MATH 2418: Linear Algebra

Assignment# 2

Due :02/03, Tuesday before 11:59pm — Term :Spring 2026

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Recommended Problems: (Do not turn in)
Sec 1.2: 1, 2, 5, 6, 7, 8, 12, 13, 14, 25, 28, 29, 31. Sec 1.3: 1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 14, 16, 18, 20, 21, 23.

1. Let $\mathbf{u} = (-4, 2, 1)$ and $\mathbf{v} = (-1, -1, 3)$ be two vectors in $\mathbb{R}^3$.

   (a) Calculate the dot product $\mathbf{u} \cdot \mathbf{v}$. What does it say about the angle between $\mathbf{u}$ and $\mathbf{v}$?

   (b) Are the two vectors $\mathbf{u}$ and $\mathbf{v}$ perpendicular, parallel or neither? $(-4,2,1) = k(-1,-1,3)$

   (c) Compute the lengths $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$ of the vectors.

   (d) Compute $\cos\theta$, where θ $(0 \le \theta \le \pi)$ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

   (e) Find the unit vector $\hat{\mathbf{u}}$ in the direction of $\mathbf{u}$.

   (f) Find two vectors parallel to $\mathbf{v}$ with length 7.

a) $u = \begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix}$ $v = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$

dot product $= -4 \cdot -1 + 2 \cdot -1 + 1 \cdot 3 =$

$4 + -2 + 3 = 5$

because dot product gets us $5 > 0$, it's an acute angle.

b) The 2 vectors u and v are neither perpendicular or parallel since the dot product is not 0, and there is no scalar multiple which makes one vector equal to the other.

c) length of $\|u\| = \sqrt{-4^2 + 2^2 + 1^2} = \sqrt{21}$

length of $\|v\| = \sqrt{-1^2 + -1^2 + 3^2} = \sqrt{11}$

d) $\cos\theta = \dfrac{u \cdot v}{\|u\| \cdot \|v\|} \Rightarrow \dfrac{5}{\sqrt{21} \cdot \sqrt{11}} \Rightarrow \dfrac{5}{\sqrt{231}}$

e) $\hat{u} = \dfrac{u}{\|u\|}$

$= \dfrac{(-4, 2, 1)}{\sqrt{21}}$

$= \left(\dfrac{-4}{\sqrt{21}}, \dfrac{2}{\sqrt{21}}, \dfrac{1}{\sqrt{21}}\right)$

2. (a) Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in $\mathbb{R}^3$ such that $\|\mathbf{u}\| = 7$ and $\|\mathbf{v}\| = 11$.

(i) Find the maximum and minimum possible values of $\mathbf{u} \cdot \mathbf{v}$.

(ii) Find the maximum and minimum possible values of $\|\mathbf{u} - \mathbf{v}\|$.

(b) Let $\mathbf{u} = (3, 1)$, $\mathbf{v} = (1, 5)$ and $\mathbf{w} = (c, d)$, $c, d \in \mathbb{R}$, be three vectors in $\mathbb{R}^2$. Find all real values c, d, such that $\mathbf{u}$ and $\mathbf{w}$ are orthogonal, and $\mathbf{v} \cdot \mathbf{w} = 3$.

a) (i) $\|u\| = 7 \quad \|v\| = 11$

$u \cdot v = \|u\|.\|v\| \cos\theta$

$\Rightarrow u \cdot v = 77 \cos\theta$

$-1 \leq \cos\theta \leq 1$

max $u \cdot v = 77\cos(1) \Rightarrow 77$

min $u \cdot v = 77\cos(-1) \Rightarrow -77$

max value of $u \cdot v = 77$

min value of $u \cdot v = -77$

(ii) $\|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2u \cdot v$

$\Rightarrow 7^2 + 11^2 - 2(u \cdot v)$

Plug in $(u \cdot v) = 77$: $7^2 + 11^2 - 2(77) = 49 + 121 - 154 = \sqrt{16} = 4 \Rightarrow \|u - v\| = 4$ min

Plug in $(u \cdot v) = -77$: $7^2 + 11^2 - 2(-77) = 170 + 154 = \sqrt{324} = 18 \Rightarrow \|u - v\| = 18$ max

b) $u \cdot w = 0$

$(3,1) \cdot (c,d) = 3c + d = 0$

$d = -3c$

$v \cdot w = 3$

$(1,5) \cdot (c,d) = c + 5d = 3$

$c + 5(-3c) = 3$

$c - 15c = 3$

$-14c = 3 \quad c = \frac{-3}{14}$

$d = -3 \cdot \frac{-3}{14}$

$= \frac{9}{14}$

$c = \frac{-3}{14}$

$d = \frac{9}{14}$

3. Given a 4×3 matrix $A = \begin{bmatrix} 7 & -2 & 0 \\ 1 & 2 & 5 \\ 0 & -1 & 2 \\ 7 & -2 & 1 \end{bmatrix}$ and the vector $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, calculate $A\mathbf{b}$

(a) as a linear combination of columns of A

(b) with the entries as dot products of rows of A with vector $\mathbf{b}$.

a)

$a_1 = (7, 1, 0, 7)$

$a_2 = (-2, 2, -1, -2)$

$a_3 = (0, 5, 2, 1)$

$b = (2, 1, 2)$

$Ab = 2a_1 + 1a_2 + 2a_3$

$2a_1 = (14, 2, 0, 14)$

$a_2 = (-2, 2, -1, -2)$

$2a_3 = (0, 10, 4, 2)$

$Ab = (12, 14, 3, 14)$

b)

$Row_1 = (7, -2, 0) \cdot (2, 1, 2) = 14 - 2 = 12$

$Row_2 = (1, 2, 5) \cdot (2, 1, 2) = 2 + 2 + 10 = 14$

$Row_3 = (0, -1, 2) \cdot (2, 1, 2) = 0 - 1 + 4 = 3$

$Row\ 4 = (7, -2, 1) \cdot (2, 1, 2) = 14 - 2 + 2 = 14$

$Ab = \begin{bmatrix} 12 \\ 14 \\ 3 \\ 14 \end{bmatrix}$

4. Let $A = \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array}\begin{bmatrix} 2 & 0 & 3 \\ 1 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

(a) write the linear system corresponding to the matrix equation $A\mathbf{x} = \mathbf{b}$.

(b) solve the linear system.

(c) Write the vector $\mathbf{b}$ as a linear combination of the columns of the matrix A.

a)

$A \cdot x = b$

$a_1 (2,0,3) \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow 2x_1 + 0x_2 + 3x_3 = b_1$

$a_2 (1,0,1) \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow 1x_1 + 0x_2 + 1x_3 = b_2$

$a_3 (-1,2,0) \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow -1x_1 + 2x_2 + 0x_3 = b_3$

$$\begin{cases} 2x_1 + 3x_3 = b_1 \\ x_1 + x_3 = b_2 \\ -1x_1 + 2x_2 = b_3 \end{cases}$$

b)

$x_1 = b_2 - x_3$

$2(b_2 - x_3) + 3x_3 = b_1$

$2b_2 - 2x_3 + 3x_3 = b_1$

$x_3 = b_1 - 2b_2$

$x_1 = b_2 - b_1 + 2b_2 \Rightarrow 3b_2 - b_1$

$-x_1 + 2x_2 = b_3$

$\Rightarrow 2x_2 = b_3 + x_1$

$x_2 = \dfrac{b_3 + 3b_2 - b_1}{2}$

$x_1 = 3b_2 - b_1$

$x_2 = \dfrac{b_3 + 3b_2 - b_1}{2}$

$x_3 = b_1 - 2b_2$

c) $b = a_1x_1 + a_2x_2 + a_3x_3$

$a_1(3b_2 - b_1) + a_2\left(\dfrac{b_3 + 3b_2 - b_1}{2}\right) + a_3(b_1 - 2b_2)$

$b = a_1(3b_2 - b_1) + a_2\left(\dfrac{b_3 + 3b_2 - b_1}{2}\right) + a_3(b_1 - 2b_2)$

5. (a) Are the columns of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \\ 6 & 0 & 6 \end{bmatrix}$ linarly dependent or independent? Justify your answer.

Is the column space of the matrix A the single point $(0,0,0)$ in $\mathbb{R}^3$, a line in $\mathbb{R}^3$ through the zero verctor, a plane in $\mathbb{R}^3$ through the zero vector or the whole space $\mathbb{R}^3$?

(b) Are the columns of the matrix $B = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{bmatrix}$ linearly dependent or independent? Justify your answer. Is the column space of the matrix B the single point $(0,0,0)$ in $\mathbb{R}^3$, a line in $\mathbb{R}^3$ through the zero verctor, a plane in $\mathbb{R}^3$ through the zero vector or the whole space $\mathbb{R}^3$?

a) The columns of the matrix A are linearly dependent because when you add the first and second column, the resultant value is the 3rd vector, which makes the 3rd vector dependant on the first 2.

It is a plane in $\mathbb{R}^3$ since colums are linearly dependant, and since the matrix is not the zero matrix it doesn't cover the zero vector making it a plane.

b) The columns of the matrix B are linearly independent since no vector column among the 3 have scalar multiples which make them equal to one another, nor are there ways to add the vectors to be another one. because of this they are all linearly independant.

Since there are 3 independent vectors, that must mean that it must span the entire space of $\mathbb{R}^3$ therefore the column space of B is the whole space $\mathbb{R}^3$

6. (a) Are the columns of the matrix $C = \begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 3 & 8 & 7 \\ 0 & 0 & 4 & 2 \end{bmatrix}$ linearly dependent or independent? Justify your answer. Is the column space of the matrix C the single point $(0, 0, 0)$ in $\mathbb{R}^3$, a line in $\mathbb{R}^3$ through the zero verctor, a plane in $\mathbb{R}^3$ through the zero vector or the whole space $\mathbb{R}^3$?

(b) Suppose that D is the matrix $D = \begin{bmatrix} 2 & 1 \\ 6 & 5 \\ 2 & 4 \end{bmatrix}$. Is the vector $\mathbf{b} = \begin{bmatrix} 8 \\ 28 \\ 14 \end{bmatrix}$ in the column space of D?

(c) Suppose that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are three linearly independent vectors in $\mathbb{R}^n$. Show that the vectors $\mathbf{u}$, $\mathbf{u} - \mathbf{v}$, and $2\mathbf{u} + 3\mathbf{v} - 4\mathbf{w}$ are also linearly independent.

a) The columns are linearly dependent because it is 4 columns in a 3d space, meaning at least 1 vector has to be dependent on another, and the zero vector automatically makes it dependent as well.

The column space of the matrix C is a plane in $\mathbb{R}^3$ because of the zero vector.

b) $x_1 \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ 28 \\ 14 \end{bmatrix}$

$\begin{cases} 2x_1 + x_2 = 8 \\ 6x_1 + 5x_2 = 28 \\ 2x_1 + 4x_2 = 14 \end{cases}$

$x_2 = 8 - 2x_1$

$6x_1 + 5(8 - 2x_1) = 28$

$6x_1 + 40 - 10x_1 = 28$

$2x_1 + 4x_2 = 2(3) + 4(2) = 14$

$-4x_1 + 40 = 28$

$-4x_1 = -12 \Rightarrow x_1 = 3$

$x_2 = 8 - 2(3) = 2$

b is in the column space of D with $x_1 = 3$, $x_2 = 2$, $x_3 = 14$

c) $au + b(u - v) + c(2u + 3v - 3w) = 0$

$a = b = c = 0$

$au + bu - bv + 2cu + 3cv - 3cw$

$(a + b + 2c)\underline{u} + (-b + 3c)\underline{v} + (-4c)\underline{w}$

$\begin{cases} a + b + 2c = 0 \\ -b + 3c = 0 \\ -4c = 0 \end{cases}$

$c = 0$

$b = 0$

$a = 0$

The vectors u, u−v, 2u+3v−4w are linearly independent