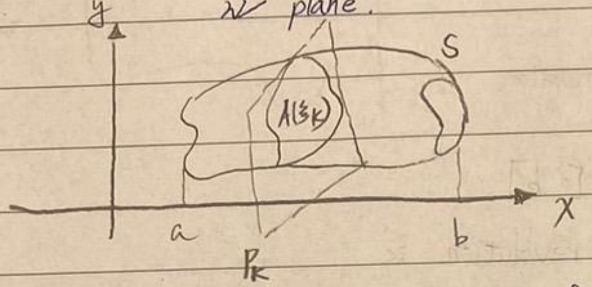
Volume of Revolution : Disk method

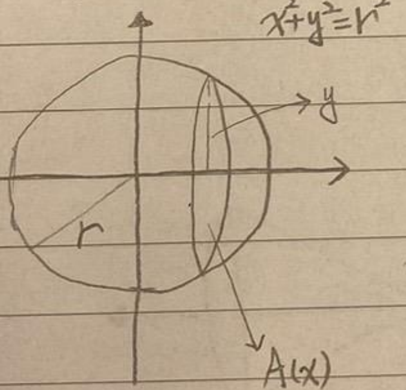
Def cross-section : A cross-section of solid is the region obtained by intersecting solid w/ plane.



- $A(\xi_k)$ is area of cross-section of S in a plane P_k

$$V = \lim_{\|P\| \to 0} \sum_{k=1}^{n} A(\xi_k)\Delta x_k = \int_a^b A(x)\,dx$$

Ex. Show volume of sphere of radius r is $\frac{4}{3}\pi r^3$



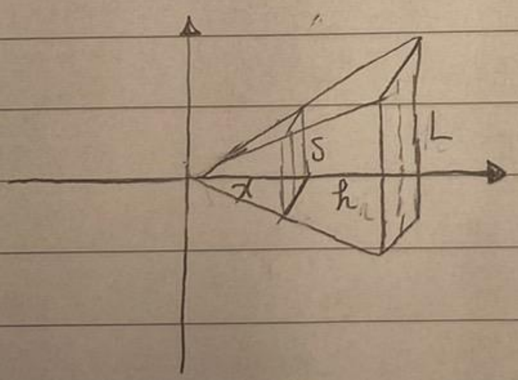
$$A(x) = \pi y^2 = \pi(r^2 - x^2)$$

$$V = \int_{-r}^{r} A(x)\,dx = \int_{-r}^{r} \pi r^2 - \pi x^2\,dx$$

$$= \pi r^2(2r) - \pi\left(\frac{x^3}{3}\Big|_{-r}^{r}\right)$$

$$= \frac{4}{3}\pi r^3$$

Ex. find volume of pyramid whose base square w/ side L and height h.

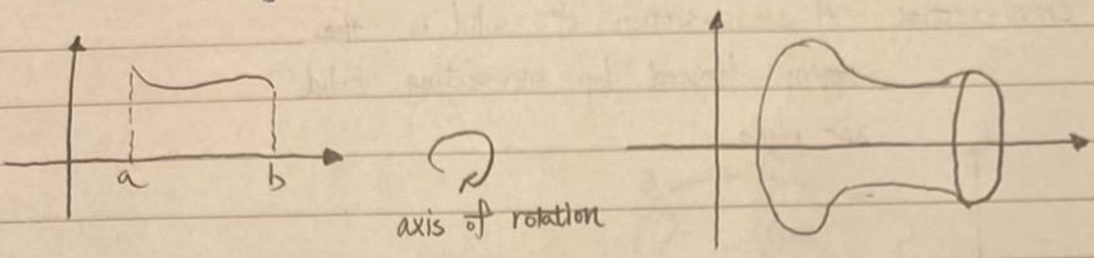


$$\frac{x}{h} = \frac{\frac{S}{2}}{\frac{L}{2}} \quad (\text{相似三角形})$$

$$A(x) = S^2 = \left(\frac{L}{h}\right)^2 x^2$$

$$\int_0^h A(x)\,dx = \frac{L^2}{h^2}\left(\frac{x^3}{3}\Big|_0^h\right) = \frac{L^2 h}{3}$$

• Disk method : Integration w.r.t x



Suppose there are n disks on $[a,b]$

the volume of the solid of revolution is

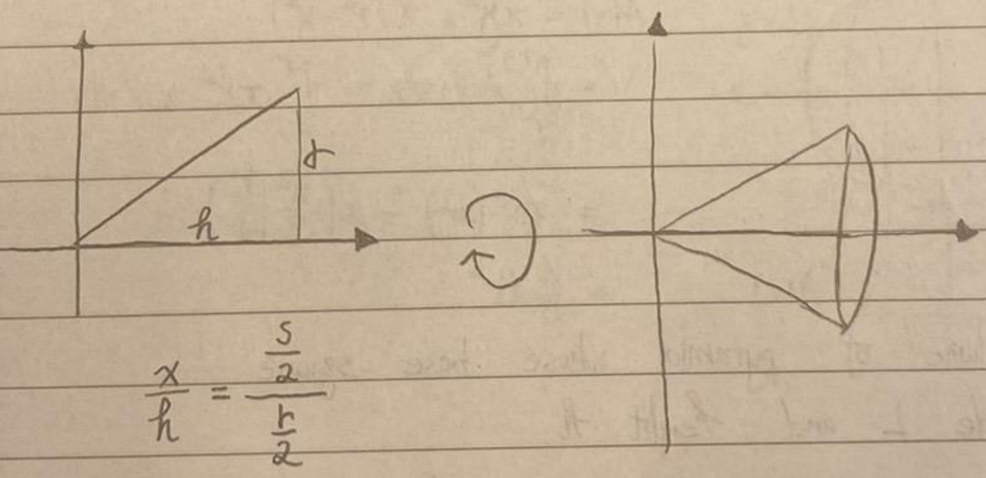
$$\sum_{k=1}^{n} \pi \big[\underbrace{f(\xi_k)}_{= r}\big]^2 \Delta x_k$$

$$\Big\downarrow \; \|P\| \to 0$$

$$V = \int_a^b \pi [f(x)]^2 \, dx$$

Ex. Volume of a right Circular Cone

w/ base radius r and height h



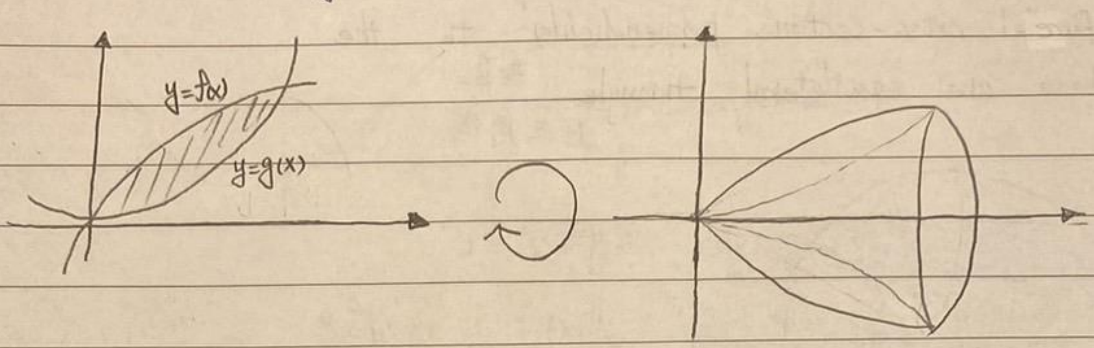
$$\frac{x}{h} = \frac{\frac{s}{2}}{\frac{r}{2}}$$

$$A(x) = \pi s^2 = \pi \left(\frac{r}{h}\right)^2 x^2$$

$$\int_0^h A(x)\, dx = \pi \left(\frac{r}{h}\right)^2 \left(\frac{x^3}{3}\Big|_0^h\right) = \frac{\pi r^2 h}{3}$$ #

(Annular ring) ☆ Möbius Ring

Σ Washer Method : Integration w.r.t x



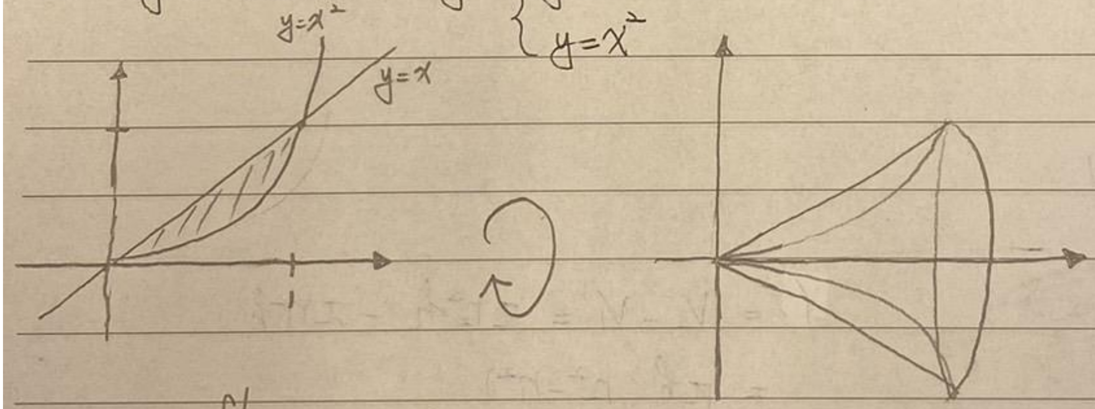
Suppose $f \geq g \geq 0 \quad \forall x \in [a,b]$

$$\sum_{k=1}^{n} \pi \left[\underbrace{f^2(\xi_k)}_{\text{outer radius}} - \underbrace{g^2(\xi_k)}_{\text{inner radius}} \right] \Delta x_k$$

$$\xrightarrow{\|P\| \to 0}$$

$$\int_a^b \pi \left[f^2(x) - g^2(x) \right] dx$$

Ex region enclosed by $\begin{cases} y = x \\ y = x^2 \end{cases}$ is rotated about x-axis



$$V = \int_0^1 \pi (x^2 - x^4)\, dx$$

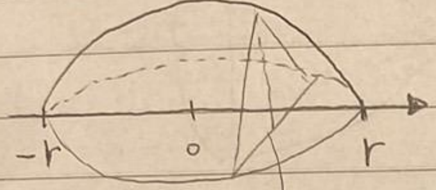
$$= \pi \left(\frac{x^3}{3} - \frac{x^5}{5} \Big|_0^1 \right) = \frac{2\pi}{15}$$

Notes. Disk method w.r.t $y \Rightarrow \int_a^b \pi [g(y)]^2 dy$

Washer method w.r.t $y \Rightarrow \int_a^b \pi [f(y)^2 - g(y)^2]\, dy$

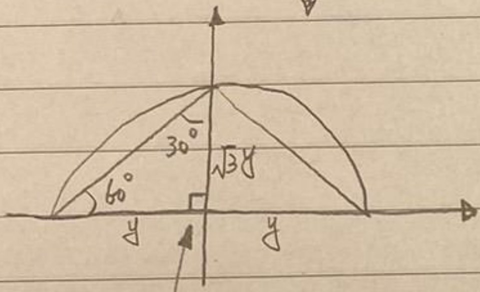
orthogonal (dim ≥ 3)

Ex. Show a solid w/ circular base of radius r
Parallel cross-sections perpendicular to the base are equilateral triangle
垂直
正三角形

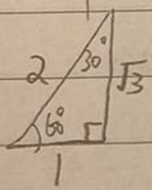


$$y^2+x^2=r^2$$

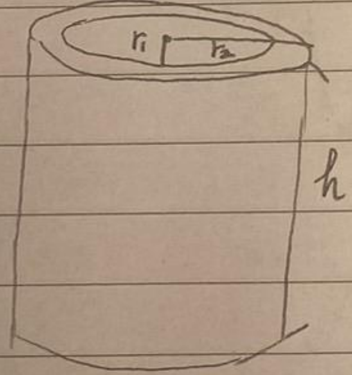
$$A(x)=\frac{1}{2}\,2y\cdot\sqrt{3}\,y$$
$$=\sqrt{3}(r^2-x^2)$$



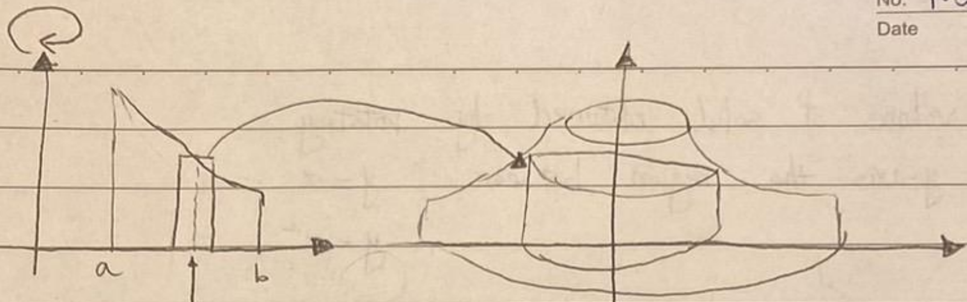
$$V=\int_{-r}^{r}A(x)\,dx$$
$$=\sqrt{3}\left[r^2(2r)-\left(\frac{x^3}{3}\Big|_{-r}^{r}\right)\right]$$
$$=\sqrt{3}\left[2r^3-\frac{2r^3}{3}\right]=\frac{4\sqrt{3}}{3}r^3$$



§ Shell Method



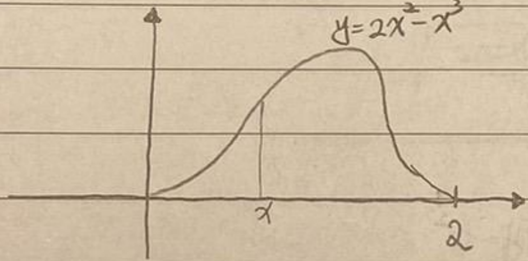
$$V=V_2-V_1=\pi r_2^2h-\pi r_1^2h$$
$$=\pi h(r_2^2-r_1^2)$$
$$=2\pi h\left(\frac{r_2+r_1}{2}\right)(r_2-r_1)$$
$$=\underbrace{2\pi\bar{r}}_{\text{circumference (圓周)}}\ \underbrace{h}_{\text{height}}\ \underbrace{\Delta r}_{\text{thickness}}$$



$$\bar{x}_k = \frac{x_{k-1}+x_k}{2}$$

$$\sum_{k=1}^{n} 2\pi \bar{x}_k f(\bar{x}_k)\,\Delta x_k \xrightarrow{\|P\| \to 0} \int_a^b \underbrace{2\pi x}_{\text{circumference}}\, \underbrace{f(x)}_{\text{height}}\, \underbrace{dx}_{\text{thickness}}$$

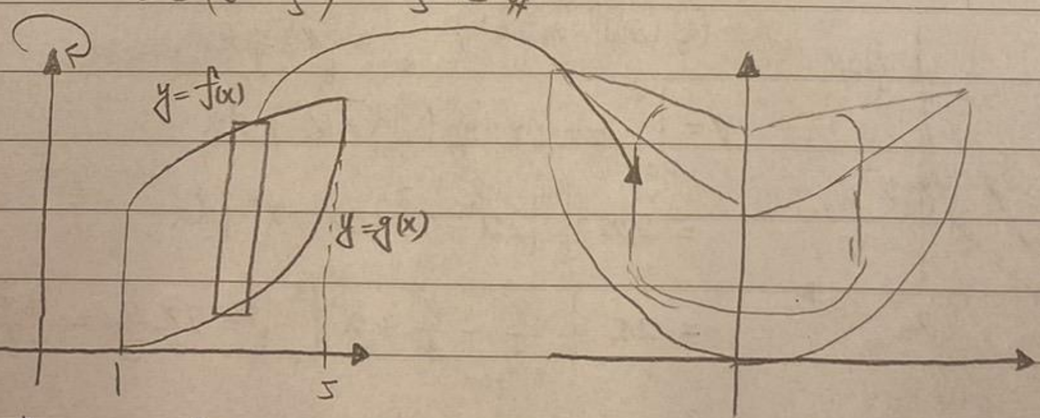
Ex. Volume of solid obtained by rotating about y-axis the region bdd by $\begin{cases} y = 2x^2 - x^3 \\ y = 0 \end{cases}$



$x^2(2-x)$

$$V = \int_0^2 2\pi x(2x^2-x^3)\,dx = 2\pi\left(\frac{2}{4}x^4 - \frac{1}{5}x^5 \Big|_0^2\right)$$

$$= 2\pi\left(8 - \frac{32}{5}\right) = \frac{16}{5}\pi \ \#$$

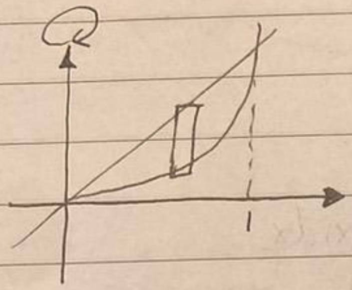


$$\sum_{k=1}^{n} 2\pi \bar{x}_k \left[f(\bar{x}_k) - g(\bar{x}_k)\right] \Delta x_k$$

$$\downarrow \ \|P\| \to 0$$

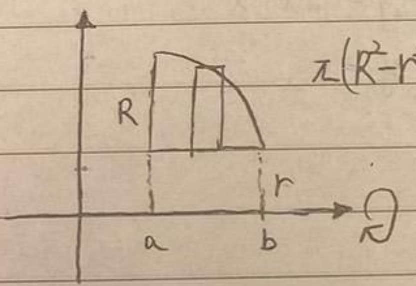
$$\int_a^b 2\pi x\left[f(x) - g(x)\right] dx$$

Ex. Find volume of solid obtained by rotating about y-axis the region between $\begin{cases} y=x \\ y=x^2 \end{cases}$

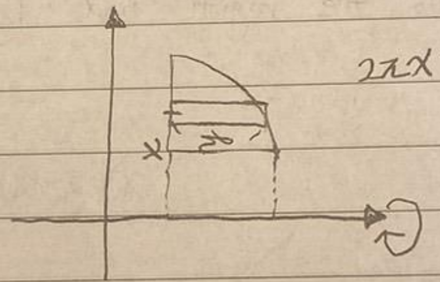


$$V=\int_0^1 2\pi x\left[x-x^2\right]dx$$
$$=2\pi\left(\frac{x^3}{3}-\frac{x^4}{4}\Big|_0^1\right)$$
$$=\frac{2\pi}{12}=\frac{\pi}{6}\ \#$$

- Disk vs Shell



$\pi(R^2-r^2)$



$2\pi x h$

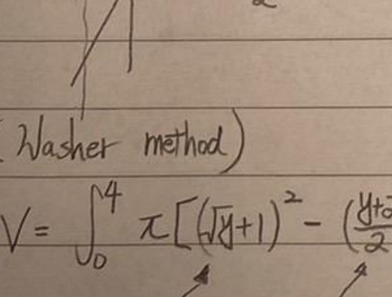
Ex. region in 1st quadrant bdd by $\begin{cases} y=x^2 \\ y=2x \end{cases}$

A solid is formed by rotating the region about $x=-1$.

$$\int_1^3 2\pi x\left[2(x-1)-(x-1)^2\right]dx$$

⇅ 平移

(a) (Shell method)



$$V=\int_0^2 2\pi(x+1)(2x-x^2)\,dx$$
$$=2\pi\int_0^2\left[2x^2-x^3+2x-x^2\right]dx$$
$$=2\pi\left[\frac{x^3}{3}-\frac{x^4}{4}+x^2\Big|_0^2\right]=2\pi\frac{8}{3}=\frac{16\pi}{3}$$

(b) (Washer method)

$$V=\int_0^4\pi\left[(\sqrt{y}+1)^2-\left(\frac{y+2}{2}\right)^2\right]dy=\int_0^4\pi\left[y+2\sqrt{y}+1-\left(1+y+\frac{y^2}{4}\right)\right]dy$$
$$=\pi\int_0^4\left(2\sqrt{y}-\frac{y^2}{4}\right)dy=\pi\left(\frac{4}{3}y^{\frac{3}{2}}-\frac{y^3}{12}\Big|_0^4\right)$$

$y=(x-1)^2$ $\quad$ $y=2(x-1)$

$$=\pi\left(\frac{32}{3}-\frac{16}{3}\right)=\frac{16}{3}\pi$$