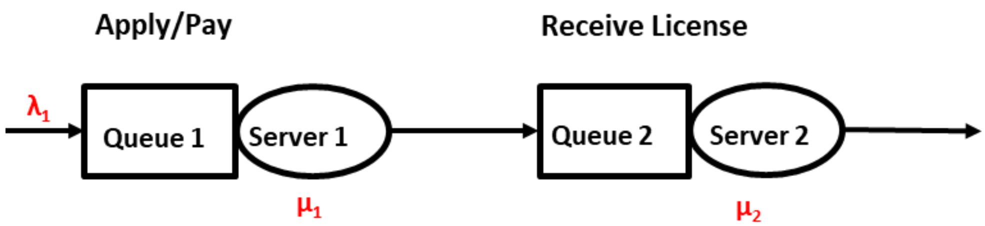
ENSE622, Spring 2018:
Homework 7: Queueing Theory
SOLUTIONS
(3/25/18)

Additional Instructions:

1. **This Homework has a maximum point value of 110 points (10 points EC).**
2. **For full credit all work must be shown.**

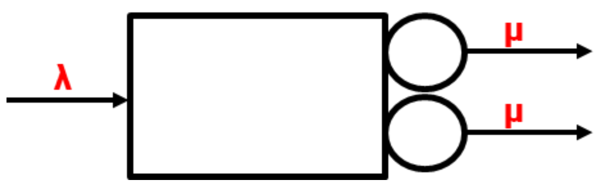
1. **Problem 1 (20 pts):** For registration at the Motor Vehicle Administration, assume one must go through two processes: application/payment and receive license. The average service times for the processes are 3 minutes and 0.57 minutes, respectively. In steady state, the average number at each process, including customers either waiting or being served, is about 20 persons for the application/payment area and 5 persons for the receive license station. The arrival rate for registration customers is 10 per hour.
   a. (5) Draw the appropriate Queuing Diagram for this system.



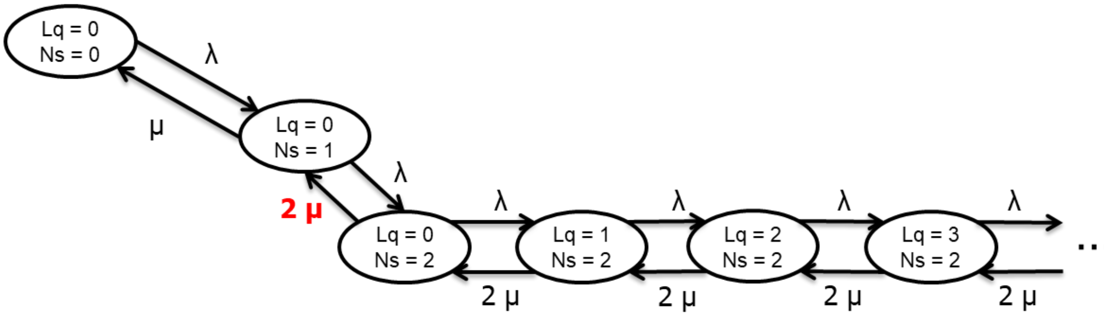
   b. (15) Find the average time a person would spend getting their registration. (Moral: use the mail!) Note that no mention of distributions is made!
      Little's Law: $W = L/\lambda = (20 + 5)/10/\text{hr} = 2.5$ hr

2. **Problem 2 (30 pts):** Consider a bank with two tellers. Assume that the arrival process is Poisson at a rate of 2 per minute and that service times are exponential with a mean of 40 seconds. It is currently 10 AM. Assume that there is sufficient capacity for any number of customers in queue and that customers are served on a first in/first out basis.
   a. (2) Use Kendall's notation to specify this system.
      M/M/2/Inf/FIFO

   b. (4) Draw the appropriate Queuing Diagram for this system.



c. (4) Draw the appropriate Markov Diagram for this system.



d. (2) Find the probability that the next customer will arrive after 10:02 AM.

Prob(tia>2 min) = $\exp(-\lambda\ t) = \exp(-2/\text{min} * 2\ \text{min}) = 0.018$

e. (2) Find the probability that between now and 10:02 AM there will be at least one arrival.

Prob(n>=1) = 1- Prob(n=0) = 1-0.018 = 0.982

f. (2) If both tellers are currently busy serving customers, find the probability that neither customer will be finished before 10:01 AM.

$P(ts1 > 1\ \&\ ts2>1) = P(ts>1)^2 = \exp(-2\ t/MST) = \exp(-2*60\ \text{sec}/40\ \text{sec}) = 0.050$

Note that you need to pay attention to units.

g. (10) In steady-state, find:

i. (6) The probability that: 1) the system is empty; 2) both servers are busy; and 3) an arriving customer has to wait;

From Dr. Fu's notes:

MM2 queue => $\rho = \lambda/(2\mu) = 2/60\ \text{sec} / (2*1/40\ \text{sec}) = 40/60 = 0.667$.

$p_o = (1-\rho)/(1+\rho) = 0.2$

$P(\text{Both busy}) = 1-(p_o+p_1) = 1 – (1+ 2\ \rho)\ p_o = 0.533$

$P(\text{Wait}) = P(\text{Both busy}) = 0.467$

ii. (2) The average # customers in the system and in queue;

From Dr. Fu's notes:

$L = 2\ \rho/(1-\rho^2) = 2.4$ cust

$Lq = 2\ p_o\ (\rho^3/(1-\rho)^2) = 2\ (\rho^3/[(1+\rho)\ (1-\rho)] = 1.07$ cust

iii. (2) The average time a customer spends in the system and in queue.

$W = L/\lambda = 1.2$ min

$Wq = Lq/\lambda = 0.533$ min

h. (4) The bank is considering reducing the number of tellers. Would this be a good idea? Explain.

For N = 1, $\rho = \lambda/(\mu) = 2/60\ \text{sec} / (1/40\ \text{sec}) = 2*40/60 = 4/3$ => unstable.

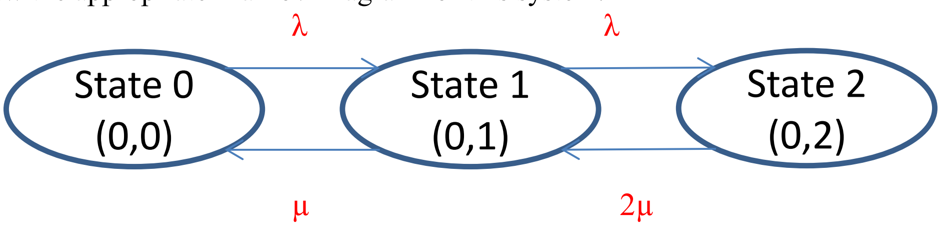
⇨ Not a good idea.

3. **Problem 3 (20 pts):** Consider Problem 2 above only now assume there is **no room (queue space) for any waiting**. Note you will likely have to do parts a.-d. prior to answering part e.

   a. (3) Use Kendall's notation to specify this system.

      M/M/2/2/FIFO

   b. (3) Draw the appropriate Markov Diagram for this system.



   c. (3) Write down the associated Steady State Kolmogorov Equations for this system.

      (1) $0 = -\lambda\, p_o + \mu\, p_1$

      (2) $0 = \lambda\, p_o - (\lambda + \mu)\, p_1 + 2\mu\, p_2 =$

      (3) $0 = \lambda\, p_1 - 2\mu\, p_2$

   d. (5) Solve these for po.

      (1) => $p_1 = \lambda\, p_o / \mu = 2 * \rho\, p_o$

      Where $\rho = \lambda/(2\mu) = 4/6$ => stable

      (3 & 1) => $p_2 = (\lambda/(2\mu)) * (2\rho\, p_o) = 2\rho^2\, p_o$

      $\sum p_i = 1$ => $p_o + 2\, p_o\, \rho + 2\rho^2\, p_o$

      => P(None busy) = **$p_o = 1/(1 + 2\rho + 2\rho^2) = 0.310$**

      Note you may use equation from Dr. Fu's notes, with $\rho' = \lambda/\mu$,

      $$po = \frac{1}{\sum_{i=0}^{2} \frac{\rho'^i}{i!}} = 0.310$$

   e. (6) Answer question 2. g. for this case.

      For this M/M/2/2 system, in Steady State:

      (i) P(Both busy) = $p_2 = 2\rho^2\, p_o = 0.276$

      P(Wait) = 0 (No queue)

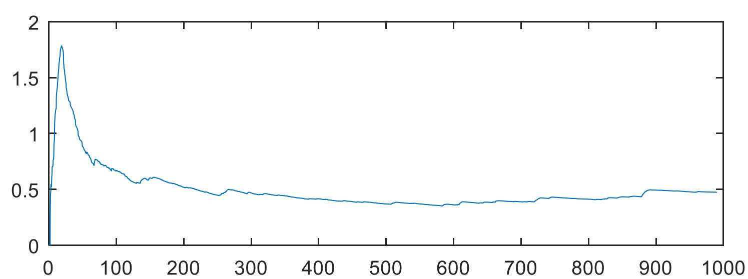
      (ii) $L = 1\, p_1 + 2\, p_2 = 2*\rho\, p_o + 2*0.276 = 0.966 \sim 1$ person

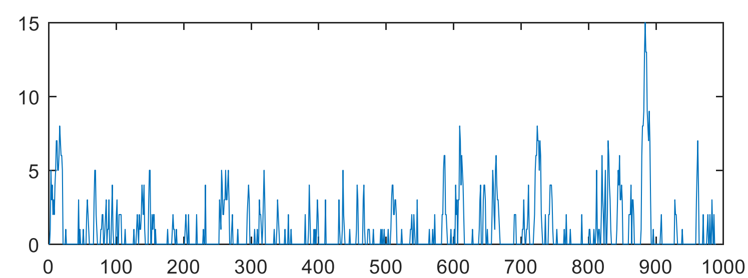
      $L_q = 0$ (No queue)

      (iii) $W = 1/u = 40$ sec, $W_q = 0$ (No queue)

4. **Problem 4 (20 pts) – Simple Queueing MC Simulation:** Consider the system described in Problem 2.
   a. (5) Develop and submit an MATLAB Discrete Event MC simulation for the queuing system described in Problem 2 that :
      i. Accepts the mean arrival and service rates (Ra & Rs), and total number of customers to be served (Nc) as inputs.
      ii. Calculates:
         1. The arrival time (Tai), queue length (Lqi) initially seen, and queue waiting time (Wqi) for each for each arriving customer.
      iii. Provides:
         1. As **text output,** the system's running mean queue waiting time (Wqmc), queue length (Lqmc), and total time in system (Wmc), as well as the associated SDs & SEs.
         2. As **graphical output,**
            a. The queue length (Lqi) initially seen and queue waiting time (Wqi) as a function of each customer's arrival time (Tai), for each for each arriving customer.
            b. The system's running mean queue waiting time (Wqmc) as a function of each customer's arrival time (Tai).

   See MATLAB Programs "MC_Event_D_MM2_Q_180403_v2" and "MC_Event_D_MM2_Q_180403_v3." V2 makes the simplifying assumption that the servers are busy with either the i-1 or i-2 customer (i.e., customers do not have "unexpectedly" long or short service times). This generally leads to shorter queues and waiting times. Version 3 corrects this simplifying assumption by adding in "memory" of a customer with an abnormally long service time.
   b. (5) Run this simulation for the values of Ra and Rs provided in Problem 2 and for Nc = 20, 200, and 2000 customers and provide the text results in a table and the associated graphical outputs.
      i. For Nc = 2000





1. rhoa = 0.6670
2. MLqa = 1.0691
3. MLqmc = 0.9636 - mean MC Q length
4. MTqa = 0.5346 min
5. MTqmc1 = 0.4729 min – mean MC time in Q (calculate using method 1)
6. MTqmc2 = 0.4729 (method 2)
7. SDmtamc = 0.8348 min – Standard Dev of MC Time in Q

8. SEmtamc = 0.0187min – Standard Error in MC MEAN Time in Q

c. (4) For each of the cases run, are the final values for Wqmc, Lqmc, and Wmc significantly different from the analytical values you obtained in Problem 2? If so, explain why. Note you may want to extend the table developed in b. to answer this question.
   i. If you used the simplifying assumption used in Version 2 of the posted code, Mlqmc and MTqmc are significantly smaller than the analytical results. This is due to the simplifying assumption.
   ii. If you did not use the simplifying assumption, you got something like the answer in b. If so you most likely found Wqmc to be more than three standard errors smaller than Wqa. This is because the system starts with no queque. If you started with the system in the steady state, it would stabilize more quickly and closer to the analytical solution.

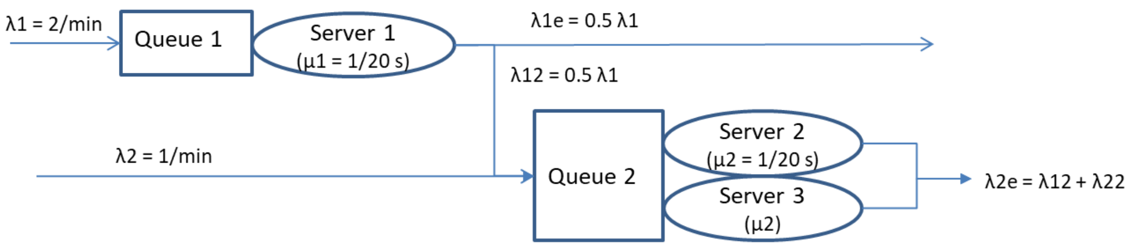
d. (3) How long does it appear to take for the system to reach steady state? Explain how you got your answer.
   i. From the graph of Wqmc, it appears to take about 2000-4000 min to get to a SS of about 0.5 (this is seen if you run it for Nc = 20,000). It takes this many minutes for Wqmc to stabilize.

e. (3) Is the graphical output you obtained of the queue length as a function of time what you expected? Explain.
   i. Probably not, you probably expected that it would show less noise and variation.
   ii. That is why MC is useful in giving you insight into how a real system behaves, that one generally does not get from the analytical results.

5. **Problem 5 (20 pts) – Simple Queueing Network:** Consider a two-station network. At station 1, there is a single server, and at station 2, there are two identical servers with a single queue. Outside arrivals to the network are Poisson at a rate 2/min to station 1 and 1/min to station 2. Half of the customers that initially arrive at station 1 go on to station 2; the others leave the network. All customers finishing at station 2 leave the network. Service times are exponentially distributed, with mean 20 sec for the station 1 server and mean 40 sec at station 2. Assume infinite capacity buffers and that the system is in steady state.

a. (4) Find the average throughput (total rate of customers served) of the network.

i. Draw the Network



ii. Verify each element is in steady state

Input to 1: 2/60 sec — Service rate: 1/20 => stable => all through

Input to 2: 1/60 + 1/60 — Service rate: 2/40 => stable => all through

iii. Look at sum of all outputs

=> Throughput = Input = 1/30 + 1/60 = 3/min.

b. (4) Find the proportion of time both stations are completely busy (i.e., all servers).

P(both busy) = P(1 busy) * P(2 busy) = (1-P(1 empty)) (1-P(2 empty)) = 2/3 * 4/5 = 8/15

$p_{1o} = 1 - \rho_1 = 1/3$

$\rho_1 = (1/30)/(1/20) = 2/3$

$p_{2o} = (1 - \rho_2) / (1 + \rho 2) = (1/3) / (5/3) = 1/5$

$\rho_2 = (1/60+1/60)/(2/40) = 2/3$

Note that technically we also need p_{21}

c. (4) Find the average number of customers at each station and in the entire network.

From formulae for 1 and 2 server systems:

$L1 = \rho_1/(1 - \rho_1) = 2.0$ cust

$L2 = 2\ \rho_2/(1 - \rho_2^2) = 2.4$ cust

For System

$L12 = L1 + L2 = 4.4$ cust

d. (4) Find the average time for a customer at each station and in the entire network.

$W = L/\lambda$ => W1 = 1 min, W2 = 1.2 min, W = (4.4/3) c/min = 1.47 min.

e. (4) Does the sum of the average times spent at each of the two stations equal the average time in the entire network? Explain

i. No.

1. Sum of time spent in two stations = W1 + W2 = 2.2 min
2. W = 1.47 min
3. W < W1+W2 because only about 1/3 of customers go through both Q1 and Q2.