ENSE 622/ENPM 642, Spring 2018
Homework 6: Markov Chain & Monte Carlo Methods
SOLUTIONS
(3/8/18)

Additional Instructions:

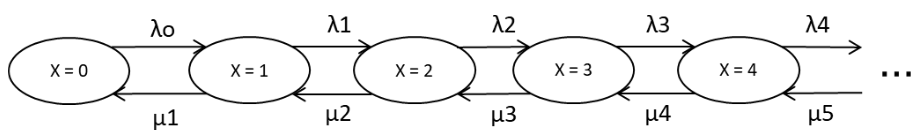
1. **This Homework has a maximum point value of 120 points (20 points EC).**
2. **For full credit all work must be shown.**
3. **Due: 7:00 pm, Monday, March 12, 2017.**

1. **(45 pts) Problem 1 (Birth-Death Process):** Consider a population of micro-organisms that can grow under certain favorable physical condition and similarly the population decrease under hostile situation. Also consider that due to space restriction the maximum number of members in the population could be N (and 0 as minimum). Let us denote the population size at time t by X_t. So at any time t, X_t will be one of the numbers from $\{0,1,\dots,N\}$. The population growth (or death) rate depends on the current population size as follows:

$$P(X_{t+1} = n+1 | X_t = n) = \lambda_n$$
$$P(X_{t+1} = n-1 | X_t = n) = \mu_n$$
$$P(X_{t+1} = n | X_t = n) = 1 - \lambda_n - \mu_n$$
$$\lambda_N = \mu_0 = 0$$

where $P(X_{t+1} = n+1 | X_t = n)$ is the probability that the population will increase by one in one time step.

a. (4 points) Draw the Markov Diagram for this process.



b. (2 points) Is this a DTMC or a CTMC process? Explain.

i. DTMC since we are solving difference equations, not differential equations (time advances incrementally by dt = 1 unit).

c. (4 points) Write down the Markov transition matrix for $N = 5$.

Ans:

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1-\lambda_0 & \lambda_0 & 0 & 0 & 0 & 0 \\ \mu_1 & 1-\mu_1-\lambda_1 & \lambda_1 & 0 & 0 & 0 \\ 0 & \mu_2 & 1-\mu_2-\lambda_2 & \lambda_2 & 0 & 0 \\ 0 & 0 & \mu_3 & 1-\mu_3-\lambda_3 & \lambda_3 & 0 \\ 0 & 0 & 0 & \mu_4 & 1-\mu_4-\lambda_4 & \lambda_4 \\ 0 & 0 & 0 & 0 & \mu_5 & 1-\mu_5 \end{bmatrix} \end{array}$$

d. (5 points) Let us denote $P(X_t = n) = p_n(t)$ for all $n = 0,1,\dots,N$. Write down the system of equations to find $p_n(t+1)$ from $p_n(t)$.

Ans:

$$\mathbf{p}(t+1) = \mathbf{p}(t) * P$$

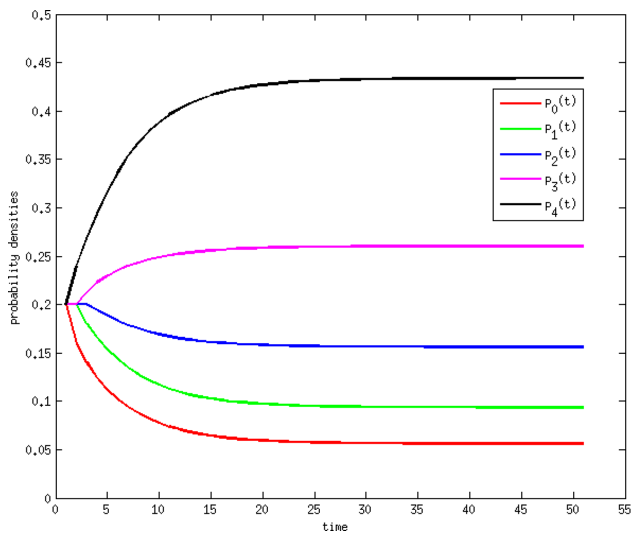
where $\mathbf{p(t)} = [p_0(t), p_1(t), \ldots, p_N(t)]$, and P is the $(N+1)\times(N+1)$ matrix given below

$$P_{ij} = \begin{cases} \lambda_i & j = i+1 \\ \mu_i & j = i-1 \quad i,j = 0,\ldots,N \\ 1-\lambda_i-\mu_i & j = i \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

or

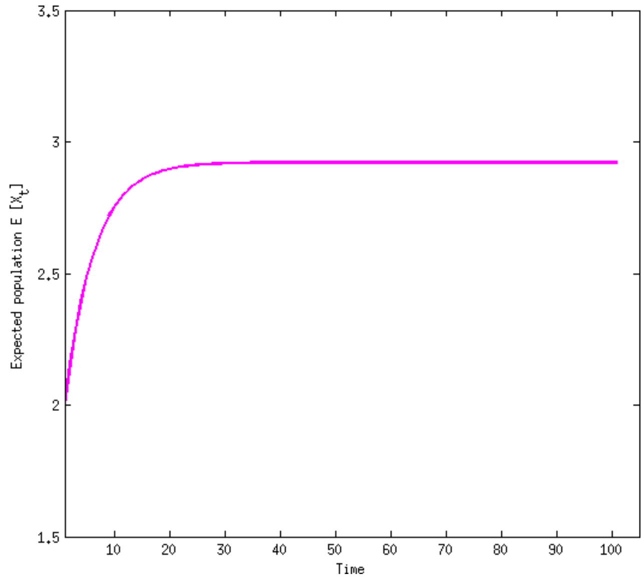
$$p_0(t+1) = (1-\lambda_0)p_0(t) + \lambda_0 p_1(t)$$
$$p_i(t+1) = \mu_{i+1}P_{i+1}(t) + (1-\lambda_i-\mu_i)p_i(t) + \lambda_{i-1}p_{i-1}(t) \quad \text{for } i = 1, \ldots N-1$$
$$p_N(t+1) = \mu_{N-1}p_{N-1}(t) + (1-\mu_N)p_N(t)$$

e. (10 points) Let us consider $N = 4$, $\lambda_n = 0.5$, $\mu_n = 0.3$ for all $n = 0,1,2, \ldots, N$ along with the condition $\lambda_N = \mu_0 = 0$ (i.e. probabilities do not depend on the population size), and the initial probabilities at time $t = 0$ are given as $p_0(0) = p_1(0) = p_2(0) = p_3(0) = p_4(0) = 0.2$ (uniformly distributed). Find (using MATLAB) $p_n(t)$ for all $n = 0,1,\ldots,N$ for $t = 1,2,\ldots,50$ and plot each $p_n(t)$ with time t. [Hint: use part b to find the one-step-ahead probabilities from the current probabilities for each t.]
   i. See MATLAB Program Problem1e.



   ii. Note: the distribution almost reached to the steady state after 50 iterations.

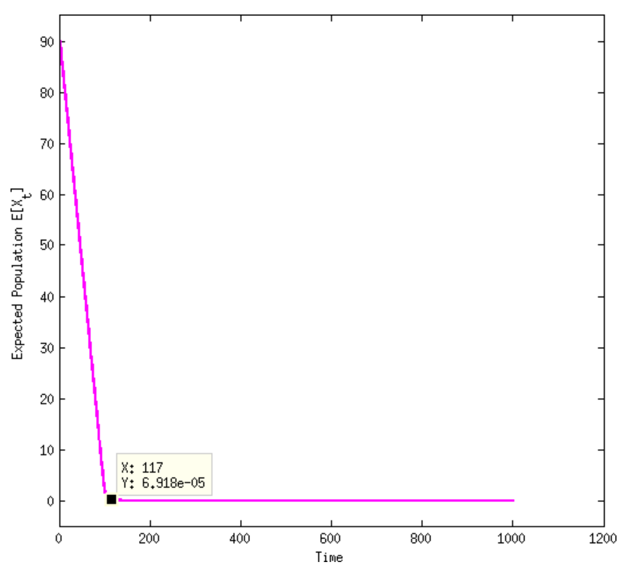
f. (10 points) Find the expected size of the population, $\mathbb{E}[X_t]$, for times $t = 0,1,2,\ldots,100$ and plot it. [Hint: if X is a random variable that takes values from $x_1, x_2, \ldots, x_m$ with probabilities $p_1, p_2, \ldots, p_m$ respectively then $\mathbb{E}[X] = \sum_{i=1}^{m} x_i p_i$.]
   i. See MATLAB Program Problem1f.



g. (10 points) Now consider that a hostile situation has occurred that decreases the probability of growth with time and increases probability of death with time as given below:

$$P(X_{t+1} = n + 1 | X_t = n) = 0.9e^{-2t}$$
$$P(X_{t+1} = n - 1 | X_t = n) = 0.9(1 - e^{-2t})$$
$$P(X_{t+1} = n | X_t = n) = 0.1$$
$$\lambda_N = \mu_0 = 0$$

Under this situation, it is expected that the population size will decrease. Take $N = 100$, initial population is 90, and plot $\mathbb{E}[Xt]$ for $t = 0,1, \ldots, 1000$. Does $\mathbb{E}[Xt]$ converge to zero? If yes, then qualitatively explain how the values of $p_n(t)$ will converge for all $n = 0,1, \ldots, 100$ as t increases.



i. See MATLAB Program Problem1g.
ii. Since population size is always non-negative and the expected value E[Xt] converges to zero, that indicates p0(t) converges to 1 and pn(t) converges to 0 as t increases.

[Note: in only 117 iterations, the population converges to zero and hence the probability distributions also converges. The Figure shows the evolution of the probability densities and it clearly shows how p0(t) converges to 1 and rest of them converges to 0].

2. (35 pts) Problem 2 (Estimating parameters of a Markov Chain): Let "`Estimate_TransitionProbabilities()`" be a function that takes a time sequence of a Markov Chain and it outputs the transition probability matrix P. The transition probability entries are estimated by the following empirical rule:

$$P_{ij} = \frac{n_{ij}}{n_i}$$

where n_{ij} is the number of transitions from state i to state j and n_i is the total number of transition from state i to any state (including itself) in the given data sequence. For example, if the given data is AABAACBCBCACB where {A,B,C} are the set of the states, then $P_{AC} = \frac{2}{5}$, $P_{BC} = \frac{2}{3}$ and $P_{CC} = 0$ etc.

a. (10 points) Write and submit a MATLAB code to implement the function `Estimate_TransitionProbabilities()` where the input data will be sequence of integers follow by a single space (for example the input `1 2 4 12 4 4 2 12' should be treated as a sequence of states `1', `2',`4',`12',`4', `4', `2', `12' respectively). The output will be the probability transition matrix

$$\text{(in this case } P = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 4 & 12 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 4 \\ 12 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & .5 & .5 \\ 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

i. See MATLAB Program Problem2

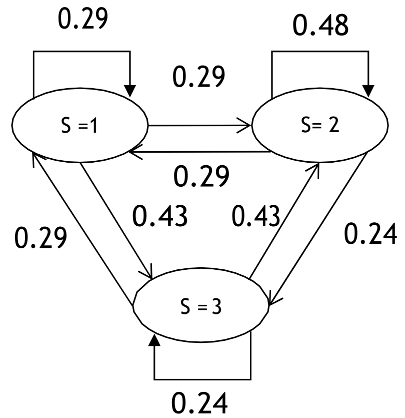
b. (10 points) Show the Matlab output from the implemented code when the input data is
" 2 3 2 2 2 1 3 1 3 1 2 2 3 1 3 3 2 2 2 1 2 2 3 3 2 2 3 2 2 3 3 2 2 2 1 1 2 1 3 1 1 1 1 2 1 3 2 1 3 3"

c. (10 points) Find the transition matrix for b. by hand and verify your MATLAB result.

Ans: states := {'2' '3' '1'}.

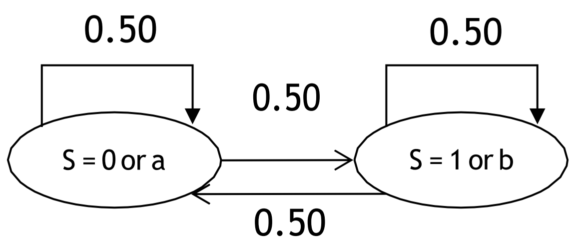
$$P = \begin{array}{c c} & \begin{array}{ccc} 2 & 3 & 1 \end{array} \\ \begin{array}{c} 2 \\ 3 \\ 1 \end{array} & \begin{bmatrix} 0.4762 & 0.2381 & 0.2857 \\ 0.4286 & 0.2857 & 0.2857 \\ 0.2857 & 0.4286 & 0.2857 \end{bmatrix} \end{array}$$

d. (5 points) Draw the Markov Diagram for the system described in b. (c)



3. (40 pts) Problem 3 (Simulating a Markov Chain using Monte Carlo): By simulating a Markov chain we mean the Monte Carlo simulation discussed in the class where the input will be the transition probability matrix (P) and the initial state (s_0) and number of transitions (N), and the output will be a sequence of length $N + 1$ of states in that Markov chain. For example if $P = \begin{array}{c} \\ 0 \\ 1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 0 & \;\;1 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \end{array}$ and initial state is ‘0’, then one such output of length 7 may be ‘0 1 1 0 1 0 0’. Note that here we want a time evolution of states not the distribution (π_t or $p_t^{(n)}$) over the time. [This is the reverse problem of Problem 2. Whereas Problem 2 identifies a Markov chain, this problem generates a Markov chain for practical applications.]

a. (4 points) Write down the Markov Diagram for this process.



b. (2 points) Is this a DTMC or a CTMC process? Explain.
   i. DTMC since we are solving difference equations, not differential equations.

c. (10 points) Write and submit a MATLAB code that simulates a Markov Chain in the way described above.
   i. See MATLAB Program Problem3

d. (8 points) Input the transition probability matrix $P = \begin{array}{c} \\ a \\ b \end{array}\!\!\begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$ and initial state `a' in the implemented code and generate a sequence of states of length 10.
   i. Ans: a b a b a b a b a b a

e. (8 points) Input the transition probability matrix $\hat{P} = \begin{array}{c c} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \end{array}$ and initial state `a', and get a sequence of length 200.

   i. Should be random sequence where a follows a, b follows a, a follows b, and b follows a ~25% of time.

f. (8 points) Use the output sequence from e. as an input to the code developed in Problem 2 and estimate the transition probability matrix. [The estimation is `good' if the estimated matrix is close the given matrix $\hat{P}$ It is interesting to see how this estimate converges to $\hat{P}$ when sequence length is increased from 200 to 2000 or even higher.]

$$\hat{P}_{\text{estimate}} = \begin{array}{c c} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{bmatrix} 0.5963 & 0.4037 \\ 0.4725 & 0.5275 \end{bmatrix} \end{array}$$

   i. When N = 2000, the estimated probabilities are:

$$\hat{P}_{\text{estimate}} = \begin{array}{c c} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{bmatrix} 0.5035 & 0.4965 \\ 0.4837 & 0.5163 \end{bmatrix} \end{array}$$

   hence the estimated probability matrix converges to the actual probability matrix P as we have more data to estimate the probability matrix.