

Multivariable Calculus

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Syllabus

Multiple integrals: Two dimensions. Informal definition and evaluation by repeated integration; example over a rectangle; properties. General domains. Change of variables. Examples. [2.5]

Volume integrals: Jacobians for cylindrical and spherical polars, examples. [1.5]

Recap on surface integrals. Flux integrals including solid angle. [1.5]

Scalar and vector fields. Vector differential operators: divergence and curl; physical interpretation. Calculation. Identities. [2.5]

Divergence theorem. Informal proof. Divergence theorem in plane – Green's theorem. Examples. Consequences: Greens 1st and second theorems. $\iiint_V \nabla \phi \, dV = \iint_{\partial V} \phi \, d\mathbf{S}$. Uniqueness of solutions of Poisson's equation. Derivation of heat equation. [4]

Stokes's theorem. Examples. Consequences. The existence of potential for a conservative force. [2]

Gauss' Flux Theorem. Examples. Equivalence with Poisson's equation. [2]

Recommended Texts

- Erwin Kreyszig, *Advanced Engineering Mathematics* (Wiley, 8th Edition, 1999).
- D. E. Bourne & P. C. Kendall, *Vector Analysis and Cartesian Tensors* (Stanley Thornes, 1992).
- H. M. Schey, *Div, Grad and Curl and all that* (W. W. Norton, Third Edition, 1996).

Further Reading

- Mary L. Boas, *Mathematical Models in the Physical Sciences* (Wiley 2nd Edition, 1983).
- Gerald B. Folland, *Advanced Calculus* (Prentice Hall, 2002)
- Murray R Spiegel, *Vector Analysis* (Schaum's Outline Series, McGraw-Hill, 1974).

Website

<https://www.maths.ox.ac.uk/courses/material>

1. MULTIPLE INTEGRALS IN 2D

1.1 Preliminaries

Definition 1 By a *scalar field* ϕ on \mathbb{R}^3 we shall mean a map $\phi: \mathbb{R}^3 \rightarrow \mathbb{R}$.

Definition 2 By a *vector field* \mathbf{F} on \mathbb{R}^3 we shall mean a map $\mathbf{F}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$.

- We will typically assume that scalar and vector fields are smooth – their partial derivatives exist with respect to x , y and z to all orders – for brevity, this will not always be stated.
- Occasionally we may consider more general scalar fields $\phi: \mathbb{R}^n \rightarrow \mathbb{R}$ and vector fields $\mathbf{F}: \mathbb{R}^n \rightarrow \mathbb{R}^m$.

Example 3 The three co-ordinates x, y, z are each scalar fields on \mathbb{R}^3 . The position vector $\mathbf{r} = (x, y, z)$ is a vector field; its magnitude $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$ is a scalar field, though note it is not smooth at $(0, 0, 0)$.

We shall also consider scalar and vector fields defined on proper subsets of \mathbb{R}^3 (or more generally \mathbb{R}^n). The domains of these fields will usually be *open*, so we can define their partial derivatives.

Definition 4 A set $U \subseteq \mathbb{R}^n$ is said to be *open* if for every $\mathbf{x} \in U$ there exists $\varepsilon > 0$ such that

$$B(\mathbf{x}, \varepsilon) = \{\mathbf{y} \in \mathbb{R}^n : |\mathbf{y} - \mathbf{x}| < \varepsilon\} \subseteq U.$$

and where

$$|\mathbf{y} - \mathbf{x}|^2 = \sum_{i=1}^n |y_i - x_i|^2.$$

We refer to $B(\mathbf{x}, \varepsilon)$ as the *open ball* of radius r , centred at \mathbf{x} .

Remark 5 (Integrals in one dimension) An informal definition of the integral

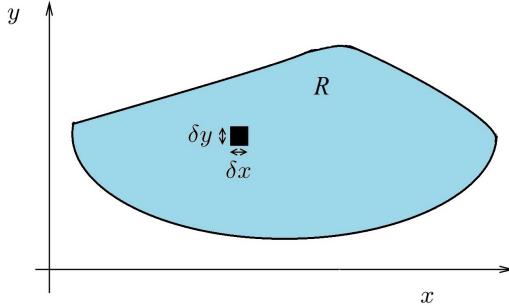
$$\int_a^b f(x) dx,$$

is as follows: Suppose $f: [a, b] \rightarrow \mathbb{R}$ is a function. Subdivide $[a, b]$ into m sub-intervals of equal length δx and let x_1, \dots, x_m be points in the respective intervals. On partitioning with smaller and smaller intervals by taking the limit $\delta x \rightarrow 0$, we have

$$\int_a^b f(x) dx = \lim_{\delta x \rightarrow 0} \sum_{r=1}^m f(x_r) \delta x,$$

provided the limit exists. This will always be the case if f is continuous (also, alternative subdivisions will not change the limit when f is continuous). A rigorous treatment of the Riemann integral will be given next term in Analysis III.

1.2 Multiple Integrals in Two Dimensions: A Brief Introduction



1. A square element of area $\delta x \delta y$ in a region R .

Consider a region of the plane, R , such as depicted in Figure 1, together with a scalar field $\psi(x, y)$.

- Partition the region into N square elements of equal area $\delta A = \delta x \delta y$; difficulties with boundary elements extending outside the region R will disappear below.
- Suppose the scalar field $\psi(\mathbf{r})$ takes the value ψ_i at the centre of the i^{th} element.
- On partitioning with smaller and smaller squares, by taking the limit $\delta A \rightarrow 0$, we have

$$\lim \sum_{i=1}^N \psi_i \delta A = \lim \sum_{i=1}^N \psi_i \delta x \delta y \stackrel{\text{def}}{=} \iint_R \psi dA = \iint_R \psi dx dy,$$

noting the region with boundary elements extending outside the region R yields no contribution to the integral in the limit.

- This is an informal definition of a **double integral**; a rigorous approach would, for example, verify the limit is independent of the details of the elements used to decompose R . If ψ is piecewise continuous and R is a suitable region (we only ever consider such cases), there are no difficulties. In general, which is not the objective here, complexities can emerge (e.g. fractal boundaries).

Note: $\psi(\mathbf{r}) = 1 \forall \mathbf{r} \Rightarrow \iint_R \psi dA = \iint_R dx dy = (\text{area of } R)$.

Properties of double integrals The following properties are inherited from integration with respect to one variable.

- *Linearity:* Let a, b be constants.

$$\iint_R (af(x, y) + bg(x, y)) dA = a \iint_R f(x, y) dA + b \iint_R g(x, y) dA.$$

- *Order:* If $f(x, y) \geq g(x, y)$ for all $(x, y) \in R$ then

$$\iint_R f(x, y) dA \geq \iint_R g(x, y) dA.$$

- *Domain splitting:* If $R = R_1 \cup R_2$ and $R_1 \cap R_2 = \emptyset$ then

$$\iint_R f(x, y) dA = \iint_{R_1} f(x, y) dA + \iint_{R_2} f(x, y) dA.$$

Interpretation.

Depending on the significance $\psi(x, y)$ the integral $\iint_R \psi(x, y) dx dy$ will signify different things. If $\psi(x, y)$ is the area density of a plate at the point (x, y) , then the integral gives the total mass of the plate. If $\psi(x, y)$, defined on all of \mathbb{R}^2 , is a probability density function then the integral represents the probability of a randomly chosen point (x, y) being in R . Alternatively, we can interpret the integral as the volume under the surface $\psi(x, y)$ (provided $\psi(x, y) \geq 0$).

Short cuts and sanity checks. Occasionally it is clear that an integral is zero because of algebraic properties of the integrand and geometric properties of the domain. For example,

$$\begin{aligned} & \iint_{\text{unit disc}} \sin(x+y) dA, & \iint_{[0,1]^2} x-y dA, & \iint_{\mathbb{R}^2} xye^{-x^2-y^2} dA, \end{aligned}$$

are all zero. The first integrand is odd. The second integrand at (y, x) takes negative the value at (x, y) , so is "odd" about the line $y = x$. The third integrand is actually even, but at $(-x, y)$ takes negative the value at (x, y) , so is "odd" about y -axis.

Even if the value of an integral is not immediately clear then estimates or bounds for it may be, and these can be useful sanity checks when an answer is found. So the integrals

$$\begin{aligned} & \iint_{\text{unit disc}} \sin^2(x+y) dA, & \iint_{\text{unit disc}} (x^2+y^2)^n dA, & \iint_{\text{unit disc}} (ax+by)^{2n} dA, \end{aligned}$$

may not be immediately clear, but the first should be between 0 and π as the integrand is bounded by 0 and 1 and π is the area of the domain. The same reasoning applies to second integral and moreover the answer should be a decreasing function of n . The third integral should give an answer that is symmetric in a and b – consider swapping the variables x and y .

Further, if an integral represents a probability then it should take a value between 0 and 1. An integral representing a mass should be positive. Etc.

Example 6 Take $R = [0, 2] \times [1, 3]$ and evaluate

$$I = \iint_R (x+y^2) dx dy.$$

Solution. We compute (first integrating with respect to x and keeping y constant)

$$\begin{aligned} I &= \int_{y=1}^{y=3} \left(\int_{x=0}^{x=2} (x+y^2) dx \right) dy = \int_{y=1}^{y=3} \left[\frac{x^2}{2} + y^2 x \right]_{x=0}^{x=2} dy, \\ &= \int_{y=1}^{y=3} \left(\frac{4}{2} + 2y^2 - 0 \right) dy = \left[2y + \frac{2y^3}{3} \right]_{y=1}^{y=3} = \frac{64}{3}, \end{aligned}$$

or (first integrating with respect to y and keeping x constant)

$$\begin{aligned} I &= \int_{x=0}^{x=2} \left(\int_{y=1}^{y=3} (x + y^2) dy \right) dx, \\ &= \int_{x=0}^{x=2} \left[xy + \frac{y^3}{3} \right]_{y=1}^{y=3} dx = \int_{x=0}^{x=2} \left(3x + \frac{27}{3} - \left(x + \frac{1}{3} \right) \right) dx, \\ &= \int_{x=0}^{x=2} \left(2x + \frac{26}{3} \right) dx = \left[\frac{2x^2}{2} + \frac{26x}{3} \right]_{x=0}^{x=2} = \frac{64}{3}. \end{aligned}$$

■

Note: We get the same answer regardless of the order in which we do the integrals (as one would expect!).

Example 7 Let R be the unit square. Determine $\iint_R y \cos^2(\pi xy) dA$.

Solution. We can represent the domain by $x \in [0, 1]$, $y \in [0, 1]$, and thus

$$\begin{aligned} \iint_R y \cos(\pi xy) dA &= \int_0^1 \left\{ \int_0^1 y \cos(\pi xy) dx \right\} dy \\ &= \frac{1}{\pi} \int_0^1 [\sin(\pi xy)]_0^1 dy \\ &= \frac{1}{\pi} \int_0^1 \sin(\pi y) dy \\ &= -\frac{1}{\pi^2} [\cos(\pi y)]_0^1 \\ &= -\frac{1}{\pi^2} (-1 - 1) = \frac{2}{\pi^2}. \end{aligned}$$

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Note: It is much easier to do the x integral first, even though the answer is independent of the order of integration.

Remark 8 Given a continuous, bounded function $f(x, y)$ on a rectangular domain $(a_1, b_1) \times (a_2, b_2)$ then it is the case that

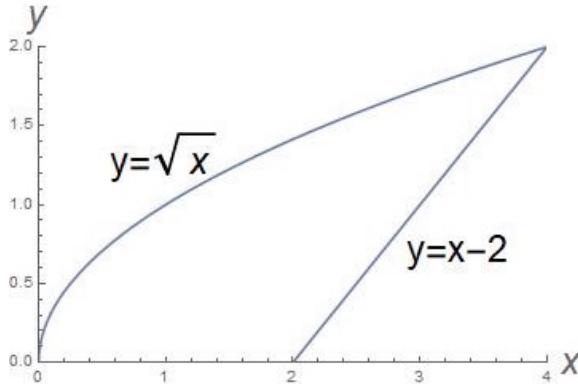
$$\int_{x=a_1}^{x=b_1} \int_{y=a_2}^{y=b_2} f(x, y) dy dx = \int_{y=a_2}^{y=b_2} \int_{x=a_1}^{x=b_1} f(x, y) dx dy, \quad (1.1)$$

but this need not be the case if the integrand is continuous, but unbounded (Sheet 1, Exercise 2). However if either of the integrals

$$\int_{x=a_1}^{x=b_1} \int_{y=a_2}^{y=b_2} |f(x, y)| dy dx \quad \text{or} \quad \int_{y=a_2}^{y=b_2} \int_{x=a_1}^{x=b_1} |f(x, y)| dx dy$$

exist, then we do have equality of the integrals in (1.1). This is a consequence of **Tonelli's** and **Fubini's theorems**.

For general domains, domains that aren't simply rectangles, then we need to be careful determining the limits of the integrals, and it may well be that one of the orders of integration is more sensible.



2. Region bounded by lines and parabola

Example 9 Calculate the area of the region R bounded by the parabola $y = \sqrt{x}$, the x -axis, and the line $y = x - 2$.

Solution. We can calculate this area using Cartesian co-ordinates in two ways, with y as the external variable and x as the internal variable, or vice versa. The first of these ways is preferable as we will see.

To work out the limits of the external variable, we simply need to determine the maximum and minimum values for that variable. In this case the variable is y and we see that $0 \leq y \leq 2$ on R .

However for a particular choice of y (in the range $0 \leq y \leq 2$) the values of x are limited in a way that depends on y . Integrating this way we are dividing R into infinitesimal, horizontal slithers. And for specific y we then have

$$y^2 \leq x \leq y + 2.$$

Hence the area is given by the multiple integral

$$\int_{y=0}^{y=2} \int_{x=y^2}^{x=y+2} 1 \, dx \, dy = \int_{y=0}^{y=2} (y + 2 - y^2) \, dy = \left[\frac{y^2}{2} + 2y - \frac{y^3}{3} \right]_0^2 = \frac{10}{3}.$$

Instead we might take x as the external variable, and on R we have $0 \leq x \leq 4$. As before, for a particular choice of x (in the range $0 \leq x \leq 4$) the values of y are limited in a way that depends on x . Integrating this way we are dividing R into infinitesimal, vertical slithers. However the formulas for these bounds differ for $0 \leq x \leq 2$ and for $2 \leq x \leq 4$. Note

$$\begin{aligned} \text{if } 0 \leq x \leq 2 & \quad \text{then } 0 \leq y \leq \sqrt{x}; \\ \text{if } 2 \leq x \leq 4 & \quad \text{then } x - 2 \leq y \leq \sqrt{x}. \end{aligned}$$

So the area is given by the two multiple integrals

$$\begin{aligned}
& \int_{x=0}^{x=2} \int_{y=0}^{y=\sqrt{x}} 1 \, dy \, dx + \int_{x=2}^{x=4} \int_{y=x-2}^{y=\sqrt{x}} 1 \, dy \, dx \\
&= \int_{x=0}^{x=2} \sqrt{x} \, dx + \int_{x=2}^{x=4} (\sqrt{x} - x + 2) \, dx \\
&= \left[\frac{2x^{3/2}}{3} \right]_0^2 + \left[\frac{2x^{3/2}}{3} - \frac{x^2}{2} + 2x \right]_2^4 \\
&= \frac{4\sqrt{2}}{3} + \left(\frac{16}{3} - \frac{16}{2} + 8 \right) - \left(\frac{4\sqrt{2}}{3} - \frac{4}{2} + 4 \right) \\
&= \frac{16}{3} - 2 = \frac{10}{3}.
\end{aligned}$$

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Example 10 Calculate the area of the disc $x^2 + y^2 \leq a^2$.

Solution. Again, we know the answer: πa^2 . If we wish to pick up all of the disc's area we can let x vary over the range $-a$ to a and, at each x , we need to let y vary from $-\sqrt{a^2 - x^2}$ to $\sqrt{a^2 - x^2}$. So we have

$$\begin{aligned}
A &= \int_{x=-a}^{x=a} \int_{y=-\sqrt{a^2-x^2}}^{y=\sqrt{a^2-x^2}} dy \, dx \\
&= \int_{x=-a}^{x=a} 2\sqrt{a^2 - x^2} \, dx \\
&= \int_{\theta=-\pi/2}^{\theta=\pi/2} 2\sqrt{a^2 - a^2 \sin^2 \theta} a \cos \theta \, d\theta \quad [x = a \sin \theta] \\
&= a^2 \int_{-\pi/2}^{\pi/2} 2 \cos^2 \theta \, d\theta = a^2 \int_{-\pi/2}^{\pi/2} 1 + \cos 2\theta \, d\theta = a^2 \left[\theta + \frac{1}{2} \sin 2\theta \right]_{-\pi/2}^{\pi/2} \\
&= \pi a^2.
\end{aligned}$$

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Definition 11 Given a plate occupying a region R of the plane, with density $\rho(x, y)$ per unit area, then the **moment of inertia** of the plate about an axis vertically through a point (x_0, y_0) equals

$$\iint_R \rho(x, y) ((x - x_0)^2 + (y - y_0)^2) \, dA.$$

Example 12 Find the moment of inertia of a uniform rectangle, with sides of length a and b and mass m about a corner of the rectangle.

Solution. Note $m = \rho ab$. Without loss of generality, we can take the corner to be $(0, 0)$ and then the moment of inertia equals

$$\begin{aligned}
& \int_{x=0}^a \int_{y=0}^b \left(\frac{m}{ab} \right) ((x-0)^2 + (y-0)^2) \, dy \, dx \\
&= \frac{m}{ab} \int_{x=0}^a \left[x^2 y + \frac{y^3}{3} \right]_0^b \, dx \\
&= \frac{m}{ab} \int_{x=0}^a \left(x^2 b + \frac{b^3}{3} \right) \, dx \\
&= \frac{m}{ab} \left[\frac{x^3 b}{3} + \frac{b^3 x}{3} \right]_0^a \\
&= \frac{m}{ab} \left(\frac{a^3 b}{3} + \frac{b^3 a}{3} \right) \\
&= \frac{m}{3} (a^2 + b^2).
\end{aligned}$$

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In the first example, it is simpler to do x -integration before y -integration (to avoid splitting the internal integral). For the example of the disc, it would be more natural to use polar co-ordinates — *if* we knew how to calculate areas with them!

1.3 Change of Variables and Jacobians.

The Jacobian, or rather its modulus, is a measure of how a general mapping stretches space locally, near a particular point, even when this stretching effect varies from point to point. The Jacobian takes its name from the German mathematician Carl Jacobi (1804-1851).

Definition 13 Given two co-ordinates $u(x, y)$ and $v(x, y)$ which depend on variables x and y , we define the **Jacobian**

$$\frac{\partial(u, v)}{\partial(x, y)}$$

to be the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}.$$

In 3D, we define the Jacobian

$$\frac{\partial(u, v, w)}{\partial(x, y, z)}$$

to be the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Example 14 Let $x = r \cos \theta$ and $y = r \sin \theta$ where r and θ are polar co-ordinates. Then

$$\begin{aligned}\frac{\partial(x, y)}{\partial(r, \theta)} &= \det \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} \\ &= \det \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} \\ &= r(\cos^2 \theta + \sin^2 \theta) = r.\end{aligned}$$

Example 15 In reverse, $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$ and

$$\begin{aligned}\frac{\partial(r, \theta)}{\partial(x, y)} &= \det \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{pmatrix} \\ &= \det \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix} \\ &= \frac{x^2+y^2}{(x^2+y^2)^{3/2}} \\ &= \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}.\end{aligned}$$

Theorem 16 Let $\mathbf{f}: R \rightarrow S$ be a bijection between two regions of \mathbb{R}^2 , which is differentiable and has differentiable inverse with

$$\frac{\partial(u, v)}{\partial(x, y)}, \quad \frac{\partial(x, y)}{\partial(u, v)},$$

defined and non-zero everywhere. Further, write $(u, v) = \mathbf{f}(x, y)$ and let $\psi(x, y) = \Psi(u, v)$. Then

$$\begin{aligned}\iint_{(u,v) \in S} \Psi(u, v) du dv &= \iint_{(x,y) \in R} \psi(x, y) \left| \frac{\partial(u, v)}{\partial(x, y)} \right| dx dy, \\ \iint_{(x,y) \in R} \psi(x, y) dx dy &= \iint_{(u,v) \in S} \Psi(u, v) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv.\end{aligned}$$

Proof. (Sketch proof.) It is sufficient to prove the first integral identity. Divide the region R into N square elements of equal area $\delta x \delta y$ as previously with ψ_i the scalar field at the centre of the i^{th} element, $i \in \{1, \dots, N\}$.

Consider the mapping of the i^{th} element, which is bounded by the co-ordinate lines $x = x_i$ and $x = x_i + \delta x$ and $y = y_i$ and $y = y_i + \delta y$. The value of the scalar field at the mapped element centre is

$$\Psi_i = \Psi(\mathbf{f}(x_i + \delta x/2, y_i + \delta y/2)) = \psi_i = \psi(x_i + \delta x/2, y_i + \delta y/2).$$

Also, given sufficiently small δx , δy the i^{th} element maps to an image region, denoted IM_i , which is a deformed parallelogram spanned by the vectors

$$\begin{aligned}\mathbf{a} &= \mathbf{f}(x_i + \delta x, y_i) - \mathbf{f}(x_i, y_i) \approx \frac{\partial \mathbf{f}}{\partial x}(x_i, y_i) \delta x, \\ \mathbf{b} &= \mathbf{f}(x_i, y_i + \delta y) - \mathbf{f}(x_i, y_i) \approx \frac{\partial \mathbf{f}}{\partial y}(x_i, y_i) \delta y,\end{aligned}$$

and, thus, of area

$$\left| \frac{\partial \mathbf{f}}{\partial x} \delta x \wedge \frac{\partial \mathbf{f}}{\partial y} \delta y \right| = \left| \frac{\partial \mathbf{f}}{\partial x} \wedge \frac{\partial \mathbf{f}}{\partial y} \right| \delta x \delta y.$$

Now $\mathbf{f} = (u, v)$, so $\mathbf{f}_x = (u_x, v_x)$, $\mathbf{f}_y = (u_y, v_y)$ and thus

$$\left| \frac{\partial \mathbf{f}}{\partial x} \wedge \frac{\partial \mathbf{f}}{\partial y} \right| \delta x \delta y = |(u_x, v_x) \wedge (u_y, v_y)| \delta x \delta y = |(u_x v_y - u_y v_x) \mathbf{k}| \delta x \delta y = \left| \frac{\partial(u, v)}{\partial(x, y)} \right| \delta x \delta y.$$

Thus, before taking limits, partitioning S by the images IM_i , we have an approximation for

$$\iint_{(u,v) \in S} \Psi(u, v) du dv$$

which is

$$\sum_{i=1}^N \Psi_i \text{Area}(IM_i) = \sum_{i=1}^N \Psi_i \left| \frac{\partial(u, v)}{\partial(x, y)} \right| \delta x \delta y = \sum_{i=1}^N \psi_i \left| \frac{\partial(u, v)}{\partial(x, y)} \right| \delta x \delta y.$$

Taking limits gives

$$\iint_{(u,v) \in S} \Psi(u, v) du dv = \iint_{(x,y) \in R} \psi(x, y) \left| \frac{\partial(u, v)}{\partial(x, y)} \right| dx dy.$$

■

If the previous motivation for this formula seems somewhat non-rigorous, note that the chain rule for Jacobians ensures that it is impossible that we might determine two different answers for a double integral by using different sets of variables.

Note: to evaluate integrals over an infinite domain (e.g., $\int_{-\infty}^{\infty} \exp(-x^2) dx$), we define

$$\int_{-\infty}^{\infty} f(x) dx = \lim_{\substack{X \rightarrow \infty \\ Y \rightarrow -\infty}} \int_Y^X f(x) dx.$$

Exercise 17 Evaluate

$$\iint_{\mathbb{R}^2} \exp[-(x^2 + y^2)] dA.$$

Hence, determine $\int_{-\infty}^{\infty} \exp[-t^2] dt$.

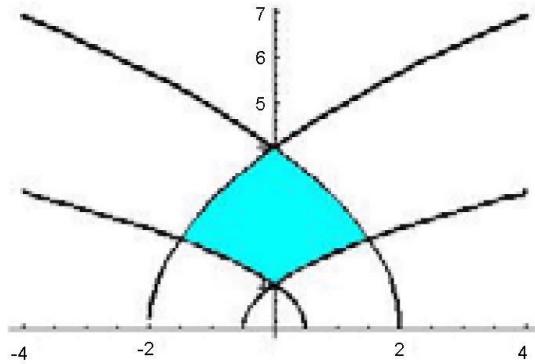
Solution.

$$\begin{aligned}
 I &= \iint_{\mathbb{R}^2} \exp[-(x^2 + y^2)] dA = \iint_{\mathbb{R}^2} \exp[-(x^2 + y^2)] dx dy \\
 &= \iint_{\mathbb{R}^2} \exp[-(r^2)] \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| dr d\theta = \int_0^\infty \left\{ \int_0^{2\pi} d\theta \right\} \exp[-(r^2)] r dr = \pi \int_0^\infty 2r \exp[-(r^2)] dr \\
 &= \pi [\exp[-(r^2)]]_0^\infty = \pi.
 \end{aligned}$$

Furthermore let $J = \int_{-\infty}^\infty \exp[-t^2] dt$. Then

$$J^2 = \int_{-\infty}^\infty \exp[-x^2] dx \int_{-\infty}^\infty \exp[-y^2] dy = \iint_{\mathbb{R}^2} \exp[-(x^2 + y^2)] dx dy = I.$$

Noting $J > 0$ to give the sign of the root, we thus have $J = \sqrt{\pi}$. ■



3. Using parabolic co-ordinates

Example 18 Calculate the area bounded by the curves

$$2x = 1 - y^2, \quad 2x = y^2 - 1, \quad 8x = 16 - y^2, \quad 8x = y^2 - 16.$$

as shown in Figure 3.

Solution. We now change to (u, v) , parabolic coordinates

$$x = \frac{1}{2} (u^2 - v^2), \quad y = uv.$$

Note that when $u = 1$ then $v = y$ and so $2x = 1 - y^2$, which is the equation of the first curves. Likewise if $u = 2$ then $v = y/2$ and so $2x = 4 - y^2/4$ which is the third curve. By symmetry the second and fourth curves correspond to $v = 1$ and $v = 2$. The region of interest is therefore $1 \leq u \leq 2$, $1 \leq v \leq 2$. The Jacobian is given by

$$\frac{\partial(x, y)}{\partial(u, v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \det \begin{pmatrix} u & -v \\ v & u \end{pmatrix} = u^2 + v^2.$$

Hence, the area is given by

$$\begin{aligned}
A &= \int_{u=1}^{u=2} \int_{v=1}^{v=2} \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dv du = \int_{u=1}^{u=2} \int_{v=1}^{v=2} (u^2 + v^2) dv du \\
&= \int_{u=1}^{u=2} \left[u^2 v + \frac{v^3}{3} \right]_{v=1}^{v=2} du = \int_{u=1}^{u=2} \left(u^2 + \frac{7}{3} \right) du \\
&= \left[\frac{u^3}{3} + \frac{7u}{3} \right]_1^2 = \frac{7}{3} + \frac{7}{3} = \frac{14}{3}.
\end{aligned}$$

■

Example 19 The cardioid with equation $r = a(1 + \cos \theta)$ bounds a region S . Find the mean value of r in S .

Solution. This mean value equals

$$\frac{1}{\text{area}(S)} \iint_S r dA.$$

The area of S equals

$$\begin{aligned}
\text{area}(S) &= \int_{\theta=0}^{2\pi} \int_{r=0}^{a(1+\cos\theta)} r dr d\theta \\
&= \frac{a^2}{2} \int_{\theta=0}^{2\pi} (1 + \cos \theta)^2 d\theta \\
&= \frac{a^2}{2} \int_{\theta=0}^{2\pi} (1 + 2 \cos \theta + \cos^2 \theta) d\theta \\
&= \frac{a^2}{2} (2\pi + 0 + \pi) = \frac{3\pi a^2}{2}.
\end{aligned}$$

And

$$\begin{aligned}
\iint_S r dA &= \int_{\theta=0}^{2\pi} \int_{r=0}^{a(1+\cos\theta)} r (r dr d\theta) \\
&= \frac{a^3}{3} \int_{\theta=0}^{2\pi} (1 + \cos \theta)^3 d\theta \\
&= \frac{a^3}{3} \int_{\theta=0}^{2\pi} (1 + 3 \cos \theta + 3 \cos^2 \theta + \cos^3 \theta) d\theta \\
&= \frac{a^3}{3} (2\pi + 0 + 3\pi + 0) = \frac{5\pi a^3}{3}.
\end{aligned}$$

Hence the mean value of r within the cardioid equals

$$\frac{5\pi a^3/3}{3\pi a^2/2} = \frac{10a}{9}.$$

2. VOLUME INTEGRALS

2.1 An Informal Definition of The Volume Integral

Consider a scalar field $\psi(x, y, z)$, and a three-dimensional region R .

- Partition R into N cubic elements, of volume $\delta V = \delta x \delta y \delta z$ and let ψ_i denote the value of ψ at the centre of the i^{th} cubic element, $i \in \{1, \dots, N\}$.
- Then, on partitioning with smaller and smaller cubes, and taking the limit $\delta V \rightarrow 0$,

$$\lim_{N \rightarrow \infty} \sum_{i=1}^N \psi_i \delta V \stackrel{\text{def}}{=} \iiint_R \psi dV = \iiint_R \psi dx dy dz$$

- This is an informal definition. For the regions R and functions ψ we shall meet there will never be any issue about whether the integral exists and we will usually determine such integrals by calculating three definite integrals separately over co-ordinates x, y, z (or other more appropriate co-ordinates). Again, complexities can emerge but these are not our focus.

Note: If the scalar field $\psi(x, y, z) = 1$ then the integral $\iiint_R \psi dV = \iiint_R dV$ gives the volume of the region R .

2.2 Examples

Example 20 A cone of height h occupies the region

$$x^2 + y^2 \leq z^2, \quad 0 \leq z \leq h.$$

and has density $\rho(x, y, z) = (x^2 + y^2)z$ at each point. Find the mass of the cone using Cartesian co-ordinates.

Solution. First, we must divide up the cone and find the corresponding limits. There are many ways to proceed. For example, we might take cross-sections of the cones, decomposing the cone into two- and then one-dimensional sections. Poorly chosen co-ordinates, whilst they could be used in principle to determine the mass, may require calculation of a complicated set of integrals.

If we use z as our first variable to divide up the cone, then the cross-section of the cone with a plane $z = z_0$ gives a disc $x^2 + y^2 \leq (z_0)^2$, $z = z_0$ (at least if $0 \leq z_0 \leq h$). We can then, say, take cross-sections of such discs with the line $y = y_0$ to produce horizontal slithers

$$x^2 \leq (z_0)^2 - (y_0)^2, \quad y = y_0, \quad z = z_0 \quad (\text{at least if } |y_0| \leq z_0)$$

and calculating their contribution to the mass is a simple one-dimensional integral. So, the mass M is given by the triple integral

$$\begin{aligned} M &= \int_{z=0}^h \int_{y=-z}^z \int_{x=-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} \rho(x, y, z) \, dx \, dy \, dz \\ &= \int_{z=0}^h \int_{y=-z}^z \int_{x=-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} (x^2 + y^2) z \, dx \, dy \, dz. \end{aligned}$$

We calculate the internal integrals in turn. We define

$$\begin{aligned} I_1(z, y) &= \int_{x=-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} (x^2 + y^2) z \, dx = \left[z \left(\frac{x^3}{3} + y^2 x \right) \right]_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} \\ &= 2z \left(\frac{(z^2 - y^2)^{3/2}}{3} + y^2 \sqrt{z^2 - y^2} \right) \\ &= \frac{2z}{3} (z^2 + 2y^2) \sqrt{z^2 - y^2}. \end{aligned}$$

The next internal integral is then

$$I_2(z) = \int_{y=-z}^z \frac{2z}{3} (z^2 + 2y^2) \sqrt{z^2 - y^2} \, dy.$$

If we make the substitution $y = z \sin t$, where $-\pi/2 \leq t \leq \pi/2$, then $I_2(z)$ becomes

$$\begin{aligned} I_2(z) &= \int_{t=-\pi/2}^{\pi/2} \frac{2z}{3} (z^2 + 2z^2 \sin^2 t) \sqrt{z^2 - z^2 \sin^2 t} (z \cos t \, dt) \\ &= \int_{t=-\pi/2}^{\pi/2} \frac{2z^3}{3} (1 + 2 \sin^2 t) |z \cos t| (z \cos t \, dt) \\ &= \frac{2z^5}{3} \int_{t=-\pi/2}^{\pi/2} (1 + 2 \sin^2 t) \cos^2 t \, dt. \end{aligned}$$

Now

$$\begin{aligned} \int_{t=-\pi/2}^{\pi/2} (\cos^2 t + 2 \sin^2 t \cos^2 t) \, dt &= \int_{t=-\pi/2}^{\pi/2} \left(\frac{1}{2} (1 + \cos 2t) + \frac{1}{2} \sin^2 2t \right) \, dt \\ &= \int_{t=-\pi/2}^{\pi/2} \left[\frac{1}{2} (1 + \cos 2t) + \frac{1}{4} (1 - \cos 4t) \right] \, dt = \left[\frac{3t}{4} + \frac{1}{4} \sin 2t - \frac{1}{8} \sin 4t \right]_{-\pi/2}^{\pi/2} = \frac{3\pi}{4}. \end{aligned}$$

Finally, we have

$$M = \int_{z=0}^h I_2(z) dz = \int_{z=0}^h \frac{2z^5}{3} \times \frac{3\pi}{4} dz = \frac{\pi}{2} \left[\frac{z^6}{6} \right]_0^h = \frac{\pi h^6}{12}.$$

Solution. (Alternative method with cylindrical polars). If $D(0, z)$ denotes the disc $x^2 + y^2 \leq z^2$, then the mass M could be written as

$$M = \int_{z=0}^h \iint_{D(0,z)} \rho(x, y, z) dx dy dz = \int_{z=0}^h \iint_{D(0,z)} (x^2 + y^2) z dx dy dz.$$

We know how to change from Cartesian co-ordinates (x, y) to planar polar co-ordinates (r, θ) : the change of variable rule is

$$\iint \phi dA = \iint \phi dx dy = \iint \phi (r dr d\theta),$$

where the r appears as it is the Jacobian $\partial(x, y) / \partial(r, \theta)$. Thus

$$\iint_{D(0,z)} (x^2 + y^2) z dx dy = \int_{r=0}^z \int_{\theta=0}^{2\pi} r^2 z (r dr d\theta) = 2\pi z \int_{r=0}^z r^3 dr = 2\pi z \left[\frac{r^4}{4} \right]_0^z = \frac{\pi z^5}{2}.$$

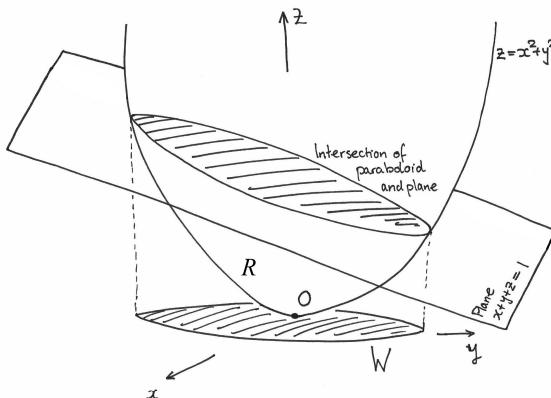
Hence,

$$M = \int_{z=0}^h \frac{\pi z^5}{2} dz = \left[\frac{\pi z^6}{12} \right]_0^h = \frac{\pi h^6}{12}.$$

Note: This change of variable makes the calculation much simpler! ■

Remark 21 In the above example, we essentially used cylindrical polar co-ordinates, appreciating that the volume element dV is given by

$$dV = dx dy dz = r dr d\theta dz.$$



4. Volume under a paraboloid

Example 22 Find the volume of the region R that lies above the paraboloid $z = x^2 + y^2$ and beneath the plane $x + y + z = 1$.

Solution. The (x, y) co-ordinates of a point (x, y, z) on the top planar surface of R satisfy

$$1 - x - y \geq x^2 + y^2$$

and, so, the top planar surface projects vertically down to the set

$$\begin{aligned} W &= \{(x, y) \in \mathbb{R}^2 : x + y + x^2 + y^2 \leq 1\} \\ &= \left\{ (x, y) \in \mathbb{R}^2 : \left(x + \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 \leq \frac{3}{2} \right\} \end{aligned}$$

which is a disc, centre $(-1/2, -1/2)$ and radius $\sqrt{3/2}$. Hence we can determine the volume of the region R as

$$\begin{aligned} V &= \iint_{(x,y) \in W} \left(\int_{z=x^2+y^2}^{1-x-y} dz \right) dA \\ &= \iint_{(x,y) \in W} (1 - x - y - x^2 - y^2) dA \\ &= \iint_{(x,y) \in W} \left(\frac{3}{2} - \left(x + \frac{1}{2}\right)^2 - \left(y + \frac{1}{2}\right)^2 \right) dA. \end{aligned}$$

Natural co-ordinates for parameterizing W are (r, θ) where

$$x = -\frac{1}{2} + r \cos \theta, \quad y = -\frac{1}{2} + r \sin \theta, \quad 0 \leq \theta < 2\pi, \quad 0 \leq r \leq \sqrt{3/2}.$$

Hence, we have

$$\begin{aligned} V &= \int_{r=0}^{\sqrt{3/2}} \int_{\theta=0}^{2\pi} \left(\frac{3}{2} - r^2 \cos^2 \theta - r^2 \sin^2 \theta \right) (r d\theta dr) \\ &= 2\pi \int_{r=0}^{\sqrt{3/2}} \left(\frac{3r}{2} - r^3 \right) dr = 2\pi \left[\frac{3r^2}{4} - \frac{r^4}{4} \right]_0^{\sqrt{3/2}} \\ &= 2\pi \left(\frac{9}{8} - \frac{9}{16} \right) \\ &= \frac{9\pi}{8}. \end{aligned}$$

Once again we have used polar coordinates. We should, therefore, consider what happens with a general change of variable.

2.3 Changing Co-ordinates in Volume Integrals

Theorem 23 Let $\mathbf{f}: R \rightarrow S$ be a bijection between subsets of \mathbb{R}^3 which is differentiable and has differentiable inverse. Take co-ordinates x_i on R and u_i on S related by the formula

$$(u_1, u_2, u_3) = \mathbf{f}(x_1, x_2, x_3),$$

and let $\psi(x_1, x_2, x_3) = \Psi(u_1, u_2, u_3)$ be scalar fields. Then

$$\iiint_S \Psi(u_1, u_2, u_3) du_1 du_2 du_3 = \iiint_R \psi(x_1, x_2, x_3) \left| \frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} \right| dx_1 dx_2 dx_3.$$

Proof. (Sketch) Partition the region R into N cubic elements and let the i^{th} cubic element be given by

$$[x_1, x_1 + \delta x_1] \times [x_2, x_2 + \delta x_2] \times [x_3, x_3 + \delta x_3]$$

which has volume $\delta x_1 \delta x_2 \delta x_3$. The value of the scalar field at the centre of the mapped element is

$$\Psi_i = \Psi(\mathbf{f}(x_1 + \delta x_1/2, x_2 + \delta x_2/2, x_3 + \delta x_3/2)) = \psi_i = \psi(x_1 + \delta x_1/2, x_2 + \delta x_2/2, x_3 + \delta x_3/2).$$

Also, under the map \mathbf{f} , the i^{th} cubic element maps to a deformed parallelepiped, denoted IM_i below, with sides

$$\begin{aligned} \mathbf{f}(x_1 + \delta x_1, x_2, x_3) - \mathbf{f}(x_1, x_2, x_3) &\approx \frac{\partial \mathbf{f}}{\partial x_1} \delta x_1; \\ \mathbf{f}(x_1, x_2 + \delta x_2, x_3) - \mathbf{f}(x_1, x_2, x_3) &\approx \frac{\partial \mathbf{f}}{\partial x_2} \delta x_2; \\ \mathbf{f}(x_1, x_2, x_3 + \delta x_3) - \mathbf{f}(x_1, x_2, x_3) &\approx \frac{\partial \mathbf{f}}{\partial x_3} \delta x_3. \end{aligned}$$

The volume of a parallelepiped with sides $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is $|\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c})|$ and, so, as a first approximation, the volume of the image of the i^{th} element is

$$\left| \left[\frac{\partial \mathbf{f}}{\partial x_1}, \frac{\partial \mathbf{f}}{\partial x_2}, \frac{\partial \mathbf{f}}{\partial x_3} \right] \right| \delta x_1 \delta x_2 \delta x_3 = \left| \frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} \right| \delta x_1 \delta x_2 \delta x_3.$$

Thus, before taking limits, partitioning S by the images IM_i , an approximation for

$$\iiint_S \Psi(u_1, u_2, u_3) du_1 du_2 du_3$$

is

$$\sum_{i=1}^N \Psi_i \text{Volume}(IM_i) = \sum_{i=1}^N \Psi_i \left| \frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} \right| \delta x_1 \delta x_2 \delta x_3 = \sum_{i=1}^N \psi_i \left| \frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} \right| \delta x_1 \delta x_2 \delta x_3.$$

Taking limits gives

$$\iiint_{(u_1, u_2, u_3) \in S} \Psi(u_1, u_2, u_3) du_1 du_2 du_3 = \iiint_{(x_1, x_2, x_3) \in R} \psi(x_1, x_2, x_3) \left| \frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} \right| dx_1 dx_2 dx_3.$$

■

If the previous motivation for this formula seems non-rigorous, note that the chain rule for Jacobians ensures that it is impossible to obtain different answers for a volume integral by using different sets of variables.

Example 24 (*Cylindrical Polar Co-ordinates*)

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z.$$

Then

$$\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \end{vmatrix} = r.$$

Example 25 (*Spherical Polar Co-ordinates*)

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta, \quad \theta \in [0, \pi], \quad \phi \in [0, 2\pi].$$

Then

$$\begin{aligned} \frac{\partial(x, y, z)}{\partial(r, \phi, \theta)} &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} \\ &= \begin{vmatrix} \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\ \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\ \cos \theta & -r \sin \theta & 0 \end{vmatrix} \\ &= \cos \theta \begin{vmatrix} r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\ r \cos \theta \sin \phi & r \sin \theta \cos \phi \end{vmatrix} + r \sin \theta \begin{vmatrix} \sin \theta \cos \phi & -r \sin \theta \sin \phi \\ \sin \theta \sin \phi & r \sin \theta \cos \phi \end{vmatrix} \\ &= r^2 \cos \theta \sin \theta \cos \theta (\cos^2 \phi + \sin^2 \phi) + r^2 \sin^3 \theta (\cos^2 \phi + \sin^2 \phi) \\ &= r^2 \sin \theta (\cos^2 \theta + \sin^2 \theta) \\ &= r^2 \sin \theta \geq 0 \text{ for } 0 \leq \theta \leq \pi. \end{aligned}$$

Thus, for spherical polar co-ordinates, we have

$$dV = dxdydz = r^2 \sin \theta dr d\theta d\phi.$$

Definition 26 *The centre of mass* of a body occupying a region R and with density $\rho(\mathbf{r})$ at the point with position vector \mathbf{r} is

$$\bar{\mathbf{r}} = (\bar{x}, \bar{y}, \bar{z}) = \frac{1}{M} \iiint_R \mathbf{r} \rho(\mathbf{r}) dV.$$

Example 27 Find the centre of mass of a uniform octant

$$\{(x, y, z) : x^2 + y^2 + z^2 < a^2, x, y, z > 0\}.$$

Solution. By symmetry, the centre of mass lies at a point $(\bar{x}, \bar{y}, \bar{z})$. Let ρ denote the density of the hemisphere. Then its mass is $\frac{1}{6}\pi a^3 \rho$ and

$$\begin{aligned}\bar{x} &= \frac{6}{\pi a^3 \rho} \int_{r=0}^a \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\theta=\pi/2} (r \sin \theta \cos \phi) (\rho r^2 \sin \theta) \, d\theta \, d\phi \, dr \\ &= \frac{6}{\pi a^3} \left(\int_{r=0}^a r^3 \, dr \right) \left(\int_{\theta=0}^{\theta=\pi/2} \sin^2 \theta \, d\theta \right) \left(\int_{\phi=0}^{\phi=\pi/2} \cos \phi \, d\phi \right) \\ &= \frac{6}{\pi a^3} \times \left[\frac{r^4}{4} \right]_0^a \times \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_0^{\pi/2} \times [\sin \phi]_0^{\pi/2} \\ &= \frac{6}{\pi a^3} \times \frac{a^4}{4} \times \frac{\pi}{4} \times 1 \\ &= \frac{3a}{8}.\end{aligned}$$

■

Definition 28 Given a function f on a region $R \subseteq \mathbb{R}^3$, the **median** of f is the value of m that satisfies

$$\text{Vol}(\{(x, y, z) : f(x, y, z) \leq m\}) = \frac{1}{2} \text{Vol}(R).$$

Example 29 Find the median value of z on the upper hemisphere of $x^2 + y^2 + z^2 \leq a^2$.

Solution. We will use cylindrical polar co-ordinates. $\text{Vol}(R) = 2\pi a^3/3$. And the volume of the hemisphere with $0 \leq z \leq h$ equals

$$\begin{aligned}&\int_{z=0}^h \int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{a^2-z^2}} r \, dr \, d\theta \, dz \\ &= 2\pi \int_{z=0}^h \left[\frac{r^2}{2} \right]_0^{\sqrt{a^2-z^2}} dz \\ &= \pi \int_{z=0}^h (a^2 - z^2) \, dz \\ &= \pi \left[a^2 z - \frac{z^3}{3} \right]_0^h \\ &= \pi \left(a^2 h - \frac{h^3}{3} \right).\end{aligned}$$

So we need that

$$\pi \left(a^2 h - \frac{h^3}{3} \right) = \frac{\pi a^3}{3}.$$

So we can rearrange to

$$0 = h^3 - 3a^2h + a^3.$$

There is one solution to this in the range $0 < h < a$ and this approximately equals $h = 0.347296a$. ■

Example 30 Evaluate the integral

$$\iiint x \, dV$$

over the intersection of the unit ball $x^2 + y^2 + z^2 \leq 1$ with the half-space $x + y \geq 3\sqrt{2}/5$.

Solution. If we take as an orthonormal basis

$$\mathbf{e}_1 = \frac{1}{\sqrt{2}}(1, -1, 0), \quad \mathbf{e}_2 = (0, 0, 1), \quad \mathbf{e}_3 = \frac{1}{\sqrt{2}}(1, 1, 0),$$

with corresponding co-ordinates X, Y, Z so that

$$X\mathbf{e}_1 + Y\mathbf{e}_2 + Z\mathbf{e}_3 = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

then we have $x = (X + Z)/\sqrt{2}$ and $x + y = Z\sqrt{2}$.

So we are left considering the integral

$$\iiint \left(\frac{X + Z}{\sqrt{2}} \right) \, dV$$

over the region $X^2 + Y^2 + Z^2 \leq 1$, $Z \geq 3/5$. By symmetry the integral

$$\iiint X \, dV = 0,$$

so we are left to calculate

$$\begin{aligned} \frac{1}{\sqrt{2}} \iiint Z \, dV &= \frac{1}{\sqrt{2}} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\cos^{-1}(3/5)} \int_{r=(3/5)\sec\theta}^1 (r \cos \theta) r^2 \sin \theta \, dr \, d\theta \, d\phi \\ &= \frac{2\pi}{\sqrt{2}} \int_{\theta=0}^{\cos^{-1}(3/5)} \int_{r=(3/5)\sec\theta}^1 r^3 \cos \theta \sin \theta \, dr \, d\theta \\ &= \pi\sqrt{2} \int_{\theta=0}^{\cos^{-1}(3/5)} \left[\frac{r^4}{4} \right]_{(3/5)\sec\theta}^1 \cos \theta \sin \theta \, d\theta \\ &= \frac{\pi\sqrt{2}}{4} \int_{\theta=0}^{\cos^{-1}(3/5)} \left(1 - \frac{81}{625} \sec^4 \theta \right) \cos \theta \sin \theta \, d\theta \\ &= \frac{\pi\sqrt{2}}{4} \int_{\theta=0}^{\cos^{-1}(3/5)} \left(\cos \theta \sin \theta - \frac{81}{625} \frac{\sin \theta}{\cos^3 \theta} \right) \, d\theta \end{aligned}$$

$$\begin{aligned}
&= \frac{\pi\sqrt{2}}{4} \left[-\frac{1}{2} \cos^2 \theta - \frac{81}{1250} \frac{1}{\cos^2 \theta} \right]_0^{\cos^{-1}(3/5)} \\
&= \frac{\pi\sqrt{2}}{4} \left[-\frac{1}{2} \times \frac{9}{25} - \frac{81}{1250} \times \frac{25}{9} + \frac{1}{2} + \frac{81}{1250} \right] \\
&= \frac{\pi\sqrt{2}}{4} \left[\frac{128}{625} \right] = \frac{32\pi\sqrt{2}}{625}.
\end{aligned}$$

■

Example 31 By a suitable rotation of co-ordinates, or otherwise, evaluate

$$\iiint (ax + by + cz)^4 \, dV$$

taken over the region $x^2 + y^2 + z^2 \leq 1$, where a, b, c are positive constants.

Solution. Consider instead this integral as

$$\iiint ((a, b, c) \cdot \mathbf{r})^4 \, dV.$$

We can rotate the axes to create new co-ordinates X, Y, Z in such a way that the Z axis points in the direction of vector (a, b, c) and under this change of co-ordinates the integral becomes

$$\iiint_{\text{unit ball}} \left(\sqrt{a^2 + b^2 + c^2} \mathbf{e}_3 \cdot \mathbf{r} \right)^4 \, dV = \iiint_{\text{unit ball}} (a^2 + b^2 + c^2)^2 Z^4 \, dV$$

where \mathbf{e}_3 is the unit vector pointing down the Z -axis; notice that we are still integrating over the unit sphere as we have simply rotated the axes about the origin.

Now, by a change to spherical polar co-ordinates,

$$\begin{aligned}
&\iiint_{\text{unit ball}} (a^2 + b^2 + c^2)^2 Z^4 \, dV \\
&= (a^2 + b^2 + c^2)^2 \int_{r=0}^1 \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} (r \cos \theta)^4 (r^2 \sin \theta) \, d\theta \, d\phi \, dr \\
&= 2\pi (a^2 + b^2 + c^2)^2 \left[\frac{r^7}{7} \right]_0^1 \left[-\frac{\cos^5 \theta}{5} \right]_0^\pi \\
&= 2\pi (a^2 + b^2 + c^2)^2 \times \frac{1}{7} \times \frac{2}{5} \\
&= \frac{4\pi}{35} (a^2 + b^2 + c^2)^2.
\end{aligned}$$

■

Example 32 Let $a, b, c \in \mathbb{R}$. Determine

$$I = \iiint_{\text{unit ball}} \cos(ax + by + cz) \, dx \, dy \, dz.$$

Show that your answer is consistent with the volume of the unit sphere being $4\pi/3$.

Solution. We can rewrite the integral as

$$I = \iiint_{\text{unit ball}} \cos(\mathbf{a} \cdot \mathbf{r}) \, dx \, dy \, dz$$

where $\mathbf{a} = (a, b, c)$.

The vector

$$\mathbf{e}_3 = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{(a, b, c)}{\sqrt{a^2 + b^2 + c^2}}$$

is of unit length and so we can extend it to an orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, with associated co-ordinates (X, Y, Z) . In terms of these co-ordinates

$$\mathbf{a} \cdot \mathbf{r} = (0, 0, |\mathbf{a}|) \cdot (X, Y, Z) = |\mathbf{a}| Z.$$

The unit sphere $x^2 + y^2 + z^2 < 1$ is still given as $X^2 + Y^2 + Z^2 < 1$ as $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are an orthonormal basis and $dV = dX \, dY \, dZ$. So

$$I = \iiint_{\text{unit ball}} \cos(|\mathbf{a}| Z) \, dX \, dY \, dZ.$$

If we now change to spherical polar co-ordinates (r, θ, ϕ) associated with (X, Y, Z) , then

$$\begin{aligned} I &= \int_{r=0}^1 \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \cos(|\mathbf{a}| r \cos \theta) r^2 \sin \theta \, d\phi \, d\theta \, dr \\ &= 2\pi \int_{r=0}^1 \int_{\theta=0}^{\pi} \cos(|\mathbf{a}| r \cos \theta) r^2 \sin \theta \, d\theta \, dr \\ &= 2\pi \int_{r=0}^1 \left[\frac{-\sin(|\mathbf{a}| r \cos \theta)}{|\mathbf{a}| r} r^2 \right]_0^\pi \, dr \\ &= 2\pi \int_{r=0}^1 \frac{r}{|\mathbf{a}|} (2 \sin(|\mathbf{a}| r)) \, dr \\ &= \frac{4\pi}{|\mathbf{a}|} \left\{ \left[\frac{-r \cos(|\mathbf{a}| r)}{|\mathbf{a}|} \right]_{r=0}^1 + \int_{r=0}^1 \frac{\cos(|\mathbf{a}| r)}{|\mathbf{a}|} \, dr \right\} \\ &= \frac{4\pi}{|\mathbf{a}|} \left[\frac{-r \cos(|\mathbf{a}| r)}{|\mathbf{a}|} + \frac{\sin(|\mathbf{a}| r)}{|\mathbf{a}|^2} \right]_{r=0}^1 \\ &= \frac{4\pi}{|\mathbf{a}|} \left[\frac{-\cos|\mathbf{a}|}{|\mathbf{a}|} + \frac{\sin(|\mathbf{a}|)}{|\mathbf{a}|^2} \right] \\ &= \frac{4\pi}{|\mathbf{a}|^3} (\sin|\mathbf{a}| - |\mathbf{a}| \cos|\mathbf{a}|). \end{aligned}$$

We determine the volume of the unit sphere by setting $a = b = c = 0$. So letting $|\mathbf{a}| \rightarrow 0$ we see

$$\begin{aligned}\frac{4\pi}{|\mathbf{a}|^3} (\sin |\mathbf{a}| - |\mathbf{a}| \cos |\mathbf{a}|) &= \frac{4\pi}{|\mathbf{a}|^3} \left(\left(|\mathbf{a}| - \frac{|\mathbf{a}|^3}{6} + O(|\mathbf{a}|^5) \right) - |\mathbf{a}| \left(1 - \frac{|\mathbf{a}|^2}{2} + O(|\mathbf{a}|^4) \right) \right) \\ &= \frac{4\pi}{|\mathbf{a}|^3} \left(\frac{|\mathbf{a}|^3}{3} + O(|\mathbf{a}|^4) \right) \\ &= \frac{4\pi}{3} + O(|\mathbf{a}|) \rightarrow \frac{4\pi}{3} \text{ as } |\mathbf{a}| \rightarrow 0.\end{aligned}$$

■

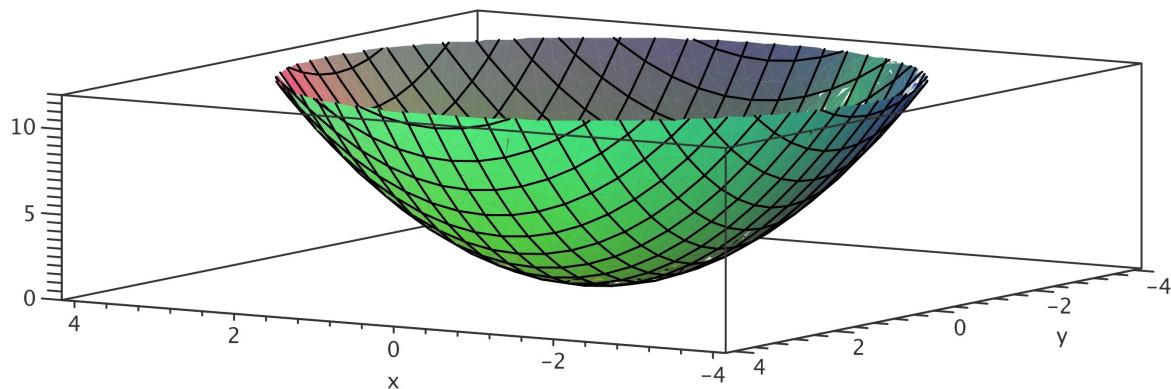
3. SURFACE INTEGRALS

3.1 Parameterized Surfaces

- Much of the following is a review from the Geometry course and will, at the most, be quickly reviewed in this course.

We will now be interested in finding integrals over surfaces. We begin by considering different ways to represent surfaces.

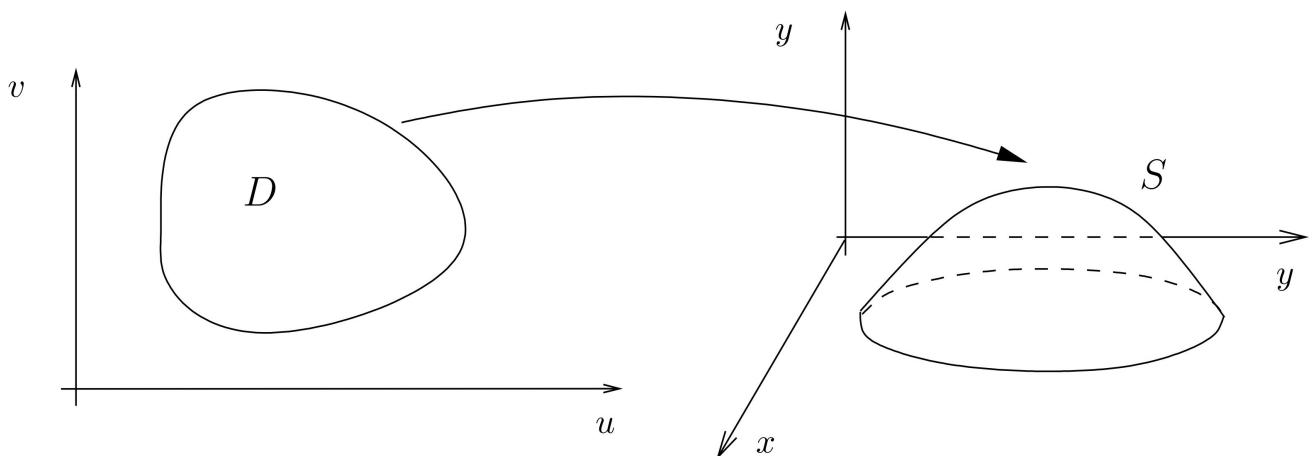
Representation of surfaces



5. A surface as a graph

Cartesian representation

In Cartesian coordinates we represent a surface by $z = f(x, y)$, e.g. the paraboloid $z = x^2 + y^2$.



6. A surface parametrized

Parametric representation

In parametric coordinates we represent a surface by $\mathbf{r} = \mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$ where $(u, v) \in D \subset \mathbb{R}^2$. So $\mathbf{r} = \mathbf{r}(u, v)$ maps D in the (u, v) plane to a surface S in \mathbb{R}^3 .

Note If $z = f(x, y)$ we can use x and y as parameters: $\mathbf{r}(x, y) = (x, y, f(x, y))$.

To represent the point $P = (x, y, z)$ in spherical polar coordinates (r, θ, ϕ) we have

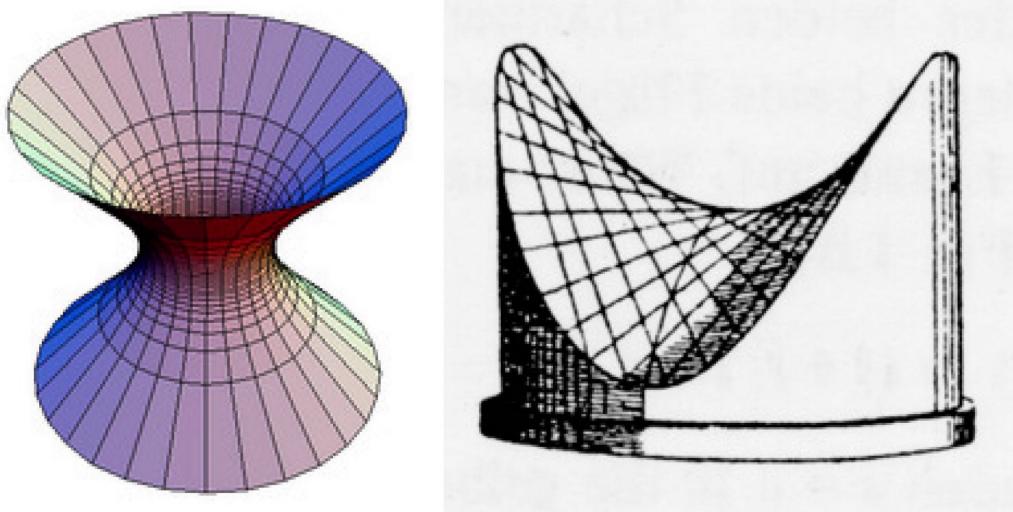
$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta,$$

where $\theta \in [0, \pi]$, $\phi \in [0, 2\pi)$. Thus we can parameterize the sphere $x^2 + y^2 + z^2 = a^2$ in $u = \theta$, $v = \phi$ by

$$\mathbf{r} = \mathbf{r}(\theta, \phi) = (a \sin \theta \cos \phi, a \sin \theta \sin \phi, a \cos \theta).$$

Example 33 The quadrics are standard parameterized surfaces:

- **Sphere:** $x^2 + y^2 + z^2 = a^2$;
- **Ellipsoid:** $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$;
- **Hyperboloid of One Sheet:** $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$;
- **Hyperboloid of Two Sheets:** $x^2/a^2 - y^2/b^2 - z^2/c^2 = 1$;
- **Paraboloid:** $z = x^2 + y^2$;
- **Hyperbolic Paraboloid:** $z = x^2 - y^2$;
- **Cone:** $x^2 + y^2 = z^2$.



7. One-sheet hyperboloid; hyperbolic paraboloid

Definition 34 A smooth parameterized surface is a map \mathbf{r} , known as the parameterization

$$\mathbf{r}: U \rightarrow \mathbb{R}^3, \quad (u, v) \mapsto (x(u, v), y(u, v), z(u, v))$$

from an open subset $U \subseteq \mathbb{R}^2$ to \mathbb{R}^3 such that

- x, y, z have continuous partial derivatives with respect to u and v of all orders;
- \mathbf{r} is a bijection between U and $\mathbf{r}(U)$ with \mathbf{r}^{-1} continuous and also possessing continuous partial derivatives of all orders;
- at each point the vectors

$$\frac{\partial \mathbf{r}}{\partial u} \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial v}$$

are linearly independent (i.e. are not scalar multiples of one another). Equivalently,

$$\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \neq \mathbf{0}.$$

We will not treat this definition with any generality. We shall simply parameterize some of the “standard” surfaces previously described.

Definition 35 Let $\mathbf{r}: U \rightarrow \mathbb{R}^3$ be a smooth parameterized surface and let \mathbf{p} be a point on the surface. The plane containing \mathbf{p} and which is parallel to the vectors

$$\frac{\partial \mathbf{r}}{\partial u}(\mathbf{p}) \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial v}(\mathbf{p})$$

$\partial \mathbf{r} / \partial u$ and $\partial \mathbf{r} / \partial v$ is called the **tangent plane** to $\mathbf{r}(U)$ at \mathbf{p} . Because these vectors are independent the tangent plane is well-defined, and can also be shown to be independent of the choice of parameterization.

Any vector in the direction

$$\frac{\partial \mathbf{r}}{\partial u}(\mathbf{p}) \wedge \frac{\partial \mathbf{r}}{\partial v}(\mathbf{p})$$

is said to be **normal** to the surface at \mathbf{p} . There are two **unit normals** of length one and we will write \mathbf{n} or $\mathbf{n}(\mathbf{p})$ for a choice of unit normal; having made that choice the other unit normal is $-\mathbf{n}$.

3.2 Surface Integrals

Let $\mathbf{r}: U \rightarrow \mathbb{R}^3$ be a smooth parameterized surface with

$$\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$$

and consider the small element of the plane that is bounded by the co-ordinate lines $u = u_0$ and $u = u_0 + \delta u$ and $v = v_0$ and $v = v_0 + \delta v$. Then \mathbf{r} maps this to a small region of the surface $\mathbf{r}(U)$ and we are interested in calculating the surface area of this small region. Note

$$\mathbf{r}(u + \delta u, v) - \mathbf{r}(u, v) \approx \frac{\partial \mathbf{r}}{\partial u}(u, v) \delta u,$$

$$\mathbf{r}(u, v + \delta v) - \mathbf{r}(u, v) \approx \frac{\partial \mathbf{r}}{\partial v}(u, v) \delta v.$$

Recall that the area of a parallelogram with sides \mathbf{a} and \mathbf{b} is $|\mathbf{a} \wedge \mathbf{b}|$. So the element of surface area we are considering is approximately

$$\left| \frac{\partial \mathbf{r}}{\partial u} \delta u \wedge \frac{\partial \mathbf{r}}{\partial v} \delta v \right| = \left| \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right| \delta u \delta v.$$

Thus, at this point we proceed as with double integrals. Partitioning U by smaller and smaller elements, of area $\delta A = \delta u \delta v$, we have in the limit $\delta A \rightarrow 0$

$$\lim_{\text{elements}} \sum \left| \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right| \delta u \delta v \stackrel{\text{def}}{=} \iint_U \left| \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right| du dv.$$

This gives the surface area of the parameterized surface; as with double integrals the pre-limit summation can be weighted with a scalar function $\psi(\mathbf{r}(u, v))$ evaluated at the centre of the elements to yield

$$\iint_U \psi(\mathbf{r}(u, v)) \left| \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right| du dv.$$

Definition 36 We will often write

$$dS = \left| \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right| du dv$$

to denote an infinitesimal part of surface area. We will also write

$$d\mathbf{S} = \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} du dv$$

This is also commonly written as $\mathbf{n} dS$ where \mathbf{n} is the choice of unit normal in the direction of $\partial \mathbf{r}/\partial u \wedge \partial \mathbf{r}/\partial v$.

Proposition 37 The surface area of $\mathbf{r}(U)$ is independent of the choice of parameterization.

Proof. Let $\Sigma = \mathbf{r}(U) = \mathbf{s}(W)$ be two different parameterizations of a surface X ; take u, v as the co-ordinates on U and p, q as the co-ordinates on W . Let $f = (f_1, f_2) : U \rightarrow W$ be the co-ordinate change map – i.e. for any $(u, v) \in U$ we have

$$\mathbf{r}(u, v) = \mathbf{s}(f(u, v)) = \mathbf{s}(f_1(u, v), f_2(u, v)) = \mathbf{s}(p, q).$$

Then

$$\frac{\partial \mathbf{r}}{\partial u} = \frac{\partial \mathbf{s}}{\partial p} \frac{\partial f_1}{\partial u} + \frac{\partial \mathbf{s}}{\partial q} \frac{\partial f_2}{\partial u}, \quad \frac{\partial \mathbf{r}}{\partial v} = \frac{\partial \mathbf{s}}{\partial p} \frac{\partial f_1}{\partial v} + \frac{\partial \mathbf{s}}{\partial q} \frac{\partial f_2}{\partial v}.$$

Hence

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} &= \frac{\partial \mathbf{s}}{\partial p} \frac{\partial f_1}{\partial u} \wedge \frac{\partial \mathbf{s}}{\partial q} \frac{\partial f_2}{\partial v} + \frac{\partial \mathbf{s}}{\partial q} \frac{\partial f_2}{\partial u} \wedge \frac{\partial \mathbf{s}}{\partial p} \frac{\partial f_1}{\partial v} \\ &= \left(\frac{\partial f_1}{\partial u} \frac{\partial f_2}{\partial v} - \frac{\partial f_1}{\partial v} \frac{\partial f_2}{\partial u} \right) \frac{\partial \mathbf{s}}{\partial p} \wedge \frac{\partial \mathbf{s}}{\partial q} \\ &= \frac{\partial(p, q)}{\partial(u, v)} \frac{\partial \mathbf{s}}{\partial p} \wedge \frac{\partial \mathbf{s}}{\partial q}. \end{aligned}$$

Finally

$$\begin{aligned} \iint_U \left| \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right| du dv &= \iint_U \left| \frac{\partial(p, q)}{\partial(u, v)} \frac{\partial \mathbf{s}}{\partial p} \wedge \frac{\partial \mathbf{s}}{\partial q} \right| du dv \\ &= \iint_U \left| \frac{\partial \mathbf{s}}{\partial p} \wedge \frac{\partial \mathbf{s}}{\partial q} \right| \left| \frac{\partial(p, q)}{\partial(u, v)} \right| du dv \\ &= \iint_W \left| \frac{\partial \mathbf{s}}{\partial p} \wedge \frac{\partial \mathbf{s}}{\partial q} \right| dp dq \end{aligned}$$

by the two-dimensional rule for the change of variables in integrals. ■

Remark 38 General method of evaluation of surface integrals

1. find a suitable parameterization, in terms of some u, v ;
2. find the domain of u, v , a set $U \subseteq \mathbb{R}^2$;
3. evaluate $\partial \mathbf{r}/\partial u \wedge \partial \mathbf{r}/\partial v$ for $(u, v) \in U$;
4. substitute into the relevant repeated integral and evaluate that integral.

Definition 39 If \mathbf{F} and ϕ are a vector field and scalar field defined on a parameterized surface $\Sigma = \mathbf{r}(U)$, then we may define the following **surface integrals**:

$$\begin{aligned} \iint_{\Sigma} \mathbf{F} \cdot d\mathbf{S} &= \iint_U \mathbf{F}(\mathbf{r}(u, v)) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right) du dv; \\ \iint_{\Sigma} \mathbf{F} dS &= \iint_U \mathbf{F}(\mathbf{r}(u, v)) \left| \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right| du dv; \\ \iint_{\Sigma} \phi d\mathbf{S} &= \iint_U \phi(\mathbf{r}(u, v)) \left(\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right) du dv; \\ \iint_{\Sigma} \phi dS &= \iint_U \phi(\mathbf{r}(u, v)) \left| \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right| du dv. \end{aligned}$$

Remark 40 When the surface Σ encloses a 3D region such as the surface of the unit sphere, the standard convention is that

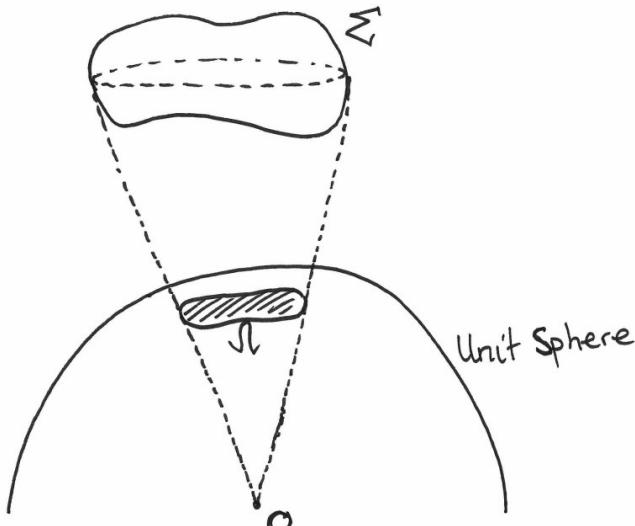
$$d\mathbf{S} = \mathbf{n} dS$$

has the normal, \mathbf{n} pointing **outwards** from the body.

We will most commonly meet integrals of the first type and fourth types. The first type of integral

$$\iint_{\Sigma} \mathbf{F} \cdot d\mathbf{S}$$

which is known as a **flux integral**.



8. Solid angle

Definition 41 The **solid angle** is the angle an object subtends at a point in three-dimensional space. More precisely, half-lines from a fixed point (or observer) will either intersect with the object in question or not; those lines of sight that are blocked by the object represent a subset of the unit sphere centred on the observer. The solid angle is the area of this subset (strictly it is the area of this subset divided by the unit of length squared to ensure the solid angle is dimensionless). The unit of solid angle is the **steradian**. Given that the surface area of a sphere is $4\pi(\text{radius})^2$ then a whole solid angle is 4π .

If Σ^* is a surface and Σ is the subset of Σ^* facing the unit sphere, then the solid angle Ω subtended at O by Σ^* equals, by definition,

$$\Omega = \iint_{\Sigma} \frac{\mathbf{e}_r \cdot d\mathbf{S}}{r^2} = \iint_{\Sigma} \frac{\mathbf{r} \cdot d\mathbf{S}}{r^3}.$$

Remark 42 If an element of surface area is viewed "full-on" at distance 1 away then it obscures a solid angle of dS . In this case \mathbf{e}_r and \mathbf{n} are parallel so that $\mathbf{e}_r \cdot d\mathbf{S} = dS$. However that same area might be oblique to the viewer and so only a fraction/component of the potential solid angle is obscured. In the extreme when \mathbf{e}_r and \mathbf{n} are perpendicular so that $\mathbf{e}_r \cdot d\mathbf{S} = 0$, and the area obscures none of the view. The term r^{-2} represents how a sphere's area grows proportionately to r^2 . The same area, at distance r now, will obscure r^{-2} of its previous solid angle.

Example 43 Find the solid angle at the apex of a right pyramid with square base of side $2d$ and height h .

Solution. Place the apex of the pyramid at the origin and orientate the axis of the pyramid along the positive z -axis so that the square base of the pyramid has vertices $(\pm d, \pm d, h)$. By symmetry the solid angle at the apex is 4 times the solid angle subtended by the smaller square with vertices

$$(0, 0, h), \quad (0, d, h), \quad (d, d, h) \quad (0, d, h).$$

Hence the solid angle is

$$\Omega = 4 \int_{y=0}^d \int_{x=0}^d \frac{(x, y, h) \cdot \mathbf{k} dx dy}{(x^2 + y^2 + h^2)^{3/2}} = 4h \int_{y=0}^d \int_{x=0}^d \frac{dx dy}{(x^2 + y^2 + h^2)^{3/2}}.$$

If we set $x = \sqrt{y^2 + h^2} \tan t$ and $\tan \tau = d / (\sqrt{y^2 + h^2})$ then

$$\begin{aligned} \Omega &= 4h \int_{y=0}^d \int_{t=0}^{\tau} \frac{\sqrt{y^2 + h^2} \sec^2 t dt dy}{((y^2 + h^2) \tan^2 t + y^2 + h^2)^{3/2}} \\ &= 4h \int_{y=0}^d \int_{t=0}^{\tau} \frac{\sqrt{y^2 + h^2} \sec^2 t dt dy}{(y^2 + h^2)^{3/2} (\tan^2 t + 1)^{3/2}} \\ &= 4h \int_{y=0}^d \int_{t=0}^{\tau} \frac{\cos t dt dy}{(y^2 + h^2)} = 4h \int_{y=0}^d \frac{\sin \tau dy}{(y^2 + h^2)} \\ &= 4hd \int_{y=0}^d \frac{dy}{(y^2 + h^2) \sqrt{y^2 + h^2 + d^2}} \end{aligned}$$

as $\sin \tau = d(y^2 + h^2 + d^2)^{-1/2}$.

If we make a similar substitution again, namely $y = \sqrt{h^2 + d^2} \tan \phi$ and $\tan \alpha = d / \sqrt{h^2 + d^2}$ then

$$\begin{aligned} \Omega &= 4hd \int_{\phi=0}^{\alpha} \frac{\sqrt{h^2 + d^2} \sec^2 \phi d\phi}{((h^2 + d^2) \tan^2 \phi + h^2) \sqrt{h^2 + d^2} \sec \phi} \\ &= 4hd \int_{\phi=0}^{\alpha} \frac{\cos \phi d\phi}{((h^2 + d^2) \sin^2 \phi + h^2 \cos^2 \phi)} \\ &= 4hd \int_{\phi=0}^{\alpha} \frac{\cos \phi d\phi}{d^2 \sin^2 \phi + h^2}. \end{aligned}$$

Our final substitution is $u = \sin \phi$ so that $\sin \alpha = d / \sqrt{h^2 + 2d^2}$ and

$$\Omega = 4hd \int_{u=0}^{\sin \alpha} \frac{du}{d^2 u^2 + h^2} = \frac{4h}{d} \left[\frac{d}{h} \tan^{-1} \left(\frac{du}{h} \right) \right]_0^{\sin \alpha} = 4 \tan^{-1} \left(\frac{d^2}{h \sqrt{h^2 + 2d^2}} \right).$$

■

Example 44 Evaluate the integral

$$\iint_{\Sigma} \mathbf{F} \wedge d\mathbf{S}$$

where in each case Σ is the closed hemispherical surface made up of points (x, y, z) such that either $x^2 + y^2 + z^2 = 1$ and $z > 0$, or $x^2 + y^2 \leq 1$ and $z = 0$; orient Σ so that $d\mathbf{S}$ is in the direction of the outward-pointing normal and $\mathbf{F} = (zx, zy, z^2)$.

Solution. We could proceed to parameterize the two parts of Σ , the upper part of the hemispherical surface Σ_1 and the planar disc Σ_2 . These can be respectively parameterized as

$$\begin{aligned}\mathbf{r}_1(\theta, \phi) &= (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \quad 0 \leq \phi \leq 2\pi, 0 \leq \theta \leq \pi/2; \\ \mathbf{r}_2(r, \theta) &= (r \cos \theta, r \sin \theta, 0) \quad 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1.\end{aligned}$$

With some further calculation we would find

$$\frac{\partial \mathbf{r}_1}{\partial \theta} \wedge \frac{\partial \mathbf{r}_1}{\partial \phi} = \sin \theta (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \quad \frac{\partial \mathbf{r}_1}{\partial r} \wedge \frac{\partial \mathbf{r}_1}{\partial \theta} = (0, 0, r).$$

In order to get the correct outward-pointing direction we need to set

$$d\mathbf{S} = \sin \theta (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) d\theta d\phi, \quad d\mathbf{S} = -r\mathbf{k},$$

respectively.

However, if we stop to think a little, we can straight away see that ... On Σ_1 we have $\mathbf{F} = z\mathbf{r}$ and $\mathbf{n} = \mathbf{r}$ so that $\mathbf{F} \wedge d\mathbf{S} = z\mathbf{r} \wedge \mathbf{n} dS = z\mathbf{r} \wedge \mathbf{r} dS = \mathbf{0}$ on all of Σ_1 , and further as $z = 0$ on all of Σ_2 then it's also true that $\mathbf{F} = \mathbf{0}$ on all of Σ_2 .

Moral of the story: take some time to consider (i) the nature of your function and the region, (ii) what integrals need to be calculated, and (iii) what co-ordinates are best for the problem under consideration. ■

Remark 45 (A point of notation in relation to sheet 3). In question 5, $\partial\Sigma$ refers to the boundary of Σ , which is formed from three curves which can be parameterized

$$\begin{aligned}C_1 &= \{(t, 0, t) : 0 \leq t \leq 2\}, \\ C_2 &= \{(2 \cos t, 2 \sin t, 2) : 0 \leq t \leq \pi\}, \\ C_3 &= \{(2 - t)(-1, 0, 1) : 0 \leq t \leq 2\}.\end{aligned}$$

Be sure to orient the curves consistently in a loop to find $\int_{\partial\Sigma} f d\mathbf{r}$ (which is defined only up to sign).

4. LINE INTEGRALS & CONSERVATIVE FIELDS

4.1 Curves

Definition 46 By a *curve* we shall mean a piecewise smooth function $\gamma: I \rightarrow \mathbb{R}^3$ defined on an interval I of \mathbb{R} . Notice that order on I also gives the curve γ an *orientation*.

We shall also use the term curve to describe the images of such maps γ . Given such an image then it will be the image of more than one such map γ and we will talk about **parameterizations** γ_1 and γ_2 of the curve. These parameterizations of the image come in two different possible orientations.

Definition 47 We say a curve $\gamma: [a, b] \rightarrow \mathbb{R}^3$ is **simple** if γ is 1–1, with the one possible exception that $\gamma(a) = \gamma(b)$ may be true; this means that the curve does not cross itself except possibly by its endpoints meeting.

Definition 48 We say a curve $\gamma: [a, b] \rightarrow \mathbb{R}^3$ is **closed** if $\gamma(a) = \gamma(b)$.

Example 49 The *line* through points \mathbf{p} and \mathbf{q} can be parameterized as

$$\gamma(t) = \mathbf{p} + t(\mathbf{q} - \mathbf{p}).$$

When $0 < t < 1$ then $\gamma(t)$ lies between \mathbf{p} and \mathbf{q} , for $t > 1$ beyond \mathbf{q} and for $t < 0$ before \mathbf{p} .

Example 50 A curve of the form

$$\gamma(t) = (a \cos t, a \sin t, ct) \tag{4.1}$$

is known as a *circular helix*.

Exercise 51 The curve Γ is parameterized by

$$\Gamma(u) = (u + \sqrt{3} \sin u, 2 \cos u, \sqrt{3}u - \sin u).$$

Find a circular helix of the form (4.1) and an isometry T such that $T\gamma = \Gamma$.

Example 52 Show that the plane with equation $Ax + By + Cz = D$ intersects the unit sphere $x^2 + y^2 + z^2 = 1$ in a circle if and only if $A^2 + B^2 + C^2 > D^2$. Parameterize the intersection of $x + y + z = 1$ with the unit sphere.

Solution. The plane $Ax + By + Cz = D$ has normal (A, B, C) and so the point closest to the origin is the point with position vector $\lambda(A, B, C)$ which lies on the plane; by substitution, we see $\lambda = D / (A^2 + B^2 + C^2)$. This point is within unit distance of the origin if and only if

$$1 > \left| \frac{D}{A^2 + B^2 + C^2} (A, B, C) \right| = \frac{|D| \sqrt{A^2 + B^2 + C^2}}{A^2 + B^2 + C^2}$$

i.e. if and only if $A^2 + B^2 + C^2 > D^2$.

By this criterion the plane $x + y + z = 1$ intersects with the unit sphere. The centre of the circle which makes the intersection is at

$$\frac{1}{1^2 + 1^2 + 1^2} (1, 1, 1) = \frac{(1, 1, 1)}{3}.$$

By Pythagoras' Theorem the radius r of the circle satisfies

$$r^2 + \left| \frac{(1, 1, 1)}{3} \right|^2 = 1 \implies r = \sqrt{\frac{2}{3}}.$$

As $\mathbf{e}_1 = (1, -1, 0) / \sqrt{2}$ and $\mathbf{e}_2 = (1, 1, -2) / \sqrt{6}$ are two orthonormal vectors parallel to the plane then every point of the circle can be written in the form $\gamma(t)$ for $0 \leq t < 2\pi$ where

$$\begin{aligned} \gamma(t) &= \frac{(1, 1, 1)}{3} + \mathbf{e}_1 \sqrt{\frac{2}{3}} \cos t + \mathbf{e}_2 \sqrt{\frac{2}{3}} \sin t \\ &= \left(\frac{1}{3} + \frac{1}{\sqrt{3}} \cos t + \frac{1}{3} \sin t, \frac{1}{3} - \frac{1}{\sqrt{3}} \cos t + \frac{1}{3} \sin t, \frac{1}{3} - \frac{2}{3} \sin t \right). \end{aligned}$$

■

4.2 Line Integrals

Definition 53 Let C be a curve in \mathbb{R}^3 , parameterized by $\gamma: [a, b] \rightarrow \mathbb{R}^3$ and let \mathbf{F} be a vector field, whose domain includes C . We define the **line integral of \mathbf{F} along C** as

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt.$$

Proposition 54 If oriented the same, the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of the choice of parameterization.

Proof. Suppose that $\gamma_1 : [a_1, b_1] \rightarrow \mathbb{R}^3$ and $\gamma_2 : [a_2, b_2] \rightarrow \mathbb{R}^3$ are two parameterizations of C with $\gamma_1(a_1) = \gamma_2(a_2)$ and $\gamma_1(b_1) = \gamma_2(b_2)$, so that γ_1 and γ_2 give C the same orientation.

Then $\gamma_2 = \gamma_1 \circ \psi$ where $\psi : [a_2, b_2] \rightarrow [a_1, b_1]$ associates the γ_2 co-ordinates of points on C with their γ_1 co-ordinate.

We now define $I : [a_1, b_1] \rightarrow \mathbb{R}$ and $J : [a_2, b_2] \rightarrow \mathbb{R}$ by

$$I(t) = \int_{a_1}^t \mathbf{F}(\gamma_1(s)) \cdot \gamma'_1(s) \, ds, \quad J(t) = \int_{a_2}^t \mathbf{F}(\gamma_2(s)) \cdot \gamma'_2(s) \, ds.$$

By the Fundamental Theorem of Calculus

$$I'(t) = \mathbf{F}(\gamma_1(t)) \cdot \gamma'_1(t), \quad J'(t) = \mathbf{F}(\gamma_2(t)) \cdot \gamma'_2(t).$$

Further, for $\psi(t)$ a function of t we have, by the chain rule,

$$\begin{aligned} \frac{d}{dt} I(\psi(t)) &= \psi'(t) I'(\psi(t)) \\ &= \psi'(t) \mathbf{F}(\gamma_1(\psi(t))) \cdot \gamma'_1(\psi(t)). \end{aligned}$$

Recall

$$\gamma_2(t) = \gamma_1(\psi(t)).$$

Thus $\gamma'_2(t) = \psi'(t)\gamma'_1(\psi(t))$ and hence

$$\begin{aligned} \frac{d}{dt} I(\psi(t)) &= \mathbf{F}(\gamma_2(t)) \cdot (\gamma'_1(\psi(t))\psi'(t)) \\ &= \mathbf{F}(\gamma_2(t)) \cdot \gamma'_2(t) = J'(t). \end{aligned}$$

It follows that $I(\psi(t))$ and $J(t)$ differ by a constant and, as they agree at $t = a_2$ (when they are both zero), then $I(\psi(t)) = J(t)$ and in particular when $t = b_2$ we have

$$\int_{a_1}^{b_1} \mathbf{F}(\gamma_1(s)) \cdot \gamma'_1(s) \, ds = \int_{a_2}^{b_2} \mathbf{F}(\gamma_2(s)) \cdot \gamma'_2(s) \, ds.$$

■

Remark 55 If we parameterized C in the reverse orientation then the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ would give negative what had been previously calculated.

Example 56 Let $\mathbf{F} = \mathbf{c} \wedge \mathbf{r}$ where \mathbf{c} is a constant vector and let C be the circular helix parameterized by

$$\mathbf{r}(t) = (\cos t, \sin t, t), \quad 0 \leq t \leq 2\pi.$$

Then

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ c_1 & c_2 & c_3 \\ \cos t & \sin t & t \end{vmatrix} \cdot (-\sin t, \cos t, 1) dt \\
&= \int_0^{2\pi} \begin{vmatrix} -\sin t & \cos t & 1 \\ c_1 & c_2 & c_3 \\ \cos t & \sin t & t \end{vmatrix} dt \\
&= \int_0^{2\pi} (-c_2 t \sin t + c_3 \cos^2 t + c_1 \sin t - c_2 \cos t + c_3 \sin^2 t - c_1 t \cos t) dt \\
&= \int_0^{2\pi} (-c_2 t \sin t + c_3 - c_1 t \cos t) dt \\
&= 2\pi c_3 - c_1 [t \sin t + \cos t]_0^{2\pi} - c_2 [-t \cos t + \sin t]_0^{2\pi} \\
&= 2\pi (c_2 + c_3).
\end{aligned}$$

Definition 57 If ϕ is a scalar field defined on a curve C with parameterization $\gamma: [a, b] \rightarrow \mathbb{R}^3$, then we also define the line integral

$$\int_C \phi ds = \int_a^b \phi(t) |\gamma'(t)| dt.$$

If $\mathbf{F} = (F_1, F_2, F_3)$ is a vector field defined on the curve C then we define

$$\int_C \mathbf{F} ds = \left(\int_C F_1 ds, \int_C F_2 ds, \int_C F_3 ds \right).$$

Remark 58 Note that if \mathbf{t} is the unit tangent vector field along C , in the same direction as the parameterization, then

$$\int_C \phi ds = \int_C (\phi \mathbf{t}) \cdot d\mathbf{r}$$

and so, by inheritance, these line integrals do not depend on the choice of parameterization. Note further that the two types of integral defined above are also independent of the choice of orientation of C , as they are independent of the sign of $\gamma'(t)$.

Definition 59 If C is a curve with parameterization $\gamma: [a, b] \rightarrow \mathbb{R}^3$ then the **arc length** of the curve is

$$\int_C ds = \int_a^b |\gamma'(t)| dt.$$

Example 60 Find the arc length of the circular helix $\mathbf{r}(t) = (\cos t, \sin t, t)$ where $0 \leq t \leq 2\pi$.

Solution. We have

$$s = \int_0^{2\pi} |(-\sin t, \cos t, 1)| dt = \int_0^{2\pi} \sqrt{1 + \sin^2 t + \cos^2 t} dt = \int_0^{2\pi} \sqrt{2} dt = 2\sqrt{2}\pi.$$

We could also parameterize the helix "in reverse" by setting $\mathbf{s}(t) = \mathbf{r}(2\pi - t) = (\cos t, -\sin t, 2\pi - t)$ where $0 \leq t \leq 2\pi$. We would still find

$$s = \int_0^{2\pi} |(-\sin t, -\cos t, -1)| dt = \int_0^{2\pi} \sqrt{1 + \sin^2 t + \cos^2 t} dt = \int_0^{2\pi} \sqrt{2} dt = 2\sqrt{2}\pi.$$

■

Notation 61 We will standardly use the notation

$$d\mathbf{r} = (dx, dy, dz) \quad \text{and} \quad ds = |\mathbf{dr}|,$$

even though this may seem a little non-rigorous and any analysis course would insist on such differentials only appearing as part of a limit or within an integral. Rest assured that these differentials can be rigorously defined, though doing so is not a primary concern of this course.

Definition 62 With notation as in the Definition 53, if the vector field \mathbf{F} represents a force on a particle then

$$\int_C \mathbf{F} \cdot d\mathbf{r} \tag{4.2}$$

is the **work** done by the force in moving the particle along C .

This is a generalization of the formula

$$\text{Work} = \text{Force} \times \text{Distance}$$

which applies to constant forces acting parallel to the direction of the motion. More generally, we have

$$\text{Work} = \text{Component of force in direction of travel} \times \text{Distance}$$

if the force and movement are not parallel. The work integral (4.2) is just an integral of such infinitesimal contributions of work.

Example 63 Show that the work done by gravity, $\mathbf{F} = -mg\mathbf{k}$, in moving a particle along a straight line from (x_1, y_1, z_1) to (x_2, y_2, z_2) equals $mg(z_1 - z_2)$.

Solution. We can parameterize the line segment as

$$\mathbf{r}(t) = (x_1, y_1, z_1) + t(x_2 - x_1, y_2 - y_1, z_2 - z_1), \quad 0 \leq t \leq 1,$$

so that

$$d\mathbf{r} = (x_2 - x_1, y_2 - y_1, z_2 - z_1) dt.$$

Then

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 -mg\mathbf{k} \cdot (x_2 - x_1, y_2 - y_1, z_2 - z_1) dt \\ &= \int_0^1 -mg(z_2 - z_1) dt \\ &= mg(z_1 - z_2). \end{aligned}$$

In fact, the work done by gravity – which is positive if $z_1 > z_2$, i.e. the particle has dropped – is the loss in gravitational potential energy which, commonly, will have been converted into kinetic energy. The work done is dependent only on the endpoints (x_1, y_1, z_1) and (x_2, y_2, z_2) and is independent of the path taken between these endpoints. This is common to certain types of field which are known as *conservative*.

4.3 Conservative Fields

Proposition 64 *If $\mathbf{F} = \nabla\phi$ and $\gamma : [a, b] \rightarrow S$ is any curve such that $\gamma(a) = \mathbf{p}$, $\gamma(b) = \mathbf{q}$ then*

$$\int_{\gamma} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \phi(\mathbf{q}) - \phi(\mathbf{p}).$$

In particular, the integral $\int_{\gamma} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ depends only on the endpoints of the curve γ .

Proof. If $\gamma(t) = (x(t), y(t), z(t))$ then

$$\begin{aligned} \int_{\gamma} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\gamma(t)) \cdot \gamma'(t) dt \\ &= \int_a^b \nabla\phi(\gamma(t)) \cdot \gamma'(t) dt \\ &= \int_a^b \left(\frac{\partial\phi}{\partial x} \frac{dx}{dt} + \frac{\partial\phi}{\partial y} \frac{dy}{dt} + \frac{\partial\phi}{\partial z} \frac{dz}{dt} \right) dt \\ &= \int_a^b \left[\frac{d}{dt} (\phi(\gamma(t))) \right] dt \quad [\text{by the chain rule}] \\ &= \phi(\gamma(b)) - \phi(\gamma(a)) \\ &= \phi(\mathbf{q}) - \phi(\mathbf{p}). \end{aligned}$$

Motivated by the above proposition, we now introduce the following definitions:

Definition 65 *Let S be an open subset of \mathbb{R}^3 . A vector field $\mathbf{F}: S \rightarrow \mathbb{R}^3$ is said to be **conservative** if there exists a scalar field $\phi: S \rightarrow \mathbb{R}$ such that $\mathbf{F} = \nabla\phi$.*

Definition 66 *If $\mathbf{F} = \nabla\phi$ then ϕ is said to be a **potential** (or **scalar potential**) for \mathbf{F} .*

Definition 67 *A subset S of \mathbb{R}^3 is said to be **path-connected** if for any points $\mathbf{p}, \mathbf{q} \in S$ there exists a curve $\gamma: [a, b] \rightarrow S$ such that $\gamma(a) = \mathbf{p}$ and $\gamma(b) = \mathbf{q}$.*

Corollary 68 *If $\nabla\phi = 0$ on a path-connected set S then ϕ is constant. In particular, if ϕ and ψ are potentials of the conservative field \mathbf{F} , defined on S , then ϕ and ψ differ by a constant.*

Proof. If $\nabla\phi = 0$ then for any curve, with endpoints \mathbf{p}, \mathbf{q} , we have

$$\phi(\mathbf{q}) - \phi(\mathbf{p}) = \int_{\gamma} \nabla\phi \cdot d\mathbf{r} = 0.$$

Given a fixed point \mathbf{p} in S , any \mathbf{q} in S is connected to \mathbf{p} by a curve, and so ϕ is constant. If $\mathbf{F} = \nabla\phi = \nabla\psi$ then $\nabla(\phi - \psi) = 0$ and the result follows. ■

Theorem 69 Let S be an open path connected subset of \mathbb{R}^3 and let $\mathbf{F} : S \rightarrow \mathbb{R}^3$ be a vector field. Then the following three statements are equivalent.

(i) \mathbf{F} is conservative – i.e. $\mathbf{F} = \nabla\phi$ for some scalar field $\phi : S \rightarrow \mathbb{R}$.

(ii) Given any two points $\mathbf{p}, \mathbf{q} \in S$ and curve γ in S , starting at \mathbf{p} and ending at \mathbf{q} , then the integral

$$\int_{\gamma} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$$

is independent of the choice of curve γ .

(iii) For any simple closed curve γ then

$$\int_{\gamma} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = 0.$$

Proof. (i) \implies (ii) : This was just proved in Proposition 64 as $\phi(\mathbf{q}) - \phi(\mathbf{p})$ is dependent only on the endpoints \mathbf{p} and \mathbf{q} and not on the path taken.

(ii) \implies (i) : Let $\mathbf{p} \in S$ be a fixed point and define for any $\mathbf{q} \in S$

$$\phi(\mathbf{q}) = \int_{\mathbf{p}}^{\mathbf{q}} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$$

which, by assumption, is independent of the curve taken and is a function only of \mathbf{q} . As S is open there is an $r > 0$ small enough that $\mathbf{q} + t\mathbf{i}$ is in S for $0 \leq t \leq r$. Then

$$\phi(\mathbf{q} + t\mathbf{i}) - \phi(\mathbf{q}) = \int_{\mathbf{q}}^{\mathbf{q}+t\mathbf{i}} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$$

where the curve from \mathbf{q} to $\mathbf{q} + t\mathbf{i}$ can be taken to be a straight line. So

$$\phi(\mathbf{q} + t\mathbf{i}) - \phi(\mathbf{q}) = \int_{s=0}^{s=t} \mathbf{F}(\mathbf{q} + s\mathbf{i}) \cdot (\mathbf{i} ds) = \int_{s=0}^{s=t} F_1(\mathbf{q} + s\mathbf{i}) ds$$

where $\mathbf{F} = (F_1, F_2, F_3)$. Hence

$$\frac{\partial\phi}{\partial x}(\mathbf{q}) = \lim_{t \rightarrow 0} \frac{\phi(\mathbf{q} + t\mathbf{i}) - \phi(\mathbf{q})}{t} = \lim_{t \rightarrow 0} \frac{1}{t} \int_{s=0}^{s=t} F_1(\mathbf{q} + s\mathbf{i}) ds = F_1(\mathbf{q})$$

by the continuity of F_1 at \mathbf{q} . Similarly

$$\frac{\partial\phi}{\partial y}(\mathbf{q}) = F_2(\mathbf{q}), \quad \frac{\partial\phi}{\partial z}(\mathbf{q}) = F_3(\mathbf{q})$$

and so $\nabla\phi = \mathbf{F}$ as required.

(ii) \implies (iii) : Suppose now that for any two points \mathbf{p} and \mathbf{q} the line integral $\int_{\mathbf{p}}^{\mathbf{q}} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ is independent of the curve taken.

Let γ be a simple closed curve in S and let \mathbf{p} and \mathbf{q} be two distinct points on γ . Then \mathbf{p} and \mathbf{q} split γ into two curves γ_1 and γ_2 , with γ_1 running from \mathbf{p} to \mathbf{q} and γ_2 running from \mathbf{q} to \mathbf{p} . So, by (ii) and Remark 55,

$$\int_{\gamma} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_{\gamma_1} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} + \int_{\gamma_2} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_{\mathbf{p}}^{\mathbf{q}} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} - \int_{\mathbf{p}}^{\mathbf{q}} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = 0$$

(iii) \implies (ii) : Let $\mathbf{p}, \mathbf{q} \in S$ and let γ_1, γ_2 be two curves in S , starting at \mathbf{p} and ending at \mathbf{q} . If γ is the curve which follows γ_1 and comes back around γ_2 , so that we have returned to \mathbf{p} then, by assumption and Remark 55,

$$0 = \int_{\gamma} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_{\gamma_1} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} - \int_{\gamma_2} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} \implies \int_{\gamma_1} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_{\gamma_2} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$$

and the line integral of \mathbf{F} from \mathbf{p} to \mathbf{q} is independent of the choice of path taken. ■

5. DIV, GRAD AND CURL

Recall by a **scalar field** ϕ on \mathbb{R}^3 we shall mean a map $\phi: \mathbb{R}^3 \rightarrow \mathbb{R}$ and by a **vector field** \mathbf{F} on \mathbb{R}^3 we shall mean a map $\mathbf{F}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$.

We shall also consider scalar and vector fields defined on proper subsets of \mathbb{R}^3 (or more generally \mathbb{R}^n). The domains of these fields will usually be *open*, so that we can define their partial derivatives.

5.1 Definitions and Properties

Definition 70 Let $\phi: \mathbb{R}^3 \rightarrow \mathbb{R}$ be a scalar field. Then the **gradient** of ϕ written $\text{grad } \phi$ or $\nabla \phi$ equals

$$\text{grad } \phi = \nabla \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right).$$

Note grad takes scalar fields to vector fields.

Definition 71 Let $\mathbf{F}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a vector field with $\mathbf{F} = (F_1, F_2, F_3)$. Then the **divergence** of \mathbf{F} written $\text{div } \mathbf{F}$ or $\nabla \cdot \mathbf{F}$ equals

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Note div takes vector fields to scalar fields.

Definition 72 Let $\mathbf{F}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a vector field with $\mathbf{F} = (F_1, F_2, F_3)$. Then the **curl** of \mathbf{F} written $\text{curl } \mathbf{F}$ or $\nabla \wedge \mathbf{F}$ equals

$$\text{curl } \mathbf{F} = \nabla \wedge \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\begin{array}{l} \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{array} \right)$$

Note curl takes vector fields to vector fields.

Definition 73 The differential operator

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

is called **del** or **nabla**.

Remark 74 Whilst ∇ looks, and very occasionally behaves, like a standard vector it is unwise to assume this behaviour of ∇ without proof; most standard vector identities are not true when they involve ∇ . For example, the commutativity of the dot product does not extend:

$$\nabla \cdot \mathbf{F} \neq \mathbf{F} \cdot \nabla.$$

The LHS is $\operatorname{div} \mathbf{F}$. At first glance, the RHS doesn't mean anything; it certainly isn't a scalar. Rather

$$\mathbf{F} \cdot \nabla = (F_1, F_2, F_3) \cdot \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = F_1 \frac{\partial}{\partial x} + F_2 \frac{\partial}{\partial y} + F_3 \frac{\partial}{\partial z}$$

which is a differential operator. In fact $\mathbf{F} \cdot \nabla$ is standard notation for it and it gives the directional derivative in the direction of \mathbf{F} if \mathbf{F} is a unit vector.

Definition 75 For any scalar field ϕ we have

$$\operatorname{div}(\operatorname{grad} \phi) = \frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial \phi}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{\partial \phi}{\partial z} \right) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2},$$

which is the **Laplacian**. As

$$\operatorname{div} \operatorname{grad} = \nabla \cdot \nabla$$

then we often write

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}.$$

Also, for vector fields $\mathbf{F} = (F_1, F_2, F_3)$ with respect to a **Cartesian** coordinate system, it is common to write

$$\nabla^2 \mathbf{F} = (\nabla^2 F_1, \nabla^2 F_2, \nabla^2 F_3).$$

Thus the Laplacian can both take scalar fields to scalar fields and vector fields to vector fields.

Remark 76 The notation grad , div , curl and ∇ , $\nabla \cdot$, $\nabla \wedge$ are equally common in mathematical texts, and so these lecture notes and the problem sheets will deliberately make use of both notations.

In summary, we have the following:

operator	maps:
$\operatorname{div} = \nabla \cdot$	vector fields to scalar fields
$\operatorname{grad} = \nabla$	scalar fields to vector fields
$\operatorname{curl} = \nabla \wedge$	vector fields to vector fields
$\operatorname{div} \operatorname{grad} = \nabla^2$	scalar fields to scalar fields vector fields to vector fields
$\mathbf{F} \cdot \nabla$	scalar fields to scalar fields vector fields to vector fields

Example 77 Let $\mathbf{r} = (x, y, z)$ and $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$. Then

$$\begin{aligned}\operatorname{div} \mathbf{r} &= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3; \\ \operatorname{curl} \mathbf{r} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \mathbf{0}.\end{aligned}$$

Example 78 With r and \mathbf{r} as above, and f a differentiable function on $(0, \infty)$ note

$$\begin{aligned}\operatorname{curl}(f(r)\mathbf{r}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(r)x & f(r)y & f(r)z \end{vmatrix} \\ &= \left(zf'(r)\frac{\partial r}{\partial y} - yf'(r)\frac{\partial r}{\partial z}, xf'(r)\frac{\partial r}{\partial z} - zf'(r)\frac{\partial r}{\partial x}, yf'(r)\frac{\partial r}{\partial x} - xf'(r)\frac{\partial r}{\partial y} \right) \\ &= f'(r)\left(\frac{zy}{r} - \frac{yz}{r}, \frac{xz}{r} - \frac{zx}{r}, \frac{yx}{r} - \frac{xy}{r}\right) = \mathbf{0}.\end{aligned}$$

Example 79 Let $\mathbf{c} = (c_1, c_2, c_3)$ be a constant vector. Then

$$\begin{aligned}\operatorname{div}(\mathbf{c} \wedge \mathbf{r}) &= \frac{\partial}{\partial x}(c_2z - c_3y) + \frac{\partial}{\partial y}(c_3x - c_1z) + \frac{\partial}{\partial z}(c_1y - c_2x) = 0; \\ \operatorname{curl}(\mathbf{c} \wedge \mathbf{r}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ c_2z - c_3y & c_3x - c_1z & c_1y - c_2x \end{vmatrix} = \begin{pmatrix} c_1 + c_1 \\ c_2 + c_2 \\ c_3 + c_3 \end{pmatrix} = 2\mathbf{c}.\end{aligned}$$

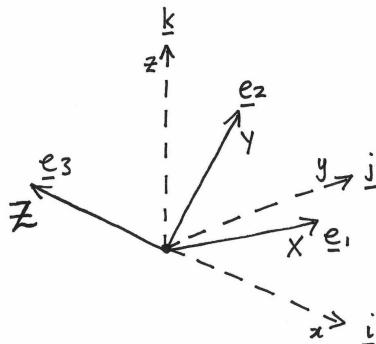
The next proposition shows that the formulae for calculating each of , grad and curl do not depend on our choice $\mathbf{i}, \mathbf{j}, \mathbf{k}$ of a Cartesian co-ordinate system. Recall that:

Definition 80 The vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are said to be an **orthonormal basis** of \mathbb{R}^3 if

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

So the vectors are all of unit length and mutually perpendicular. Further it is said to be **right-handed** if $\mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{e}_3$ and **left-handed** if $\mathbf{e}_1 \wedge \mathbf{e}_2 = -\mathbf{e}_3$.

$\mathbf{e}_1 = \mathbf{i}, \mathbf{e}_2 = \mathbf{j}, \mathbf{e}_3 = \mathbf{k}$ is an example of a right-handed orthonormal basis.



9. An orthonormal basis

Proposition 81 Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be a right-handed orthonormal basis of \mathbb{R}^3 with associated coordinates X, Y, Z defined by the identity

$$X\mathbf{e}_1 + Y\mathbf{e}_2 + Z\mathbf{e}_3 = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}. \quad (5.1)$$

Then

$$\mathbf{e}_1 \frac{\partial}{\partial X} + \mathbf{e}_2 \frac{\partial}{\partial Y} + \mathbf{e}_3 \frac{\partial}{\partial Z} = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}.$$

Proof. From (5.1) we have that

$$\begin{aligned} x &= (\mathbf{e}_1 \cdot \mathbf{i}) X + (\mathbf{e}_2 \cdot \mathbf{i}) Y + (\mathbf{e}_3 \cdot \mathbf{i}) Z, \\ X &= (\mathbf{e}_1 \cdot \mathbf{i}) x + (\mathbf{e}_1 \cdot \mathbf{j}) y + (\mathbf{e}_1 \cdot \mathbf{k}) z, \end{aligned}$$

and four other similar equations for y, z, Y, Z . It follows from the chain rule that

$$\frac{\partial}{\partial X} = \frac{\partial x}{\partial X} \frac{\partial}{\partial x} + \frac{\partial y}{\partial X} \frac{\partial}{\partial y} + \frac{\partial z}{\partial X} \frac{\partial}{\partial z} = (\mathbf{e}_1 \cdot \mathbf{i}) \frac{\partial}{\partial x} + (\mathbf{e}_1 \cdot \mathbf{j}) \frac{\partial}{\partial y} + (\mathbf{e}_1 \cdot \mathbf{k}) \frac{\partial}{\partial z}$$

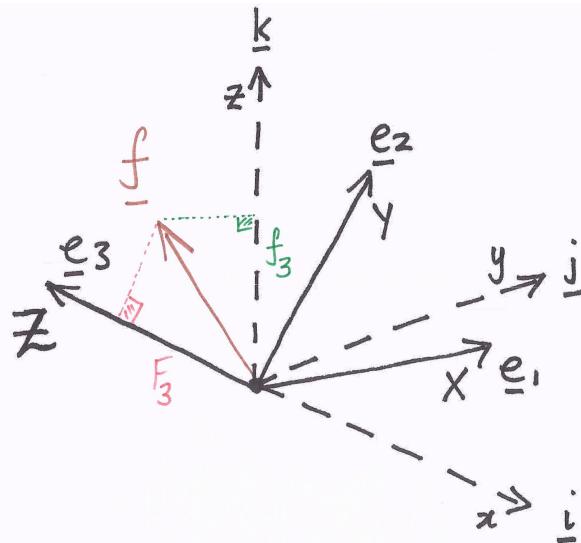
and hence

$$\begin{aligned} &\mathbf{e}_1 \frac{\partial}{\partial X} + \mathbf{e}_2 \frac{\partial}{\partial Y} + \mathbf{e}_3 \frac{\partial}{\partial Z} \\ &= \mathbf{e}_1 \left[(\mathbf{e}_1 \cdot \mathbf{i}) \frac{\partial}{\partial x} + (\mathbf{e}_1 \cdot \mathbf{j}) \frac{\partial}{\partial y} + (\mathbf{e}_1 \cdot \mathbf{k}) \frac{\partial}{\partial z} \right] + \mathbf{e}_2 [\dots] + \mathbf{e}_3 [\dots] \\ &= [\mathbf{e}_1 (\mathbf{e}_1 \cdot \mathbf{i}) + \mathbf{e}_2 (\mathbf{e}_2 \cdot \mathbf{i}) + \mathbf{e}_3 (\mathbf{e}_3 \cdot \mathbf{i})] \frac{\partial}{\partial x} + [\dots] \frac{\partial}{\partial y} + [\dots] \frac{\partial}{\partial z} \\ &= \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}; \end{aligned}$$

this last line follows from the orthonormality of the bases as

$$\mathbf{i} = \alpha \mathbf{e}_1 + \beta \mathbf{e}_2 + \gamma \mathbf{e}_3$$

leads to $\alpha = \mathbf{e}_1 \cdot \mathbf{i}$, $\beta = \mathbf{e}_2 \cdot \mathbf{i}$, $\gamma = \mathbf{e}_3 \cdot \mathbf{i}$, by dotting the equation with $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ respectively. ■



10. Components for the two bases

Corollary 82 Let $\mathbf{f} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a vector field given by $\mathbf{f} = (f_1, f_2, f_3)$, let $\phi : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a scalar field and let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be a right-handed orthonormal basis of \mathbb{R}^3 . Suppose that

$$\begin{aligned} F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3 &= f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}; \\ \Phi(X, Y, Z) &= \phi(x, y, z). \end{aligned}$$

That is, \mathbf{F} and Φ represent \mathbf{f} and ϕ in the new co-ordinates X, Y, Z . Then

$$\begin{aligned} \text{grad } \phi &= \frac{\partial \Phi}{\partial X}\mathbf{e}_1 + \frac{\partial \Phi}{\partial Y}\mathbf{e}_2 + \frac{\partial \Phi}{\partial Z}\mathbf{e}_3; \\ \text{div } \mathbf{f} &= \frac{\partial F_1}{\partial X} + \frac{\partial F_2}{\partial Y} + \frac{\partial F_3}{\partial Z}; \\ \text{curl } \mathbf{f} &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial X} & \frac{\partial}{\partial Y} & \frac{\partial}{\partial Z} \\ F_1 & F_2 & F_3 \end{vmatrix}. \end{aligned}$$

That is, we calculate grad, div and curl using the same formulae, irrespective of what right-handed orthonormal co-ordinate system we are using.

Proof. *Sketch Proof* The previous proposition showed that the formula for grad does not change under an orthonormal change of co-ordinates. From Geometry, we have that the formula for the dot product is similarly invariant as is the formula for the cross product provided the change is to another right-handed system. Hence the formulae for $\text{div} = \nabla \cdot$ and $\text{curl} = \nabla \wedge$ do not change when we change to another right-handed orthonormal system. ■

Adjustments to the above formulae need to be made for co-ordinate systems that aren't orthonormal – e.g. cylindrical polar co-ordinates or spherical polar co-ordinates. The relevant formulae below are for reference only; certainly you are not expected to memorize them. *The main point to take away from this is that the grad, div, curl formulae are different for co-ordinate systems that are not Cartesian.*

Definition 83 *Cylindrical polar co-ordinates* r, θ, z are defined by the identity

$$\mathbf{r} = (x, y, z) = (r \cos \theta, r \sin \theta, z).$$

If we note

$$d\mathbf{r} = (\cos \theta, \sin \theta, 0) dr + (-r \sin \theta, r \cos \theta, 0) d\theta + (0, 0, 1) dz$$

then we may write

$$d\mathbf{r} = \mathbf{e}_r dr + r \mathbf{e}_\theta d\theta + \mathbf{e}_z dz$$

where

$$\mathbf{e}_r = (\cos \theta, \sin \theta, 0), \quad \mathbf{e}_\theta = (-\sin \theta, \cos \theta, 0), \quad \mathbf{e}_z = (0, 0, 1).$$

Note, for any values of r, θ, z , the vectors $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$ make a right-handed orthonormal basis of \mathbb{R}^3 .

Proposition 84 *The formulae for grad, div, curl in terms of cylindrical polar co-ordinates of fields $\psi(r, \theta, z)$ and $\mathbf{F} = F_r \mathbf{e}_r + F_\theta \mathbf{e}_\theta + F_z \mathbf{e}_z$ are*

$$\begin{aligned}\nabla\psi &= \frac{\partial\psi}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial\psi}{\partial\theta} \mathbf{e}_\theta + \frac{\partial\psi}{\partial z} \mathbf{e}_z; \\ \nabla \cdot \mathbf{F} &= \frac{1}{r} \frac{\partial}{\partial r} (r F_r) + \frac{1}{r} \frac{\partial F_\theta}{\partial\theta} + \frac{\partial F_z}{\partial z}; \\ \nabla \wedge \mathbf{F} &= \frac{1}{r} \begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ F_r & rF_\theta & F_z \end{vmatrix}.\end{aligned}$$

Proof. By way of example we shall prove only the first identity here. Recall that Cartesian co-ordinates are given in terms of cylindrical polar ones by

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z$$

Say $\phi(x, y, z) = \psi(r, \theta, z)$. Then

$$\begin{aligned}\frac{\partial\psi}{\partial r} &= \frac{\partial\phi}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial\phi}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial\phi}{\partial z} \frac{\partial z}{\partial r} = \frac{\partial\phi}{\partial x} \cos \theta + \frac{\partial\phi}{\partial y} \sin \theta, \\ \frac{\partial\psi}{\partial\theta} &= \frac{\partial\phi}{\partial x} \frac{\partial x}{\partial\theta} + \frac{\partial\phi}{\partial y} \frac{\partial y}{\partial\theta} + \frac{\partial\phi}{\partial z} \frac{\partial z}{\partial\theta} = -\frac{\partial\phi}{\partial x} r \sin \theta + \frac{\partial\phi}{\partial y} r \cos \theta, \\ \frac{\partial\psi}{\partial z} &= \frac{\partial\phi}{\partial z}.\end{aligned}$$

Hence

$$\begin{aligned}&\frac{\partial\psi}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial\psi}{\partial\theta} \mathbf{e}_\theta + \frac{\partial\psi}{\partial z} \mathbf{e}_z \\ &= \left(\frac{\partial\phi}{\partial x} \cos \theta + \frac{\partial\phi}{\partial y} \sin \theta \right) (\cos \theta, \sin \theta, 0) + \frac{1}{r} \left(-\frac{\partial\phi}{\partial x} r \sin \theta + \frac{\partial\phi}{\partial y} r \cos \theta \right) (-\sin \theta, \cos \theta, 0) + \frac{\partial\phi}{\partial z} \mathbf{k} \\ &= \left(\frac{\partial\phi}{\partial x} \cos^2 \theta + \frac{\partial\phi}{\partial y} \sin^2 \theta \right) \mathbf{i} + \left(\frac{\partial\phi}{\partial y} \sin^2 \theta + \frac{\partial\phi}{\partial x} \cos^2 \theta \right) \mathbf{j} + \frac{\partial\phi}{\partial z} \mathbf{k} \\ &= \nabla\phi.\end{aligned}$$

■

Definition 85 *Spherical polar co-ordinates r, θ, ϕ are defined by the identity*

$$\mathbf{r} = (x, y, z) = (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta).$$

If we note $d\mathbf{r}$ equals

$$(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) dr + r (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta) d\theta + r \sin \theta (-\sin \phi, \cos \phi, 0) d\phi$$

then we may write

$$d\mathbf{r} = \mathbf{e}_r dr + r \mathbf{e}_\theta d\theta + r \sin \theta \mathbf{e}_\phi d\phi$$

where

$$\mathbf{e}_r = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \quad \mathbf{e}_\theta = (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta), \quad \mathbf{e}_\phi = (-\sin \phi, \cos \phi, 0).$$

Note, for any values of r, θ, ϕ the vectors $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi$, make a right-handed orthonormal basis.

Proposition 86 *The formulae for grad, div, curl in terms of spherical polar co-ordinates of fields $\psi(r, \phi, \theta)$ and $\mathbf{F} = F_r \mathbf{e}_r + F_\phi \mathbf{e}_\phi + F_\theta \mathbf{e}_\theta$ are*

$$\begin{aligned}\nabla\psi &= \frac{\partial\psi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\mathbf{e}_\phi; \\ \nabla \cdot \mathbf{F} &= \frac{1}{r^2\sin\theta} \left[\frac{\partial}{\partial r}(r^2\sin\theta F_r) + \frac{\partial}{\partial\theta}(r\sin\theta F_\theta) + \frac{\partial}{\partial\phi}(rF_\phi) \right]; \\ \nabla \wedge \mathbf{F} &= \frac{1}{r^2\sin\theta} \begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta\mathbf{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix}.\end{aligned}$$

Those interested in reading up on the more general theory of curvilinear co-ordinates (which explains the similarities in the previous two propositions) should see Boas (pp.427-435) or Kreyszig (pp.498-504).

5.2 Identities

Proposition 87 *Let \mathbf{F} be a vector field on \mathbb{R}^3 and ϕ be a scalar field on \mathbb{R}^3 . Then*

$$\operatorname{curl} \operatorname{grad} \phi = \mathbf{0}; \quad \operatorname{div} \operatorname{curl} \mathbf{F} = 0.$$

Proof. With the notation $\nabla\phi = (\phi_x, \phi_y, \phi_z)$ where subscripts now denote partial differentiation, we have

$$\operatorname{curl}(\nabla\phi) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi_x & \phi_y & \phi_z \end{vmatrix} = (\phi_{zy} - \phi_{yz}, \phi_{xz} - \phi_{zx}, \phi_{yx} - \phi_{xy}) = \mathbf{0}.$$

Again with subscripts denoting partial differentiation, we also have

$$\begin{aligned}\operatorname{div}(\operatorname{curl} \mathbf{F}) &= \operatorname{div} \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \\ &= [(F_3)_{yx} - (F_2)_{zx}] + [(F_1)_{zy} - (F_3)_{xy}] + [(F_2)_{xz} - (F_1)_{yz}] = 0.\end{aligned}$$

■

Remark 88 *These are in fact aspects of the same theorem, though a detailed explanation would stray into graduate level work on differential forms and topology. But it may be worth saying that the set-up on \mathbb{R}^3*

$$\text{scalar} \xrightarrow{\operatorname{grad}} \text{vector} \xrightarrow{\operatorname{curl}} \text{vector} \xrightarrow{\operatorname{div}} \text{scalar}$$

can be extended to the idea of differentiating differential forms on \mathbb{R}^n

$$\text{scalars} \xrightarrow{d} \text{degree 1 forms} \xrightarrow{d} \text{degree 2 forms} \xrightarrow{d} \dots \xrightarrow{d} \text{degree } n \text{ forms}.$$

at each stage of which we have the identity $d \circ d = 0$. Those interested should check Folland, section 5.9.

Proposition 89 (*Product Rules for div and curl*) Let \mathbf{F} be a vector field on \mathbb{R}^3 and ϕ, ψ be scalar fields on \mathbb{R}^3 . Then

$$\begin{aligned}\operatorname{div}(\phi\mathbf{F}) &= \nabla\phi \cdot \mathbf{F} + \phi \operatorname{div} \mathbf{F}; \\ \operatorname{curl}(\phi\mathbf{F}) &= \phi \operatorname{curl} \mathbf{F} + \nabla\phi \wedge \mathbf{F}.\end{aligned}$$

Proof. (i) With subscripts denoting partial differentiation we have

$$\begin{aligned}\operatorname{div}(\phi\mathbf{F}) &= (\phi F_1)_x + (\phi F_2)_y + (\phi F_3)_z \\ &= \phi(F_1)_x + \phi_x F_1 + \phi(F_2)_y + \phi_y F_2 + \phi(F_3)_z + \phi_z F_3 \\ &= \phi \left[(F_1)_x + (F_2)_y + (F_3)_z \right] + (\phi_x, \phi_y, \phi_z) \cdot (F_1, F_2, F_3) \\ &= \phi \operatorname{div} \mathbf{F} + (\nabla\phi) \cdot \mathbf{F}.\end{aligned}$$

(ii) $\operatorname{curl}(\phi\mathbf{F})$ equals

$$\begin{aligned}&\left(\frac{\partial(\phi F_3)}{\partial y} - \frac{\partial(\phi F_2)}{\partial z}, \frac{\partial(\phi F_1)}{\partial z} - \frac{\partial(\phi F_3)}{\partial x}, \frac{\partial(\phi F_2)}{\partial x} - \frac{\partial(\phi F_1)}{\partial y} \right) \\ &= \phi \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) + (\phi_y F_3 - \phi_z F_2, \phi_z F_1 - \phi_x F_3, \phi_x F_2 - \phi_y F_1) \\ &= \phi \operatorname{curl} \mathbf{F} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \phi_x & \phi_y & \phi_z \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \phi \operatorname{curl} \mathbf{F} + (\nabla\phi) \wedge \mathbf{F}.\end{aligned}$$

■

Proposition 90 (*Further Identities*) For vector fields \mathbf{F}, \mathbf{G} :

$$\begin{aligned}\nabla(\mathbf{F} \cdot \mathbf{G}) &= (\mathbf{F} \cdot \nabla) \mathbf{G} + (\mathbf{G} \cdot \nabla) \mathbf{F} + \mathbf{F} \wedge (\nabla \wedge \mathbf{G}) + \mathbf{G} \wedge (\nabla \wedge \mathbf{F}) \\ \nabla \cdot (\mathbf{F} \wedge \mathbf{G}) &= \mathbf{G} \cdot (\nabla \wedge \mathbf{F}) - \mathbf{F} \cdot (\nabla \wedge \mathbf{G}) \\ \nabla \wedge (\mathbf{F} \wedge \mathbf{G}) &= \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla) \mathbf{F} - (\mathbf{F} \cdot \nabla) \mathbf{G} \\ \nabla \wedge (\nabla \wedge \mathbf{F}) &= \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}\end{aligned}$$

Remark 91 All vector calculus can be proven by simply grinding out the calculations. But there are neater ways to write out the formula for ,grad and curl as given below:

$$\nabla\phi = \sum_i \frac{\partial\phi}{\partial x_i} \mathbf{e}_i, \quad \nabla \cdot \mathbf{F} = \sum_i \mathbf{e}_i \cdot \frac{\partial \mathbf{F}}{\partial x_i}, \quad \nabla \wedge \mathbf{F} = \sum_i \mathbf{e}_i \wedge \frac{\partial \mathbf{F}}{\partial x_i}, \quad \mathbf{F} \cdot \nabla = \sum_i (\mathbf{F} \cdot \mathbf{e}_i) \frac{\partial}{\partial x_i},$$

where the dummy variable i ranges over 1, 2, 3 in each case and $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is any right-handed

orthonormal basis. For example

$$\begin{aligned}
\sum_i \mathbf{e}_i \wedge \frac{\partial \mathbf{F}}{\partial x_i} &= \mathbf{e}_1 \wedge \frac{\partial \mathbf{F}}{\partial x_1} + \mathbf{e}_2 \wedge \frac{\partial \mathbf{F}}{\partial x_2} + \mathbf{e}_3 \wedge \frac{\partial \mathbf{F}}{\partial x_3} \\
&= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 1 & 0 & 0 \\ \frac{\partial F_1}{\partial x_1} & \frac{\partial F_2}{\partial x_1} & \frac{\partial F_3}{\partial x_1} \end{vmatrix} + \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 0 & 1 & 0 \\ \frac{\partial F_1}{\partial x_2} & \frac{\partial F_2}{\partial x_2} & \frac{\partial F_3}{\partial x_2} \end{vmatrix} + \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 0 & 0 & 1 \\ \frac{\partial F_1}{\partial x_3} & \frac{\partial F_2}{\partial x_3} & \frac{\partial F_3}{\partial x_3} \end{vmatrix} \\
&= -\mathbf{e}_2 \frac{\partial F_3}{\partial x_1} + \mathbf{e}_3 \frac{\partial F_2}{\partial x_1} + \mathbf{e}_1 \frac{\partial F_3}{\partial x_2} - \mathbf{e}_3 \frac{\partial F_1}{\partial x_2} - \mathbf{e}_1 \frac{\partial F_2}{\partial x_3} + \mathbf{e}_2 \frac{\partial F_1}{\partial x_3} \\
&= \nabla \wedge \mathbf{F}.
\end{aligned}$$

With these formulae the proofs of identities involving grad and curl are much shorter; on the other hand the formulae need to be applied with much more care and attention to detail than was previously the case.

Proof. The second and third identities appear in the Exercise Sheets. To prove the first identity recall that $\mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w}) \mathbf{v} - (\mathbf{u} \cdot \mathbf{v}) \mathbf{w}$ and hence

$$\begin{aligned}
&(\mathbf{F} \cdot \nabla) \mathbf{G} + (\mathbf{G} \cdot \nabla) \mathbf{F} + \mathbf{F} \wedge (\nabla \wedge \mathbf{G}) + \mathbf{G} \wedge (\nabla \wedge \mathbf{F}) \\
&= \sum_i (\mathbf{F} \cdot \mathbf{e}_i) \frac{\partial \mathbf{G}}{\partial x_i} + \sum_i (\mathbf{G} \cdot \mathbf{e}_i) \frac{\partial \mathbf{F}}{\partial x_i} + \mathbf{F} \wedge \left(\sum_i \mathbf{e}_i \wedge \frac{\partial \mathbf{G}}{\partial x_i} \right) + \mathbf{G} \wedge \left(\sum_i \mathbf{e}_i \wedge \frac{\partial \mathbf{F}}{\partial x_i} \right) \\
&= \sum_i (\mathbf{F} \cdot \mathbf{e}_i) \frac{\partial \mathbf{G}}{\partial x_i} + \sum_i (\mathbf{G} \cdot \mathbf{e}_i) \frac{\partial \mathbf{F}}{\partial x_i} + \sum_i \mathbf{F} \wedge \left(\mathbf{e}_i \wedge \frac{\partial \mathbf{G}}{\partial x_i} \right) + \sum_i \mathbf{G} \wedge \left(\mathbf{e}_i \wedge \frac{\partial \mathbf{F}}{\partial x_i} \right) \\
&= \sum_i \left\{ (\mathbf{F} \cdot \mathbf{e}_i) \frac{\partial \mathbf{G}}{\partial x_i} + (\mathbf{G} \cdot \mathbf{e}_i) \frac{\partial \mathbf{F}}{\partial x_i} + \left[\left(\mathbf{F} \cdot \frac{\partial \mathbf{G}}{\partial x_i} \right) \mathbf{e}_i - (\mathbf{F} \cdot \mathbf{e}_i) \frac{\partial \mathbf{G}}{\partial x_i} \right] + \left[\left(\mathbf{G} \cdot \frac{\partial \mathbf{F}}{\partial x_i} \right) \mathbf{e}_i - (\mathbf{G} \cdot \mathbf{e}_i) \frac{\partial \mathbf{F}}{\partial x_i} \right] \right\} \\
&= \sum_i \mathbf{e}_i \left(\mathbf{F} \cdot \frac{\partial \mathbf{G}}{\partial x_i} + \mathbf{G} \cdot \frac{\partial \mathbf{F}}{\partial x_i} \right) \\
&= \sum_i \mathbf{e}_i \frac{\partial}{\partial x_i} (\mathbf{F} \cdot \mathbf{G}) = \nabla (\mathbf{F} \cdot \mathbf{G}).
\end{aligned}$$

To prove the fourth identity recall again that $\mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w}) \mathbf{v} - (\mathbf{u} \cdot \mathbf{v}) \mathbf{w}$ and then

$$\begin{aligned}
\nabla \wedge (\nabla \wedge \mathbf{F}) &= \sum_i \mathbf{e}_i \wedge \frac{\partial}{\partial x_i} \left(\sum_j \mathbf{e}_j \wedge \frac{\partial \mathbf{F}}{\partial x_j} \right) \\
&= \sum_i \sum_j \mathbf{e}_i \wedge \left(\mathbf{e}_j \wedge \frac{\partial^2 \mathbf{F}}{\partial x_i \partial x_j} \right) \\
&= \sum_i \sum_j \left[\left(\mathbf{e}_i \cdot \frac{\partial^2 \mathbf{F}}{\partial x_i \partial x_j} \right) \mathbf{e}_j \right] - \sum_i \sum_j \left[(\mathbf{e}_i \cdot \mathbf{e}_j) \frac{\partial^2 \mathbf{F}}{\partial x_i \partial x_j} \right] \\
&= \sum_j \mathbf{e}_j \frac{\partial}{\partial x_j} \left[\sum_i \mathbf{e}_i \cdot \frac{\partial \mathbf{F}}{\partial x_i} \right] - \sum_i \sum_j \left[\delta_{ij} \frac{\partial^2 \mathbf{F}}{\partial x_i \partial x_j} \right]
\end{aligned}$$

$$= \sum_j \mathbf{e}_j \frac{\partial}{\partial x_j} \left[\sum_i \mathbf{e}_i \cdot \frac{\partial \mathbf{F}}{\partial x_i} \right] - \sum_i \frac{\partial^2 \mathbf{F}}{\partial x_i^2} = \nabla (\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}.$$

■

5.3 Physical Interpretation

Example 92 (Fourier's Law) In an isotropic medium with constant thermal conductivity k , with temperature $T(\mathbf{x}, t)$ at position \mathbf{x} and at time t , Fourier's Law states that

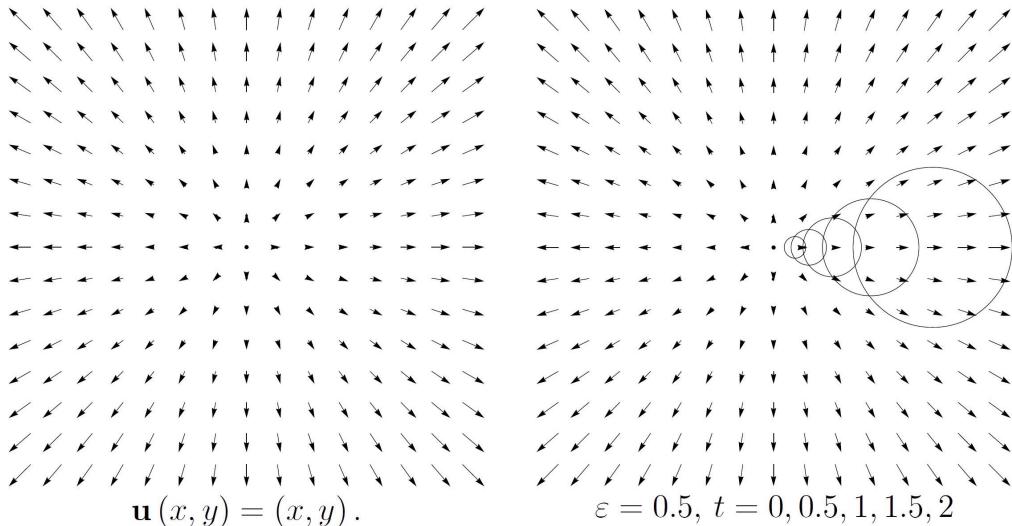
$$\mathbf{q} = -k \nabla T$$

where \mathbf{q} is the heat flux – that is the amount of energy that flows through a particular surface per unit area per unit time.

This law contains much of the essence of grad. Recall from the Michaelmas Calculus course that, for a given scalar function ϕ and a unit vector \mathbf{v} , the directional derivative of ϕ in the direction \mathbf{v} is $\nabla \phi \cdot \mathbf{v}$. In particular, ϕ increases fastest in the direction of $\nabla \phi$ and decreases fastest in the direction $-\nabla \phi$. So Fourier's law, loosely put, states that at each point the heat energy moves towards the coolest nearby points.

What about the physical motivation behind divergence?

Example 93 Consider fluid particles in the xy -plane flowing according to the steady flow



11. A source with $\operatorname{div} \mathbf{u} = 2$.

We have a source at $(0, 0)$ from which the fluid emanates. However, at every (x, y) , we have that

$$\operatorname{div} \mathbf{u} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} = 2.$$

Suppose a particle starts off at (x_0, y_0) ; then the particle's position $(x(t), y(t))$ at time t is determined by the differential equations

$$\frac{dx}{dt} = x, \quad \frac{dy}{dt} = y, \quad x(0) = x_0, \quad y(0) = y_0$$

and, hence,

$$(x(t), y(t)) = (x_0 e^t, y_0 e^t).$$

Consider now a blob of fluid of area $\pi\varepsilon^2$ which at $t = 0$ occupies

$$(x - 1)^2 + y^2 < \varepsilon^2.$$

Where will the particles which make up this blob have moved by time t ? Initially (at $t = 0$) the particles were at $x_0 = 1 + r \cos \theta$, $y = r \sin \theta$ where $r < \varepsilon$; at time $t > 0$ they have moved to

$$(x(t), y(t)) = (e^t + re^t \cos \theta, re^t \sin \theta) \quad r < \varepsilon,$$

which comprises a disc with centre $(e^t, 0)$ and area $A(t) = \pi\varepsilon^2 e^{2t}$. So, throughout the motion we have

$$\frac{dA}{dt} = (\operatorname{div} \mathbf{u}) A.$$

More generally, for a two dimensional flow, it is the case that

$$\nabla \cdot \mathbf{u}(\mathbf{x}) = \lim_{\varepsilon \rightarrow 0} \left[\frac{1}{\operatorname{area}(\operatorname{Flow image of } B(\mathbf{x}, \varepsilon) \text{ at time } t)} \frac{d}{dt} \operatorname{area}(\operatorname{Flow image of } B(\mathbf{x}, \varepsilon) \text{ at time } t) \right],$$

with the analogous result for 3D flows in terms of volume rather than area. In physical terms, then, the divergence of a vector field is a measure of the flow's source density, the extent to which the vector field flow behaves like a source or a sink at a given point.

If you choose to take the second year option in fluid dynamics you will meet:

Example 94 Euler's Equations for an ideal fluid (Second year course in Fluid Dynamics). Euler's equations for an incompressible, inviscid fluid state that

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{\nabla p}{\rho} + \mathbf{g}, \quad \nabla \cdot \mathbf{u} = 0,$$

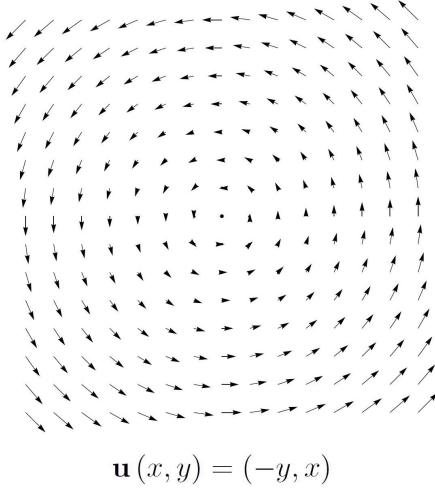
where $\mathbf{u}(\mathbf{x}, t)$ is the flow velocity, p is the pressure, ρ is the density and \mathbf{g} is acceleration due to gravity. It is the equation

$$\nabla \cdot \mathbf{u} = 0$$

which encodes the incompressibility of the fluid.

Such vector calculus are found in many applications of mathematics, including Maxwell's equations of electromagnetism, fluid and solid mechanics, and cellular behaviour.

Roughly speaking the curl of a vector field relates to the rotation of the vector field, *but very much in a local sense*.



12. Globally rotating field. $\operatorname{curl} \mathbf{u} = 2\mathbf{k}$

Example 95 (Rotation of a rigid body) Recall from the Michaelmas Dynamics course that if $\mathbf{r} = r\mathbf{e}_r$ then

$$\dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta.$$

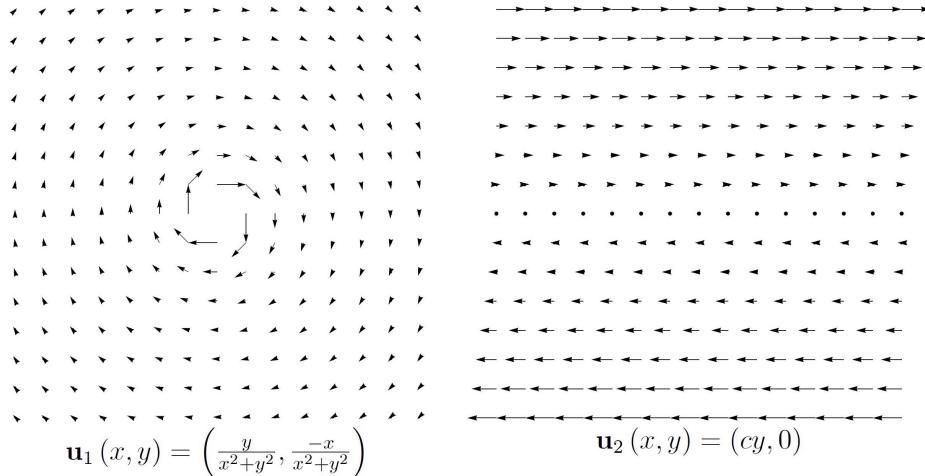
In particular, if a rigid body is rotating with constant angular velocity ω about the z -axis then the distance of a point r from the z -axis will remain constant and $\dot{\theta} = \omega$ so that

$$\dot{\mathbf{r}} = r\omega\mathbf{e}_\theta = \omega\mathbf{k} \wedge (r\mathbf{e}_r) = \omega\mathbf{k} \wedge \mathbf{r}.$$

We have already seen from Example 79 that

$$\operatorname{curl} \dot{\mathbf{r}} = \operatorname{curl} (\omega\mathbf{k} \wedge \mathbf{r}) = 2\omega\mathbf{k}.$$

Example 96 It is important to stress though that curl is very subtly a measure of local rotation. Consider the following two examples:



13. Rotating flow with zero curl; linear flow with non-zero curl.

A simple calculation shows that $\text{curl } \mathbf{u}_2 = -c\mathbf{k}$ yet there appears to be no sense of global rotation, whilst in Exercise 5 on Sheet 4 you are asked to show that $\text{curl } \mathbf{u}_1 = \mathbf{0}$, yet there appears to be a clear sense of global rotation. Despite particles moving in circles in the flow \mathbf{u}_1 different circles are moving around the origin with different angular velocities and with a greater angular velocity towards the origin; in fact this differential is such that a circular paddle wheel inserted into the fluid would not spin as it moves around the origin. In contrast, with the shear flow \mathbf{u}_2 , whilst the particles are moving in lines, a small paddle wheel inserted into the flow would spin as the flow for greater y is moving faster. In general, the axis of this spinning is in the direction of curl .

6. DIVERGENCE AND STOKES' THEOREMS

6.1 The Divergence Theorem

Theorem 97 (Divergence Theorem – Lagrange 1762, Gauss 1813, Green 1825) Let R be a region of \mathbb{R}^3 with a piecewise smooth boundary ∂R , and let \mathbf{F} be a differentiable vector field on R . Then

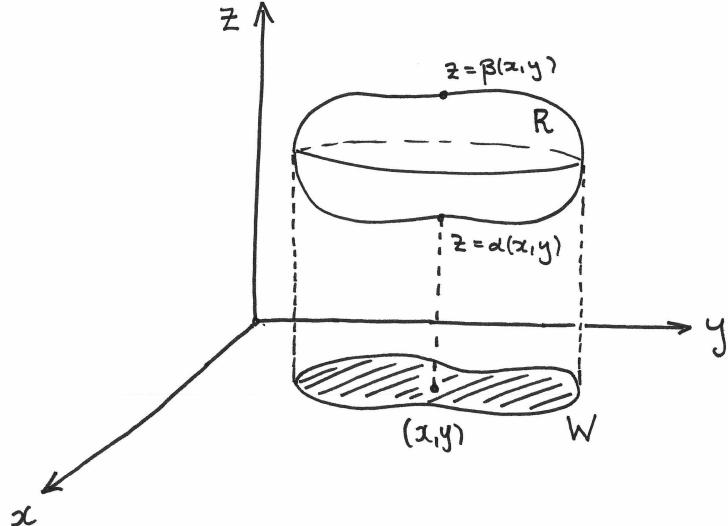
$$\iiint_R \operatorname{div} \mathbf{F} dV = \iint_{\partial R} \mathbf{F} \cdot d\mathbf{S},$$

where $d\mathbf{S}$ is oriented in the direction of the outward pointing normal from R .

Definition 98 We say that a region R is **convex** if for any $\mathbf{p}, \mathbf{q} \in R$ the line segment connecting \mathbf{p} and \mathbf{q} is contained in R .

We will only prove the Divergence Theorem for convex regions, though the proof extends with very little extra work to any region that can be decomposed into convex regions.

Proof. (Not examinable.) Let W be a subset of the xy -plane and α, β continuous functions on W .



14. A region between two graphs.

Let

$$R = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in W, \alpha(x, y) \leq z \leq \beta(x, y)\}$$

and $\mathbf{F} = F\mathbf{k}$ is a differentiable vector field on R . (That is R is z -convex in the sense that the intersection of R with any line parallel with the z -axis is an interval.) The Divergence Theorem in this case reads as

$$\iiint_R \frac{\partial F}{\partial z} dV = \iint_{\partial R} F\mathbf{k} \cdot d\mathbf{S}.$$

Note that ∂R is made up of three surfaces:

$$\begin{aligned}\Sigma^+ &= \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in W, z = \beta(x, y)\}, \\ \Sigma &= \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in \partial W, \alpha(x, y) \leq z \leq \beta(x, y)\}, \\ \Sigma^- &= \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in W, z = \alpha(x, y)\}.\end{aligned}$$

Then

$$\iiint_R \frac{\partial F}{\partial z} dV = \iint_W \int_{\alpha(x,y)}^{\beta(x,y)} \frac{\partial F}{\partial z} dz dA = \iint_W (F(x, y, \beta(x, y)) - F(x, y, \alpha(x, y))) dA.$$

Regarding the boundary integral:

$$\begin{aligned}\text{on } \Sigma^+ : \quad d\mathbf{S} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \beta_x \\ 0 & 1 & \beta_y \end{vmatrix} dx dy = (-\beta_x, -\beta_y, 1) dx dy; \\ \text{on } \Sigma^- : \quad d\mathbf{S} &= - \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \alpha_x \\ 0 & 1 & \alpha_y \end{vmatrix} dx dy = (\alpha_x, \alpha_y, -1) dx dy; \\ \text{on } \Sigma : \quad \mathbf{k} \cdot d\mathbf{S} &= 0.\end{aligned}$$

Hence we have

$$\begin{aligned}\iint_{\Sigma^+} F \mathbf{k} \cdot d\mathbf{S} &= \iint_W F(x, y, \beta(x, y)) dx dy; \\ \iint_{\Sigma^-} F \mathbf{k} \cdot d\mathbf{S} &= - \iint_W F(x, y, \alpha(x, y)) dx dy; \\ \iint_{\Sigma} F \mathbf{k} \cdot d\mathbf{S} &= 0;\end{aligned}$$

and we have shown

$$\iiint_R \frac{\partial F}{\partial z} dV = \iint_{\partial R} F \mathbf{k} \cdot d\mathbf{S}$$

for the given region R . However, any convex body has the property of being in the form of R when viewed from each of the x, y, z directions. Hence we have shown

$$\iiint_R \operatorname{div} \mathbf{F} dV = \iint_{\partial R} \mathbf{F} \cdot d\mathbf{S}$$

for any convex body. ■

Corollary 99 *Let ϕ be a smooth scalar field defined on a region $R \subseteq \mathbb{R}^3$ with a piecewise smooth boundary. Then*

$$\iiint_R \nabla \phi dV = \iint_{\partial R} \phi d\mathbf{S}.$$

Proof. Let \mathbf{c} be a constant vector and $\mathbf{F} = \mathbf{c}\phi$. Then

$$\operatorname{div}(\mathbf{c}\phi) = \mathbf{c} \cdot \nabla\phi + \phi \operatorname{div} \mathbf{c} = \mathbf{c} \cdot \nabla\phi$$

as \mathbf{c} is constant. By the Divergence Theorem

$$\iiint_R \mathbf{c} \cdot \nabla\phi \, dV = \iint_{\partial R} \mathbf{c}\phi \cdot d\mathbf{S}, \implies \mathbf{c} \cdot \left(\iiint_R \nabla\phi \, dV \right) = \mathbf{c} \cdot \left(\iint_{\partial R} \phi \, d\mathbf{S} \right).$$

As \mathbf{c} is an arbitrary vector then

$$\iiint_R \nabla\phi \, dV = \iint_{\partial R} \phi \, d\mathbf{S}.$$

■

Corollary 100 (Green's Theorem in the plane) *Let D be a closed bounded region in the (x, y) plane, whose boundary C is a piecewise smooth simple closed curve, and that $p(x, y), q(x, y)$ have continuous first-order derivatives in D . Then*

$$\iint_D \left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} \right) dx dy = \int_C (p, q) \cdot \mathbf{n} \, ds = \int_C p \, dy - q \, dx, \quad (6.1)$$

where \mathbf{n} is the outward pointing unit normal to C in the (x, y) plane. (Recall that a simple closed curve is a curve that has its endpoints meet but otherwise does not cross itself.)

Proof. The first equality is a simple corollary of the divergence theorem and is effectively the divergence theorem in a plane. Let the vector field \mathbf{F} be given by

$$\mathbf{F} = (p(x, y), q(x, y), 0)$$

and define the three dimensional region R to be

$$R = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in D, z \in [0, 1]\},$$

with boundary ∂R . Noting $\nabla \cdot \mathbf{F}$ has no dependence on z , we have

$$\iint_D \left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} \right) dx dy = \iint_D \nabla \cdot \mathbf{F} \, dx dy \int_0^1 dz = \iint_R \nabla \cdot \mathbf{F} \, dV = \int_{\partial R} \mathbf{F} \cdot \mathbf{n} \, dS.$$

The contribution to the surface integral from the surfaces at $z = 0, 1$ are zero as the integrand is zero. For the surface

$$\{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in D, z \in [0, 1]\}$$

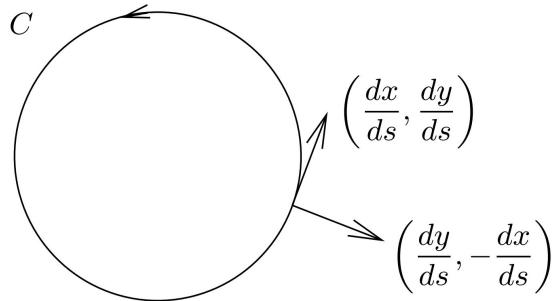
the outward pointing unit normal to R coincides with the outward pointing unit normal to C ; also we can write the surface element dS as

$$dS = ds \, dz,$$

where ds is the arclength element of the curve C since the \mathbf{k} direction and the plane of the region D are perpendicular. Hence

$$\int_{\partial R} \mathbf{F} \cdot \mathbf{n} dS = \int_0^1 \left\{ \int_C \mathbf{F} \cdot \mathbf{n} ds \right\} dz = \int_0^1 \left\{ \int_C (p, q) \cdot \mathbf{n} ds \right\} dz = \int_C (p, q) \cdot \mathbf{n} ds,$$

with the final equality from the fact the integrand $(p, q) \cdot \mathbf{n}$ has no dependence on z .



15. Unit tangent and normal vectors to a curve.

For the second equality, we parameterize the curve C in the form $(x(s), y(s))$, where s is arc length, increasing on moving *anticlockwise* around C when viewed from above. The unit tangent, \mathbf{t} , and the outward unit normal \mathbf{n} are given by

$$\mathbf{t} = \left(\frac{dx}{ds}, \frac{dy}{ds} \right), \quad \mathbf{n} = \left(\frac{dy}{ds}, -\frac{dx}{ds} \right). \quad (6.2)$$

Using the expression for \mathbf{n} , we have

$$\int_C (p, q) \cdot \mathbf{n} ds = \int_C (p, q) \cdot \left(\frac{dy}{ds}, -\frac{dx}{ds} \right) ds = \int_C p dy - q dx \quad (6.3)$$

and so we have the second equality, namely

$$\iint_D \left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} \right) dx dy = \int_C p dy - q dx,$$

where C is positively oriented, i.e. the integration around C is anticlockwise. ■

Example 101 Verify the Divergence Theorem when R is the finite region bounded by the paraboloid $z = x^2 + y^2$ and the plane $x + y + z = 1$, and $\mathbf{F} = (x, y, z)$.

Solution. We already met this region in Example 22. We showed that the top surface of R projects vertically down to the set

$$W = \{(x, y) \in \mathbb{R}^2 : (x + 1/2)^2 + (y + 1/2)^2 \leq 3/2\}$$

which is a disc, centre $(-1/2, -1/2)$ and radius $\sqrt{3/2}$. Further we calculated the volume of the region to be $9\pi/8$. As $\operatorname{div} \mathbf{F} = 3$ then

$$\iiint_R \operatorname{div} F dV = 3 \operatorname{Vol}(R) = \frac{27\pi}{8}.$$

The boundary ∂R is made up of two surfaces

$$\begin{aligned}\Sigma_1 &= \{(x, y, z) : z \geq x^2 + y^2, x + y + z = 1\} = \{(x, y, z) : (x, y) \in W, x + y + z = 1\}; \\ \Sigma_2 &= \{(x, y, z) : z = x^2 + y^2, x + y + z \leq 1\} = \{(x, y, z) : (x, y) \in W, z = x^2 + y^2\}.\end{aligned}$$

We can parameterize Σ_1 and Σ_2 as

$$\mathbf{r}_1(u, v) = (u, v, 1 - u - v); \quad \mathbf{r}_2(u, v) = (u, v, u^2 + v^2).$$

Note

$$(\mathbf{r}_1)_u \wedge (\mathbf{r}_1)_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = (1, 1, 1)$$

which is outward-pointing from R , and

$$(\mathbf{r}_2)_u \wedge (\mathbf{r}_2)_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = (-2u, -2v, 1)$$

which is inward-pointing and so we will take its negative. So

$$\iint_{\Sigma_1} \mathbf{F} \cdot d\mathbf{S} = \iint_W (u, v, 1 - u - v) \cdot (1, 1, 1) du dv = \iint_W du dv = \pi (\sqrt{3/2})^2 = 3\pi/2.$$

For the second surface

$$\begin{aligned}\iint_{\Sigma_2} \mathbf{F} \cdot d\mathbf{S} &= \iint_W (u, v, u^2 + v^2) \cdot (2u, 2v, -1) dA \\ &= \iint_W (u^2 + v^2) dA \\ &= \int_{r=0}^{\sqrt{3/2}} \int_{\theta=0}^{2\pi} \{(-1/2 + r \cos \theta)^2 + (-1/2 + r \sin \theta)^2\} r d\theta dr \\ &= \int_{r=0}^{\sqrt{3/2}} \int_{\theta=0}^{2\pi} \{r^2 - r \cos \theta - r \sin \theta + 1/2\} r d\theta dr \\ &= 2\pi \int_{r=0}^{\sqrt{3/2}} \left\{ r^3 + \frac{r}{2} \right\} d\theta dr \\ &= 2\pi \left[\frac{r^4}{4} + \frac{r^2}{4} \right]_0^{\sqrt{3/2}} \\ &= 2\pi (9/16 + 3/8) = \frac{15\pi}{8}.\end{aligned}$$

Hence, in total, we have

$$\iint_{\partial R} \mathbf{F} \cdot d\mathbf{S} = \frac{3\pi}{2} + \frac{15\pi}{8} = \frac{12\pi}{8} + \frac{15\pi}{8} = \frac{27\pi}{8}.$$

■

Example 102 The twice continuously differentiable function $f(x, y, z)$ is homogeneous of degree n , so that

$$f(tx, ty, tz) = t^n f(x, y, z) \quad \text{for all } t \in \mathbb{R}. \quad (6.4)$$

Prove **Euler's Theorem**, which states that

$$\mathbf{r} \cdot \nabla f = nf.$$

Let B be the unit ball $x^2 + y^2 + z^2 < 1$ and let ∂B be its boundary. Show that

$$\iiint_B f \, dV = \frac{1}{n(n+3)} \iiint_B \nabla^2 f \, dV,$$

and hence show that

$$\iiint_B (x - y + z)^4 \, dV = \frac{36\pi}{35}.$$

Solution. If we differentiate the identity (6.4) with respect to t then we find

$$x \frac{\partial f}{\partial x}(tx, ty, tz) + y \frac{\partial f}{\partial y}(tx, ty, tz) + z \frac{\partial f}{\partial z}(tx, ty, tz) = nt^{n-1} f(x, y, z).$$

If we set $t = 1$ then we arrive at Euler's Theorem $\mathbf{r} \cdot \nabla f = nf$.

By the Divergence Theorem, with B as the unit ball, and ∂B its boundary, then

$$\iiint_B \nabla^2 f \, dV = \iiint_B \nabla \cdot \nabla f \, dV = \iint_{\partial B} \nabla f \cdot \mathbf{n} \, dS = \iint_{\partial B} \nabla f \cdot \mathbf{r} \, dS = n \iint_{\partial B} f \, dS$$

as $\mathbf{n} = \mathbf{r}$ on ∂B . On the other hand

$$\nabla \cdot (f\mathbf{r}) = (\nabla f) \cdot \mathbf{r} + f(\nabla \cdot \mathbf{r}) = nf + 3f = (n+3)f.$$

and so, by the Divergence Theorem again,

$$(n+3) \iint_{\partial B} f \, dS = \iint_{\partial B} \nabla \cdot (f\mathbf{r}) \, dS = \iint_{\partial B} f\mathbf{r} \cdot \mathbf{n} \, dS = \iint_{\partial B} f \, dS$$

as ∂B is the unit sphere. Hence

$$\iiint_B f \, dV = \frac{1}{n(n+3)} \iiint_B \nabla^2 f \, dV,$$

The function $(x - y + z)^4$ is homogeneous of degree 4 in x, y, z and so that

$$\begin{aligned}
& \iiint_B (x - y + z)^4 \, dV \\
= & \frac{1}{4 \times 7} \iiint_B \nabla^2 (x - y + z)^4 \, dV \\
= & \frac{1}{4 \times 7} \iiint_B 36(x - y + z)^2 \, dV \quad [\text{which is again homogeneous}] \\
= & \frac{36}{4 \times 7} \times \frac{1}{2 \times 5} \iiint_B \nabla^2 (x - y + z)^2 \, dV \\
= & \frac{36}{4 \times 7} \times \frac{1}{2 \times 5} \iiint_B 6 \, dV \\
= & \frac{36}{4 \times 7} \times \frac{6}{2 \times 5} \times \frac{4\pi}{3} \\
= & \frac{36\pi}{35}.
\end{aligned}$$

■

6.2 Applications of The Divergence Theorem

6.2.1 Boundary value problems

The Divergence Theorem also has important implications for certain boundary-value problems.

Corollary 103 (Uniqueness of the Dirichlet Problem) *Let $R \subseteq \mathbb{R}^3$ be a path-connected region with piecewise smooth boundary ∂R and let f be a continuous function defined on ∂R . Suppose that ϕ_1 and ϕ_2 are such that*

$$\begin{aligned}
\nabla^2 \phi_1 &= 0 = \nabla^2 \phi_2 \quad \text{in } R; \\
\phi_1 &= f = \phi_2 \quad \text{on } \partial R.
\end{aligned}$$

Then $\phi_1 = \phi_2$ in R .

Proof. Let $\psi = \phi_1 - \phi_2$, so that

$$\nabla^2 \psi = 0 \quad \text{in } R, \quad \psi = 0 \quad \text{on } \partial R.$$

If we consider the function ψ^2 then, by the Divergence Theorem and as $\nabla^2 = \nabla \cdot \nabla$,

$$\iint_{\partial R} \nabla(\psi^2) \cdot d\mathbf{S} = \iiint_R \nabla^2(\psi^2) \, dV.$$

Looking at the LHS,

$$\iint_{\partial R} \nabla(\psi^2) \cdot d\mathbf{S} = \iint_{\partial R} 2\psi \nabla \psi \cdot d\mathbf{S} = 0 \quad \text{as } \psi = 0 \text{ on } \partial R.$$

On the RHS we have

$$\nabla^2(\psi^2) = 2\psi \nabla^2 \psi + 2\nabla \psi \cdot \nabla \psi = 2|\nabla \psi|^2 \quad \text{as } \nabla^2 \psi = 0 \text{ in } R.$$

Hence

$$\iiint_R 2|\nabla \psi|^2 dV = 0 \implies \nabla \psi = \mathbf{0} \text{ in } R.$$

As $\nabla \psi = \mathbf{0}$ in R and R is path-connected then ψ is constant throughout R . But $\psi = 0$ on ∂R and so by continuity $\psi = 0$ throughout R . ■

Corollary 104 (Uniqueness, up to a constant, of the Neumann Problem) Let $R \subseteq \mathbb{R}^3$ be a path-connected region with piecewise smooth boundary ∂R and let f be a continuous function defined on ∂R . Suppose that ϕ_1 and ϕ_2 are such that

$$\begin{aligned} \nabla^2 \phi_1 &= 0 = \nabla^2 \phi_2 \quad \text{in } R; \\ \frac{\partial \phi_1}{\partial n} &= f = \frac{\partial \phi_2}{\partial n} \quad \text{on } \partial R. \end{aligned}$$

Then $\phi_1 - \phi_2$ is constant in R .

Proof. Let $\psi = \phi_1 - \phi_2$, so that

$$\nabla^2 \psi = 0 \text{ in } R, \quad \frac{\partial \psi}{\partial n} = \nabla \psi \cdot \mathbf{n} = 0 \text{ on } \partial R.$$

If we consider the function ψ^2 , then by the Divergence Theorem and as $\nabla^2 = \nabla \cdot \nabla$,

$$\iint_{\partial R} \nabla(\psi^2) \cdot d\mathbf{S} = \iiint_R \nabla^2(\psi^2) dV.$$

Looking at the LHS,

$$\iint_{\partial R} \nabla(\psi^2) \cdot d\mathbf{S} = \iint_{\partial R} 2\psi \nabla \psi \cdot \mathbf{n} dS = 0 \quad \text{as } \nabla \psi \cdot \mathbf{n} = 0 \text{ on } \partial R.$$

On the RHS we have

$$\nabla^2(\psi^2) = 2\psi \nabla^2 \psi + 2\nabla \psi \cdot \nabla \psi = 2|\nabla \psi|^2 \quad \text{as } \nabla^2 \psi = 0 \text{ in } R.$$

Hence

$$\iiint_R 2|\nabla \psi|^2 dV = 0 \implies \nabla \psi = \mathbf{0} \text{ in } R.$$

As $\nabla \psi = \mathbf{0}$ in R then ψ is constant throughout R . ■

Corollary 105 (Robin (or Mixed) Boundary Problem) Let $R \subseteq \mathbb{R}^3$ be a path-connected region with piecewise smooth boundary ∂R and let f be a continuous function defined on ∂R . Further let β be a real constant. Suppose that ϕ_1 and ϕ_2 are such that

$$\begin{aligned}\nabla^2 \phi_1 &= 0 = \nabla^2 \phi_2 \quad \text{in } R; \\ \beta \phi_1 + \frac{\partial \phi_1}{\partial n} &= f = \beta \phi_2 + \frac{\partial \phi_2}{\partial n} \quad \text{on } \partial R.\end{aligned}$$

- (i) If $\beta > 0$ then $\phi_1 = \phi_2$ in R .
- (ii) If $\beta = 0$ then $\phi_1 - \phi_2$ is constant in R .
- (iii) If $\beta < 0$ then the solution need not be unique, even up to a constant.

Proof. Let $\psi = \phi_1 - \phi_2$, so that

$$\nabla^2 \psi = 0 \quad \text{in } R, \quad \beta \psi + \frac{\partial \psi}{\partial n} = \beta \psi + \nabla \psi \cdot \mathbf{n} = 0 \quad \text{on } \partial R.$$

If we consider the function ψ^2 then, by the Divergence Theorem and as $\nabla^2 = \nabla \cdot \nabla$,

$$\iint_{\partial R} \nabla (\psi^2) \cdot d\mathbf{S} = \iiint_R \nabla^2 (\psi^2) dV.$$

Looking at the LHS and noting $\beta \psi + \nabla \psi \cdot \mathbf{n} = 0$ on ∂R we have

$$\iint_{\partial R} \nabla (\psi^2) \cdot d\mathbf{S} = \iint_{\partial R} 2\psi (\nabla \psi \cdot \mathbf{n}) dS = \iint_{\partial R} 2\psi (-\beta \psi) dS = -2\beta \iint_{\partial R} \psi^2 dS.$$

On the RHS we have as before

$$\nabla^2 (\psi^2) = 2\psi \nabla^2 \psi + 2\nabla \psi \cdot \nabla \psi = 2 |\nabla \psi|^2 \quad \text{as } \nabla^2 \psi = 0 \text{ in } R.$$

Hence

$$-2\beta \iint_{\partial R} \psi^2 dS = \iiint_R 2 |\nabla \psi|^2 dV.$$

If $\beta > 0$ then the LHS is non-positive and the RHS is non-negative – consequently both are zero and $\nabla \psi = \mathbf{0}$ in R and $\psi = 0$ on ∂R . Hence $\psi = 0$ throughout R .

If $\beta = 0$ then the Robin Boundary Problem is just the Neumann Problem, with which we have already dealt.

The following example shows that solutions need not be unique if $\beta < 0$. ■

Example 106 Let R be the interior of the cylinder $x^2 + y^2 = 1$. Find two solutions to the Robin Boundary Problem

$$\nabla^2 \psi = 0 \quad \text{in } R, \quad \psi = \frac{\partial \psi}{\partial n} \quad \text{on } \partial R.$$

Solution. Laplace's equation in cylindrical polars reads

$$\nabla^2 \psi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2} = 0.$$

We know for $\psi = r^n \sin n\theta$, where n is a non-negative integer, that $\nabla^2 \psi = 0$ and on the boundary $r = 1$ we have

$$\frac{\partial \psi}{\partial n} = \frac{\partial \psi}{\partial r} = nr^{n-1} \sin n\theta = n \sin n\theta; \quad \psi = \sin n\theta.$$

Hence when $n = 1$, we see $r \sin \theta$ satisfies the given boundary problem. Other solutions are 0, $r \cos \theta$ and $r \sin(\theta - \alpha)$ for arbitrary α so we see this solution is far from unique. ■

6.2.2 Derivation of balance equations in 3D: the heat equation.

Finally, as a last application, we use the Divergence Theorem to rederive the heat equation, but now in 3D. We need to following lemma first.

Lemma 107 *If $\phi : \mathbb{R}^3 \rightarrow \mathbb{R}$ is a continuous scalar field such that*

$$\iiint_R \phi \, dV = 0$$

for all bounded subsets R , then $\phi \equiv 0$.

Proof. Suppose, for a contradiction that $\phi(\mathbf{p}) \neq 0$ for some point p . Without any loss of generality let's suppose $\phi(\mathbf{p}) > 0$. Let $\varepsilon = \frac{1}{2}\phi(\mathbf{p}) > 0$. Then, by the continuity of ϕ at \mathbf{p} , there exists $\delta > 0$ such that

$$|\phi(\mathbf{x}) - \phi(\mathbf{p})| < \varepsilon \quad \text{when} \quad |\mathbf{x} - \mathbf{p}| < \delta.$$

In particular, $\phi(\mathbf{x}) > \frac{1}{2}\phi(\mathbf{p})$ on all of $B(\mathbf{p}, \delta) = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x} - \mathbf{p}| < \delta\}$ and so

$$\iiint_{B(\mathbf{p}, \delta)} \phi \, dV > \frac{1}{2}\phi(\mathbf{p}) \times \text{Vol}(B(\mathbf{p}, \delta)) > 0$$

which is a contradiction. ■

Theorem 108 *Let $T(\mathbf{x}, t)$ denote the temperature at position \mathbf{x} and at time t in an isotropic medium R with thermal conductivity k , density ρ , specific heat c with heat flow determined by Fourier's Law which states that*

$$\mathbf{q} = -k\nabla T$$

*where \mathbf{q} is the **heat flux**. Then T satisfies the **heat equation***

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c} \nabla^2 T.$$

Proof. Let $S \subseteq R$ be an arbitrary subset of R . The total heat energy in S equals

$$\iiint_S \rho c T \, dV.$$

The amount of heat flowing *out* of S is given by the flux integral

$$\iint_{\partial S} \mathbf{q} \cdot d\mathbf{S}.$$

So if no heat is generated within S then the only heat loss or gain is via the boundary ∂S . Hence we have

$$\begin{aligned} \frac{d}{dt} \iiint_S \rho c T \, dV &= - \iint_{\partial S} \mathbf{q} \cdot d\mathbf{S} \\ &= \iint_{\partial S} k \nabla T \cdot d\mathbf{S} \quad [\text{by Fourier's Law}] \\ &= \iiint_S \nabla \cdot (k \nabla T) \, dV \quad [\text{by the Divergence Theorem}] \\ &= \iiint_S k \nabla^2 T \, dV. \end{aligned}$$

So

$$\begin{aligned} \iiint_S \rho c \frac{\partial T}{\partial t} \, dV &= \iiint_S k \nabla^2 T \, dV, \\ \implies \iiint_S \left(\rho c \frac{\partial T}{\partial t} - k \nabla^2 T \right) \, dV &= 0. \end{aligned}$$

As this is true for an arbitrary subset S of R then it follows, from Lemma 107, that

$$\rho c \frac{\partial T}{\partial t} - k \nabla^2 T = 0 \quad \text{throughout } R.$$

■

6.3 Stokes' Theorem

There is another *equivalent* form of Green's theorem in the plane is a precursor for Stokes' Theorem, which is our next subject of study. Recall that Green's Theorem states that:

Let D be a closed bounded region in the (x, y) plane, whose boundary C is a piecewise smooth simple closed curve, and that $p(x, y), q(x, y)$ have continuous first-order derivatives in D . Then

$$\iint_D \left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} \right) dx dy = \int_C p dy - q dx,$$

where C is positively oriented, i.e. the integration around C is anticlockwise.

If we replace (p, q) with $(q, -p)$ then this instead reads

$$\iint_D \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} \right) dx dy = \int_C p dx + q dy.$$

The RHS becomes

$$\int_C p dx + q dy = \int_C (p, q) \cdot d\mathbf{r},$$

whilst on the LHS note that

$$\text{curl}(p, q) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ p(x, y) & q(x, y) & 0 \end{vmatrix} = \left(\frac{\partial p}{\partial x} - \frac{\partial q}{\partial y} \right) \mathbf{k},$$

and so

$$\iint_D \left(\frac{\partial p}{\partial x} - \frac{\partial q}{\partial y} \right) dA = \iint_D \text{curl } \mathbf{F} \cdot (\mathbf{k} dA) = \iint_D \text{curl } \mathbf{F} \cdot d\mathbf{S}.$$

Thus Greens theorem in the plane can be rephrased as

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \text{curl } \mathbf{F} \cdot d\mathbf{S}, \quad (6.5)$$

where $\mathbf{F} = (p, q)$. Here

$$d\mathbf{S} = \mathbf{k} dS = \mathbf{n} dS, \quad (6.6)$$

where now, \mathbf{n} is the normal to the planar region D in \mathbb{R}^3 (rather than the in-plane normal to the boundary C as used in the previous section deducing Green's theorem in the plane).

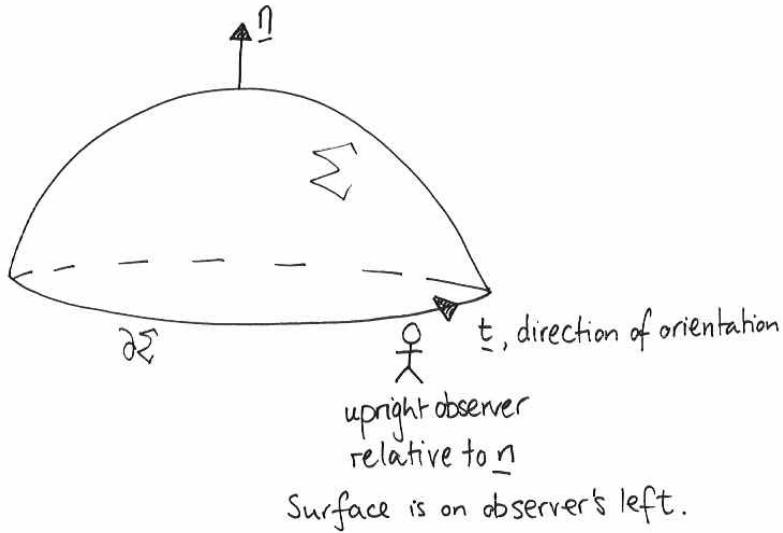
The identity 6.5 is purely a rewriting of Green's Theorem, but it is part of a larger identity known as *Stokes' Theorem* which applies more generally to surfaces in \mathbb{R}^3 with boundary. However, before we can properly understand the identity in its fullest generality, we need to appreciate how to consistently orient a curve C which bounds a surface Σ .

Definition 109 At each point of a smooth surface there are two unit normal vectors $\pm \mathbf{n}$. Given a smooth surface Σ , then by an **orientation** of Σ we shall mean a continuous choice of unit normal \mathbf{n} for the whole surface Σ .

For the bounding curve, which we shall denote as $\partial\Sigma$, there are two possible orientations. Given a certain orientation of $\partial\Sigma$ and an oriented tangent vector \mathbf{t} along $\partial\Sigma$ then the vector $\mathbf{t} \wedge \mathbf{n}$ lies in the tangent plane of Σ at the boundary point and is normal to the bounding curve $\partial\Sigma$. So $\mathbf{t} \wedge \mathbf{n}$ points either towards the surface Σ or away from Σ .

We shall always choose to orient a surface Σ and its boundary $\partial\Sigma$ in such a way that $\mathbf{t} \wedge \mathbf{n}$ points away from Σ .

Remark 110 Note that this is consistent in the plane with taking positively oriented curves and normal \mathbf{k} , so that equations (6.5, 6.6) are equivalent to Green's theorem in the plane, equation (??).



16. Orienting surface and boundary.

In practice it is fairly clear how to consistently orient a surface and its boundary. If one imagines an observer moving along the boundary $\partial\Sigma$ (in the direction of the orientation) and upright (in the sense of the surface normal \mathbf{n}) then the surface Σ will be to his/her left.

Example 111 Given the hemisphere $\Sigma = \{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$ with boundary $\partial\Sigma = \{(x, y, 0) : x^2 + y^2 = 1\}$ then we can either orient Σ as

$$\mathbf{n}(x, y, z) = (x, y, z) \quad \text{or} \quad -\mathbf{n}(x, y, z) = (-x, -y, -z).$$

We can parameterize $\partial\Sigma$ as $(\cos\theta, \sin\theta, 0)$. So one choice of unit tangent is

$$\mathbf{t} = (-\sin\theta, \cos\theta, 0).$$

At $(\cos\theta, \sin\theta, 0)$ we have

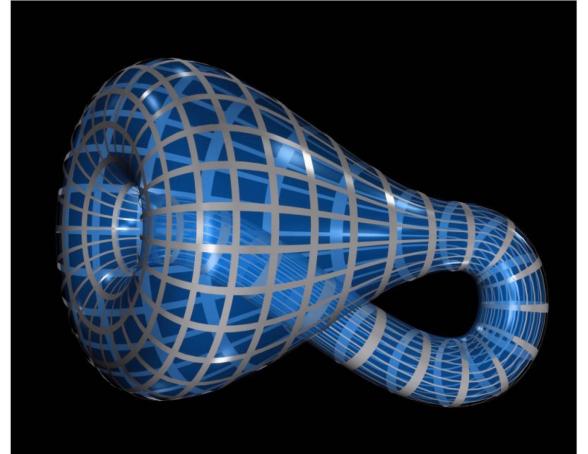
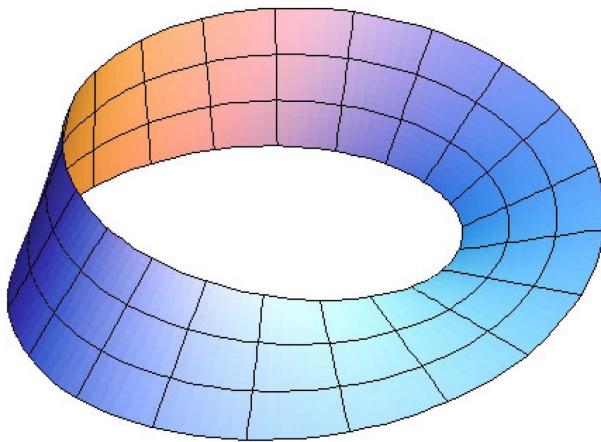
$$\mathbf{t} \wedge \mathbf{n} = (-\sin\theta, \cos\theta, 0) \wedge (\cos\theta, \sin\theta, 0) = (0, 0, -1)$$

which points down, away from the hemisphere. The two consistent ways to orient the surface and boundary are

$$(\mathbf{n} \text{ on } \Sigma \text{ and } \mathbf{t} \text{ on } \partial\Sigma) \quad \text{or} \quad (-\mathbf{n} \text{ on } \Sigma \text{ and } -\mathbf{t} \text{ on } \partial\Sigma).$$

Example 112 Not all surfaces have an orientation, such surfaces being called **non-orientable**. Examples of non-orientable surfaces are the **Möbius strip** and the **Klein bottle**. We shall

only be interested in **orientable** surfaces though.



17. Möbius strip and Klein bottle

Theorem 113 (Stokes' Theorem c. 1850.) *The first known appearance of this result though is in fact in a letter from Kelvin to Stokes.) Let Σ be a smooth oriented surface in \mathbb{R}^3 whose boundary is the curve $\partial\Sigma$. Let \mathbf{F} be a smooth vector field defined on $\Sigma \cup \partial\Sigma$. Then*

$$\int_{\partial\Sigma} \mathbf{F} \cdot d\mathbf{r} = \iint_{\Sigma} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}. \quad (6.7)$$

Remark 114 *The following proof applies only to patches of surface – we shall see in later in Example (121) that Stokes' Theorem applies generally to more complicated surfaces and their boundaries. In any event the proof of Stokes' Theorem is not examinable.*

Proof. Suppose that Σ is parameterized as $\mathbf{r}(u, v)$ where (u, v) ranges over some $S \subseteq \mathbb{R}^2$, with $\partial\Sigma$ being the image of ∂S under \mathbf{r} . We shall first prove Stokes Theorem for the vector field $\mathbf{F} = (F, 0, 0)$, noting that similar proofs apply for fields $(0, F, 0)$ and $(0, 0, F)$ and then the more general (6.7) follows by linearity.

So, if $\mathbf{F} = (F, 0, 0)$, then (6.7) reads as

$$\int_{\partial\Sigma} F dx = \iint_{\Sigma} (0, F_z, -F_y) \cdot d\mathbf{S}. \quad (6.8)$$

Now using the parameterization $\mathbf{r}(u, v)$ we have

$$\begin{aligned} \iint_{\Sigma} (0, F_z, -F_y) \cdot d\mathbf{S} &= \iint_S (0, F_z, -F_y) \cdot (\mathbf{r}_u \wedge \mathbf{r}_v) du dv \\ &= \iint_S (0, F_z, -F_y) \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} du dv \\ &= \iint_S [F_z(z_u x_v - x_u z_v) - F_y(x_u y_v - y_u x_v)] du dv. \end{aligned}$$

On the other hand if we apply Green's Theorem to the LHS of (6.8) we have

$$\begin{aligned}
\int_{\partial\Sigma} F \, dx &= \int_{\partial S} F (x_u du + x_v dv) \\
&= \iint_S [(Fx_v)_u - (Fx_u)_v] \, du \, dv \\
&= \iint_S [F_u x_v + F x_{vu} - F_v x_u - F x_{uv}] \, du \, dv \\
&= \iint_S [F_u x_v - F_v x_u] \, du \, dv \\
&= \iint_S [(F_x x_u + F_y y_u + F_z z_u) x_v - (F_x x_v + F_y y_v + F_z z_v) x_u] \, du \, dv \\
&= \iint_S [F_z (z_u x_v - x_u z_v) - F_y (x_u y_v - y_u x_v)] \, du \, dv.
\end{aligned}$$

The result follows. ■

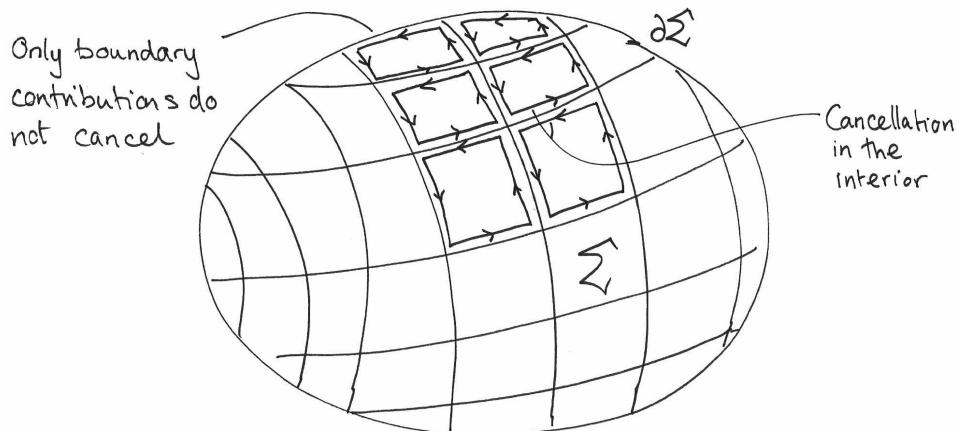
Remark 115 *Stokes' Theorem and the Divergence Theorem are in fact statements of a more general theorem, also known as Stokes' Theorem, which applies in all dimensions. It more generally reads:*

$$\int_M d\omega = \int_{\partial M} \omega$$

where M is a (compact) oriented n -dimensional manifold (i.e. the n -dimensional equivalent of a surface) with boundary ∂M and ω is a differential form of degree $n-1$. A proper appreciation of this result is approximately at fourth year undergraduate level. Note, though, that when $n=1$ Stokes' Theorem reads

$$\int_a^b f'(x) \, dx = f(b) - f(a),$$

when $M = [a, b]$ and $\partial M = \{a, b\}$ oriented to sum positively the contribution at b and negatively at a .



18. Informal explanation of Stokes' theorem

Remark 116 For an intuitive appreciation of why Stokes' Theorem is true, recall that $\operatorname{curl} \mathbf{F}$, when physically interpreted, is a measure of the local spin of the field \mathbf{F} . So one might envisage summing up all the curl over the surface Σ with most contributions to this total spin cancelling out, in a manner akin to intermeshing cogs. The only contributions that wouldn't be cancelled out are those at the boundary, these remaining contributions making up the integral around the boundary.

Note that Stokes' Theorem is a scalar identity, that is both sides of the identity are scalar quantities. There is also a vector identity related to Stokes' Theorem. But firstly we note:

Lemma 117 If

$$\mathbf{c} \cdot \mathbf{v} = \mathbf{c} \cdot \mathbf{w} \text{ for all } \mathbf{c} \in \mathbb{R}^3$$

then $\mathbf{v} = \mathbf{w}$.

Proof. We have $\mathbf{c} \cdot (\mathbf{v} - \mathbf{w}) = 0$ for all \mathbf{c} and thus for all basis vectors. Thus all the components of \mathbf{v} , \mathbf{w} are equal, hence so are the vectors. ■

Corollary 118 Let Σ be a smooth oriented surface in \mathbb{R}^3 whose boundary is the curve $\partial\Sigma$. Let ψ be a smooth scalar field defined on $\Sigma \cup \partial\Sigma$. Then

$$\iint_{\Sigma} \nabla\psi \wedge d\mathbf{S} = - \int_{\partial\Sigma} \psi d\mathbf{r}.$$

Proof. Let \mathbf{c} be an arbitrary constant vector and set $\mathbf{F} = \psi\mathbf{c}$. In this case, and using the curl product rule, Stokes Theorem reads

$$\iint_{\Sigma} \operatorname{curl}(\psi\mathbf{c}) \cdot d\mathbf{S} = \iint_{\Sigma} (\psi \operatorname{curl} \mathbf{c} + \nabla\psi \wedge \mathbf{c}) \cdot d\mathbf{S} = \int_{\partial\Sigma} \psi\mathbf{c} \cdot d\mathbf{r}.$$

As \mathbf{c} is constant then $\operatorname{curl} \mathbf{c} = \mathbf{0}$ and we have

$$\mathbf{c} \cdot \left(- \iint_{\Sigma} \nabla\psi \wedge d\mathbf{S} \right) = \iint_{\Sigma} \nabla\psi \wedge \mathbf{c} \cdot d\mathbf{S} = \mathbf{c} \cdot \left(\int_{\partial\Sigma} \psi d\mathbf{r} \right).$$

As \mathbf{c} is arbitrary then by Lemma 117 we have

$$- \iint_{\Sigma} \nabla\psi \wedge d\mathbf{S} = \int_{\partial\Sigma} \psi d\mathbf{r}$$

and the result follows. ■

Example 119 Verify Stokes' Theorem when $\mathbf{F} = (y, z, x)$ on the hemisphere

$$\Sigma = \{(x, y, z) : x^2 + y^2 + z^2 = a^2, z \geq 0\}.$$

Solution. We can parameterize the sphere as

$$\mathbf{r}(\theta, \phi) = (a \sin \theta \cos \phi, a \sin \theta \sin \phi, a \cos \theta) \quad 0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi.$$

Then

$$\frac{\partial \mathbf{r}}{\partial \theta} \wedge \frac{\partial \mathbf{r}}{\partial \phi} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a \cos \theta \cos \phi & a \cos \theta \sin \phi & -a \sin \theta \\ -a \sin \theta \sin \phi & a \sin \theta \cos \phi & 0 \end{vmatrix} = a^2 \sin \theta (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta).$$

Now $\operatorname{curl} \mathbf{F} = (-1, -1, -1)$ and hence

$$\begin{aligned} \iint_{\Sigma} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} (-1, -1, -1) \cdot a^2 \sin \theta (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) d\phi d\theta \\ &= -a^2 \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} \sin \theta (\sin \theta \cos \phi + \sin \theta \sin \phi + \cos \theta) d\phi d\theta \\ &= -2\pi a^2 \int_{\theta=0}^{\pi/2} \sin \theta \cos \theta d\theta = -2\pi a^2 \left[\frac{\sin^2 \theta}{2} \right]_0^{\pi/2} \\ &= -\pi a^2. \end{aligned}$$

On the other hand, once we parameterize and orient $\partial\Sigma$ as $\mathbf{r}(\theta) = (a \cos \theta, a \sin \theta, 0)$, we have

$$\begin{aligned} \int_{\partial\Sigma} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (a \sin \theta, 0, a \cos \theta) \cdot (-a \sin \theta, a \cos \theta, 0) d\theta \\ &= -a^2 \int_0^{2\pi} \sin^2 \theta d\theta = -\pi a^2. \end{aligned}$$

■

Example 120 Verify Stokes' Theorem when $\mathbf{F} = (2y, 3x, x^2 + y^2 + z^2)$ and Σ is the lower half of the ellipsoid $x^2/4 + y^2/9 + z^2/27 = 1$.

Solution. Here $\partial\Sigma$ is the ellipse $x^2/4 + y^2/9 = 1$ in the xy -plane. If $\partial\Sigma$ is oriented anti-clockwise then we need to take the upward pointing normal on Σ .

We may parameterize $\partial\Sigma$ by $\mathbf{r}(t) = (2 \cos t, 3 \sin t, 0)$ and so

$$\begin{aligned} \int_{\partial\Sigma} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (6 \sin t, 6 \cos t, 4 \cos^2 t + 9 \sin^2 t) \cdot (-2 \sin t, 3 \cos t, 0) dt \\ &= \int_0^{2\pi} (-12 \sin^2 t + 18 \cos^2 t) dt \\ &= \int_0^{2\pi} ((-6 - 6 \cos 2t) + (9 + 9 \cos 2t)) dt \\ &= 6\pi. \end{aligned}$$

On the other hand we can parameterize the lower half of the ellipsoid as

$$\mathbf{r}(\theta, \phi) = (2 \sin \theta \cos \phi, 3 \sin \theta \sin \phi, 3\sqrt{3} \cos \theta).$$

So

$$\begin{aligned}\mathbf{r}_\theta \wedge \mathbf{r}_\phi &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 \cos \theta \cos \phi & 3 \cos \theta \sin \phi & -3\sqrt{3} \sin \theta \\ -2 \sin \theta \sin \phi & 3 \sin \theta \cos \phi & 0 \end{vmatrix} \\ &= (9\sqrt{3} \sin^2 \theta \cos \phi, 6\sqrt{3} \sin^2 \theta \sin \phi, 6 \sin \theta \cos \theta),\end{aligned}$$

but this points downwards, so we will take its negative. Also

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & 3x & x^2 + y^2 + z^2 \end{vmatrix} = (2y, -2x, 1).$$

Finally, then

$$\begin{aligned}&\iint_{\Sigma} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} \\ &= \int_{\theta=\pi/2}^{\pi} \int_{\phi=0}^{2\pi} (6 \sin \theta \sin \phi, -4 \sin \theta \cos \phi, 1) \cdot (-9\sqrt{3} \sin^2 \theta \cos \phi, -6\sqrt{3} \sin^2 \theta \sin \phi, -6 \sin \theta \cos \theta) d\phi d\theta \\ &= \int_{\theta=\pi/2}^{\pi} \int_{\phi=0}^{2\pi} (-54\sqrt{3} \sin^2 \theta \sin \phi \cos \phi + 24\sqrt{3} \sin^3 \theta \cos \phi \sin \phi - 6 \sin \theta \cos \theta) d\phi d\theta \\ &= -2\pi \int_{\theta=\pi/2}^{\pi} 6 \sin \theta \cos \theta d\theta \\ &= 2\pi \left[\frac{3 \cos 2\theta}{2} \right]_{\pi/2}^{\pi} \\ &= 3\pi(1+1) = 6\pi.\end{aligned}$$

■

Example 121 Verify Stokes' Theorem for $\mathbf{F} = (x^2 + 3yz, -2xz, 7y)$ on Σ , where Σ is that part of the cylindrical surface $x^2 + y^2 = a^2$ between the planes $x + y + z = 0$ and $x + y + z = h$, where $h \geq 0$.

Solution. We may parameterize the cylinder as

$$\mathbf{r}(\theta, z) = (a \cos \theta, a \sin \theta, z)$$

and orient it with outward pointing normal $d\mathbf{S} = (\cos \theta, \sin \theta, 0) a d\theta dz$. So

$$\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + 3yz & -2xz & 7y \end{vmatrix} = (7 + 2x, 3y, -5z).$$

$$\begin{aligned}
\iint_{\Sigma} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \int_{\theta=0}^{2\pi} \int_{z=-a \cos \theta - a \sin \theta}^{h-a \cos \theta - a \sin \theta} (7 + 2a \cos \theta, 3a \sin \theta, -5z) \cdot (\cos \theta, \sin \theta, 0) \, a \, dz \, d\theta \\
&= \int_{\theta=0}^{2\pi} \int_{z=-a \cos \theta - a \sin \theta}^{h-a \cos \theta - a \sin \theta} (7 \cos \theta + 2a \cos^2 \theta + 3a \sin^2 \theta) \, a \, dz \, d\theta \\
&= \int_{\theta=0}^{2\pi} (7 \cos \theta + 2a \cos^2 \theta + 3a \sin^2 \theta) \, ah \, d\theta \\
&= (2a\pi + 3a\pi) \, ah \\
&= 5\pi a^2 h.
\end{aligned}$$

On the other side, we have a disconnected boundary $\partial\Sigma$ made up of

$$\begin{aligned}
C_1 &= \{(a \cos \theta, a \sin \theta, -a \cos \theta - a \sin \theta) : 0 \leq \theta \leq 2\pi\}; \\
C_2 &= \{(a \cos \theta, a \sin \theta, h - a \cos \theta - a \sin \theta) : 0 \leq \theta \leq 2\pi\}.
\end{aligned}$$

As we used the outward pointing normal to orient Σ then we need to orient C_1 in the direction of increasing θ but orient C_2 in the direction of decreasing θ .

If we write $x(\theta) = a \cos \theta$, $y(\theta) = a \sin \theta$, $z(\theta) = -a \cos \theta - a \sin \theta$ then we have

$$\begin{aligned}
\int_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (x(\theta)^2 + 3y(\theta)z(\theta), -2x(\theta)z(\theta), 7y(\theta)) \cdot d\mathbf{r}(\theta) \\
\int_{C_2} \mathbf{F} \cdot d\mathbf{r} &= - \int_0^{2\pi} (x(\theta)^2 + 3y(\theta)[z(\theta) + h], -2x(\theta)[z(\theta) + h], 7y(\theta)) \cdot d\mathbf{r}(\theta).
\end{aligned}$$

Hence

$$\begin{aligned}
\int_{\partial\Sigma} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (-3hy(\theta), 2hx(\theta), 0) \cdot d\mathbf{r}(\theta) \\
&= ah \int_0^{2\pi} (-3a \sin \theta, 2a \cos \theta, 0) \cdot (-\sin \theta, \cos \theta, \sin \theta - \cos \theta) \, d\theta \\
&= a^2 h \int_0^{2\pi} (3 \sin^2 \theta + 2 \cos^2 \theta) \, d\theta \\
&= 5\pi a^2 h.
\end{aligned}$$

■

Example 122 Let C be a closed curve bounding a smooth surface Σ . Show that

$$\int_C \mathbf{r} \wedge d\mathbf{r} = 2 \iint_{\Sigma} d\mathbf{S}.$$

Hence, or otherwise, evaluate $\iint_{\Sigma} d\mathbf{S}$ where Σ is the surface

$$\Sigma = \left\{ (x, y, z) : z = \frac{x^2}{a^2} + \frac{y^2}{b^2}, x^2 + y^2 < 1 \right\}.$$

[You may assume that $\nabla \wedge (\mathbf{u} \wedge \mathbf{v}) = (\mathbf{v} \cdot \nabla) \mathbf{u} - (\nabla \cdot \mathbf{u}) \mathbf{v} + (\nabla \cdot \mathbf{v}) \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{v}$.]

Solution. If we set $\mathbf{F} = \mathbf{c} \wedge \mathbf{r}$ where \mathbf{c} is an arbitrary constant vector, then we have

$$\begin{aligned}\nabla \wedge (\mathbf{c} \wedge \mathbf{r}) &= (\mathbf{r} \cdot \nabla) \mathbf{c} - (\nabla \cdot \mathbf{c}) \mathbf{r} + (\nabla \cdot \mathbf{r}) \mathbf{c} - (\mathbf{c} \cdot \nabla) \mathbf{r} \\ &= (\nabla \cdot \mathbf{r}) \mathbf{c} - (\mathbf{c} \cdot \nabla) \mathbf{r} \quad [\text{as } \mathbf{c} \text{ is constant}] \\ &= 3\mathbf{c} - \mathbf{c} \\ &= 2\mathbf{c}\end{aligned}$$

as

$$(\mathbf{c} \cdot \nabla) \mathbf{r} = \left(c_1 \frac{\partial}{\partial x} + c_2 \frac{\partial}{\partial y} + c_3 \frac{\partial}{\partial z} \right) (x, y, z) = (c_1, c_2, c_3) = \mathbf{c}.$$

So, by Stokes' Theorem,

$$\begin{aligned}\int_C (\mathbf{c} \wedge \mathbf{r}) \cdot d\mathbf{r} &= \iint_{\Sigma} 2\mathbf{c} \cdot d\mathbf{S} \\ \implies \mathbf{c} \cdot \left(\int_C \mathbf{r} \wedge d\mathbf{r} \right) &= \mathbf{c} \cdot \left(\iint_{\Sigma} 2d\mathbf{S} \right).\end{aligned}$$

As \mathbf{c} is arbitrary then by Lemma 117

$$\int_C \mathbf{r} \wedge d\mathbf{r} = 2 \iint_{\Sigma} d\mathbf{S}. \quad (6.9)$$

The bounding curve of the given surface Σ is

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}, \quad x^2 + y^2 = 1$$

which can naturally be parameterized as

$$\mathbf{r}(\theta) = \left(\cos \theta, \sin \theta, \frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} \right), \quad 0 \leq \theta \leq 2\pi.$$

By (6.9) we have (writing $c = \cos \theta$, $s = \sin \theta$)

$$\begin{aligned}\iint_{\Sigma} d\mathbf{S} &= \frac{1}{2} \int_0^{2\pi} \left(c, s, \frac{c^2}{a^2} + \frac{s^2}{b^2} \right) \wedge \left(-s, c, \left(\frac{-1}{a^2} + \frac{1}{b^2} \right) 2cs \right) d\theta \\ &= \frac{1}{2} \int_0^{2\pi} \left(\frac{cs^2}{b^2} + \frac{(-2cs^2 - c^3)}{a^2}, \frac{c^2s}{a^2} + \frac{(-s^3 - 2c^2s)}{b^2}, 1 \right) d\theta \\ &= \frac{1}{2} \int_0^{2\pi} \left(\frac{cs^2}{b^2} + \frac{(-cs^2 - c)}{a^2}, \frac{c^2s}{a^2} + \frac{(-s - c^2s)}{b^2}, 1 \right) d\theta \\ &= \frac{1}{2} \left[\left(\frac{s^3}{3b^2} + \frac{(-s^3/3 - s)}{a^2}, \right), \frac{-c^3}{3a^2} + \frac{(c + c^3/3)}{b^2}, \theta \right]_0^{2\pi} \\ &= \frac{1}{2} [(0, 0, 2\pi)] \\ &= \pi \mathbf{k}.\end{aligned}$$

■

6.4 Stokes' Theorem and Conservative Fields

Definition 123 A region $R \subseteq \mathbb{R}^3$ is said to be **simply connected** if every simple, closed curve C can be continuously deformed to a single point.

Specifically, if $\mathbf{r}: [0, 1] \rightarrow R$ is a simple closed curve beginning and ending in \mathbf{p} then there exists a map $H: [0, 1] \times [0, 1] \rightarrow R$ such that $H(t, 1) = \mathbf{r}(t)$ and $H(t, 0) = \mathbf{p}$.

Example 124 \mathbb{R}^3 is simply connected, as is any plane or line. $\mathbb{R}^2 - \{\mathbf{0}\}$ is not simply connected, nor is the cylinder $x^2 + y^2 = a^2$, nor a torus. $\mathbb{R}^3 - \{\mathbf{0}\}$ is simply connected though.

Corollary 125 (Existence of a Potential) Let R be a simply connected region and let \mathbf{F} be a smooth vector field on R for which $\operatorname{curl} \mathbf{F} = \mathbf{0}$. Then \mathbf{F} is conservative, i.e. there exists a potential ϕ on R such that $\mathbf{F} = \nabla\phi$.

Remark 126 Firstly note that if \mathbf{F} is conservative then $\mathbf{F} = \nabla\phi$ and hence $\operatorname{curl} \mathbf{F} = \mathbf{0}$. Combining this observation with corollary 125 gives, by use of Theorem 69, that with a restriction to simply connected regions, $\operatorname{curl} \mathbf{F} = \mathbf{0}$ is equivalent to (i), (ii) and (iii):

(i) \mathbf{F} is conservative – i.e. $\mathbf{F} = \nabla\phi$ for some scalar field $\phi: S \rightarrow \mathbb{R}$.

(ii) Given any two points $\mathbf{p}, \mathbf{q} \in S$ and curve γ in S , starting at \mathbf{p} and ending at \mathbf{q} , then the integral

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$$

is independent of the choice of curve C .

(iii) For any simple closed curve C then

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = 0.$$

Proof. Let \mathbf{p} be a fixed point in R and define for any $\mathbf{q} \in R$,

$$\phi(\mathbf{q}) = \int_{C_1} \mathbf{F} \cdot d\mathbf{r}$$

where C_1 is a curve in S from \mathbf{p} to \mathbf{q} . If C_2 is a second such curve then there is a simple closed curve C formed by C_1 followed by C_2 in reverse orientation. As R is simply connected then C is the boundary of a surface Σ . By Stokes' Theorem then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_{\Sigma} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0.$$

This means that

$$\phi(\mathbf{q}) = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

and so ϕ is a well-defined function on R , dependent only on the variable \mathbf{q} and not on the choice of curve from \mathbf{p} to \mathbf{q} . Further, arguing as in Theorem 69, we can show that $\nabla\phi = \mathbf{F}$. ■

But this is **not** generally equivalent – for example,

$$\mathbf{F} = \left(\frac{y}{x^2 + y^2}, \frac{-x}{x^2 + y^2} \right) \quad (x, y) \neq (0, 0)$$

satisfies $\operatorname{curl} \mathbf{F} = \mathbf{0}$ but there is no ϕ on $\mathbb{R}^2 - \{(0, 0)\}$ such that $\mathbf{F} = \nabla\phi$. See the Exercise Sheets.

7. GAUSS' FLUX THEOREM, POISSON'S EQUATION AND GRAVITY

Definition 127 (Newton's Law of Gravity) The gravitational field associated with a point mass M at the origin O is

$$\mathbf{f} = -\frac{GM}{r^3}\mathbf{r} = -\frac{GM}{r^2}\mathbf{e}_r = \nabla \left(\frac{GM}{r} \right).$$

The gravitational force on a particle of mass m at the point \mathbf{r}

$$\mathbf{F} = -\frac{GMm}{r^2}\mathbf{e}_r.$$

Here G is the gravitational constant $6.67300 \times 10^{-11}\text{m}^3\text{kg}^{-1}\text{s}^{-2}$.

Note that the gravitational field \mathbf{f} is conservative with gravitational potential

$$\phi = \frac{GM}{r}.$$

Recall that potentials are determined by the field only up to a constant. This choice of potential is such that $\phi \rightarrow 0$ as $r \rightarrow \infty$.

In addition to $\operatorname{curl} \mathbf{f} = \mathbf{0}$, we also have that that

$$\operatorname{div} \mathbf{f} = \nabla^2 \phi = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi}{dr} \right) = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \left(\frac{-GM}{r^2} \right) \right) = 0, \quad r \neq 0,$$

and

$$\nabla^2 \phi = 0 \quad \text{for gravitational potential in vacuo.}$$

This is physically intuitive as, aside from at the origin, there is no other mass contributing to the gravitational field.

Proposition 128 The gravitational potential $\phi = GM/r$ is the amount of work gravity does per unit mass to bring a point object from ∞ to \mathbf{r} .

Proof. The work done by gravity to bring a unit point mass from ∞ to \mathbf{r} is

$$\frac{1}{m} \int_{\infty}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r} = \int_{\infty}^{\mathbf{r}} \mathbf{f} \cdot d\mathbf{r} = [\phi(\mathbf{r}) - \phi(\infty)] = \phi(\mathbf{r}).$$

■

Definition 129 The gravitational potential energy of a point mass m at \mathbf{r} equals $-m\phi(\mathbf{r})$.

Remark 130 In many texts the gravitational potential Φ is instead defined to be the gravitational energy per unit mass; in such texts then the potential used is $\Phi = -\phi$ and $\mathbf{f} = -\nabla\Phi$.

Given n particles of mass M_i at points with position vectors \mathbf{r}_i , the gravitational potential ϕ and field \mathbf{f} are then given by

$$\phi(\mathbf{p}) = \sum_{i=1}^n \frac{GM_i}{|\mathbf{p} - \mathbf{r}_i|}, \quad \mathbf{f}(\mathbf{p}) = - \sum_{i=1}^n \frac{GM_i(\mathbf{p} - \mathbf{r}_i)}{|\mathbf{p} - \mathbf{r}_i|^3}.$$

Suppose instead, in the continuous case, we have matter of density $\rho(\mathbf{r})$ occupying a region R ; then the potential ϕ and field \mathbf{f} are given by

$$\phi(\mathbf{p}) = \iiint_R \frac{G\rho(\mathbf{r}) dV}{|\mathbf{p} - \mathbf{r}|}, \quad \mathbf{f}(\mathbf{p}) = - \iiint_R \frac{G\rho(\mathbf{r})(\mathbf{p} - \mathbf{r}) dV}{|\mathbf{p} - \mathbf{r}|^3}.$$

In the above we have an integral relationship between the mass density ρ and the potential ϕ . Our objective is find an equivalent relationship in terms of a partial differential equation. To proceed we first need to informally consider the Dirac delta function.

The Dirac Delta Function Note that, for a ball B radius a centred at the origin, and with $\mathbf{f}(\mathbf{r}) = \mathbf{r}/r^3 = \mathbf{e}_r/r^2$,

$$\iint_{\partial B} \mathbf{f} \cdot d\mathbf{S} = \iint_{\partial B} \frac{\mathbf{e}_r}{a^2} \cdot (\mathbf{e}_r dS) = \frac{1}{a^2} \iint_{\partial B} dS = \frac{4\pi a^2}{a^2} = 4\pi.$$

A direct calculation shows that

$$\nabla \cdot \left(\frac{\mathbf{r}}{r^3} \right) = 0 \quad \text{for } r \neq 0,$$

but the Divergence Theorem (improperly applied to a discontinuous function) "should" give

$$\iiint_B \nabla \cdot \left(\frac{\mathbf{r}}{r^3} \right) dV = \iint_{\partial B} \frac{\mathbf{r}}{r^3} \cdot d\mathbf{S} = 4\pi,$$

and this would be true for any size sphere (or indeed region) containing the origin. So, for a general region R , we have that

$$\iiint_R \nabla \cdot \left(\frac{\mathbf{r}}{r^3} \right) dV = \begin{cases} 4\pi & \text{if } 0 \in R, \\ 0 & \text{if } 0 \notin R. \end{cases}$$

The shorthand for this is to write

$$\nabla \cdot \left(\frac{\mathbf{r}}{r^3} \right) = 4\pi\delta(\mathbf{r})$$

where δ is the **Dirac Delta Function**. This function has the important sifting property that

$$\iiint_R \phi(\mathbf{r}) \delta(\mathbf{r} - \mathbf{a}) dV = \phi(\mathbf{a}).$$

(All the above can, in fact, be made rigorous as part of the theory of **distributions** or **generalized functions** which were developed by Schwartz in the 1940s.)

Theorem 131 (Poisson's Equation) Let ϕ be the gravitational potential associated with a non-uniform material of density $\rho(\mathbf{r})$ occupying a region R . Then

$$\nabla^2 \phi = -4\pi G \rho(\mathbf{r}).$$

Proof. (Non-examinable for the general case.) The gravitational field associated with this matter is given by

$$\mathbf{f}(\mathbf{p}) = \nabla \phi(\mathbf{p}) = - \iiint_R \frac{G\rho(\mathbf{r})(\mathbf{p} - \mathbf{r})}{|\mathbf{p} - \mathbf{r}|^3} dV.$$

Then $\nabla^2 \phi(\mathbf{p})$ equals

$$\nabla \cdot \mathbf{f}(\mathbf{p}) = - \iiint_R G\rho(\mathbf{r}) \nabla \cdot \left(\frac{(\mathbf{p} - \mathbf{r})}{|\mathbf{p} - \mathbf{r}|^3} \right) dV = - \iiint_R G\rho(\mathbf{r}) 4\pi \delta(\mathbf{r} - \mathbf{p}) dV = -4\pi G \rho(\mathbf{p}).$$

■

Example 132 A spherical shell, centred at O and with internal and external radii a and b respectively, has matter distribution with density ρ given by

$$\rho(r) = \begin{cases} 1/r & 0 < a \leq r \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

Find the gravitational potential at all points in space.

[You may assume that the spherically symmetric form of Laplace's equation is such that $\nabla^2 \phi(r) = \frac{1}{r^2} \frac{d}{dr} (r^2 \frac{d\phi}{dr})$.]

Solution. Method One – Poisson's Equation.

By Poisson's equation gravitational potential ϕ satisfies

$$\nabla^2 \phi = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi}{dr} \right) = \begin{cases} 0 & r < a, \\ -4\pi G/r & a < r < b, \\ 0 & b < r. \end{cases}$$

Solving $\nabla^2 \phi = 0$ in the region $r < a$ we get

$$\begin{aligned} \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi}{dr} \right) &= 0, \\ \implies r^2 \frac{d\phi}{dr} &= -A \implies \frac{d\phi}{dr} = \frac{-A}{r^2}, \\ \implies \phi &= \frac{A}{r} + B. \end{aligned}$$

Similarly $\phi = E/r + F$ in the region $r > b$ for constants E and F .

In the middle region $a < r < b$ we have

$$\begin{aligned} \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi}{dr} \right) &= \frac{-4\pi G}{r} \\ \implies \frac{d}{dr} \left(r^2 \frac{d\phi}{dr} \right) &= -4\pi Gr \implies r^2 \frac{d\phi}{dr} = -2\pi Gr^2 - C \\ \implies \frac{d\phi}{dr} &= -2\pi G - \frac{C}{r^2} \\ \implies \phi &= -2\pi Gr + \frac{C}{r} + D. \end{aligned}$$

So, for constants A, B, C, D, E, F , we have

$$\phi(r) = \begin{cases} \frac{A}{r} + B & r < a, \\ -2\pi Gr + \frac{C}{r} + D & a < r < b, \\ \frac{E}{r} + F & b < r. \end{cases}$$

We can use certain conditions now to determine these constants:

- (i) because of physical considerations ϕ is finite at $r = 0$,
- (ii) ϕ is both continuous and differentiable at both $r = a$ and at $r = b$,
- (iii) ϕ is unique only up to addition by a constant – to specify ϕ uniquely we typically stipulate that $\phi(r) \rightarrow 0$ as $r \rightarrow \infty$.

From condition (i) we have that $A = 0$ and from (iii) that $F = 0$.

The two boundary conditions at $r = a$, namely that $\phi(a_-) = \phi(a_+)$ and $\phi'(a_-) = \phi'(a_+)$, tell us that

$$B = -2\pi Ga + \frac{C}{a} + D, \quad 0 = -2\pi G - \frac{C}{a^2}.$$

Similar conditions at $r = b$ give us that

$$-2\pi Gb + \frac{C}{b} + D = \frac{E}{b}, \quad -2\pi G - \frac{C}{b^2} = -\frac{E}{b^2}.$$

We have four linear equations in the four unknowns B, C, D, E in terms of the radii a, b and the gravitational constant G .

The second equation immediately gives C and thus the fourth equation gives E . Then the third equation gives D and the first finally gives B . Thus one can then deduce:

$$\phi(r) = \begin{cases} 4\pi G(b-a) & r < a, \\ -2\pi Gr - \frac{2\pi Ga^2}{r} + 4\pi Gb & a < r < b, \\ \frac{2\pi G(b^2-a^2)}{r} & b < r. \end{cases}$$

Because of the symmetry of the above problem, we can take another approach using Gauss' Flux Theorem.

Theorem 133 (Gauss' Flux Theorem) For a smooth and bounded region R , which contains matter of total mass M , then

$$\iint_{\partial R} \mathbf{f} \cdot d\mathbf{S} = -4\pi GM.$$

This result is in fact equivalent to Poisson's equation.

Proof. Poisson's equation gives us that

$$\nabla^2 \phi = -4\pi G \rho$$

where $\rho(\mathbf{r})$ is the density of the matter. Applying the Divergence Theorem we have

$$\iint_{\partial R} \mathbf{f} \cdot d\mathbf{S} = \iiint_R \nabla \cdot \mathbf{f} dV = \iiint_R \nabla^2 \phi dV = -4\pi G \iiint_R \rho dV = -4\pi GM.$$

Conversely suppose that we know

$$\iint_{\partial R} \mathbf{f} \cdot d\mathbf{S} = -4\pi GM$$

for any bounded region R . Then

$$\iiint_R \nabla^2 \phi dV = -4\pi G \iiint_R \rho dV$$

and so

$$\iiint_R (\nabla^2 \phi + 4\pi G \rho) dV = 0 \text{ for any bounded region } R.$$

Hence (at least if $\nabla^2 \phi$ and ρ are piecewise continuous) we have

$$\nabla^2 \phi + 4\pi G \rho \equiv 0.$$

■ **Solution.** (to Example 132) Method Two - Gauss' Flux Theorem.

Alternatively, we may use Gauss' Flux Theorem applied to concentric spheres centred on the shell's centre.

Now, ϕ is only dependent on r and, so, is constant on the sphere $r = R$, which has surface area $4\pi R^2$.

So, if we apply the flux theorem to the region $r \leq R$ we have

$$\iint_{r=R} \nabla \phi \cdot d\mathbf{S} = \iint_{r=R} \phi'(R) \mathbf{e}_r \cdot d\mathbf{S} = \iint_{r=R} \phi'(R) dS = 4\pi R^2 \phi'(R) = -4\pi GM(R)$$

where $M(R)$ is the total mass within the region $r \leq R$. For $a \leq R \leq b$ we have

$$\begin{aligned} M(R) &= \int_{r=a}^R \int_{\theta=0}^{\pi} \int_{\alpha=0}^{2\pi} \frac{1}{r} r^2 \sin \theta d\alpha d\theta dr \\ &= 2\pi \times [-\cos \theta]_0^\pi \times \int_{r=a}^R r dr \\ &= 2\pi (R^2 - a^2). \end{aligned}$$

Hence

$$\phi'(R) = \begin{cases} 0 & R < a, \\ 2\pi G \left(\frac{a^2}{R^2} - 1 \right) & a < R < b, \\ \frac{2\pi G (a^2 - b^2)}{R^2} & b < R. \end{cases}$$

If we integrate we have

$$\phi(R) = \begin{cases} A & R < a, \\ B - 2\pi G \left(\frac{a^2}{R} + R \right) & a < R < b, \\ \frac{2\pi G (b^2 - a^2)}{R} + C & b < R. \end{cases}$$

As $\phi(\infty) = 0$ then $C = 0$. As $\phi(b_-) = \phi(b_+)$ then

$$B - 2\pi G \left(\frac{a^2}{b} + b \right) = \frac{2\pi G (b^2 - a^2)}{b} \implies B = 4\pi G b.$$

Finally as $\phi(a_-) = \phi(a_+)$ then

$$A = 4\pi G b - 2\pi G \left(\frac{a^2}{a} + a \right) = 4\pi G (b - a),$$

and we have the same expressions for ϕ as were achieved by the previous method. ■