

Higher-order moments in structural VAR models

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Structural VAR model

$$\begin{array}{ll} \text{output} & \rightarrow \begin{bmatrix} y_t \\ \pi_t \\ i_t \end{bmatrix} = \underbrace{\sum_{i=1}^p A_i \begin{bmatrix} y_{t-1} \\ \pi_{t-1} \\ i_{t-1} \end{bmatrix}}_{\text{autoregressive relation (easy)}} + \underbrace{B_0 \begin{bmatrix} \varepsilon_{s,t} \\ \varepsilon_{d,t} \\ \varepsilon_{i,t} \end{bmatrix}}_{\text{simultaenous relation (hard)}} \\ \text{inflation} & \leftarrow \begin{array}{l} \text{supply shock} \\ \text{demand shock} \end{array} \\ \text{FFR} & \leftarrow \text{monetary policy shock} \end{array}$$

Structural VAR model

$$\begin{array}{lll} \text{output} & \rightarrow & \underbrace{\begin{bmatrix} y_t \\ \pi_t \\ i_t \end{bmatrix}}_{\substack{\text{observable} \\ \text{variables}}} \\ \text{inflation} & \rightarrow & = \underbrace{\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}}_{\substack{\text{simultaenous} \\ \text{impact}}} \underbrace{\begin{bmatrix} \varepsilon_{s,t} \\ \varepsilon_{d,t} \\ \varepsilon_{i,t} \end{bmatrix}}_{\substack{\text{structural} \\ \text{shocks}}} \\ \text{FFR} & \rightarrow & \leftarrow \begin{array}{l} \text{supply shock} \\ \text{demand shock} \\ \text{monetary policy shock} \end{array} \end{array}$$

(reduced form
shock)

Assumptions:

- A1) Serially independent (identically distributed) shocks $\rightarrow \epsilon_t$ independent of $\epsilon_{\tilde{t}}$
- A2) Mutually independent shocks $\rightarrow \epsilon_{j,t}$ independent of $\epsilon_{\tilde{j},t}$
- A3) Normalization: Zero mean, unit variance shocks

Moment conditions

SVAR:

$$\textcolor{blue}{u} := \textcolor{red}{B}_0 \epsilon$$

Unmixed innovations

$$e(B) = B^{-1} \textcolor{blue}{u}$$

$$\text{Note: } e(B_0) = \epsilon$$

Independent structural shocks:

$$E[\epsilon_1 \epsilon_2] = 0$$

$$E[\epsilon_1^3 \epsilon_2] = 0$$

⋮

Independent unmixed innovations:

$$E[e(B)_1 e(B)_2] \stackrel{!}{=} 0$$

$$E[e(B)_1^3 e(B)_2] \stackrel{!}{=} 0$$

⋮

→ moment conditions

Literature: Lanne and Luoto (2021), Keweloh (2021), or Guay (2020)

$$\hat{B}_T := \arg \min_B g_T(B)' W g_T(B), \quad (1)$$

with

$$g_T(B) = \frac{1}{T} \sum_{t=1}^T f(B, u_t) \quad \text{and} \quad f(B, u_t) = \begin{bmatrix} e(B)_{1,t} & e(B)_{2,t} \\ e(B)_{1,t}^3 & e(B)_{2,t} \\ \vdots & \end{bmatrix}. \quad (2)$$

Consistency and asymptotic normality

$$\begin{aligned} \hat{B}_T &\xrightarrow{P} B_0 \\ \sqrt{T}(\hat{B}_T - B_0) &\xrightarrow{d} \mathcal{N}(0, M_0 S_0 M_0') \end{aligned} \quad \text{with} \quad \begin{aligned} M_0 &:= (G_0' S_0^{-1} G_0)^{-1} G_0' W \\ G_0 &:= E \left[\frac{\partial f(B_0, u_t)}{\partial \text{vec}(B)'} \right] \\ S_0 &:= \lim_{T \rightarrow \infty} E [T g_T(B_0) g_T(B_0)'] \end{aligned}$$

Optimal weighting

$$W_0 := S_0^{-1}$$

$$S_0 := \lim_{T \rightarrow \infty} E [T g_T(B_0) g_T(B_0)'] \quad (3)$$

$$= \lim_{T \rightarrow \infty} E \left[T \left(\frac{1}{T} \sum_{t=1}^T \begin{bmatrix} \epsilon_{1,t} \epsilon_{2,t} \\ \epsilon_{1,t}^3 \epsilon_{2,t} \\ \dots \end{bmatrix} \right) \left(\frac{1}{T} \sum_{t=1}^T \begin{bmatrix} \epsilon_{1,t} \epsilon_{2,t} \\ \epsilon_{1,t}^3 \epsilon_{2,t} \\ \dots \end{bmatrix} \right)' \right] \quad (4)$$

A1) serially independent

$$E \left[\begin{bmatrix} \epsilon_1 \epsilon_2 \\ \epsilon_1^3 \epsilon_2 \\ \vdots \end{bmatrix} \begin{bmatrix} \epsilon_1 \epsilon_2 & \epsilon_1^3 \epsilon_2 & \dots \end{bmatrix} \right] = E [f(B_0, u_t) f(B_0, u_t)] =: S_0^{SI} \quad (5)$$

$$S_0^{SI} = E \left[\begin{bmatrix} \epsilon_1 \epsilon_2 \\ \epsilon_1^3 \epsilon_2 \\ \vdots \end{bmatrix} \begin{bmatrix} \epsilon_1 \epsilon_2 & \epsilon_1^3 \epsilon_2 & \dots \end{bmatrix} \right] \quad (6)$$

$$= \begin{bmatrix} E[\epsilon_1^2 \epsilon_2^2] & E[\epsilon_1^4 \epsilon_2^2] & \dots \\ E[\epsilon_1^4 \epsilon_2^2] & E[\epsilon_1^6 \epsilon_2^2] & \dots \\ \dots & \dots & \dots \end{bmatrix} \quad (7)$$

A2) mutually independent

$$= \begin{bmatrix} E[\epsilon_1^2]E[\epsilon_2^2] & E[\epsilon_1^4]E[\epsilon_2^2] & \dots \\ E[\epsilon_1^4]E[\epsilon_2^2] & E[\epsilon_1^6]E[\epsilon_2^2] & \dots \\ \dots & \dots & \dots \end{bmatrix} := S_0^{SI-MI} \quad (8)$$

$$S_0 = \lim_{T \rightarrow \infty} E [T g_T(B_0) g_T(B_0)'] \quad (9)$$

$$S_0^{SI} = \begin{bmatrix} E[\epsilon_1^2 \epsilon_2^2] & E[\epsilon_1^4 \epsilon_2^2] & \dots \\ E[\epsilon_1^4 \epsilon_2^2] & E[\epsilon_1^6 \epsilon_2^2] & \dots \\ \dots & \dots & \dots \end{bmatrix} \rightarrow \hat{S}_T^{SI}(\hat{B}_T) \quad (10)$$

$$S_0^{SI-MI} = \begin{bmatrix} E[\epsilon_1^2]E[\epsilon_2^2] & E[\epsilon_1^4]E[\epsilon_2^2] & \dots \\ E[\epsilon_1^4]E[\epsilon_2^2] & E[\epsilon_1^6]E[\epsilon_2^2] & \dots \\ \dots & \dots & \dots \end{bmatrix} \rightarrow \hat{S}_T^{SI-MI}(\hat{B}_T) \quad (11)$$

If the shocks are serially uncorrelated and $\hat{B}_T \xrightarrow{P} B_0$:

$$\hat{S}_T^{SI}(\hat{B}_T) \xrightarrow{P} S_0.$$

If the shocks are serially uncorrelated, mutually independent, and $\hat{B}_T \xrightarrow{P} B_0$:

$$\hat{S}_T^{SI-MI}(\hat{B}_T) \xrightarrow{P} S_0.$$

Initial weighting

$$1) W = I$$

$$2) W = S_{SI}^{-1} = \begin{bmatrix} E[\epsilon_1^2 \epsilon_2^2] & E[\epsilon_1^4 \epsilon_2^2] & \dots \\ E[\epsilon_1^4 \epsilon_2^2] & E[\epsilon_1^6 \epsilon_2^2] & \dots \\ \dots & \dots & \dots \end{bmatrix}^{-1} = E \begin{bmatrix} e(\hat{B})_1^2 e(\hat{B})_2^2 & e(\hat{B})_1^4 e(\hat{B})_2^2 & \dots \\ e(\hat{B})_1^4 e(\hat{B})_2^2 & e(\hat{B})_1^6 e(\hat{B})_2^2 & \dots \\ \dots & \dots & \dots \end{bmatrix}^{-1} = ?$$

$$3) W = S_{SI-MI}^{-1} = \begin{bmatrix} E[\epsilon_1^2]E[\epsilon_2^2] & E[\epsilon_1^4]E[\epsilon_2^2] & \dots \\ E[\epsilon_1^4]E[\epsilon_2^2] & E[\epsilon_1^6]E[\epsilon_2^2] & \dots \\ \dots & \dots & \dots \end{bmatrix}^{-1} \stackrel{\epsilon \sim \mathcal{N}(0,1)}{\approx} \begin{bmatrix} 1 & 3 & \dots \\ 3 & 15 & \dots \\ \dots & \dots & \dots \end{bmatrix}^{-1}$$

Monte Carlo Simulation

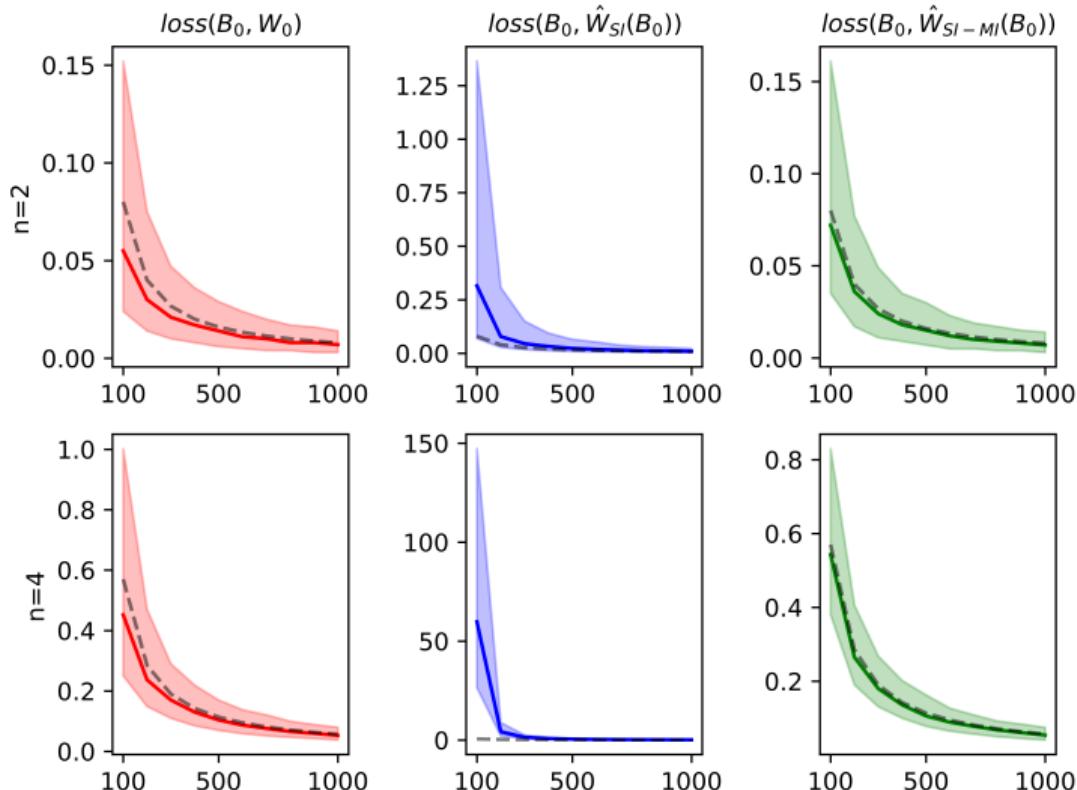
SVAR $u = B_0\epsilon$ with $n = 2$ and $n = 4$ variables where

$$B_0 = \begin{bmatrix} 1 & 0 \\ 0.5 & 1 \end{bmatrix} \text{ and } B_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ 0.5 & 0.5 & 1 & 0 \\ 0.5 & 0.5 & 0.5 & 1 \end{bmatrix}.$$

Table 1: Number of moment conditions

	$n = 2$	$n = 4$
(Co-)variance	3	10
Coskewness	2	16
Cokurtosis	3	31

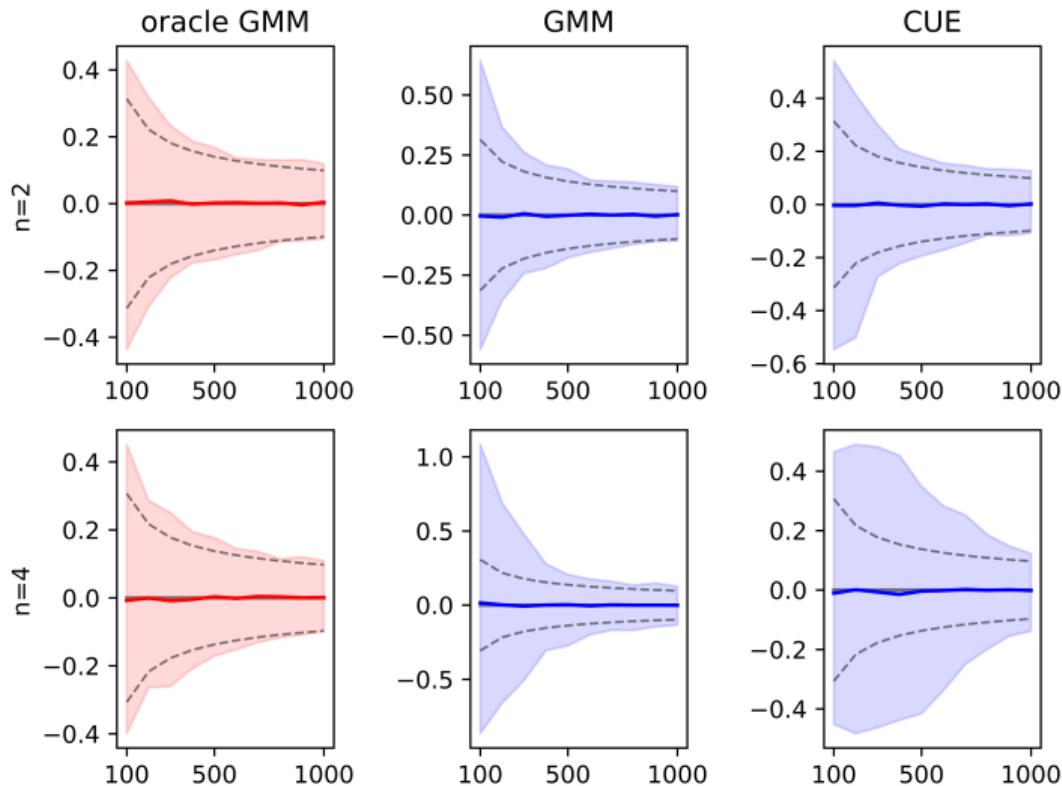
Figure 1: Median and (10%, 90%) quantiles of GMM loss at B_0 for different weightings



In grey: Expected loss
at B_0 and W_0 in finite
sample.

Red/Blue/Green:
Median and quantiles
of GMM loss at B_0 for
different weightings.

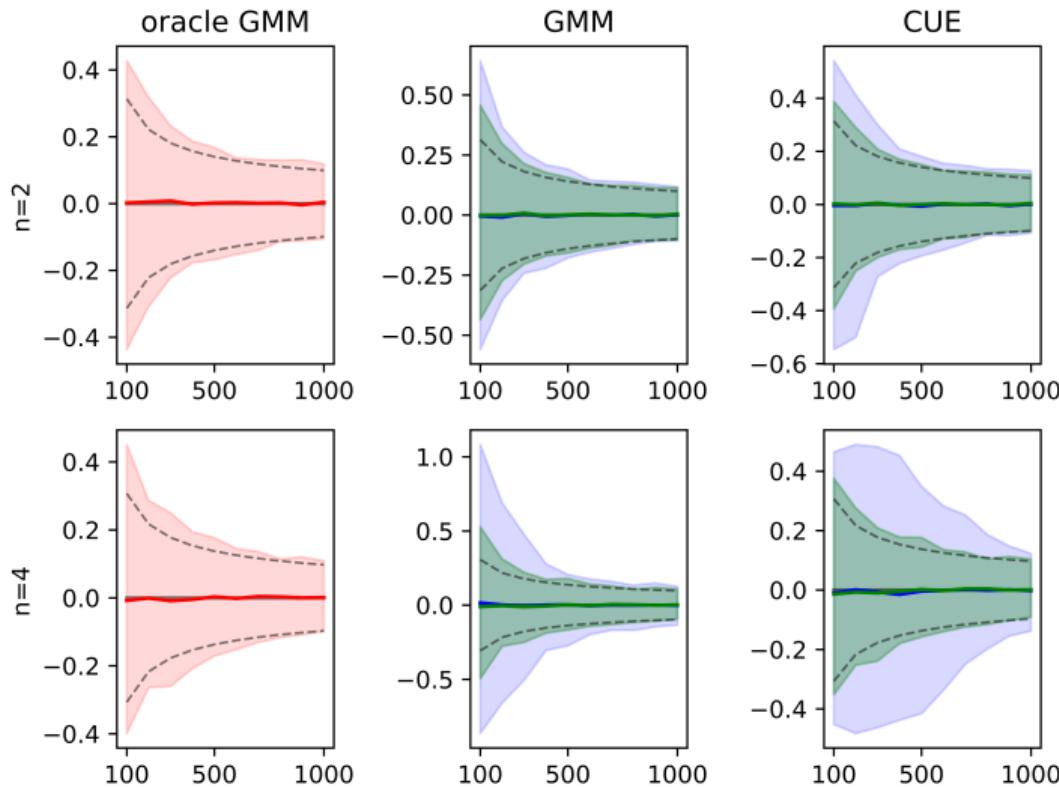
Figure 2: Median and (5%, 95%) quantiles of estimated element $\hat{B}_{1,n}$



Grey: $B_{1,n}$ and 90% confidence intervals $(B_{1,n} \pm z^* \frac{\sigma}{\sqrt{T}})$

Median & quantiles $\hat{B}_{1,1}$
Red: W_0
Blue: \hat{W}_{SI}

Figure 3: Median and (5%, 95%) quantiles of estimated element $\hat{B}_{1,n}$



Grey: $B_{1,n}$ and 90% confidence intervals $(B_{1,n} \pm z^* \frac{\sigma}{\sqrt{T}})$

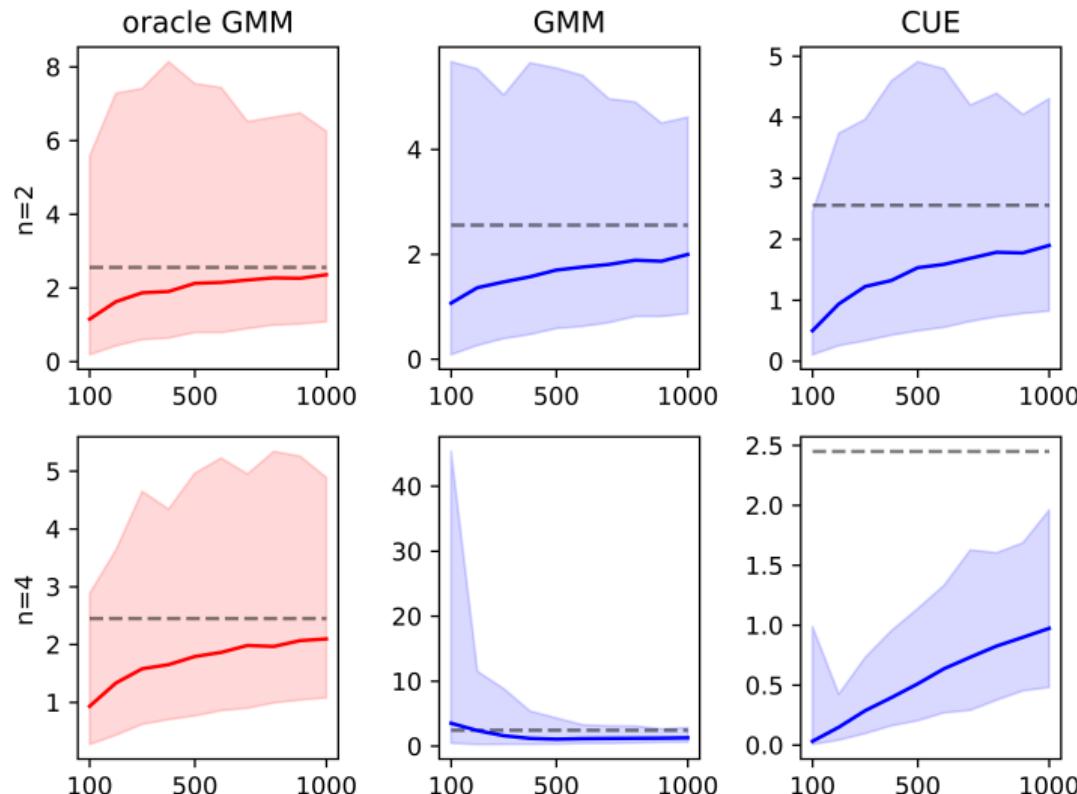
Median & quantiles $\hat{B}_{1,1}$

Red: W_0

Blue: \hat{W}_{SI}

Green: \hat{W}_{SI-MI}

Figure 4: Median and (5%, 95%) quantiles of estimated variance of the element $\hat{B}_{1,n}$



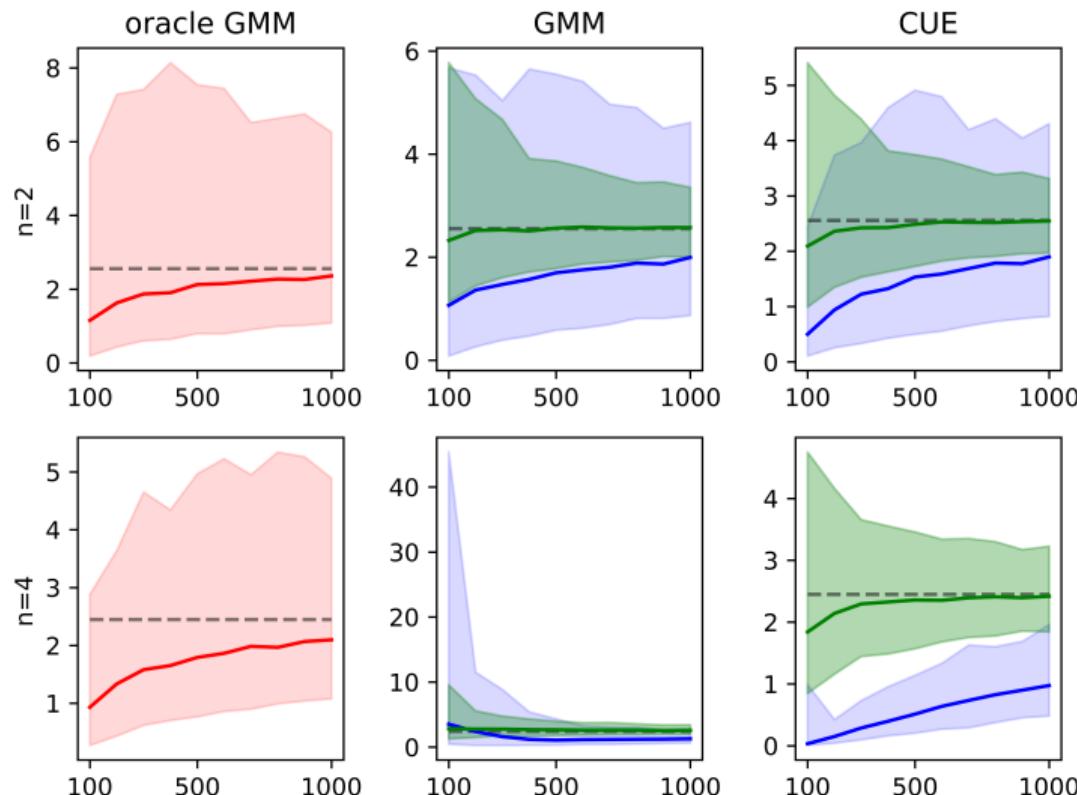
Grey: Avar of $\hat{B}_{1,n}$ using $(M_0 S_0 M_0')$.

Median & quantiles
of estimated Avar

Red: W_0 and $\hat{\sigma}_{SI}$

Blue: \hat{W}_{SI} and $\hat{\sigma}_{SI}$.

Figure 5: Median and (5%, 95%) quantiles of estimated variance of the element $\hat{B}_{1,n}$



Grey: Avar of $\hat{B}_{1,n}$ using $(M_0 S_0 M_0')$.

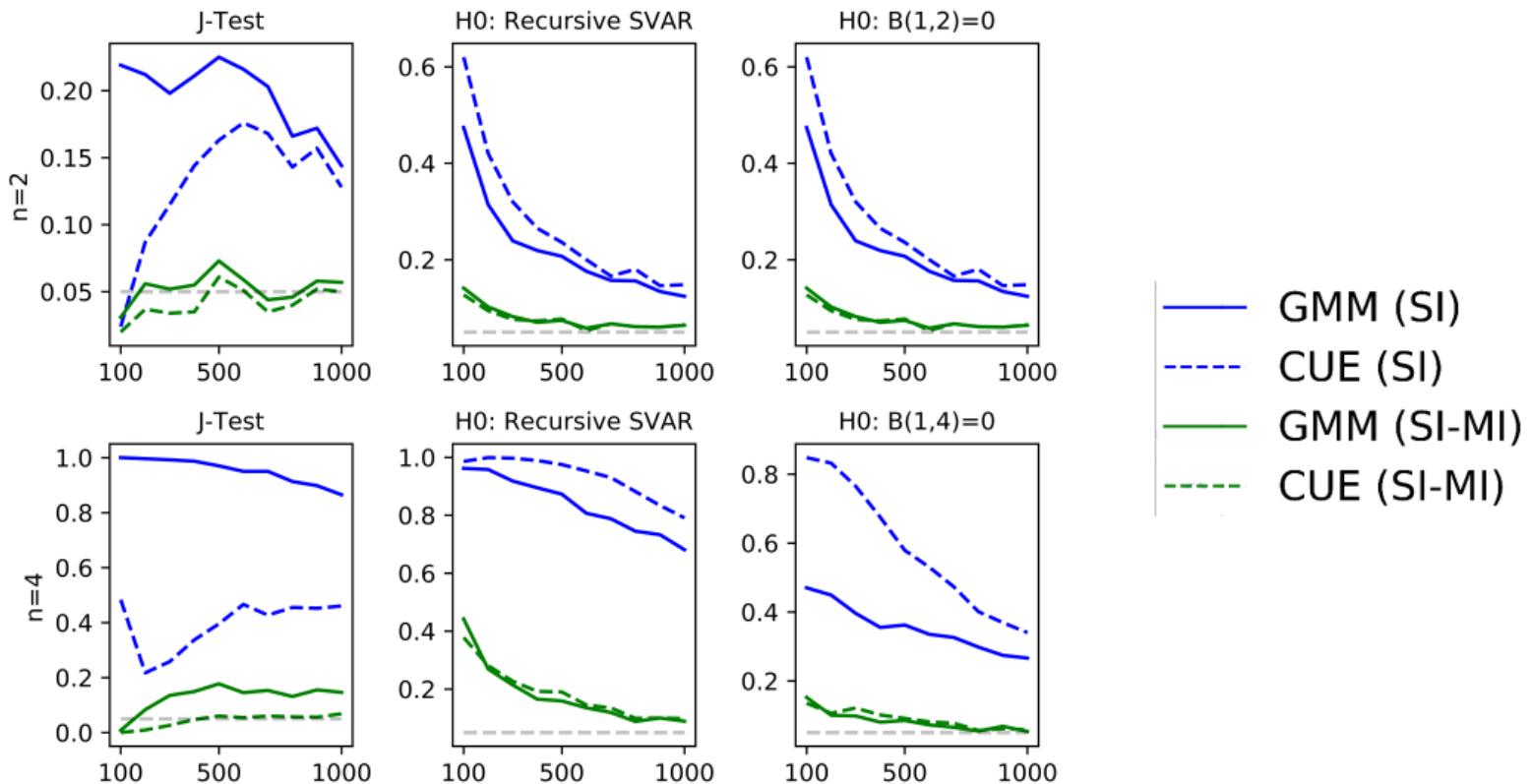
Median & quantiles
of estimated Avar

Red: W_0 and $\hat{\sigma}_{SU}(\hat{B})$

Blue: \hat{W}_{SI} and $\hat{\sigma}_{SI}$

Green: \hat{W}_{SI-MI} and
 $\hat{\sigma}_{SI-MI}$

Figure 6: 10% rejection rate for J-Test and Wald tests



Conclusion

Conclusion:

Exploiting independence to estimate S_0 and G_0 improves finite sample performance.

Next steps:

- Relax assumptions: serial uncorrelated and independent shocks
- Room for improvements regarding inference: Newey and Windmeijer (2009)
- Whitening and optimal weighting

Whitening and optimal weighting

Consider a bivariate SVAR

$$\begin{bmatrix} u_{1,t} \\ u_{2,t} \end{bmatrix} = \begin{bmatrix} 1 & \textcolor{red}{b_{12}} \\ \textcolor{red}{b_{21}} & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{bmatrix}$$

and the GMM estimator

$$\hat{B} := \arg \min \left(\frac{1}{T} \sum_{t=1}^T \begin{bmatrix} e(B)_{1,t} e(B)_{2,t} \\ e(B)_{1,t}^3 e(B)_{2,t} \\ e(B)_{1,t} e(B)_{2,t}^3 \end{bmatrix} \right)' W \left(\frac{1}{T} \sum_{t=1}^T \begin{bmatrix} e(B)_{1,t} e(B)_{2,t} \\ e(B)_{1,t}^3 e(B)_{2,t} \\ e(B)_{1,t} e(B)_{2,t}^3 \end{bmatrix} \right).$$

Whitening and optimal weighting

What is the optimal weighting matrix of the following whitened estimator?

$$\hat{B}^{white} := \arg \min \left(\frac{1}{T} \sum_{t=1}^T \begin{bmatrix} e(B)_{1,t}^3 e(B)_{2,t} \\ e(B)_{1,t} e(B)_{2,t}^3 \end{bmatrix} \right)' W \left(\frac{1}{T} \sum_{t=1}^T \begin{bmatrix} e(B)_{1,t}^3 e(B)_{2,t} \\ e(B)_{1,t} e(B)_{2,t}^3 \end{bmatrix} \right)$$

$$s.t. \frac{1}{T} \sum_{t=1}^T e(B)_{1,t} e(B)_{2,t} = 0$$

Can the whitened estimator be as efficient as the original estimator?

References

- Guay, A. (2020). Identification of structural vector autoregressions through higher unconditional moments. *Journal of Econometrics*.
- Keweloh, S. A. (2021). A generalized method of moments estimator for structural vector autoregressions based on higher moments. *Journal of Business & Economic Statistics* 39(3), 772–782.
- Lanne, M. and J. Luoto (2021). Gmm estimation of non-gaussian structural vector autoregression. *Journal of Business & Economic Statistics* 39(1), 69–81.
- Newey, W. K. and F. Windmeijer (2009). Generalized method of moments with many weak moment conditions. *Econometrica* 77(3), 687–719.