Non-Gaussian Structural Vector Autoregressions

Sascha Alexander Keweloh

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TU Dortmund University

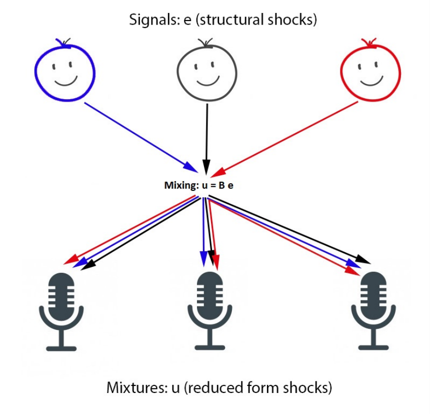
Structural Vector Autoregression

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix}$$

In matrix notation:

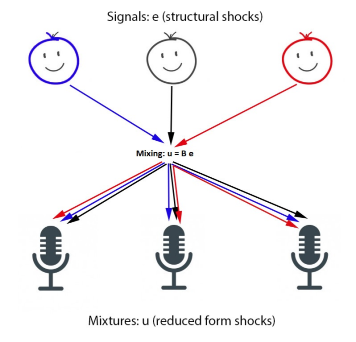
$$\underbrace{u}_{\text{observed mixtures}} = \underbrace{B_0}_{\text{mixing matrix}} \underbrace{\epsilon}_{\text{signals}}$$

The Cocktail-Party-Problem



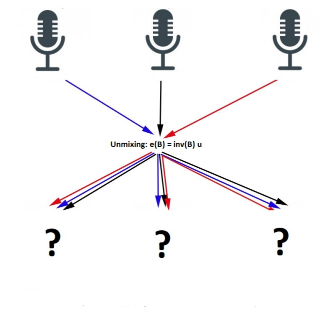
Structural Vector Autoregression

$$u = B_0 \epsilon$$



Unmixed innovations

$$e_t(B) := B^{-1} u_t \tag{1}$$



Recursive SVAR

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ b_{21} & 1 & 0 \\ b_{31} & b_{32} & 1 \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix}$$

Assumptions:
Mixing B_0 is recursive

Shocks ϵ are uncorrelated

Moment conditions

$$E[e_1(B)e_2(B)] = 0$$
$$E[e_1(B)e_3(B)] = 0$$
$$E[e_2(B)e_3(B)] = 0$$

Identified ✓

Keweloh, Sascha. "A Generalized Method of Moments Estimator for Structural Vector Autoregressions Based on Higher Moments." *Journal of Business & Economic Statistics* (2019)

Non-Recursive SVAR

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 & b_{12} & b_{13} \\ b_{21} & 1 & b_{23} \\ b_{31} & b_{32} & 1 \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix}$$

Assumptions:
Shocks ϵ are independent

Shocks ϵ are non-Gaussian

Moment conditions

$$E[e_1(B)e_2(B)] = 0$$
$$E[e_1(B)e_3(B)] = 0$$
$$E[e_2(B)e_3(B)] = 0$$

$$E[e_1^2(B)e_2(B)] = 0$$
$$E[e_1(B)e_2^2(B)] = 0$$
$$E[e_1^2(B)e_3(B)] = 0$$
$$E[e_1(B)e_3^2(B)] = 0$$
$$E[e_2^2(B)e_3(B)] = 0$$
$$E[e_2(B)e_3^2(B)] = 0$$
$$E[e_1(B)e_2(B)e_3(B)] = 0$$

Identified $\checkmark$

Keweloh, "Block-Recursive SVARs."

Block-Recursive SVAR

$$
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ b_{21} & 1 & b_{23} \\ b_{31} & b_{32} & 1 \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix}
$$

Assumptions:
Shocks ϵ are independent

Shocks ϵ are non-Gaussian

Moment conditions

$$E[e_1(B)e_2(B)] = 0$$
$$E[e_1(B)e_3(B)] = 0$$
$$E[e_2(B)e_3(B)] = 0$$

$$\cancel{E[e_1^2(B)e_2(B)] = 0}$$
$$\cancel{E[e_1(B)e_2^2(B)] = 0}$$
$$\cancel{E[e_1^2(B)e_3(B)] = 0}$$
$$\cancel{E[e_1(B)e_3^2(B)] = 0}$$
$$E[e_2^2(B)e_3(B)] = 0$$
$$E[e_2(B)e_3^2(B)] = 0$$
$$\cancel{E[e_1(B)e_2(B)e_3(B)] = 0}$$

Identified ✓

Simulated recursive SVAR

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ 0.5 & 0.5 & 1 & 0 \\ 0.5 & 0.5 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \end{bmatrix}$$

$$\varepsilon_i = z\phi_1 + (1-z)\phi_2 \text{ with } \phi_1 \sim \mathcal{N}(-0.2, 0.7),\ \phi_2 \sim \mathcal{N}(0.75, 1.5),\ z \sim \mathcal{B}(0.79),$$

$\mathcal{B}(p)$ indicates a Bernoulli distribution and $\mathcal{N}(\mu, \sigma^2)$ indicates a normal distribution. The structural shocks have mean zero, unit variance, skewness equal to 0.91 and an excess kurtosis of 2.51.

Keweloh "Block-Recursive SVARs."

	Non-Recursive SVAR w. higher moments	Recursive SVAR w. second moments (Cholesky)
$\underset{(Avar)}{Limit}$	$\begin{bmatrix} \underset{(1.033)}{1} & \underset{(2.450)}{0} & \underset{(2.450)}{0} & \underset{(2.450)}{0} \\ \underset{(2.585)}{0.5} & \underset{(1.522)}{1} & \underset{(3.192)}{0} & \underset{(3.192)}{0} \\ \underset{(3.265)}{0.5} & \underset{(3.265)}{0.5} & \underset{(2.076)}{1} & \underset{(3.999)}{0} \\ \underset{(4.010)}{0.5} & \underset{(4.010)}{0.5} & \underset{(4.010)}{0.5} & \underset{(2.695)}{1} \end{bmatrix}$	$\begin{bmatrix} \underset{(1.128)}{1} & \underset{(0)}{0} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(1.282)}{0.5} & \underset{(1.128)}{1} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(1.532)}{0.5} & \underset{(1.282)}{0.5} & \underset{(1.128)}{1} & \underset{(0)}{0} \\ \underset{(1.782)}{0.5} & \underset{(1.532)}{0.5} & \underset{(1.282)}{0.5} & \underset{(1.128)}{1} \end{bmatrix}$

$$
\begin{array}{rc}
\text{output} & \rightarrow \\
\text{investment} & \rightarrow \\
\text{inflation} & \rightarrow \\
\text{federal funds rate} & \rightarrow \\
\text{real stock returns} & \rightarrow
\end{array}
\begin{bmatrix} u_t^y \\ u_t^I \\ u_t^\pi \\ u_t^i \\ u_t^s \end{bmatrix}
=
\begin{bmatrix}
b_{11} & b_{12} & b_{13} & 0 & 0 \\
b_{21} & b_{22} & b_{23} & 0 & 0 \\
b_{31} & b_{32} & b_{33} & 0 & 0 \\
b_{41} & b_{42} & b_{43} & b_{44} & b_{45} \\
b_{51} & b_{52} & b_{53} & b_{54} & b_{55}
\end{bmatrix}
\begin{bmatrix} \varepsilon_t^y \\ \varepsilon_t^I \\ \varepsilon_t^\pi \\ \varepsilon_t^s \\ \varepsilon_t^i \end{bmatrix}, \qquad (2)
$$

Keweloh, Seepe "Monetary policy and the stock market"

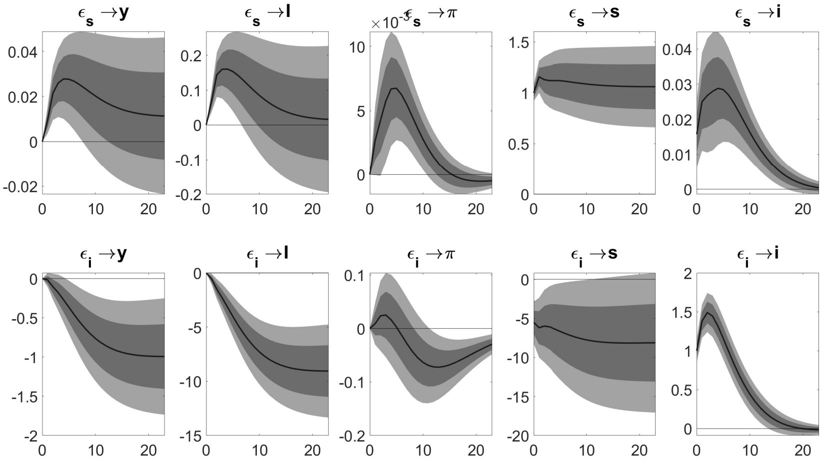


Figure 1: ....

Keweloh "Block-Recursive SVARs."

Block-Recursive SVAR

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ b_{21} & 1 & 0 \\ b_{31} & b_{32} & 1 \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix}$$

Moment conditions

$$E[e_1(B)e_2(B)] = 0$$
$$E[e_1(B)e_3(B)] = 0$$
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Moment conditions

$$E[e_1^2(B)e_2(B)] = 0$$
$$E[e_1(B)e_2^2(B)] = 0$$
$$E[e_1(B)e_2(B)] = 0 \qquad E[e_1^2(B)e_3(B)] = 0$$
$$E[e_1(B)e_3(B)] = 0 \qquad E[e_1(B)e_3^2(B)] = 0$$
$$E[e_2(B)e_3(B)] = 0 \qquad E[e_2^2(B)e_3(B)] = 0$$
$$E[e_2(B)e_3^2(B)] = 0$$
$$E[e_1(B)e_2(B)e_3(B)] = 0$$

Question:
Are higher moments non-redundant?

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Are higher moments non-redundant?

Proposition
In a block-recursive SVAR with non-Gaussian and independent shocks, higher moments are non-redundant.

Corollary
In a non-Gaussian recursive SVAR with independent shocks, the frequently used estimator obtained by applying the Cholesky decomposition to the variance covariance matrix of the reduced form shocks is inefficient.

Keweloh "Block-Recursive SVARs."

Asymptotic performance

	Recursive SVAR w. second moments (Cholesky)	Recursive SVAR w. higher moments
Limit (*Avar*)	$\begin{bmatrix} \underset{(1.128)}{1} & \underset{(0)}{0} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(1.282)}{0.5} & \underset{(1.128)}{1} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(1.532)}{0.5} & \underset{(1.282)}{0.5} & \underset{(1.128)}{1} & \underset{(0)}{0} \\ \underset{(1.782)}{0.5} & \underset{(1.532)}{0.5} & \underset{(1.282)}{0.5} & \underset{(1.128)}{1} \end{bmatrix}$	$\begin{bmatrix} \underset{(1.021)}{1} & \underset{(0)}{0} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(1.033)}{0.5} & \underset{(1.019)}{1} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(1.232)}{0.5} & \underset{(1.033)}{0.5} & \underset{(1.017)}{1} & \underset{(0)}{0} \\ \underset{(1.438)}{0.5} & \underset{(1.232)}{0.5} & \underset{(1.033)}{0.5} & \underset{(1.016)}{1} \end{bmatrix}$

Keweloh "Block-Recursive SVARs."

Small sample performance - Many moments problem

T=150	Recursive SVAR w. second moments (Cholesky)	Recursive SVAR w. higher moments 2-step
mean (*std*)	$\begin{bmatrix} \underset{(1.138)}{1} & \underset{(0)}{0} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(1.300)}{0.5} & \underset{(1.048)}{1} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(1.499)}{0.5} & \underset{(1.263)}{0.5} & \underset{(1.126)}{1} & \underset{(0)}{0} \\ \underset{(1.784)}{0.5} & \underset{(1.539)}{0.5} & \underset{(1.275)}{0.5} & \underset{(1.096)}{0.99} \end{bmatrix}$	$\begin{bmatrix} \underset{(7.50)}{0.95} & \underset{(0)}{0} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(49.2)}{0.47} & \underset{(828)}{1.00} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(-)}{-19.9} & \underset{(-)}{-26.2} & \underset{(-)}{158} & \underset{(0)}{0} \\ \underset{(-)}{-287} & \underset{(-)}{31} & \underset{(-)}{-145} & \underset{(-)}{506} \end{bmatrix}$

Optimal weighting

The GMM estimator with $W = S^{-1}$ has the smallest asymptotic variance.

$$S = E\left[\begin{bmatrix} e_1(B_0)e_2(B_0) \\ e_1(B_0)e_3(B_0) \\ e_2(B_0)e_3(B_0) \\ \vdots \end{bmatrix} \begin{bmatrix} e_1(B_0)e_2(B_0) \\ e_1(B_0)e_3(B_0) \\ e_2(B_0)e_3(B_0) \\ \vdots \end{bmatrix}'\right] \tag{3}$$

$$\rightarrow \hat{S}_T = \frac{1}{T}\sum_{t=1}^{T} \begin{bmatrix} e_{1,t}(\hat{B})e_{2,t}(\hat{B}) \\ e_{1,t}(\hat{B})e_{3,t}(\hat{B}) \\ e_{2,t}(\hat{B})e_{3,t}(\hat{B}) \\ \vdots \end{bmatrix} \begin{bmatrix} e_{1,t}(\hat{B})e_{2,t}(\hat{B}) \\ e_{1,t}(\hat{B})e_{3,t}(\hat{B}) \\ e_{2,t}(\hat{B})e_{3,t}(\hat{B}) \\ \vdots \end{bmatrix}' \tag{4}$$

The true mixing matrix B_0 is unknown, but a B_0 it holds: $e(B_0) = \epsilon$

$$S = E\left[\begin{bmatrix} \epsilon_1\epsilon_2 \\ \epsilon_1\epsilon_3 \\ \epsilon_2\epsilon_3 \\ \epsilon_1^2 - 1 \\ \vdots \end{bmatrix}\begin{bmatrix} \epsilon_1\epsilon_2 \\ \epsilon_1\epsilon_3 \\ \epsilon_2\epsilon_3 \\ \epsilon_1^2 - 1 \\ \vdots \end{bmatrix}'\right] \overset{\epsilon \text{ independent}}{=} \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & E[\epsilon_1^4] - 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \tag{5}$$

Parametric estimator

$\hat{S}_T^{para} \rightarrow$ use (5) and replace values like $E[\epsilon_1^4]$ with estimates $1/T \sum_{t=1}^{T} e_1(\hat{B}^4)$.

If ϵ has mutually independent components with zero mean and unit variance, it holds that

$$\hat{S}_T^{para} \overset{T\rightarrow\infty}{\rightarrow} S. \tag{6}$$

Small sample performance - Many moments problem (Solution part 1)

T=150	Recursive SVAR w. second moments (Cholesky)	Recursive SVAR w. higher moments 2-step parametric
mean (*std*)	$\begin{bmatrix} \underset{(1.138)}{1} & \underset{(0)}{0} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(1.300)}{0.5} & \underset{(1.048)}{1} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(1.499)}{0.5} & \underset{(1.263)}{0.5} & \underset{(1.126)}{1} & \underset{(0)}{0} \\ \underset{(1.784)}{0.5} & \underset{(1.539)}{0.5} & \underset{(1.275)}{0.5} & \underset{(1.096)}{0.99} \end{bmatrix}$	$\begin{bmatrix} \underset{(2.609)}{1.14} & \underset{(0)}{0} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(1.914)}{0.57} & \underset{(2.687)}{1.13} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(2.157)}{0.57} & \underset{(1.785)}{0.56} & \underset{(2.289)}{1.12} & \underset{(0)}{0} \\ \underset{(2.686)}{0.56} & \underset{(2.204)}{0.56} & \underset{(1.786)}{0.56} & \underset{(2.307)}{1.12} \end{bmatrix}$

Small sample performance - Many moments problem (Solution part 2)

T=150	Recursive SVAR w. second moments (Cholesky)	Recursive SVAR w. higher moments 2-step parametric (scaled)
mean (*std*)	$\begin{bmatrix} \underset{(1.138)}{1} & \underset{(0)}{0} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(1.300)}{0.5} & \underset{(1.048)}{1} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(1.499)}{0.5} & \underset{(1.263)}{0.5} & \underset{(1.126)}{1} & \underset{(0)}{0} \\ \underset{(1.784)}{0.5} & \underset{(1.539)}{0.5} & \underset{(1.275)}{0.5} & \underset{(1.096)}{0.99} \end{bmatrix}$	$\begin{bmatrix} \underset{(1.136)}{1.00} & \underset{(0)}{0} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(1.209)}{0.50} & \underset{(1.051)}{1.00} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(1.414)}{0.50} & \underset{(1.185)}{0.50} & \underset{(1.131)}{1.00} & \underset{(0)}{0} \\ \underset{(1.726)}{0.50} & \underset{(1.478)}{0.50} & \underset{(1.208)}{0.50} & \underset{(1.104)}{0.99} \end{bmatrix}$

Summary

Keweloh, Hetzenecker ”Soft restrictions in SVARs via Lasso”

T=5000	Recursive SVAR w. second moments (Cholesky)	non-Recursive SVAR w. higher moments	Soft-Restrictions SVAR w. higher moments (Lasso)
mean (*std*)	$\begin{bmatrix} \underset{(1.1)}{1} & \underset{(0)}{0} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(1.3)}{0.5} & \underset{(1.1)}{1} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(1.5)}{0.5} & \underset{(1.3)}{0.5} & \underset{(1.1)}{1} & \underset{(0)}{0} \\ \underset{(1.8)}{0.5} & \underset{(1.5)}{0.5} & \underset{(1.3)}{0.5} & \underset{(1.1)}{1} \end{bmatrix}$	$\begin{bmatrix} \underset{(1.0)}{1} & \underset{(2.7)}{0} & \underset{(2.6)}{0} & \underset{(2.6)}{0} \\ \underset{(2.7)}{0.5} & \underset{(1.6)}{1} & \underset{(3.4)}{0} & \underset{(3.6)}{0} \\ \underset{(3.4)}{0.5} & \underset{(3.4)}{0.5} & \underset{(2.2)}{1} & \underset{(4.4)}{0} \\ \underset{(4.2)}{0.5} & \underset{(4.3)}{0.5} & \underset{(4.4)}{0.5} & \underset{(2.9)}{1} \end{bmatrix}$	$\begin{bmatrix} \underset{(1.0)}{1} & \underset{(0.1)}{0} & \underset{(0.1)}{0} & \underset{(0.1)}{0} \\ \underset{(1.1)}{0.5} & \underset{(1.1)}{1} & \underset{(0.1)}{0} & \underset{(0.0)}{0} \\ \underset{(1.4)}{0.5} & \underset{(1.3)}{0.5} & \underset{(1.0)}{1} & \underset{(0.1)}{0} \\ \underset{(1.6)}{0.5} & \underset{(1.3)}{0.5} & \underset{(1.2)}{0.5} & 1 \end{bmatrix}$

References