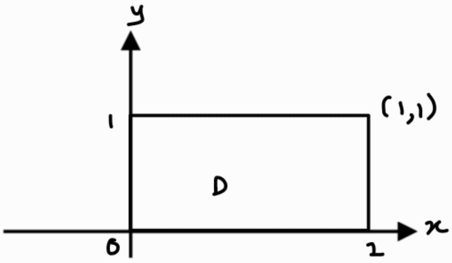
Double Integration:

$$\iint_R A \; \partial A$$

Iterated Integrals & Fubini's Theorem

- The function has to be continuous.

e.g. $f(x,y) = xy$



$$\iint_D f(x,y) \; \partial A$$

$$= \int_{y=0}^{1}\int_{x=0}^{2} xy \quad \partial x \, dy$$

dA can be either dy dx or dx dy
this is known as order of integration.

$$= \int_0^1 y \left[\frac{x^2}{2}\right]_0^1 dy$$

$$= \int_0^1 y \left[\frac{1^2}{2}\right] dy$$

$$= \frac{1}{2}\int_0^1 y \; dy$$

$$= \frac{1}{2}\int_0^1 \frac{y^2}{2}$$

$$= \frac{1}{2}\left[\frac{1}{2} - 0\right]$$

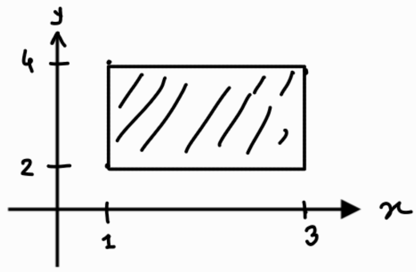
$$= \frac{1}{4}$$

(2) $f(x,y) = e^{x+y}$

$1 \le x \le 3$

$2 \le y \le 4$

$e^{x+y} = e^{x} \cdot e^{y}$



$$= \int_{1}^{3} \int_{2}^{4} e^{x} \cdot e^{y} \, dy \, dx$$

$$= \int_{1}^{3} e^{x} \left[e^{y} \right]_{2}^{4}$$

$$= \int_{1}^{3} e^{x} \left[e^{4} - e^{2} \right] dx$$

$$= \int_{1}^{3} e^{x+4} - e^{x+2}$$

$$= \left[e^{x+4} - e^{x+2} \right]_{1}^{3}$$

$$= \left[e^{3+4} - e^{3+2} - e^{1+4} + e^{1+2} \right]$$

$$= e^{7} - e^{5} - e^{5} + e^{3}$$

$= 819.89$ units

H.W

$$\int_{0}^{\ln 2} \int_{1}^{\ln 5} e^{2x+y} \, dy \, dx \Rightarrow \int_{0}^{\ln 2} e^{2x} \int_{1}^{\ln 5} e^{y} \, dy \, dx$$

$$= \int_{0}^{\ln 2} e^{2x} \left[e^{y} \right]_{1}^{\ln 5} dx$$

$$= \int_0^{\ln 2} e^{2x} \left[\cancel{e}^{\cancel{\ln} 5} - e^{1} \right] dx$$

$$= [5 - e] \int_0^{\ln(2)} e^{2x}$$

$$= \frac{2.28}{2} \left[e^{2x} \right]_0^{\ln(2)}$$

$$= 1.14 \times \left[\cancel{e}^{\cancel{\ln}(2^2)} - e^0 \right]$$

$$= 1.14 \times [4 - 1]$$

$$= 1.14 \times 3$$

$$= 3.42$$

(2) $$\int_{-1}^{2} \int_{0}^{\pi/2} y \sin x \; \partial x \, dy$$

$$\int_{-1}^{2} y \left[-\cos x \right]_0^{\pi/2} dy$$

$$\int_{-1}^{2} y \left[-\cos(90) - (-\cos(0)) \right] \partial y$$

$$\int_{-1}^{2} y \left[-0 + 1 \right] \partial y$$

$$= \int_{-1}^{2} y \, \partial y$$

$$= \left[\frac{y^2}{2} \right]_{-1}^{2}$$

$$= \left[\frac{2^2}{2} - \frac{-1^2}{2} \right]$$

$$= \left[\frac{4}{2} - \frac{1}{2} \right]$$

$$= \frac{3}{2}$$

$$= 1.5$$

(3)

$$V = \int_{1}^{e^2} \int_{\ln(y)}^{2} 1 \, dx \, dy$$

$$= \int_{1}^{e^2} 2 - \ln(y) \, dy$$

$$\boxed{\int \ln(x) = x\ln(x) - x}$$

$$= \left[3y - y\ln(y) \right]_{1}^{e^2}$$

$$= \left[\underline{3e^2 - 2e^2} \cancel{\ln(e^2)} - 3(1) + 1\ln(1) \right]$$

$$V = \int_{0}^{2} \int_{1}^{e^x} 1 \, dy \, dx$$

$$= \int_{0}^{2} e^x - 1$$

$$= \left[e^x - x \right]_{0}^{2}$$

$$= e^2 - 2 - e^0 + 0$$

$$= e^2 - 2 - 1$$

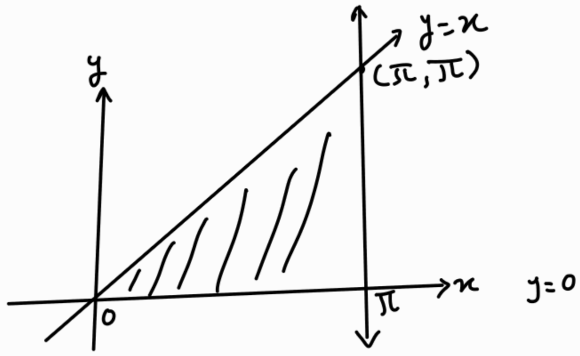
$$= e^2 - 3$$

$$= e^2 - 3$$

2) $f(x,y) = x \sin(y)$

$R: \; y = x \; , y = 0 , \; x = \pi$

$$\int_0^{\pi}\int_0^{x} x \sin(y) \; dy\, dx$$



$dxdy \Rightarrow \displaystyle\int_0^{\pi}\int_y^{\pi} x \sin(y) \; dx\, dy$

$$\int_0^{\pi}\int_0^{x} x \sin(y) \; dy\, dx \Rightarrow \int_0^{\pi} -x\cos x + x$$

$u = x \qquad u' = 1$

$v = -\sin x \qquad v' = \cos x$

$$\Rightarrow \left[\frac{x^2}{2}\right]_0^{\pi} - \int_0^{\pi} x\cos x$$

$$\Rightarrow \frac{\pi^2}{2} - \frac{0}{2} \left[- x\sin x - \cos x\right]_0^{\pi}$$

$$\Rightarrow \frac{\pi^2}{2} - \cancel{\pi \sin \pi} - \cancel{\cos \pi} + \underset{0}{\cancel{0 \sin 0}} + \cancel{\cos 0}$$

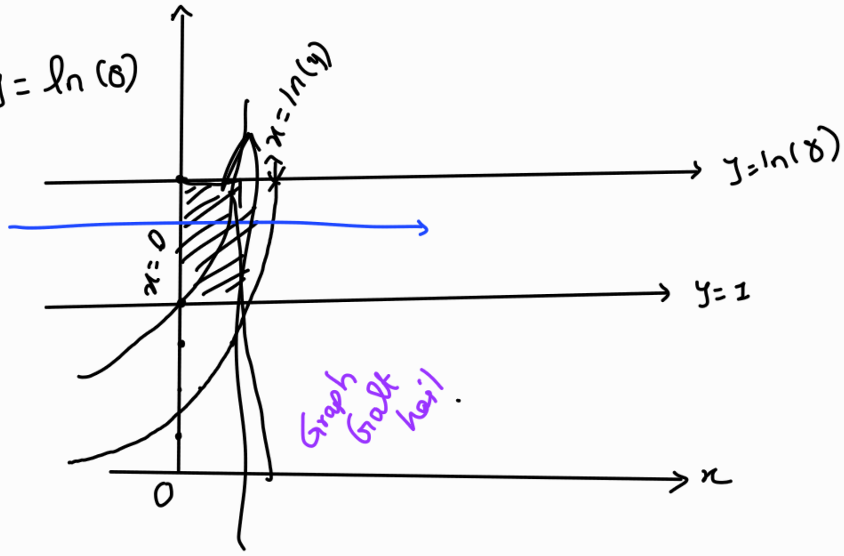
$$\Rightarrow \frac{\pi^2}{2} - (-1) + 1$$

$$= \frac{\pi^2}{2} + 2$$

$$= 6.93$$

(3) $f(x,y) = e^{x+y}$

$R: x = 0,\ x = \ln(y)\quad y = 1,\ y = \ln(8)$



$$\int_0^{\ln 8}\int_x^{\ln 8} e^{x+y}\, dy\, dx$$

$$\int_1^{\ln(8)}\int_0^{\ln(y)} e^{x+y}\, dx\, dy \Rightarrow \int_1^{\ln(8)}\int_0^{\ln(y)} e^x e^y\, dx\, dy$$

$$\Rightarrow \int_1^{\ln(8)} e^y \left[e^x\right]_0^{\ln(y)}$$

$$\Rightarrow \int_1^{\ln(8)} e^y \left[e^{\ln(y)} - e^0\right]$$

$$\Rightarrow \int_1^{\ln(8)} y \cdot e^y - e^y$$

$u = y \qquad u' = 1$

$v = e^y \qquad v' = e^y$

$ye^y - \int e^y$

$ye^y - e^y$

$$\Rightarrow \int_1^{\ln(8)} ye^y - \int_1^{\ln(8)} e^y$$

$$\Rightarrow \left[ye^y - e^y\right]_1^{\ln(8)} - \left[e^y\right]_1^{\ln(8)}$$

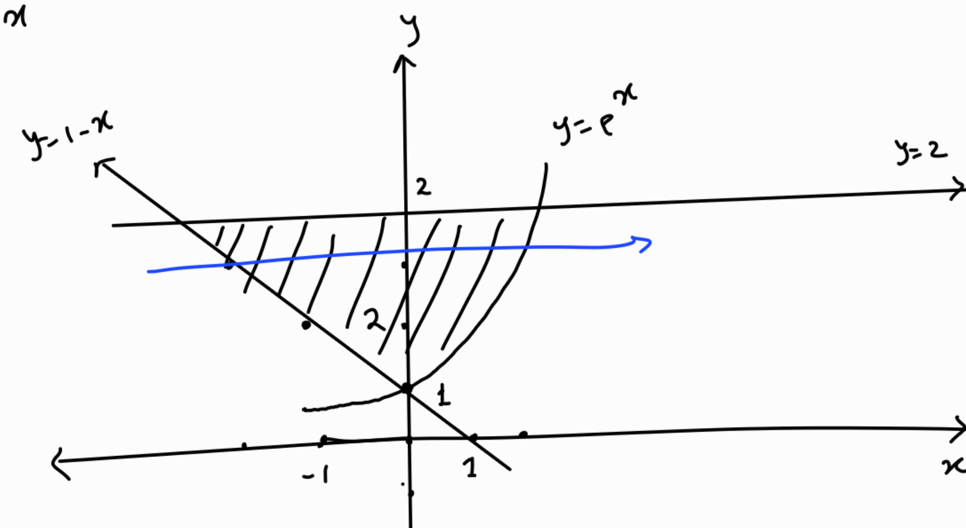
$$\Rightarrow \left[\ln(8)e^{\ln(8)} - e^{\ln(8)} - \cancel{1e^1} + \cancel{e^1}\right]$$

$- e^{\ln(8)} + e^1$

$\Rightarrow 2.079 \times 8 - 8 - 8 + e^1$

$\Rightarrow 3.35$

1) $y = 1 - x, \quad y = 2, \; y = e^x$



$$\int_1^{2} \int_{1-y}^{\ln(y)} 1 \, dx \, dy$$

$= \int_0^2 \ln(y) - 1 + y \; dy$

$= \left[y\ln(y) - y - y + \frac{y^2}{2} \right]_1^2$

$= \left[y\ln(y) - 2y + \frac{y^2}{2} \right]_1^2$

$= 2\ln(2) - 4 + \frac{4}{2} - 0 + 2 - \frac{1}{2}$

$= 2\ln(2) - 4 + 2 + 2 - \frac{1}{2}$

$= 2\ln(2) - \frac{1}{2}$

$1 - x = e^x$

$e^x + x = 1$

at $x = 0$,

$e^0 + 0 = 1$

$= 1$

$\therefore x = 0$ at pt of intersection.

Polar Co-ordinates $\longrightarrow (r, \theta)$

$x^2+y^2=r^2$ ... (1)

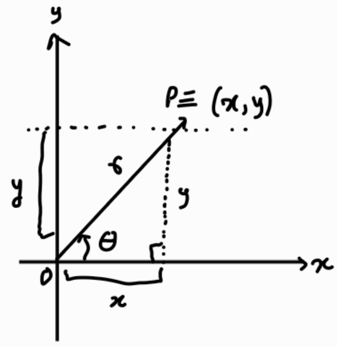
$\cos\theta = \dfrac{x}{r} \longrightarrow$ adj/hypo.

$x = r\cos\theta$ ... (2)

$\sin\theta = \dfrac{y}{r}$

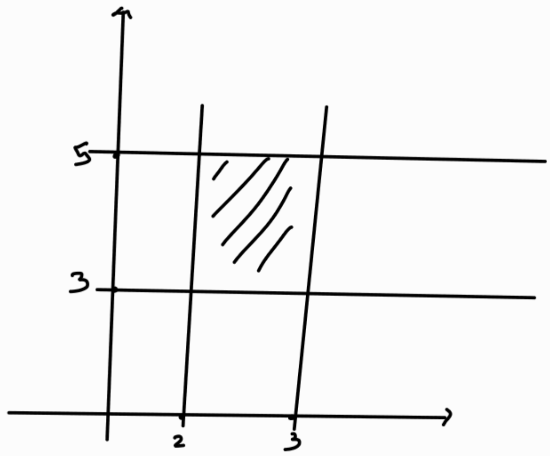
$y = r\sin\theta$ ... (3)

$\dfrac{y}{x} = \tan\theta$ ... (4)

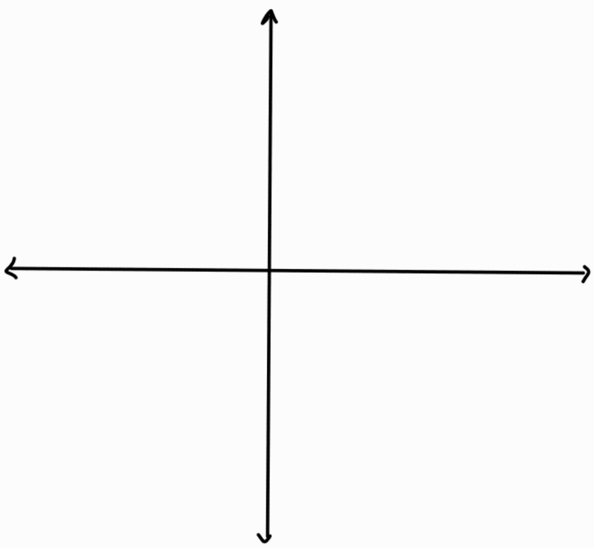


Q) $2 \leq x \leq 3$

$3 \leq y \leq 5$



Cartesian Co-ords



Double integration & polar co-ordinates : ∴ these make solving easily.

$x \xrightarrow{\text{becomes}} r\cos\theta$

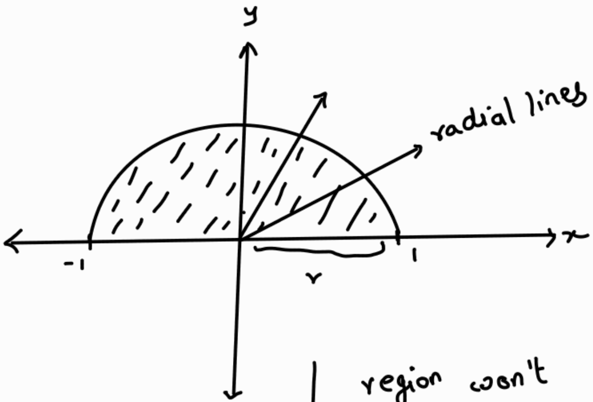
$y \longrightarrow r\sin\theta$

$A \longrightarrow r\,dr\,d\theta$

↑ Magnification / Jacobian factor.

e.g. $V = \int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx$

$\Downarrow$



$y = 0$ or $y = \sqrt{1-x^2}$

$y^2 = 1 - x^2$

$x^2 + y^2 = 1 \quad \cdots ①$

$\boxed{\therefore r = 1}$

$= \int_{\theta=0}^{\pi}\int_{r=0}^{1} r^2\, r\,dr\,d\theta$

region won't be in negative as y was starting from x=0 and going till $y = \sqrt{1-x^2}$ which cannot be -ve.

$\int_{0}^{\pi} \left[\frac{r^4}{4}\right] d\theta$

$\frac{1}{4}\int_{0}^{\pi} d\theta$

$\frac{1}{4}\left[\pi - 0\right]$

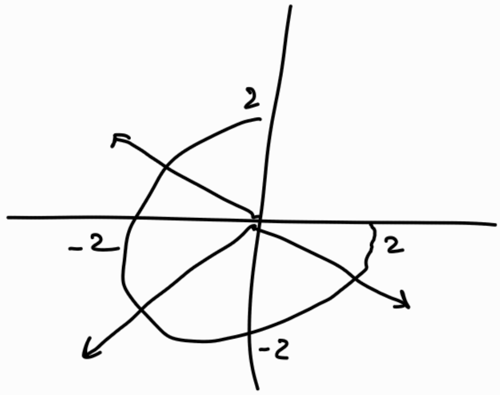
$= \frac{\pi}{4}$

r- limits taken by drawing radial lines

Q.2]

$0 \le r \le 2$

$\frac{\pi}{2} \le \theta \le 2\pi$



$x^2 + y^2 = 4$

$r^2 = 4$

$r = 2$

$$\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} y \; \partial x \, dy = \int_0^2 y \left[\sqrt{4-y^2} + \sqrt{4-y^2} \right] \partial y$$

$$= \int_0^2 y \left[2\sqrt{4-y^2} \right]$$

$$= 2\int_0^2 y\sqrt{4-y^2} \; dy$$

$4 - y^2 = u$

$-2y = \frac{du}{\partial y}$

$\partial y = \frac{du}{-2y}$

let $4 - y^2 = u$

$$= 2\int_0^2 \cancel{y}\sqrt{u} \cdot \frac{du}{-2\cancel{y}}$$

$$= -\frac{2}{2} \int_0^2 \sqrt{u} \; du$$

$$= -1\int_0^2 u^{1/2}\,du$$

$$= -1\left[\frac{2u^{3/2}}{3}\right]_0^2$$

$$= -1\left[\frac{2(\sqrt{4-y^2})^3}{3}\right]_0^2$$

$$= -1\left[\frac{2(0)^3}{3} - \frac{2(\sqrt{4-0})^3}{3}\right]$$

$$= 0 - \frac{16}{3} = \frac{-16}{3}$$

$$= \frac{16}{3}$$

e)

$$r^2 = x^2 + y^2$$

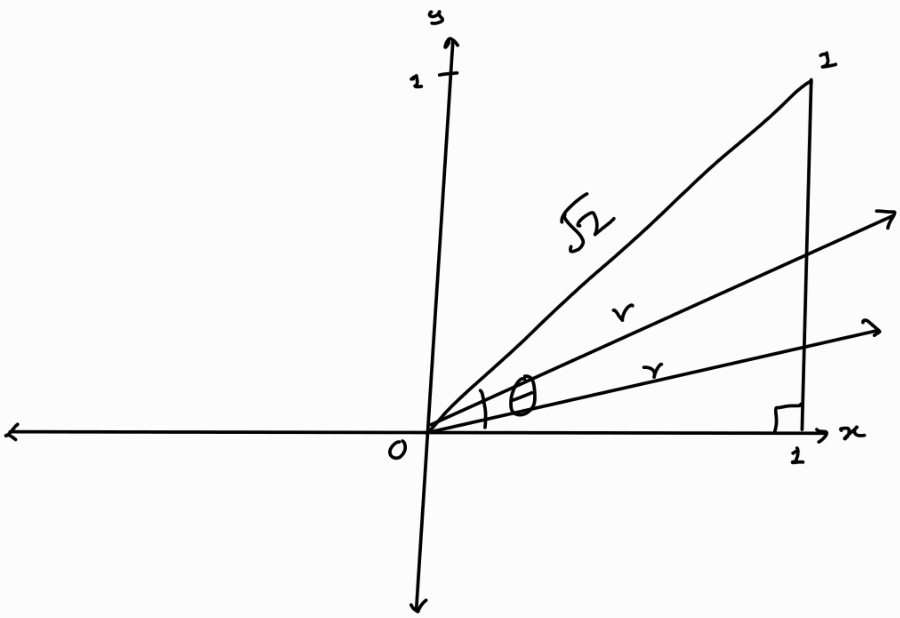
$$\tan\theta = \frac{1}{1}$$

$$\theta = \tan^{-1}(1)$$

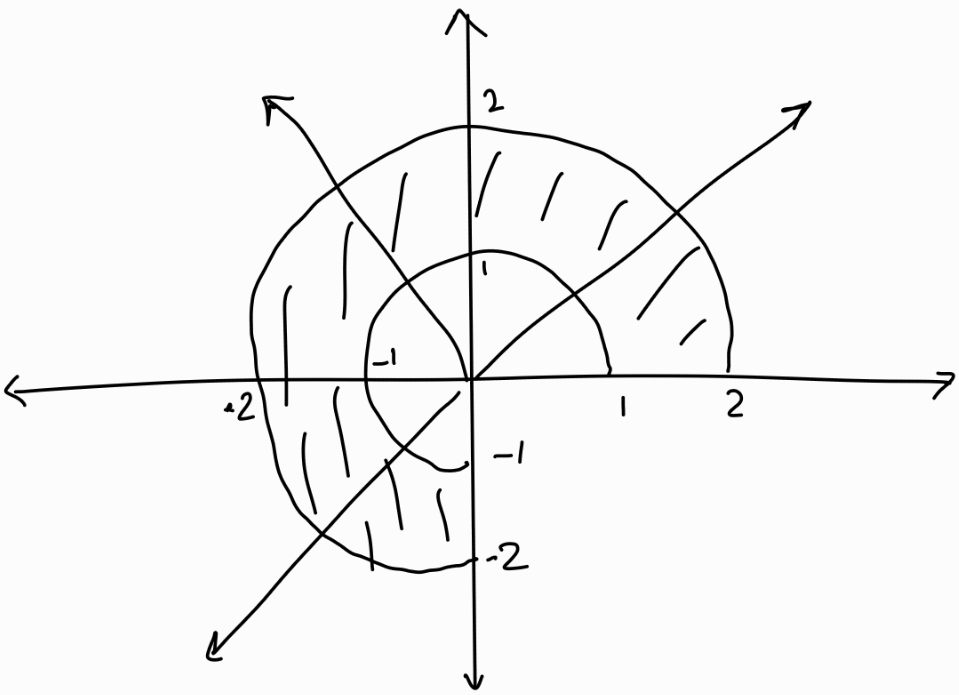
$$\theta = 45 = \pi/4$$

$$\cos\theta = \frac{1}{r}$$

$$r = \sec\theta$$



2)



$$0 \le \theta \le \frac{3\pi}{2}$$

$$1 \le r \le 2$$

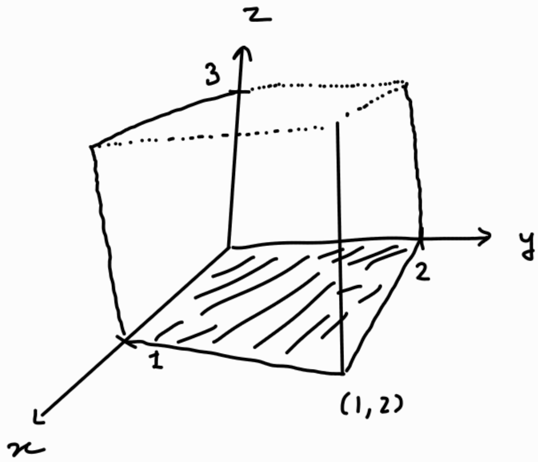
Triple Integrations: [3-fold integration]

- used when we have a function as follows: $\mathbb{R}^3 \longrightarrow \mathbb{R}$

$$\iiint_D f(x,y,z)\ \partial V$$

- we have change in x, y, z $\therefore$ $dV \to$ i.e. change in volume.

e.g. $f(x,y,z) = xyz$



$$\int_0^1 \int_0^2 \int_0^3 xyz\ dz\,dy\,dx$$

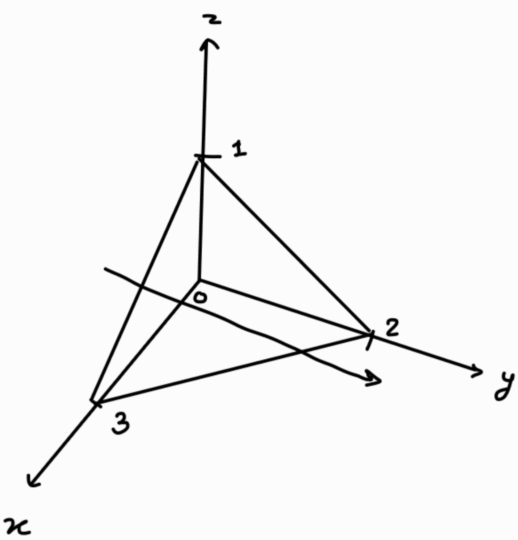
$$= \int_0^1 \int_0^2 xy\frac{9}{2}\ \partial y\,dx$$

$$= \int_0^1 \frac{9x}{\cancel{2}} \times \cancel{2}$$

$$= \frac{9}{2}$$

2) Find volume of surface in first octant and bounded by $2x + 3y + 6z = 6$.

$$\int_0^3 \int_0^{\frac{6-2x}{3}} \int_0^{\frac{6-2x-3y}{6}} 1\ dz$$



$$\int_0^3 \int_0^{\frac{6-2x}{3}} \frac{6-2x-3y}{6}\ dy\,dx$$

$$\frac{1}{6}\int_0^3 \left[6y - 2xy - \frac{3y^2}{2}\right]_0^{\frac{6-2x}{3}}$$

$$\frac{1}{6}\int_0^3 \left[\frac{6(6-2x)}{3} - 2x\frac{(6-2x)}{3} - \frac{3\left(\frac{6-2x}{3}\right)^2}{2}\right] - \left[0-0-0\right]$$

$$\frac{1}{6}\int_0^3 \left[\frac{36-12x}{3} - \frac{12x+4x^2}{3} - \frac{\overset{1}{\cancel{3}}\left(\frac{\overset{18}{\cancel{36}}-\overset{12}{\cancel{24}}x+\overset{2}{\cancel{4}}x^2}{\cancel{9}\;3}\right)}{\cancel{2}}\right]$$

$$\frac{1}{6}\int_0^3 \left[\frac{\cancel{36}-12x-12x+\cancel{4x^2}-\cancel{18}+12x+\cancel{2x^2}}{3}\right]$$

$$\frac{1}{6}\times\frac{1}{3}\int_0^3 \left[6x^2-12x+18\right]$$

$$\frac{1}{18}\left[\frac{\overset{2}{\cancel{6}}x^3}{\underset{1}{\cancel{3}}} - \frac{\overset{6}{\cancel{12}}x^2}{\underset{1}{\cancel{2}}} + 18x\right]_0^3$$

$$\frac{1}{18}\left[2(3)^3 - 6(3)^2 + 18(3)\right]$$

$$\frac{1}{18}\left[2\times 27 - 45 + 54\right]$$

$$\frac{1}{18}\left[54-45+54\right]$$

$$\frac{1}{18}\left[63\right] = 63/18$$

Cylinderical Co-ordinate System

Cylinderical co-ordinate system:

Cartesian $\qquad\qquad$ Polar

$(x, y) \longrightarrow (r, \theta)$

$(x, y, z) \longrightarrow (r, \theta, z)$ $\leftarrow$ Cylinderical co-ordinate system.

here,

$x \longrightarrow r\cos\theta$ $\qquad\qquad r = \sqrt{x^2+y^2}$

$y \longrightarrow r\sin\theta$

$z \longrightarrow z$

$\partial V \longrightarrow r\, \partial z\, \partial r\, \partial\theta$

$$V(r) \longrightarrow \iiint_R r\, \partial z\, \partial r\, \partial\theta$$

e.g.) $r = 2\sin\theta$ $\quad$ as base

$z = 4 - y$ $\qquad\qquad$ $y = r\sin\theta$

// to know the region, convert to cartesian, $\quad \therefore \sin\theta = \frac{y}{r}$

$r = 2\sin\theta$

$r = 2\frac{y}{r}$

$r^2 = 2y$

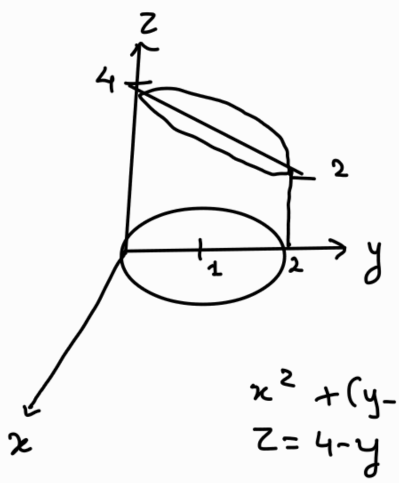
but, $r^2 = x^2+y^2$

$\therefore x^2+y^2 = 2y$

$x^2 + y^2 - 2y = 0$

$x^2 + y^2 - 2y + 1 = 1$

$x^2 + (y-1)^2 = 1$



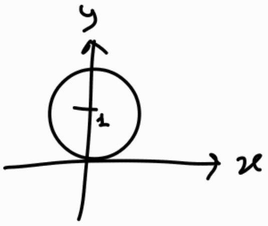
$x^2 + (y-1)^2 = 1$

$z = 4 - y$

$\therefore$ Centre of circle $= (0, 1)$ $\quad r = 1$

$$\int_0^{\pi}\int_0^{2\sin\theta}\int_0^{4-r\sin\theta} r\,dz\,dr\,d\theta$$

Since circle is in first quadrant



$$\int_0^{\pi}\int_0^{2\sin\theta} r\,[z]_0^{4-r\sin\theta} \Rightarrow r\left[4-r\sin\theta\right]$$

$$\int_0^{\pi}\int_0^{2\sin\theta} 4r - r^2\sin\theta$$

$$\int_0^{\pi}\left[\frac{\cancel{4}^{2} r^2}{\cancel{2}_{1}} - \frac{r^3\sin\theta}{3}\right]_0^{2\sin\theta}$$

$$\int_0^{\pi}\left[8\sin^2\theta - \frac{8\sin^4\theta}{3}\right]$$

$$8\int_0^{\pi}\frac{1-\cos 2\theta}{2}$$

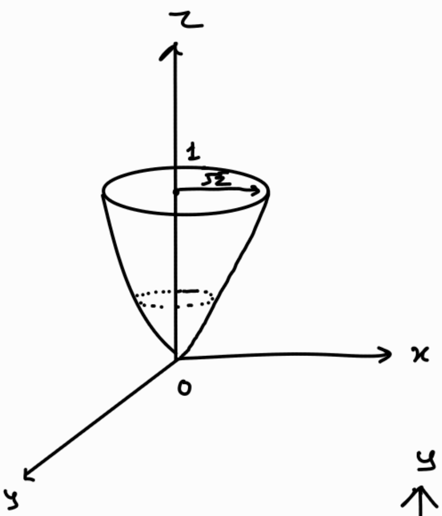
<u>skip</u>

$$\int_0^1 \int_0^{\sqrt{2}} \int_0^{2\pi} \left(r^3 \cos^2\theta + r z^2\right) \partial\theta \, \partial r \, \partial z$$

$0 \leq \theta \leq 2\pi$

$0 \leq r \leq \sqrt{2}$

$0 \leq z \leq 1$

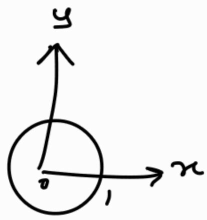


$z = 1$

$r = \sqrt{2} \Rightarrow z = r^2$

$x^2 + y^2 = z$

$\theta = 2\pi$



$$\int_0^{2\pi} \int_0^1 \int_{r^2}^1 r \quad dz \, dr \, d\theta$$

$$\int_0^{2\pi} \int_0^1 r - r^3 \quad \partial r \, \partial\theta$$

$$\int_0^{2\pi} \left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1$$

$$\int_0^{2\pi} \left[\frac{1}{2} - \frac{1}{4}\right) \quad \partial\theta$$

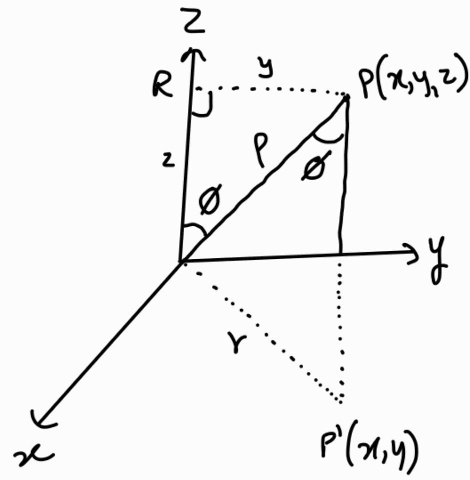
$$\left[\frac{\theta}{2} - \frac{\theta}{4}\right]_0^{2\pi} \Rightarrow \frac{2\pi}{2} - \frac{2\pi}{4} = \frac{4\pi - 2\pi}{4}$$

$$= \pi/2$$

Spherical Co-ordinates:

$(x, y, z) \longrightarrow (r, \theta, z)$ cylinder.

$(x,y,z) \longrightarrow (\rho, \phi, \theta)$ spherical $\quad 0 \le \phi \le \pi$, $\quad 0 \le \theta \le 2\pi$



$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\theta$

here,

$\cos\phi = \frac{\text{adj}}{\text{hypo}} = \frac{z}{\rho}$

$\boxed{z = \rho\cos\phi}$

$\sin\phi = \frac{r}{\rho}$

$\therefore r = \rho\sin\phi$

$$\partial v = \iiint_R \rho^2 \sin\phi \; \partial\rho \, \partial\phi \, \partial\theta$$

$\rho = \sqrt{x^2+y^2+z^2}$

$\rho \le 2$, $\phi = \frac{\pi}{3}$ and $\frac{2\pi}{3}$,

$$V = \int_0^{2\pi} \int_{\pi/3}^{2\pi/3} \int_0^{2} \rho^2 \sin\phi \; \partial\rho \, \partial\phi \, \partial\theta$$

$$= \int_0^{2\pi} \frac{8}{3} \int_{\pi/3}^{2\pi/3} \sin\phi \; \partial\phi \, \partial\theta$$

$$= \int_0^{2\pi} -\cos\left(\frac{2\pi}{3}\right) + \cos(\pi/3)$$

$$= \int_0^{2\pi} \frac{8}{3}\left(\frac{1}{2} + \frac{1}{2}\right)$$

$$= \frac{8}{3} 2\pi$$

$$= \frac{16}{3}\pi$$

Q.) $\displaystyle\int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\phi}^{2} \rho^2 \sin\phi \; \partial\rho \, \partial\phi \, d\theta$

$$\int_0^{2\pi}\int_0^{\pi/3} \sin\phi \left[\frac{8}{3} - \frac{\sec^3\theta}{} \right.$$

Jacobian Matrix

A matrix with entries which are Partial valued function.

$f : \mathbb{R}^2 \to \mathbb{R}^2$

$(x, y) \to (\underbrace{2x+y}_{u}, \underbrace{y-x}_{v})$

$u = 2x + y$

$v = y - x$

$J = \dfrac{\partial(u,v)}{\partial(x,y)}$

$$J = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} = 2 \times 1 - 1 \times (-1)$$

$= 2 + 1$

$= 3$

$J^{-1} = \dfrac{\partial(x,y)}{\partial(u,v)}$ = Inverse i.e. Reciprocal of J

$J J^{-1} = 1$

$\mathbb{R}^2 \to \mathbb{R}$ ← Scaler valued function

$\mathbb{R}^2 \to \mathbb{R}^2$

$\mathbb{R}^n \to \mathbb{R}^m$ where $m > 2$

vector valued function

Substitution in Multiple Integrals:

Steps:

1] Sketch region in xy plane.

2] Choose a substitution

3] Find x and y in terms of u and v.

4] Find $J = \dfrac{\partial(x,y)}{\partial(u,v)}$

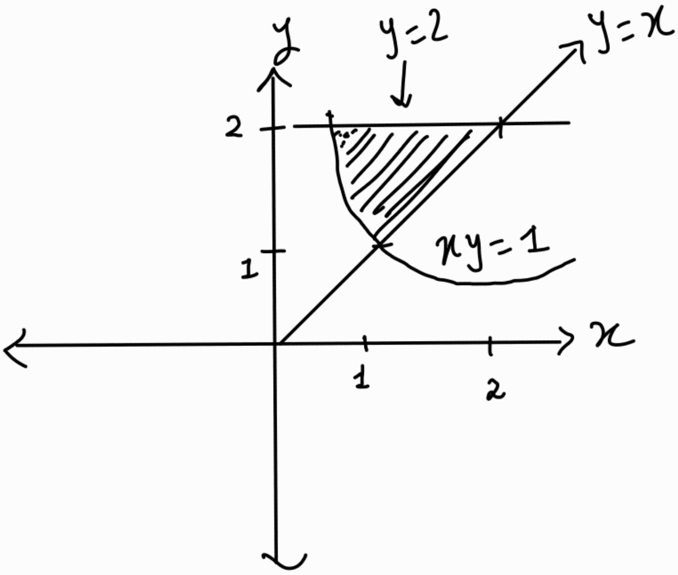
5] Find equations in terms of u and v [to draw region for u-v plane with help of step 1 to 3.

6] Sketch region in uv plane

7] Find limits for u and v then solve.

Eg.

$$\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy$$



$\frac{1}{y} \le x \le y$

$1 \le y \le 2$

Let

$\sqrt{y/x} = u \quad \cdots (1)$

$\sqrt{xy} = v$

y

(1) $\Rightarrow \quad y/x = u^2 \Rightarrow y = xu^2 \Rightarrow y = \frac{v}{u}u^2 = \boxed{vu}$

(2) $\quad xy = v^2 \Rightarrow x(xu^2) = v^2 \Rightarrow x^2 = \frac{v^2}{u^2} \Rightarrow x = \frac{v}{u}$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} -v/u^2 & 1/u \\ v & u \end{vmatrix}$$

$$= \frac{-v}{u} - \frac{v}{u} = -2v/u$$

$$|J| = \frac{2v}{u} \quad \ldots (*)$$

Step 5:

Equations in terms of x and y:

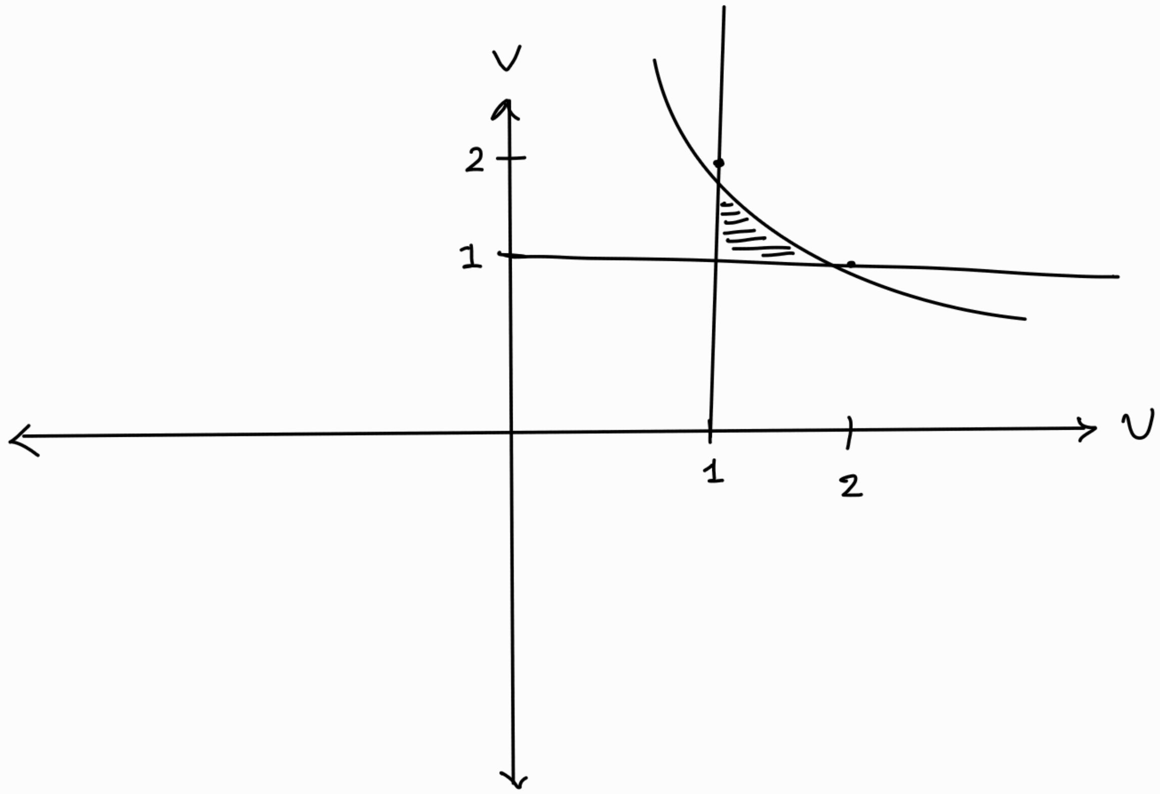
(1) $y = x \quad \Rightarrow \quad vu = \frac{v}{u} \Rightarrow u^2 = \frac{v}{v} \Rightarrow u^2 = 1$

$\Rightarrow u = 1$

(2) $y = 2 \quad \Rightarrow \quad uv = 2$

$\hookrightarrow v = \frac{2}{u}$

(3) $xy = 1 \quad \Rightarrow \quad v\cancel{u} \cdot \frac{v}{\cancel{u}} = 1 \Rightarrow v^2 = 1 \Rightarrow v = 1$



$$V = \iint u \cdot e^{v} \cdot \frac{2v}{u} \; \partial v \, du$$

$$= \int_1^2 \int_1^{2/u} \cancel{u} \cdot e^v \cdot \frac{2v}{\cancel{u}} \, dv \, du$$

Apply product rule

or reverse order.

$$V = \int_1^2 \int_1^{2/v} \cancel{u} \cdot e^v \cdot \frac{2v}{\cancel{u}} \, du \, dv$$

$$uv = 2$$

$$= \int_1^2 e^v 2v \left(\frac{2}{v} - 1 \right) dv$$

$$= \int_1^2 2e^v \cancel{v} \cdot \frac{2}{\cancel{v}} - e^v 2v \, dv$$

$$= \int_1^2 4e^v - e^v 2v \, dv$$

$$= 2\left[2e^{v} - ve^{v} + e^{v}\right]_1^2$$

$$= 2\left[3e^{2} - 2e^{2} - 2e^{1} - e^{1} + e^{1}\right]$$

$$= 2\left[e^{2} - 2e^{1}\right]$$

$$= 2e^{2} - 4e^{1}$$

Q) $u = 2x + y$

$v = x - y$

$y = u - 2x$	$y = x - v$
$x = v + y$	$y = \frac{u+v}{3} - v$
$x = v + u - 2x$	
$3x = v + u$	$y = \frac{u + v - 3v}{3} = \frac{u - 2v}{3}$
$x = \frac{v+u}{3}$	

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{vmatrix}$$

$2x+y=4$ $\quad u=4$

$2x+y=7$ $\quad u=7$

$x-y=2$ $\quad v=2$

$x-y=-1$ $\quad v=-1$

$$= -\frac{2}{9} - \left(+1/3 \times \frac{1}{3}\right)$$

$$= \frac{-2}{9} - \frac{1}{9}$$

$$= \frac{-3}{9} = \frac{-1}{3}$$

$$V = \int_{-1}^{2}\int_{4}^{7} \frac{uv}{3}\,du\,dv$$

$$= \frac{1}{3}\int_{-1}^{2} v\left[\frac{49}{2} - \frac{16}{2}\right]$$

$$= \frac{1}{\cancel{3}}\,\frac{\cancel{33}\,11}{2}\int_{-1}^{2} v\,dv$$

$$= \frac{11}{2}\left[\frac{v^2}{2}\right]_{-1}^{2} \Rightarrow \frac{11}{2}\left[\frac{2^2}{2} - \frac{(-1)^2}{2}\right]$$

$$= \frac{11}{2}\left[\frac{4}{2} - \frac{1}{2}\right]$$

$$= \frac{33}{2}$$

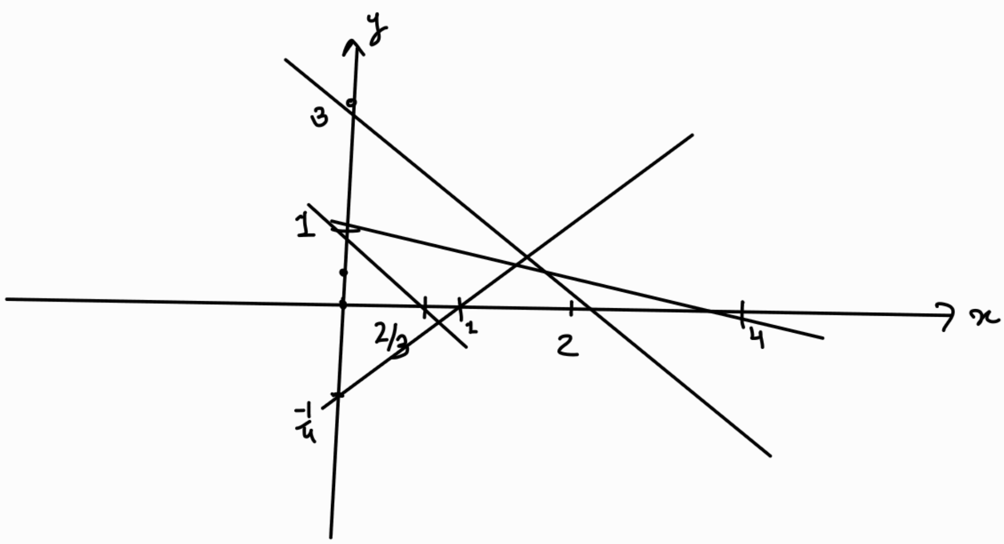
2) $V = \iint_D \underline{3x^2 + 14xy + 8y^2} \, \partial A$

$\downarrow$

$3x^2 + 12xy + 2xy + 8y^2$

$3x(x+4y) + 2y(x+4y)$

$(3x+2y)(x+4y)$



$u = 3x + 2y$

$v = x + 4y$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} = 12 - 2 = 10$$

$\therefore J = 10.$ $\quad J = 1/10$

$y = 1 + \left(-\frac{3}{2}\right)x \Rightarrow \underbrace{3x + 2y}_{u} = 2$

$y = 3 + \left(-\frac{3}{2}\right)x \Rightarrow \underbrace{3x + 2y}_{u} = 6$

$y = -\frac{x}{4} \Rightarrow \underbrace{x + 4y}_{v} = 0$

$y = -\frac{x}{4} + 1 = \underbrace{x + 4y}_{v} = 4$

$$\frac{1}{10}\int_2^6\int_0^4 uv \quad dv\,du$$

$$\frac{1}{10}\int_2^6 \frac{16}{2} u \ du$$

$$\frac{4}{5}\cancel{\frac{8}{10}}\left[\frac{u^2}{2}\right]_2^6$$

$$\frac{4}{5}\left[\frac{36}{2} - \frac{4}{2}\right]$$

$$\frac{4}{5}\left[18 - 2\right]$$

$$\frac{64}{5}$$

$$\frac{1}{2}\int_1^4\int_1^4 \frac{u}{v^2} + u \quad du\, dv$$

$$\frac{1}{2}\int_1^4 \left[\frac{u^2}{2v^2} + \frac{u^2}{2}\right]_1^4 \quad \partial v$$

$$\frac{1}{2}\int_1^4 \left[\frac{8}{v^2} + 8 - \frac{1}{2v^2} - \frac{1}{2}\right]$$

$$\frac{1}{2}\int_1^4 \left[\frac{15}{2v^2} + \frac{15}{2}\right]$$

$$\frac{1}{2} \times \frac{15}{2} \int_1^4 \left(\frac{1}{v^2} + 1\right)$$

$$\int_1^4 \frac{-1}{v} + v$$

$$\frac{15}{4}\left[-\frac{1}{v}+v\right]_1^4$$

$$\frac{15}{4}\left[-\frac{1}{4}+4+\cancel{1}-\cancel{1}\right]$$

$$\frac{15}{4}\left[\frac{15}{4}\right)$$

$$\frac{225}{16}$$

e) $V = \int_1^2 \int_0^1 \int_0^2 v(u+3w)\, \partial v\, \partial u\, dw$

$$= 2\int_1^2 \int_0^1 u+3w \quad \partial u\, dw$$

$$= 2\int_1^2 \left[\frac{u^2}{2}+3wu\right]_0^1$$

$$= 2 \int_1^2 \left[\frac{1}{2} + 3\omega \right] \partial z$$

$$= 2 \left[\frac{1}{2}\omega + \frac{3\omega^2}{2} \right]_1^2$$

$$= 2 \left[\frac{2}{2} + \frac{3 \times 2 \times \cancel{2}}{\cancel{2}} - \frac{1}{2} - \frac{3}{2} \right]$$

$$= 2 \left[1 + 6 - 2 \right]$$

$$= 2 \left[5 \right]$$

$$= 10$$