

Random Numbers

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Abstract—This manual provides a simple introduction to the generation of random numbers

1 UNIFORM RANDOM NUMBERS

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution: Download the following C code.

```
wget https://raw.githubusercontent.com/
Sasank-2004/Random-Numbers/main/
codes/1.1.c
wget https://raw.githubusercontent.com/
Sasank-2004/Random-Numbers/main/
codes/coeffs.h
```

Use the following commands to execute the C code.

```
gcc 1.1.c -lm
./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \quad (1.1)$$

Solution: The following code plots Fig. 1.2

```
wget https://raw.githubusercontent.com/
Sasank-2004/Random-Numbers/main/
codes/1.2.py
```

Use the following command to execute the python code.

```
python3 1.2.py
```

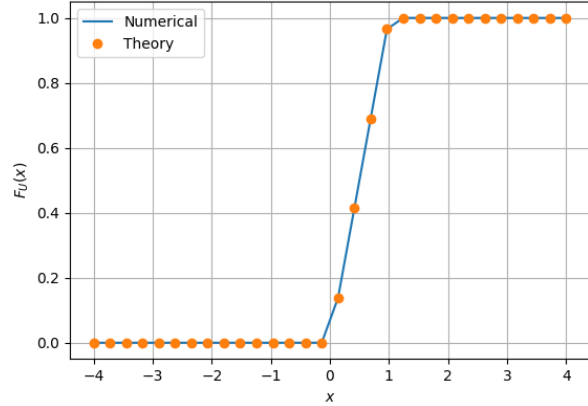


Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.

Solution: U is a Uniform random variable between 0 and 1. So the p.d.f of U is given by

$$p_U(x) = \begin{cases} 0 & x < 0 \\ 1 & 0 \leq x \leq 1 \\ 0 & x > 1 \end{cases} \quad (1.2)$$

We Know That

$$F_U(x) = \int_{-\infty}^x p_U(x) dx \quad (1.3)$$

$$\Rightarrow F_U(x) = \begin{cases} 0 & x < 0 \\ \int_0^x 1 dx & 0 \leq x \leq 1 \\ \int_0^1 1 dx & x > 1 \end{cases} \quad (1.4)$$

$$\Rightarrow F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases} \quad (1.5)$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (1.6)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \quad (1.7)$$

Write a C program to find the mean and variance of U .

Solution: The following C Code gives the mean and variance.

```
wget https://raw.githubusercontent.com/
Sasank-2004/Random-Numbers/main/
codes/1.4.c
```

Use the following commands to execute the C code.

```
gcc 1.4.c -lm
./a.out
```

1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.8)$$

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.9)$$

Solution: We know that

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases} \quad (1.10)$$

$$\Rightarrow dF_U(x) = \begin{cases} 0 & x < 0 \\ dx & 0 \leq x \leq 1 \\ 0 & x > 1, \end{cases} \quad (1.11)$$

$$\Rightarrow E[U] = \int_0^1 x dx = 0.5 \quad (1.12)$$

$$\Rightarrow E[U^2] = \int_0^1 x^2 dx = 0.33 \quad (1.13)$$

Mean = $E[U] = 0.5$

Variance = $E[U^2] - (E[U])^2 = 0.08$

2 CENTRAL LIMIT THEOREM

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: Download the following C code.

```
wget https://raw.githubusercontent.com/
Sasank-2004/Random-Numbers/main/
codes/2.1.c
```

Use the following commands to execute the C code.

```
gcc 2.1.c -lm
./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: The following code plots Fig. 2.2

```
wget https://raw.githubusercontent.com/
Sasank-2004/Random-Numbers/main/
codes/2.2.py
```

Use the following command to execute the python code.

```
python3 2.2.py
```

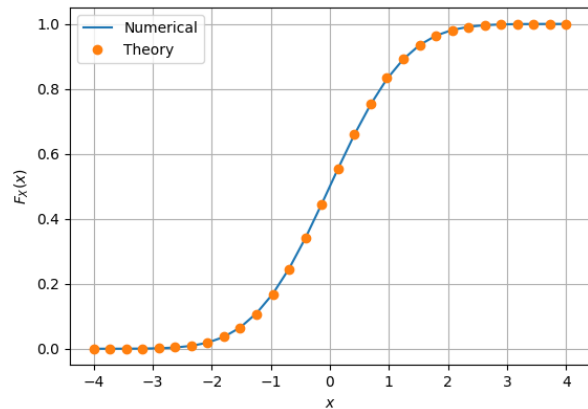


Fig. 2.2: The CDF of X

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.2)$$

What properties does the PDF have?

Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
wget https://raw.githubusercontent.com/
Sasank-2004/Random-Numbers/main/
codes/2.3.py
```

Use the following command to execute the python code.

```
python3 2.3.py
```

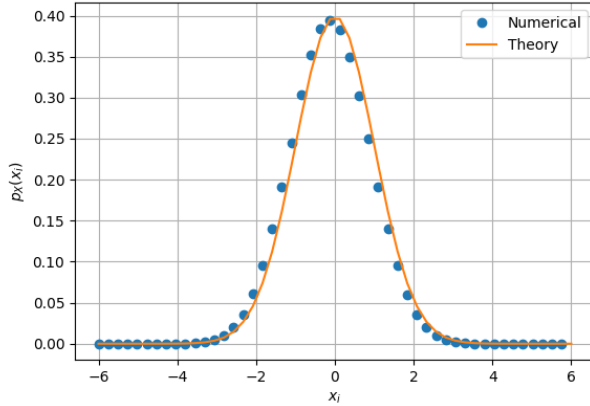


Fig. 2.3: The PDF of X

2.4 Find the mean and variance of X by writing a C program. **Solution:** The following C Code gives the mean and variance.

```
wget https://raw.githubusercontent.com/
Sasank-2004/Random-Numbers/main/
codes/2.4.c
```

Use the following commands to execute the C code.

```
gcc 2.4.c -lm
./a.out
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.3)$$

repeat the above exercise theoretically.

Solution: CDF of the distribution is given by

$$F_X(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \quad (2.4)$$

Mean of the distribution is

$$E[X] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-x^2/2} dx \quad (2.5)$$

The function $x e^{-x^2/2}$ is odd function

$$\therefore E[X] = 0 \quad (2.6)$$

Variance of the distribution is

$$\text{var}[X] = E[X^2] - (E[X])^2 \quad (2.7)$$

$$\Rightarrow \text{var}[X] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2/2} dx \quad (2.8)$$

$$\Rightarrow \text{var}[X] = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} x^2 e^{-x^2/2} dx \quad (2.9)$$

Let $x^2 = t$

$$\Rightarrow \text{var}[X] = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} t e^{-t/2} \frac{dt}{2\sqrt{t}} \quad (2.10)$$

$$\Rightarrow \text{var}[X] = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \sqrt{t} e^{-t/2} dt \quad (2.11)$$

If we apply Integration by parts we get

$$\text{var}[X] = \frac{1}{\sqrt{2\pi}} \left(\sqrt{t} (-2e^{-t/2}) \Big|_0^{\infty} + \int_0^{\infty} \frac{e^{-t/2}}{\sqrt{t}} dt \right) \quad (2.12)$$

$$\Rightarrow \text{var}[X] = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{e^{-t/2}}{\sqrt{t}} dt \quad (2.13)$$

Let $t = p^2$

$$\Rightarrow \text{var}[X] = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} 2e^{-p^2/2} dp \quad (2.14)$$

$$\Rightarrow \text{var}[X] = \frac{2}{\sqrt{2\pi}} \times \frac{\sqrt{2\pi}}{2} \quad (2.15)$$

$$\Rightarrow \text{var}[X] = 1 \quad (2.16)$$

3 FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF.

Solution: Download the following codes

```
wget https://raw.githubusercontent.com/
Sasank-2004/Random-Numbers/main/
codes/3.1.c
wget https://raw.githubusercontent.com/
Sasank-2004/Random-Numbers/main/
codes/3.2.c
```

Use the following commands to execute the C code and Python code.

```
gcc 3.1.c -lm
./a.out
python3 3.2.py
```

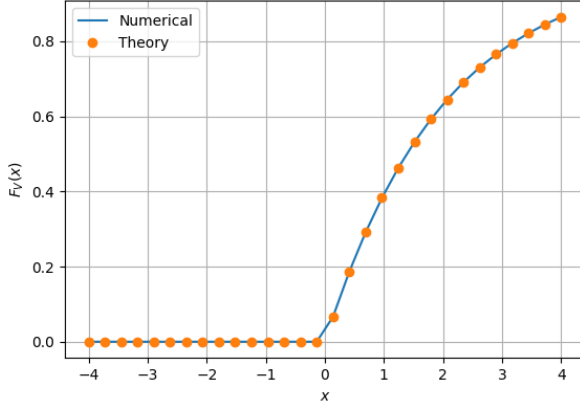


Fig. 3.1: The PDF of X

3.2 Find a theoretical expression for $F_V(x)$.

Solution: We know that if $Y = g(X)$ then

$$F_Y(x) = F_X(g^{-1}(x))$$

Given $V = -2 \ln(1 - U)$

$$\Rightarrow e^{-V/2} = 1 - U \quad (3.2)$$

$$\Rightarrow U = 1 - e^{-V/2} \quad (3.3)$$

$$\Rightarrow F_V(x) = F_U(1 - e^{-x/2}) \quad (3.4)$$

$$F_U(1 - e^{-x/2}) = \begin{cases} 0 & 1 - e^{-x/2} < 0 \\ 1 - e^{-x/2} & 0 \leq 1 - e^{-x/2} \leq 1 \\ 1 & 1 - e^{-x/2} > 1 \end{cases} \quad (3.5)$$

$$\Rightarrow F_U(1 - e^{-x/2}) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x/2} & x \geq 0 \end{cases} \quad (3.6)$$

$$\therefore F_V(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x/2} & x \geq 0 \end{cases} \quad (3.7)$$