MUSIC Algorithm for Direction of Arrival Estimation of Multiple Sources

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***Abstract*—Processing the signals received on an array of sensors for the location of the source is of great interest considering the general physical signal phenomena. The general problem considers sensors with arbitrary locations and arbitrary directional characteristics (gain/phase/polarization) in a noise environment of arbitrary covariance matrix.**
A description is given of the multiple signal classification (MUSIC) algorithm, which provides asymptotically unbiased estimates of 1) number of incident wavefronts present; 2) directions of arrival (DOA) (or emitter locations); 3) strengths and cross correlations among the incident waveforms; 4) noise strength. Examples and comparisons with methods based on maximum likelihood (ML) and maximum entropy (ME), as well as conventional beamforming are included. A simulation for directions of arrival (DOA) Estimation is performed to show the efficiency and parameter performance analysis of the proposed algorithm.

***Index Terms*—direction of arrival, maximum likelihood, maximum entropy, beamforming**

I. Problem Statement

The general problem considers sensors with arbitrary locations and arbitrary directional characteristics in a noise environment of arbitrary covariance matrix [1]. The multiple signal classification approach is described; it can be implemented as an algorithm to provide asymptotically unbiased estimates of

1) Number of signals
2) Direction of Arrival (DOA)
3) Strengths and Cross Correlations among the directional waveforms

and many more such applications.
The term Multiple signal classification (MUSIC) is used to describe experimental and theoretical techniques involved in determining the parameters of multiple wavefronts arriving at sensor array from measurements made on the signals received at the array elements.

II. Mathematical Model

We consider an array of M elements, receiving D wavefronts corresponding to the D signal sources, and additive noise. Thus, the multiple signal classification approach begins with the following model for characterizing the received M vector X as:

$$X = AF + W \tag{1}$$

or

$$\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_M \end{bmatrix} = \begin{bmatrix} | & | & & | \\ a(\theta_1) & a(\theta_2) & \cdots & a(\theta_D) \\ | & | & & | \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_D \end{bmatrix} + \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_M \end{bmatrix} \tag{2}$$

Here, X is the observed vector at the M array elements. Columns of A represent the "steering vectors" or "mode vectors" which are based on signal arrival angles and array element locations. The element A_{ij} represents the signal due to j^{th} source on i^{th} array element.
The complex quantities F_1, F_2, ..., F_D which are the elements of F matrix represent the incident signals, namely amplitude and phase representations of obtained wavefronts. The W vector represents noise.
In geometric terms, the problem of solving for Direction of Arrival (DOA) of multiple incident waveforms consists of locating intersections of $a(\theta)$ continuum with the range space of A.

III. MUSIC Algorithm

The algorithm can be briefly divided into

1) Finding the Covariance matrix S of X
2) Eigen decomposition of S
3) Estimating number of signals/sources ($\hat{D}$)
4) Separating Signal and Noise Subspaces
5) Obtaining the $P_{MU}(\theta)$ matrix and evaluating its value vs θ
6) Picking D peaks of $P_{MU}(\theta)$
7) Calculate other parameters

A. *S Matrix*

S matrix is defined as the Covariance matrix of the observed vector X. It is mathematically denoted by

$$S \triangleq \overline{XX^*} = A\overline{FF^*}A^* + \overline{WW^*} \tag{3}$$

or

$$S = APA^* + \lambda S_0 \tag{4}$$

Where, we make the assumption that $P = \overline{FF^*}$. It is also assumed above that the signal and noise are uncorrelated.

B. Obtaining a Solution with Eigen Decomposition

If number of array elements are less than the number of incident signals (equivalent to number of sources) ($D < M$), the matrix APA^* will be singular with rank $< M$. Therefore

$$|APA^*| = |S - \lambda S_0| = 0 \tag{5}$$

This equation resembles the eigen decomposition of matrix S in the metric of S_0. But, as A is full rank and P is positive definite, APA^* must be non-negative definite. This constraints the λ to be the minimum eigen value λ_{min}. Using this information, we can rewrite S as

$$S = APA^* + \lambda_{min} S_0 \tag{6}$$

with λ_{min} being the minimum solution to $|S - \lambda S_0| = 0$. Here, we also look at the special case where the elements of Noise vector W are mean 0, and variance σ^2. In this case, $\lambda_{min} S_0 = \sigma^2 I$

C. Estimating number of sources/signals

From above, we obtain the rank of $APA^* = D$. Thus, we can conclude that λ_{min} occurs $N = M - D$ times in the eigen values. This is true because the eigen values of S and $S - \lambda_{min} S_0 = APA^*$ differ by λ_{min} always. Since the minimum eigen value of APA^* is zero, λ_{min} must repeat N times, which corresponds to the noise subspace. Therefore, the number of incident signals can be estimated as

$$\hat{D} = M - \hat{N} \tag{7}$$

Here, $\hat{N}$ is the multiplicity of $\lambda_{min}(S, S_0)$. In practice, it is generally observed that the multiple λ_{min} will occur in a cluster rather than all precisely equal. The "spread" on this cluster decreases as more data is processed.

D. Signal and Noise Subspaces

The M eigen vectors of S in metric of S_0 must satisfy the eigen decomposition expression

$$Se_i = \lambda_i S_0 e_i,\ \forall i = 1, 2, ..., M$$

Since $S = APA^* + \lambda_{min} S_0$, we can rewrite above equation as

$$APA^* e_i = (\lambda_i - \lambda_{min}) S_0 e_i$$

From the above equation, we can observe that each time $\lambda_i = \lambda_{min}$, we must have $A^* e_i = 0$. This clearly implies that the eigen vectors associated with $\lambda_{min}(S, S_0)$ are orthogonal to the space spanned by the columns of A, the incident mode vectors. Thus, we can refer to the $N - dim$ subspace spanned by N noise vectors and the $D - dim$ subspace spanned by D incident signal mode vectors.

E. Obtaining $P_{MU}(\theta)$ vs θ

Assuming E_N denotes the N noise eigen vectors, we can obtain the normal Euclidean distance of the vector $a(\theta)$ from E_N to be $d^2 = a^*(\theta) E_N E_N^* a(\theta)$. We use the inverse of this to obtain

$$P_{MU}(\theta) = \frac{1}{a^*(\theta) E_N E_N^* a(\theta)} \tag{8}$$

We can see that this is an asymptotically unbiased estimator, even for multiple incident wavefronts.

IV. Intuition behind Peaks of $P_{MU}(\theta)$

The final step involves the estimation of the direction of arrival (DOA) using the $P_{MU}(\theta)$ matrix. The estimate is the choice of θ for which the $P_{MU}(\theta)$ function is maximized. It can be mathematically represented as

$$\hat{\theta}_{MUSIC} = \underset{\theta}{\operatorname{argmax}} P_{MU}(\theta) \tag{9}$$

It can be interpreted as the maximum over the array of values obtained by conducting a grid substitution into the function $P_{MU}(\theta)$.

By intuition, we can see that the denominator in the function $P_{MU}(\theta)$ is the Euclidean Distance between $a(\theta)$ and E_N. By convention, we obtain this to be minimum when the angle θ is the actual direction of arrival. Thus, we pick the first D peaks of the function $P_{MU}(\theta)$, to obtain an estimate to the actual direction of arrival.

An analytic solution for the function $P_{MU}(\theta)$ can be obtained to be

$$P = (A^*A)^{-1} A^* (S - \lambda_{min} S_0) A (A^*A)^{-1} \tag{10}$$

In the case of colored noise, the LS Estimate requires whitening and the final solution can be obtained to be

$$P = (A^* S_0^{-1} A)^{-1} A^* S_0^{-1} (S - \lambda_{min} S_0) S_0^{-1} A (A^* S_0^{-1} A)^{-1} \tag{11}$$

V. Comparison with Previous Estimators

Comparison is done with other estimators like ordinary beamforming ($P_{BF}(.)$) [2], ML(maximum likelihood) $P_{ML}(.)$ [3] and ME (maximum entropy) ($P_{ME}(.)$) [4]. The mathematical representations of these estimators with their descriptions are as below

$$P_{BF}(\theta) = a^* s a(\theta) \tag{12}$$

This gives the power one would measure at the output of beamformer.

$$P_{ML}(\theta) = \frac{1}{\lambda_{min}(A^* S^{-1} A(\theta))} \tag{13}$$

This gives the log likelihood function maximization, but requires a $D - dim$ search and plot. It assumes the noise vector to statistically follow $N(0, \mathbf{C})$.

$$P_{ME}(\theta) = \frac{1}{a^* C C^* a(\theta)} \tag{14}$$

Here, C are the columns of S^{-1}. we select one of the M array elements as reference and find weights to be applied to the remaining $M-1$ received signals to permit their sum with a MMSE fit to reference.

Comparing the proposed algorithm with all the above, we observe that the proposed algorithm takes lesser computational time, and gives a good estimate even in multiple signal scenario, with no specification on array structure.

VI. Simulations

Considering a ULA (Uniform Linear Array), the model can be described as described in Section-II, with some extra parameters like snapshots (number of time instants of data), Spacing between the array elements (this quantity will be a single number as the array is uniform and linear) and frequency of incident signals.

The simulation of Direction of Arrival Estimation for a $D = 2$, $M = 10$, snapshots=20, wavelength(λ)=150, spacing(d) $= \frac{\lambda}{2}$, incident signals are sinusoidal with amplitude 2 and frequencies $\frac{\pi}{4}$ and $\frac{\pi}{3}$ at angles of -30^o and 70^o, and SNR=20. Different experiments and corresponding observations have been added in the next section [5].

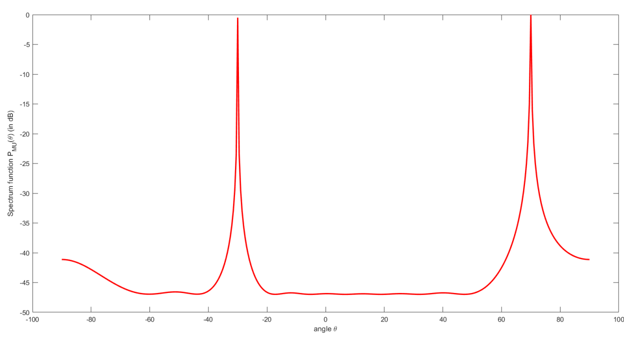

Fig. 1. MUSIC Algorithm for DOA with angles -30^o and 70^o

Figure-1 shows the plot for the function $P_{MU}(\theta)$ plotted against θ.

VII. Experiments and Observations

Considering the same model as in the previous section, parameter-based experiments [6] of the working of the MUSIC algorithm for Direction of Arrival (DOA) Estimation can be conducted, some of which include

1) The Spacing between Array Elements
2) Increasing Number of Antenna Array
3) Increasing the Number of Snapshots
4) Increasing the Signal Incidence Angle Difference

A. The Spacing between Array Elements

Figure-2 shows the simulation for $d = \frac{\lambda}{6}$ (blue) and $d = \frac{\lambda}{6}$ (red). It can be observed that we obtain better peaks when the distance increases from $\frac{\lambda}{6}$ to $\frac{\lambda}{2}$. But the spectrum losses efficiency when the element spacing is increased beyond $\frac{\lambda}{2}$. So the maximum allowable spacing in order to have the best efficiency is $\frac{\lambda}{2}$.

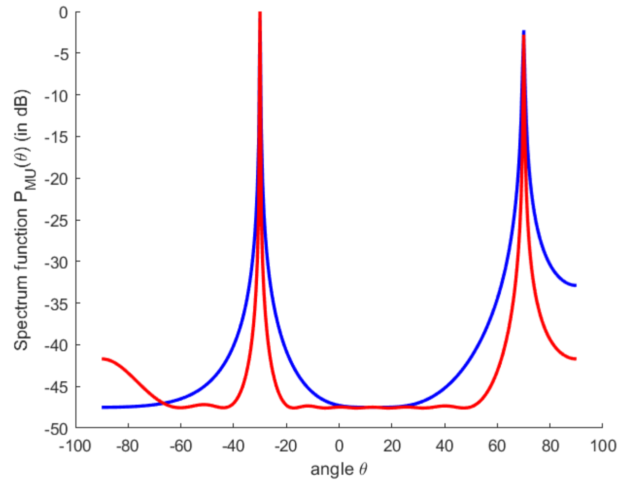

Fig. 2. MUSIC Algorithm for $d = \frac{\lambda}{6}$ (blue) and $d = \frac{\lambda}{6}$ (red)

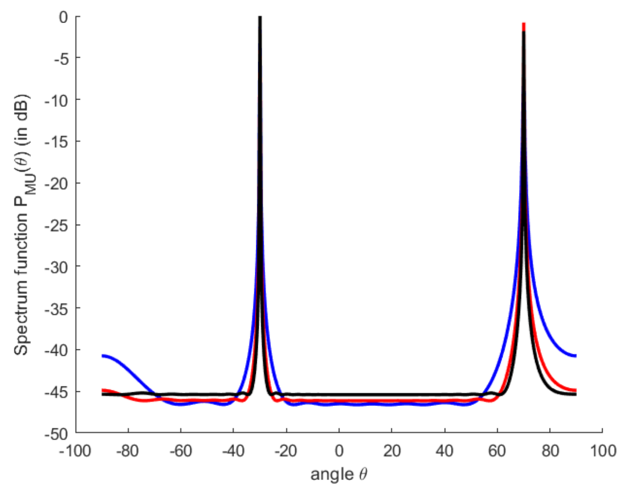

Fig. 3. MUSIC Algorithm for $M = 10$ (blue), $M = 20$ (red), $M = 30$ (black)

B. Increasing Number of Antenna Array

Figure-3 shows the simulation for $M = 10$ (blue), $M = 20$ (red), $M = 30$ (black). We find better results on increasing the number of array elements.

C. Increasing the Number of Snapshots

Figure-4 shows the simulation for $snapshots = 10$ (blue), $snapshots = 20$ (red) and $snapshots = 30$ (black). We observe that increasing the number of snapshots can increase the efficiency of the estimation algorithm.

D. Increasing the Signal Incidence Angle Difference

Figure-5 shows the simulation for difference in angles 10^o (blue), 15^o (red) and 20^o (black). It can be observed that the estimation results are best when the difference between the incident angles is highest.

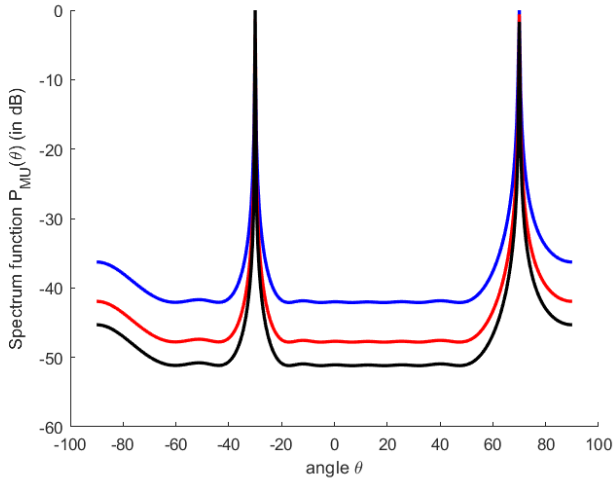

Fig. 4. MUSIC Algorithm for $snapshots = 10$ (blue), $snapshots = 20$ (red) and $snapshots = 30$ (black)

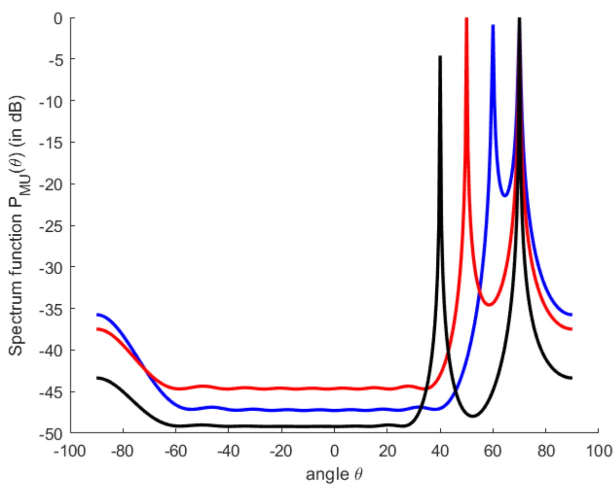

Fig. 5. MUSIC Algorithm for difference in angles 10^o (blue), 15^o (red) and 20^o (black)

VIII. Conclusion

The approach presented here for multiple signal classification is very general and of wide application. MUSIC provides asymptotically unbiased estimates of a general set of signal parameters approaching the Cramer-Rao accuracy bound.

MUSIC models the data as the sum of point source emissions and noise rather than the convolution of an all pole transfer function driven by a white noise (i.e., autoregressive modeling, maximum entropy) or maximizing a probability under the assumption that the X vector is zero mean, Gaussian (maximum likelihood for Gaussian data).

As there are no constraints placed on array geometry, this algorithm can be scaled to a variety of array-structure independent problems. Also, with no constraints placed on the statistical model of noise, it can be generalized to various problems with non-AWGN noise processes also.

References

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[5] https://github.com/Sasanka-GRS/SDET-Project , Codes and Plots are available here.

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