• Cramer Rule

A system of n linear equations in n variables

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n = b_n \end{cases} \iff A\vec{x} = \vec{b}$$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

Thm. Assume A is invertible ($\det(A) \neq 0$)
then the only solution of system
is given by

$$x_i = \frac{\det(A_i)}{\det(A)} \quad i = 1, \ldots, n$$

where A_i is obtained substituting the vector $\vec{b}$ for the ith column of A.

Ex.

$$\begin{cases} A + B + C = 0 \\ 5A + 4B + 3C = -3 \\ 6A + 3B + 2C = 4 \end{cases} \quad H = \begin{bmatrix} 1 & 1 & 1 \\ 5 & 4 & 3 \\ 6 & 3 & 2 \end{bmatrix} \quad \det(H) = (8+18+15) - (24+9+10) = 41 - 43 = -2$$

$$H_1 = \begin{bmatrix} 0 & 1 & 1 \\ -3 & 4 & 3 \\ 4 & 3 & 2 \end{bmatrix} \quad \det(H_1) = (0+12-9) - (16-6) = -7$$

$$H_2 = \begin{bmatrix} 1 & 0 & 1 \\ 5 & -3 & 3 \\ 6 & 4 & 2 \end{bmatrix} \quad \det(H_2) = (-6+20) - (-18+12) = 20$$

$$H_3 = \begin{bmatrix} 1 & 1 & 0 \\ 5 & 4 & -3 \\ 6 & 3 & 4 \end{bmatrix} \quad \det(H_3) = (16-18) - (-9+20) = -13$$

$$\Rightarrow \quad x_1 = \frac{\det(H_1)}{\det(H)} = \frac{-7}{-2} = \frac{7}{2}, \quad x_2 = \frac{\det(H_2)}{\det(H)} = \frac{20}{-2} = -10$$

$$x_3 = \frac{\det(H_3)}{\det(H)} = \frac{-13}{-2} = \frac{13}{2}$$

Chryv

Define augmented matrix of A by including $\vec{b}$ into A as last column

$$\hat{A} = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & b_n \end{array}\right]$$

- Gauss Elimination

To solve the system we need following steps

1. Forward Elimination : to find

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{22}^* x_2 + a_{23}^* x_3 + \cdots + a_{2n}^* x_n &= b_2^* \\ a_{33}^* x_3 + \cdots + a_{3n}^* x_n &= b_3^* \\ &\vdots \end{aligned}$$

2. Backward Substitution

Ex
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 5 & 4 & 3 & -3 \\ 6 & 3 & 2 & 4 \end{array}\right] \xRightarrow{(-5),\,(-6)} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & -3 \\ 0 & -3 & -4 & 4 \end{array}\right] \xRightarrow{(-3)} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 2 & 13 \end{array}\right]$$

then
$$\begin{aligned} A + B + C &= 0 \\ -B - 2C &= -3 \\ 2C &= 13 \end{aligned}$$

$\Rightarrow$ $C = \frac{13}{2}$

$-B = -3 + 13 = 10$

$A = 10 - \frac{13}{2} = \frac{7}{2}$

Ex $\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}\,dx = \int 2x + \frac{5x-3}{x^2-2x-3}\,dx$

$$\begin{array}{r} 2x \\ x^2-2x-3 \overline{\smash{)}\,2x^3-4x^2-x-3} \\ \underline{2x^3-4x^2-6x} \\ 5x-3 \end{array}$$

$= \int 2x\,dx + 3\int \frac{1}{x-3}\,dx + 2\int \frac{1}{x+1}\,dx$

$= x^2 + 3\ln|x-3| + 2\ln|x+1| + C$

$$\frac{5x-3}{x^2-2x-3} = \frac{5x-3}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$$

$$A(x+1) + B(x-3) = 5x-3$$

$$\begin{array}{l} A+B=5 \\ A-3B=-3 \end{array} \Rightarrow \begin{array}{l} A=3 \\ B=2 \end{array}$$

D. Integration by Part (分部積分)

★ Motivation

$$\frac{d}{dx}\{f(x)g(x)\} = f'(x)g(x) + g'(x)f(x)$$

$$\int \frac{d}{dx}\{f(x)g(x)\}\,dx = \int [f'(x)g(x) + g'(x)f(x)]\,dx$$

$$\underbrace{f(x)}_{u}\,\underbrace{g(x)}_{v} = \int f'(x)g(x)\,dx + \int g'(x)f(x)\,dx$$

$$\Rightarrow \int u\,dv = uv - \int v\,du$$

or $\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$

In definite integral

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)\,dx$$

Remark. 可以用來降階或是利用
等價積分求解

Ex. $\int \frac{3x^4-1}{x^4+2x^2+1}dx = \int \frac{3x^4-1}{(x^2+1)^2}dx$ ...... (1)

$$\begin{array}{r} 3x^2-3 \\ x^2+1\overline{)\,3x^4-1} \\ \underline{3x^4+3x^2} \\ -3x^2-1 \\ \underline{-3x^2-3} \\ 2 \end{array} \qquad \begin{array}{r} 3 \\ x^2+1\overline{)\,3x^2-3} \\ \underline{3x^2+3} \\ -6 \end{array}$$

$$3x^4-1 = (x^2+1)\left[3(x^2+1)-6\right]+2$$
$$= 3(x^2+1)^2-6(x^2+1)+2$$

$$\Rightarrow (1) = \int 3 - \frac{6}{x^2+1} + \frac{2}{(x^2+1)^2}dx$$

$$\int \frac{1}{(x^2+1)^2}dx = \int \frac{1}{x^2+1}dx - \int \frac{x^2}{(x^2+1)^2}dx$$

$$\int \frac{x^2}{(x^2+1)^2}dx = \frac{1}{2}\int (x^2+1)^{-2}\cdot x \, d(x^2+1)$$
$$= \frac{-1}{2}\int x \, d(x^2+1)^{-1}$$
$$= -\frac{1}{2}\left(\frac{x}{x^2+1} - \int \frac{1}{x^2+1}dx\right) = \frac{\tan^{-1}(x)}{2} - \frac{x}{2(x^2+1)}$$

$$\Rightarrow (1) = 3x - 6\tan^{-1}(x) + 2\left[\frac{\tan^{-1}(x)}{2} - \frac{x}{2(x^2+1)}\right] + C$$
$$= 3x - 5\tan^{-1}(x) - \frac{x}{x^2+1} + C \quad \#$$

Ex $\int \ln x \, dx = x\ln x - \int x \frac{1}{x}dx$
$$= x\ln x - x + C \quad \#$$

★ $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$, $\alpha > 0$ (Gamma fn)

show $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$

<pf>

$$\int_0^\infty x^{\alpha-1}e^{-x}dx = -\int_0^\infty x^{\alpha-1}de^{-x}$$
$$= -\left[x^{\alpha-1}e^{-x}\Big|_0^\infty - \int_0^\infty (\alpha-1)x^{(\alpha-1)-1}e^{-x}dx\right]$$
$$= (\alpha-1)\Gamma(\alpha-1)$$
$$\#$$

Let $n \in \mathbb{Z}^+$, $\Gamma(n) = (n-1)!$

$$\int \frac{x^2}{(x^2+1)^2}\,dx$$

Let $y = (x^2+1)^{-1}$

$$dy = -\frac{2x}{(x^2+1)}\,dx \Rightarrow \frac{-(x^2+1)^2}{2x}\,dy = dx$$

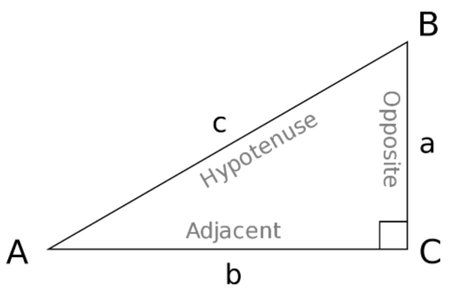
$$\int \frac{x^2}{(x^2+1)^2}\,dx = \int \left[\frac{x^2}{(x^2+1)^2}\right]\left[\frac{-(x^2+1)^2}{2x}\,dy\right] = -\frac{1}{2}\int x\,dy$$

Using integration by part

$$\int x\,dy = xy - \int y\,dx = \frac{x}{x^2+1} - \int \frac{1}{1+x^2}\,dx = \frac{x}{x^2+1} - \tan^{-1}(x) + C$$

Hence

$$\int \frac{x^2}{(x^2+1)^2}\,dx = -\frac{1}{2}\left(\frac{x}{x^2+1} - \tan^{-1}(x)\right) + C = \frac{\tan^{-1}(x)}{2} - \frac{x}{2(x^2+1)} + C$$



sine

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

cosecant

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$$

cosine

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

secant

$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

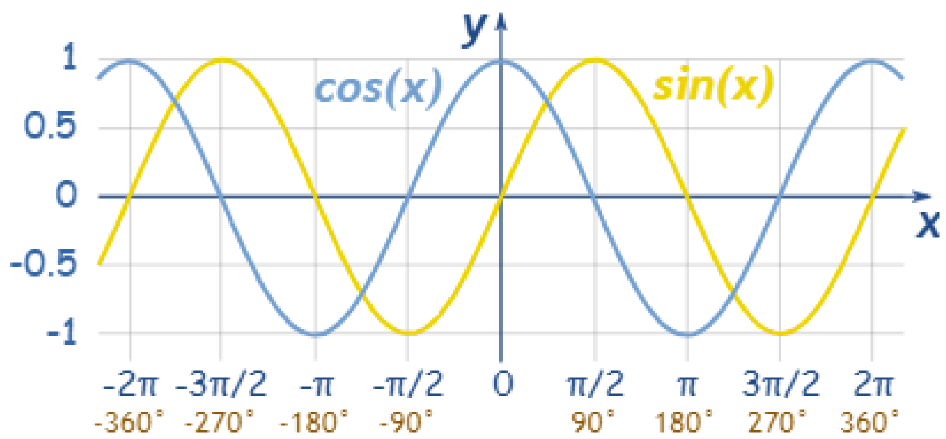
tangent

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

cotangent

$$\cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta},\quad \cot\theta = \frac{\cos\theta}{\sin\theta},\quad \sec\theta = \frac{1}{\cos\theta},\quad \csc\theta = \frac{1}{\sin\theta}.$$



Half-angle formulae [edit]

$$\sin\frac{\theta}{2} = \operatorname{sgn}\left(\sin\frac{\theta}{2}\right)\sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\frac{\theta}{2} = \operatorname{sgn}\left(\cos\frac{\theta}{2}\right)\sqrt{\frac{1+\cos\theta}{2}}$$

E. Trigonometric substitution (三角代換)

Integral feature

$\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{x^2-a^2}$

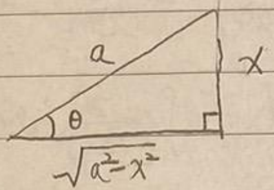
recall

$\cos^2(x)+\sin^2(x)=1$

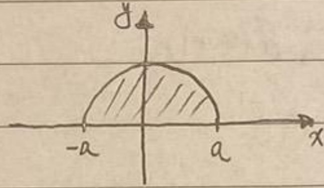
$1+\tan^2(x)=\sec^2(x)$

(i) let $x=a\sin(\theta)$

$\sqrt{a^2-x^2}=a\cos(\theta)$



Ex. $\int_{-a}^{a}\sqrt{a^2-x^2}\,dx$



let $x=a\sin(\theta)$

$dx=a\cos(\theta)\,d\theta$

$-a<x<a$

$-1<\frac{x}{a}<1$

$-\frac{\pi}{2}<\sin^{-1}(\frac{x}{a})<\frac{\pi}{2}$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a\cos(\theta)\cdot a\cos(\theta)\,d\theta$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^2\cos^2(\theta)\,d\theta$$

半角公式 $\cos(\frac{\theta}{2})=\sqrt{\frac{1+\cos(2\theta)}{2}}$

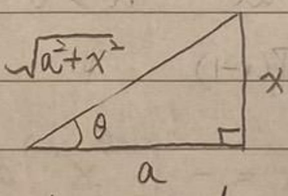
$$=a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1+\cos(2\theta)}{2}\,d\theta$$

$$=a^2\left[\frac{1}{2}\left[\frac{\pi}{2}+\frac{\pi}{2}\right]+\left(\frac{1}{4}\sin(2\theta)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right)\right]$$

$$=\frac{a^2\pi}{2} \quad \#$$

(ii) let $x=a\tan(\theta)$

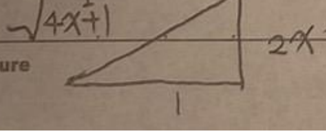
$\sqrt{a^2+x^2}=a\sec(\theta)$



Ex. $\int\frac{1}{\sqrt{4x^2+1}}dx=\int\frac{1}{\sqrt{1+\tan^2\theta}}\frac{1}{2}\sec^2(\theta)\,d\theta$

let $2x=\tan(\theta)$

$2dx=\sec^2(\theta)\,d\theta$

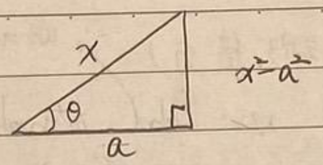


$$=\frac{1}{2}\int\sec(\theta)\,d\theta=\frac{1}{2}\ln|\sec(\theta)+\tan(\theta)|+C$$

$$=\ln\left|\sqrt{4x^2+1}+2x\right|+C \quad \#$$

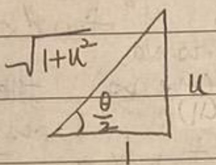
(iii) let $x = a\sec(\theta)$

$\sqrt{x^2-a^2} = a\tan(\theta)$



Ex $\displaystyle\int_0^{\pi} \frac{1}{\alpha - \cos(\theta)}\,d\theta = \int_0^{\infty} \frac{1}{\alpha - \left(\frac{1-u^2}{1+u^2}\right)} \frac{2}{1+u^2}\,du$ ------ (1)

let $u = \tan\left(\frac{\theta}{2}\right)$



$2\tan^{-1}(u) = \theta$

$\frac{2}{1+u^2}\,du = d\theta$

$\cos(\theta) = \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)$

$= \left(\frac{1}{\sqrt{1+u^2}}\right)^2 - \left(\frac{u}{\sqrt{1+u^2}}\right)^2 = \frac{1-u^2}{1+u^2}$

$$(1) = 2\int_0^{\infty} \frac{1}{\alpha(1+u^2) - (1-u^2)}\,du = 2\int_0^{\infty} \frac{1}{(\alpha-1) + (\alpha+1)u^2}\,du$$

$$= \frac{2}{\alpha-1}\int_0^{\infty} \frac{1}{1+\left(\sqrt{\frac{\alpha+1}{\alpha-1}}\,u\right)^2}\,du = \frac{2}{\alpha-1}\sqrt{\frac{\alpha-1}{\alpha+1}}\left(\tan^{-1}\left(\sqrt{\frac{\alpha+1}{\alpha-1}}\,u\right)\Big|_0^{\infty}\right)$$

$$= \frac{\pi}{\sqrt{\alpha^2-1}} \qquad \alpha > 1$$

§ Some identities

$\sin^2(\theta) + \cos^2(\theta) = 1$, $\tan^2(\theta) + 1 = \sec^2(\theta)$, $\cot^2(\theta) + 1 = \csc^2(\theta)$

$$\left.\begin{aligned} \sin(\alpha \pm \beta) &= \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \\ \cos(\alpha \mp \beta) &= \cos(\alpha)\cos(\beta) \pm \sin(\alpha)\sin(\beta) \end{aligned}\right\} \text{和差化積}$$

if let $\alpha = \beta = \theta$

(倍角公式) $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 1 - 2\sin^2(\theta) = 2\cos^2(\theta) - 1$

$$\Rightarrow \left.\begin{aligned} \sin(\theta) &= \pm\sqrt{\frac{1-\cos(2\theta)}{2}} \\ \cos(\theta) &= \pm\sqrt{\frac{1+\cos(2\theta)}{2}} \end{aligned}\right\} \text{半角公式}$$