Random Numbers

Gollapudi Sasank CS21BTECH11019

Contents

***Abstract*—This manual provides a simple introduction to the generation of random numbers**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following C code.

```
wget https://raw.githubusercontent.com/
    Sasank−2004/Random−Numbers/main/
    codes/1.1.c
wget https://raw.githubusercontent.com/
    Sasank−2004/Random−Numbers/main/
    codes/coeffs.h
```

Use the following commands to execute the C code.

```
gcc 1.1.c −lm
./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{1.1}$$

Solution: The following code plots Fig. 1.2

```
wget https://raw.githubusercontent.com/
    Sasank−2004/Random−Numbers/main/
    codes/1.2.py
```

Use the following command to execute the python code.

```
python3 1.2.py
```

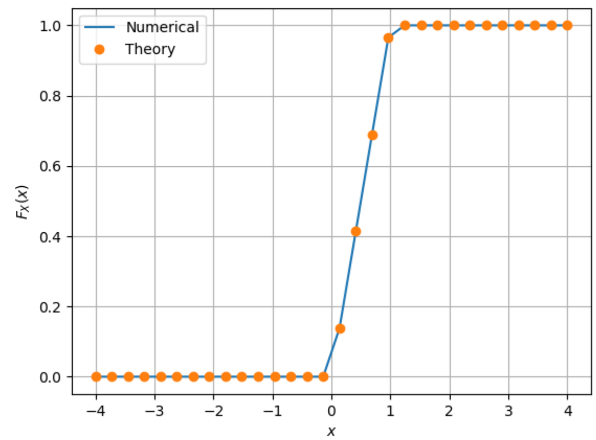

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution: U is a Uniform random variable between 0 and 1.So the p.d.f of U is given by

$$p_U(x) = \begin{cases} 0 & x < 0 \\ 1 & 0 \le x \le 1 \\ 0 & x > 1 \end{cases} \tag{1.2}$$

We Know That

$$F_U(x) = \int_{-\infty}^{x} p_U(x)dx \tag{1.3}$$

$$\implies F_U(x) = \begin{cases} 0 & x < 0 \\ \int_0^x 1dx & 0 \le x \le 1 \\ \int_0^1 1dx & x > 1 \end{cases} \tag{1.4}$$

$$\implies F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases} \tag{1.5}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.6}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.7}$$

Write a C program to find the mean and variance of U.

Solution: The following C Code gives the mean and variance.

```
wget https://raw.githubusercontent.com/
    Sasank−2004/Random−Numbers/main/
    codes/1.4.c
```

Use the following commands to execute the C code.

```
gcc 1.4.c −lm
./a.out
```

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.8}$$

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.9}$$

Solution: We know that

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases} \tag{1.10}$$

$$\Rightarrow dF_U(x) = \begin{cases} 0 & x < 0 \\ dx & 0 \le x \le 1 \\ 0 & x > 1, \end{cases} \tag{1.11}$$

$$\Rightarrow E[U] = \int_0^1 x dx = 0.5 \tag{1.12}$$

$$\Rightarrow E[U^2] = \int_0^1 x dx = 0.33 \tag{1.13}$$

Mean = E[U] = 0.5

$$var[U] = E[U - E[U]]^2 \tag{1.14}$$

$$\Rightarrow var[U] = E[U^2 + E[U]^2 - 2UE[U]] \tag{1.15}$$

$$\Rightarrow var[U] = E[U^2] + E[U]^2 - 2E[U]^2 \tag{1.16}$$

$$\Rightarrow var[U] = E[U^2] - E[U]^2 \tag{1.17}$$

$\Rightarrow$ Variance $= E[U^2] - E[U]^2 = 0.08$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: Download the following C code.

```
wget https://raw.githubusercontent.com/
    Sasank−2004/Random−Numbers/main/
    codes/2.1.c
```

Use the following commands to execute the C code.

```
gcc 2.1.c −lm
./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: The following code plots Fig. 2.2

```
wget https://raw.githubusercontent.com/
    Sasank−2004/Random−Numbers/main/
    codes/2.2.py
```

Use the following command to execute the python code.

```
python3 2.2.py
```

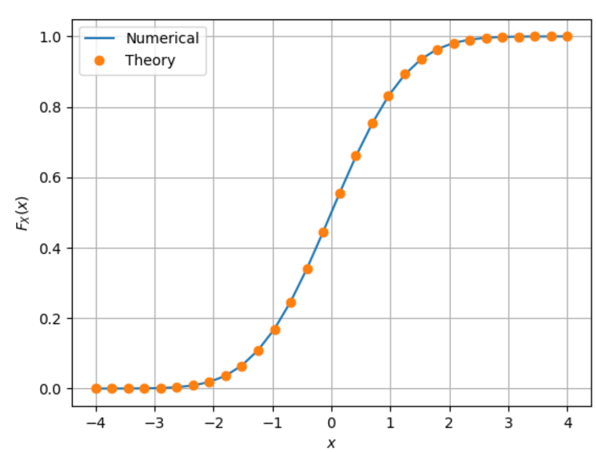

Fig. 2.2: The CDF of X

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx}F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
wget https://raw.githubusercontent.com/
    Sasank−2004/Random−Numbers/main/
    codes/2.3.py
```

Use the following command to execute the python code.

```
python3 2.3.py
```

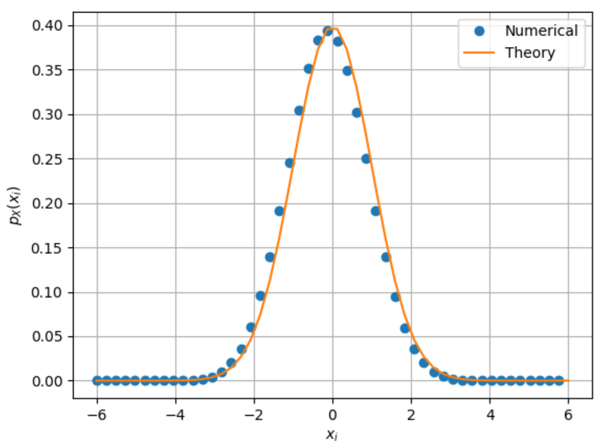

Fig. 2.3: The PDF of X

2.4 Find the mean and variance of X by writing a C program. **Solution:** The following C Code gives the mean and variance.

```
wget https://raw.githubusercontent.com/
    Sasank−2004/Random−Numbers/main/
    codes/2.4.c
```

Use the following commands to execute the C code.

```
gcc 2.4.c −lm
./a.out
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.3}$$

repeat the above exercise theoretically.
Solution: CDF of the distribution is given by

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx \tag{2.4}$$

Mean of the distribution is

$$E[X] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} xe^{-x^2/2}dx \tag{2.5}$$

The function $xe^{-x^2/2}$ is odd function

$$\therefore E[X] = 0 \tag{2.6}$$

Variance of the distribution is

$$var[X] = E[X^2] - E[X]^2 \tag{2.7}$$
$$\Rightarrow var[X] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^2e^{-x^2/2}dx \tag{2.8}$$
$$\Rightarrow var[X] = \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} x^2e^{-x^2/2}dx \tag{2.9}$$

Let $x^2 = t$

$$\Rightarrow var[X] = \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} te^{-t/2}\frac{dt}{2\sqrt{t}} \tag{2.10}$$
$$\Rightarrow var[X] = \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} \sqrt{t}e^{-t/2}dt \tag{2.11}$$

If we apply Integration by parts we get

$$var[X] = \frac{1}{\sqrt{2\pi}}\left(\sqrt{t}\left(-2e^{-t/2}\right)\Big|_0^{\infty} + \int_0^{\infty}\frac{e^{-t/2}}{\sqrt{t}}dt\right) \tag{2.12}$$

$$\Rightarrow var[X] = \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} \frac{e^{-t/2}}{\sqrt{t}}dt \tag{2.13}$$

Let $t = p^2$

$$\Rightarrow var[X] = \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} 2e^{-p^2/2}dp \tag{2.14}$$
$$\Rightarrow var[X] = \frac{2}{\sqrt{2\pi}} \times \frac{\sqrt{2\pi}}{2} \tag{2.15}$$
$$\Rightarrow var[X] = 1 \tag{2.16}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1-U) \tag{3.1}$$

and plot its CDF.
Solution: Download the following codes

```
wget https://raw.githubusercontent.com/
    Sasank-2004/Random-Numbers/main/
    codes/3.1.c
wget https://raw.githubusercontent.com/
    Sasank-2004/Random-Numbers/main/
    codes/3.2.c
```

Use the following commands to execute the C code and Python code.

```
gcc 3.1.c -lm
./a.out
python3 3.2.py
```

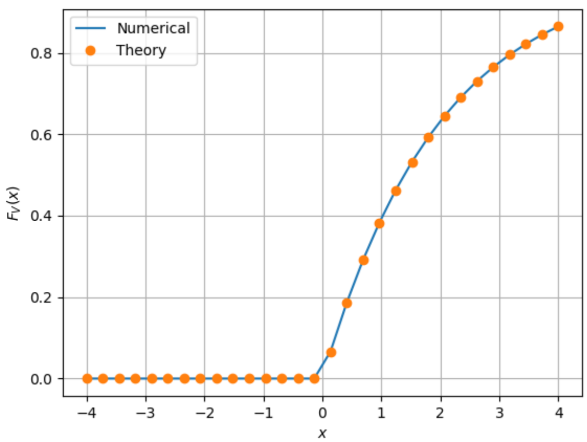

Fig. 3.1: The CDF of V

3.2 Find a theoretical expression for $F_V(x)$.
Solution: We know that if $Y = g(X)$ then $F_Y(x) = F_X(g^{-1}(x))$
Given $V = -2\ln(1-U)$

$$\Rightarrow e^{-V/2} = 1 - U \tag{3.2}$$

$$\Rightarrow U = 1 - e^{-V/2} \tag{3.3}$$

$$\Rightarrow F_V(x) = F_U(1 - e^{-x/2}) \tag{3.4}$$

$$F_U(1-e^{-x/2}) = \begin{cases} 0 & 1-e^{-x/2} < 0 \\ 1-e^{-x/2} & 0 \le 1-e^{-x/2} \le 1 \\ 1 & 1-e^{-x/2} > 1 \end{cases} \tag{3.5}$$

$$\Rightarrow F_U(1-e^{-x/2}) = \begin{cases} 0 & x < 0 \\ 1-e^{-x/2} & x \ge 0 \end{cases} \tag{3.6}$$

$$\therefore F_V(x) = \begin{cases} 0 & x < 0 \\ 1-e^{-x/2} & x \ge 0 \end{cases} \tag{3.7}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: Download the following C code

```
wget https://raw.githubusercontent.com/
    Sasank-2004/Random-Numbers/main/
    codes/4.1.c
```

Use the following commands to execute the C code.

```
gcc 4.1.c -lm
./a.out
```

4.2 Find the CDF of T.
Solution: The following code plots Fig. 4.2

```
wget https://raw.githubusercontent.com/
    Sasank-2004/Random-Numbers/main/
    codes/4.2.py
```

Use the following command to execute the python code.

```
python3 4.2.py
```

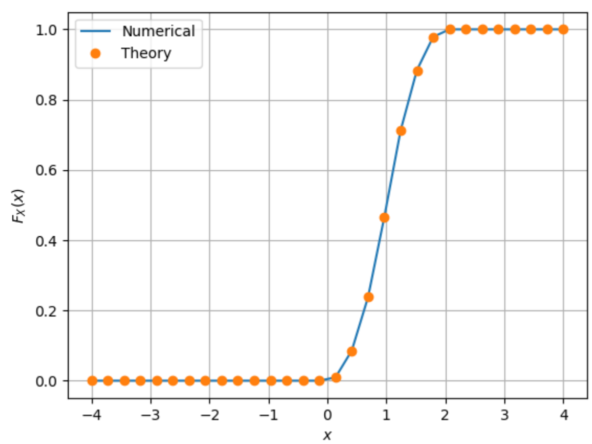

Fig. 4.2: The CDF of T

4.3 Find the PDF of T.
Solution: The PDF of T is plotted in Fig. 4.3 using the code below

```
wget https://raw.githubusercontent.com/
    Sasank-2004/Random-Numbers/main/
    codes/4.3.py
```

Use the following command to execute the python code.

```
python3 4.3.py
```

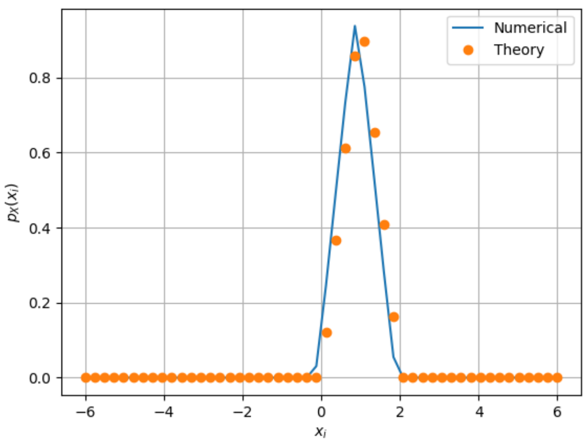

Fig. 4.3: The PDF of T

4.4 Find the theoretical expressions for the PDF and CDF of T.
Solution: Given $T = U_1 + U_2$
Let x : Represent values of U_1
and y : Represent values of U_2
We Know that $x, y \in (0, 1) \implies 0 < t < 2$
For $0 < t \leq 1$

$$F_T(t) = \int_{y=0}^{t} \int_{x=0}^{t-y} 1dxdy \tag{4.2}$$

$$F_T(t) = \int_{y=0}^{t} (t-y)dy \tag{4.3}$$

$$F_T(t) = \left(ty - \frac{y^2}{2}\right)\Bigg|_0^z \tag{4.4}$$

$$F_T(t) = \frac{t^2}{2} \tag{4.5}$$

For $1 < t \leq 2$

$$F_T(t) = 1 - \int_{y=t-1}^{1} \int_{x=t-y}^{1} 1dxdy \tag{4.6}$$

$$F_T(t) = 1 - \int_{y=t-1}^{1} (1-t+y)dy \tag{4.7}$$

$$F_T(t) = 1 - \left(y - ty + \frac{y^2}{2}\right)\Bigg|_{t-1}^{1} \tag{4.8}$$

$$F_T(t) = 1 - \left(1 - t + \frac{1}{2}\right) + \left(t - 1 - t(t-1) + \frac{(t-1)^2}{2}\right) \tag{4.9}$$

$$F_T(t) = 1 - 1 + t - \frac{1}{2} + t - 1 - t^2 + t + \frac{t^2}{2} + \frac{1}{2} - t \tag{4.10}$$

$$F_T(t) = -\frac{t^2}{2} + 2t - 1 \tag{4.11}$$

For $t < 0$ $F_T(t) = 0$
For $t > 2$ $F_T(t) = F_T(2) = 1$

$$\therefore F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t \leq 1 \\ -\frac{t^2}{2} + 2t - 1 & 1 < t \leq 2 \\ 1 & t > 2 \end{cases} \tag{4.12}$$

We know that

$$f_T(t) = \frac{d}{dt}(F_T(t)) \tag{4.13}$$

$$\implies f_t(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \leq 1 \\ 2 - t & 1 < t \leq 2 \\ 0 & t > 2 \end{cases} \tag{4.14}$$

4.5 Verify your results through a plot.
Solution: From the figures Fig:4.2 and Fig:4.3 we can verify the theoretical and experimental plots.