

$\Rightarrow v_1, v_2, v_3, v_4$  spans  $V$

Show that  $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$  also spans  $V$

$$L = (v_1, v_2, v_3, v_4)$$

$$\text{Span of } L = \{ a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 : a_1, a_2, a_3, a_4 \in \mathbb{R} \}$$

$$L' = (v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$$

$$\text{Span of } L' = \{ b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4 v_4 : b_1, b_2, b_3, b_4 \in \mathbb{R} \}$$

$$= b_1 v_1 - b_1 v_2 + b_2 v_2 - b_2 v_3 + b_3 v_3 - b_3 v_4 + b_4 v_4$$

$$= b_1 v_1 + (b_2 - b_1) v_2 + (b_3 - b_2) v_3 + (b_4 - b_3) v_4$$

$$\text{Since } b_1, b_2, b_3, b_4 \in \mathbb{R}$$

$$b_2 - b_1, b_3 - b_2, b_4 - b_3 \in \mathbb{R}$$

$$\therefore L' \text{ also spans } V$$

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Alternatively,

if  $v \in V$

$$v = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 \text{ for some } a_1, a_2, a_3, a_4 \in \mathbb{R}$$

$$\text{Let } b_1, b_2, b_3, b_4 \in \mathbb{R}$$

s.t.

$$b_1 = a_1, \quad b_2 - b_1 = a_2, \quad b_3 - b_2 = a_3$$

$$b_4 - b_3 = a_4$$

such values of  $b$  always exist in  $\mathbb{R}$  for any  $a$ 's.

Q.E.D.