CHAPTER

3 Magnetostatics and Electrodynamics

Static magnetic fields are produced by permanent magnets and steady currents flowing in conductors. Magnetostatics deals with magnetic fields produced by steady currents. Faraday discovered the method of generation of electric and magnetic fields which vary with time. Electrodynamics deals with the study of varying electric and magnetic fields. Maxwell unified the important laws of electricity and magnetism and formulated a unified theory in 1861. He formulated four equations that are regarded as the basis of all electric and magnetic phenomena. The consequences of Maxwell's equations are very far reaching. Maxwell predicted the existence of electromagnetic waves and that light is a form of electromagnetic radiation.

3.1 MAGNETIC FIELD

In 1820 Oersted discovered that electric currents create magnetic fields. A steady current I flowing in a straight conductor produces a magnetic field around it. The *magnetic lines of force* exist in the form of a series of concentric circles with conductor as the centre. The direction of the field can be found by the right hand rule. If the current carrying conductor is gripped with the right hand so that the thumb points the direction of current flow, then the curled fingers point in the direction of the magnetic field. Fig. 3.1 shows the method of finding the direction of magnetic field.

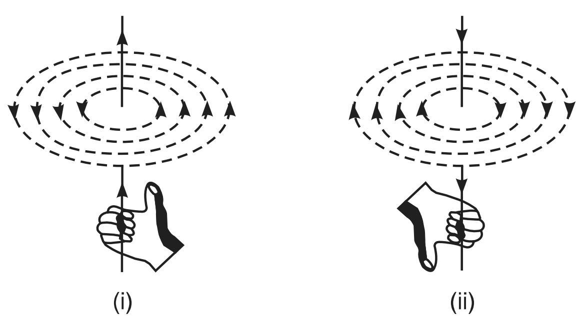

Fig. 3.1

The region around a current carrying conductor or a permanent magnet where magnetic effects are experienced is called a *magnetic field*. A magnetic field is schematically represented by magnetic lines of force, which are also known as *field lines* or lines of *magnetic induction*. A magnetic field is described either by magnetic field strength **H** or by the *magnetic induction* (or *magnetic flux density*), **B**.

Relation between B & H:

3.2 MAGNETIC FLUX DENSITY

Magnetic flux

The lines of induction are collectively called **flux**. The magnetic flux through a region is the number of lines of induction passing normally through the region. The concentration of lines of induction is an indication of magnetic field strength. It is defined in terms of the flux density.

Magnetic induction

The number of field lines passing through a unit area of cross-section is called the **magnetic flux density** or **magnetic induction**. It is denoted by magnetic induction vector, **B**. Thus,

$$\mathbf{B} = \frac{\text{Magnetic flux}}{\text{area}} = \frac{\varphi}{A} \tag{3.1}$$

Therefore, magnetic flux is given by $\varphi = BA$. In a more general way, let the area be inclined at an angle to the magnetic field. Let θ be the angle between the normal to the area and the direction of magnetic field. Then,

$$\varphi = \mathrm{B\,A} \cos\theta = \mathbf{B} \cdot \mathbf{A} \tag{3.2}$$

Thus, magnetic flux through an area is equal to the dot product of magnetic field **B** and area **A**.

The unit of magnetic flux is weber (Wb) and the unit of magnetic induction is weber per square metre (Wb/m^2) or tesla (T).

The magnetic flux through any surface may also be given by the surface integral of the normal component of B. Thus,

$$\varphi = \int_S \mathbf{B} \cdot d\mathbf{s} \tag{3.3}$$

where $d\mathbf{S}$ is the elemental surface.

3.3 BIOT-SAVART LAW

Let a conductor of an arbitrary shape carry a steady current I. Let P be a point in the magnetic field produced by the current. Let a small element AB of length dl produce magnetic field dB at P. Let r be the distance of P from the current element $I\,dl$ and θ be the angle between dl and r. According to Biot-Savart law, the magnitude of magnetic field dB is directly proportional to the product $I\,dl \sin\theta$ and is inversely proportional to the square of distance between current element and the point P. Thus,

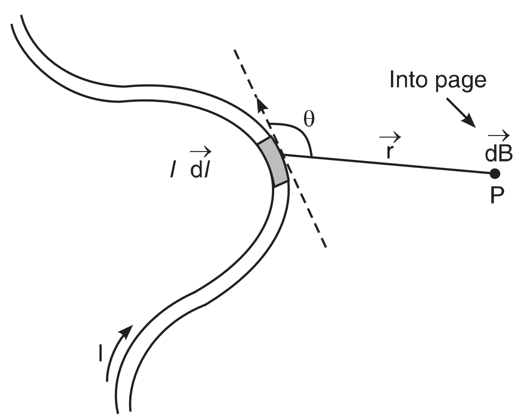

Fig. 3.2

$$dB \propto I\,dl \sin\theta \quad \text{and} \quad dB \propto \frac{1}{r^2}$$

Combining these relations, $$dB \propto \frac{I\,dl \sin\theta}{r^2}$$

or $$dB = k\,\frac{I\,dl \sin\theta}{r^2}$$

where k is a constant proportionality. The value of k depends on the medium in which the conductor is situated and the system of units adopted. In SI units, its value for free space is

$$k = \frac{\mu_0}{4\pi} \quad \text{where } \mu_0 = 4\pi \times 10^{-7} \text{ Tm / A}$$

$$\therefore \quad dB = \frac{\mu_0}{4\pi}\frac{I\,dl \sin\theta}{r^2} \quad \text{Biot-Savart law} \tag{3.4}$$

The Biot-Savart law holds only for steady currents. The current element $I\,dl$ is the source of static magnetic field, just as a charge q is the source of static electric field. The above law is written in the vector form as

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{(d\mathbf{l} \times \mathbf{r})}{r^3} \tag{3.5}$$

The direction of the magnetic field is given by the right hand thumb rule (see Fig. 3.1). The direction of $d\mathbf{B}$ is into the plane of the paper.

The total magnetic field at P due to the conductor is obtained by summing up the contributions of all current elements.

$$\therefore \quad \mathbf{B} = \int d\mathbf{B} = \int \frac{\mu_0}{4\pi}\frac{I\,(d\mathbf{l} \times \mathbf{r})}{r^3} \tag{3.6}$$

3.4 AMPERE'S LAW

Ampere's law states that *the line integral of the tangential component of the magnetic field over any closed path is equal to the amount of the current enclosed by the loop*. Thus,

$$\oint \mathbf{B} \cdot d\boldsymbol{l} = \mu_0 I \tag{3.7}$$

Both Ampere's law and the Biot-Savart law are relations between a current distribution and the magnetic field that it generates. We can apply Biot-Savart law to calculate the magnetic field caused by any current distribution. On the other hand, Ampere's law allows us to calculate magnetic field with ease in case of symmetry.

Let us consider an infinitely long wire along the z-axis carrying a current I amp. The magnetic flux density due to this wire is directed everywhere circular to the wire and its magnitude is dependent only on the distance from the wire. Let us consider a circular path C of radius r in the plane normal to the wire and centered at the wire. The current enclosed by an arbitrarily closed path C is given by the surface integral of the current density over any surface S bounded by the closed path C.

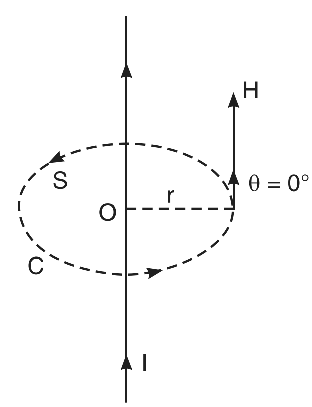

Fig. 3.3

The total current flowing through the surface area S is given by

$$I = \int \mathbf{J} \cdot ds \tag{3.8}$$

Therefore, we can write

$$\oint \mathbf{H} \cdot d\boldsymbol{l} = \int \mathbf{J} \cdot ds$$

$$\therefore \quad \oint_C \mathbf{B} \cdot d\boldsymbol{l} = \mu_0 \int_S \mathbf{J} \cdot ds \tag{3.9}$$

This is known as *Ampere's circuital law*.

3.4.1 Ampere's Circuital Law in Differential Form

If we now shrink the path C to a very small size ΔC so that the surface area bounded by it becomes very small, ΔS, we can write equ. (3.6) as

$$\oint_{\Delta C} \mathbf{B} \cdot dl = \mu_0 \int_{\Delta S} \mathbf{J} \cdot ds \tag{3.10}$$

Since the surface area ΔS is very small we can consider the current density to be uniform over the surface so that

$$\int_{\Delta S} \mathbf{J} \cdot ds \approx \mathbf{J} \cdot \Delta S$$

The relation becomes exact in the limit $\Delta S \to 0$. Dividing both the sides of equ. (3.10) by ΔS and letting $\Delta S \to 0$, we have

$$\begin{aligned} \lim_{\Delta S \to 0} \frac{\oint_{\Delta C} \mathbf{B} \cdot d\boldsymbol{l}}{\Delta S} &= \lim_{\Delta S \to 0} \frac{\mu_0 \int_{\Delta S} \mathbf{J} \cdot ds}{\Delta S} \\ &= \mu_0 \lim_{\Delta S \to 0} \frac{\mathbf{J} \cdot \Delta \mathbf{S}}{\Delta S} \\ &= \mu_0 \, \mathbf{J} \cdot \mathbf{n} \end{aligned} \tag{3.11}$$

Now, the curl of **B** is defined as the vector having the magnitude given by the maximum value of the quantity on the left side of equ. (3.11). We note that this maximum value occurs for an orientation of ΔS for which the direction of its normal coincides with the direction of **J** and it is equal to μ_0 times the magnitude of **J**. Thus

$$|\nabla \times \mathbf{B}| = \text{maximum value of} \left(\lim_{\Delta S \to 0} \frac{\oint_{\Delta C} \mathbf{B} \cdot dl}{\Delta \mathbf{S}} \right) = \mu_0 \, |\mathbf{J}| \tag{3.12}$$

or

$$\nabla \times \mathbf{B} = \mu_0 \, \mathbf{J} \tag{3.13}$$

Equ. (3.13) is Ampere's circuital law in *differential form*.

3.5 GAUSS'S LAW FOR MAGNETISM

Just as in the case of electrostatics, the magnetic flux through an element of area d**s** is given by the dot product of **B** with d**s**. For an arbitrary surface *ds* bounded by a closed contour *S*, total magnetic flux passing through the surface is given by

$$\phi = \oint_S \mathbf{B} \cdot ds \tag{3.14}$$

The lines of vector **B** have neither beginning nor ending. The number of lines emerging from any volume bounded by a closed surface S is always equal to the number of lines entering the volume. Hence, the flux of B through any closed surface is equal to zero. Thus,

$$\oint_S \mathbf{B} \cdot ds = 0$$

Dividing both sides of the above equation by an incremental volume $\Delta \upsilon$ over which the surface is to be considered closed, we get

$$\frac{\oint \mathbf{B} \cdot ds}{\Delta \upsilon} = 0$$

The limit of the left side of the equation, as $\Delta\upsilon \to 0$, is the divergence of the vector **B**.

$$\therefore \qquad \nabla \cdot \mathbf{B} = 0 \tag{3.15}$$

3.6 MAGNETIC SCALAR POTENTIAL

In electrostatics, the potential V is a **scalar**. It is related to the magnetic field **E** as

$$\mathbf{E} = -\nabla V$$

The scalar potential is related to the sources, i.e. charge distribution and can be easily calculated. In a similar way, we may define magnetic scalar potential by the following relation.

$$\mathbf{B} = -\mu_0 \nabla\phi$$

where ϕ is the **magnetic scalar potential** due to the sources.

According to Ampere's law

$$\oint \mathbf{B} \cdot d\boldsymbol{l} = \mu_0 i \tag{3.16}$$

In general the right hand side of the above equation is not zero and **B** is a **solenoidal** (not a conservative) **field**. In regions outside the sources of **B** (see Fig. 3.4), a closed path is not linked with a current and therefore, the line integral of **B** over such a closed path vanishes. In such cases, we may consider **B** to be a *conservative field* and therefore, express it as

$$\mathbf{B} = -\mu_0 \nabla\phi \tag{3.17}$$

This does not mean that **B** changes its character from a solenoidal field to a conservative field. All that we say is that in the regions outside the sources, we can simplify the mathematical formulation by expressing **B** in terms of the gradient of ϕ. Let us obtain now an expression for ϕ.

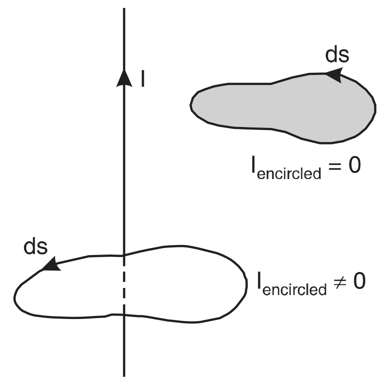

Fig. 3.4

The magnetic scalar potential at P due to a current loop is the sum of the potentials due to the individual small loops. A current loop is equivalent to a magnetic dipole. The magnetic moment of the small current loop is given by

$$dm = i\, dA$$

where dA is the area of the small loop. The potential due to the dipole is given by an expression similar to equ. (1.40) derived in case of an electric dipole. Thus,

$$\text{The scalar potential} \qquad d\phi = \frac{1}{4\pi}\frac{i\, dA \cos\theta}{r^2}$$

where r is the distance of the elemental loop dA from P and θ is the angle between r and the vector **dA**. But $dA \cos\theta = r^2\, d\Omega$, where $d\Omega$ is the solid angle subtended at P by the boundary of the given current loop.

$$\therefore \qquad \phi = \int d\phi = \frac{i}{4\pi}\int d\Omega = \frac{i\Omega}{4\pi} \tag{3.18}$$

3.7 MAGNETIC VECTOR POTENTIAL

Unlike the electric field, the magnetic field is a **solenoidal** field. In spite of that we can define the magnetic induction **B** in terms of some potential function. Two space derivatives are possible with a vector field. They are divergence and curl. We know from equ. (3.15) that

$$\nabla \cdot \mathbf{B} = 0$$

Since the divergence of a curl is always zero, the second possibility is of expressing **B** as the curl of some vector potential function. Thus, we write