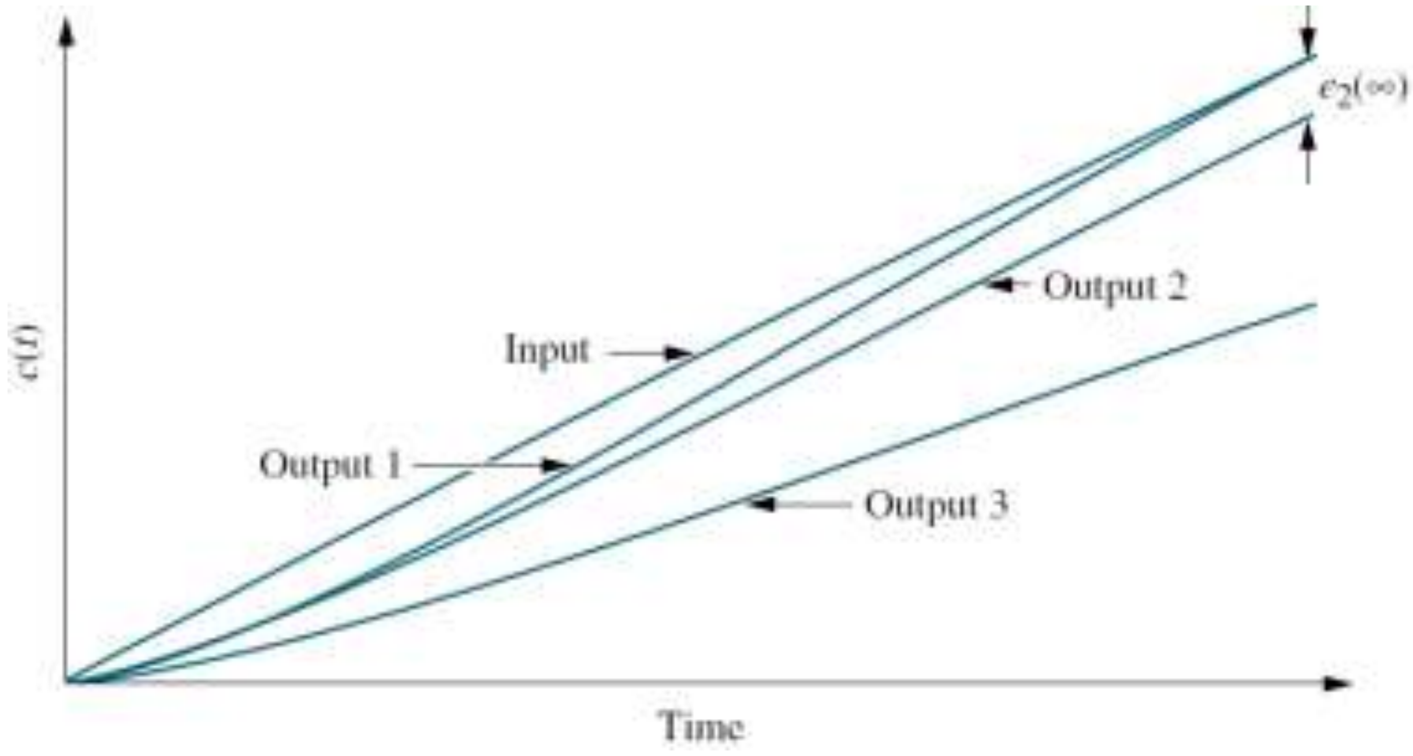


Steady-State Error in Terms of G(s): Ramp Input

If there are no integrations in the forward path, then

$$\lim_{s \to 0} sG(s) = 0$$

and the steady-state error would be infinite and lead to diverging ramps. (Output 3)



Steady-State Error in Terms of G(s): Parabola Input

Using Eq. (4) with $R(s) = 1/s^3$, we find

$$e(\infty) = e_{parabola}(\infty) = \lim_{s\to 0} \frac{s(1/s^3)}{1 + G(s)} = \lim_{s\to 0} \frac{1}{s^2 + s^2 G(s)} = \frac{1}{\lim_{s\to 0} s^2 G(s)} \qquad \dots 11$$

In order to have zero steady-state error for parabolic input

$$\lim_{s\to 0} s^2 G(s) = \infty \qquad \dots 12$$

Hence, to satisfy Eq. (12), $G(s)$ must take on the following form

$$G(s) = \frac{(s + z_1)(s + z_2)\dots}{S^n (s + p_1)(s + p_2)\dots}$$

Here the s must be grater than 3. i.e. there must be at least three integrations in the forward path.

Steady-State Error in Terms of G(s): Parabola Input

If only two integrations exist in the forward path, then we have

$$\lim_{s \to \infty} s^2 G(s) = \frac{z_1 z_2 z_3 \dots}{p_1 p_2 p_3 \dots}$$

which is finite rather hence this configuration leads to a constant error.

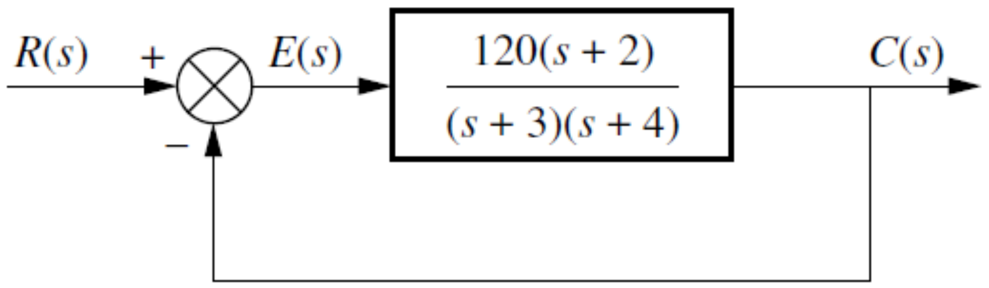
If there is only one or no integration in the forward path, then

$$\lim_{s \to 0} s^2 G(s) = 0$$

and the steady-state error would be infinite.

Steady-State Error in Terms of G(s): Numerical

Find the steady-state errors for inputs of 5u(t), 5tu(t), and $5t^2$u(t) to the system as shown in following Figure. Check whether the closed loop system is stable system or not. $s_1 = -124.9837; s_2 = -2.01626$



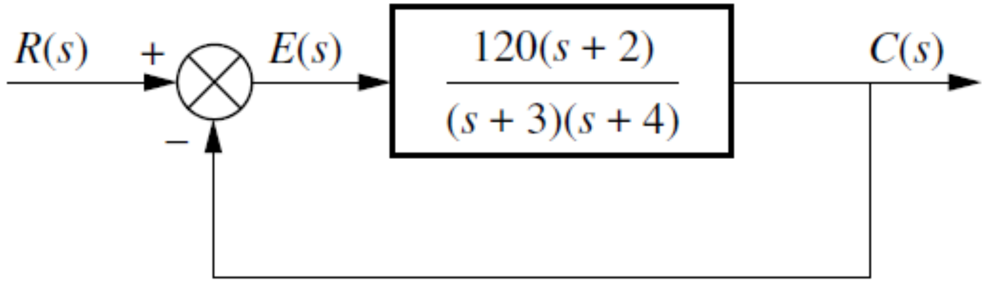
For the input **5u(t)** whose Laplace transform is **5/s**, the steady state error is

$$e(\infty) = e_{step}(\infty) = \lim_{s\to 0}\frac{s(5/s)}{1+G(s)} = \frac{5}{1+\lim_{s\to 0}G(s)} = \frac{5}{1+20} = \frac{5}{21}$$

Finite steady state error.

Steady-State Error in Terms of G(s): Numerical

Find the steady-state errors for inputs of 5u(t), 5tu(t), and $5t^2$u(t) to the system as shown in following Figure.



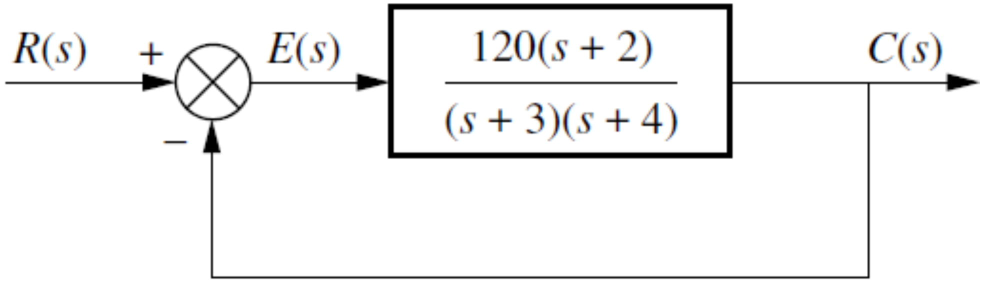
For the input **5tu(t)** whose Laplace transform is **5/s²**, the steady state error is

$$e(\infty) = e_{ramp}(\infty) = \lim_{s\to 0} \frac{s(5/s^2)}{1 + G(s)} = \lim_{s\to 0} \frac{5}{s + G(s)} = \frac{5}{\lim_{s\to 0} sG(s)} = \frac{5}{0} = \infty$$

Infinite steady state error.

Steady-State Error in Terms of G(s): Numerical

Find the steady-state errors for inputs of 5u(t), 5tu(t), and $5t^2$u(t) to the system as shown in following Figure.



For the input **5t²u(t)** whose Laplace transform is **5/s³**, the steady state error is

$$e(\infty) = e_{parabola}(\infty) = \lim_{s\to 0} \frac{s(10/s^3)}{1 + G(s)} = \lim_{s\to 0} \frac{10}{s^2 + s^2G(s)} = \frac{10}{\lim_{s\to 0} s^2G(s)} = \frac{10}{0} = \infty$$

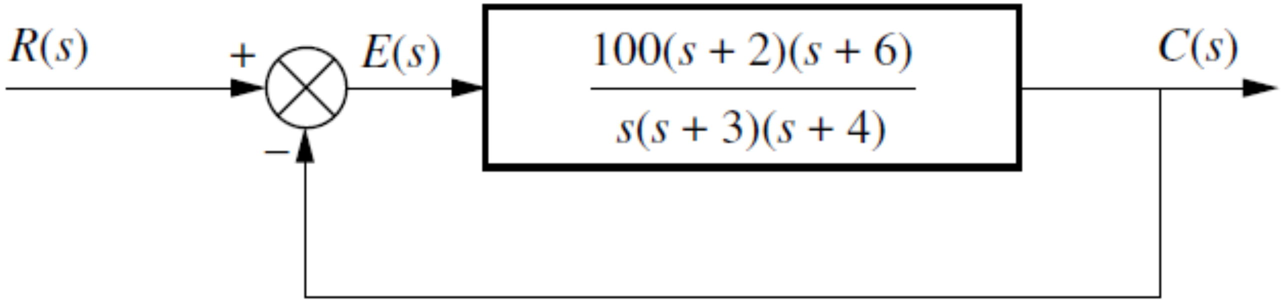
Infinite steady state error.

$s \approx -98.847$

Steady-State Error in Terms of G(s): Numerical 2

Find the steady-state errors for inputs of 5u(t), 5tu(t), and $5t^2u(t)$ to the system shown in Figure. The function u(t) is the unit step.

Check whether the closed loop system is stable system or not.



$$s_1 = -98.847$$

$$s_2 = -7.3246$$

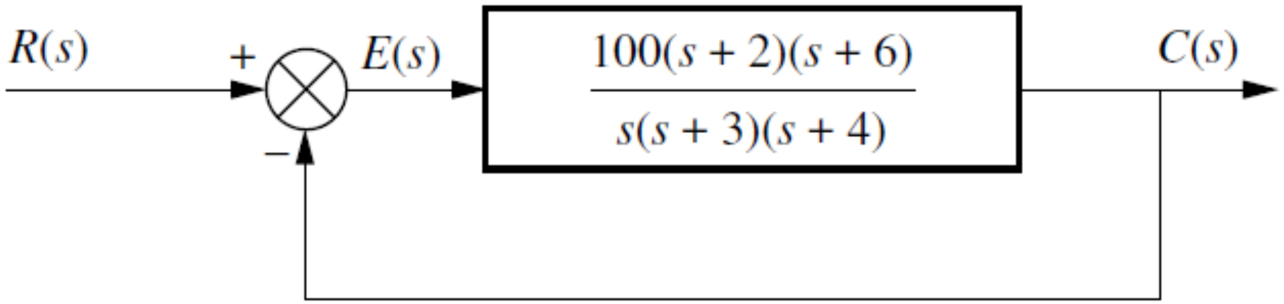
$$s3 = -0.82871$$

For the input **5u(t)** whose Laplace transform is **5/s**, the steady state error is

$$e(\infty) = e_{step}(\infty) = \lim_{s \to 0} \frac{s(5/s)}{1 + G(s)} = \frac{5}{1 + \lim_{s \to 0} G(s)} = \frac{5}{\infty} = 0$$

Steady-State Error in Terms of G(s): Numerical 2

Find the steady-state errors for inputs of 5u(t), 5tu(t), and $5t^2u(t)$ to the system shown in Figure. The function u(t) is the unit step.



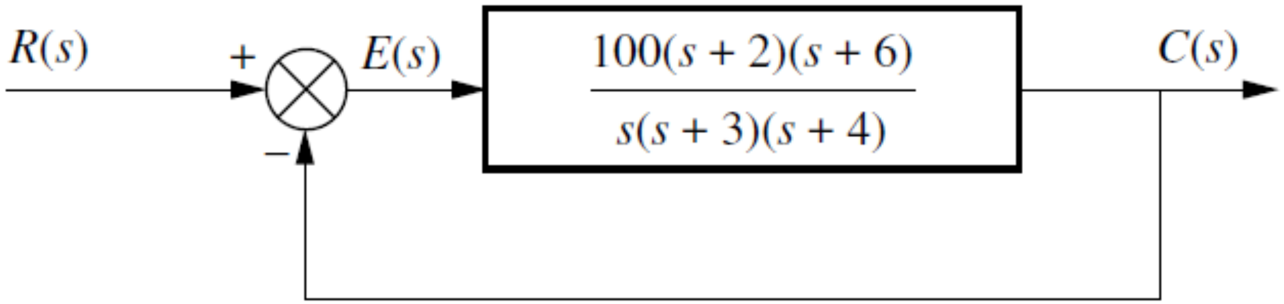
For the input **5tu(t)** whose Laplace transform is **$5/s^2$**, the steady state error is

$$e(\infty) = e_{ramp}(\infty) = \lim_{s\to 0} \frac{s(5/s^2)}{1+G(s)} = \lim_{s\to 0} \frac{5}{s+G(s)} = \frac{5}{\lim_{s\to 0} sG(s)} = \frac{5}{100} = \frac{1}{20}$$

Finite steady state error.

Steady-State Error in Terms of G(s): Numerical 2

Find the steady-state errors for inputs of 5u(t), 5tu(t), and $5t^2u(t)$ to the system shown in Figure. The function u(t) is the unit step.



For the input $\mathbf{5t^2u(t)}$ whose Laplace transform is $\mathbf{5/s^3}$, the steady state error is

$$e(\infty) = e_{parabola}(\infty) = \lim_{s\to 0} \frac{s(10/s^3)}{1+G(s)} = \lim_{s\to 0} \frac{10}{s^2+s^2G(s)} = \frac{10}{\lim_{s\to 0} s^2G(s)} = \frac{10}{0} = \infty$$

Infinite steady state error.

Static Error Constants and System Type

We know that

For step input u(t), ramp input tu(t) and parabolic input t^2u(t) the steady state error are

$$e(\infty) = e_{step}(\infty) = \frac{1}{1 + \lim_{s\to 0} G(s)}, e(\infty) = e_{ramp}(\infty) = \frac{1}{\lim_{s\to 0} sG(s)} \; and$$

$$e(\infty) = e_{ramp}(\infty) = \frac{1}{\lim_{s\to 0} s^2 G(s)}$$ respectively.

The three terms in the denominator that are taken to the limit determine the steady-state error. We call these limits static error constants.

Static Error Constants and System Type

The three terms in the denominator that are taken to the limit determine the steady-state error. We call these limits static error constants.

Position constant, $\text{Kp} = \lim_{s\to 0} G(s)$

Velocity constant, $\text{Kv} = \lim_{s\to 0} sG(s)$

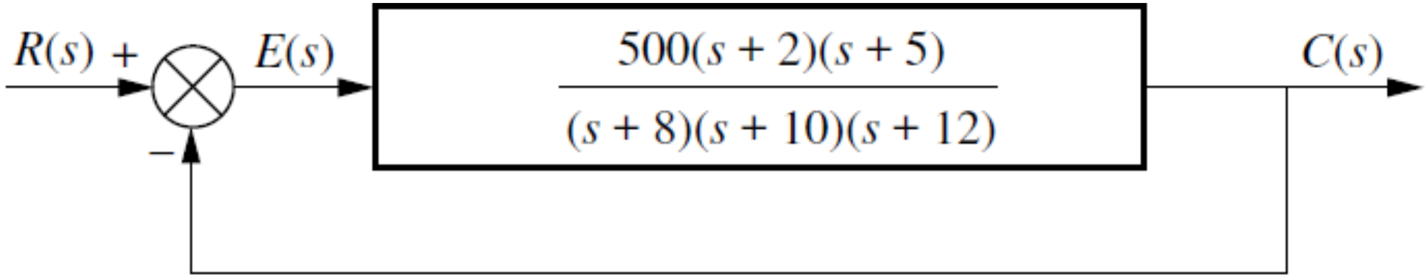
Acceleration constant, $\text{K}_{\text{a}} = \lim_{s\to 0} sG(s)$

These quantities depend upon the form of G(s) and may have values of zero, finite constant, or infinity.

Since the static error constant appears in the denominator of the steady-state error, the value of the steady-state error decreases as the static error constant increases.

Static Error Constants and System Type

For system shown evaluate the static error constants and find the expected error for the standard step, ramp, and parabolic inputs.



First verify that the closed-loop system is stable.

Position constant, $\mathrm{Kp} = \lim_{s \to 0} G(s) = \frac{500 \times 2 \times 5}{8 \times 10 \times 12} = 5.208$

Velocity constant, $\mathrm{Kv} = \lim_{s \to 0} sG(s) = 0$

Acceleration constant, $\mathrm{K_a} = \lim_{s \to 0} sG(s) = 0$

Static Error Constants and System Type

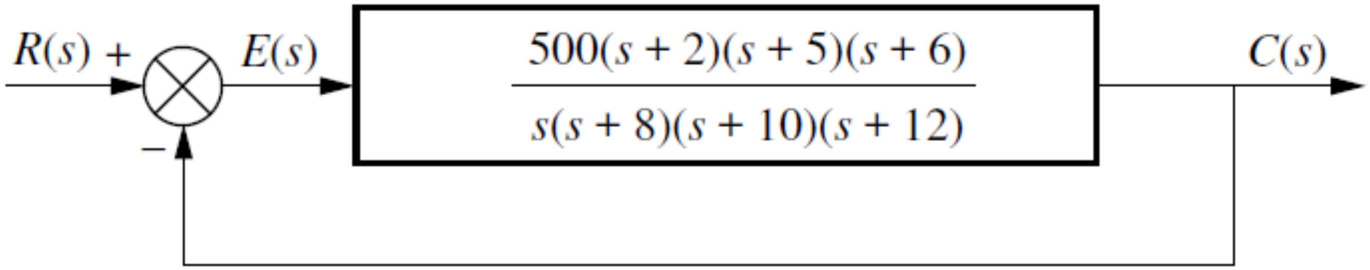
For step input, $e(\infty) = \frac{1}{1+Kp} = \frac{1}{5.208} = 0.161$

For ramp input, $e(\infty) = \frac{1}{K_v} = \infty$

For parabolic input, $e(\infty) = \frac{1}{K_a} = \infty$

Static Error Constants and System Type

For system shown evaluate the static error constants and find the expected error for the standard step, ramp, and parabolic inputs.



First verify that the closed-loop system is stable.

Position constant, $\mathrm{Kp} = \lim_{s\to 0} G(s) = \infty$

Velocity constant, $\mathrm{Kv} = \lim_{s\to 0} sG(s) = \frac{500\times 2\times 5\times 6}{8\times 10\times 12} = 31.25$

Acceleration constant, $\mathrm{K_a} = \lim_{s\to 0} sG(s) = 0$

Static Error Constants and System Type

For step input, $$e(\infty) = \frac{1}{1+Kp} = 0$$

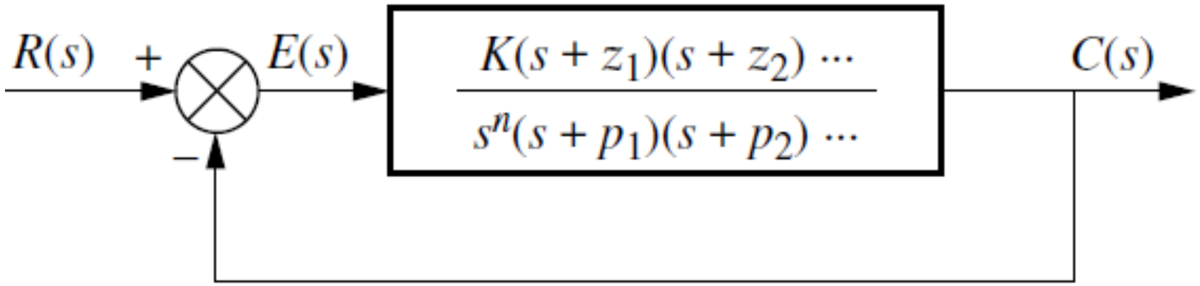
For ramp input, $$e(\infty) = \frac{1}{K_v} = \frac{1}{31.25} = 0.032$$

For parabolic input, $$e(\infty) = \frac{1}{K_a} = \infty$$

System Type

The values of the static error constants, again, depend upon the form of G(s), especially the number of pure integrations in the forward path.

Since steady-state errors are dependent upon the number of integrations in the forward path, we give a name to this system attribute.



Consider the system shown, we define system type to be the value of **n** in the denominator i.e. the number of pure integrations in the forward path. Therefore, a system with **n = 0** is a **Type 0** system. If **n = 1** or **n = 2**, the corresponding system is a **Type 1** or **Type 2** system, respectively.

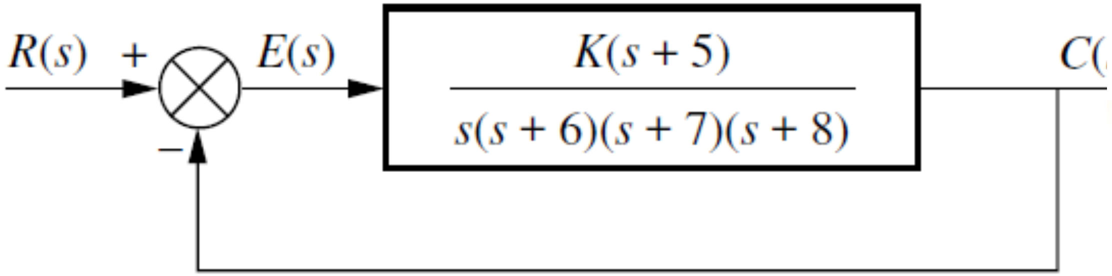
System Type

Relationships between input, system type, static error constants, and steady-state errors

Input	Steady-state error formula	Type 0 Static error constant	Type 0 Error	Type 1 Static error constant	Type 1 Error	Type 2 Static error constant	Type 2 Error
Step, $u(t)$	$\frac{1}{1+K_p}$	$K_p = \text{Constant}$	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v = \text{Constant}$	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a = \text{Constant}$	$\frac{1}{K_a}$

System Type

For the given the control system, find the value of K so that there is 10% error in the steady state.



$K_v = 10 = $

$$K_v = 10 = \lim_{s \to 0} sG(s) = \frac{K \times 5}{6 \times 7 \times 8}$$

$$K = 672$$

Since the system is Type 1, the error stated in the problem must apply to a ramp input; only a ramp yields a finite error in a Type 1 system. Thus, $e(\infty) = \frac{1}{Kv} = 0.1$

$$K_v = 10 = \lim_{s \to 0} sG(s) = \frac{K \times 5}{6 \times 7 \times 8}$$

$$\therefore K = 675$$

Feedback Control System
Unit 4
Stability

Stability: Definitions

The total response of a system is the sum of the forced and natural responses

$$c(t) = c_{\text{forced}}(t) + c_{\text{natural}}(t)$$

A LTI system is **stable** if the **natural response** approaches **zero** as **time** approaches **infinity**.

A LTI system is **unstable** if the **natural response grows** without **bound** as **time** approaches **infinity**.

A LTI system is **marginally stable** if the **natural response** neither **decays** nor **grows** but remains **constant** or **oscillates** as **time** approaches **infinity**.

Stability: Definitions

Therefore, for the stable system only the forced response remains as the natural response approaches zero.

Now if the input is bounded and the total response is not approaching infinity as time approaches infinity, then the natural response is obviously not approaching infinity.

If the input is unbounded, we see an unbounded total response, and we cannot arrive at any conclusion about the stability of the system; we cannot tell whether the total response is unbounded because the forced response is unbounded or because the natural response is unbounded.

Stability: Definitions

Thus, the alternate definition of stability is

A system is **stable** if **every bounded input yields a bounded output**.

This is the bounded-input, bounded-output (BIBO) definition of stability.

Stability: Poles locations

How do we determine if a system is stable?

Let us focus on the natural response definitions of stability.

We know the system poles that poles in the left half-plane (lhp) yield either pure exponential decay or damped sinusoidal natural responses.

These natural responses decay to zero as time approaches infinity. Thus, if the closed-loop system poles are in the left half of the plane and hence have a negative real part, the system is stable.

That is, stable systems have closed-loop transfer functions with poles only in the left half-plane.

Stability: Poles locations

Poles in the right half-plane (rhp) yield either pure exponentially increasing or exponentially increasing sinusoidal natural responses.

These natural responses approach infinity as time approaches infinity.

Thus, if the closed-loop system poles are in the right half of the s-plane and hence have a positive real part, the system is unstable.

Stability: Poles locations

The poles of multiplicity greater than 1 on the imaginary axis lead to the sum of responses of the form

$At^n \cos(\omega t + \Phi)$, where $n = 1; 2; \ldots ;$

which also approaches infinity as time approaches infinity.

Thus, unstable systems have closed loop transfer functions with

at least one pole in the right half-plane and/or

poles of multiplicity greater than 1 on the imaginary axis

Stability: Poles locations

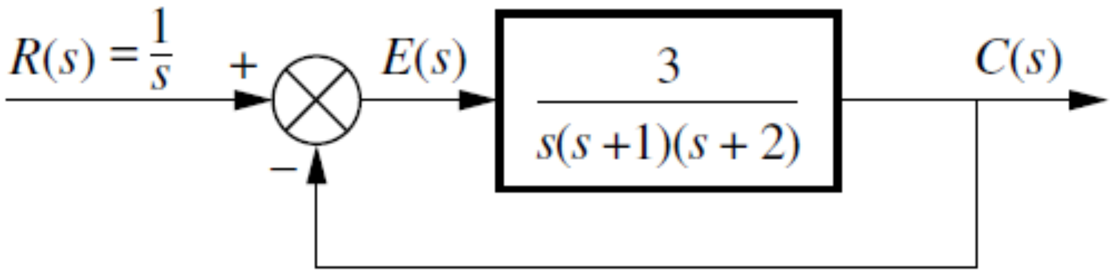
A system that has imaginary axis poles of multiplicity 1 yields pure sinusoidal oscillations as a natural response.

These responses neither increase nor decrease in amplitude.

Thus, marginally stable systems have closed-loop transfer functions with only imaginary axis poles of multiplicity 1 and poles in the left half-plane.

Stability: Poles locations

Comment on the stabilty of the system by finding the poles of closed loop system



It is not always a simple matter to determine if a feedback control system is stable.

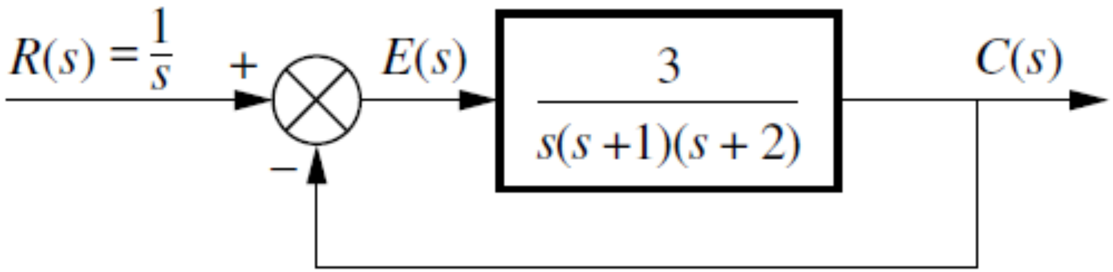
Here we know the poles of the forward transfer function in Figure, now we have to find the closed loop poles

$G(s) = \frac{3}{s(s+1)(s+3)}$ the closed loop transfer function is

$$T(s) = \frac{G(s)}{1+G(s)} = \frac{\frac{3}{s(s+1)(s+3)}}{1+\frac{3}{s(s+1)(s+3)}} =$$

Stability: Poles locations

Comment on the stability of the system by finding the poles of closed loop system



It is not always a simple matter to determine if a feedback control system is stable.

Here we know the poles of the forward transfer function in Figure, now we have to find the closed loop poles

$G(s) = \frac{3}{s(s+1)(s+2)}$ the closed loop transfer function is $T(s) = \frac{G(s)}{1+G(s)}$

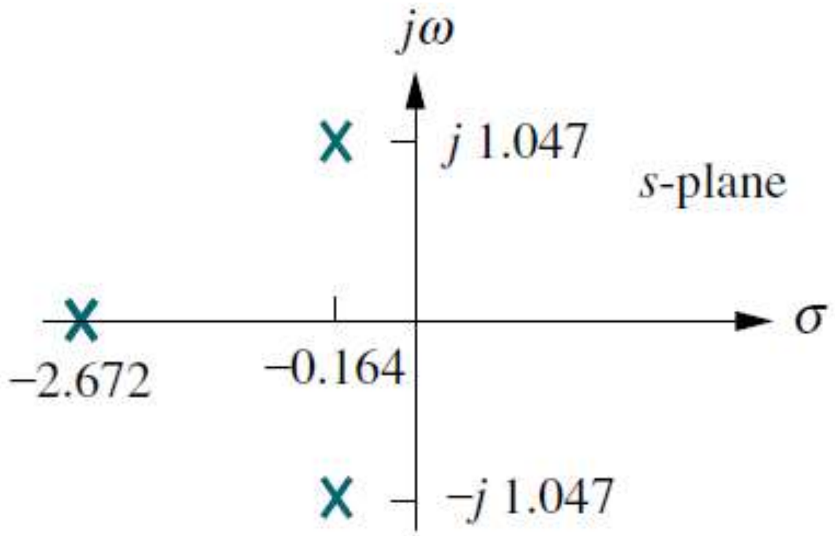
$$\therefore T(s) = \frac{\frac{3}{s(s+1)(s+2)}}{1+\frac{3}{s(s+1)(s+2)}} = \frac{3}{s(s+1)(s+2)+3} = \frac{3}{s^3+3s^2+2s+3}$$

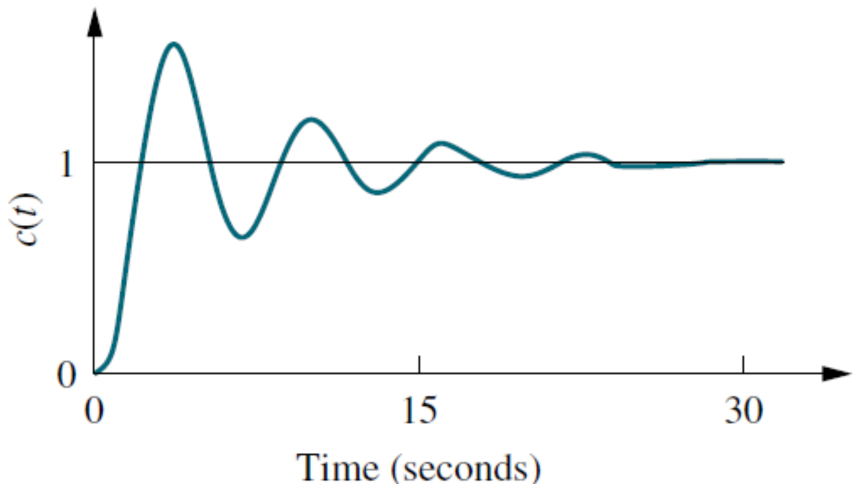
Stability: Poles locations

The roots of $s^3 + 3s^2 + 2s + 3$ gives us the closed loop poles

$s_1 = -2.6717$ and $s_{2,3} = -.16415 \pm j1.04687$

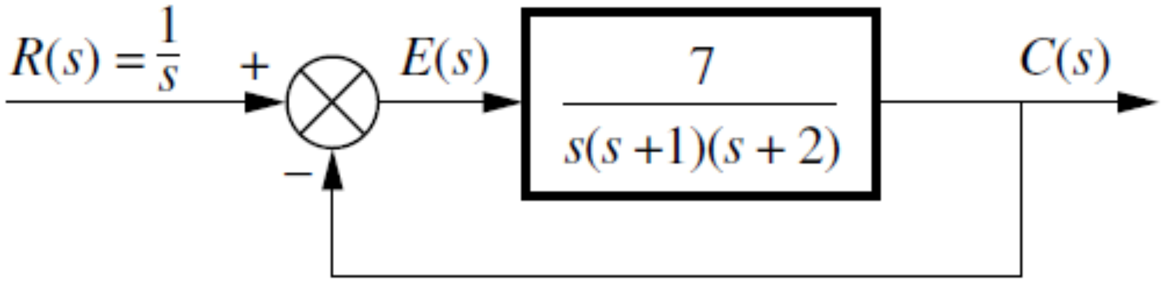
As all three poles are in left half of s plane the system is stable system.





Stability: Poles locations

Comment on the stability of the system by finding the poles of closed loop system



Here we know the poles of the forward transfer function in Figure, now we have to find the closed loop poles

$G(s) = \frac{7}{s(s+1)(s+2)}$ the closed loop transfer function is $T(s) = \frac{G(s)}{1+G(s)}$

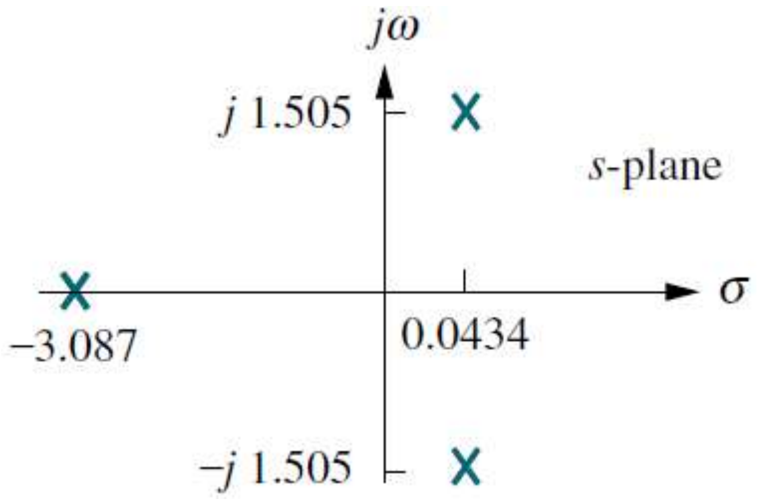
$$\therefore T(s) = \frac{\frac{7}{s(s+1)(s+2)}}{1 + \frac{7}{s(s+1)(s+2)}} = \frac{7}{s(s+1)(s+2)+7} = \frac{7}{s^3+3s^2+2s+7}$$

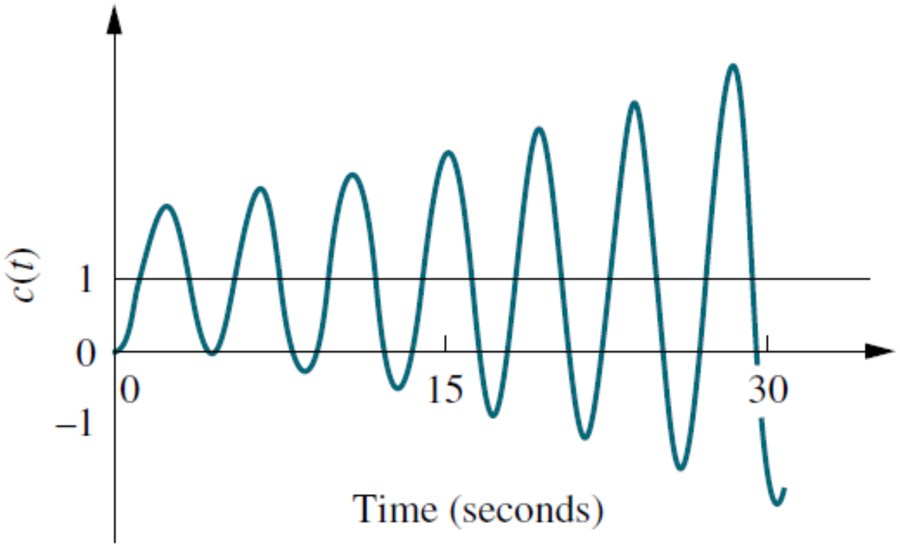
Stability: Poles locations

The roots of $s^3 + 3s^2 + 2s + 7$ gives us the closed loop poles

$s_1 = -3.0867$ and $s_{2,3} = 0.0434 \pm j1.5053$

As two poles are in right half of s plane the system is unstable system.





Routh-Hurwitz Criterion

This method gives stability information without finding the closed-loop system poles.

Using this method gives us

how many closed-loop system poles are in the left half-plane, in the right half-plane, and on the $j\omega$-axis.

Thus, we can find the number of poles in each section of the s-plane, but we cannot find their coordinates.

The method is called the Routh-Hurwitz criterion for stability (Routh, 1905).

Routh-Hurwitz Criterion

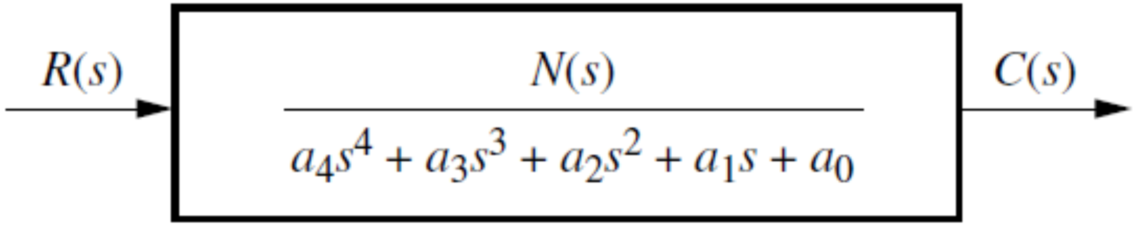
The method requires two steps:

(1) Generate a data table called a Routh table and

(2) interpret the Routh table to tell how many closed-loop system poles are inthe left half-plane, the right half-plane, and on the jω-axis.

Routh-Hurwitz Criterion

Consider a system having the closed loop transfer function as shown



We first create the Routh table shown in Table (next slide).

1. Begin by labeling the rows with powers of s from the highest power of the denominator of the closed-loop transfer function to s^0
2. Start with the coefficient of the highest power of s in the denominator and list, horizontally in the first row, every other coefficient.
3. In the second row, list horizontally, starting with the next highest power of s, every coefficient that was skipped in the first row

Routh-Hurwitz Criterion

Initial layout for Routh table

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2			
s^1			
s^0			

Routh-Hurwitz Criterion

The remaining entries are filled in as follows.

- Each entry is a negative determinant of entries in the previous two rows divided by the entry in the first column directly above the calculated row.
- The left-hand column of the determinant is always the first column of the previous two rows, and
- the right-hand column is the elements of the column above and to the right.
- The table is complete when all of the rows are completed down to s^0.

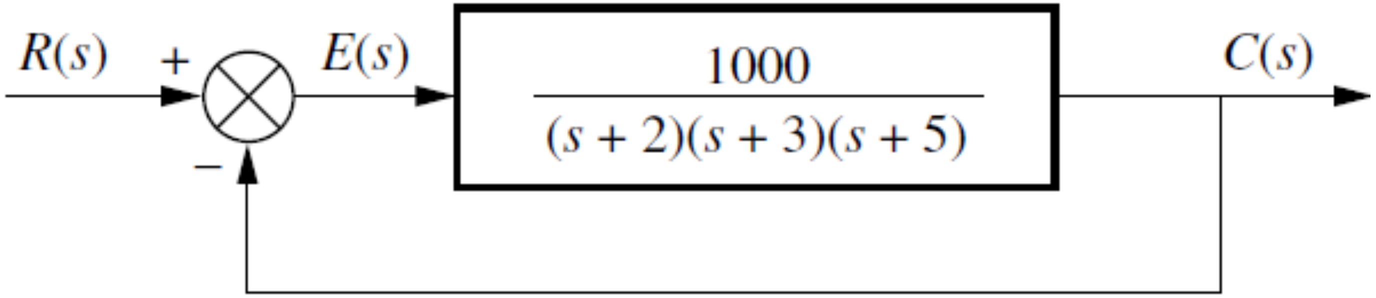
Routh-Hurwitz Criterion

Find all the elements of Routh table

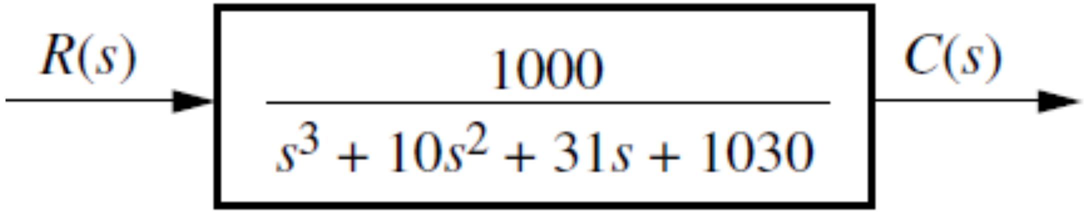
s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\dfrac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\dfrac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\dfrac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\dfrac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\dfrac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

Routh-Hurwitz Criterion

Make the Routh table for the system shown in Figure



The first step is to find the equivalent closed-loop system because we want to test locations of the closed loop poles. So, the closed loop system as shown below



Now form the Routh table

Routh-Hurwitz Criterion

First label the rows with powers of s from s^3 down to s^0 in a vertical column,

Next form the first row of the table, using the coefficients of the denominator of the closed-loop transfer function.

Start with the coefficient of the highest power and skip every other power of s.

Now form the second row with the coefficients of the denominator skipped in the previous step.

Subsequent rows are formed with determinants, as discussed.

For convenience, any row of the Routh table can be multiplied by a positive constant without changing the values of the rows below. In this example, 2nd the row was multiplied by 1/10.

Routh-Hurwitz Criterion

Completed Routh table

$$
\begin{array}{llll}
\hline
s^3 & 1 & 31 & 0 \\
s^2 & \cancel{10}\quad 1 & \cancel{1030}\quad 103 & 0 \\
s^1 & \dfrac{-\begin{vmatrix} 1 & 31 \\ 1 & 103 \end{vmatrix}}{1} = -72 & \dfrac{-\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}}{1} = 0 & \dfrac{-\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0 \\
s^0 & \dfrac{-\begin{vmatrix} 1 & 103 \\ -72 & 0 \end{vmatrix}}{-72} = 103 & \dfrac{-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0 & \dfrac{-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0 \\
\hline
\end{array}
$$

Interpreting the Basic Routh Table

The Routh-Hurwitz criterion declares that

the number of roots of the polynomial that are in the right half-plane is equal to the number of sign changes in the first column.

If the closed-loop transfer function has all poles in the left half of the s-plane, the system is stable. Thus, a system is stable if there are no sign changes in the first column of the Routh table.

In example which we are discussing has two sign changes in the first column.

The first sign change occurs from 1 in the s^2 row to 72 in the s^1 row.

The second occurs from 72 in the s^1 row to 103 in the s^0 row.

Thus, the system is unstable since two poles exist in the right half-planey.

$$\begin{array}{ll}
s^3 & 1 \\
s^2 & \cancel{10}\ \ 1 \\
s^1 & \dfrac{-\begin{vmatrix} 1 & 31 \\ 1 & 103 \end{vmatrix}}{1} = -72 \\
s^0 & \dfrac{-\begin{vmatrix} 1 & 103 \\ -72 & 0 \end{vmatrix}}{-72} = 103
\end{array}$$

Routh Table: Zero Only in the First Column (Epsilon method)

Determine the stability of the closed-loop transfer function

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

Form the Routh table by using the denominator. Begin by assembling the Routh table down to the row where a zero appears only in the first column (the s^3 row).

Next replace the zero by a small number, **ε**, and complete the table.

To begin the interpretation, we must first assume a sign, positive or negative, for the quantity **ε**.

Routh Table: Zero Only in the First Column (Epsilon method)

Completed Routh table

s^5	1	3	5
s^4	2	6	3
s^3	$\cancel{0}\ \ \epsilon$	$\frac{7}{2}$	0
s^2	$\frac{6\epsilon - 7}{\epsilon}$	3	0
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	0	0
s^0	3	0	0

Routh Table: Zero Only in the First Column (Epsilon method)

Determining signs in first column of a Routh table with zero as first element in a row

Label	**First column**	$\epsilon = +$	$\epsilon = -$
s^5	1	$+$	$+$
s^4	2	$+$	$+$
s^3	$\cancel{0}$ ϵ	$+$	$-$
s^2	$\dfrac{6\epsilon - 7}{\epsilon}$	$-$	$+$
s^1	$\dfrac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	$+$	$+$
s^0	3	$+$	$+$

Routh Table: Zero Only in the First Column (Epsilon method)

If **ε** is chosen **positive**, table will show a sign change from the s^3 row to the s^2 row, and there will be another sign change from the s^2 row to the s^1 row.

Hence, the system is unstable and has two poles in the right half-plane.

If **ε** is chosen **negative**. Table would then show a sign change from the s^4 row to the s^3 row.

Another sign change would occur from the s^3 row to the s^2 row.

So the result would be exactly the same as that for a positive choice for **ε**.

Thus, the system is unstable, with two poles in the right half-plane

Routh Table: Zero Only in the First Column (Alternative method)

Another method that can be used when a zero appears only in the first column of a row is derived from the fact that

a polynomial that has the reciprocal roots of the original polynomial has its roots distributed the same—right half-plane, left half plane, or imaginary axis—because taking the reciprocal of the root value does not move it to another region.

Thus, if we can find the polynomial that has the reciprocal roots of the original, it is possible that the Routh table for the new polynomial will not have a zero in the first column.

This method is usually computationally easier than the epsilon method just discussed.

Routh Table: Zero Only in the First Column (Alternative method)

We now show that the new polynomial (the one with the reciprocal roots) is simply the original polynomial with its coefficients written in reverse order.

Assume the equation.

$$s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0 \qquad \ldots 1$$

If s is replaced by 1/d, then d will have roots which are the reciprocal of s. Making this substitution in Eq. (1)

$$\left(\frac{1}{d}\right)^n + a_{n-1}\left(\frac{1}{d}\right)^{n-1} + \cdots + a_1\left(\frac{1}{d}\right) + a_0 = 0 \qquad \ldots 2$$

Factoring out $(1/d)^n$

Routh Table: Zero Only in the First Column (Alternative method)

$$\left(\frac{1}{d}\right)^n\left[1+a_{n-1}\left(\frac{1}{d}\right)^{-1}+\cdots+a_1\left(\frac{1}{d}\right)^{(1-n)}+a_0\left(\frac{1}{d}\right)^{-n}\right]$$

$$=\left(\frac{1}{d}\right)^n\left[1+a_{n-1}d+\cdots+a_1d^{(n-1)}+a_0d^n\right]=0 \qquad \ldots 3$$

Thus, the polynomial with reciprocal roots is a polynomial with the coefficients written in reverse order.

Routh Table: Zero Only in the First Column (Alternative method)

Determine the stability of the closed-loop transfer function

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

First write a polynomial that has the reciprocal roots of the denominator of T(s). This polynomial is formed by writing the denominator of T(s) in reverse order.

Hence $D(s) = 3s^5 + 5s^4 + 6s^3 + 3s^2 + 2s + 1$

Routh Table: Zero Only in the First Column (Alternative method)

Routh table

s^5	3	6	2
s^4	5	3	1
s^3	4.2	1.4	
s^2	1.33	1	
s^1	−1.75		
s^0	1		

Since there are two sign changes, the system is unstable and has two right-half-plane poles.

This is the same as the result obtained before by the epsilon method.

Routh Table: Entire Row is Zero

Determine the number of right-half-plane poles in the closed-loop transfer function

$$T(s) = \frac{10}{s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56}$$

The Routh table is

s^5			1			6			8
s^4		$\cancel{7}$	1		$\cancel{42}$	6		$\cancel{56}$	8
s^3	$\cancel{0}$	$\cancel{4}$	1	$\cancel{0}$	$\cancel{12}$	3	$\cancel{0}$	$\cancel{0}$	0
s^2			3			8			0
s^1			$\frac{1}{3}$			0			0
s^0			8			0			0

Routh Table: Entire Row is Zero

At the second row we multiply through by 1/7 for convenience.

$$s^4 \qquad \cancel{7} \quad 1 \qquad \cancel{42} \quad 6 \qquad \cancel{56} \quad 8$$

We stop at the third row, since the entire row consists of zeros, and use the following procedure. First we return to the row immediately above the row of zeros and form an auxiliary polynomial, using the entries in that row as coefficients.

The polynomial will start with the power of s in the label column and continue by skipping every other power of s. Thus, the polynomial formed for this example is

$$P(s) = s^4 + 6s^2 + 8$$

Next we differentiate the polynomial with respect to s and obtain

$$\frac{dP(s)}{ds} = 4s^3 + 12s + 0 \qquad \ldots 1$$

Routh Table: Entire Row is Zero

Use the coefficients of Eq. (1) to replace the row of zeros.

Again, for convenience, the third row is multiplied by 1/4 after replacing the zeros.

s^3	$\cancel{0}$ $\cancel{4}$	1	$\cancel{0}$ $\cancel{12}$	3	$\cancel{0}$ $\cancel{0}$	0			

The remainder of the table is formed in a straightforward manner by following the standard procedure. Table shows that all entries in the first column are positive. Hence, there are no right–half-plane poles.

Routh Table: Entire Row is Zero

An entire row of zeros:

An entire row of zeros will appear in the Routh table when a purely even or purely odd polynomial is a factor of the original polynomial.

For example, $s^4 + 5s^2 + 7$ is an even polynomial; it has only even powers of s. Even polynomials only have roots that are symmetrical about the origin.

The polynomial $s^5 + 5s^3 + 7s$ is an example of an odd polynomial; it has only odd powers of **s**. Odd polynomials are the product of an even polynomial and an odd power of **s**. Thus, the constant term of an odd polynomial is always missing.

Routh Table: Entire Row is Zero

An entire row of zeros:

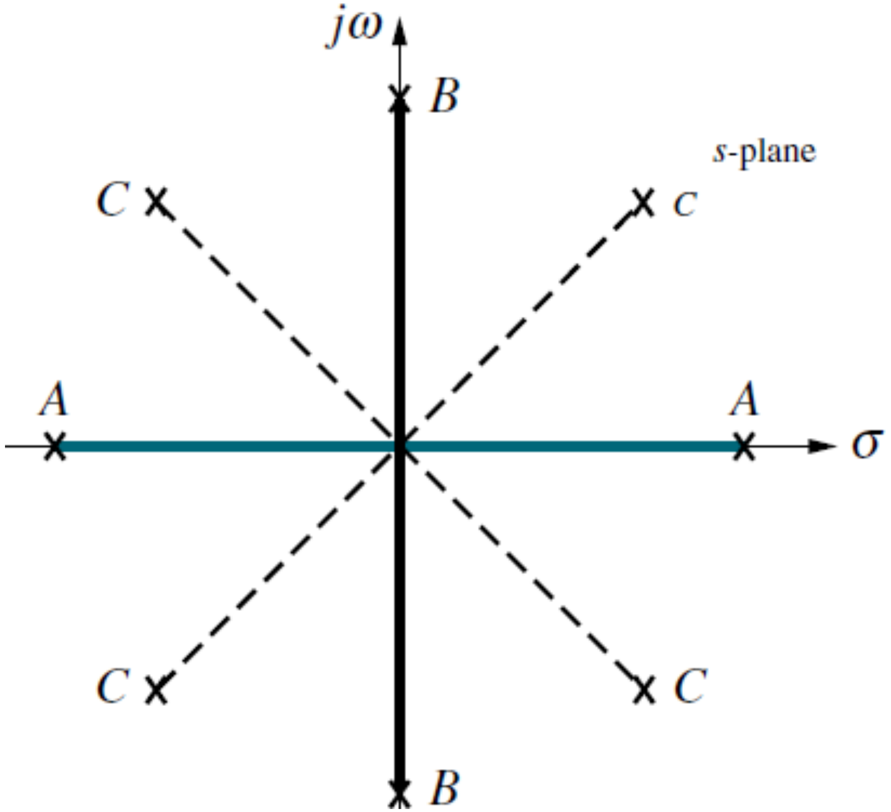
(1) The roots are symmetrical and real,

(2) the roots are symmetrical and imaginary, or

(3) the roots are quadrantal.

Each case or combination of these cases will generate an even polynomial.

Routh Table: Entire Row is Zero



A: Real and symmetrical about the origin

B: Imaginary and symmetrical about the origin

C: Quadrantal and symmetrical about the origin

Routh Table: Entire Row is Zero

An entire row of zeros:

(1) The roots are symmetrical and real,

(2) the roots are symmetrical and imaginary, or

(3) the roots are quadrantal.

Each case or combination of these cases will generate an even polynomial.

Routh Table: Entire Row is Zero

It is this even polynomial that causes the row of zeros to appear.

Thus, the row of zeros tells us of the existence of an even polynomial whose roots are symmetric about the origin.

Some of these roots could be on the $j\omega$-axis. On the other hand, since $j\omega$ roots are symmetric about the origin, if we do not have a row of zeros, we cannot possibly have $j\omega$ roots.

Another characteristic of the Routh table is that the row previous to the row of zeros contains the even polynomial that is a factor of the original polynomial.

Finally, everything from the row containing the even polynomial down to the end of the Routh table is a test of only the even polynomial.

Routh Table: Entire Row is Zero

For the transfer function find how many poles are in the right half-plane, in the left half-plane, and on the jω-axis.

$$T(s) = \frac{20}{s^8 + s^7 + 12s^6 + 22s^5 + 39s^4 + 59s^3 + 48s^2 + 38s + 20}$$

Use the denominator and form the Routh table

For convenience the s^6 row is multiplied by 1/10, and

$$s^6 \quad -\cancel{10}\ -1 \quad -\cancel{20}\ -2 \quad \cancel{10}\ \ 1 \quad \cancel{20}\ \ 2 \quad 0$$

the s^5 row is multiplied by 1/20.

$$s^5 \quad \cancel{20}\ \ 1 \quad \cancel{60}\ \ 3 \quad \cancel{40}\ \ 2 \quad 0 \quad 0$$

Routh Table: Entire Row is Zero

Routh Table

s^8	1	12	39	48	20
s^7	1	22	59	38	0
s^6	$-\cancel{10}\ -1$	$-\cancel{20}\ -2$	$\cancel{10}\ \ 1$	$\cancel{20}\ \ 2$	0
s^5	$\cancel{20}\ \ 1$	$\cancel{60}\ \ 3$	$\cancel{40}\ \ 2$	0	0
s^4	1	3	2	0	0
s^3	$\cancel{0}\ \ \cancel{4}\ \ 2$	$\cancel{0}\ \ \cancel{6}\ \ 3$	$\cancel{0}\ \ \cancel{0}\ \ 0$	0	0
s^2	$\cancel{\frac{3}{2}}\ \ 3$	$\cancel{2}\ \ 4$	0	0	0
s^1	$\frac{1}{3}$	0	0	0	0
s^0	4	0	0	0	0

Routh Table: Entire Row is Zero

At the s^3 row we obtain a row of zeros. Moving back one row to s^4, we extract the even polynomial, P(s), as $P(s) = s^4 + 3s^2 + 2$

s^3	$\cancel{0}$ $\cancel{4}$ 2	$\cancel{0}$ $\cancel{6}$ 3	$\cancel{0}$ $\cancel{0}$ 0	0	0

This polynomial will divide evenly into the denominator and thus is a factor. Taking the derivative with respect to s to obtain the coefficients that replace the row of zeros in the s^3 row, we find $dP(s)/ds = 4s^3 + 6^s + 0$

Replace the row of zeros with 4, 6, and 0 and multiply the row by 1/2 for convenience. Finally, continue the table to the s^0 row, using the standard procedure.

Routh Table: Entire Row is Zero

At the s^3 row we obtain a row of zeros. Moving back one row to s^4, we extract the even polynomial, P(s), as $P(s) = s^4 + 3s^2 + 2$

$$s^3 \qquad \cancel{0} \quad \cancel{4} \quad 2 \qquad \cancel{0} \quad \cancel{6} \quad 3 \qquad \cancel{0} \quad \cancel{0} \quad 0 \qquad 0 \qquad 0$$

This polynomial will divide evenly into the denominator and thus is a factor. Taking the derivative with respect to s to obtain the coefficients that replace the row of zeros in the s^3 row, we find $dP(s)/ds = 4s^3 + 6^s + 0$

Replace the row of zeros with 4, 6, and 0 and multiply the row by 1/2 for convenience. Finally, continue the table to the s^0 row, using the standard procedure.

Routh Table: Entire Row is Zero

Interpretation

Since all entries from the even polynomial at the s^4 row down to the s^0 row are a test of the even polynomial,

we begin to draw some conclusions about the roots of the even polynomial.

No sign changes exist from the s^4 row down to the s^0 row.

Thus, the even polynomial does not have right–half-plane poles. Since there are no right–half-plane poles, no left–half-plane poles are present because of the requirement for symmetry. Hence, the even polynomial must have all four of its poles on the $j\omega$-axis.

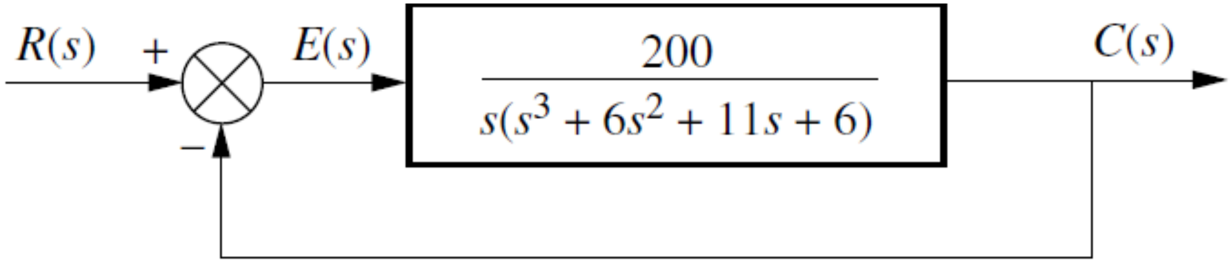
Routh Table: Entire Row is Zero

Polynomial			
Location	Even (fourth order)	Other (fourth order)	Total (Eight Order)
Right half plane	0	2	2
Left half plane	0	2	2
jω	4	0	4

.

Routh Table: Numerical

Find the number of poles in the left half-plane, the right half-plane, and on the jω-axis for the system



First, find the closed-loop transfer function as $T(s) = \dfrac{200}{s^4 + 6s^3 + 11s^2 + 6s + 200}$

Routh Table

s^4		1		11	200
s^3	$\cancel{6}$	1	$\cancel{6}$	1	
s^2	$\cancel{10}$	1	$\cancel{200}$	20	
s^1		-19			
s^0		20			

Routh Table: Numerical

Routh Table

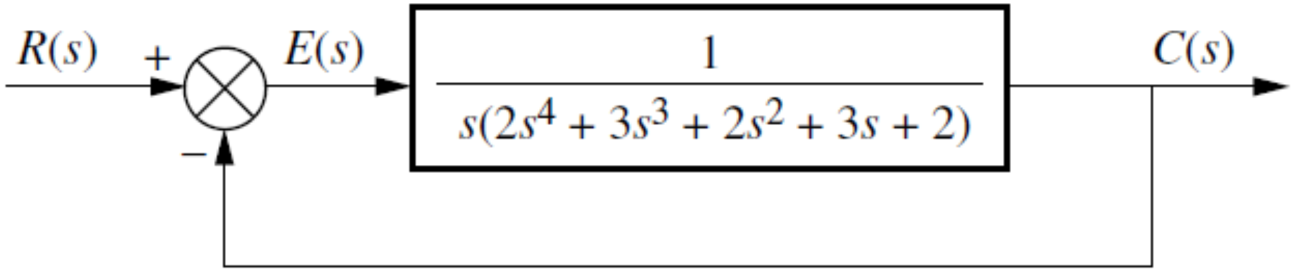
s^4		1		11	200
s^3	$\cancel{6}$	1	$\cancel{6}$	1	
s^2	$\cancel{10}$	1	$\cancel{200}$	20	
s^1		-19			
s^0		20			

At the s_1 row there is a negative coefficient; thus, there are two sign changes. The system is unstable, since it has two right–half plane poles and two left–half-plane poles.

The system cannot have $j\omega$ poles since a row of zeros did not appear in the Routh table.

Routh Table: Numerical

Find the number of poles in the left half-plane, the right half-plane, and on the jω-axis for the system



The closed-loop transfer function is $T(s) = \dfrac{1}{2s^5 + 3s^4 + 2s^3 + 3s^2 + 2s + 1}$

A zero appears in the first column of the s^3 row.

Since the entire row is not zero, simply replace the zero with a small quantity, ε, and continue the table. Permitting ε to be a small, positive quantity, we find that the first term of the s^2 row is negative. Thus, there are two sign changes, and the system is unstable, with two poles in the right half-plane. The remaining poles are in the left half-plane

Routh Table: Numerical

Routh Table

A zero appears in the first column of the s^3 row.

Since the entire row is not zero, simply replace the zero with a small quantity, ε, and continue the table. Permitting ε to be a small, positive quantity, we find that the first term of the s^2 row is negative. Thus, there are two sign changes, and the system is unstable, with two poles in the right half-plane. The remaining poles are in the left half-plane.

s^5	2	2	2
s^4	3	3	1
s^3	$\cancel{0}\ \ \epsilon$	$\frac{4}{3}$	
s^2	$\frac{3\epsilon - 4}{\epsilon}$	1	
s^1	$\frac{12\epsilon - 16 - 3\epsilon^2}{9\epsilon - 12}$		
s^0	1		

Routh Table: Numerical

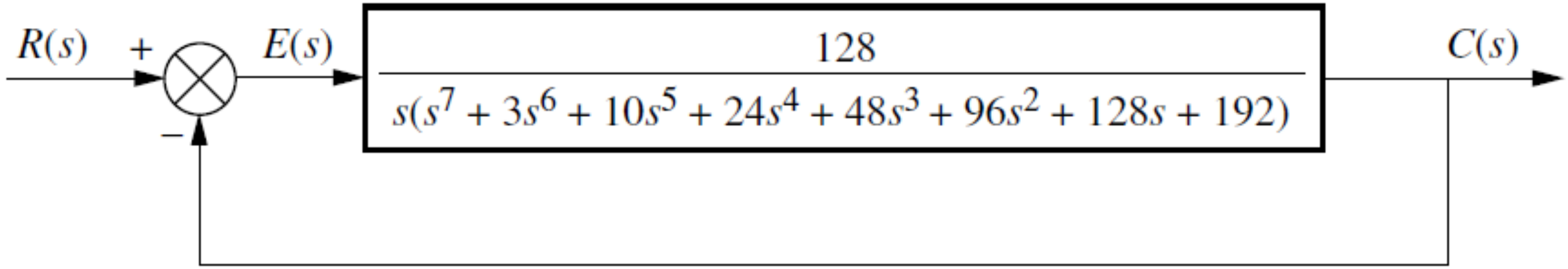
We also can use the alternative approach, where we produce a polynomial whose roots are the reciprocal of the original. $T(s) = \dfrac{1}{2s^5 + 3s^4 + 2s^3 + 3s^2 + 2s + 1}$ polynomial by writing the coefficients in reverse order

$$s^5 + 2s^4 + 3s^3 + 2s^2 + 3s + 2$$

s^5	1	3	3
s^4	2	2	2
s^3	2	2	
s^2	$\cancel{0}$ ϵ	2	
s^1	$\dfrac{2\epsilon - 4}{\epsilon}$		
s^0	2		

Routh Table: Numerical

Find the number of poles in the left half-plane, the right half-plane, and on the jω-axis for the system of Figure 6.8. Draw conclusions about the stability of the closed-loop system



The closed-loop transfer function for the system is

$$T(s) = \frac{128}{s^8 + 3s^7 + 10s^6 + 24s^5 + 48s^4 + 96s^3 + 128s^2 + 192s + 128}$$

Using the denominator, form the Routh table. A row of zeros appears in the s^5 row. Thus, the closed-loop transfer function denominator must have an even polynomial as a factor. Return to the s^6 row and form the even polynomial.

Routh Table: Numerical

Routh table

s^8	1	10	48	128	128
s^7	$\cancel{3}\quad 1$	$\cancel{24}\quad 8$	$\cancel{96}\quad 32$	$\cancel{192}\quad 64$	
s^6	$\cancel{2}\quad 1$	$\cancel{16}\quad 8$	$\cancel{64}\quad 32$	$\cancel{128}\quad 64$	
s^5	$\cancel{0}\quad \cancel{6}\quad 3$	$\cancel{0}\quad \cancel{32}\quad 16$	$\cancel{0}\quad \cancel{64}\quad 32$	$\cancel{0}\quad \cancel{0}\quad 0$	
s^4	$\cancel{\frac{8}{3}}\quad 1$	$\cancel{\frac{64}{3}}\quad 8$	$\cancel{64}\quad 24$		
s^3	$\cancel{-8}\quad -1$	$\cancel{-40}\quad -5$			
s^2	$\cancel{3}\quad 1$	$\cancel{24}\quad 8$			
s^1	3				
s^0	8				

Routh Table: Numerical

$$P(s) = s^6 + 8s^4 + 32s^2 + 64$$

Differentiate this polynomial with respect to s to form the coefficients that will replace the row of zeros.

$$\frac{dP(s)}{ds} = 6s^5 + 32s^3 + 64s + 0$$

Replace the row of zeros at the s^5 row by the coefficients of above equation and multiply through by 1/2 for convenience. Then complete the table.

We note that there are two sign changes from the even polynomial at the s^6 row down to the end of the table. Hence, the even polynomial has two right–half plane poles. Because of the symmetry about the origin, the even polynomial must have an equal number of left–half-plane poles. Therefore, the even polynomial has two left–half-plane poles. Since the even polynomial is of sixth order, the two remaining poles must be on the jω-axis.

Routh Table: Numerical

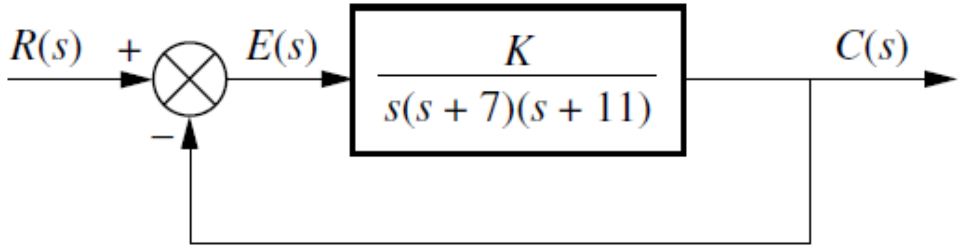
There are no sign changes from the beginning of the table down to the even polynomial at the s^6 row. Therefore, the rest of the polynomial has no right–half plane poles.

The system has two poles in the right half-plane, four poles in the left half-plane, and two poles on the jω-axis, which are of unit multiplicity. The closed-loop system is unstable because of the right–half-plane poles

Polynomial			
Location	Even (sixth order)	Other (fourth order)	Total (Eight Order)
Right half plane	2	0	2
Left half plane	2	2	4
jω	2	0	2

Routh Table: Numerical

Find the range of gain, K, for the system that will cause the system to be stable, unstable, and marginally stable. Assume K > 0.



First find the closed-loop transfer function as $T(s) = \dfrac{K}{s^3 + 18s^2 + 77s + K}$

Routh Table

s^3	1	77
s^2	18	K
s^1	$\dfrac{1386 - K}{18}$	
s^0	K	

Routh Table: Numerical

Since K is assumed positive, we see that all elements in the first column are always positive except the s^1 row. This entry can be positive, zero, or negative, depending upon the value of K. If $K < 1386$, all terms in the first column will be positive, and since there are no sign changes, the system will have three poles in the left half-plane and be stable.

Routh Table

s^3	1	77
s^2	18	K
s^1	$\frac{1386 - K}{18}$	
s^0	K	

Routh Table: Numerical

If K > 1386, the s^1 term in the first column is negative. There are two sign changes, indicating that the system has two right–half-plane poles and one left–half-plane pole, which makes the system unstable.

Routh Table

s^3	1	77
s^2	18	K
s^1	$\dfrac{1386 - K}{18}$	
s^0	K	

Routh Table: Numerical

If K = 1386, we have an entire row of zeros, which could signify jω poles. Returning to the s^2 row and replacing K with 1386, we form the even polynomial $P(s) = 18s^2 + 1386$

Differentiating with respect to s, we have $\frac{dP(s)}{ds} = 36s + 0$

Replacing the row of zeros with the coefficients, we obtain the modified Routh Hurwitz table for the case of K = 1386

Routh Table

Routh Table: Numerical

Routh Table

s^3	1	77
s^2	18	1386
s^1	$\cancel{0}$ 36	
s^0	1386	

Since there are no sign changes from the even polynomial (s^2 row) down to the bottom of the table, the even polynomial has its two roots on the jω-axis of unit multiplicity.

Since there are no sign changes above the even polynomial, the remaining root is in the left half-plane. Therefore, the system is marginally stable.