§ Improper Integral (瑕設積分)

Def. Improper integral w/ ubd intervals. unbounded

(i) Let f be conti on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$$

(ii) Let f be conti on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$$

(iii) Let f be conti on $(-\infty, \infty)$ then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$$

(i) is conv when $\lim_{b\to\infty}\int_a^b f(x)dx$ exist

(ii) 〃 when $\lim_{a\to-\infty}\int_a^b f(x)\,dx$ exist

(iii) 〃 when $\lim_{\lambda\to\infty}\int_c^\lambda f(x)\,dx$ and $\lim_{\lambda\to-\infty}\int_\lambda^c f(x)\,dx$ exist

Ex $\int_0^\infty e^{-x}\,dx$

$$= \lim_{\lambda\to\infty}\int_0^\lambda e^{-x}dx = \lim_{\lambda\to 0}\left(-e^{-x}\Big|_0^\lambda\right) = \lim_{\lambda\to 0}\left(1-e^{-\lambda}\right) = 1$$

Ex $\int_{-\infty}^\infty \frac{e^x}{1+e^{2x}}\,dx$

$$= \int_{-\infty}^0 \frac{e^x}{1+e^{2x}}\,dx + \int_0^\infty \frac{e^x}{1+e^{2x}}\,dx$$

$$= \lim_{\lambda\to-\infty}\int_\lambda^0 \frac{e^x}{1+e^{2x}}\,dx + \lim_{\lambda\to\infty}\int_0^\lambda \frac{e^x}{1+e^{2x}}\,dx$$

$$= \lim_{\lambda\to-\infty}\left(\tan^{-1}(e^x)\Big|_\lambda^0\right) + \lim_{\lambda\to\infty}\left(\tan^{-1}(e^x)\Big|_0^\lambda\right)$$

$$= \left(\frac{\pi}{4}-0\right) + \left(\frac{\pi}{2}-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

thm.
$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1} & p>1 \\ \text{div} & p\le 1 \end{cases}$$

is called p-integral

<pf>

$p=1$, $\lim\limits_{\lambda\to\infty} \int_1^{\lambda} \frac{1}{x}\,dx = \lim\limits_{\lambda\to\infty} \ln(\lambda) - \ln(1)$ div.

$p\neq 1$, $$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{\lambda\to\infty} \int_1^{\lambda} \frac{1}{x^p}\,dx = \lim_{\lambda\to\infty}\left(\frac{x^{-p+1}}{1-p} \Big|_1^{\lambda} \right)$$

$$= \lim_{\lambda\to\infty} \frac{1}{1-p}\left[\frac{1}{\lambda^{p-1}} - 1\right]$$

if $p>1$, $p-1>0$,

as $\lambda\to\infty$, $\lambda^{p-1}\longrightarrow\infty$, then

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \frac{1}{p-1} \quad \text{if } p>1$$

if $p<1$, $p-1<0$, $\lambda^{1-p}\longrightarrow\infty$, then

$$\int_1^{\infty} \frac{1}{x^p}\,dx \quad \text{div} \quad ※$$

Def Improper Integral w/ infinite discontinuities

(i) let f be conti on $[a,b)$ and $\lim\limits_{x\to b^-} f(x) = \pm\infty$, then

$$\int_a^b f(x)\,dx = \lim_{\lambda\to b^-} \int_a^{\lambda} f(x)\,dx \quad (\text{瑕點為右})$$

(ii) let f be conti on $(a,b]$, and $\lim\limits_{x\to a^+} f(x) = \pm\infty$, then

$$\int_a^b f(x)\,dx = \lim_{\lambda\to a^+} \int_{\lambda}^{b} f(x)\,dx \quad (\text{瑕點為左})$$

(iii) let f be conti on $[a,b]$,

except for some $c\in(a,b)$ and $\lim\limits_{x\to c} f(x) = \pm\infty$ then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

(i),(ii) If $\int_a^b f(x)\,dx = \lim\limits_{\substack{\alpha\to a^+ \\ \beta\to b^-}} \int_{\alpha}^{\beta} f(x)\,dx$ exist, then $\int_a^b f(x)\,dx$ conv

(iii) both $\int_a^c f(x)\,dx$, $\int_c^b f(x)\,dx$ conv

Ex $\int_{-\infty}^{\infty} \frac{1}{x}\,dx$

<X>

$\frac{1}{x}$ is odd $\Rightarrow \int_{-\infty}^{\infty} \frac{1}{x}\,dx = 0$

<O>

$$\int_{-\infty}^{\infty} \frac{1}{x}\,dx = \int_{-\infty}^{-1} + \int_{-1}^{0} + \int_{0}^{1} + \int_{1}^{\infty} \Rightarrow \text{div}$$

$$\int_{1}^{\infty} \frac{1}{x}\,dx = \lim_{\lambda\to\infty} \int_{1}^{\lambda} \frac{1}{x}\,dx = \lim_{\lambda\to\infty} \ln(\lambda) = \infty$$

§ Comparison Test

thm. Suppose f and g are conti on $[a,\infty)$ $(-\infty, c]$

$f(x) \geq g(x) \geq 0$ then

$$\int_{a}^{\infty} g(x)\,dx \ \text{div} \Rightarrow \int_{a}^{\infty} f(x)\,dx \ \text{div} \qquad \left(\int_{-\infty}^{c}\right)$$

$$\int_{a}^{\infty} f(x)\,dx \ \text{conv} \Rightarrow \int_{a}^{\infty} g(x)\,dx \ \text{conv} \qquad \left(\int_{-\infty}^{c}\right)$$

Ex. $\int_{1}^{\infty} e^{-x^2}\,dx$

Note that $e^{-x^2} < e^{-x} \quad \forall x \in [1,\infty)$

$$\int_{1}^{\lambda} e^{-x^2}\,dx \leq \int_{1}^{\lambda} e^{-x}\,dx = \left(-e^{-x}\Big|_{1}^{\lambda}\right) = e^{-1} - e^{-\lambda} \leq e^{-1}$$

$\Rightarrow \int_{1}^{\infty} e^{-x^2}\,dx$ conv

Ex. $$\int_{0}^{\infty} \frac{1}{x^{\frac{3}{2}} + x^{\frac{1}{2}}}\,dx = \int_{0}^{\infty} \frac{1}{\sqrt{x}(x+1)}\,dx = \int_{0}^{1} + \int_{1}^{\infty}$$

for $0 < x < 1 \Rightarrow \frac{1}{(1+x)\sqrt{x}} \leq \frac{1}{\sqrt{x}} \Rightarrow \int_{0}^{1} \frac{1}{\sqrt{x}}\,dx = \left(2\sqrt{x}\Big|_{0}^{1}\right) = 2$

$1 < x < \infty \Rightarrow \frac{1}{(1+x)\sqrt{x}} \leq \frac{1}{x^{\frac{3}{2}}} \Rightarrow \int_{1}^{\infty} \frac{1}{x^{\frac{3}{2}}}\,dx$ conv

p-integral

then $\int_{0}^{\infty} \frac{1}{\sqrt{x}(x+1)}\,dx$ conv positive

紅色case 不用這條件

§ Limit Comparison Test

thm. Let f and g conti on $(a,b]$, $[\ )$, $[a,\infty)$ $\left(\lim\limits_{x\to a^+,\, b^-} f(x) = \infty = \lim\limits_{x\to a^+,\, b^-} g(x) \right)$

and if $\lim\limits_{x\to a^+,\, b^-,\, \infty} \dfrac{f(x)}{g(x)} = L \qquad 0 < L < \infty$

a). $L \neq \infty$, then $\int_a^b f(x)\,dx$ conv $\Longleftrightarrow$ $\int_a^b g(x)\,dx$ conv

b) $L = 0$, then $\int_a^b g(x)\,dx$ conv $\Longrightarrow$ $\int_a^b f(x)\,dx$ conv

c) $L = \infty$, then $\int_a^b g(x)\,dx$ div $\Longrightarrow$ $\int_a^b f(x)\,dx$ div

Ex $\int_1^\infty \frac{1}{1+x^2}\,dx$

since $\int_1^\infty \frac{1}{x^2}\,dx$ conv

$$\lim_{x\to\infty} \frac{\frac{1}{1+x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{x^2}{1+x^2} = 1$$

by limit comparison test $\Longrightarrow$ $\int_1^\infty \frac{1}{1+x^2}\,dx$ conv

Ex. $\int_1^\infty \frac{\ln x}{x}\,dx$

$$\lim_{x\to\infty} \frac{\frac{\ln x}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \ln x = \infty$$ and $\int_1^\infty \frac{1}{x}\,dx$ div

by limit comparison test $\Longrightarrow$ $\int_1^\infty \frac{\ln x}{x}\,dx$ div

Ex. $\int_1^\infty \frac{\ln x}{x^2}\,dx$

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$$\lim_{x\to\infty} \frac{\frac{\ln x}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \ln x = \infty$$ and $\int_1^\infty \frac{1}{x^2}\,dx$ conv

<O>

$$\lim_{x\to\infty} \frac{\frac{\ln x}{x^2}}{\frac{1}{x^{\frac{3}{2}}}} = \lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = 0$$ and $\int_1^\infty \frac{1}{x^{\frac{3}{2}}}\,dx$ conv

by limit comparison test $\Longrightarrow$ $\int_1^\infty \frac{\ln x}{x^2}\,dx$ conv.

Chuyu

§ Gamma and Beta function

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t}\, dt \quad , \quad \alpha > 0$$

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha) \quad (\alpha \in \mathbb{Z}^{+},\ \Gamma(\alpha+1) = \alpha!\ )$$

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

$$\Gamma(1) = 1$$

- $\Gamma(\alpha)\Gamma(1-\alpha) = \dfrac{\pi}{\sin(\alpha\pi)} \quad 0<\alpha<1$

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\, dx \quad , \quad \alpha,\beta > 0$$

- Alternative form

$$B(\alpha,\beta) = \int_0^{\infty} \frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta}}\, dt = B(\beta,\alpha)$$

<pf> let $x = \dfrac{1}{1+t}$ $\quad 0<x<1$

$t = \dfrac{1}{x} - 1$ $\quad \infty < t < 0$

$$dx = \frac{-1}{(1+t)^2}\, dt$$

$$B(\alpha,\beta) = \int_{\infty}^{0} \left(\frac{1}{1+t}\right)^{\alpha-1}\left(\frac{t}{1+t}\right)^{\beta-1}\frac{-1}{(1+t)^2}\, dt$$

$$= \int_0^{\infty} \frac{t^{\beta-1}}{(1+t)^{\alpha+\beta}}\, ds$$

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- $B(\alpha, 1-\alpha) = \dfrac{\pi}{\sin(\pi\alpha)}$

Ex. $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2(x)\cos^4(x)\, dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} [\sin^2(x)]^{\frac{1}{2}}[\cos^2(x)]^{\frac{3}{2}} \cdot 2\sin(x)\cos(x)\, dx$

$\left(\begin{array}{l} \text{let } y = \sin^2(x) \\ dy = 2\sin(x)\cos(x)\,dx \end{array}\right)$ $\displaystyle = \frac{1}{2}\int_0^1 y^{\frac{3}{2}-1}(1-y)^{\frac{5}{2}-1}\, dy$

$$= \frac{1}{2}B\left(\frac{3}{2},\frac{5}{2}\right) = \frac{1}{2}\frac{\Gamma(\frac{3}{2})\Gamma(\frac{5}{2})}{\Gamma(4)}$$

$$= \frac{\frac{1}{2}\sqrt{\pi}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{2\cdot 3!} = \frac{\pi}{32}$$

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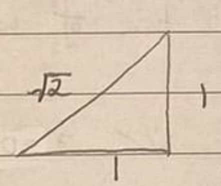
Ex. $\int_0^{\infty} \frac{1}{1+x^4}\,dx = \int_0^{\infty} \frac{1}{1+y} \frac{1}{4y^{\frac{3}{4}}}\,dy$

$\left(\begin{array}{l} \text{let } y = x^4 \\ dy = 4x^3\,dx \end{array}\right) = \frac{1}{4}\int_0^{\infty} \frac{y^{\frac{1}{4}-1}}{(1+y)^{\frac{1}{4}+\frac{3}{4}}}\,dy = \frac{B(\frac{1}{4},\frac{3}{4})}{4}$



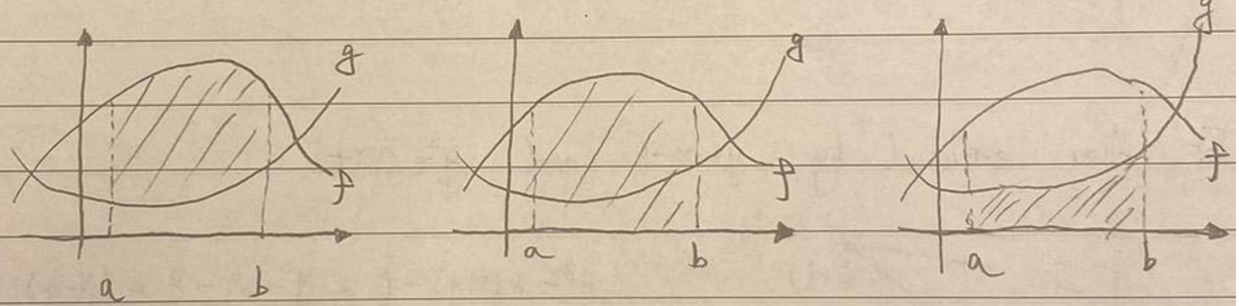
$= \frac{\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})}{4} = \frac{\Gamma(\frac{1}{4})\Gamma(1-\frac{1}{4})}{4} = \frac{\pi}{4\sin(\frac{\pi}{4})}$

$= \frac{\pi}{4}\sqrt{2}$ ※

Ex $\int \frac{1}{1+x^4}\,dx$ (hard)

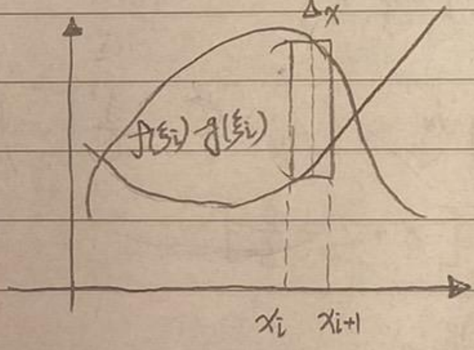
§ Application of Integration.

Area between two curves "w.r.t. x"



Area between f & g — Area under f — Area under g

$\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$



$[f(\xi_1)-g(\xi_1)]\Delta x_1 + \cdots + [f(\xi_n)-g(\xi_n)]\Delta x_n = \sum_{k=1}^{n}[f(\xi_k)-g(\xi_k)]\Delta x_k$

$\Rightarrow$

$A = \int_a^b [f(x)-g(x)]\,dx$ $\quad \|T\| \to 0$

Remark $f(x) \geq g(x) \quad \forall x \in [a,b]$

Ex. Area of region bdd $\begin{cases} \text{above by} & y = e^x \\ \text{below by} & y = x \end{cases}$

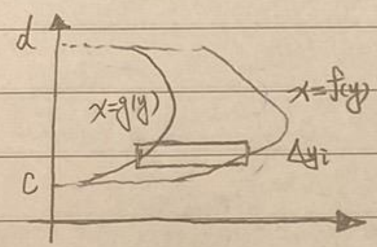
w/ $x = 0$ to $x = 1$

$$A = \int_0^1 (e^x - x)\,dx = \left(e^x - \frac{x^2}{2} \Big|_0^1 \right)$$

$$= \left[e - \frac{1}{2}\right] - [1 - 0] = e - \frac{3}{2} \ast$$

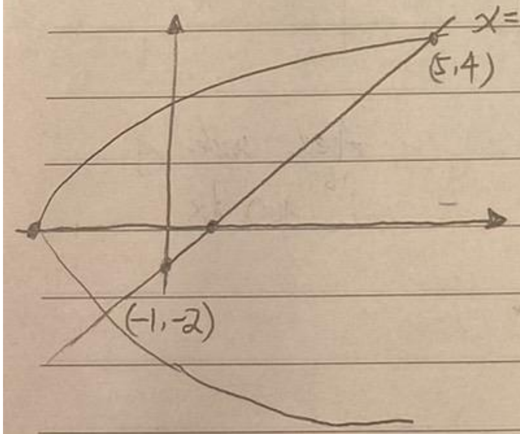
Area between two curves "w.r.t y"

$f(y) \geq g(y) \quad \forall y \in [c, d]$



$$A = \int_c^d [f(y) - g(y)]\,dy$$

Ex. Area enclosed by $y = x - 1$ and $y^2 = 2x + 6$



$y^2 - 2(1+y) - 6 = y^2 - 2y - 8 = (y-4)(y+2)$

$\Rightarrow y = 4$ or -2

$$\int_{-2}^{4} \left(1 + y - \left(\frac{y^2 - 6}{2}\right)\right) dy$$

$$= \int_{-2}^{4} \left(4 + y - \frac{y^2}{2}\right) dy = \left(4y + \frac{y^2}{2} - \frac{y^3}{6} \Big|_{-2}^{4} \right)$$

$$= \left[16 + 8 - \frac{64}{6}\right] - \left[-8 + 2 + \frac{8}{6}\right]$$

$$= 30 - \left(\frac{64 + 8}{6}\right) = 18 \ast$$

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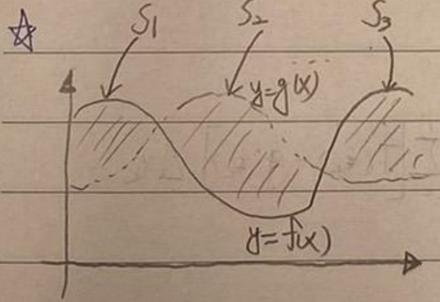
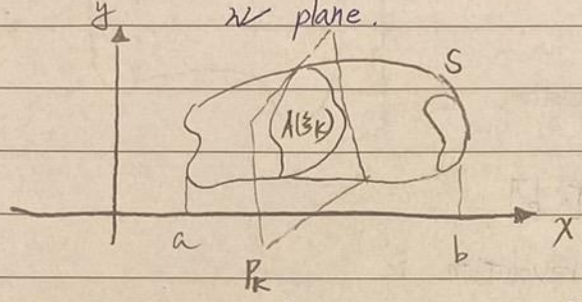


$$A = \int_a^b |f(x) - g(x)|\,dx = A_1 + \cdots + A_n$$

§ Volume of Revolution : Disk method

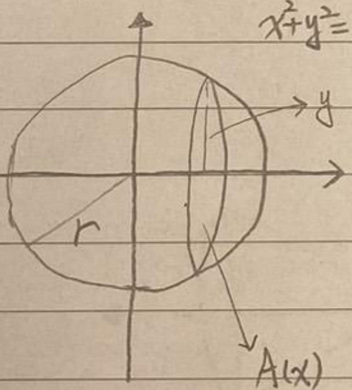
Def cross-section : A cross-section of solid is the region obtained by intersecting solid w/ plane.



- $A(\xi_K)$ is area of cross-section of S in a plane P_K

$$V = \lim_{\|P\| \to 0} \sum_{k=1}^{n} A(\xi_K)\, \Delta x_K = \int_a^b A(x)\, dx$$

Ex. Show volume of sphere of radius r is $\frac{4}{3}\pi r^3$



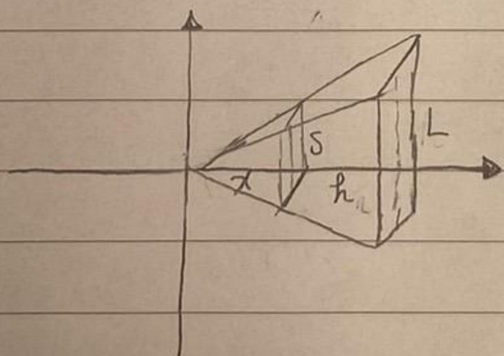
$$A(x) = \pi y^2 = \pi(r^2 - x^2)$$

$$V = \int_{-r}^{r} A(x)\, dx = \int_{-r}^{r} \pi r^2 - \pi x^2\, dx$$

$$= \pi r^2 (2r) - \pi \left(\frac{x^3}{3} \Big|_{-r}^{r} \right)$$

$$= \frac{4}{3}\pi r^3$$

Ex. find volume of pyramid whose base square w/ side L and height h.



$$\frac{x}{h} = \frac{\frac{S}{2}}{\frac{L}{2}} \quad (相似三角形)$$

$$A(x) = S^2 = \left(\frac{L}{h}\right)^2 x^2$$

$$\int_0^h A(x)\, dx = \frac{L^2}{h^2}\left(\frac{x^3}{3}\Big|_0^h\right) = \frac{L^2 h}{3}$$