Higher-order moments in structural VAR models

Sascha Alexander Keweloh

July 20, 2021

TU Dortmund University

Structural VAR model

$$
\begin{array}{rc}
\text{output} & \rightarrow \\
\text{inflation} & \rightarrow \\
\text{FFR} & \rightarrow
\end{array}
\begin{bmatrix} y_t \\ \pi_t \\ i_t \end{bmatrix} = \underbrace{\sum_{i=1}^{p} A_i \begin{bmatrix} y_{t-1} \\ \pi_{t-1} \\ i_{t-1} \end{bmatrix}}_{\substack{\textit{autoregressive} \\ \textit{relation} \\ \text{(easy)}}} + \underbrace{B_0 \begin{bmatrix} \varepsilon_{s,t} \\ \varepsilon_{d,t} \\ \varepsilon_{i,t} \end{bmatrix}}_{\substack{\textit{simultaenous} \\ \textit{relation} \\ \text{(hard)}}}
\begin{array}{lc}
\leftarrow & \text{supply shock} \\
\leftarrow & \text{demand shock} \\
\leftarrow & \text{monetary policy shock}
\end{array}
$$

Structural VAR model

$$\begin{array}{rc} \text{output} & \rightarrow \\ \text{inflation} & \rightarrow \\ \text{FFR} & \rightarrow \end{array} \underbrace{\begin{bmatrix} y_t \\ \pi_t \\ i_t \end{bmatrix}}_{\substack{\textit{observable} \\ \textit{variables} \\ \text{(reduced form} \\ \text{shock)}}} = \underbrace{\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}}_{\substack{\textit{simultaenous} \\ \textit{impact}}} \underbrace{\begin{bmatrix} \varepsilon_{s,t} \\ \varepsilon_{d,t} \\ \varepsilon_{i,t} \end{bmatrix}}_{\substack{\textit{structural} \\ \textit{shocks}}} \begin{array}{lc} \leftarrow & \text{supply shock} \\ \leftarrow & \text{demand shock} \\ \leftarrow & \text{monetary policy shock} \end{array}$$

Assumptions:
A1) Serially independent (identically distributed) shocks $\rightarrow$ ϵ_t independent of $\epsilon_{\tilde{t}}$
A2) Mutually independent shocks $\rightarrow$ $\epsilon_{j,t}$ independent of $\epsilon_{\tilde{j},t}$
A3) Normalization: Zero mean, unit variance shocks

Moment conditions

SVAR:

$$u := B_0 \epsilon$$

Independent structural shocks:

$$E[\epsilon_1 \epsilon_2] = 0$$
$$E[\epsilon_1^3 \epsilon_2] = 0$$
$$\vdots$$

Unmixed innovations

$$e(B) = B^{-1} u$$

Note: $e(B_0) = \epsilon$

Independent unmixed innovations:

$$E[e(B)_1 e(B)_2] \overset{!}{=} 0$$
$$E[e(B)_1^3 e(B)_2] \overset{!}{=} 0$$
$$\vdots$$

$\rightarrow$ moment conditions

SVAR GMM

Literature: Lanne and Luoto (2021), Keweloh (2021), or Guay (2020)

$$\hat{B}_T := \arg\min_B g_T(B)' W g_T(B), \tag{1}$$

with

$$g_T(B) = \frac{1}{T}\sum_{t=1}^{T} f(B, u_t) \qquad \textit{and} \qquad f(B, u_t) = \begin{bmatrix} e(B)_{1,t} e(B)_{2,t} \\ e(B)_{1,t}^3 e(B)_{2,t} \\ \vdots \end{bmatrix}. \tag{2}$$

Consistency and asymptotic normality

$$\hat{B}_T \xrightarrow{p} B_0$$
$$\sqrt{T}(\hat{B}_T - B_0) \xrightarrow{d} \mathcal{N}(0, M_0 S_0 M_0')$$

with

$$M_0 := \left(G_0' S_0^{-1} G_0\right)^{-1} G_0' W$$
$$G_0 := E\left[\frac{\partial f(B_0, u_t)}{\partial vec(B)'}\right]$$
$$S_0 := \underset{T\to\infty}{lim} E\left[T g_T(B_0) g_T(B_0)'\right]$$

Optimal weighting

$$W_0 := S_0^{-1}$$

$$S_0 := \lim_{T\to\infty} E\left[Tg_T(B_0)g_T(B_0)'\right] \tag{3}$$

$$= \lim_{T\to\infty} E\left[T\left(\frac{1}{T}\sum_{t=1}^{T}\begin{bmatrix}\epsilon_{1,t}\epsilon_{2,t}\\ \epsilon_{1,t}^{3}\epsilon_{2,t}\\ \dots\end{bmatrix}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\begin{bmatrix}\epsilon_{1,t}\epsilon_{2,t}\\ \epsilon_{1,t}^{3}\epsilon_{2,t}\\ \dots\end{bmatrix}\right)'\right] \tag{4}$$

$$\overset{\substack{A1)serially\\ independent}}{=} E\left[\begin{bmatrix}\epsilon_1\epsilon_2\\ \epsilon_1^3\epsilon_2\\ \vdots\end{bmatrix}\begin{bmatrix}\epsilon_1\epsilon_2 & \epsilon_1^3\epsilon_2 & \dots\end{bmatrix}\right] = E\left[f(B_0,u_t)f(B_0,u_t)\right] =: S_0^{SI} \tag{5}$$

$$S_0^{SI} = E\left[\begin{bmatrix}\epsilon_1\epsilon_2\\ \epsilon_1^3\epsilon_2\\ \vdots\end{bmatrix}\begin{bmatrix}\epsilon_1\epsilon_2 & \epsilon_1^3\epsilon_2 & \dots\end{bmatrix}\right] \tag{6}$$

$$= \begin{bmatrix}E[\epsilon_1^2\epsilon_2^2] & E[\epsilon_1^4\epsilon_2^2] & \dots\\ E[\epsilon_1^4\epsilon_2^2] & E[\epsilon_1^6\epsilon_2^2] & \dots\\ \dots & & \end{bmatrix} \tag{7}$$

$$\overset{A2)mutually\ independent}{=} \begin{bmatrix}E[\epsilon_1^2]E[\epsilon_2^2] & E[\epsilon_1^4]E[\epsilon_2^2] & \dots\\ E[\epsilon_1^4]E[\epsilon_2^2] & E[\epsilon_1^6]E[\epsilon_2^2] & \dots\\ \dots & & \end{bmatrix} := S_0^{SI-MI} \tag{8}$$

$$S_0 = \lim_{T\to\infty} E\left[Tg_T(B_0)g_T(B_0)'\right] \tag{9}$$

$$S_0^{SI} = \begin{bmatrix} E[\epsilon_1^2\epsilon_2^2] & E[\epsilon_1^4\epsilon_2^2] & \dots \\ E[\epsilon_1^4\epsilon_2^2] & E[\epsilon_1^6\epsilon_2^2] & \dots \\ \dots & & \end{bmatrix} \qquad \to \hat{S}_T^{SI}(\hat{B}_T) \tag{10}$$

$$S_0^{SI-MI} = \begin{bmatrix} E[\epsilon_1^2]E[\epsilon_2^2] & E[\epsilon_1^4]E[\epsilon_2^2] & \dots \\ E[\epsilon_1^4]E[\epsilon_2^2] & E[\epsilon_1^6]E[\epsilon_2^2] & \dots \\ \dots & & \end{bmatrix} \qquad \to \hat{S}_T^{SI-MI}(\hat{B}_T) \tag{11}$$

If the shocks are serially uncorrelated and $\hat{B}_T \xrightarrow{p} B_0$:
$\hat{S}_T^{SI}(\hat{B}_T) \xrightarrow{p} S_0$.

If the shocks are serially uncorrelated, mutually independent, and $\hat{B}_T \xrightarrow{p} B_0$:
$\hat{S}_T^{SI-MI}(\hat{B}_T) \xrightarrow{p} S_0$.

Initial weighting

1) $W = I$

2) $W = S_{SI}^{-1} = \begin{bmatrix} E[\epsilon_1^2\epsilon_2^2] & E[\epsilon_1^4\epsilon_2^2] & ... \\ E[\epsilon_1^4\epsilon_2^2] & E[\epsilon_1^6\epsilon_2^2] & ... \\ ... & & \end{bmatrix}^{-1} = E \begin{bmatrix} e(\hat{B})_1^2 e(\hat{B})_2^2 & e(\hat{B})_1^4 e(\hat{B})_2^2 & ... \\ e(\hat{B})_1^4 e(\hat{B})_2^2 & e(\hat{B})_1^6 e(\hat{B})_2^2 & ... \\ ... & & \end{bmatrix}^{-1} = \; ?$

3) $W = S_{SI-MI}^{-1} = \begin{bmatrix} E[\epsilon_1^2]E[\epsilon_2^2] & E[\epsilon_1^4]E[\epsilon_2^2] & ... \\ E[\epsilon_1^4]E[\epsilon_2^2] & E[\epsilon_1^6]E[\epsilon_2^2] & ... \\ ... & & \end{bmatrix}^{-1} \overset{\epsilon \sim \mathcal{N}(0,1)}{\approx} \begin{bmatrix} 1 & 3 & ... \\ 3 & 15 & ... \\ ... & & \end{bmatrix}^{-1}$

Monte Carlo Simulation

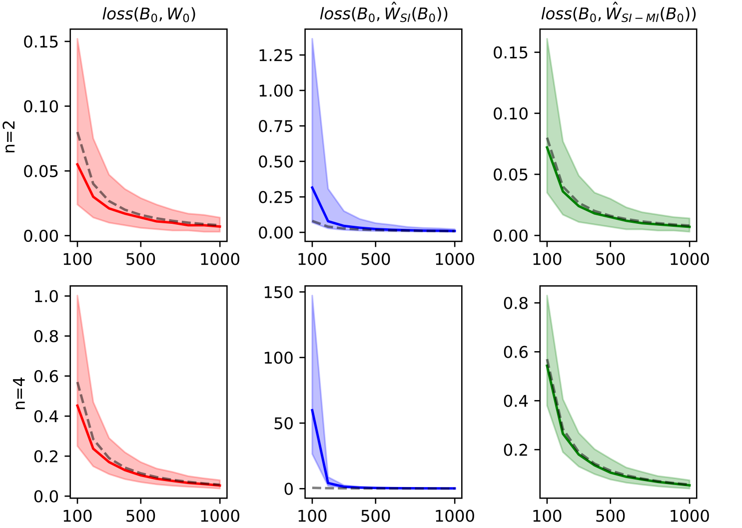
SVAR $u = B_0\epsilon$ with $n = 2$ and $n = 4$ variables where

$$B_0 = \begin{bmatrix} 1 & 0 \\ 0.5 & 1 \end{bmatrix} \text{ and } B_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ 0.5 & 0.5 & 1 & 0 \\ 0.5 & 0.5 & 0.5 & 1 \end{bmatrix}.$$

Table 1: Number of moment conditions

	$n = 2$	$n = 4$
(Co-)variance	3	10
Coskewnes	2	16
Cokurtosis	3	31

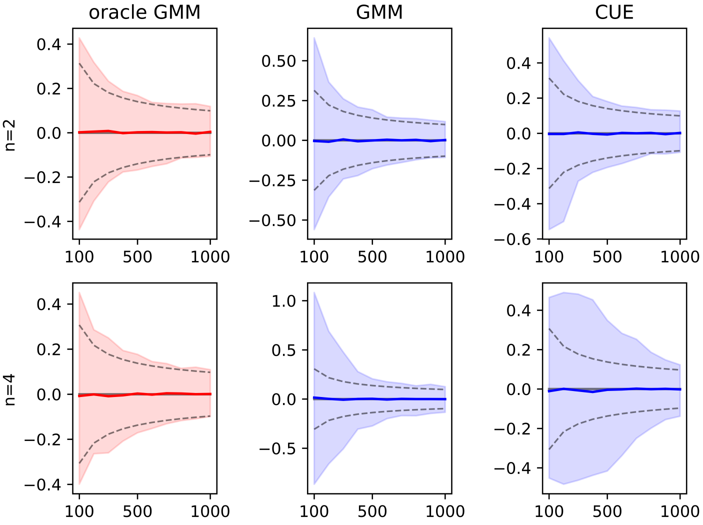
Figure 1: Median and (10%, 90%) quantiles of GMM loss at B_0 for different weightings



In grey: Expected loss at B_0 and W_0 in finite sample.

Red/Blue/Green: Median and quantiles of GMM loss at B_0 for different weightings.

Figure 2: Median and (5%, 95%) quantiles of estimated element $\hat{B}_{1,n}$



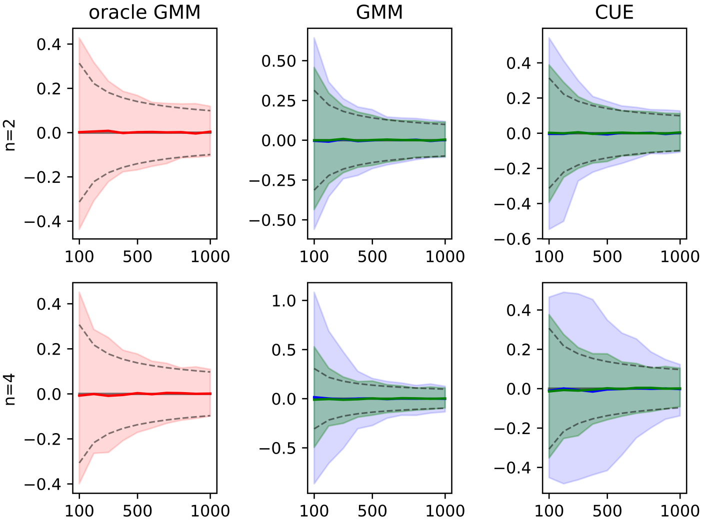
Grey: $B_{1,n}$ and 90% confidence intervals $(B_{1,n} \pm z^* \frac{\sigma}{\sqrt{T}})$

Median & quantiles $\hat{B}_{1,1}$

Red: W_0

Blue: $\hat{W}_{SI}$

Figure 3: Median and (5%, 95%) quantiles of estimated element $\hat{B}_{1,n}$



Grey: $B_{1,n}$ and 90% confidence intervals $(B_{1,n} \pm z^* \frac{\sigma}{\sqrt{T}})$

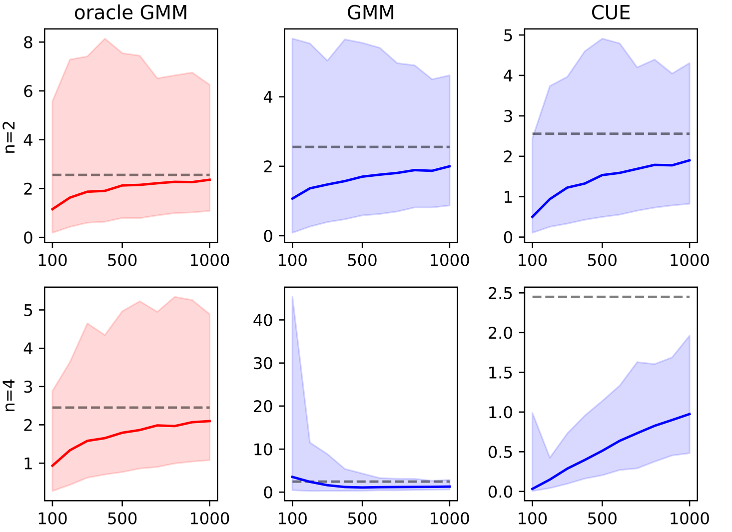
Median & quantiles $\hat{B}_{1,1}$

Red: W_0

Blue: $\hat{W}_{SI}$

Green: $\hat{W}_{SI-MI}$

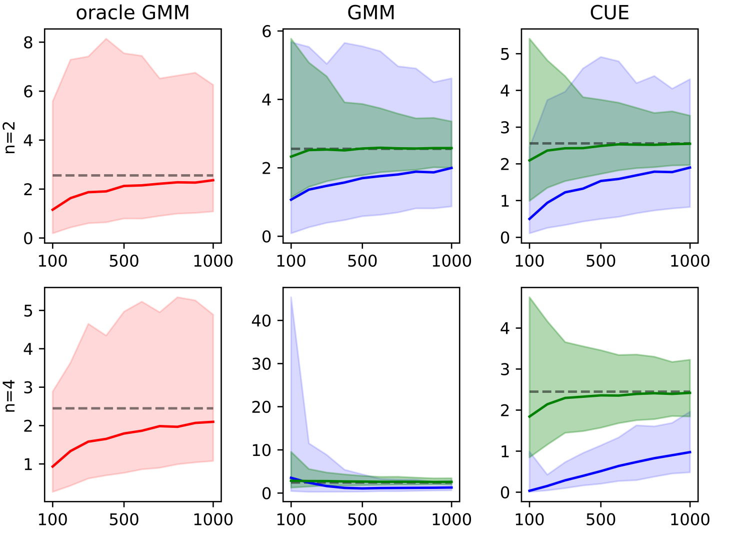
Figure 4: Median and (5%, 95%) quantiles of estimated variance of the element $\hat{B}_{1,n}$



Grey: Avar of $\hat{B}_{1,n}$ using $(M_0 S_0 M_0')$.

Median & quantiles of estimated Avar
Red: W_0 and $\hat{\sigma}_{SI}$
Blue: $\hat{W}_{SI}$ and $\hat{\sigma}_{SI}$.

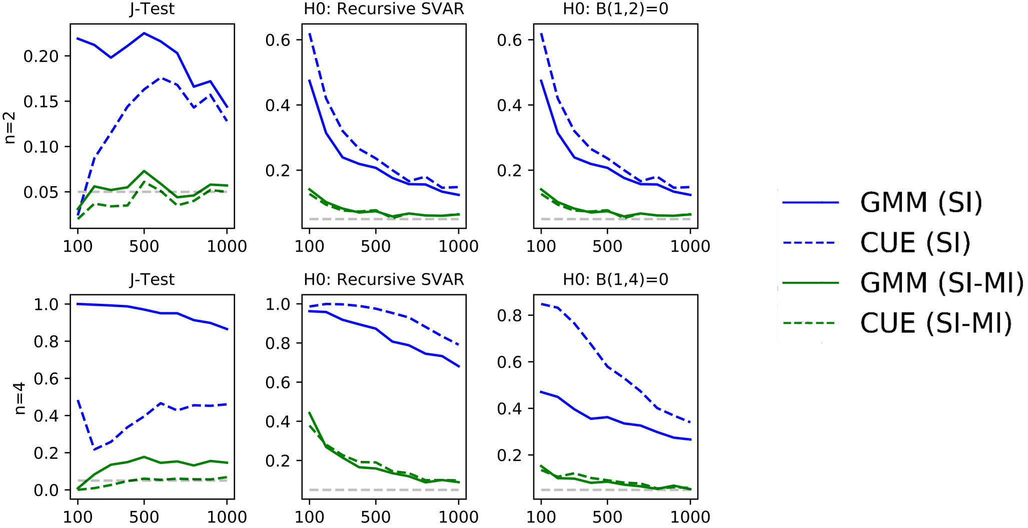
Figure 5: Median and (5%, 95%) quantiles of estimated variance of the element $\hat{B}_{1,n}$



Grey: Avar of $\hat{B}_{1,n}$ using $(M_0 S_0 M_0')$.

Median & quantiles of estimated Avar
Red: W_0 and $\hat{\sigma}_{SU}(\hat{B})$
Blue: $\hat{W}_{SI}$ and $\hat{\sigma}_{SI}$
Green: $\hat{W}_{SI-MI}$ and $\hat{\sigma}_{SI-MI}$

Figure 6: 10% rejection rate for J-Test and Wald tests



Conclusion

Conclusion:
Exploiting independence to estimate S_0 and G_0 improves finite sample performance.

Next steps:

- Relax assumptions: serial uncorrelated and independent shocks
- Room for improvements regarding inference: Newey and Windmeijer (2009)
- Whitening and optimal weighting

Whitening and optimal weighting

Consider a bivariate SVAR

$$\begin{bmatrix} u_{1,t} \\ u_{2,t} \end{bmatrix} = \begin{bmatrix} 1 & b_{12} \\ b_{21} & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{bmatrix}$$

and the GMM estimator

$$\hat{B} := \arg\min \left(\frac{1}{T} \sum_{t=1}^{T} \begin{bmatrix} e(B)_{1,t} e(B)_{2,t} \\ e(B)_{1,t}^3 e(B)_{2,t} \\ e(B)_{1,t} e(B)_{2,t}^3 \end{bmatrix} \right)' W \left(\frac{1}{T} \sum_{t=1}^{T} \begin{bmatrix} e(B)_{1,t} e(B)_{2,t} \\ e(B)_{1,t}^3 e(B)_{2,t} \\ e(B)_{1,t} e(B)_{2,t}^3 \end{bmatrix} \right).$$

Whitening and optimal weighting

What is the optimal weighting matrix of the following whitened estimator?

$$\hat{B}^{white} := \arg\min \left(\frac{1}{T} \sum_{t=1}^{T} \begin{bmatrix} e(B)_{1,t}^3 e(B)_{2,t} \\ e(B)_{1,t} e(B)_{2,t}^3 \end{bmatrix} \right)' W \left(\frac{1}{T} \sum_{t=1}^{T} \begin{bmatrix} e(B)_{1,t}^3 e(B)_{2,t} \\ e(B)_{1,t} e(B)_{2,t}^3 \end{bmatrix} \right)$$

$$s.t. \frac{1}{T} \sum_{t=1}^{T} e(B)_{1,t} e(B)_{2,t} = 0$$

Can the whitened estimator be as efficient as the original estimator?

References

Guay, A. (2020). Identification of structural vector autoregressions through higher unconditional moments. *Journal of Econometrics*.

Keweloh, S. A. (2021). A generalized method of moments estimator for structural vector autoregressions based on higher moments. *Journal of Business & Economic Statistics 39*(3), 772–782.

Lanne, M. and J. Luoto (2021). Gmm estimation of non-gaussian structural vector autoregression. *Journal of Business & Economic Statistics 39*(1), 69–81.

Newey, W. K. and F. Windmeijer (2009). Generalized method of moments with many weak moment conditions. *Econometrica 77*(3), 687–719.