→ Suppose V is a vector space and

$S, T \in L(V, V)$

such that

range S $\subset$ null T

Prove that $(ST)^2 = 0$

Proof: Let $v \in V$

$$
\begin{aligned}
& (ST)(ST)(v) \\
&= (STS)(Tv) \\
&= (STS)(v_1) \quad [v_1 \in V] \\
&= (ST)(Sv_1) \\
&= (ST)(v_2) \quad [v_2 \in V] \text{ and } v_2 \in \text{ranges} \\
&\qquad \therefore v_2 \in \text{null } T \\
&= (S)(Tv_2) \quad (\because v_2 \in \text{null } T) \\
&= (S)(0) \\
&= 0
\end{aligned}
$$

QED.