Mathematical Modelling of System (Cont.)

Development of an Inverted Pendulum test-bench

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1 Non-linear System Model

We have the the following two non-linear equations describing the dynamic model of the inverted pendulum:

$$F = (M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta \tag{1}$$

$$(I+ml^2)\ddot{\theta} + m\ddot{x}l\cos\theta - mgl\sin\theta = 0 \tag{2}$$

Making $\ddot{\theta}$ and $\ddot{x}$ subject from (2), we get:

$$\ddot{\theta} = \frac{mgl\sin\theta}{I+ml^2} - \frac{ml\cos\theta}{I+ml^2}\ddot{x} \tag{3}$$

$$\ddot{x} = \frac{mgl\sin\theta}{ml\cos\theta} - \frac{(I+ml^2)\ddot{\theta}}{ml\cos\theta} = \frac{g\sin\theta}{\cos\theta} - \frac{(I+ml^2)\ddot{\theta}}{ml\cos\theta} \tag{4}$$

Substituting (3) into (1), we get:

$$F = (M+m)\ddot{x} + ml\cos\theta(\frac{mgl\sin\theta}{I+ml^2} - \frac{ml\cos\theta}{I+ml^2}\ddot{x}) - ml\sin\theta\dot{\theta}^2$$

$$F = (M+m)\ddot{x} + \frac{m^2gl^2\sin\theta\cos\theta}{I+ml^2} - \frac{m^2l^2\cos\theta^2}{I+ml^2}\ddot{x} - ml\sin\theta\dot{\theta}^2$$

$$(M+m-\frac{m^2l^2\cos\theta^2}{I+ml^2})\ddot{x} = F + ml\sin\theta\dot{\theta}^2 - \frac{m^2gl^2\sin\theta\cos\theta}{I+ml^2}$$

Let $a = M+m-\frac{m^2l^2\cos\theta^2}{I+ml^2}$, the final equation for $\ddot{x}$ becomes:

$$\therefore \ddot{x} = \frac{F}{a} - \frac{m^2gl^2\sin\theta\cos\theta}{a(I+ml^2)} + \frac{ml\sin\theta\dot{\theta}^2}{a} \tag{5}$$

Now, substituting (4) into (1):

$$F = (M+m)\frac{g\sin\theta}{\cos\theta} - (M+m)\frac{(I+ml^2)\ddot{\theta}}{ml\cos\theta} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta$$

$$F = (ml\cos\theta - (M+m)\frac{(I+ml^2)}{ml\cos\theta})\ddot{\theta} + (M+m)\frac{g\sin\theta}{\cos\theta} - ml\dot{\theta}^2\sin\theta$$

Let $b = ml\cos\theta - (M+m)\dfrac{(I+ml^2)}{ml\cos\theta}$, the final equation for $\ddot{\theta}$ becomes:

$$\therefore \ddot{\theta} = \frac{F}{b} - (M+m)\frac{g\sin\theta}{b\cos\theta} + \frac{ml\dot{\theta}^2\sin\theta}{b} \tag{6}$$

2 Non-linear System Model incorporating Viscous Friction

$$F = (M+m)\ddot{x} - B_1\dot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta \tag{7}$$

$$(I+ml^2)\ddot{\theta} - B_2\dot{\theta} + m\ddot{x}l\cos\theta - mgl\sin\theta = 0 \tag{8}$$

From (8), making $\ddot{\theta}$ and $\ddot{x}$ subject:

$$\ddot{\theta} = \frac{mgl\sin\theta - ml\ddot{x}\cos\theta + B_2\dot{\theta}}{I+ml^2} \tag{9}$$

$$\ddot{x} = \frac{mgl\sin\theta - (I+ml^2)\ddot{\theta} + B_2\dot{\theta}}{ml\cos\theta} \tag{10}$$

Substituting (9) in (7):

$$F = (M+m)\ddot{x} - B_1\dot{x} + ml(\frac{mgl\sin\theta - ml\ddot{x}\cos\theta + B_2\dot{\theta}}{I+ml^2})\cos\theta - ml\dot{\theta}^2\sin\theta$$

$$F = (M+m-\frac{m^2l^2\cos\theta^2}{I+ml^2})\ddot{x} - B_1\dot{x} + \frac{m^2gl^2\sin\theta\cos\theta}{I+ml^2} + \frac{mlB_2\dot{\theta}\cos\theta}{I+ml^2} - ml\dot{\theta}^2\sin\theta$$

Again, let $a = M+m-\dfrac{m^2l^2\cos\theta^2}{I+ml^2}$, and make $\ddot{x}$ subject:

$$\therefore \ddot{x} = \frac{F}{a} + \frac{B_1\dot{x}}{a} - \frac{m^2gl^2\sin\theta\cos\theta}{a(I+ml^2)} + \frac{ml\dot{\theta}^2\sin\theta}{a} \tag{11}$$

Now, substituting (10) in (7), we get:

$$F = (M+m)(\frac{mgl\sin\theta - (I+ml^2)\ddot{\theta} + B_2\dot{\theta}}{ml\cos\theta}) - B_1\dot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta$$

$$F = (ml\cos\theta - \frac{(M+m)(I+ml^2)}{ml\cos\theta})\ddot{\theta} + \frac{(M+m)B_2\dot{\theta}}{ml\cos\theta} + \frac{(M+m)g\sin\theta}{\cos\theta} - ml\dot{\theta}^2\sin\theta$$

Here, let $b = ml\cos\theta - \dfrac{(M+m)(I+ml^2)}{ml\cos\theta}$. Hence, making $\ddot{\theta}$ subject, we get:

$$\therefore \ddot{\theta} = \frac{F}{b} - \frac{(M+m)B_2\dot{\theta}}{bml\cos\theta} - \frac{(M+m)g\sin\theta}{b\cos\theta} + \frac{ml\dot{\theta}^2\sin\theta}{b} \tag{12}$$

3 Linearized System Model

We make the following assumptions to linearize our equations at point $\theta = 180$ degrees:

$$\sin\theta \approx \theta$$
$$\cos\theta \approx -1$$
$$\theta\dot{\theta}^2 \approx 0$$

Then, our system equations become:

$$F = (M+m)\ddot{x} - B_1\dot{x} - ml\ddot{\theta} \tag{13}$$
$$(I+ml^2)\ddot{\theta} - B_2\dot{\theta} - ml\ddot{x} - mgl\theta = 0 \tag{14}$$

From (14), we make $\ddot{\theta}$ and $\ddot{x}$ subject:

$$\ddot{\theta} = \frac{mgl\theta + ml\ddot{x} + B_2\dot{\theta}}{I+ml^2} \tag{15}$$

$$\ddot{x} = \frac{(I+ml^2)\ddot{\theta} - mgl\theta - B_2\dot{\theta}}{ml} \tag{16}$$

Substituting (15) in (13):

$$F = (M+m)\ddot{x} - B_1\dot{x} - ml(\frac{mgl\theta + ml\ddot{x} + B_2\dot{\theta}}{I+ml^2})$$
$$F = (M+m-\frac{m^2l^2}{I+ml^2})\ddot{x} - \frac{mlB_2\dot{\theta}}{I+ml^2} - \frac{m^2gl^2\theta}{I+ml^2} - B_1\dot{x}$$

Let $a = M+m-\frac{m^2l^2}{I+ml^2}$, the linearized expression for $\ddot{x}$ becomes:

$$\therefore \ddot{x} = \frac{F}{a} + \frac{mlB_2\dot{\theta}}{a(I+ml^2)} + \frac{m^2gl^2\theta}{a(I+ml^2)} + \frac{B_1\dot{x}}{a} \tag{17}$$

Now, substituting (16) in (13):

$$F = (M+m)\frac{(I+ml^2)\ddot{\theta} - mgl\theta - B_2\dot{\theta}}{ml} - B_1\dot{x} - ml\ddot{\theta}$$
$$F = \big(\frac{(M+m)(I+ml^2)}{ml} - ml\big)\ddot{\theta} - \frac{(M+m)(mgl\theta)}{ml} - \frac{(M+m)B_2\dot{\theta}}{ml} - B_1\dot{x}$$

Let $b = \big(\frac{(M+m)(I+ml^2)}{ml} - ml\big)$, the linearized expression for $\ddot{\theta}$ becomes:

$$\therefore \ddot{\theta} = \frac{F}{b} + \frac{(M+m)g\theta}{b} + \frac{(M+m)B_2\dot{\theta}}{mlb} + \frac{B_1\dot{x}}{b} \tag{18}$$

4 State Space Representation of the Linearized Model

The State-Space variables are as follows:

$$\mathbf{x}_1 = \theta$$
$$\mathbf{x}_2 = \dot{\theta}$$
$$\mathbf{x}_3 = x$$
$$\mathbf{x}_4 = \dot{x}$$

This can be written in the form:

$$\dot{\mathbf{x}} = \frac{d}{dt}\begin{bmatrix}\mathbf{x_1}\\ \mathbf{x_2}\\ \mathbf{x_3}\\ \mathbf{x_4}\end{bmatrix} = \frac{d}{dt}\begin{bmatrix}\theta\\ \dot{\theta}\\ x\\ \dot{x}\end{bmatrix} = \begin{bmatrix}\dot{\theta}\\ \ddot{\theta}\\ \dot{x}\\ \ddot{x}\end{bmatrix} = \begin{bmatrix}\dot{\mathbf{x_1}}\\ \dot{\mathbf{x_2}}\\ \dot{\mathbf{x_3}}\\ \dot{\mathbf{x_4}}\end{bmatrix}$$

Hence, utilizing equations (17) and (18):

$$\begin{bmatrix}\dot{\mathbf{x_1}}\\ \dot{\mathbf{x_2}}\\ \dot{\mathbf{x_3}}\\ \dot{\mathbf{x_4}}\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0\\ \frac{(M+m)g}{b} & \frac{(M+m)B_2}{mlb} & 0 & \frac{B_1}{b}\\ 0 & 0 & 0 & 1\\ \frac{m^2l^2g}{a(I+ml^2)} & \frac{mlB_1}{a(I+ml^2)} & 0 & \frac{B_1}{a}\end{bmatrix}\begin{bmatrix}\mathbf{x_1}\\ \mathbf{x_2}\\ \mathbf{x_3}\\ \mathbf{x_4}\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{b}\\ 0\\ \frac{1}{a}\end{bmatrix}\mathbf{F}$$