Higher-order moments in structural VAR models

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Structural VAR model

$$
\begin{array}{rc}
\text{output} & \rightarrow \\
\text{inflation} & \rightarrow \\
\text{FFR} & \rightarrow
\end{array}
\underbrace{\begin{bmatrix} y_t \\ \pi_t \\ i_t \end{bmatrix}}_{\substack{u_t \\ \text{reduced form} \\ \text{shocks}}}
=
\underbrace{\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}}_{\substack{B_0 \\ \textit{simultaenous} \\ \textit{impact}}}
\underbrace{\begin{bmatrix} \varepsilon_{s,t} \\ \varepsilon_{d,t} \\ \varepsilon_{i,t} \end{bmatrix}}_{\substack{\varepsilon_t \\ \textit{structural} \\ \textit{shocks}}}
\begin{array}{cl}
\leftarrow & \text{supply shock} \\
\leftarrow & \text{demand shock} \\
\leftarrow & \text{monetary policy shock}
\end{array}
$$

Assumptions:

A1) Serially independent (identically distributed) shocks $\rightarrow$ ϵ_t independent of $\epsilon_{\tilde{t}}$

A2) Mutually independent shocks $\rightarrow$ $\epsilon_{j,t}$ independent of $\epsilon_{\tilde{j},t}$

A3) Zero mean, unit variance, and non-Gaussian shocks

SVAR:

$$u = B_0 \epsilon \iff \epsilon = B_0^{-1} u$$

Unmixed innovations

$$e(B) := B^{-1} u$$

Note: $e(B_0) = \epsilon$

Independent structural shocks:

$$E[\epsilon_1 \epsilon_2] = 0$$
$$E[\epsilon_1^3 \epsilon_2] = 0$$
$$\vdots$$

Independent unmixed innovations:

$$E[e(B)_1 e(B)_2] \overset{!}{=} 0$$
$$E[e(B)_1^3 e(B)_2] \overset{!}{=} 0$$
$$\vdots$$

Literature: Lanne and Luoto (2021), Keweloh (2021), or Guay (2020)

SVAR GMM

$$\hat{B}_T := \arg\min_B g_T(B)' W g_T(B), \tag{1}$$

with

$$g_T(B) = \frac{1}{T}\sum_{t=1}^{T} f(B, u_t) \qquad \textit{and} \qquad f(B, u_t) = \begin{bmatrix} e(B)_{1,t} e(B)_{2,t} \\ e(B)_{1,t}^3 e(B)_{2,t} \\ \vdots \end{bmatrix}. \tag{2}$$

Consistency and asymptotic normality

$$\hat{B}_T \xrightarrow{p} B_0$$

$$\sqrt{T}(\hat{B}_T - B_0) \xrightarrow{d} \mathcal{N}(0, MSM')$$

with

$$\begin{aligned} M &:= \left(G'S^{-1}G\right)^{-1} G'W \\ G &:= E\left[\frac{\partial f(B_0,u_t)}{\partial vec(B)'}\right] \\ S &:= \lim_{T\to\infty} E\left[T g_T(B_0) g_T(B_0)'\right] \\ &\overset{ser.unc.}{=} E\left[f(B_0,u_t) f(B_0,u_t)'\right] \end{aligned}$$

Optimal weighting

$$W_0 := S^{-1}$$

Problem
Higher-order moment conditions make it difficult to estimate S (and G).

$$S := \lim_{T\to\infty} E\left[Tg_T(B_0)g_T(B_0)'\right] \tag{3}$$

$$= \lim_{T\to\infty} E\left[T\left(\frac{1}{T}\sum_{t=1}^{T}\begin{bmatrix}\epsilon_{1,t}\epsilon_{2,t}\\ \epsilon_{1,t}^3\epsilon_{2,t}\\ \dots\end{bmatrix}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\begin{bmatrix}\epsilon_{1,t}\epsilon_{2,t}\\ \epsilon_{1,t}^3\epsilon_{2,t}\\ \dots\end{bmatrix}\right)'\right] \tag{4}$$

$$\overset{\substack{A1)serially\\ independent}}{=} E\left[\begin{bmatrix}\epsilon_1\epsilon_2\\ \epsilon_1^3\epsilon_2\\ \vdots\end{bmatrix}\begin{bmatrix}\epsilon_1\epsilon_2 & \epsilon_1^3\epsilon_2 & \dots\end{bmatrix}\right] = E\left[f(B_0,u_t)f(B_0,u_t)'\right] =: S_{SI} \tag{5}$$

$$... = E\left[\begin{bmatrix}\epsilon_1\epsilon_2\\ \epsilon_1^3\epsilon_2\\ \vdots\end{bmatrix}\begin{bmatrix}\epsilon_1\epsilon_2 & \epsilon_1^3\epsilon_2 & ...\end{bmatrix}\right] = \begin{bmatrix}E[\epsilon_1^2\epsilon_2^2] & E[\epsilon_1^4\epsilon_2^2] & ...\\ E[\epsilon_1^4\epsilon_2^2] & E[\epsilon_1^6\epsilon_2^2] & ...\\ ... & & \end{bmatrix} \tag{6}$$

$$\overset{\substack{A2) mutually\\ independent}}{=} \begin{bmatrix}E[\epsilon_1^2]E[\epsilon_2^2] & E[\epsilon_1^4]E[\epsilon_2^2] & ...\\ E[\epsilon_1^4]E[\epsilon_2^2] & E[\epsilon_1^6]E[\epsilon_2^2] & ...\\ ... & & \end{bmatrix} := S_{SI-MI} \tag{7}$$

$$General: \quad S = \lim_{T\to\infty} E\left[Tg_T(B_0)g_T(B_0)'\right]$$

$$Traditional: \quad S_{SI} = E[f(B_0)f(B_0)'] = \begin{bmatrix} E[\epsilon_1^2\epsilon_2^2] & E[\epsilon_1^4\epsilon_2^2] & \ldots \\ E[\epsilon_1^4\epsilon_2^2] & E[\epsilon_1^6\epsilon_2^2] & \ldots \\ \ldots & & \end{bmatrix} \to \hat{S}_{SI}(\hat{B}_T)$$

$$Proposed: \quad S_{SI-MI} = \begin{bmatrix} E[\epsilon_1^2]E[\epsilon_2^2] & E[\epsilon_1^4]E[\epsilon_2^2] & \ldots \\ E[\epsilon_1^4]E[\epsilon_2^2] & E[\epsilon_1^6]E[\epsilon_2^2] & \ldots \\ \ldots & & \end{bmatrix} \to \hat{S}_{SI-MI}(\hat{B}_T)$$

Claim

Traditional estimator $\hat{S}_{SI}(\hat{B}_T)$ performs poorly in small samples.

Proposed estimator $\hat{S}_{SI-MI}(\hat{B}_T)$ performs better.

Monte Carlo Simulation

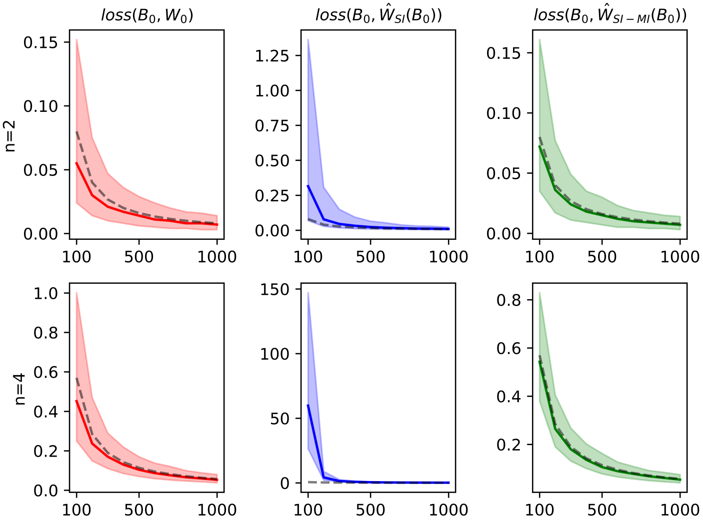
SVAR $u = B_0\epsilon$ with $n = 2$ and $n = 4$ variables where

$$B_0 = \begin{bmatrix} 1 & 0 \\ 0.5 & 1 \end{bmatrix} \text{ and } B_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ 0.5 & 0.5 & 1 & 0 \\ 0.5 & 0.5 & 0.5 & 1 \end{bmatrix}.$$

Table 1: Number of moment conditions

	$n=2$	$n=4$
(Co-)variance	3	10
Coskewnes	2	16
Cokurtosis	3	31

Figure 1: Median and (10%, 90%) quantiles of GMM loss at B_0 for different weightings



In grey: Expected loss at B_0 and W_0 in finite sample.

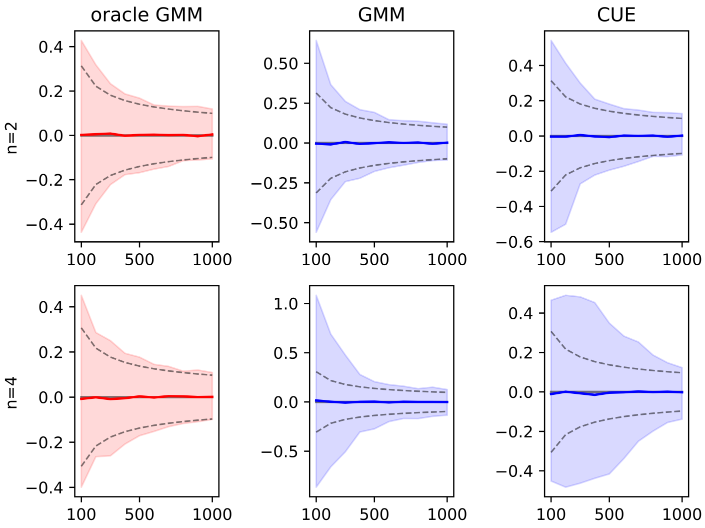
Median and quantiles of GMM loss at B_0 for different weightings.

Red: $W_0 = S^{-1}$

Blue: $\hat{W}_{SI} = \hat{S}_{SI}^{-1}$

Green: $\hat{W}_{SI-MI} = \hat{S}_{SI-MI}^{-1}$

Figure 2: Median and (5%, 95%) quantiles of estimated element $\hat{B}_{1,n}$



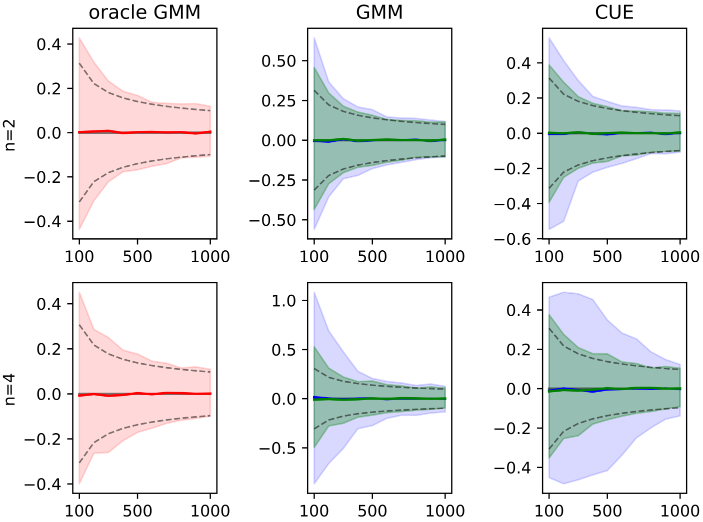
Grey: $B_{1,n}$ and 90% confidence intervals $(B_{1,n} \pm z^* \frac{\sigma}{\sqrt{T}})$

Median & quantiles $\hat{B}_{1,1}$

Red: W_0

Blue: $\hat{W}_{SI}$

Figure 3: Median and (5%, 95%) quantiles of estimated element $\hat{B}_{1,n}$



Grey: $B_{1,n}$ and 90% confidence intervals $(B_{1,n} \pm z^* \frac{\sigma}{\sqrt{T}})$

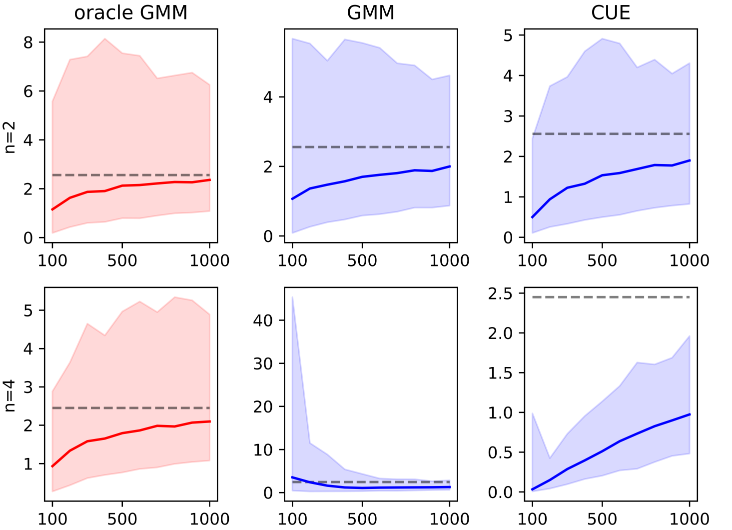
Median & quantiles $\hat{B}_{1,1}$

Red: W_0

Blue: $\hat{W}_{SI}$

Green: $\hat{W}_{SI-MI}$

Figure 4: Median and (5%, 95%) quantiles of estimated variance of the element $\hat{B}_{1,n}$



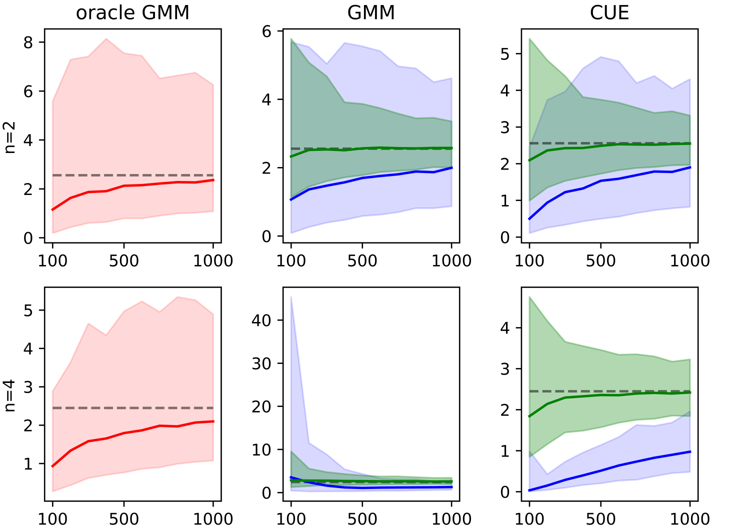
Grey: Avar of $\hat{B}_{1,n}$ using $(M_0 S_0 M_0')$.

Median & quantiles of estimated Avar

Red: W_0 and $\hat{\sigma}_{SI}$

Blue: $\hat{W}_{SI}$ and $\hat{\sigma}_{SI}$.

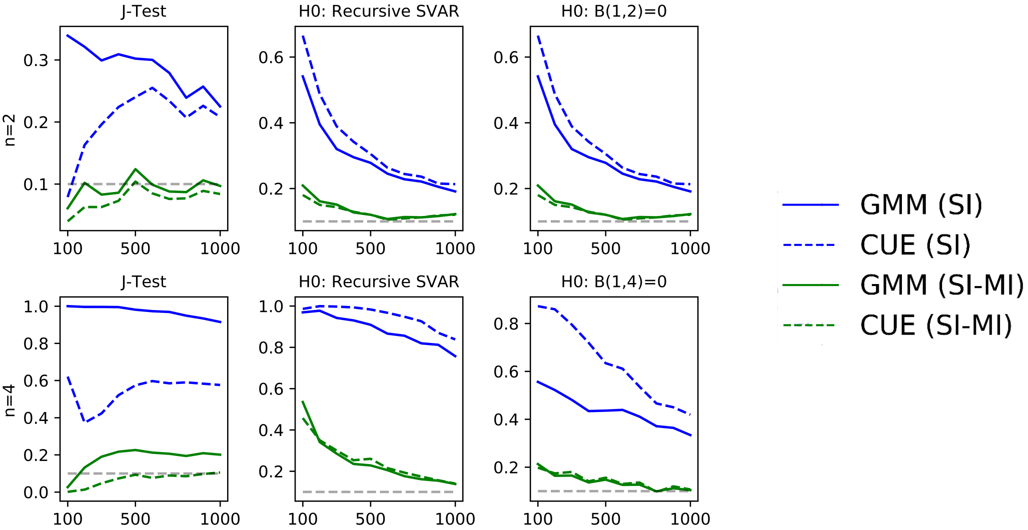
Figure 5: Median and (5%, 95%) quantiles of estimated variance of the element $\hat{B}_{1,n}$



Grey: Avar of $\hat{B}_{1,n}$ using $(M_0 S_0 M_0')$.

Median & quantiles of estimated Avar
Red: W_0 and $\hat{\sigma}_{SU}(\hat{B})$
Blue: $\hat{W}_{SI}$ and $\hat{\sigma}_{SI}$
Green: $\hat{W}_{SI-MI}$ and $\hat{\sigma}_{SI-MI}$

Figure 6: 10% rejection rate for J-Test and Wald tests



Conclusion:
Exploiting independence to estimate S and G improves finite sample performance.

Next steps:

- Relax assumptions: serially and mutually independent shocks
- Room for improvements regarding inference: Newey and Windmeijer (2009)
- Whitening and optimal weighting

Whitening and optimal weighting

Consider a bivariate SVAR

$$\begin{bmatrix} u_{1,t} \\ u_{2,t} \end{bmatrix} = \begin{bmatrix} 1 & b_{12} \\ b_{21} & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{bmatrix}$$

and the GMM estimator

$$\hat{B} := \arg\min \left(\frac{1}{T} \sum_{t=1}^{T} \begin{bmatrix} e(B)_{1,t} e(B)_{2,t} \\ e(B)_{1,t}^3 e(B)_{2,t} \\ e(B)_{1,t} e(B)_{2,t}^3 \end{bmatrix} \right)' W \left(\frac{1}{T} \sum_{t=1}^{T} \begin{bmatrix} e(B)_{1,t} e(B)_{2,t} \\ e(B)_{1,t}^3 e(B)_{2,t} \\ e(B)_{1,t} e(B)_{2,t}^3 \end{bmatrix} \right).$$

Whitening and optimal weighting

What is the optimal weighting matrix of the following whitened estimator?

$$\hat{B}^{white} := \arg\min \left(\frac{1}{T} \sum_{t=1}^{T} \begin{bmatrix} e(B)_{1,t}^3 e(B)_{2,t} \\ e(B)_{1,t} e(B)_{2,t}^3 \end{bmatrix} \right)' W \left(\frac{1}{T} \sum_{t=1}^{T} \begin{bmatrix} e(B)_{1,t}^3 e(B)_{2,t} \\ e(B)_{1,t} e(B)_{2,t}^3 \end{bmatrix} \right)$$
$$s.t. \frac{1}{T} \sum_{t=1}^{T} e(B)_{1,t} e(B)_{2,t} = 0$$

Can the whitened estimator be as efficient as the original estimator?

Initial weighting

1) $W = I$

2) $W = S_{SI}^{-1} = \begin{bmatrix} E[\epsilon_1^2\epsilon_2^2] & E[\epsilon_1^4\epsilon_2^2] & ... \\ E[\epsilon_1^4\epsilon_2^2] & E[\epsilon_1^6\epsilon_2^2] & ... \\ ... & & \end{bmatrix}^{-1} = \ ?$

3) $W = S_{SI-MI}^{-1} = \begin{bmatrix} E[\epsilon_1^2]E[\epsilon_2^2] & E[\epsilon_1^4]E[\epsilon_2^2] & ... \\ E[\epsilon_1^4]E[\epsilon_2^2] & E[\epsilon_1^6]E[\epsilon_2^2] & ... \\ ... & & \end{bmatrix}^{-1} \overset{\epsilon\sim\mathcal{N}(0,1)}{\approx} \begin{bmatrix} 1 & 3 & ... \\ 3 & 15 & ... \\ ... & & \end{bmatrix}^{-1}$

References

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