Laplace Transform

→ Transforms function from T-space to S-space

$$\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

Proof 1: Laplace is a linear transformation.

$$\mathcal{L}\{f+g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}$$

$$\mathcal{L}\{f+g\} = \int_0^\infty (f+g)(t) e^{-st}\, dt$$

$$= \lim_{a\to\infty} \int_0^a (f(t)+g(t))\, e^{-st}\, dt$$

$$= \lim_{a\to\infty} \int_0^a f(t) e^{-st} dt + \lim_{a\to\infty} \int_0^a g(t) e^{-st} dt$$

$$= \mathcal{L}\{f\} + \mathcal{L}\{g\}$$

Proof 2: Scaler Multiplication.

$$\mathcal{L}\{\alpha F\} = \int_0^\infty (\alpha f)(t)\, e^{-st}\, dt$$

$$= \alpha \cdot \lim_{a\to\infty} \int_0^a f(t) e^{-st} dt$$

$$= \alpha \cdot \mathcal{L}\{f(t)\}$$

Refer Mathaholic vids

Examples:

① $\mathcal{L}\{1\} = \int_0^\infty 1\, e^{-st}\, dt$

$$= \lim_{a\to\infty} \int_0^a e^{-st}\, dt$$

$$= \lim_{a\to\infty} \left. \frac{e^{-st}}{-s} \right|_0^a$$

$$= \frac{e^{-s\cdot\infty}}{-s} - \frac{e^{-s\cdot 0}}{-s}$$

$$= \frac{-e^{-\infty}}{-s} - \frac{e^0}{-s}$$

$$= 0 - \frac{1}{-s}$$

$$\mathcal{L}\{1\} = \frac{1}{s}$$

② $\mathcal{L}\{2020\} \Rightarrow \mathcal{L}\{2020\cdot 1\}$

$$\Rightarrow 2020 \cdot \mathcal{L}\{1\}$$

$$= 2020 \cdot 1/s = 2020/s$$

③ $\mathcal{L}\{e^{at}\}$

$$\Rightarrow \int_0^\infty e^{at} \cdot e^{-st}\, dt$$

$$\Rightarrow \int_0^\infty e^{(a-s)t}\, dt$$

$\ldots s > a$
else L.T not exist

$$\Rightarrow \lim_{b\to\infty} \left(\frac{e^{(a-s)t}}{a-s} \right)_0^b$$

$$= \frac{e^{-\infty}}{a-s} - \frac{e^0}{a-s}$$

$$= 0 - \frac{1}{a-s}$$

if $s < a$,
e would have gone to positive infinity.

$$= -\frac{1}{-(s-a)}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

④ $\mathcal{L}\{t\} = \int_0^\infty t\, e^{-st}\, dt$

$= \lim_{a\to\infty} \int_0^a t\, e^{-st}\, dt$

$= \frac{e^{-s\infty}}{-s}\left(\infty + \frac{1}{s}\right) - \frac{e^{-s0}}{-s}\left(0 + \frac{1}{s}\right)$

$= 0\left(\infty + \frac{1}{s}\right) - \frac{1}{-s}\left(\frac{1}{s}\right)$

$= 0 - \frac{1}{-s^2}$

$\mathcal{L}\{t\} = \frac{1}{s^2}$

$\int t\, e^{-st} \Rightarrow \begin{array}{l} u = t \\ dv = e^{-st} \end{array} \quad \begin{array}{l} du = 1 \\ v = \frac{e^{-st}}{-s} \end{array}$

$\int u\,dv = uv - \int v\,du$

$= \frac{t \cdot e^{-st}}{-s} - -\int \frac{e^{-st}}{s} \cdot 1$

$= \frac{t \cdot e^{-st}}{-s} + \frac{e^{-st}}{-s^2}$

$= \frac{t \cdot e^{-st}}{-s} + \frac{e^{-st}}{-s^2}$

$= \frac{e^{-st}}{-s}\left(t + \frac{1}{s}\right)$

⑤ $\mathcal{L}\{t^2\} = \int_0^\infty t^2 e^{-st}\, dt$

$= \lim_{a\to\infty} \int_0^a t^2 e^{-st}\, dt$

$\Rightarrow \lim_{a\to\infty} \left(\frac{t^2 e^{-st}}{-s} + \frac{2t e^{-st}}{-s^2} + \frac{2 e^{-st}}{-s^3}\right)_0^a$

$\left(\frac{e^{-st}}{-s}\left(t^2 + \frac{2t}{s} + \frac{2}{s^2}\right)\right)_0^\infty$

$0 - \frac{e^0}{-s}\left(0 + \frac{2\times 0}{s} + \frac{2}{s^2}\right)$

$\Rightarrow \frac{-1}{-s} \times \frac{2}{s^2}$

$\mathcal{L}\{t^2\} \Rightarrow \frac{2}{s^3}$

$\therefore \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

$\mathcal{L}\{t^a\} = \frac{\Gamma(a+1)}{s^{a+1}}$

Gamma function

$\int t^2 e^{-st} \qquad \begin{array}{l} u = t^2 \\ dv = e^{-st} \end{array} \qquad \begin{array}{l} du = 2t \\ v = \frac{e^{-st}}{-s} \end{array}$

$\Rightarrow \frac{t^2 e^{-st}}{-s} - \int \frac{2t\, e^{-st}}{-s}$

$\Rightarrow \frac{t^2 e^{-st}}{-s} - \frac{2}{-s}\int t e^{-st} \quad \ldots ①$

$\begin{array}{ll} u = t & du = 1 \\ dv = e^{-st} & v = \frac{e^{-st}}{-s} \end{array}$

$\int t e^{-st} = \frac{t\, e^{-st}}{-s} - \frac{1}{-s}\int e^{-st}$

$= \frac{t e^{-st}}{-s} - \frac{1}{-s}\frac{e^{-st}}{-s}$

$= \frac{t e^{-st}}{-s} + \frac{e^{-st}}{-s^2}$

① $\rightarrow \frac{t^2 e^{-st}}{-s} - \frac{2}{-s}\left(\frac{t e^{-st}}{-s} + \frac{e^{-st}}{-s^2}\right)$

$\rightarrow \frac{t^2 e^{-st}}{-s} + \frac{2t e^{-st}}{-s^2} + \frac{2e^{-st}}{-s^3}$

⑥ $\mathcal{L}\{\cosh(\omega t)\} = \mathcal{L}\left\{\dfrac{e^{\omega t} + e^{-\omega t}}{2}\right\}$

$$= \frac{1}{2}\left[\mathcal{L}\{e^{\omega t}\} + \mathcal{L}\{e^{-\omega t}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-\omega} + \frac{1}{s+\omega}\right]$$

$$= \frac{1}{2}\left[\frac{s+\omega + s-\omega}{s^2-\omega^2}\right]$$

$$= \frac{1}{2}\left[\frac{2s}{s^2-\omega^2}\right]$$

$$\mathcal{L}\{\cosh(\omega t)\} = \frac{s}{s^2-\omega^2}$$

⑦ $\mathcal{L}\{\sinh(\omega t)\} = \mathcal{L}\left\{\dfrac{e^{\omega t} - e^{-\omega t}}{2}\right\}$

$$= \frac{1}{2}\left[\mathcal{L}\{e^{\omega t}\} - \mathcal{L}\{e^{-\omega t}\}\right.$$

$$= \frac{1}{2}\left[\frac{1}{s-\omega} - \frac{1}{s+\omega}\right] \quad \cdots \because \mathcal{L}\{e^{-at}\} = \frac{1}{s-a}$$

$$= \frac{1}{2}\left[\frac{s+\omega - s+\omega}{s^2-\omega^2}\right]$$

$$= \frac{1}{2}\left[\frac{2\omega}{s^2-\omega^2}\right]$$

$$\mathcal{L}\{\sinh(\omega t)\} = \frac{\omega}{s^2-\omega^2}$$

⑧ $\mathcal{L}\{\cos(\omega t)\} = \mathcal{L}\left\{\dfrac{e^{i\omega t} + e^{-i\omega t}}{2}\right\}$

$$= \frac{1}{2}\left[\frac{1}{s-i\omega} + \frac{1}{s+i\omega}\right]$$

$$= \frac{1}{2}\left[\frac{2s}{s^2-i\omega^2}\right]$$

$$= \frac{s}{s^2+\omega^2}$$

⑨ $\mathcal{L}\{\sin \omega t\} = \mathcal{L}\left\{\dfrac{e^{i\omega t} - e^{-i\omega t}}{2i}\right\}$

$$= \frac{1}{2i}\left[\frac{1}{s-i\omega} - \frac{1}{s+i\omega}\right]$$

$$= \frac{1}{\cancel{2i}}\left[\frac{\cancel{2i}\omega}{s^2-i\omega^2}\right]$$

$$\mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2+\omega^2}$$

HW:

① $\mathcal{L}\{t^2+t+1\} = \mathcal{L}\{t^2\} + \mathcal{L}\{t\} + \mathcal{L}\{1\}$

$$= \frac{2}{s^3} + \frac{1}{s^2} + \frac{1}{s}$$

② $\mathcal{L}\{t^{1/2}\} =$

③ $L\{\sin(3t) + \cosh(4t)\}$

$$\Rightarrow \mathcal{L}\{\sin(3t)\} + \mathcal{L}\{\cosh(4t)\}$$

$$\Rightarrow \frac{3}{s^2+9} + \frac{s}{s^2-16}$$

$$\Rightarrow \frac{3s^2 - 48 + s^3 + 9s}{s^4 - 16s^2 + 9s^2 - 144} \Rightarrow \frac{s^3 + 3s^2 + 9s - 48}{s^4 - 7s^2 - 144}$$

Gamma function:

$$\Gamma : D \rightarrow \mathbb{R}^+$$

$$D \rightarrow \{1,2,3\ldots\}$$

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} \cdot e^{-t}\, dt$$

Gamma

e.g. $\Gamma(1) = \int_0^{\infty} t^{1-1} \cdot e^{-t}\, dt$

$$= \int_0^{\infty} t^0 \cdot e^{-t}\, dt$$

$$= \int_0^{\infty} e^{-t}\, dt$$

$$= \lim_{a \to \infty} \int_0^{a} e^{-t}\, dt$$

$$= \left(-e^{-t}\right)_0^{\infty}$$

$$= -e^{-\infty} - (-e^{-0})$$

$$= 0 + e^0$$

$$= 0 + 1$$

$$\Gamma(1) = 1$$

Gamma Relations:

① $\Gamma(\alpha+1) = \alpha\, \Gamma(\alpha)$

$\Gamma(n+1) = n\,\Gamma(n)$

$* \Gamma(n+1) = n!$ ... only for natural numbers

e.g. $\Gamma(1000) = 999!$

② $\Gamma(1/2) = \sqrt{\pi}$

e.g. $\Gamma(3/2) = \Gamma\left(\frac{1}{2}+1\right)$

$$= \frac{1}{2}\Gamma(1/2)$$

$$= \frac{1}{2} \times \sqrt{\pi} = \frac{\sqrt{\pi}}{2}$$

e.g. 2) $\Gamma(5/2) = \Gamma(3/2 + 1)$

$$= 3/2\, \Gamma(3/2)$$

$$= 3/2\, \Gamma(1/2 + 1)$$

$$= 3/2 \cdot 1/2\, \Gamma(1/2)$$

$$= \frac{3\sqrt{\pi}}{4}$$

* $\mathcal{L}\{f(t)\} = F(s)$

$\mathcal{L}^{-1}\{F(s)\} = f(t)$

$\mathcal{L}\{g(t)\} = G(s)$

$\mathcal{L}^{-1}\{G(s)\} = g(t)$

$$\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha L^{-1}\{F(s)\} + \beta \mathcal{L}^{-1}\{G(s)\}$$

$$= \mathcal{L}\{\alpha L^{-1}\{F(s)\} + \beta \mathcal{L}^{-1}\{G(s)\}$$

$$= \mathcal{L}\{\alpha f(s) + \beta g(s)\}$$

$$= \alpha F(s) + \beta G(s)$$

$$= \alpha L^{-1}\{F(s)\} + \beta \mathcal{L}^{-1}\{G(s)\}$$

* Functions of exponential order

Session 4

- A function $f(t)$ is said to be of exponential order if there exists constants $M > 0$, $\alpha \geq 0$ s.t. $|f(t)| \leq Me^{\alpha t}$, $\forall\, t \geq t_0$

Session 5:

Existence Theorem:

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt$$

Piecewise Continuous:

A function 'f' is piecewise continuous if:

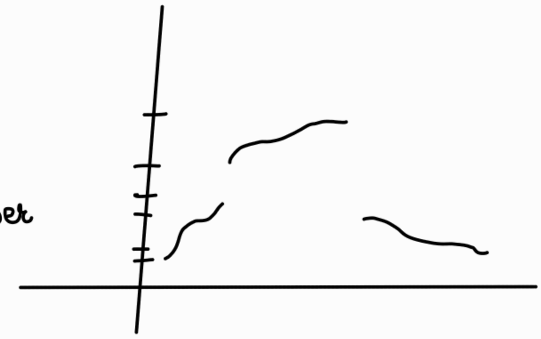
1] interval $[a,b]$ can broken into a finite number of sub-intervals,

$$a = t_0 < t_1 < t_2 \ldots < t_n = b \text{, such that,}$$

f is continuous on each $[t_i, t_{i+1}] \rightarrow \{i = 0 \ldots n-1\}$

2] f has jump discontinuity at t_i,

$$\left| \lim_{t \to t_i} f(t) \right| < \infty, \; i = 1 \ldots n$$



Piece wise continuous functions (Continuous in pieces.)

Existence Theorem:

If $f(t)$ is piecewise continuous and of exponential growth the $\mathcal{L}\{f\}$ exists for all $s > \alpha$.

Proof:

$$|\mathcal{L}\{f\}| = \left| \int_0^\infty e^{-st} f(t)\, dt \right| \le \int_0^\infty |f(t)|\, e^{-st}\, dt$$

... by property of integration.

$$\le \int_0^\infty M e^{\alpha t} e^{-st}\, dt = \frac{M}{s-\alpha}$$

Constant

exponential growth function

$$\mathcal{L}\{1/\sqrt{t}\} = \int_0^\infty e^{-st}\, t^{-1/2}\, dt$$

$$\Gamma(\alpha) = \int_0^\infty e^{-u}\, u^{\alpha-1}\, du$$

let $u = st$ $\quad \therefore t = u/s$

$\therefore du = s\, dt$

$$\int_0^\infty e^{-u} \left(\frac{u}{s}\right)^{-1/2} \frac{du}{s} \Rightarrow \int_0^\infty e^{-u}\, \frac{\cancel{s}^{1/2}}{u}\, \frac{du}{\cancel{s}\, s^{1/2}}$$

$$= \frac{1}{\sqrt{s}} \int_0^\infty e^{-u} \frac{1}{\sqrt{u}} \, du$$

$$= \frac{1}{\sqrt{s}} \int_0^\infty e^{-u} u^{-1/2} \, du$$

$$= \frac{1}{\sqrt{s}} \int_0^\infty e^{-u} u^{1/2 - 1} \, du$$

$\Rightarrow$ Gamma function,

$$= \frac{1}{\sqrt{s}} \Gamma\left(1/2\right)$$

$$= \frac{1}{\sqrt{s}} \sqrt{\pi}$$

$$= \sqrt{\pi} / \sqrt{s}$$

* You can have two different functions with the same Laplace as Integration doesnot affect at countably many points.

First Shifting Theorem:

$$\mathcal{L}\{f(t)\} = F(s) \quad \text{then,}$$

$$\mathcal{L}\{e^{at}F(t)\} = F(s-a)$$

$$e^{at}f(t) = L^{-1}\{F(s-a)\}$$

E.g.

① $\mathcal{L}\{t \cdot e^{1.7t}\}$

Steps:

① Whenever exponential think first shifting.

② Ignore expo. find laplace.

③ Shift s in laplace by 'a'.

by first shift theorem,

$$\mathcal{L}\{t\} = \frac{1}{s^2} \quad , a = 1.7$$

$$\mathcal{L}\{t.e^{1.7t}\} = \mathcal{L}\{F(s-a)\}$$

$$= \frac{1}{(s-1.7)^2}$$

$$= \frac{1}{s^2 - 3.4s + 2.89}$$

② $L\{e^{7t}. \cos wt\}$

$$L\{\cos wt\} = \frac{s}{s^2+w^2}$$

$$L\{e^{7t}. \cos wt\} = \frac{(s-7)}{(s-7)^2+w^2}$$

$$= \frac{s-7}{s^2-14s+49+w^2}$$

③ $L^{-1}\left\{\frac{6}{(s-1)^2}\right\}$

① Take consts out, if any

② Ignore the 'a' wala part

③ Find what's the laplace of whatever's left.

by first shifting theorem,

$$= 6\,\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}$$

$$= 6\,e^{t}.t$$

④ $L^{-1}\left\{\frac{s-6}{(s-1)^2+9}\right\} \Rightarrow \mathcal{L}^{-1}\left\{\frac{s-1-5}{(s-1)^2+3^2}\right\} \Rightarrow \mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+3^2}\right\} - \mathcal{L}^{-1}\left\{\frac{5}{(s-1)^2+3^2}\right\}$

$$\Downarrow$$

$$\Rightarrow e^{t}\cos 3t - \mathcal{L}^{-1}\left\{\frac{3\times}{3}\frac{5}{(s-1)^2+3^2}\right\}$$

$$\Rightarrow e^{t}\cos 3t - \frac{5}{3}\sin 3t \cdot e^{t}$$

HW

1) $\mathcal{L}\{ e^{-4t} \cos(11t)\}$

$$\mathcal{L}\{\cos(11t)\} = \frac{s}{s^2 + 11^2}$$

$$\mathcal{L}\{e^{-4t}\cos(11t)\} = \frac{s--4}{(s--4)^2 + 11^2}$$

$$= \frac{s+4}{s^2+8s+16+121}$$

$$= \frac{s+4}{s^2+8s+137}$$

2) $\mathcal{L}\{ e^{3t} \sinh(t)\}$

$$\mathcal{L}\{\sinh(t)\} = \frac{1}{s^2 - 1^2} \qquad \because \omega = 1$$

$$\mathcal{L}\{e^{3t}\sinh(t)\} = \frac{1}{(s-3)^2 - 1^2}$$

$$= \frac{1}{s^2-6s+9-1}$$

$$= \frac{1}{s^2-6s+8}$$

3) $\mathcal{L}^{-1}\left\{\frac{4s-2}{s^2-6s+18}\right\}$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{4s-2+10-10}{s^2-3s-3s+9+9}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{4(s-3)+10}{(s-3)^2+3^2}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{4(s-3)}{(s-3)^2+3^2}\right\} + \mathcal{L}^{-1}\left\{\frac{10}{(s-3)^2+3^2}\right\}$$

$$\Rightarrow 4e^{3t}\cos 3t + \frac{10}{3}e^{3t}\sin 3t$$

4) $\mathcal{L}^{-1}\left\{\frac{\sqrt{8}}{(s+\sqrt{2})^3}\right\}$

$$\Rightarrow \left\{\frac{2\sqrt{2}}{(s+\sqrt{2})^3}\right\}$$

$$\Rightarrow \sqrt{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+\sqrt{2})^3}\right\}$$

$$\mathcal{L}^{-1}\left\{2/s^3\right\} = t^2$$

$$\therefore \sqrt{2}\mathcal{L}^{-1}\left\{2/(s+\sqrt{2})^3\right\} = \sqrt{2}\, e^{-\sqrt{2}} t^2$$

- $f(t) \rightsquigarrow F(s)$

$f(at) \rightsquigarrow \frac{1}{a} F(s/a)$

Given : $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

$$F(s/a) = \int_0^\infty e^{-\frac{s}{a}t} f(t)\, dt \quad \ldots \text{ we know } (1)$$

$$\mathcal{L}\{f(at)\} = \int_0^\infty e^{-st} f(at)\, dt$$

let $u = at$, $\therefore t = \frac{u}{a}$ $\quad \frac{du}{dt} = a \Rightarrow dt = \frac{du}{a}$

$$\Rightarrow \int_0^\infty e^{-s\left(\frac{u}{a}\right)} f(u) \frac{du}{a}$$

$$F(as) \Rightarrow \frac{1}{a} \int_0^\infty e^{-s\frac{u}{a}} f(u)\, du$$

$$\Rightarrow \frac{1}{a} \int_0^\infty e^{-\left(\frac{s}{a}\right)u} f(u)\, du$$

$$\Rightarrow \frac{1}{a} F\left(\frac{s}{a}\right) \quad \ldots \text{ we know } (1)$$

Differential Equations using laplace

* Initial condition must be given

$$\mathcal{L}\{y'\} = s\,\mathcal{L}\{y\} - y(0)$$

- Assumptions:
  - $f(t)$ is continuous on $[0, \infty)$
  - f is of exponential growth
  - f' is piece wise continuous on $[0, A]$ $\forall A > 0$

} Nice function :)

- $\mathcal{L}\{y''\} = s^2 \mathcal{L}\{y\} - s\,y(0) - y'(0)$
- $\mathcal{L}\{y'''\} = s^3 \mathcal{L}\{y\} - s^2 y(0) - s y'(0) - y''(0)$
- $\mathcal{L}\{y^{iv}\} = s^4 \mathcal{L}\{y\} - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0)$

e.g. 1) $y' + 4y = 1$, $y(0) = 1$

$$\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \mathcal{L}\{1\}$$

$$sY(s) - y(0) + 4Y(s) = \frac{1}{s}$$

$$Y(s)(s+4) = \frac{1}{s} + 1$$

$$Y(s) = \frac{1}{s(s+4)} + \frac{1}{s+4}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{1/4}{s} - \frac{1/4}{s+4} + e^{-4t}$$

$$= \frac{1}{4}\cdot\frac{1}{s} - \frac{1}{4}\frac{1}{s+4} + e^{-4t}$$

$$= \frac{1}{4}\cdot 1 - \frac{1}{4}e^{-4t} + e^{-4t}$$

$$= \frac{1}{4}\left(1 - e^{-4t} + 4e^{-4t}\right)$$

$$= \frac{1}{4}\left(1 + 3e^{-4t}\right)$$

$$\frac{1}{s(s+4)} = \frac{A}{s} + \frac{B}{s+4}$$

$$\frac{1}{s+4} = A + \frac{B\cancel{s}}{\cancel{s+4}}$$

$$\boxed{\frac{1}{4} = A}$$

$$\frac{1}{s} = \frac{A(s+4)}{s} + B$$

$$s = -4$$

$$\boxed{\frac{1}{-4} = B}$$

② $y'' + 7y' + 12y = e^{3t}$ $\quad y(0) = 0$, $y'(0) = 0$

$$s^2Y(s) - \cancel{y(0)} - \cancel{y'(0)} + 7sY(s) - \cancel{y(0)} + 12Y(s) = \frac{1}{s-3}$$

$$Y(s)\left(s^2 + 7s + 12\right) = \frac{1}{s-3}$$

$$Y(s) = \frac{1}{(s+4)(s+3)(s-3)}$$

Partial equation,

$$\frac{1}{(s+4)(s+3)(s-3)} = \frac{A}{s+4} + \frac{B}{s+3} + \frac{C}{s-3}$$

Mul by $(s+4)$, $s=-4$

$$\boxed{A = \frac{1}{7}}$$

Mul by $s+3$, $s=-3$

$$\boxed{B = -\frac{1}{6}}$$

Mul by $s-3$, $s=+3$

$$\boxed{C = \frac{1}{42}}$$

$$Y(s) = \frac{1}{7}\cdot\frac{1}{s+4} - \frac{1}{6}\cdot\frac{1}{s+3} + \frac{1}{42}\cdot\frac{1}{s-3}$$

$$y(t) = \frac{1}{7}e^{-4t} - \frac{1}{6}e^{-3t} + \frac{1}{42}e^{3t}$$

HW ① $y'' + y' - 2y = 4$ $\quad y(0) = 2$, $y'(0) = 1$

$$\Rightarrow s^2Y(s) - sy(0) - y'(0) + sY(s) - y(0) - 2Y(s) = \frac{4}{s}$$

$$s^2Y(s) - 2s - 1 + sY(s) - 2 - 2Y(s) = \frac{4}{s}$$

$$Y(s)\left(s^2 + s - 2\right) = \frac{4}{s} + 2s + 1 + 2$$

$$Y(s)(s+2)(s-1) = \frac{2s^2 + 3s + 4}{s}$$

$$Y(s) = \frac{2s^2 + 3s + 4}{s(s+2)(s-1)}$$

$$\frac{2s^2 + 3s + 4}{s(s+2)(s-1)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s-1}$$

$\rightarrow$ $A = -2$, $B = 1$, $C = 3$

$s = 1$

$$C = \frac{2+3+4}{3} = \frac{9}{3} = 3$$

$$Y(s) = \frac{-2}{s} + \frac{1}{s+2} + \frac{3}{s-1}$$

$$Y(t) = -2(1) + e^{-2t} + 3e^{t}.$$

Differential equations if initial conditions are NOT at 0.

e.g $y' - 6y = 1$, $y(2) = 4$

$$sY(s) - y(0) - 6Y(s) = \frac{1}{s}$$

$$Y(s)(s-6) = Y(0) + \frac{1}{s} \qquad \text{let } Y(0) = C$$

$$Y(s) = \frac{C}{s-6} + \frac{1}{s(s-6)}$$

$$\frac{1}{s(s-6)} = \frac{A}{s} + \frac{B}{s-6}$$

$$-\frac{1}{6} = A$$

$$B = \frac{1}{6}$$

$$Y(t) = Ce^{6t} + \mathcal{L}^{-1}\left(-\frac{1}{6}\cdot\frac{1}{s} + \frac{1}{6}\cdot\frac{1}{s-6}\right)$$

$$-\text{ll}- \quad + \frac{1}{6}\left(-\mathcal{L}^{-1}\left(\frac{1}{s}\right) + \mathcal{L}^{-1}\left\{\frac{1}{s-6}\right\}\right)$$

$$-\text{ll}- + \frac{1}{6}\left(-1 + e^{6t}\right)$$

$$Y(t) = Ce^{6t} - \frac{1}{6} + \frac{e^{6t}}{6}$$

Given

$$y(t) = y_0$$

$$y(2) = 4$$

$$4 = Ce^{6\cdot 2} - \frac{1}{6} + \frac{e^{6\cdot 2}}{6}$$

$$4 = Ce^{12} - \frac{1}{6} + \frac{e^{12}}{6}$$

$$24 = 6Ce^{12} - 1 + e^{12}$$

$$24 + 1 = 6Ce^{12} + e^{12}$$

$$25 = 6Ce^{12} + e^{12}$$

$$\frac{25}{e^{12}} = 6C + 1$$

$$25e^{-12} - 1 = 6c$$

$$c = \frac{25e^{-12} - 1}{6}$$

$$y(t) = \left(\frac{25e^{12} - 1}{6}\right) e^{6t} - \frac{1}{6} + \frac{e^{6t}}{6}$$

$$= \left(\frac{25e^{12} - 1 + 1}{6}\right) e^{6t} - \frac{1}{6}$$

$$= \left(\frac{25e^{12}}{6}\right) e^{6t} - \frac{1}{6}$$

2) $y'' + 3y' - 4y = 6e^{2t-2}$, $y(1) = 4$, $y'(1) = 5$

$\eta = t - 1 \Rightarrow t = \eta + 1$ $\quad \ldots$ $\underline{\eta = \text{'eta'}}$

$y''(t-1) + 3y'(t-1) - 4y(t-1) = 6e^{2(t-1)}$

let $u(\eta) = y(t+1)$

$$u'(\eta) = \frac{\partial u}{\partial \eta} = \frac{dy}{d\eta} = \frac{\partial y}{\partial t} \times \frac{\partial t}{\partial \eta} = y'(t) \cdot 1 = y'(\eta + 1)$$

$$\therefore u'(\eta) = y'(\eta + 1)$$

$$u''(\eta) = y''(\eta + 1)$$

$$u(0) = y(0+1)$$

$$u(0) = y(1) = 4$$

$$u'(0) = y'(0+1) = y'(1) = 5$$

$$\Rightarrow u''(\eta) + 3u'(\eta) - 4u(\eta) = 6e^{2n}$$

$$\underline{s^2 u(s)} - s u(0) - u'(0) + \underline{3s u(s)} - u(0) \underline{- 4u(s)} = \frac{6}{e-2}$$

$$u(s)\left(s^2 + 3s - 4\right) = \frac{6}{s-2} + 4s + 5 + 4$$

$$Y(s)(s+4)(s-1) = \frac{6+4s^2-8s+5s-10+4s-8}{s-2}$$

$$Y(s) = \frac{4s^2+s-12}{(s-2)(s-1)(s+4)}$$

$$\frac{4s^2+s-12}{(s-2)(s-1)(s-4)} = \frac{A}{s-2} + \frac{B}{s-1} + \frac{C}{s+4)}$$

let $s=2$ $\quad A = -3$

$$A = \frac{4\times4+2-12}{(2-1)(2-4)} = \frac{6}{-2} = -3$$

let $s=1$

$$B = \frac{4(1)+1-12}{(1-2)(1-4)} = \frac{-7}{3} = -\frac{7}{3}$$

let $s=-4$

$$C = \frac{64-4-12}{-6\times-5} = \frac{48}{30} = \frac{8}{5}$$

$$\Rightarrow Y(s) = \frac{-3}{s-2} - \frac{7}{3}\left(\frac{1}{s-1}\right) + \frac{8}{5}\left(\frac{1}{s+4}\right)$$

$$Y(\eta) = -3e^{2\eta} - \frac{7}{3}e^{\eta} + \frac{8}{5}e^{-4\eta}$$

$$y(t+1) = Y(\eta)$$

$$y(t+1) = -3e^{2t+2} - \frac{7}{3}e^{t+1} + \frac{8}{5}e^{-4(t+1)}$$

HW $\quad y'' + 2y' + 5y = 50(t-3)$

$y(3) = -4$
$y'(3) = -14$

let $\eta = t-3 \quad , t = n+3$

$\therefore y''(t-3) + 2y'(t-3) + 5y(t-3) = 50(t-3)$

let $u(n) = y(t-3)$

$\eta = n$

$u''(\eta) + 2u'(n) + 5y(\eta) = 50\eta$

$u'(0) = y'(0+3) = -14 \quad \dots (A)$

$u(0) = y(0+3) = -4 \quad \dots (B)$

Aqqqprd

$s^2 u(s) - s\,u(0) - u'(0) + s2U(s) - u(0) + 5u(s)$

$= \dfrac{50}{s^2}$

$U(s)(s^2 + 2s + 5) - (-4s - 14 - 4) = \dfrac{50}{s^2}$

$u(s)(s^2 + 2s + 5) = \dfrac{50}{s^2} + (-4s - 18)$

$= \dfrac{50}{s^2} - \dfrac{4s^3}{s^2} - \dfrac{18s^2}{s^2}$

$= \dfrac{50 - 4s^3 - 18s^2}{s^2}$

$u(s) = \dfrac{50 - 4s^3 - 18s^2}{s^2(s^2 + 2s + 5)}$

X

If $\mathcal{L}\{f(t)\} = F(s)$

Then, $\mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\} = \dfrac{F(s)}{s}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s} F(s)\right\} = \int_0^t f(\tau)\, d\tau$$

e.g. ① $\mathcal{L}\left\{\int_0^t e^{2\tau} \sin(7\tau)\, d\tau\right\}$

$$= \frac{1}{s} \mathcal{L}\left\{e^{2t} \sin 7t\right\}$$

$$= \frac{1}{s} \frac{7}{(s-2)^2 + 49}$$

② $y' + 8y = \int_0^t \sin(3t)\cos(3t)\, dt \;; \quad y(0) = 0$

$$sY(s) + 8Y(s) = \frac{1}{s} \mathcal{L}\left(\frac{\sin 6t}{2}\right) \qquad \because \underline{\sin 2\theta =}\ 2\sin\theta\cos\theta$$

$$= \frac{1}{2s} \times \frac{6}{s^2 + 36}$$

$$Y(s)\,(s+8) = \frac{3}{s(s^2+36)(s+8)} = \frac{A}{s} + \frac{Bs + C}{s^2+36} + \frac{D}{s+8}$$

$s = 0$, mul by s

$$A = \frac{\cancel{3}\,1}{(\cancel{36}_{12})8} = \quad A = \frac{1}{96}$$

$s = -8$, mul by $(s+8)$,

$$D = \frac{3}{(-8)(100)}$$

$$= \frac{-3}{800},$$

$$Y(s) = \frac{3}{s(s^2+36)(s+8)} = \frac{1}{96s} - \frac{3}{800(s+8)} - \frac{5}{150(s^2+36)} - \frac{3}{100(s^2+36)}$$

$$y(t) = \frac{1}{96} - \frac{3}{8} e^{-8t} - \frac{1}{150}\cos 6t - \frac{1}{200}\sin 6t$$

③ $\mathcal{L}^{-1}\left\{\frac{1}{s(s-9)}\right\}$

the $1/s$ here can be considered as

$$\mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\} = \int_0^t f(\tau)\,d\tau$$

$$= \left(\int_0^t \frac{1}{s-9}\right)$$

$$= \int_0^t e^{9\tau}\,d\tau$$

$$= \frac{e^{9t}}{9} - \frac{e^{0}}{9}$$

$$= \frac{1}{9}\left(e^{9t} - 1\right)$$

④ $\mathcal{L}^{-1}\left\{\frac{1}{s^3+16s}\right\} \Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+16}\right\}$

$$\Rightarrow \int_0^t \mathcal{L}\left\{\frac{1}{s^2+16}\right\}$$

$$= \int_0^t \frac{1}{4}\mathcal{L}\left\{\frac{4}{s^2+16}\right\}$$

$$= \frac{1}{4}\int_0^t \sin 4\tau\,d\tau$$

$$= \frac{1}{4}\left(-\frac{\cos 4\tau}{4}\right)_0^t$$

$$= \frac{1}{4}\left(-\frac{\cos 4t}{4} - \frac{-1}{4}\right)$$

$$= \frac{1}{16}\left(-\cos 4t + 1\right)$$

First Shifting Theorem

- $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$

$e^{at} f(t) = L^{-1}\{F(s-a)\}$

- Mul. by expo in T-space, there's a shift in T-space.

Second Shifting Theorem

- $\mathcal{L}\{f(t-a)\cdot u(t-a)\} = e^{-as}F(s)$

- If shift in T-space multiply by exponential in S-space

e.g. $f(t) = \sin(3t)$, $\underline{0 < t < \pi}$

$\downarrow$

$u(t) - u(t-\pi)$

but $u(t) = 1$, $\Rightarrow 1 - u(t-\pi)$

$$f(t) = \sin(3t)\left(1 - u(t-\pi)\right)$$

$$= \sin 3t - \sin(3t)\cdot u(t-\pi)$$

$$\mathcal{L}\{f(t)\} = \frac{3}{s^2+9} - \mathcal{L}\{\sin(3t)\cdot u(t-\pi)\}$$

$$= \frac{3}{s^2+9} - \mathcal{L}\{\sin 3(t-\pi+\pi) \cdot u(t-\pi)\}$$

$$= -11- \quad - \mathcal{L}\{\sin 3(t-\pi)\cos(3\pi) \cdot u(t-\pi)\}$$

$$= \frac{3}{s^2+9} + \mathcal{L}\{\sin 3(t-\pi) \cdot u(t-\pi)\}$$

$$= \frac{3}{s^2+9} + e^{-\pi s} \cdot \frac{3}{s^2+9}$$

$$= \frac{3}{s^2+9}\left(1 + e^{-\pi s}\right)$$

$\Rightarrow \sin(A+B) = \sin A \cos B + \cos A \sin B$

$A = 3(t-\pi)$

$B = 3\pi$

$\Rightarrow \sin 3(t-\pi) \cos 3\pi + \cos 3(t-\pi) \sin 3\pi$

but $\sin 3\pi = 0$

$\therefore \Rightarrow \sin 3(t-\pi)\cos 3\pi$

Also, $\cos 3\pi = -1$,

$\Rightarrow -\sin 3(t-\pi)$

2) $t = (5 < t < 10)$

$$f(t) = t\left(u(t-5) - u(t-10)\right)$$

$$= t\,u(t-5) - t\,u(t-10)$$

$$= (t-5+5)\,u(t-5) - (t-10+10)\,u(t-10)$$

$$= (t-5)\,u(t-5) + 5u(t-5) - (t-10)\,u(t-10) - 10u(t-10)$$

$$F(s) = \frac{e^{-5s}}{s^2} + \frac{5e^{-5s}}{s} - \frac{e^{-10s}}{s^2} - \frac{10e^{-10s}}{s}$$

③ $\mathcal{L}^{-1}\left\{\dfrac{e^{-s}s}{s^2+\omega^2}\right\} \Rightarrow \mathcal{L}^{-1}\left\{e^{-s}\cdot\dfrac{s}{s^2+\omega^2}\right\}$

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)(u-a)$$

on comparison we get,

$a = 1,$

$F(s) = \dfrac{s}{s^2+\omega^2} \Rightarrow \therefore f(t) = \cos\omega t$

$$\Rightarrow \therefore \mathcal{L}^{-1}\left\{\frac{e^{-s}s}{s^2+\omega^2}\right\} = \cos(\omega(t-1))\cdot u(t-1)$$

4) $\mathcal{L}^{-1}\left\{\dfrac{e^{-2\pi s}-e^{-8\pi s}}{s^2+1}\right\}$

$\Rightarrow \mathcal{L}^{-1}\left\{e^{-2\pi s}\times\dfrac{1}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{e^{-8\pi s}\cdot\dfrac{1}{s^2+1}\right\}$

$a = 2\pi$ | $a = 8\pi$

$\Rightarrow F(s) = 1/(s^2+1)$

$f(t) = \sin t$

$$\Rightarrow \sin(t-2\pi)\cdot u(t-2\pi) - \sin(t-8\pi)\cdot u(t-8\pi)$$

$$\Rightarrow \sin(t)\,u(t-2\pi) - \sin(t)\cdot u(t-8\pi)$$

$\because \sin(t-2\pi) = \sin t$ $\quad\because 2\pi = 360°$

$\sin(t-8\pi) = \sin t$ $\quad\because 8\pi = 4(360°)$

$$\Rightarrow \sin t\,(u(t-2\pi) - u(t-8\pi))$$

Q) e^{t}, $(0<t<2)$

$f(t) = e^t(u(t) - u(t-2))$

$= e^t(1 - u(t-2))$

$= e^t - e^t u(t-2)$

$e^{at} = \dfrac{1}{s-a}$

$F(s) = \dfrac{1}{s-1} - \mathcal{L}\{e^{t-2+2}u(t-2)\}$

$= \dfrac{1}{s-1} - \mathcal{L}\{e^{t-2}e^{2}u(t-2)\}$

$$= \frac{1}{s-1} - e^{2}\,\mathcal{L}\{ e^{t-2} u(t-2)$$

$$= \frac{1}{s-1} - e^{2} \cdot e^{2s} \cdot \frac{1}{s-1}$$

$$= \frac{1}{s-1} - \frac{e^{2} e^{2s}}{s-1}$$

$$= \frac{1 - e^{2+2s}}{s-1}$$

a) $\mathcal{L}^{-1}\left\{ \frac{e^{-2s}}{s^5} \right\}$

$$\Rightarrow \mathcal{L}^{-1}\left\{ e^{-2s} \cdot \frac{1}{s^5} \right\}$$

$$t^4 = \frac{4!}{s^{4+1}} = \frac{4\times3\times2\times1}{s^5} = \frac{24}{s^5}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{ e^{-2s} \cdot \frac{24}{s^5} \cdot \frac{1}{24} \right\}$$

$$\Rightarrow \frac{1}{24}\,\mathcal{L}^{-1}\left\{ e^{-2s} \cdot \frac{24}{s^5} \right\}$$

$a = 2$

$F(s) = 24/s^5$

$\therefore f(t) = t^4$

$$\Rightarrow \frac{1}{24} (t-2)^4 \cdot u(t-2)$$

Convolution product:

$$\underset{\text{Convolution}}{\mathcal{L}\{f * g\}} = \mathcal{L}\{f\} \cdot \mathcal{L}\{g\}$$ and $$\boxed{f * g = g * f}$$

Defn:

$$(f * g)(t) = \int_0^t f(\tau) \cdot g(t-\tau)\, d\tau$$

e.g. $f(t) = t$,

$$\underbrace{t}_{f} * \underbrace{1}_{g} = \int_0^t f(\tau) \cdot g(t-\tau)\, d\tau$$

$$= \int_0^t \tau \cdot 1 \, d\tau$$

$$= \frac{t^2}{2}$$

Convolution Theorem :

Laplace of $f * g$ is nothing but product of Laplaces of $F(s)$ and $G(s)$.

$$\mathcal{L}\{(f * g)(t)\} = \mathcal{L}\{F(s)\} \cdot \mathcal{L}\{G(s)\}$$

$$(f * g)(t) = \mathcal{L}^{-1}\{F(s) \cdot G(s)\}$$

1) $1 * \sin(t)$:

$f(\tau) = 1$

$g(t-\tau) = \sin(t-\tau)$

$$1 * \sin(t) = \int_0^t 1 \cdot \sin(t-\tau)\, d\tau$$ $\ldots\ldots \to$ $\sin(A-B) = \sin A \cos B - \cos A \sin B$

but, we know,

Convolution is commutative,

$\therefore \; 1 * \sin(t) = \sin(t) * 1$

$f(t) = \sin(t)$

$g(t) = 1$

$$1 * \sin(t) = \int_0^t \sin(\tau) \cdot 1 \, d\tau$$ $\ldots \because g(t)$ is 1, it's a constant function.

$$1 * \sin(t) = -\cos(\tau)\Big|_0^t$$

$$= -\cos(t) + \cos(0)$$

$$= 1 - \cos(t)$$

---

2) $L^{-1}\left\{\frac{1}{s(s-1)}\right\} \Rightarrow$ $F(s) = 1/s$

$\therefore f(t) = 1$

$G(s) = 1/(s-1)$

$g(t) = e^t$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{s(s-1)}\right\} = 1 * e^t \Rightarrow e^t * 1$$

... always choose 'g' as simplest function.

$\therefore f(t) = e^t \, ; \, g(t) = 1$

$$\Rightarrow \int_0^t f(\tau)\cdot g(t-\tau)\, d\tau$$

$$= \int_0^t e^\tau \cdot 1 \, d\tau$$

$$= \int_0^t e^\tau \, d\tau$$

$$= e^t - e^0$$

$$= e^t - 1$$

Q. $\mathcal{L}^{-1}\left\{\frac{5}{(s^2+1)(s^2+25)}\right\}$

$F(s) = \frac{1}{s^2+1}$

$G(s) = 5/s^2+25$

$f(t) = \sin t$

$g(t) = \sin 5t$

$$\sin t * \sin 5t = \sin 5t * \sin t$$

$$= \int_0^t \sin 5\tau \cdot \sin(t-\tau)\, d\tau$$

$\sin A = \sin 5\tau$

$\sin B = \sin(t-\tau)$

$\sin A \sin B = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)$

$$= \frac{1}{2}\int_0^t \cos(5\tau - t + \tau) - \cos(5\tau + t - \tau)\, d\tau$$

$$= \frac{1}{2}\int_0^t \cos(6\tau - t) - \cos(4\tau + t)\, d\tau$$

$$= \frac{1}{2}\int_0^t \cos(6\tau) - \cos(t) - \cos(4\tau) - \cos(t)\, d\tau$$

$$= \frac{1}{2}\left[\frac{\sin 6\tau}{6} - \cos(t)\tau - \frac{\sin(4\tau)}{4} - \cos(t)\tau\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin 6t}{6} - t\cos t - \frac{\sin 4t}{4} - t\cos t - 0\right]$$

$$= \frac{1}{2}\left[\frac{\sin 6t}{6} - 2t\cos t - \frac{\sin 4t}{4}\right]$$

H.W

① $\mathcal{L}\{t * t\} \Rightarrow f(t) = t$
$g(t) = t$

$$\mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\{f(t)\} \cdot \mathcal{L}\{g(t)\}$$

$$= \frac{1}{s^2} \cdot \frac{1}{s^2}$$

$$= \frac{1}{s^4}$$

② $\mathcal{L}\{t * t * t \ldots t\}$

Assume t occurs n times,

$$\mathcal{L}\{f_1(t) * f_2(t) * f_3(t) \ldots * f_n(t)\} = \mathcal{L}\{f_1(t)\} \cdot \mathcal{L}\{f_2(t)\} \ldots \mathcal{L}\{f_n(t)\}$$

$$= \prod_{i=1}^{n} \frac{1}{s^2}$$

$$= \text{OR} \Rightarrow \frac{1}{s^{2n}}$$

③ $\mathcal{L}^{-1}\left\{\frac{1}{s^2(\ ^2+9)}\right\}$

$\Rightarrow \quad F(s) = \frac{1}{s^2} \qquad \therefore f(t) = t$

$G(s) = \frac{1}{s^2+9} \Rightarrow \frac{3}{s^2+9}\cdot\frac{1}{3} \qquad g(t) = \frac{1}{3}\sin 3t$

If $\mathcal{L}\{f(t)\} = F(s)$

Then, $\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{F(s)}{s}$

$$\mathcal{L}\left\{\int_0^t\int_0^\tau f(u)\,du\right\} = \frac{F(s)}{s^2} = \frac{1}{s^2(s^2+9)}$$

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\cdot\frac{1}{s(s^2+9)}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+9)}\right\}$$

$$\mathcal{L}\left\{\int_0^t g(\tau)\,d\tau\right\} = \frac{1}{s(s^2+9)}$$

$$g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$g(t) = \frac{1}{3}\sin 3t$$

$$f(t) = \int_0^t g(t)\,dt = \int_0^t \frac{1}{3}\sin 3t\,dt$$

$$= -\frac{1}{9}\left[\cos 3t\right]_0^t$$

$$= -\frac{1}{9}\left[\cos 3t - 1\right]$$

$$= \frac{1}{9}\left[1 - \cos 3t\right]$$

$$\int_0^t f(t)\,dt = \int_0^t \frac{1}{9}\left[1 - \cos 3t\right]dt$$

$$= \frac{1}{9}\left[t - \frac{\sin 3t}{3}\right]_0^t$$

$$= \frac{1}{9}\left[t - \frac{\sin 3t}{3}\right]$$

(3) $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+9)}\right\} = \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+9)}\right\}$

let $\frac{1}{s(s^2+9)} = F(s)$

$$F(s) = \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$= \int_0^t \frac{1}{3}\sin 3t$$

$$= \frac{1}{3}\left[\frac{-\cos 3t}{3}\right]_0^t$$

$$= \frac{1}{9}\left[-\cos 3t - (-1)\right]$$

$$= \frac{1}{9}\left[-\cos 3t + 1\right]$$

$$f(t) = \frac{1}{9}\left[1 - \cos 3t\right]$$

$$\Rightarrow \int_0^t f(t)$$

$$= \frac{1}{9}\int_0^t 1 - \cos 3t$$

$$= \frac{1}{9}\left(\frac{3t}{3} - \frac{\sin 3t}{3}\right)_0^t$$

$$= \frac{1}{27}\left[3t - \sin 3t - 0 + 0\right]$$

$$= \frac{1}{27}\left[3t - \sin 3t\right]$$

(4) $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+4)(s^2+1)}\right\}$

$F(s) = \dfrac{2}{s^2+2^2}\cdot\dfrac{1}{2}$; $G(s) = \dfrac{1}{s^2+1}$

$f(t) = \dfrac{\sin 2t}{2}$; $g(t) = \cos t$

$$\Rightarrow f(t) * g(t)$$

$$= \int_0^t \frac{\sin 2\tau}{2}\cdot\cos(t-\tau)\,d\tau$$

$$= \frac{1}{\cancel{2}}\int_0^t \cancel{2}\sin\tau\cos\tau\cdot\cos(t-\tau)\,d\tau$$

$$= \int_0^t \sin\tau\cos\tau\cdot\cos t - \int_0^t \sin\tau\cos\tau\cos\tau$$

$$= \qquad\qquad - \int_0^t \sin\tau\cos^2\tau$$

$$- \int_0^t \sin\tau\,(1-\sin^2\tau)$$

$$- \int_0^t \sin\tau - \sin^3\tau$$

$$- \left[-\cos t - 0\right] - \left[\cos^3 t\right.$$

Convolution:

$$L^{-1}\left[\frac{1}{(s-1)(s+2)}\right]$$

$$f(s) = \frac{1}{s-1} \qquad f(t) = e^{t}$$

$$g(s) = \frac{1}{s+2} \qquad g(t) = e^{-2t}$$

$$L^{-1}\left[F(s)\ g(s)\right] = e^{t} * e^{-2t}$$

$$= \int_0^t e^{u}\, e^{-2(t-u)}\, du$$

$$= \int_0^t e^{u} \cdot e^{-2t} \cdot e^{2u}$$

$$= e^{-2t} \int_0^t e^{u} \cdot e^{2u}\, du$$

$$= e^{-2t} \int_0^t e^{3u}$$

$$= e^{-2t} \left[\frac{e^{3u}}{3}\right]_0^t$$

$$= e^{-2t} \cdot \left[\frac{e^{3t}}{3} - \frac{1}{3}\right]$$

$$= \frac{e^{-2t} \cdot e^{3t}}{3} - \frac{e^{-2t}}{3}$$

Q. $L^{-1}\left[\dfrac{s}{(s^2+a^2)^2}\right]$

$$F(s) = \frac{1}{s^2+a^2} \qquad f(t) = \frac{1}{a}\sin at$$

$$G(s) = \frac{s}{s^2+a^2} \qquad g(t) = \cos at$$

$$\mathcal{L}^{-1}\left\{F(s)\,G(s)\right\} = f(t) * g(t)$$

$$= \int_0^t \cos a\tau \cdot \frac{1}{a}\sin a(t-\tau)\,d\tau$$

$$= \frac{1}{a}\int_0^t \cos a\tau \sin a(t-\tau)\,d\tau$$

$$= \frac{1}{a}\int_0^t \frac{1}{2}\left(\sin a(\cancel{\tau}+t-\cancel{\tau}) - \sin(au - at + a\tau)\right) d\tau$$

$$= \frac{1}{2a}\int_0^t \sin at - \sin(2au - at)\,du \qquad \cos A \sin B = \sin(A+B) - \sin(A-B)$$

$$= \frac{1}{2a}\left[\sin at(u) \; \frac{-(-\cos(2au - at)}{2a}\right]_0^t$$

$$= \frac{1}{2a}\left\{\sin at(u) + \frac{\cos 2au - at}{2a}\right.$$

$$= \frac{1}{2a}\left[t\sin at + \frac{\cos 2at - at}{2a} - 0 - \frac{\cos 2a(0) - at}{2a}\right]$$

$$= \frac{1}{2a}\left[t\sin at + \frac{\cos \cancel{at}}{\cancel{2a}} - 0 - \cancel{\frac{\cos(-at)}{2a}}\right]$$

but $\cos(-\theta) = \cos\theta$

$$= \frac{1}{2a}\left[t\sin at\right]$$

$$= \frac{t \sin at}{2a}$$