College of Engineering Pune

Linear Algebra and Univariate Calculus(D.S.Y)

Tutorial 4

Linear Mappings, Kernel and image of a linear map, Rank nullity theorem

1. Let $T : V \to W$ be a linear transformation. Show that:
   (a) $T(0) = 0$.
   (b) $T(-v) = -T(v)$ for all $v \in V$

2. Determine which of the following mappings F are linear. If linear, then find its kernel and image space. Also find Nullity and rank and hence verify Rank-Nullity theorem.
   (a) $F : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $F(x, y, z) = (x, z)$
   (b) $F : \mathbb{R}^4 \to \mathbb{R}^4$ defined by $F(x, y, z, w) = (-x, -y, -z, -w)$
   (c) $F : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $F(x, y, z) = (x, y, z) + (0, 1, 0)$
   (d) $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x, y) = (x - y, 2y)$
   (e) $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x, y) = (xy, x + y)$
   (f) $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x, y) = (y, x)$
   (g) $F : \mathbb{R}^2 \to \mathbb{R}$ defined by $F(x, y, z) = xy$
   (h) $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x, y) = (x, y + 1)$
   (i) $F : \mathbb{R}^3 \to \mathbb{R}$ defined by $F(x, y, z) = 3x - 2y + z$
   (j) Let D be a derivative map from set of differentiable functions to set of differentiable functions i.e., $D(f) = \frac{df}{dx}$.
   (k) Let D^2 be a double derivative map from set of twice differentiable functions to set of twice differentiable functions i.e., $D^2(f) = \frac{d^2f}{dx^2}$.
   (l) Let M be the space of all 2 x 2 matrices. Let, $P : M \to M$ be a map such that $P(A) = \frac{A+{}^tA}{2}$. Generalize to nxn matrices.
   (m) Let M be the space of all 2 x 2 matrices. Let, $P : M \to M$ be a map such that $P(A) = \frac{A-{}^tA}{2}$. Generalize to nxn matrices.
   (n) Let M be the space of all 2 x 2 matrices. Let, $P : M \to M$ be a map such that $P(A) = trace(A)$. Generalize to n x n matrices.

3. Using Kernel classify whether above functions are one-one or not. Further, using Rank-Nullity theorem conclude whether function is onto or not.

4. What is the dimension of the space of solutions of the following systems of lineat equations? In each case, find a basis for the space of solutions.

(a)

$$\begin{aligned} 2x + y - z &= 0 \\ 2x + y + z &= 0 \end{aligned}$$

(b)

$$\begin{aligned} x + y + z &= 0 \\ x - y &= 0 \\ y + z &= 0 \end{aligned}$$

(c)

$$\begin{aligned} 2x - 3y + z &= 0 \\ x + y - z &= 0 \\ 3x + 4y &= 0 \\ 5x + y + z &= 0 \end{aligned}$$

(d)

$$\begin{aligned} 4x + 7y - \pi z &= 0 \\ 2x - y + z &= 0 \end{aligned}$$

5. Let A be a fixed mxn matrix. Let. $T : \mathbb{R}^n \to \mathbb{R}^m$ be a map defined as: $T(X) = AX$ where X is a n x 1 vector in $\mathbb{R}^n$. Show that T is a linear transformation.

6. In above example, Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 7 \end{bmatrix}$. Find Null space of T, Image space of T. Hence conclude Nullity(T) and Rank(T). Further verify Rank-Nullity theorem.

7. Take a 3 x 4 matrix of your choice and do the above things. (Don't take a null matrix :) . Also try to take distinct entries!)

8. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $T(1, 0) = (1, 1)$ and $T(0, 1) = (2, 3)$. Find $T(a, b)$ for any $(a, b) \in \mathbb{R}^2$. Hence calculate image of $(3, 7)$.

9. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $T(1, 0, 0) = (1, 1, 0)$, $T(0, 1, 0) = (2, 3, 0)$ and $T(0, 0, 1) = (1, 0, 5)$. Find $T(a, b, c)$ for any $(a, b, c) \in \mathbb{R}^3$. Hence calculate image of $(3, 7, 1)$.