# Random Numbers

Contents




***Abstract*—This manual provides a simple introduction to the generation of random numbers**


## 1 Uniform Random Numbers

Let $U$ be a uniform random variable between 0 and 1.

1.1 Generate $10^6$ samples of $U$ using a C program and save into a file called uni.dat .
**Solution:** Download the following C code.

```
https://github.com/Sasank−2004/Random−
    Numbers/blob/main/codes/uniform.c
```

Use the following commands to execute the C code.

```
gcc uniform.c
./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of $U$ using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

**Solution:** The following code plots Fig. 1.2

```
https://github.com/Sasank−2004/Random−
    Numbers/blob/main/codes/
    uniform_cdf_plot.py
```

Use the following command to execute the python code.

```
python3 uniform_cdf_plot.py
```

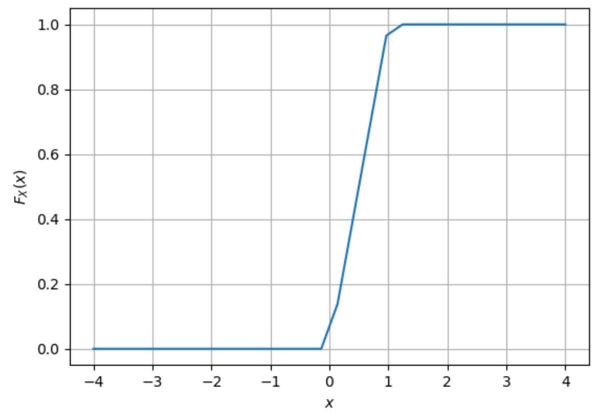


Fig. 1.2: The CDF of $U$

1.3 Find a theoretical expression for $F_U(x)$.
**Solution:** $U$ is a Uniform random variable between 0 and 1.So the p.d.f of $U$ is given by

$$p_U(x) = \begin{cases} 0 & x < 0 \\ 1 & 0 \leq x \leq 1 \\ 0 & x > 1 \end{cases} \tag{1.2}$$

We Know That

$$F_U(x) = \int_{-\infty}^{x} p_U(x)dx \tag{1.3}$$

$$\implies F_U(x) = \begin{cases} 0 & x < 0 \\ \int_0^x 1dx & 0 \leq x \leq 1 \\ \int_0^1 1dx & x > 1 \end{cases} \tag{1.4}$$

$$\implies F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases} \tag{1.5}$$

1.4 The mean of $U$ is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.6}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.7}$$

Write a C program to find the mean and variance of $U$.
**Solution:** The following C Code gives the mean and variance.

```
https://github.com/Sasank-2004/Random-
    Numbers/blob/main/codes/
    uniform_mean_and_variance.c
```

Use the following commands to execute the C code.

```
gcc uniform_mean_and_variance.c
./a.out
```

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.8}$$

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.9}$$

**Solution:** We know that

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases} \tag{1.10}$$

$$\Rightarrow dF_U(x) = \begin{cases} 0 & x < 0 \\ dx & 0 \le x \le 1 \\ 0 & x > 1, \end{cases} \tag{1.11}$$

$$\Rightarrow E[U] = \int_0^1 xdx = 0.5 \tag{1.12}$$

$$\Rightarrow E[U^2] = \int_0^1 xdx = 0.33 \tag{1.13}$$

Mean = E[U] = 0.5
Variance = $E[U^2] - (E[U])^2 = 0.08$

## 2 Central Limit Theorem

2.1 Generate $10^6$ samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
**Solution:** Download the following C code.

```
https://github.com/Sasank-2004/Random-
    Numbers/blob/main/codes/gaussian.c
```

Use the following commands to execute the C code.

```
gcc gaussian.c
./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of $X$ using the samples in gau.dat. What properties does a CDF have?
**Solution:** The following code plots Fig. 2.2

```
https://github.com/Sasank-2004/Random-
    Numbers/blob/main/codes/
    uniform_cdf_plot.py
```

Use the following command to execute the python code.

```
python3 gaussian_cdf_plot.py
```

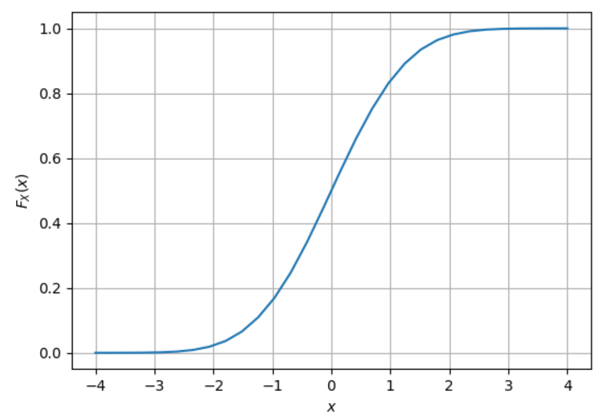


Fig. 2.2: The CDF of $X$

2.3 Load gau.dat in python and plot the empirical PDF of $X$ using the samples in gau.dat. The PDF of $X$ is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

What properties does the PDF have?
**Solution:** The PDF of $X$ is plotted in Fig. 2.3 using the code below

```
https://github.com/Sasank-2004/Random-
    Numbers/blob/main/codes/
    gaussian_pdf_plot.py
```

Use the following command to execute the python code.

```
python3 gaussian_pdf_plot.py
```

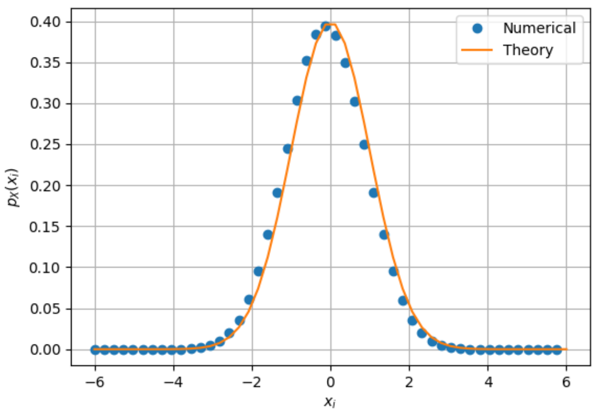


Fig. 2.3: The PDF of $X$

2.4 Find the mean and variance of $X$ by writing a C program. **Solution:** The following C Code gives the mean and variance.

```
https://github.com/Sasank−2004/Random−
    Numbers/blob/main/codes/
    gaussian_mean_and_variance.c
```

Use the following commands to execute the C code.

```
gcc gaussian_mean_and_variance.c
./a.out
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.3)$$

repeat the above exercise theoretically.
**Solution:** CDF of the distribution is given by

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \quad (2.4)$$

Mean of the distribution is

$$E[X] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-x^2/2} dx \quad (2.5)$$

The function $xe^{-x^2/2}$ is odd function

$$\therefore E[X] = 0 \quad (2.6)$$

Variance of the distribution is

$$var[X] = E[X^2] - (E[X])^2 \quad (2.7)$$

$$\Rightarrow var[X] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2/2} dx \quad (2.8)$$

$$\Rightarrow var[X] = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} x^2 e^{-x^2/2} dx \quad (2.9)$$

Let $x^2 = t$

$$\Rightarrow var[X] = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} t e^{-t/2} \frac{dt}{2\sqrt{t}} \quad (2.10)$$

$$\Rightarrow var[X] = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \sqrt{t} e^{-t/2} dt \quad (2.11)$$

If we apply Integration by parts we get

$$var[X] = \frac{1}{\sqrt{2\pi}} \left( \sqrt{t} \left(-2e^{-t/2}\right)\Big|_0^{\infty} + \int_0^{\infty} \frac{e^{-t/2}}{\sqrt{t}} dt \right) \quad (2.12)$$

$$\Rightarrow var[X] = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{e^{-t/2}}{\sqrt{t}} dt \quad (2.13)$$

Let $t = p^2$

$$\Rightarrow var[X] = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} 2e^{-p^2/2} dp \quad (2.14)$$

$$\Rightarrow var[X] = \frac{2}{\sqrt{2\pi}} \times \frac{\sqrt{2\pi}}{2} \quad (2.15)$$

$$\Rightarrow var[X] = 1 \quad (2.16)$$

## 3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1 - U) \quad (3.1)$$

and plot its CDF.
**Solution:** Download the following codes

```
https://github.com/Sasank−2004/Random−
    Numbers/blob/main/codes/variable.c
https://github.com/Sasank−2004/Random−
    Numbers/blob/main/codes/
    variable_cdf_plot.py
```

Use the following commands to execute the C code and Python code.

```
gcc variable.c −lm
./a.out
python3 variable_cdf_plot.py
```

3.2 Find a theoretical expression for $F_V(x)$.
**Solution:** We know that if $Y = g(X)$ then

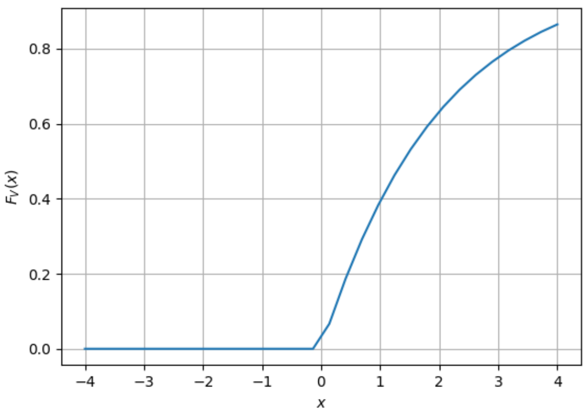


Fig. 3.1: The PDF of $X$

$F_Y(x) = F_X(g^{-1}(x))$
Given $V = -2\ln(1-U)$

$$\Rightarrow e^{-V/2} = 1 - U \tag{3.2}$$

$$\Rightarrow U = 1 - e^{-V/2} \tag{3.3}$$

$$\Rightarrow F_V(x) = F_U(1 - e^{-x/2}) \tag{3.4}$$

$$F_U(1 - e^{-x/2}) = \begin{cases} 0 & 1 - e^{-x/2} < 0 \\ 1 - e^{-x/2} & 0 \le 1 - e^{-x/2} \le 1 \\ 1 & 1 - e^{-x/2} > 1 \end{cases} \tag{3.5}$$

$$\Rightarrow F_U(1 - e^{-x/2}) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x/2} & x \ge 0 \end{cases} \tag{3.6}$$

$$\therefore F_V(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x/2} & x \ge 0 \end{cases} \tag{3.7}$$