Non-Gaussian Structural Vector Autoregressions

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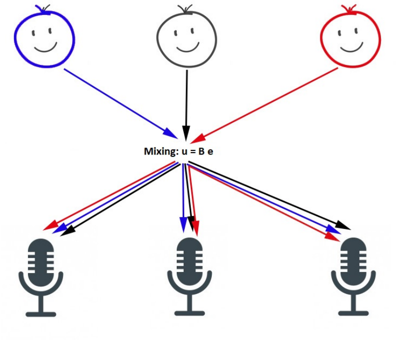
The Cocktail-Party-Problem



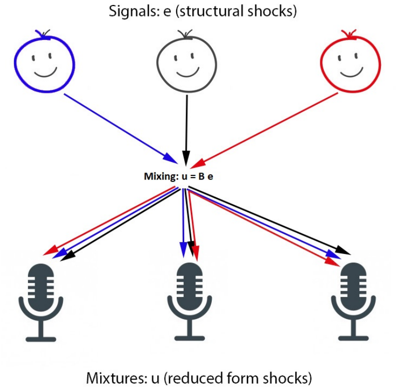
Structural Vector Autoregression

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix}$$

In matrix notation:

$$\underbrace{u}_{\text{observed mixtures}} = \underbrace{B_0}_{\text{mixing matrix}} \underbrace{\epsilon}_{\text{signals}}$$

The Cocktail-Party-Problem



Simulated SVAR

$$\underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}}_{u} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ 0.5 & 0.5 & 1 & 0 \\ 0.5 & 0.5 & 0.5 & 1 \end{bmatrix}}_{B_0} \underbrace{\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \end{bmatrix}}_{\epsilon}$$

$$\varepsilon_i = z\phi_1 + (1-z)\phi_2$$

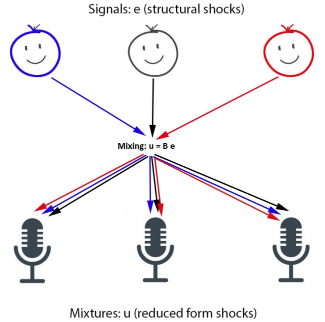
$$\phi_1 \sim \mathcal{N}(-0.2, 0.7), \;\; \phi_2 \sim \mathcal{N}(0.75, 1.5)$$

$$z \sim \mathcal{B}(0.79)$$

$\mathcal{B}(p)$ is Bernoulli and $\mathcal{N}(\mu, \sigma^2)$ is normal. The shocks have skewness 0.91 and excess kurtosis 2.51.

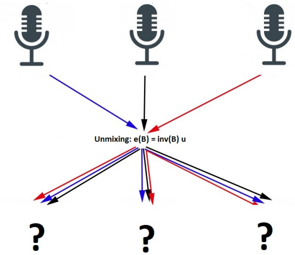
Structural Vector Autoregression

$$u = B_0 \epsilon$$



Unmixed innovations

$$e(B) := B^{-1} u$$



Recursive SVAR

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ b_{21} & 1 & 0 \\ b_{31} & b_{32} & 1 \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix}$$

Assumptions:
Mixing B_0 is recursive

Shocks ϵ are uncorrelated

Moment conditions

$$E[e_1(B)e_2(B)] = 0$$
$$E[e_1(B)e_3(B)] = 0$$
$$E[e_2(B)e_3(B)] = 0$$

Identified ✓

$\rightarrow$ Cholesky decomposition

Keweloh, Sascha. "A Generalized Method of Moments Estimator for Structural Vector Autoregressions Based on Higher Moments." *Journal of Business & Economic Statistics* (2019)

Non-Recursive SVAR

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 & b_{12} & b_{13} \\ b_{21} & 1 & b_{23} \\ b_{31} & b_{32} & 1 \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix}$$

Assumptions:
Shocks ϵ are independent

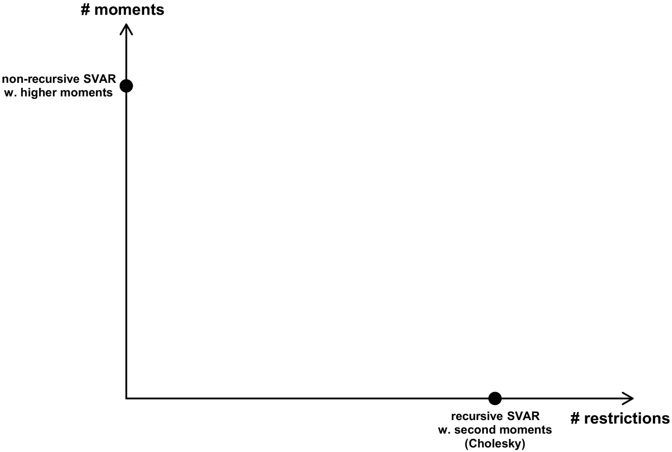
Shocks ϵ are non-Gaussian

Moment conditions

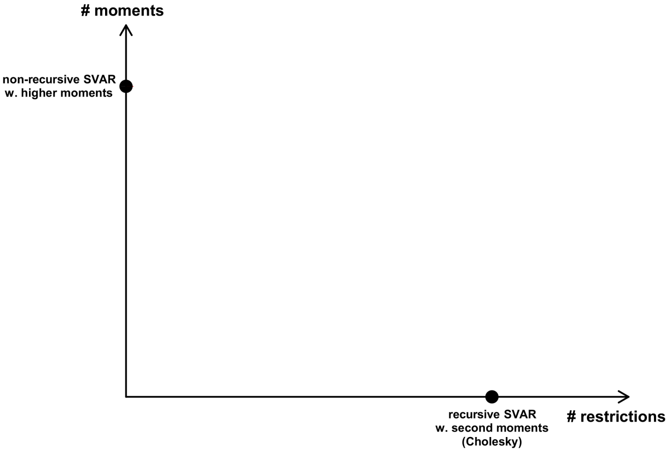
$$E[e_1(B)e_2(B)] = 0$$
$$E[e_1(B)e_3(B)] = 0$$
$$E[e_2(B)e_3(B)] = 0$$

$$E[e_1^2(B)e_2(B)] = 0$$
$$E[e_1(B)e_2^2(B)] = 0$$
$$E[e_1^2(B)e_3(B)] = 0$$
$$E[e_1(B)e_3^2(B)] = 0$$
$$E[e_2^2(B)e_3(B)] = 0$$
$$E[e_2(B)e_3^2(B)] = 0$$
$$E[e_1(B)e_2(B)e_3(B)] = 0$$

Identified ✓

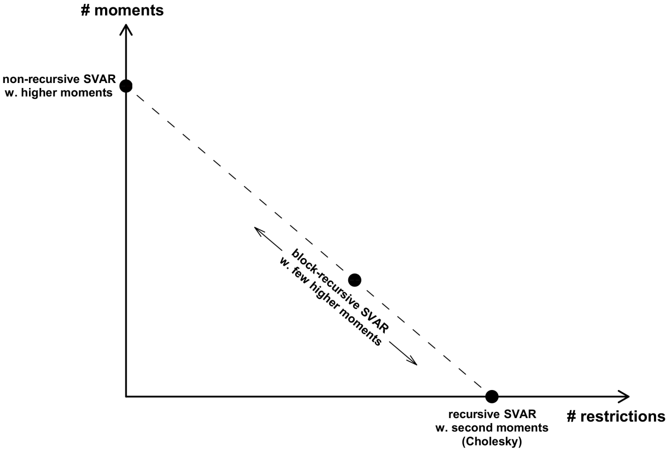

moments
non-recursive SVAR
w. higher moments
recursive SVAR
w. second moments
(Cholesky)
restrictions

	Non-Recursive SVAR w. higher moments	Recursive SVAR w. second moments (Cholesky)
Limit (*Avar*)	$\begin{bmatrix} \underset{(1.033)}{1} & \underset{(2.450)}{0} & \underset{(2.450)}{0} & \underset{(2.450)}{0} \\ \underset{(2.585)}{0.5} & \underset{(1.522)}{1} & \underset{(3.192)}{0} & \underset{(3.192)}{0} \\ \underset{(3.265)}{0.5} & \underset{(3.265)}{0.5} & \underset{(2.076)}{1} & \underset{(3.999)}{0} \\ \underset{(4.010)}{0.5} & \underset{(4.010)}{0.5} & \underset{(4.010)}{0.5} & \underset{(2.695)}{1} \end{bmatrix}$	$\begin{bmatrix} \underset{(1.128)}{1} & \underset{(0)}{0} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(1.282)}{0.5} & \underset{(1.128)}{1} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(1.532)}{0.5} & \underset{(1.282)}{0.5} & \underset{(1.128)}{1} & \underset{(0)}{0} \\ \underset{(1.782)}{0.5} & \underset{(1.532)}{0.5} & \underset{(1.282)}{0.5} & \underset{(1.128)}{1} \end{bmatrix}$



moments
non-recursive SVAR
w. higher moments
recursive SVAR
w. second moments
(Cholesky)
restrictions

Keweloh "Block-Recursive SVARs."



Keweloh "Block-Recursive SVARs."

Block-Recursive SVAR

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ b_{21} & 1 & b_{23} \\ b_{31} & b_{32} & 1 \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix}$$

Assumptions:
Shocks ϵ are independent

Shocks ϵ are non-Gaussian

Moment conditions

$$E[e_1(B)e_2(B)] = 0$$
$$E[e_1(B)e_3(B)] = 0$$
$$E[e_2(B)e_3(B)] = 0$$

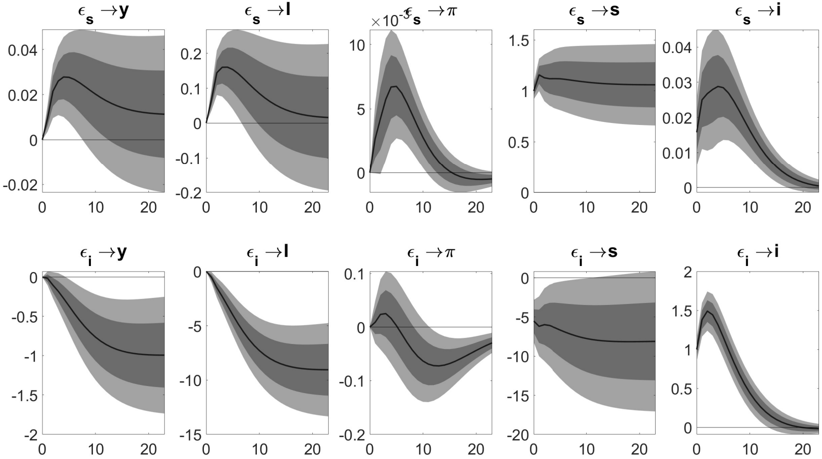
$$\cancel{E[e_1^2(B)e_2(B)] = 0}$$
$$\cancel{E[e_1(B)e_2^2(B)] = 0}$$
$$\cancel{E[e_1^2(B)e_3(B)] = 0}$$
$$\cancel{E[e_1(B)e_3^2(B)] = 0}$$
$$E[e_2^2(B)e_3(B)] = 0$$
$$E[e_2(B)e_3^2(B)] = 0$$
$$\cancel{E[e_1(B)e_2(B)e_3(B)] = 0}$$

Identified ✓

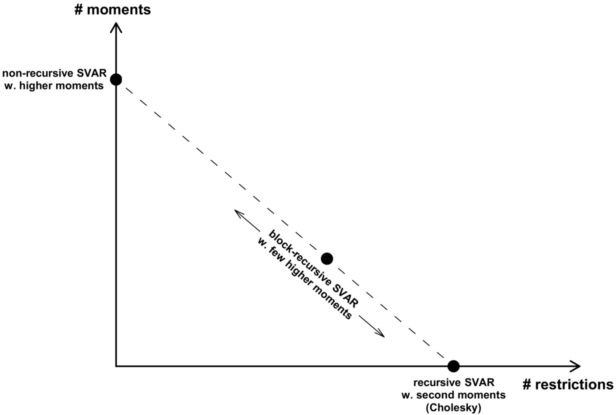
Keweloh, Seepe "Monetary policy and the stock market"

$$
\begin{array}{lc}
\text{output} & \rightarrow \\
\text{investment} & \rightarrow \\
\text{inflation} & \rightarrow \\
\text{federal funds rate} & \rightarrow \\
\text{real stock returns} & \rightarrow
\end{array}
\begin{bmatrix} u_t^y \\ u_t^I \\ u_t^\pi \\ u_t^i \\ u_t^s \end{bmatrix}
=
\begin{bmatrix}
b_{11} & b_{12} & b_{13} & 0 & 0 \\
b_{21} & b_{22} & b_{23} & 0 & 0 \\
b_{31} & b_{32} & b_{33} & 0 & 0 \\
b_{41} & b_{42} & b_{43} & b_{44} & b_{45} \\
b_{51} & b_{52} & b_{53} & b_{54} & b_{55}
\end{bmatrix}
\begin{bmatrix} \varepsilon_t^y \\ \varepsilon_t^I \\ \varepsilon_t^\pi \\ \varepsilon_t^s \\ \varepsilon_t^i \end{bmatrix}, \tag{1}
$$

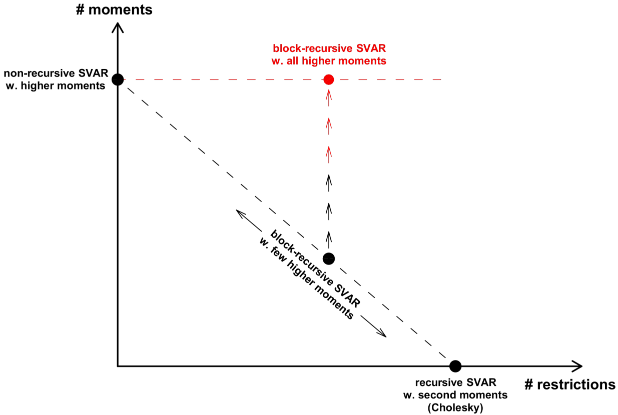
Keweloh, Seepe "Monetary policy and the stock market"



Keweloh "Block-Recursive SVARs."



Are higher moments non-redundant?



Keweloh "Block-Recursive SVARs."

Recursive SVAR

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ b_{21} & 1 & 0 \\ b_{31} & b_{32} & 1 \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix}$$

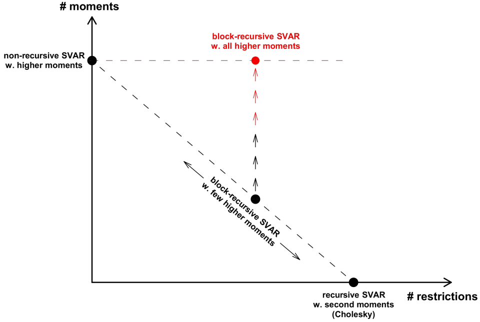
Identifying conditions

$$E[e_1(B)e_2(B)] = 0$$
$$E[e_1(B)e_3(B)] = 0$$
$$E[e_2(B)e_3(B)] = 0$$

Overidentifying conditions

$$E[e_1^2(B)e_2(B)] = 0$$
$$E[e_1(B)e_2^2(B)] = 0$$
$$E[e_1^2(B)e_3(B)] = 0$$
$$E[e_1(B)e_3^2(B)] = 0$$
$$E[e_2^2(B)e_3(B)] = 0$$
$$E[e_2(B)e_3^2(B)] = 0$$
$$E[e_1(B)e_2(B)e_3(B)] = 0$$

Are higher moments non-redundant?



Proposition
In a block-recursive SVAR with non-Gaussian and independent shocks, higher moments are non-redundant.

Corollary
In a non-Gaussian recursive SVAR with independent shocks, the frequently used estimator obtained by applying the Cholesky decomposition to the variance covariance matrix of the reduced form shocks is inefficient.

Asymptotic performance

	Recursive SVAR w. second moments (Cholesky)	Recursive SVAR w. higher moments
Limit (*Avar*)	$\begin{bmatrix} \underset{(1.128)}{1} & \underset{(0)}{0} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(1.282)}{0.5} & \underset{(1.128)}{1} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(1.532)}{0.5} & \underset{(1.282)}{0.5} & \underset{(1.128)}{1} & \underset{(0)}{0} \\ \underset{(1.782)}{0.5} & \underset{(1.532)}{0.5} & \underset{(1.282)}{0.5} & \underset{(1.128)}{1} \end{bmatrix}$	$\begin{bmatrix} \underset{(1.021)}{1} & \underset{(0)}{0} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(1.033)}{0.5} & \underset{(1.019)}{1} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(1.232)}{0.5} & \underset{(1.033)}{0.5} & \underset{(1.017)}{1} & \underset{(0)}{0} \\ \underset{(1.438)}{0.5} & \underset{(1.232)}{0.5} & \underset{(1.033)}{0.5} & \underset{(1.016)}{1} \end{bmatrix}$

Small sample performance - Many moments problem

T=150	Recursive SVAR w. second moments (Cholesky)	Recursive SVAR w. higher moments 2-step GMM
$\underset{(std)}{\textit{mean}}$	$\begin{bmatrix} \underset{(1.138)}{1} & \underset{(0)}{0} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(1.300)}{0.5} & \underset{(1.048)}{1} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(1.499)}{0.5} & \underset{(1.263)}{0.5} & \underset{(1.126)}{1} & \underset{(0)}{0} \\ \underset{(1.784)}{0.5} & \underset{(1.539)}{0.5} & \underset{(1.275)}{0.5} & \underset{(1.096)}{0.99} \end{bmatrix}$	$\begin{bmatrix} \underset{(7.50)}{0.95} & \underset{(0)}{0} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(49.2)}{0.47} & \underset{(828)}{1.00} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(-)}{-19.9} & \underset{(-)}{-26.2} & \underset{(-)}{158} & \underset{(0)}{0} \\ \underset{(-)}{-287} & \underset{(-)}{31} & \underset{(-)}{-145} & \underset{(-)}{506} \end{bmatrix}$

Optimal weighting

The GMM estimator with $W = S^{-1}$ has the smallest asymptotic variance.

$$S = E\left[\begin{bmatrix} e_1(B_0)e_2(B_0) \\ e_1(B_0)e_3(B_0) \\ e_2(B_0)e_3(B_0) \\ \vdots \end{bmatrix} \begin{bmatrix} e_1(B_0)e_2(B_0) \\ e_1(B_0)e_3(B_0) \\ e_2(B_0)e_3(B_0) \\ \vdots \end{bmatrix}'\right] \tag{2}$$

$$\rightarrow \hat{S}_T = \frac{1}{T}\sum_{t=1}^{T} \begin{bmatrix} e_{1,t}(\hat{B})e_{2,t}(\hat{B}) \\ e_{1,t}(\hat{B})e_{3,t}(\hat{B}) \\ e_{2,t}(\hat{B})e_{3,t}(\hat{B}) \\ \vdots \end{bmatrix} \begin{bmatrix} e_{1,t}(\hat{B})e_{2,t}(\hat{B}) \\ e_{1,t}(\hat{B})e_{3,t}(\hat{B}) \\ e_{2,t}(\hat{B})e_{3,t}(\hat{B}) \\ \vdots \end{bmatrix}' \tag{3}$$

The true mixing matrix B_0 is unknown, but a B_0 it holds: $e(B_0) = \epsilon$

$$S = E\left[\begin{bmatrix} \epsilon_1\epsilon_2 \\ \epsilon_1\epsilon_3 \\ \epsilon_2\epsilon_3 \\ \epsilon_1^2 - 1 \\ \vdots \end{bmatrix}\begin{bmatrix} \epsilon_1\epsilon_2 \\ \epsilon_1\epsilon_3 \\ \epsilon_2\epsilon_3 \\ \epsilon_1^2 - 1 \\ \vdots \end{bmatrix}'\right] \overset{\epsilon \text{ independent}}{=} \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & E[\epsilon_1^4] - 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \tag{4}$$

Parametric estimator of S

Use (4) and replace values like $E[\epsilon_1^4]$ with estimates $1/T\sum_{t=1}^T e_1^4(\hat{B})$.

If ϵ has mutually independent components with zero mean and unit variance, it holds that

$$\hat{S}_T^{para} \overset{T\to\infty}{\to} S. \tag{5}$$

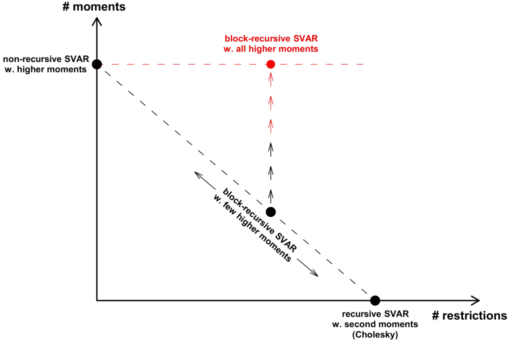
Small sample performance - Many moments problem (Solution part 1)

T=150	Recursive SVAR w. second moments (Cholesky)	Recursive SVAR w. higher moments 2-step GMM parametric
mean (*std*)	$\begin{bmatrix} \underset{(1.138)}{1} & \underset{(0)}{0} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(1.300)}{0.5} & \underset{(1.048)}{1} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(1.499)}{0.5} & \underset{(1.263)}{0.5} & \underset{(1.126)}{1} & \underset{(0)}{0} \\ \underset{(1.784)}{0.5} & \underset{(1.539)}{0.5} & \underset{(1.275)}{0.5} & \underset{(1.096)}{0.99} \end{bmatrix}$	$\begin{bmatrix} \underset{(2.609)}{1.14} & \underset{(0)}{0} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(1.914)}{0.57} & \underset{(2.687)}{1.13} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(2.157)}{0.57} & \underset{(1.785)}{0.56} & \underset{(2.289)}{1.12} & \underset{(0)}{0} \\ \underset{(2.686)}{0.56} & \underset{(2.204)}{0.56} & \underset{(1.786)}{0.56} & \underset{(2.307)}{1.12} \end{bmatrix}$

Small sample performance - Many moments problem (Solution part 2)

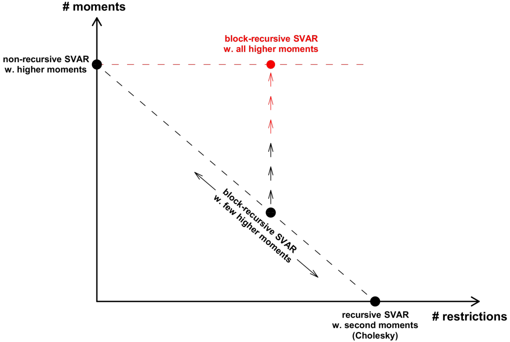
T=150	Recursive SVAR w. second moments (Cholesky)	Recursive SVAR w. higher moments 2-step parametric (scaled)
$\underset{(std)}{mean}$	$\begin{bmatrix} \underset{(1.138)}{1} & \underset{(0)}{0} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(1.300)}{0.5} & \underset{(1.048)}{1} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(1.499)}{0.5} & \underset{(1.263)}{0.5} & \underset{(1.126)}{1} & \underset{(0)}{0} \\ \underset{(1.784)}{0.5} & \underset{(1.539)}{0.5} & \underset{(1.275)}{0.5} & \underset{(1.096)}{0.99} \end{bmatrix}$	$\begin{bmatrix} \underset{(1.136)}{1.00} & \underset{(0)}{0} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(1.209)}{0.50} & \underset{(1.051)}{1.00} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(1.414)}{0.50} & \underset{(1.185)}{0.50} & \underset{(1.131)}{1.00} & \underset{(0)}{0} \\ \underset{(1.726)}{0.50} & \underset{(1.478)}{0.50} & \underset{(1.208)}{0.50} & \underset{(1.104)}{0.99} \end{bmatrix}$

Summary



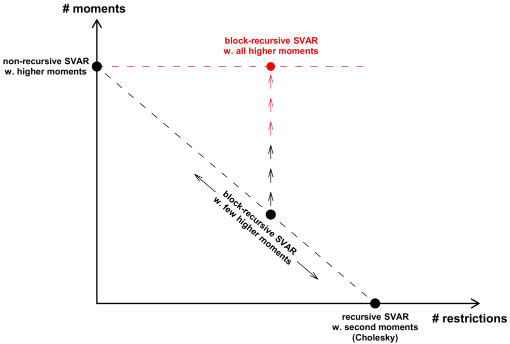
- Block-recursive framework generalizes between unrestricted and recursive SVAR.
- Restrictions (derived from economic theory) decrease dependence of identification on higher moments and increase the performance of the estimator.
- Higher moments are non-redundant.

Critique



Restrictions decrease dependence on higher moments and increase the performance of the estimator...

Critique

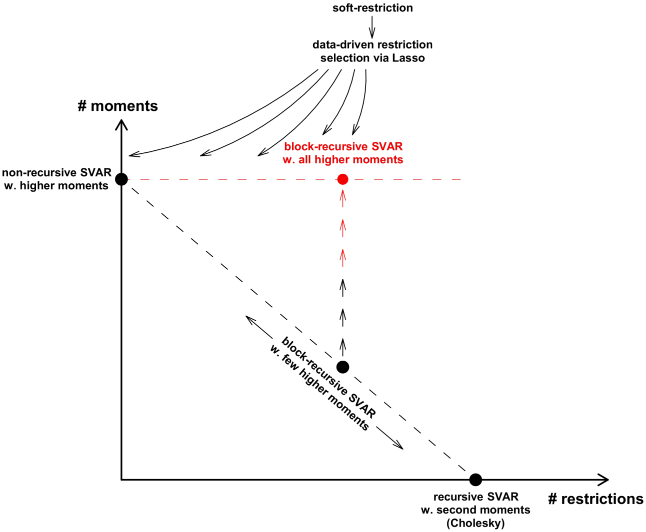


Restrictions decrease dependence on higher moments and increase the performance of the estimator...

... if the restrictions are correct or otherwise, all bets are off.

A restriction is prejudice, which we can never overcome.

Keweloh, Hetzenecker "Soft restrictions in SVARs via Lasso"



Restrictions via adaptive Lasso:

$$\underset{B \in \mathbb{R}^{n \times n}}{\arg\min} \; g(B)' W g(B) + \lambda \sum_{i,k \in R} w_{ij} |b_{ij}|$$

$\rightarrow$ Oracle property

T=5000	Recursive SVAR w. second moments (Cholesky)	non-Recursive SVAR w. higher moments	Soft-Restrictions SVAR w. higher moments (Post Lasso)
$\underset{(std)}{mean}$	$\begin{bmatrix} \underset{(1.1)}{1} & \underset{(0)}{0} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(1.3)}{0.5} & \underset{(1.1)}{1} & \underset{(0)}{0} & \underset{(0)}{0} \\ \underset{(1.5)}{0.5} & \underset{(1.3)}{0.5} & \underset{(1.1)}{1} & \underset{(0)}{0} \\ \underset{(1.8)}{0.5} & \underset{(1.5)}{0.5} & \underset{(1.3)}{0.5} & \underset{(1.1)}{1} \end{bmatrix}$	$\begin{bmatrix} \underset{(1.0)}{1} & \underset{(2.7)}{0} & \underset{(2.6)}{0} & \underset{(2.6)}{0} \\ \underset{(2.7)}{0.5} & \underset{(1.6)}{1} & \underset{(3.4)}{0} & \underset{(3.6)}{0} \\ \underset{(3.4)}{0.5} & \underset{(3.4)}{0.5} & \underset{(2.2)}{1} & \underset{(4.4)}{0} \\ \underset{(4.2)}{0.5} & \underset{(4.3)}{0.5} & \underset{(4.4)}{0.5} & \underset{(2.9)}{1} \end{bmatrix}$	$\begin{bmatrix} \underset{(1.0)}{1} & \underset{(0.1)}{0} & \underset{(0.1)}{0} & \underset{(0.1)}{0} \\ \underset{(1.1)}{0.5} & \underset{(1.1)}{1} & \underset{(0.1)}{0} & \underset{(0.0)}{0} \\ \underset{(1.4)}{0.5} & \underset{(1.3)}{0.5} & \underset{(1.0)}{1} & \underset{(0.1)}{0} \\ \underset{(1.6)}{0.5} & \underset{(1.3)}{0.5} & \underset{(1.2)}{0.5} & \underset{(1.1)}{1} \end{bmatrix}$