<pf. FTC part I>

$$F(x+h) - F(x) = \int_x^{x+h} f(t)\,dt$$

by M.V.T $\exists\ c(h) \in [x, x+h]$

s.t $\dfrac{F(x+h)-F(x)}{h} = \dfrac{\int_x^{x+h} f(t)\,dt}{h} = f(c(h))$

$\downarrow h \to 0$ $\downarrow h \to 0$

$F'(x)$ $f(x)$

Thus $F'(x) = f(x)$

Ex. find $\frac{d}{dx}\left\{\int_0^x \sqrt{1+t^2}\,dt\right\}$

Ans : $\sqrt{1+x^2}$

Ex. find $\frac{d}{dx}\left\{\int_1^{x^4} \sec(\theta)\,d\theta\right\}$

Ans $\dfrac{d\{\int_1^{x^4}\sec(\theta)d\theta\}}{dx^4} \cdot \dfrac{dx^4}{dx}$ (by chain rule)

$= \sec(x^4) \cdot 4x^3$

Thm (FTC + Chain Rule)

$$\frac{d}{dx}\left\{\int_{v(x)}^{u(x)} f(t)\,dt\right\} = f(u(x))\,u'(x) - f(v(x))\,v'(x)$$

Ex $\frac{d}{dx}\left\{\int_{x^3}^{2x} 1+\cos(t)\,dt\right\}$

$= 2\,[1+\cos(2x)] - 3x^2[1+\cos(x^3)]$

Ex $\frac{d}{dx}\left\{\int_{10x}^{x^2} t^3\sin(1+t)\,dt\right\}$

$= 2x\,[x^6\sin(1+x^2)] - 10\,[(10x)^3\sin(1+10x)]$

Ex $\frac{d}{dx}\left\{\int_x^{2x} t^3\,dt\right\}$

$= 2\,(2x)^3 - x^3 = 15x^3$

thm (FTC, part II)

Let $f:[a,b]\longrightarrow\mathbb{R}$ be conti fn., then

$$\int_a^b f(x)\,dx = F(b)-F(a)$$

where F is any anti-derivative of f s.t. $F'=f$

<pf FTC part II>

Let $P=\{x_0,x_1,\dots,x_n\}$ be any partition of $[a,b]$

$$F(b)-F(a) = [F(x_n)-F(x_{n-1})]+\cdots+[F(x_1)-F(x_0)]$$
$$=\sum_{k=1}^{n}[F(x_k)-F(x_{k-1})]$$

for each k, by MVT $\exists\ \xi_k\in[x_{k-1},x_k]$

s.t. $F'(\xi_k)(x_k-x_{k-1})=F(x_k)-F(x_{k-1})$

since $F'(\xi_k)=f(\xi_k)$ and let $\Delta x_k = x_k-x_{k-1}$

$$F(b)-F(a)=\sum_{k=1}^{n}f(\xi_k)\Delta x_k \xrightarrow[\|P\|\to 0]{} \int_a^b f(x)\,dx$$

(Remark. $F(b)-F(a)$ denote as $F(x)\Big|_a^b$)

Ex. $\int_3^6 \frac{1}{x}\,dx = \ln 6-\ln 3=\ln 2$

$f(x)\longleftrightarrow F(x)=\ln|x|$

Ex. $\int_1^4 3\sqrt{x}\,dx = 3\int_1^4\sqrt{x}\,dx = 3\left(\frac{2}{3}x^{\frac{3}{2}}\Big|_1^4\right)=3\left[\frac{2}{3}\cdot 8-\frac{2}{3}\right]=14$

Ex $\int_1^3 \frac{y^3-2y^2-y}{y^2}\,dy = \int_1^3\left[y-2-\frac{1}{y}\right]dy = \underbrace{\int_1^3 y\,dy}_{\frac{y^2}{2}\Big|_1^3} - 2(3-1) - \underbrace{\int_1^3\frac{1}{y}\,dy}_{\ln y\Big|_1^3}$

$$=\left(\frac{9}{2}-\frac{1}{2}\right)-4-\ln 3 = -\ln 3$$

Ex. $\int_0^1(x^e+e^x)\,dx = \int_0^1 x^e\,dx+\int_0^1 e^x\,dx$

$$=\left(\frac{x^{e+1}}{e+1}\Big|_0^1\right)+\left(e^x\Big|_0^1\right)=\frac{1}{e+1}+(e-1)=\frac{e^2}{e+1}\ \#$$

§ Indefinite integral

Ex. what fn did we diff to get $f(x) = x^4 + 3x + 9$

$$F(x) = \frac{x^5}{5} + \frac{3x^2}{2} + 9x$$

how about $F_1(x) = \frac{x^5}{5} + \frac{3x^2}{2} + 9x + 940857$?

In fact any fn of the form $F(x) + C$ will give $f(x)$ upon differentiating

thm. F is anti-derivative of f on I, then G is 〃

$$\Leftrightarrow G(x) = F(x) + \underbrace{C}_{\text{constant}} \quad \forall x \in I$$

(Remark. 導函數相同的函數相差一常數)

Recall FTC I $\Rightarrow$ f conti then $\int_a^x f(t)\,dt$ is anti-derivative of f

FTC II $\Rightarrow$ $\int_a^b f(x)\,dx$ can be evaluating by $F(b) - F(a)$ where F is anti-derivative of f.

Def. The collection of all anti-derivative of f is called indefinite integral of f w.r.t x, denote by

$$\int f(x)\,dx$$

☆ anti-derivative $\overset{\text{FTC}}{\Longleftrightarrow}$ definite integral

$$\int_a^b f(x)\,dx = F(b) - F(a) = [F(b) + C] - [F(a) + C]$$

$$= F(x) + C \Big|_a^b = \int f(x)\,dx \Big|_a^b$$

☆ $\int_a^b f(x)\,dx$ is a number

but $\int f(x)\,dx$ is a family of fn.

- Properties

$$\int af(x) + bg(x)\,dx = a\int f(x)\,dx + b\int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

$$\int x^{-1}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C$$

$$\int b^x\,dx = \frac{b^x}{\ln b} + C \quad \left(b^x = e^{x\ln b}\right)$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1}(x) + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + C$$

$$\left(\begin{aligned} y &= \tan^{-1}(x) \\ x &= \tan(y) \\ \frac{dx}{dy} &= \sec^2(y) = 1+\tan^2(y) = 1+x^2 \\ \frac{dy}{dx} &= \frac{1}{1+x^2} \end{aligned}\right)$$

- Trigonometric part

$$\int \sin(x)\,dx = -\cos(x) + C$$

$$\int \cos(x)\,dx = \sin(x) + C$$

$$\int \sec^2(x)\,dx = \tan(x) + C$$

$$\int \csc^2(x)\,dx = \cot(x) + C$$

$$\int \sec(x)\tan(x)\,dx = \sec(x) + C$$

$$\int \csc(x)\cot(x)\,dx = -\csc(x) + C$$

Ex. $\displaystyle\int \frac{\sin(x)}{\cos^2(x)}\,dx = \int \frac{1}{\cos(x)}\cdot\frac{\sin(x)}{\cos(x)}\,dx = \int \sec(x)\tan(x)\,dx = \sec(x) + C$

Ex. $\displaystyle\int (1-2\sin^2(x))\sin(2x)\,dx \overset{\sin^2(x)+\cos^2(x)=1}{=} \int [\cos^2(x)-\sin^2(x)]\sin(2x)\,dx$

$\cos(2x) = \cos(x+x) = \cos^2(x) - \sin^2(x)$

$$= \int \cos(2x)\sin(2x)\,dx$$

$\sin(2x) = 2\sin x\cos x$

$$= \frac{1}{2}\int \sin(4x)\,dx$$

$$= -\frac{1}{8}\cos(4x) + C$$

(変数変換)

No. P.13

Date :

§ Change of Variable / substitution rule

u-substitution

☆ Motivation (if $F' = f$)

$$\int f(g(x))g'(x)\,dx = \int F'(g(x))g'(x)\,dx = F(g(x)) + C$$

若 $u = g(x)$

$$\int f(g(x))g'(x)dx = F(g(x)) + C = F(u) + C = \int f(u)\,du$$

thm (Indefinite integral)

Let $y = g(x)$ is diff fn whose range is I and f is conti on I, then

$$\int f(g(x))g'(x)\,dx = \int f(y)\,dy$$

Ex. $\int \sqrt{2x-1}\,dx = \int \sqrt{y}\,\frac{1}{2}\,dy = \frac{1}{2}\left(\frac{2}{3}y^{\frac{3}{2}}\right) + C = \frac{y^{\frac{3}{2}}}{3} + C$

$\begin{pmatrix} \text{let } y = 2x-1 \\ dy = 2dx \end{pmatrix}$

$= \frac{(2x-1)^{\frac{3}{2}}}{3} + C$ ＃

Ex $\int \frac{x}{\sqrt{1-4x^2}}\,dx = \int \frac{1}{\sqrt{y}}\,\frac{1}{-8}\,dy = -\frac{1}{8}\left(2y^{\frac{1}{2}}\right) + C = \frac{-y^{\frac{1}{2}}}{4} + C$

$\begin{pmatrix} \text{let } y = 1-4x^2 \\ dy = -8x\,dx \end{pmatrix}$

$= \frac{-\sqrt{1-4x^2}}{4} + C$

Ex. $\int \frac{e^x}{e^{2x}+1}\,dx$, $\int \frac{1}{e^x+1}\,dx$, ☆$\int \frac{1+x}{(1+xe^x)x}\,dx$

$\int \frac{e^x}{e^{2x}+1}\,dx = \tan^{-1}(e^x) + C$

$\int \frac{1}{e^x+1}\,dx = -\ln|1+e^{-x}| + C$

<pf ☆>

$$\int \frac{(1+x)e^x}{(1+xe^x)xe^x}\,dx = \int \frac{1}{(1+y)y}\,dy = \int \frac{1}{y} - \frac{1}{y+1}\,dy = \ln|y| - \ln|y+1| + C$$

$$= \ln\frac{|xe^x|}{|1+xe^x|} + C$$ ＃

$\begin{pmatrix} \text{let } y = xe^x \\ dy = (e^x + xe^x)dx = e^x(1+x) \end{pmatrix}$

Chryv culture

(definite integral)

thm. Let $y = g(x)$ has conti derivative g' on $[a,b]$
and f is conti on the range of g, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(y)\,dy$$

<pf>

let F is anti-derivative of f

$F(g(x))$ is anti-derivative of $f(g(x))g'(x)$

by FTC II

$$\int_a^b f(g(x))g'(x)\,dx = F(g(x))\Big|_a^b = F(g(b)) - F(g(a))$$

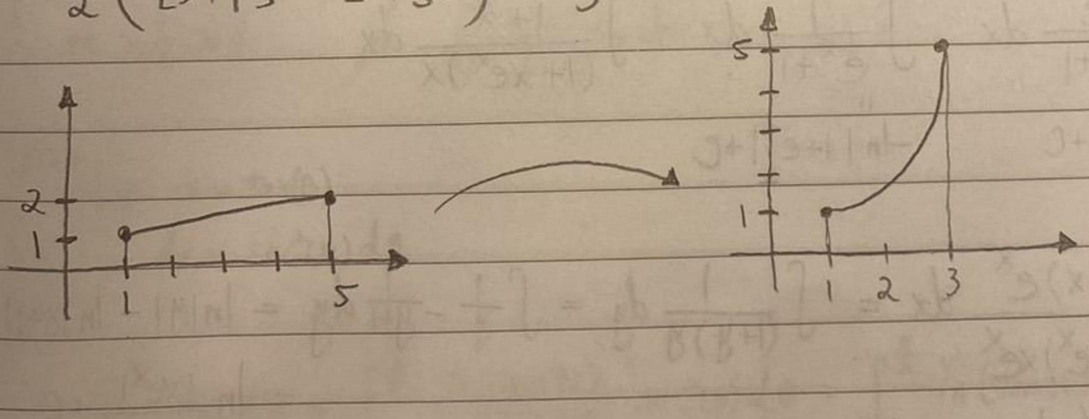
$$\int_{g(a)}^{g(b)} f(u)\,du = F(u)\Big|_{g(a)}^{g(b)}$$

Ex. $\displaystyle\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx$

$$\left(\begin{array}{ll} \text{let } y = \sqrt{2x-1} & 1 < x < 5 \\ y^2 = 2x-1 & 2 < 2x < 10 \\ 2y\,dy = 2\,dx & 1 < 2x-1 < 9 \\ & 1 < \sqrt{2x-1} < 3 \end{array}\right)$$

$$= \int_1^3 \frac{1}{y}\,\frac{1+y^2}{2}\,y\,dy = \frac{1}{2}\int_1^3 1+y^2\,dy = \frac{1}{2}\left(y + \frac{y^3}{3}\Big|_1^3\right)$$

$$= \frac{1}{2}\left([3+9] - [1+\tfrac{1}{3}]\right) = \frac{16}{3}$$



(odd & even)

thm. Suppose f conti on $[-a, a]$

(a) if f is even then
$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

(b) if f is odd then
$$\int_{-a}^{a} f(x)\,dx = 0$$

<pf>

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = \int_{0}^{a}[f(y)+f(-y)]\,dy$$

$\left(\begin{array}{l} \text{let } x=-y \\ dx=-dy \\ -a<x<0 \\ a>-x>0 \end{array}\right)$

$$\int_{a}^{0} f(-y)\,(-dy) = \int_{0}^{a} f(-y)\,dy$$

$\Rightarrow$ if f is even $f(-y)=f(y)$

then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

$\Rightarrow$ if f is odd $f(-y)=-f(y)$

then $\int_{-a}^{a} f(x)\,dx = 0$

Ex. $\displaystyle\int_{-1}^{1} \frac{x|x|}{x^4+1}\,dx = 0$

Ex $\displaystyle\int_{0}^{\infty} \frac{\ln x}{1+x^2}\,dx = \underbrace{\int_{0}^{1} \frac{\ln x}{1+x^2}\,dx}_{} + \int_{1}^{\infty} \frac{\ln x}{1+x^2}\,dx = 0$

(M1)

$\left(\begin{array}{l} \text{let } x=\frac{1}{t} \\ dx = -\frac{1}{t^2}dt \\ 0<x<1 \\ \infty > \frac{1}{x} > 1 \end{array}\right)$

$$\int_{0}^{1} \frac{\ln x}{1+x^2}\,dx = \int_{\infty}^{1} \frac{\ln(\frac{1}{t})}{1+(\frac{1}{t})^2}\left(-\frac{1}{t^2}\right)dt = \int_{1}^{\infty} \frac{-\ln(t)}{1+t^2}\,dt$$

(M2) let $x=\tan(\theta)$

$dx = \sec^2(\theta)\,d\theta$

$$\int_{0}^{\infty} \frac{\ln x}{1+x^2}\,dx = \int_{0}^{\frac{\pi}{2}} \frac{\ln(\tan(\theta))}{1+\tan^2(\theta)}\sec^2(\theta)\,d\theta = \int_{0}^{\frac{\pi}{2}} \ln(\tan(\theta))\,d\theta$$

$$= \int_{0}^{\frac{\pi}{2}} \ln(\sin(\theta))\,d\theta - \int_{0}^{\frac{\pi}{2}} \ln(\cos(\theta))\,d\theta = 0$$

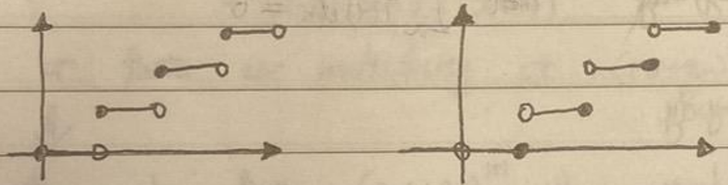
§ Technique of Integration

A. absolute value

$$\int_0^2 |x-1|\,dx = \int_0^1 (1-x)\,dx + \int_1^2 (x-1)\,dx$$
$$= \left(x-\frac{x^2}{2}\Big|_0^1\right) + \left(\frac{x^2}{2}-x\Big|_1^2\right)$$
$$= \left(1-\frac{1}{2}\right) - \left(\frac{1}{2}-1\right) = 1$$

Ex. $\int_{-2}^{2} |x^2-1|\,dx$

B. floor fn / greatest integer fn (下取整函數, $\lfloor x \rfloor$)
ceiling fn / least integer fn (上 〃 , $\lceil x \rceil$)



Ex. $\int_0^2 \lfloor x \rfloor\,dx = \int_0^1 0\,dx + \int_1^2 1\,dx = 1$

Ex. $\int_0^2 \lfloor x^2 \rfloor\,dx = \int_0^1 0\,dx + \int_1^{\sqrt{2}} 1\,dx + \int_{\sqrt{2}}^{\sqrt{3}} 2\,dx + \int_{\sqrt{3}}^{2} 3\,dx$

$$= (\sqrt{2}-1) + 2(\sqrt{3}-\sqrt{2}) + 3(2-\sqrt{3})$$
$$= 5-\sqrt{2}-\sqrt{3}$$

C. partial fraction (部分分式)

Def. A rational fn is a fn of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where P、Q are polynomials and $D(f) = \{x : Q(x) = 0\}$

if $\deg(P) < \deg(Q)$, then f is proper

$\deg(P) \geq \deg(Q)$, then f is improper

If f is improper, divide Q to P by long division until a reminder $R(x)$ is obtained s.t $\deg(R) < \deg(Q)$

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

It can be shown that Q can factor as product of

(i) Linear factors s.t $px+q$

(ii) Irreducible quadratic factors s.t ax^2+bx+c w/ $b^2-4ac<0$

Ex. $Q(x) = x^4-16 = (x^2-4)(x^2+4)$
$= (x-2)(x+2)(x^2+4)$

or factor w/ multiplicity s.t $(px+q)^m \cdot (ax^2+bx+c)^n$

$\Downarrow$

for each factor $(px+q)^m$, the partial fraction decomp must include

$$\frac{A_1}{px+q} + \cdots + \frac{A_m}{(px+q)^m}$$

Similar, for $(ax^2+bx+c)^n$, the partial fraction decomp must include

$$\frac{B_1x+C_1}{ax^2+bx+c} + \cdots + \frac{B_nx+C_n}{(ax^2+bx+c)^n}$$

Ex. $\displaystyle\int \frac{3x+4}{(x-1)(x-2)(x-3)}dx = \frac{7}{2}\int \frac{1}{x-1}dx - 10\int\frac{1}{x-2}dx + \frac{13}{2}\int\frac{1}{x-3}dx$

$$\frac{3x+4}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

$\Rightarrow 3x+4 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$
$= A(x^2-5x+6) + B(x^2-4x+3) + C(x^2-3x+2)$

$\Rightarrow A+B+C=0$
$5A+4B+3C=3$
$6A+3B+2C=4$

$\Rightarrow A=\frac{7}{2}$
$B=-10$
$C=\frac{13}{2}$