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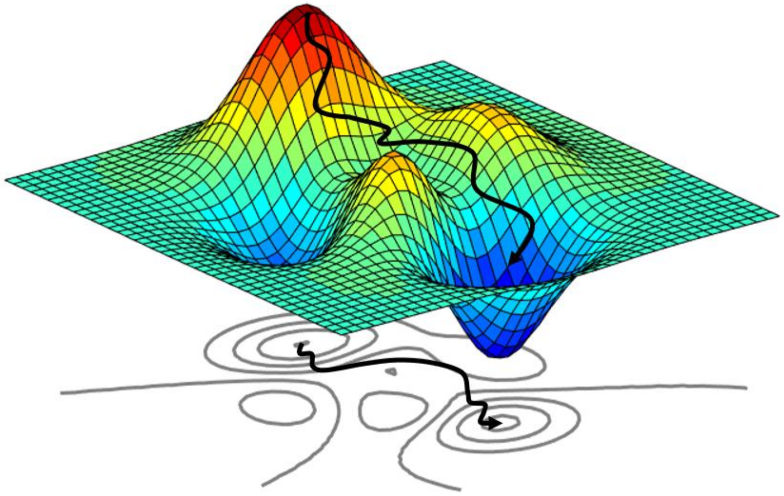
(U4284) Python程式設計
Linear Program



Speaker: 吳淳硯

Behavioral Attenuation

- In Benjamin Enke and colleagues' study in 2024, "Behavioral Attenuation"
  - They investigate how cognitive limitations affect economic decision-making. They conducted over 30 experiments covering areas like investment, taxation, and fairness. The findings reveal that in 93% of cases, people's responsiveness to economic fundamentals diminishes as their cognitive uncertainty increases. This phenomenon, termed "Behavioral attenuation" suggests that individuals often underreact to information, leading to suboptimal decisions.

- Indeed, mathematical optimization has become an important tool for a decision maker. It is used as an aid to decision maker, system designer or system operator, who supervises the process, checks the results, and modifies the problem when necessary.

Mathematical Optimization

- A mathematical optimization problem, has the form

$$\begin{aligned} \min_{\boldsymbol{x}} \ & \phi_0(\boldsymbol{x}) \\ \text{s.t.} \ & \phi_i(\boldsymbol{x}) \le 0, \qquad i = 1, \cdots, m \\ & \psi_i(\boldsymbol{x}) = 0, \qquad i = 1, \cdots, p \end{aligned}$$

  - $\boldsymbol{x} \in \mathbb{R}^n$ is the optimization/decision variable
  - $\phi_0 : \mathbb{R}^n \to \mathbb{R}$ is the objective/cost function.
  - $\phi_i : \mathbb{R}^n \to \mathbb{R}$ are called inequality constraint functions.
  - $\psi_i : \mathbb{R}^n \to \mathbb{R}$ are called equality constraint functions.

- The set of points for which are defined above

$$\mathcal{D} = \left\{ \bigcap_{i=0}^{m} \operatorname{dom}(\phi_i) \right\} \cap \left\{ \bigcap_{i=1}^{p} \operatorname{dom}(\psi_i) \right\}$$

  is called domain of the optimization problem. A point $\boldsymbol{x} \in \mathcal{D}$ is feasible if it satisfies the all constraints. The set of all feasible points is called feasible/constraint set.

Optimality - 1

- The optimal value $\mathcal{V}^*$ of the optimization problem is defined as
$$\mathcal{V}^* = \inf\left\{\phi_0(\boldsymbol{x}) \middle| \begin{matrix}\phi_i(\boldsymbol{x}) \leq 0, i = 1, \cdots, m \\ \psi_i(\boldsymbol{x}) = 0, i = 1, \cdots, p\end{matrix}\right\}$$
We allow $\mathcal{V}^*$ to take on the extended values $\pm\infty$.
  - If the problem is infeasible, $\mathcal{V}^* = \infty$.
  - If there are feasible points $\{\boldsymbol{x}_k\}$ with $\phi_0(\boldsymbol{x}_k) \to \infty$ as $k \to \infty$, then $\mathcal{V}^* = -\infty$. The problem is unbounded below.

- We say $\boldsymbol{x}^*$ is an optimal point if $\boldsymbol{x}^*$ is feasible and $\phi_0(\boldsymbol{x}^*) = \mathcal{V}^*$. The set of all optimal points is the optimal set, denoted
$$\boldsymbol{x}_{\text{opt}} = \left\{\boldsymbol{x} \middle| \begin{matrix}\phi_i(\boldsymbol{x}) \leq 0, i = 1, \cdots, m \\ \psi_i(\boldsymbol{x}) = 0, i = 1, \cdots, p \\ \phi_0(\boldsymbol{x}) = \mathcal{V}^*\end{matrix}\right\}$$

- If there exists an optimal point for the optimization problem, we say the optimal value is attained/achieved, and the problem is solvable.

Optimality - 2

- A feasible point $\boldsymbol{x}$ with
$$\phi_0(\boldsymbol{x}) \leq \mathcal{V}^* + \varepsilon, \qquad \varepsilon > 0$$
is called ε-suboptimal, and the set of all ε-suboptimal points is called the ε-suboptimal set.

- A feasible point $\boldsymbol{x}$ is locally optimal if $\exists r > 0$ such that
$$\phi_0(\boldsymbol{x}) = \inf\left\{ \phi_0(\boldsymbol{z}) \,\middle|\, \begin{array}{c} \phi_i(\boldsymbol{z}) \leq 0, i = 1, \cdots, m \\ \psi_i(\boldsymbol{z}) = 0, i = 1, \cdots, p \\ \|\boldsymbol{z} - \boldsymbol{x}\|_{\ell^2} \leq r \end{array} \right\}$$

- Ex. We illustrate these definition with a few simple unconstraint optimization problems with $x \in \mathbb{R}$ and $\mathrm{dom}(\phi_0) = \mathbb{R}^+$.
  - $\phi_0(x) = 1/x \Rightarrow \mathcal{V}^* = 0$, but optimal value is not achieve.
  - $\phi_0(x) = -\ln x \Rightarrow \mathcal{V}^* = -\infty$, so this is unbounded below.
  - $\phi_0(x) = x \ln x \Rightarrow \mathcal{V}^* = -1/e$, achieved at the unique optimal points $x^* = 1/e$.

Change of variables

- Let $h: \mathbb{R}^n \to \mathbb{R}$ is 1-1 with
$$h\big(\mathrm{dom}(h)\big) \supseteq \mathcal{D}$$
- Consider the problem
$$\begin{aligned} &\min_x \phi_0(\boldsymbol{x}) \\ &\text{s.t. } \phi_i(\boldsymbol{x}) \leq 0, \qquad i = 1, \cdots, m \\ &\qquad \psi_i(\boldsymbol{x}) = 0, \qquad i = 1, \cdots, p \end{aligned}$$
- Change variable to $\boldsymbol{z}$ by $\boldsymbol{x} = h(\boldsymbol{z})$
$$\begin{aligned} &\min_x \tilde{\phi}_0(\boldsymbol{z}) \\ &\text{s.t. } \tilde{\phi}_i(\boldsymbol{z}) \leq 0, \qquad i = 1, \cdots, m \\ &\qquad \tilde{\psi}_i(\boldsymbol{z}) = 0, \qquad i = 1, \cdots, p \end{aligned}$$
where $\tilde{\phi}_i(\boldsymbol{z}) = \phi_i\big(h(\boldsymbol{z})\big)$ and $\tilde{\psi}_i(\boldsymbol{z}) = \psi\big(h(\boldsymbol{z})\big)$
- The two problems are clearly equivalent:
$$\boldsymbol{x}^* = h(\boldsymbol{z}^*)$$

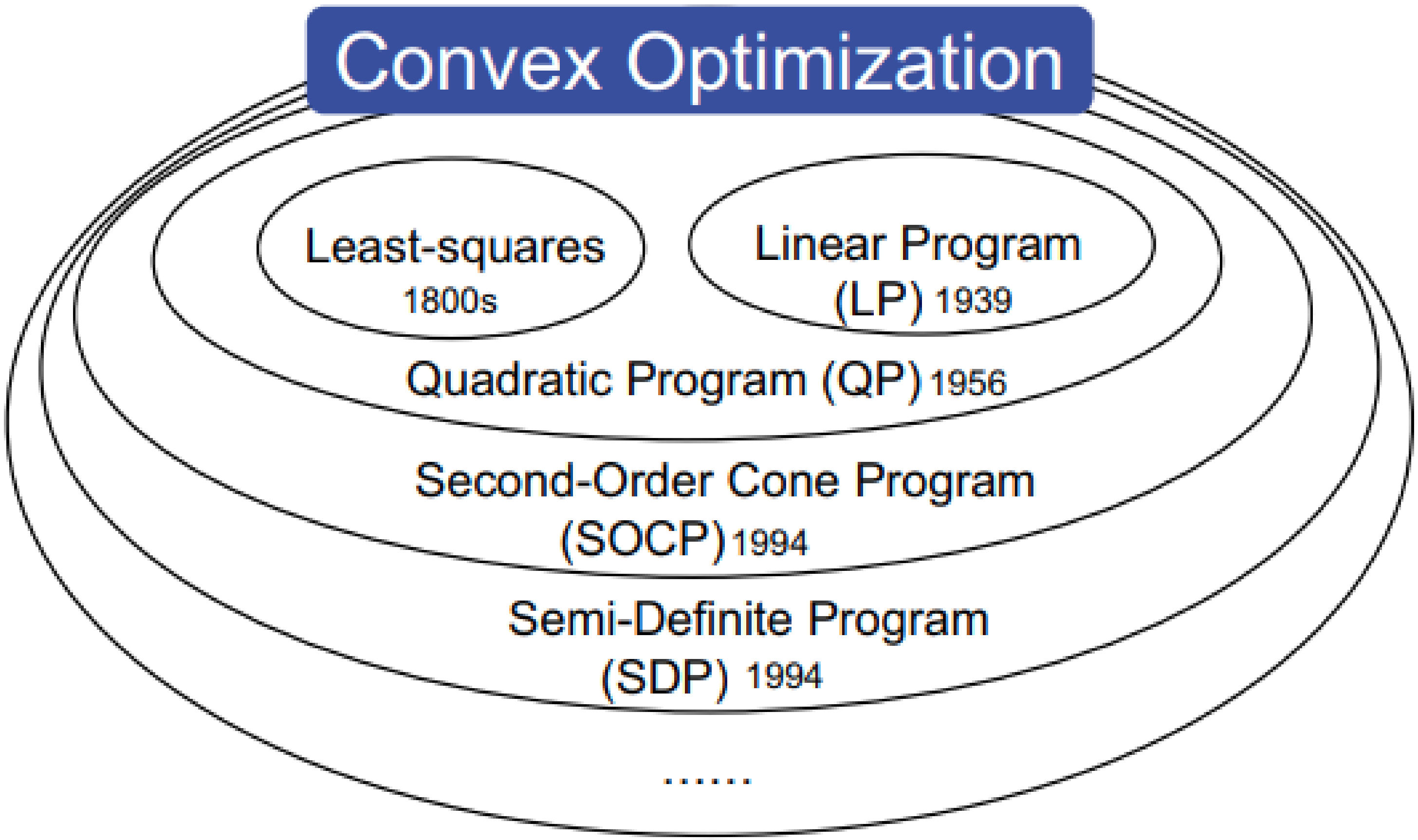
Application

- Portfolio optimization:
  Seek the best way to invest some capital in a set of n assets.
  - x_i is the investment in the i^{th} asset.
  - $\boldsymbol{x}$ is the overall portfolio allocation across the set of assets.
  - The constraints $\{\phi_i, i = 1, \dots, m\}$ represent a limit on the budget.
  - Investments are nonnegative (short position is not allowed).
  - The objective ϕ_0 might be a risk measure of the portfolio return.

- Data fitting:
  Find a model that best fit some observed data and prior information.
  - $\boldsymbol{x}$ are the parameters in the model.
  - The constraints $\{\phi_i, i = 1, \dots, m\}$ represent required limits on the parameters.
  - The objective ϕ_0 might be a measure of misfit or prediction error between the observed data and the values predicted by the model.

A class of tractable optimization problems

- More and more tractable problems have been developed so far. It turns out that all of the tractable problems share the common property, and this property established the class of tractable problems, named convex optimization.

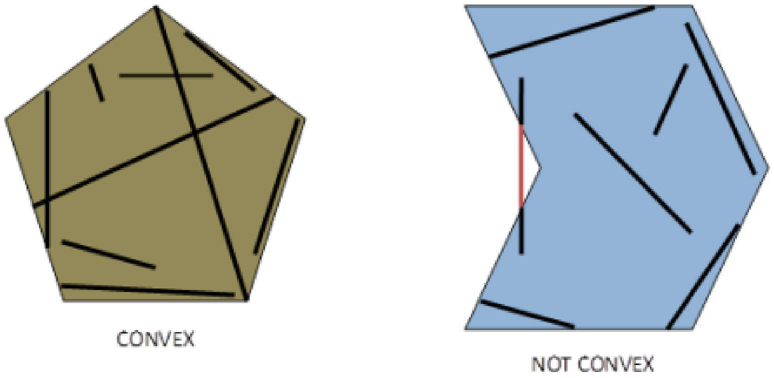


Some Basic - 1

- Let $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_k \in \mathbb{R}^n$ and $\theta_1, \cdots, \theta_k \in \mathbb{R}$. Then
$$\sum_{i=1}^{k} \theta_k \boldsymbol{x}_k$$
is called a linear combination of vectors $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_k$. It is further a
  - Conic combination ($\theta_i \geq 0$).
  - Affine combination ($\theta_1 + \cdots + \theta_k = 1$).
  - Convex combination = Conic combination + Affine combination.
- Affine set $\mathcal{A}$
  - A set is affine if and only if contains every affine combination of its points.
- The affine hull of a set A
$$\mathbf{aff}(A) = \{\theta_1 \boldsymbol{x}_1 + \cdots + \theta_k \boldsymbol{x}_k | \boldsymbol{x}_1, \cdots, \boldsymbol{x}_k \in A, \theta_1 + \cdots + \theta_k = 1\}$$
It is the smallest affine set that contains A. If $\mathcal{A}$ is any affine set with $A \subseteq \mathcal{A}$ then $\mathbf{aff}(A) \subseteq \mathcal{A}$.

Some Basic - 2

- Convex set $\mathcal{C}$
  - A set is convex if and only if contains every convex combination of its points.
  - Ex.



- The convex hull of a set C

$$\mathbf{conv}(C) = \left\{ \theta_1 \boldsymbol{x}_1 + \cdots + \theta_k \boldsymbol{x}_k \,\middle|\, \begin{matrix} \boldsymbol{x}_k \in C, \theta_i \geq 0, i = 1, \cdots, k \\ \theta_1 + \cdots + \theta_k = 1 \end{matrix} \right\}$$

It is the smallest convex set that contains C. If $\mathcal{C}$ is any convex set that contains C then $\mathbf{conv}(C) \subseteq \mathcal{C}$.

- Similarly the conic hull of a set C

$$\mathbf{cone}(C) = \{\theta_1 \boldsymbol{x}_1 + \cdots + \theta_k \boldsymbol{x}_k | \boldsymbol{x}_k \in C, \theta_i \geq 0, i = 1, \cdots, k\}$$

Some Basic - 3

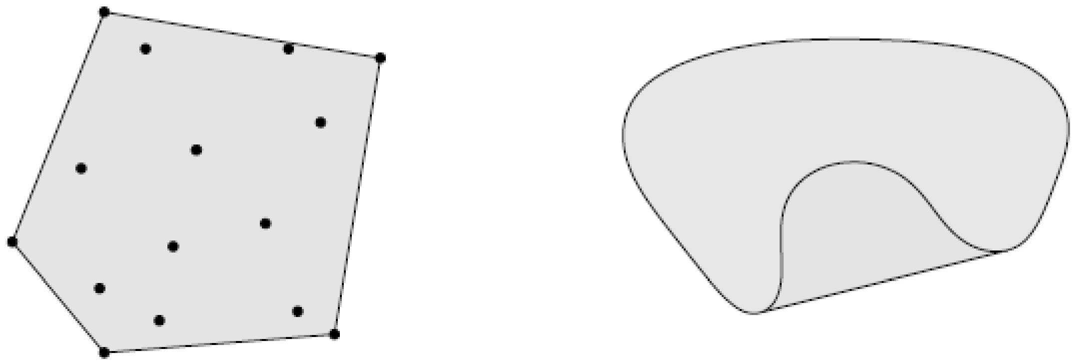


Figure 2.3 The convex hulls of two sets in $\mathbf{R}^2$. *Left.* The convex hull of a set of fifteen points (shown as dots) is the pentagon (shown shaded). *Right.* The convex hull of the kidney shaped set in figure 2.2 is the shaded set.

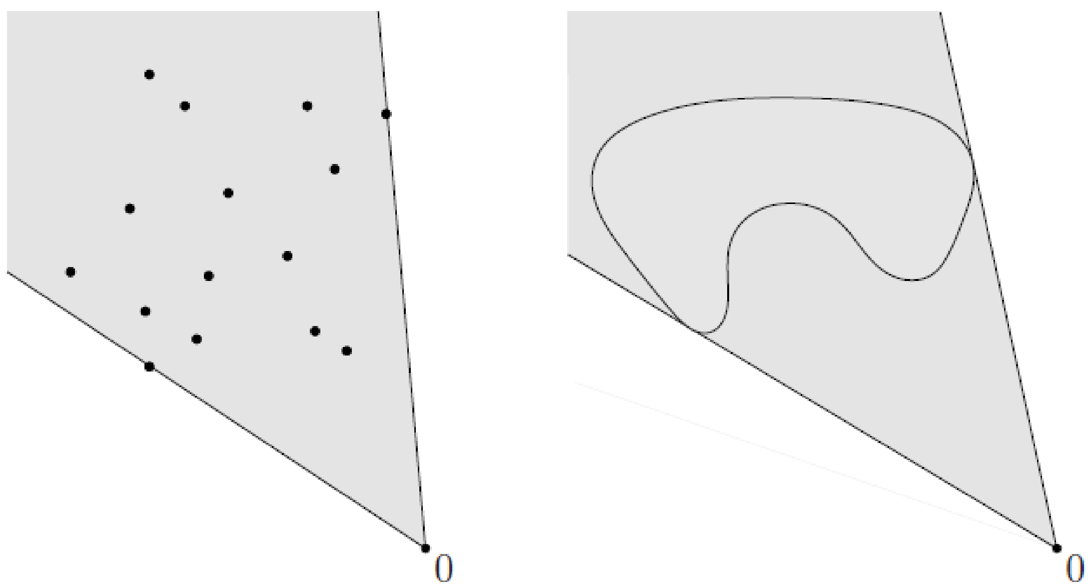
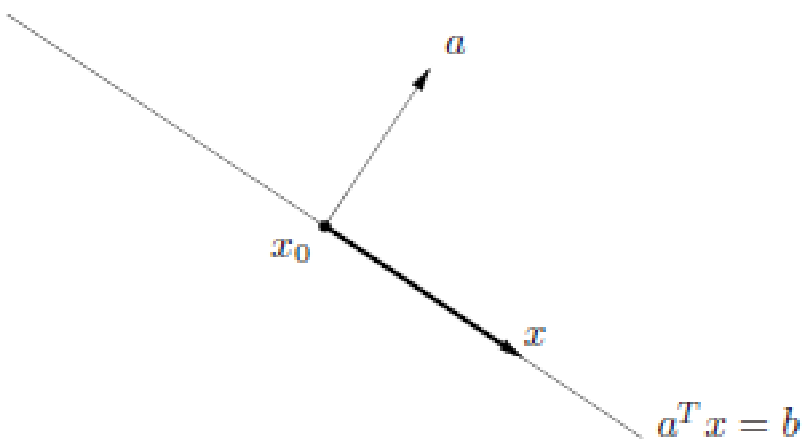
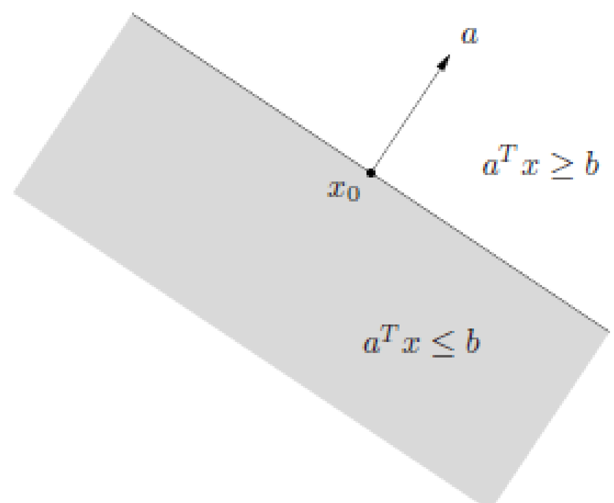


Figure 2.5 The conic hulls (shown shaded) of the two sets of figure 2.3.

Some Basic - 4

- $\boldsymbol{a} \in \mathbb{R}^n, \boldsymbol{a} \neq 0$ and $b \in \mathbb{R}$
  - (closed) Halfspaces is set of the form
    $$\mathcal{H}^- = \{\boldsymbol{x} | \boldsymbol{a}^\top \boldsymbol{x} \leq b\}, \qquad \mathcal{H}^+ = \{\boldsymbol{x} | \boldsymbol{a}^\top \boldsymbol{x} \geq b\}$$
  - A Hyperplane divides $\mathbb{R}^n$ into two halfspaces
    $$\mathcal{H} = \{\boldsymbol{x} | \boldsymbol{a}^\top \boldsymbol{x} = b\} = \mathcal{H}^+ \cap \mathcal{H}^-$$
  - The boundary of halfspace is hyperplane.
- They are all convex sets, and $\boldsymbol{a}$ is called the normal vector.



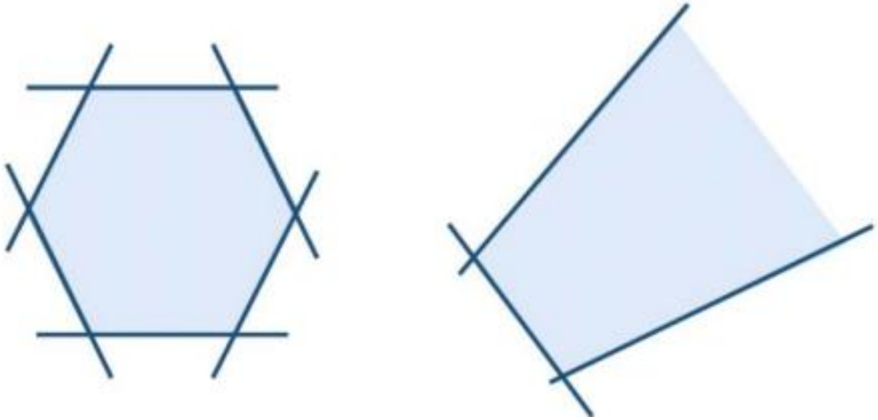


Polyhedron

- A Polyhedron is solution set of finitely many linear inequalities and equalities

$$\mathcal{P} = \left\{ \boldsymbol{x} \,\middle|\, \begin{matrix} \boldsymbol{a}_i^\top \boldsymbol{x} \le b_i, i = 1, \cdots, m \\ \boldsymbol{c}_i^\top \boldsymbol{x} \le d_i, i = 1, \cdots, p \end{matrix} \right\} = \{\boldsymbol{x} | \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b}, \boldsymbol{C}\boldsymbol{x} = \boldsymbol{d}\}$$

- A polyhedron is thus the intersection of finite number of half spaces and hyperplanes. Polyhedron are convex sets. A bounded $\mathcal{P}$ is sometimes called polytope.



- Ex. Nonnegative orthant is polyhedral cone.

$$\mathbb{R}_+^n = \{\boldsymbol{x} \in \mathbb{R}^n | \boldsymbol{x} \ge 0\}$$

Preserve Convexity

- The intersection of (any number of) convex sets is convex.

$$\mathcal{S}_\alpha \text{ is convex for every } \alpha \in \mathbb{A} \Rightarrow \bigcap_{\alpha \in \mathbb{A}} \mathcal{S}_\alpha \text{ convex}$$

- Affine mappings:

  $f : \mathbb{R}^n \to \mathbb{R}^m$ and $f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$ with $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$

  - The image of a convex set under f is convex.

    $\mathcal{S} \subseteq \mathbb{R}^n$ convex $\Rightarrow f(\mathcal{S}) = \{f(\boldsymbol{x}) | \boldsymbol{x} \in \mathcal{S}\}$ convex

  - The inverse image f^{-1} of a convex set under f is convex.

    $\mathcal{S}^* \subseteq \mathbb{R}^m$ convex $\Rightarrow f^{-1}(\mathcal{S}^*) = \{\boldsymbol{x} \in \mathbb{R}^n | f(\boldsymbol{x}) \in \mathcal{S}^*\}$ convex

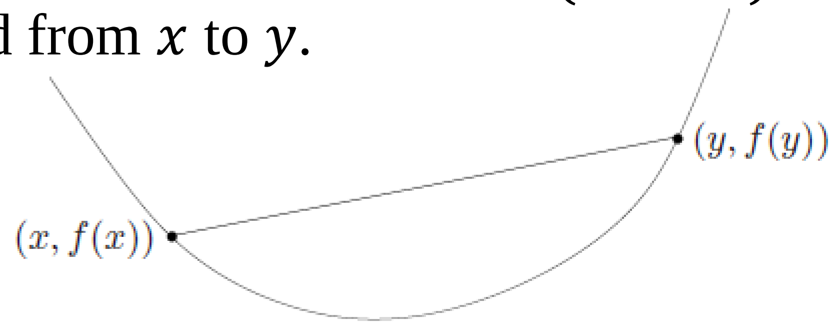
- Ex. $\mathcal{S} \subseteq \mathbb{R}^n$ is convex, $\alpha \in \mathbb{R}$, $\boldsymbol{a} \in \mathbb{R}^n$
  - $\alpha\mathcal{S}$ is convex. (Scaling)
  - $\mathcal{S} + \boldsymbol{a}$ is convex. (Translation)
  - $\{x | (x, y) \in \mathcal{S}\}$ is convex. (Projection onto some coordinates)

Convex function

- $f: \mathbb{R}^n \to \mathbb{R}$ is convex if $\mathrm{dom}(f)$ is a convex set and $\forall x, y \in \mathrm{dom}(f)$, $\theta \in [0,1]$

$$f(\theta \boldsymbol{x} + (1-\theta)\boldsymbol{y}) \leq \theta f(\boldsymbol{x}) + (1-\theta) f(\boldsymbol{y})$$

- Geometrically, the line segment between $\big(x, f(x)\big)$ and $\big(y, f(y)\big)$, which is the chord from x to y.



- f is concave if $-f$ is convex.
- Ex.
  - Affine. ($ax + b$ on $\mathbb{R}$, for any $a, b \in \mathbb{R}$)
  - Exponential. (e^{ax} for any $a \in \mathbb{R}$)
  - Powers. (x^{α} on $\mathbb{R}^{+}$ for $\alpha \geq 1$ or $\alpha \leq 0$)
  - Positive part (Relu). ($\max\{0, x\}$)

Convex Optimization - 1

- The convex problem has three additional requirement:
  - ϕ_0 must be convex.
  - ϕ_i must be convex.
  - $\psi_i(\boldsymbol{x}) = \boldsymbol{a}_i^\top \boldsymbol{x} - b_i$ must be affine.
- The feasible and optimal set of a convex optimization are convex, since it is the intersection of the domain of the problem
$$\mathcal{D} = \bigcap_{i=0}^{m} \mathrm{dom}(\phi_i)$$
Thus, the convex optimization minimize a convex objective function over a convex set.
- The form of convex optimization is
$$\begin{aligned} &\min_{x} \phi_0(\boldsymbol{x}) \\ &\text{s.t. } \phi_i(\boldsymbol{x}) \leq 0, i = 1, \cdots, m \\ &\qquad \boldsymbol{a}_i^\top \boldsymbol{x} = b_i, i = 1, \cdots, p \end{aligned}$$

Convex Optimization - 2

- Ex

$$\begin{aligned} &\min \phi_0(x) = x_1^2 + x_2^2 \\ &\text{s.t. } \phi_1(x) = x_1/(1 + x_2^2) \le 0 \\ &\qquad \psi_1(x) = (x_1 + x_2)^2 = 0 \end{aligned}$$

  - ϕ_0 is convex, feasible set $\{(x_1, x_2): x_1 = -x_2 \le 0\}$ is convex.
  - It is not a convex problem: ϕ_1 is not convex, ψ_1 is not affine.

- Reformulated as

$$\begin{aligned} &\min \phi_0(x) = x_1^2 + x_2^2 \\ &\text{s.t. } \phi_1(x) = x_1 \le 0 \\ &\qquad \psi_1(x) = x_1 + x_2 = 0 \end{aligned}$$

  which is in standard convex optimization form, since ϕ_0 and ϕ_1 are convex, ψ_1 is affine.

- A convex optimization problem is not just one of minimizing a convex function over a convex set; it is also required that the feasible set be described specifically by a set of inequalities involving convex functions, and a set of linear equality constraints.

Local and Global optima

- Any locally optimal point of convex problem is optimal.
  (Proof)

  Suppose $\boldsymbol{x}$ is locally optimal and $\boldsymbol{y}$ is optimal
  $$\phi_0(\boldsymbol{y}) < \phi_0(\boldsymbol{x})$$
  $\exists r > 0$ such that
  $$\|\boldsymbol{z} - \boldsymbol{x}\|_{\ell^2} \leq r \Rightarrow \phi_0(\boldsymbol{z}) \geq \phi_0(\boldsymbol{x})$$
  Consider
  $$\boldsymbol{z} = \theta \boldsymbol{y} + (1-\theta)\boldsymbol{x}, \qquad \theta = \frac{r}{2\|\boldsymbol{y} - \boldsymbol{x}\|_{\ell^2}}$$

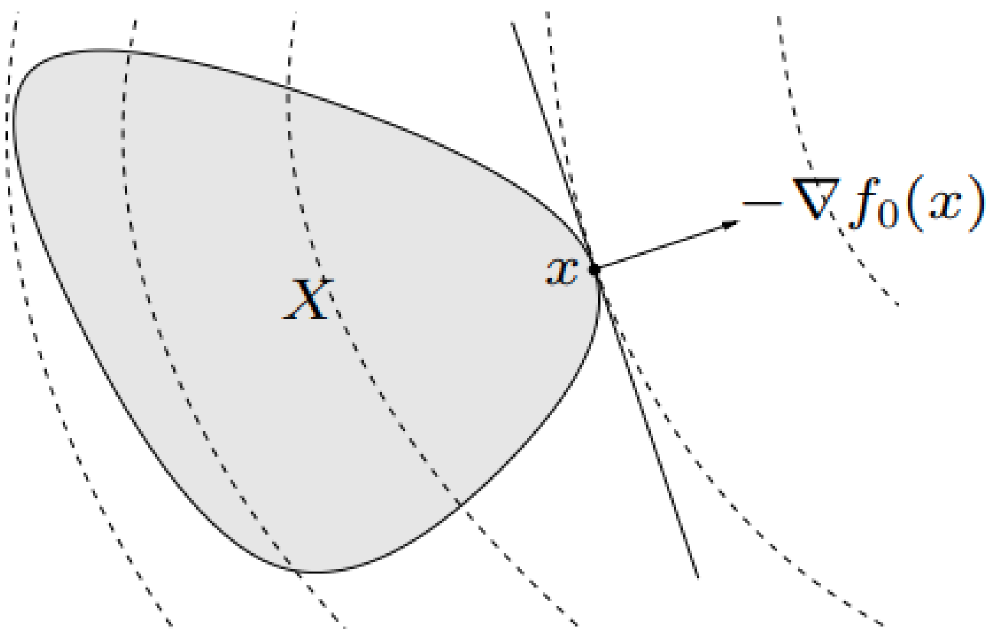
  - $\|\boldsymbol{y} - \boldsymbol{x}\|_{\ell^2} > r$, so $\theta \in (0, 1/2)$
  - $\boldsymbol{z}$ is a convex combination of two feasible points, hence also feasible.
  - $\|\boldsymbol{z} - \boldsymbol{x}\|_{\ell^2} = r/2$

  $$\phi_0(\boldsymbol{z}) \leq \theta \phi_0(\boldsymbol{y}) + (1-\theta)\phi_0(\boldsymbol{x}) < \phi_0(\boldsymbol{x})$$
  which contradicts our assumption that $\boldsymbol{x}$ is locally optimal.

Optimality criterion for differentiable ϕ_0

- $\boldsymbol{x}$ is optimal for a convex problem $\Leftrightarrow$ it is feasible and

$$\nabla\phi_0(\boldsymbol{x})^\top(\boldsymbol{y}-\boldsymbol{x}) \geq 0 \; \forall \text{ feasible } \boldsymbol{y}$$



- If nonzero, $\nabla\phi_o(\boldsymbol{x})$ defines a supporting hyperplane to feasible set $\boldsymbol{X}$ at $\boldsymbol{x}$

What is CVXPY?

- CVXPY is a python-embedded modeling language for convex optimization problems. It converts problems into standard form known as conic form, a generalization of a linear program.
- Note that for a minimization problem the

$$\text{optimal value is } \begin{cases} \text{inf if infeasible} \\ -\text{inf if unbounded} \end{cases}$$

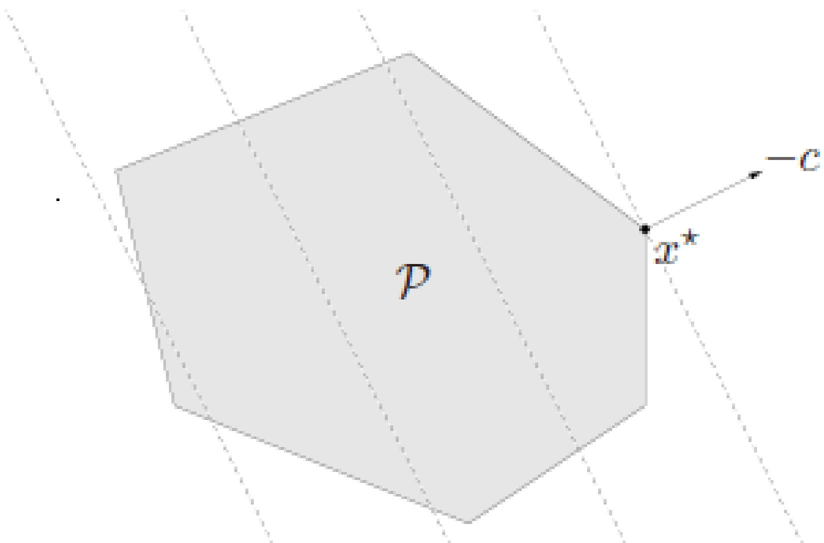
  for maximization problems the opposite is true.
- If the solver called by CVXPY solves the problem but to a lower accuracy than desired, the problem status indicate the lower accuracy achieved.
- It provides the different status strings.
  - optimal, optimal_inaccurate.
  - infeasible, infeasible_inaccurate.
  - unbounded, unbounded_inaccurate.
  - infeasible_or_unbounded.

Setting

- Variables:
  - scalars, vectors or matrices. (but not higher dimensions)
- Constraints:
  - Elementwise ==, >=, <=.
  - Semidefinite >>, <<.
- Parameters: constants for a given problem instance: no need to rewrite the code when the values change.
- Computing trade-off curves is a common use of parameters. Ex. Lasso.
- Operators: the same as NumPy. Ex. +, -, *, /, @, .T, **.
- Function: most usual one are already in-built.
- Solver: it supports several solvers
  - ECOS, OSQP, SCS, GLPK, MOSEK, GUROBI…etc.

Linear Program

- The objective and all constraint functions are all affine, the problem is called linear program (LP). A general LP has the form
$$\begin{aligned} \min\ & \boldsymbol{c}^\top \boldsymbol{x} + d \\ \text{s.t.}\ & \boldsymbol{G}\boldsymbol{x} \leq \boldsymbol{h} \\ & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{aligned}$$
where $\boldsymbol{G} \in \mathbb{R}^{m \times n}$ and $\boldsymbol{A} \in \mathbb{R}^{p \times n}$. LP are, of course, convex optimization. It is common to omit the constant d.
- The feasible set of the LP is a polyhedron; the problem is to minimize the affine function $\boldsymbol{c}^\top \boldsymbol{x} + d$ over the polyhedron.



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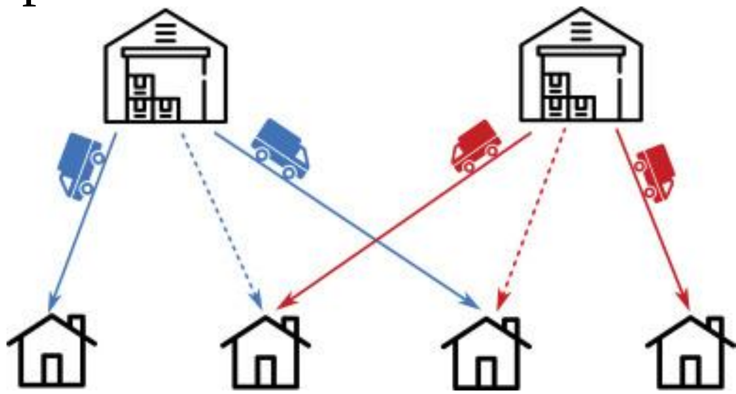
Application of LP

- Goal: Minimize the total cost of transporting product A while still fulfilling the demand from the warehouses and without exceeding the supply produced by the plants.
  - $x_{i,j}$: Quantity transported from i to j.
    $c_{i,j}$: The cost of transporting one unit of product A
  - Objective function to minimize

$$\phi_0(\boldsymbol{x}) = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j}x_{i,j}$$



  - Not exceed the supply in any factory:

$$x_{i,1} + \cdots + x_{i,n} \leq s_i \ \forall i = 1, \cdots, m$$

  - Fulfill needs of the warehouses:

$$x_{1,j} + \cdots + x_{m,j} \geq d_j \ \forall j = 1, \cdots, n$$

  - Quantity transported must be nonnegative:

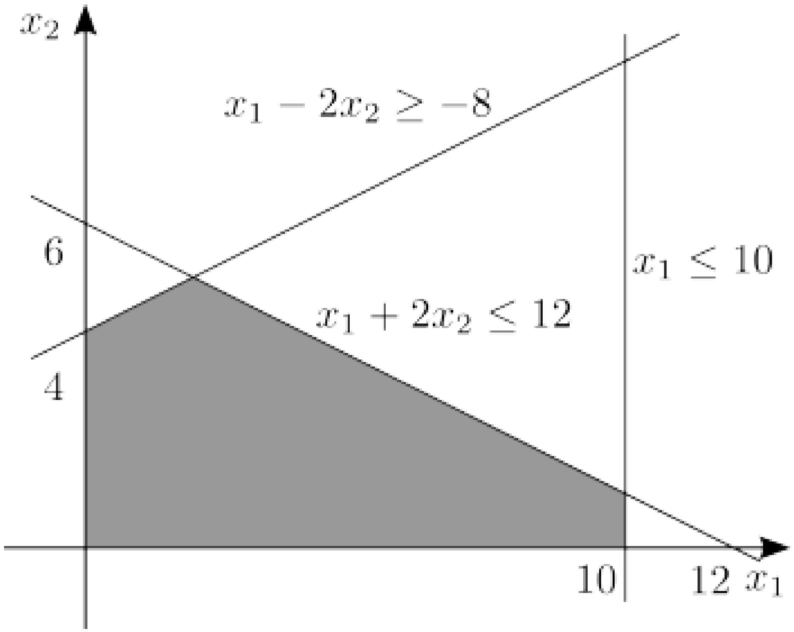
$$x_{i,j} \geq 0 \ \forall i, j$$

Graphical approach - 0

- For LP with only two decision variables, we may solve them with graphical approach.

- Consider the following example.
  Step 1 : Draw the feasible region

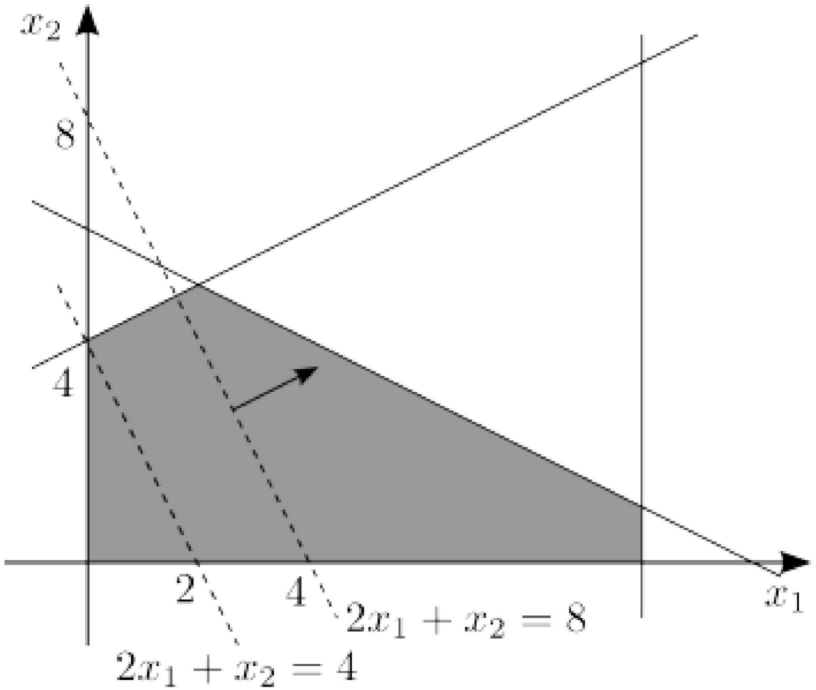
$$\begin{array}{llcrcr} \max & 2x_1 & + & x_2 & & \\ \text{s.t.} & x_1 & & & \leq & 10 \\ & x_1 & + & 2x_2 & \leq & 12 \\ & x_1 & - & 2x_2 & \geq & -8 \\ & x_1 & & & \geq & 0 \\ & & & x_2 & \geq & 0. \end{array}$$



Graphical approach - 1

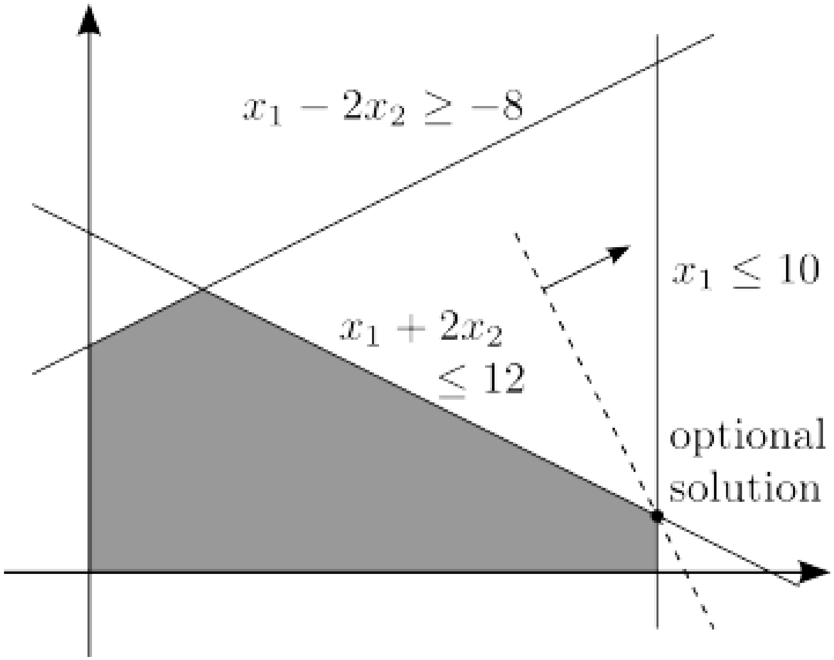
- Step 2 : Draw some isoquant lines
  - A line such that all points on it result in the same objective value.
- Step 3 : Indicate the direction to push the isoquant line
  - The direction that decreases/increases the objective value for a minimization/maximization problem.

$$\begin{array}{lrcrcr} \max & 2x_1 & + & x_2 & & \\ \text{s.t.} & x_1 & & & \leq & 10 \\ & x_1 & + & 2x_2 & \leq & 12 \\ & x_1 & - & 2x_2 & \geq & -8 \\ & x_1 & & & \geq & 0 \\ & & & x_2 & \geq & 0. \end{array}$$



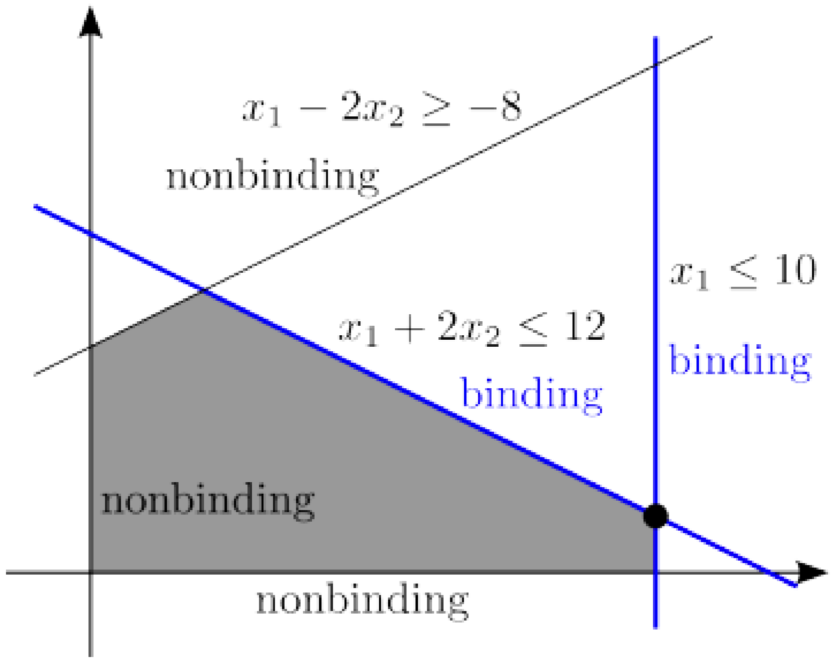
Graphical approach - 2

- Step 4 : Push the isoquant line to the end of the feasible region.
  - Stop when any further step makes all points on the isoquant line infeasible



Graphical approach - 3

- Step 5 : Identify the binding constraints at the optimal solution.



- Set the binding constraints to equalities and the solve the linear system.

$$\left[\begin{array}{cc|c} 1 & 0 & 10 \\ 1 & 1 & 12 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 10 \\ 0 & 2 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 10 \\ 0 & 1 & 1 \end{array}\right]$$

and obtain an optimal solution $(x_1^*, x_2^*) = (10,1)$

Re-formulate

- Some applications lead directly to LP form. Some cases the original problem does not have a standard LP form but can transformed to an equivalent LP form.
- Consider the Chebyshev approximation problem with its nondifferentiable objective

$$\min_{\boldsymbol{x}} \left\{ \max_{i=1,\cdots,k} \left| \boldsymbol{a}_i^\top \boldsymbol{x} - \boldsymbol{b}_i \right| \right\}$$

Here $\boldsymbol{x} \in \mathbb{R}^n$ is the variable, $a_1, \cdots, a_k \in \mathbb{R}^n$, $b_1, \cdots, b_k \in \mathbb{R}$ are parameters.

- Chebyshev approximation problem can be formulated and solved as a LP form.

$$\min_{\boldsymbol{t}} \boldsymbol{t}$$

$$\text{subject to} \begin{cases} \boldsymbol{a}_i^\top \boldsymbol{x} - \boldsymbol{t} \leq \boldsymbol{b}_i \\ -\boldsymbol{a}_i^\top \boldsymbol{x} - \boldsymbol{t} \leq -\boldsymbol{b}_i \end{cases}, \qquad i = 1, \cdots, k$$

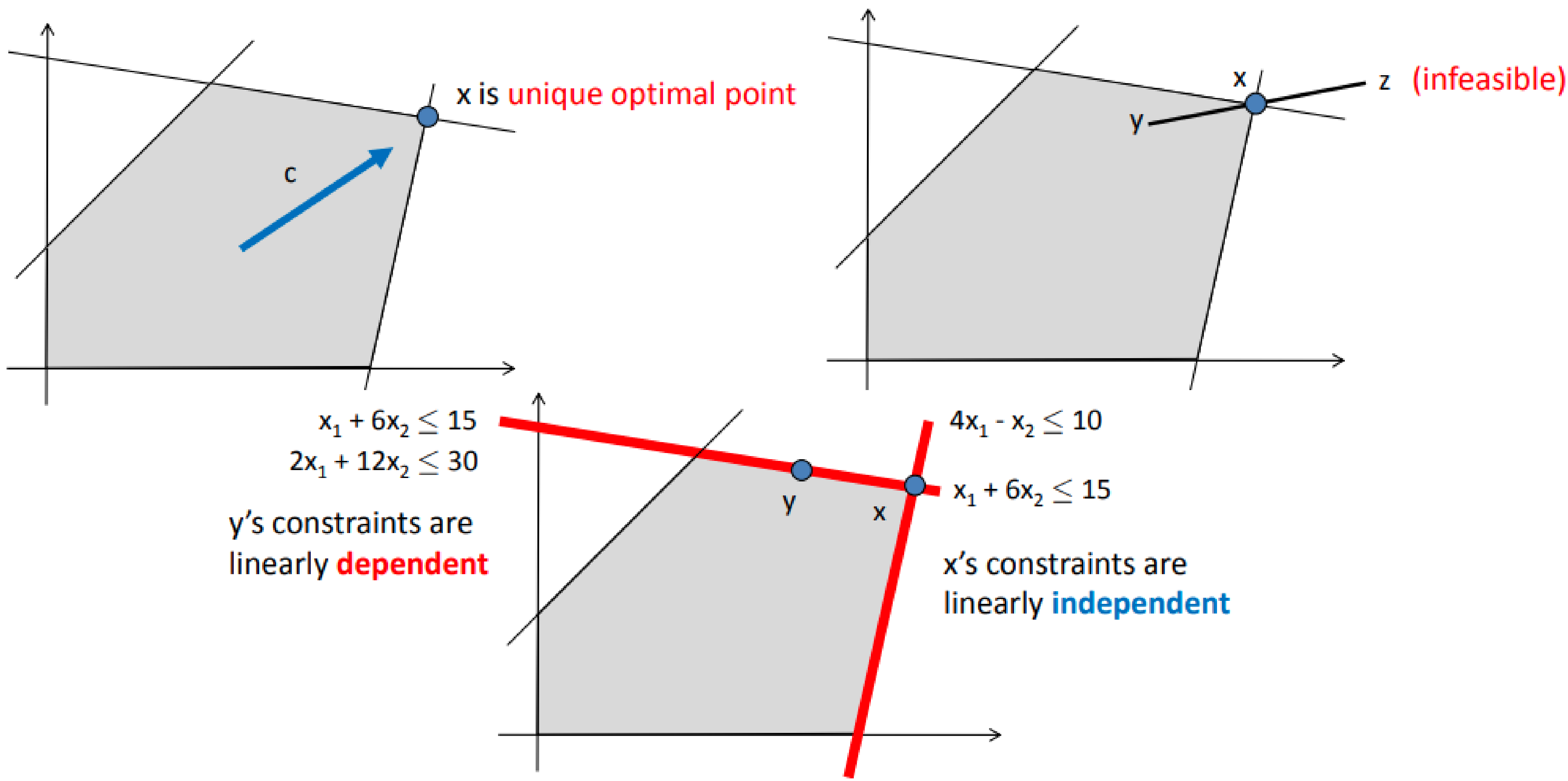
with variable $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{t} \in \mathbb{R}$.

Corner Points

- (Convex Analysis)
  A point $x \in \mathcal{P}$ is a extreme point if x is not a interior point of any line segment contained in $\mathcal{P}$.
  - $\nexists \boldsymbol{y}, \boldsymbol{z} \in \mathcal{P} \setminus \{\boldsymbol{x}\}, \boldsymbol{y} \neq \boldsymbol{x}, \boldsymbol{z} \neq \boldsymbol{x}$ and $\theta \in [0,1]$ such that
  $$\boldsymbol{x} = \theta \boldsymbol{y} + (1 - \theta)\boldsymbol{z}$$
- (Linear Program)
  A point $x \in \mathcal{P}$ is a Basic Feasible Solution (BFS) if there exist n linearly independent $\boldsymbol{a}_i$ with $\boldsymbol{a}_i^\top \boldsymbol{x} = b_i$.
- (Polyhedral Geometry)
  A point $\boldsymbol{x} \in \mathcal{P}$ is a vertex if there exists a linear function $\boldsymbol{c}^\top \boldsymbol{x}$ for some $\boldsymbol{c} \in \mathbb{R}^n$ such that
  $$\boldsymbol{c}^\top \boldsymbol{x} < \boldsymbol{c}^\top \boldsymbol{y} \; \forall \boldsymbol{y} \in \mathcal{P} \setminus \{\boldsymbol{x}\}$$
- Let $\mathcal{P}$ be polyhedron. The following are equivalent
  $$\text{vertex} \Leftrightarrow \text{extreme point} \Leftrightarrow \text{BFS}$$

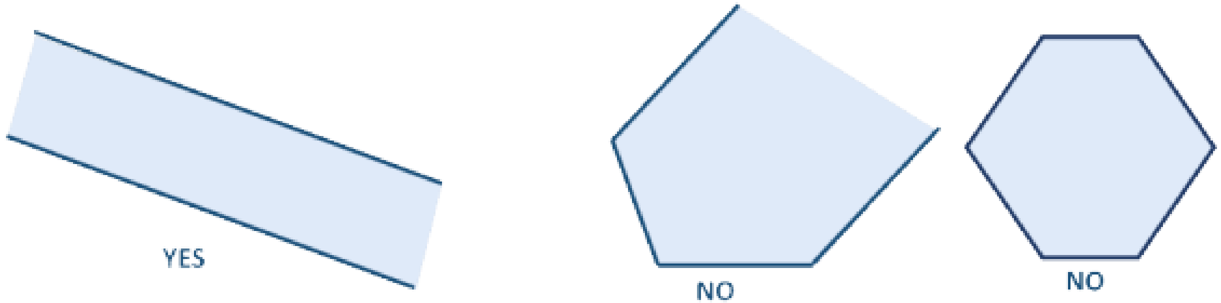
Visual of corner point

- Vertex: $\boldsymbol{x}$ is the farthest point in some direction.
- Extreme point: there is no feasible line segment that goes through $\boldsymbol{x}$ in both direction
- BFS: $\boldsymbol{x}$ lies on the boundary of many linearly independent constraints.



Fundamental theorem of LP

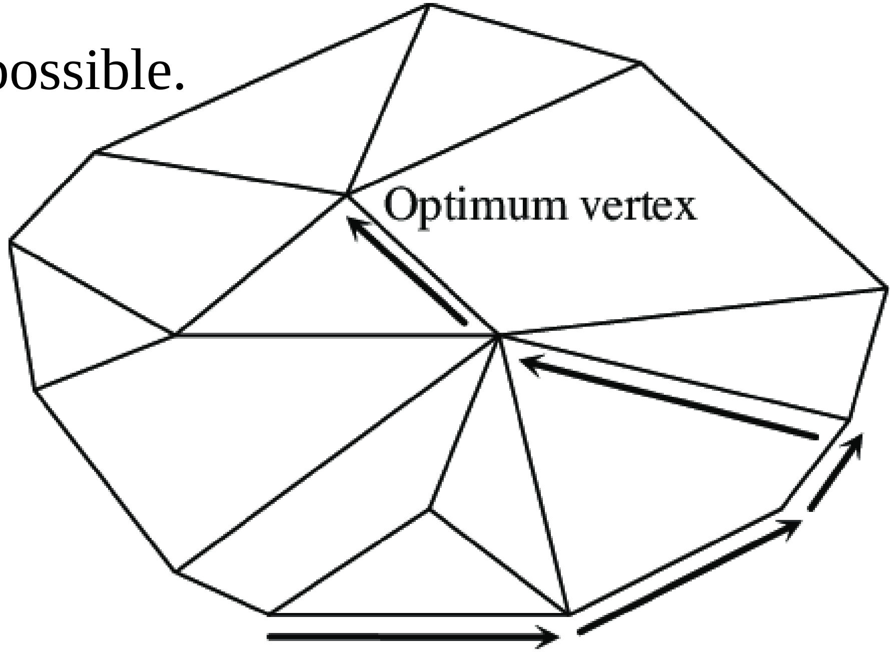
- A polyhedron contains a line if $\exists \boldsymbol{x} \in \mathcal{P}$ and $d \in \mathbb{R}^n, d \neq 0$ such that
$$\boldsymbol{x} + \lambda \boldsymbol{d} \in \mathcal{P} \ \forall \lambda \in \mathbb{R}$$



- Consider a nonempty polyhedron $\mathcal{P}$. The following are equivalent
  - $\mathcal{P}$ does not contain a line.
  - $\mathcal{P}$ has at least one vertex.
- (Fundamental theorem of linear program)
Consider the problem
$$\min_{\boldsymbol{x}} \boldsymbol{c}^\top \boldsymbol{x} \ \text{ subject to } \boldsymbol{x} \in \mathcal{P}$$
where $\mathcal{P}$ is a bounded polyhedron (polytope) and x^* is an optimal solution to the problem, then $\boldsymbol{x}^*$ is either an extreme point (vertex) of $\mathcal{P}$.

Simplex method

- Consequences
  - We only need to search among vertices not the entire polyhedron.
  - The number of vertices is finite, so Simplex can navigate this discrete set.
- Simplex method strategy
  - Start at a feasible vertex.
  - Identifying a neighboring vertex that improves the objective.
  - Move to that vertex.
  - Repeat until no improvement is possible.



Brewery problem

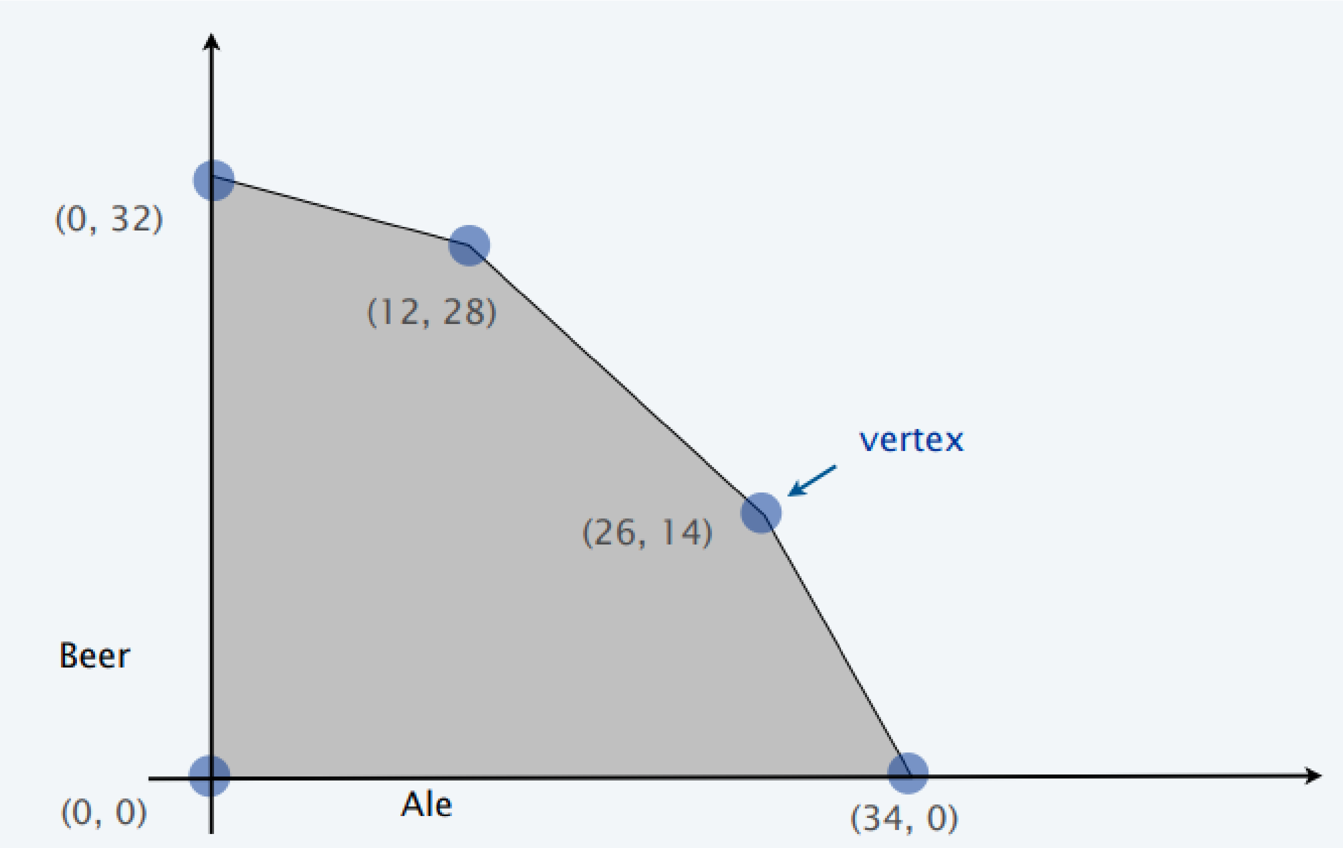
- Small brewery produces ale and beer.
  - Production limit by scarce resources : corn, hops, barley malt
  - Recipes for ale and beer require different proportions of resources

Beverage	Corn (pounds)	Hops (ounces)	Malt (pounds)	Profit ($)
Ale (barrel)	5	4	35	13
Beer (barrel)	15	4	20	23
constraint	480	160	1190	

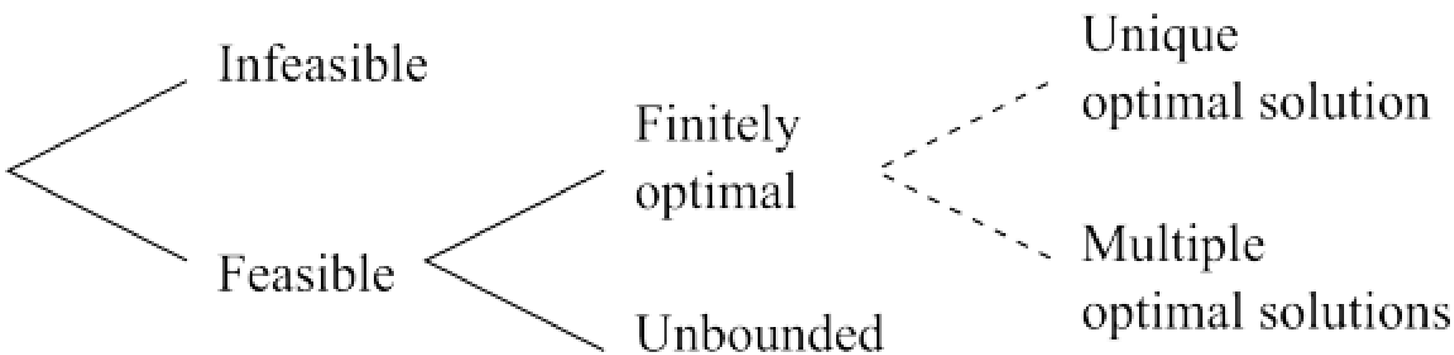
- Find maximum ?
  - All resource to ale : 34 barrels of ale = $442
  - All resource to beer : 32 barrels of beer = $736
  - 7.5 barrels of ale + 29.5 barrels of beer = $776
  - 12 barrels of ale + 28 barrels of beer = $800

Converting to standard form

$$\begin{array}{lrcrcr} \max & 13A & + & 23B & & \\ \text{s.t.} & 5A & + & 15B & \leq & 480 \\ & 4A & + & 4B & \leq & 160 \\ & 35A & + & 20B & \leq & 1190 \\ & A & , & B & \geq & 0 \end{array}$$



Summary



- In solving an LP (or any mathematical program) in practice, we only want to find an optimal solution, not all.
  - All we want is to make an optimal decision