

# Higher-order moments in structural VAR models

---

Sascha Alexander Keweloh

July 21, 2021

TU Dortmund University



## Structural VAR model

$$\begin{array}{lll} \text{output} & \rightarrow & \underbrace{\begin{bmatrix} y_t \\ \pi_t \\ i_t \end{bmatrix}}_{u_t} \\ \text{inflation} & \rightarrow & = \underbrace{\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}}_{B_0} \underbrace{\begin{bmatrix} \varepsilon_{s,t} \\ \varepsilon_{d,t} \\ \varepsilon_{i,t} \end{bmatrix}}_{\varepsilon_t} \\ \text{FFR} & \rightarrow & \leftarrow \begin{array}{l} \text{supply shock} \\ \text{demand shock} \\ \text{monetary policy shock} \end{array} \end{array}$$

reduced form      simultaneous      structural  
                      shocks            impact            shocks

### Assumptions:

- A1) Serially independent (identically distributed) shocks       $\rightarrow \varepsilon_t$  independent of  $\varepsilon_{\tilde{t}}$
- A2) Mutually independent shocks       $\rightarrow \varepsilon_{j,t}$  independent of  $\varepsilon_{\tilde{j},t}$
- A3) Zero mean, unit variance, and non-Gaussian shocks

## Moment conditions

SVAR:

$$\textcolor{blue}{u} = \textcolor{red}{B}_0 \epsilon \iff \epsilon = \textcolor{red}{B}_0^{-1} \textcolor{blue}{u}$$

Unmixed innovations

$$e(B) := B^{-1} \textcolor{blue}{u}$$

$$\text{Note: } e(B_0) = \epsilon$$

Independent structural shocks:

$$E[\epsilon_1 \epsilon_2] = 0$$

$$E[\epsilon_1^3 \epsilon_2] = 0$$

⋮

Independent unmixed innovations:

$$E[e(B)_1 e(B)_2] \stackrel{!}{=} 0$$

$$E[e(B)_1^3 e(B)_2] \stackrel{!}{=} 0$$

⋮

## SVAR GMM

$$\hat{B}_T := \arg \min_B g_T(B)' W g_T(B), \quad (1)$$

with

$$g_T(B) = \frac{1}{T} \sum_{t=1}^T f(B, u_t) \quad \text{and} \quad f(B, u_t) = \begin{bmatrix} e(B)_{1,t} e(B)_{2,t} \\ e(B)_{1,t}^3 e(B)_{2,t} \\ \vdots \end{bmatrix}. \quad (2)$$

## Consistency and asymptotic normality

$$M := (G' S^{-1} G)^{-1} G' W$$

$$\hat{B}_T \xrightarrow{P} B_0$$

with

$$G := E \left[ \frac{\partial f(B_0, u_t)}{\partial \text{vec}(B)'} \right]$$

$$\sqrt{T}(\hat{B}_T - B_0) \xrightarrow{d} \mathcal{N}(0, MSM')$$

$$S := \lim_{T \rightarrow \infty} E [T g_T(B_0) g_T(B_0)'] \\ \underset{\text{ser.unc.}}{=} E [f(B_0, u_t) f(B_0, u_t)']$$

## Optimal weighting

$$W_0 := S^{-1}$$

## Problem

Higher-order moment conditions make it difficult to estimate  $S_0$  (and  $G_0$ ).

$$S := \lim_{T \rightarrow \infty} E [ T g_T(B_0) g_T(B_0)' ] \quad (3)$$

$$= \lim_{T \rightarrow \infty} E \left[ T \left( \frac{1}{T} \sum_{t=1}^T \begin{bmatrix} \epsilon_{1,t} \epsilon_{2,t} \\ \epsilon_{1,t}^3 \epsilon_{2,t} \\ \dots \end{bmatrix} \right) \left( \frac{1}{T} \sum_{t=1}^T \begin{bmatrix} \epsilon_{1,t} \epsilon_{2,t} \\ \epsilon_{1,t}^3 \epsilon_{2,t} \\ \dots \end{bmatrix} \right)' \right] \quad (4)$$

$$\stackrel{\text{A1)}{\substack{\text{serially} \\ \text{independent}}} = E \left[ \begin{bmatrix} \epsilon_1 \epsilon_2 \\ \epsilon_1^3 \epsilon_2 \\ \vdots \end{bmatrix} \begin{bmatrix} \epsilon_1 \epsilon_2 & \epsilon_1^3 \epsilon_2 & \dots \end{bmatrix} \right] = E [ f(B_0, u_t) f(B_0, u_t)' ] =: S_{SI} \quad (5)$$

$$\dots = E \left[ \begin{bmatrix} \epsilon_1 \epsilon_2 \\ \epsilon_1^3 \epsilon_2 \\ \vdots \end{bmatrix} \begin{bmatrix} \epsilon_1 \epsilon_2 & \epsilon_1^3 \epsilon_2 & \dots \end{bmatrix} \right] = \begin{bmatrix} E[\epsilon_1^2 \epsilon_2^2] & E[\epsilon_1^4 \epsilon_2^2] & \dots \\ E[\epsilon_1^4 \epsilon_2^2] & E[\epsilon_1^6 \epsilon_2^2] & \dots \\ \dots & \dots & \dots \end{bmatrix} \quad (6)$$

*A2) mutually independent*

$$= \begin{bmatrix} E[\epsilon_1^2]E[\epsilon_2^2] & E[\epsilon_1^4]E[\epsilon_2^2] & \dots \\ E[\epsilon_1^4]E[\epsilon_2^2] & E[\epsilon_1^6]E[\epsilon_2^2] & \dots \\ \dots & \dots & \dots \end{bmatrix} := S_{SI-MI} \quad (7)$$

*General :*  $S = \lim_{T \rightarrow \infty} E [ T g_T(B_0) g_T(B_0)' ]$

*Traditional :*  $S_{SI} = E[f(B_0)f(B_0)'] = \begin{bmatrix} E[\epsilon_1^2 \epsilon_2^2] & E[\epsilon_1^4 \epsilon_2^2] & \dots \\ E[\epsilon_1^4 \epsilon_2^2] & E[\epsilon_1^6 \epsilon_2^2] & \dots \\ \dots & \dots & \dots \end{bmatrix} \rightarrow \hat{S}_{SI}(\hat{B}_T)$

*Proposed :*  $S_{SI-MI} = \begin{bmatrix} E[\epsilon_1^2]E[\epsilon_2^2] & E[\epsilon_1^4]E[\epsilon_2^2] & \dots \\ E[\epsilon_1^4]E[\epsilon_2^2] & E[\epsilon_1^6]E[\epsilon_2^2] & \dots \\ \dots & \dots & \dots \end{bmatrix} \rightarrow \hat{S}_{SI-MI}(\hat{B}_T)$

## Claim

Traditional estimator  $\hat{S}_{SI}(\hat{B}_T)$  performs poorly in small samples.

Proposed estimator  $\hat{S}_{SI-MI}(\hat{B}_T)$  performs better.

## Monte Carlo Simulation

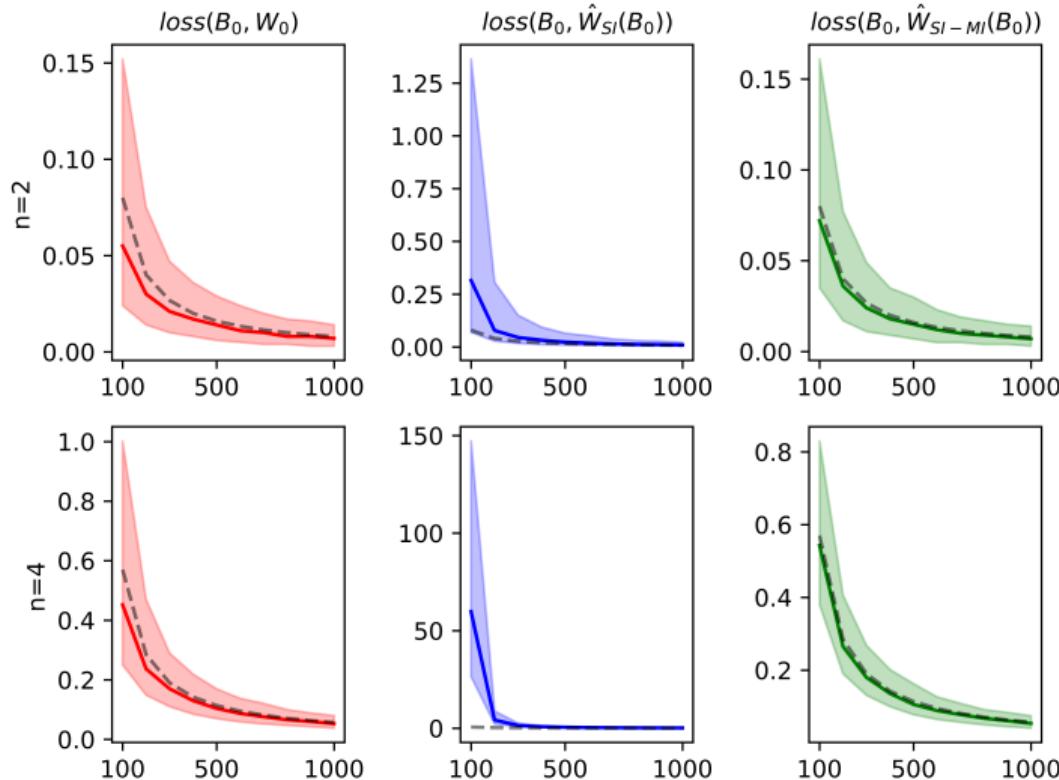
SVAR  $u = B_0\epsilon$  with  $n = 2$  and  $n = 4$  variables where

$$B_0 = \begin{bmatrix} 1 & 0 \\ 0.5 & 1 \end{bmatrix} \text{ and } B_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ 0.5 & 0.5 & 1 & 0 \\ 0.5 & 0.5 & 0.5 & 1 \end{bmatrix}.$$

**Table 1:** Number of moment conditions

	$n = 2$	$n = 4$
(Co-)variance	3	10
Coskewness	2	16
Cokurtosis	3	31

**Figure 1:** Median and (10%, 90%) quantiles of GMM loss at  $B_0$  for different weightings



In grey: Expected loss at  $B_0$  and  $W_0$  in finite sample.

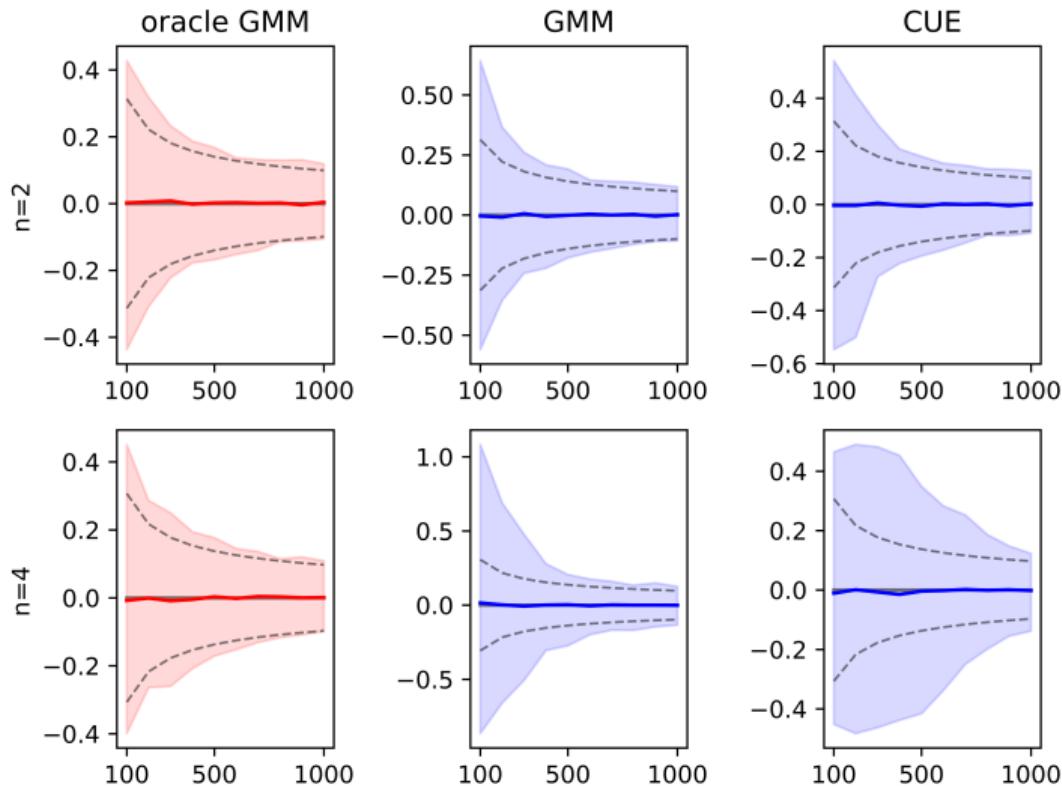
Median and quantiles of GMM loss at  $B_0$  for different weightings.

Red:  $W_0 = S^{-1}$

Blue:  $\hat{W}_{SI} = \hat{S}_{SI}^{-1}$

Green:  $\hat{W}_{SI-MI} = \hat{S}_{SI-MI}^{-1}$

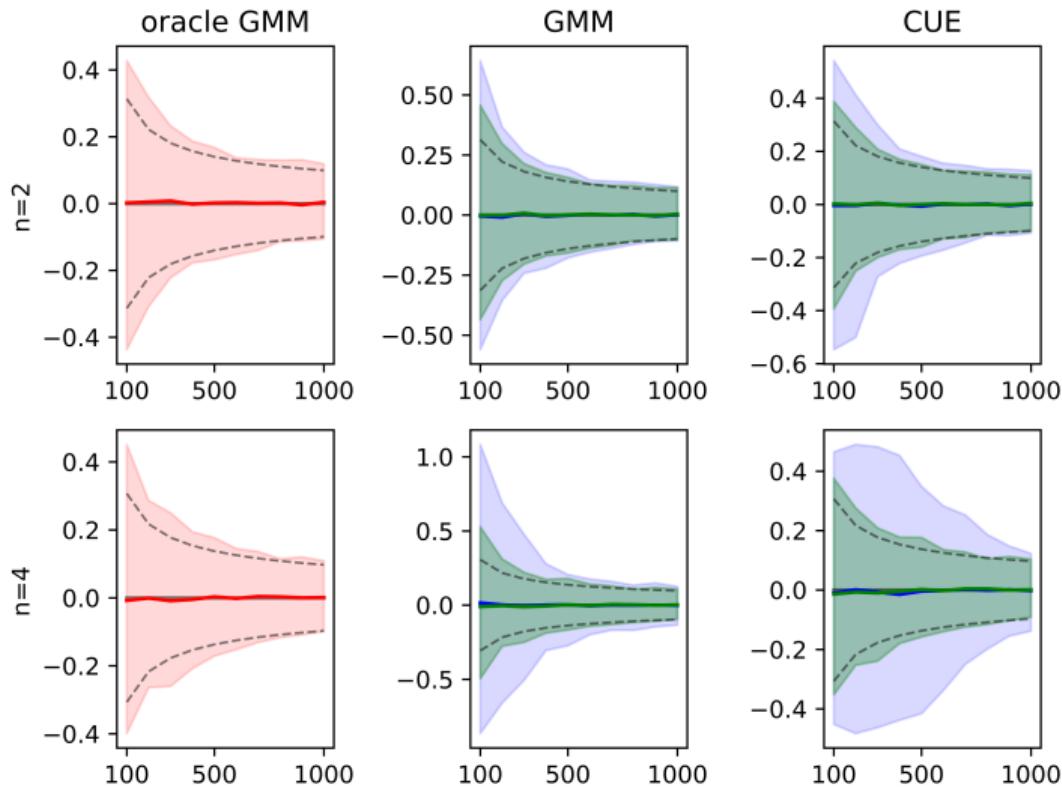
**Figure 2:** Median and (5%, 95%) quantiles of estimated element  $\hat{B}_{1,n}$



Grey:  $B_{1,n}$  and 90% confidence intervals  $(B_{1,n} \pm z^* \frac{\sigma}{\sqrt{T}})$

Median & quantiles  $\hat{B}_{1,1}$   
Red:  $W_0$   
Blue:  $\hat{W}_{SI}$

**Figure 3:** Median and (5%, 95%) quantiles of estimated element  $\hat{B}_{1,n}$



Grey:  $B_{1,n}$  and 90% confidence intervals  $(B_{1,n} \pm z^* \frac{\sigma}{\sqrt{T}})$

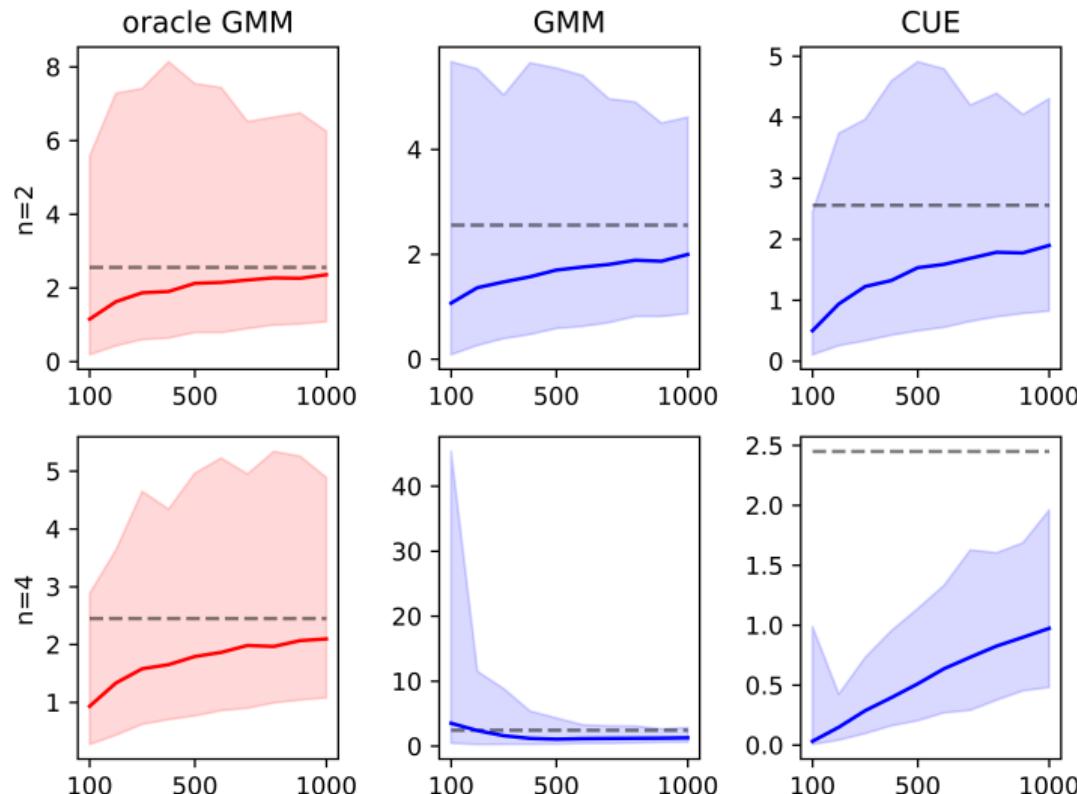
Median & quantiles  $\hat{B}_{1,1}$

Red:  $W_0$

Blue:  $\hat{W}_{SI}$

Green:  $\hat{W}_{SI-MI}$

**Figure 4:** Median and (5%, 95%) quantiles of estimated variance of the element  $\hat{B}_{1,n}$



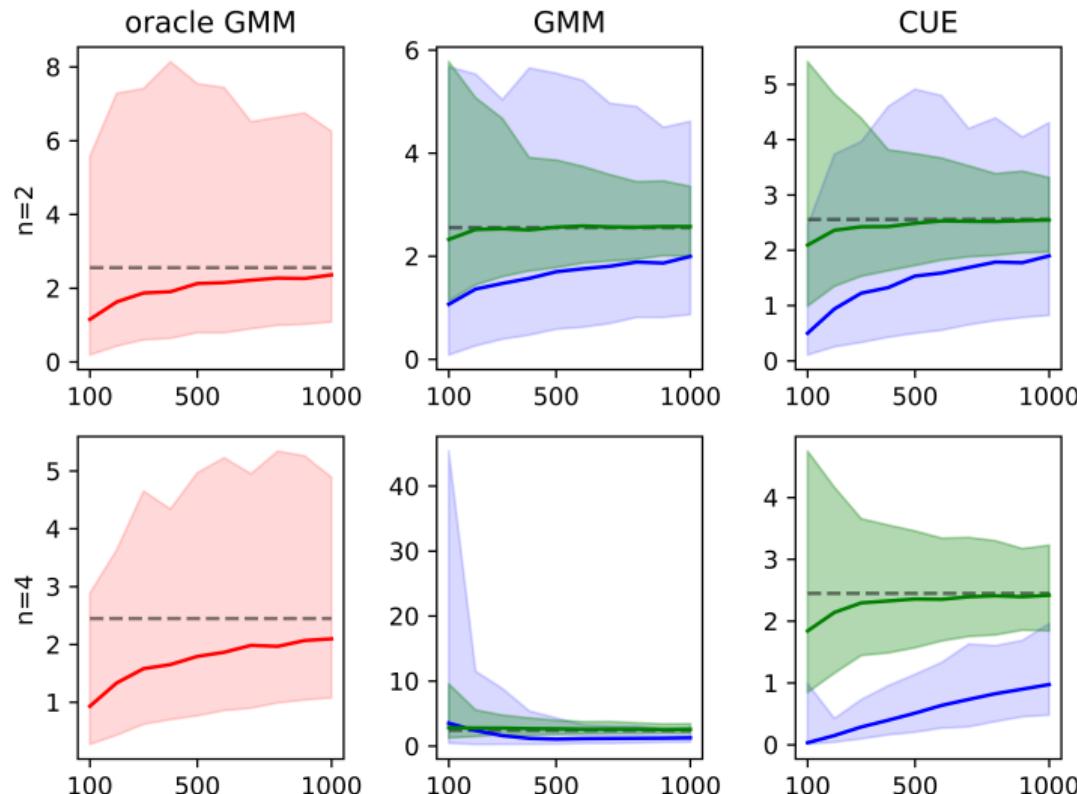
Grey: Avar of  $\hat{B}_{1,n}$  using  $(M_0 S_0 M_0')$ .

Median & quantiles  
of estimated Avar

Red:  $W_0$  and  $\hat{\sigma}_{SI}$

Blue:  $\hat{W}_{SI}$  and  $\hat{\sigma}_{SI}$ .

**Figure 5:** Median and (5%, 95%) quantiles of estimated variance of the element  $\hat{B}_{1,n}$



Grey: Avar of  $\hat{B}_{1,n}$  using  $(M_0 S_0 M_0')$ .

Median & quantiles of estimated Avar

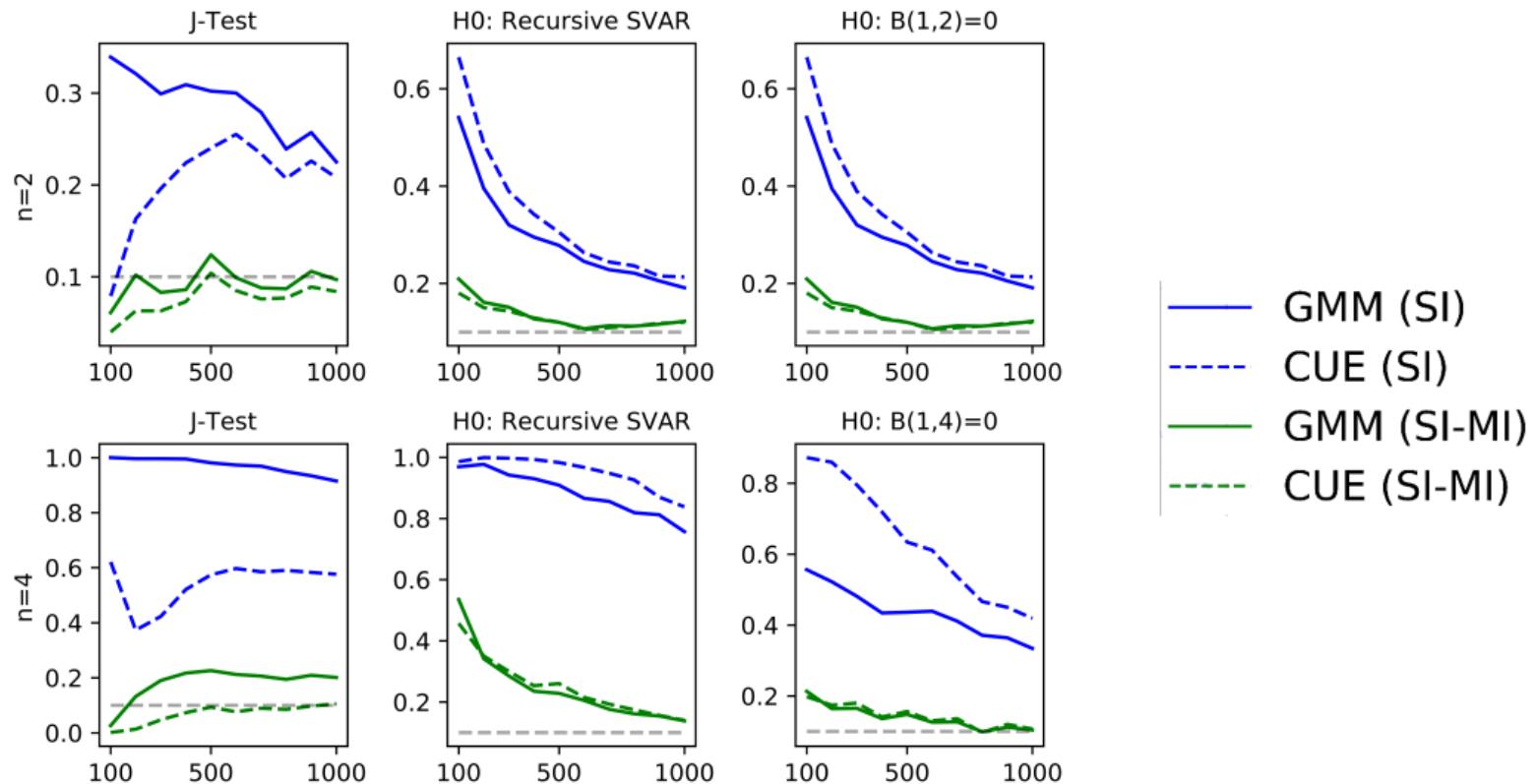
Red:  $W_0$  and  $\hat{\sigma}_{SU}(\hat{B})$

Blue:  $\hat{W}_{SI}$  and  $\hat{\sigma}_{SI}$

Green:  $\hat{W}_{SI-MI}$  and

$\hat{\sigma}_{SI-MI}$

**Figure 6:** 10% rejection rate for J-Test and Wald tests



# Conclusion

## Conclusion:

Exploiting independence to estimate  $S_0$  and  $G_0$  improves finite sample performance.

## Next steps:

- Relax assumptions: serially and mutually independent shocks
- Room for improvements regarding inference: Newey and Windmeijer (2009)
- Whitening and optimal weighting

## Whitening and optimal weighting

Consider a bivariate SVAR

$$\begin{bmatrix} u_{1,t} \\ u_{2,t} \end{bmatrix} = \begin{bmatrix} 1 & \textcolor{red}{b_{12}} \\ \textcolor{red}{b_{21}} & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{bmatrix}$$

and the GMM estimator

$$\hat{B} := \arg \min \left( \frac{1}{T} \sum_{t=1}^T \begin{bmatrix} e(B)_{1,t} e(B)_{2,t} \\ e(B)_{1,t}^3 e(B)_{2,t} \\ e(B)_{1,t} e(B)_{2,t}^3 \end{bmatrix} \right)' W \left( \frac{1}{T} \sum_{t=1}^T \begin{bmatrix} e(B)_{1,t} e(B)_{2,t} \\ e(B)_{1,t}^3 e(B)_{2,t} \\ e(B)_{1,t} e(B)_{2,t}^3 \end{bmatrix} \right).$$

## Whitening and optimal weighting

What is the optimal weighting matrix of the following whitened estimator?

$$\hat{B}^{white} := \arg \min \left( \frac{1}{T} \sum_{t=1}^T \begin{bmatrix} e(B)_{1,t}^3 e(B)_{2,t} \\ e(B)_{1,t} e(B)_{2,t}^3 \end{bmatrix} \right)' W \left( \frac{1}{T} \sum_{t=1}^T \begin{bmatrix} e(B)_{1,t}^3 e(B)_{2,t} \\ e(B)_{1,t} e(B)_{2,t}^3 \end{bmatrix} \right)$$

$$s.t. \frac{1}{T} \sum_{t=1}^T e(B)_{1,t} e(B)_{2,t} = 0$$

Can the whitened estimator be as efficient as the original estimator?

## Initial weighting

$$1) W = I$$

$$2) W = S_{SI}^{-1} = \begin{bmatrix} E[\epsilon_1^2 \epsilon_2^2] & E[\epsilon_1^4 \epsilon_2^2] & \dots \\ E[\epsilon_1^4 \epsilon_2^2] & E[\epsilon_1^6 \epsilon_2^2] & \dots \\ \dots & \dots & \dots \end{bmatrix}^{-1} = ?$$

$$3) W = S_{SI-MI}^{-1} = \begin{bmatrix} E[\epsilon_1^2]E[\epsilon_2^2] & E[\epsilon_1^4]E[\epsilon_2^2] & \dots \\ E[\epsilon_1^4]E[\epsilon_2^2] & E[\epsilon_1^6]E[\epsilon_2^2] & \dots \\ \dots & \dots & \dots \end{bmatrix}^{-1} \stackrel{\epsilon \sim \mathcal{N}(0,1)}{\approx} \begin{bmatrix} 1 & 3 & \dots \\ 3 & 15 & \dots \\ \dots & \dots & \dots \end{bmatrix}^{-1}$$

## References

---

- Guay, A. (2020). Identification of structural vector autoregressions through higher unconditional moments. *Journal of Econometrics*.
- Keweloh, S. A. (2021). A generalized method of moments estimator for structural vector autoregressions based on higher moments. *Journal of Business & Economic Statistics* 39(3), 772–782.
- Lanne, M. and J. Luoto (2021). Gmm estimation of non-gaussian structural vector autoregression. *Journal of Business & Economic Statistics* 39(1), 69–81.
- Newey, W. K. and F. Windmeijer (2009). Generalized method of moments with many weak moment conditions. *Econometrica* 77(3), 687–719.