Random Numbers

CONTENTS

Abstract—**This manual provides a simple introduction to the generation of random numbers**

1 UNIFORM RANDOM NUMBERS

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following C code.

> https://github.com/Sasank−2004/Random−
>   Numbers/blob/main/codes/uniform.c

Use the following commands to execute the C code.

```
gcc uniform.c
./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \qquad (1.1)$$

Solution: The following code plots Fig. 1.2

> https://github.com/Sasank−2004/Random−
>   Numbers/blob/main/codes/
>   uniform_cdf_plot.py

Use the following command to execute the python code.

```
python3 uniform_cdf_plot.py
```

1.3 Find a theoretical expression for $F_U(x)$.
Solution: U is a Uniform random variable

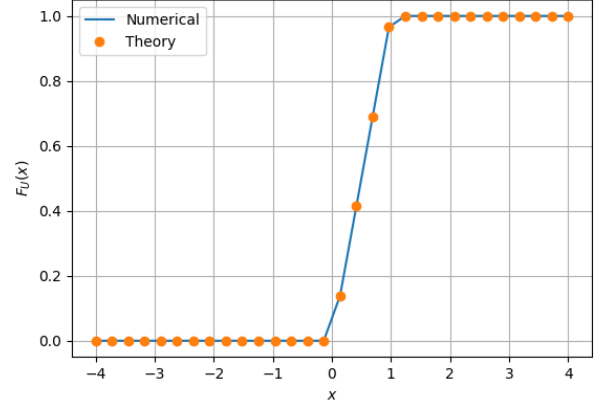Fig. 1.2: The CDF of U

between 0 and 1.So the p.d.f of U is given by

$$p_U(x) = \begin{cases} 0 & x < 0 \\ 1 & 0 \leq x \leq 1 \\ 0 & x > 1 \end{cases} \qquad (1.2)$$

We Know That

$$F_U(x) = \int_{-\infty}^{x} p_U(x)dx \qquad (1.3)$$

$$\Rightarrow F_U(x) = \begin{cases} 0 & x < 0 \\ \int_0^x 1dx & 0 \leq x \leq 1 \\ \int_0^1 1dx & x > 1 \end{cases} \qquad (1.4)$$

$$\Rightarrow F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases} \qquad (1.5)$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^{N} U_i \qquad (1.6)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \qquad (1.7)$$

Write a C program to find the mean and variance of U.
Solution: The following C Code gives the mean and variance.

```
https://github.com/Sasank−2004/Random−
    Numbers/blob/main/codes/
    uniform_mean_and_variance.c
```

Use the following commands to execute the C code.

```
gcc uniform_mean_and_variance.c
./a.out
```

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.8)$$

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.9)$$

Solution: We know that

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases} \quad (1.10)$$

$$\Rightarrow dF_U(x) = \begin{cases} 0 & x < 0 \\ dx & 0 \le x \le 1 \\ 0 & x > 1, \end{cases} \quad (1.11)$$

$$\Rightarrow E[U] = \int_0^1 x\,dx = 0.5 \quad (1.12)$$

$$\Rightarrow E[U^2] = \int_0^1 x\,dx = 0.33 \quad (1.13)$$

Mean = E[U] = 0.5
Variance = $E[U^2] - (E[U])^2 = 0.08$

2 CENTRAL LIMIT THEOREM

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution: Download the following C code.

```
https://github.com/Sasank−2004/Random−
    Numbers/blob/main/codes/gaussian.c
```

Use the following commands to execute the C code.

```
gcc gaussian.c
./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The following code plots Fig. 2.2

```
https://github.com/Sasank−2004/Random−
    Numbers/blob/main/codes/
    uniform_cdf_plot.py
```

Use the following command to execute the python code.

```
python3 gaussian_cdf_plot.py
```
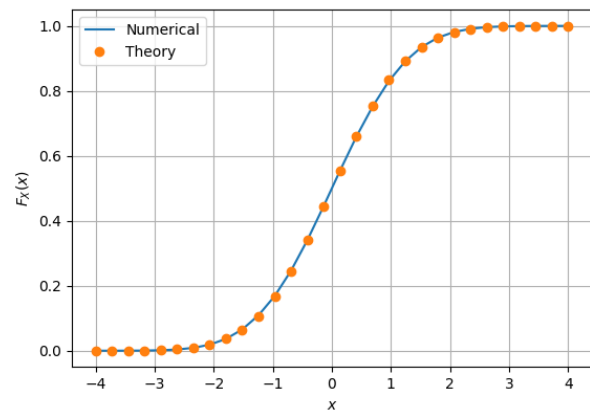

Fig. 2.2: The CDF of X

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.2)$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
https://github.com/Sasank−2004/Random−
    Numbers/blob/main/codes/
    gaussian_pdf_plot.py
```

Use the following command to execute the python code.
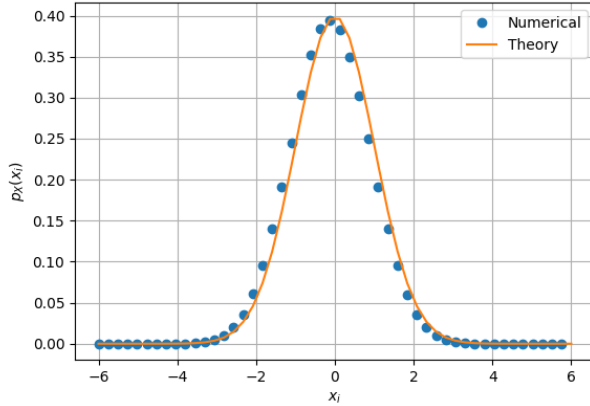
```
python3 gaussian_pdf_plot.py
```

Fig. 2.3: The PDF of X

2.4 Find the mean and variance of X by writing a C program. **Solution:** The following C Code gives the mean and variance.

```
https://github.com/Sasank−2004/Random−
    Numbers/blob/main/codes/
    gaussian_mean_and_variance.c
```

Use the following commands to execute the C code.

```
gcc gaussian_mean_and_variance.c
./a.out
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.3)$$

repeat the above exercise theoretically.
Solution: CDF of the distribution is given by

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \quad (2.4)$$

Mean of the distribution is

$$E[X] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} xe^{-x^2/2} dx \quad (2.5)$$

The function $xe^{-x^2/2}$ is odd function

$$\therefore E[X] = 0 \quad (2.6)$$

Variance of the distribution is

$$var[X] = E[X^2] - (E[X])^2 \quad (2.7)$$

$$\Rightarrow var[X] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2/2} dx \quad (2.8)$$

$$\Rightarrow var[X] = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} x^2 e^{-x^2/2} dx \quad (2.9)$$

Let $x^2 = t$

$$\Rightarrow var[X] = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} te^{-t/2} \frac{dt}{2\sqrt{t}} \quad (2.10)$$

$$\Rightarrow var[X] = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \sqrt{t} e^{-t/2} dt \quad (2.11)$$

If we apply Integration by parts we get

$$var[X] = \frac{1}{\sqrt{2\pi}} \left(\sqrt{t}\left(-2e^{-t/2}\right)\Big|_{0}^{\infty} + \int_{0}^{\infty} \frac{e^{-t/2}}{\sqrt{t}} dt \right) \quad (2.12)$$

$$\Rightarrow var[X] = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{e^{-t/2}}{\sqrt{t}} dt \quad (2.13)$$

Let $t = p^2$

$$\Rightarrow var[X] = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} 2e^{-p^2/2} dp \quad (2.14)$$

$$\Rightarrow var[X] = \frac{2}{\sqrt{2\pi}} \times \frac{\sqrt{2\pi}}{2} \quad (2.15)$$

$$\Rightarrow var[X] = 1 \quad (2.16)$$

3 FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2\ln(1 - U) \quad (3.1)$$

and plot its CDF.
Solution: Download the following codes

```
https://github.com/Sasank−2004/Random−
    Numbers/blob/main/codes/variable.c
https://github.com/Sasank−2004/Random−
    Numbers/blob/main/codes/
    variable_cdf_plot.py
```

Use the following commands to execute the C code and Python code.

```
gcc variable.c −lm
./a.out
python3 variable_cdf_plot.py
```

3.2 Find a theoretical expression for $F_V(x)$.
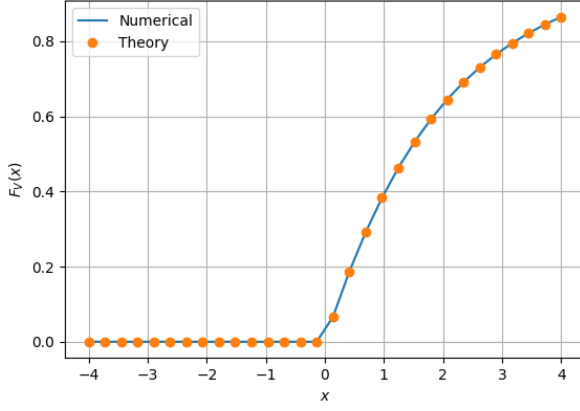**Solution:** We know that if $Y = g(X)$ then

Fig. 3.1: The PDF of X

$F_Y(x) = F_X(g^{-1}(x))$

Given $V = -2\ln(1-U)$

$$\Rightarrow e^{-V/2} = 1 - U \tag{3.2}$$

$$\Rightarrow U = 1 - e^{-V/2} \tag{3.3}$$

$$\Rightarrow F_V(x) = F_U(1 - e^{-x/2}) \tag{3.4}$$

$$F_U(1 - e^{-x/2}) = \begin{cases} 0 & 1 - e^{-x/2} < 0 \\ 1 - e^{-x/2} & 0 \le 1 - e^{-x/2} \le 1 \\ 1 & 1 - e^{-x/2} > 1 \end{cases}$$
$$\tag{3.5}$$

$$\Rightarrow F_U(1 - e^{-x/2}) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x/2} & x \ge 0 \end{cases} \tag{3.6}$$

$$\therefore F_V(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x/2} & x \ge 0 \end{cases} \tag{3.7}$$