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Slope of secant line

$$m_{PQ} = \frac{\Delta y}{\Delta x} = \frac{f(x+h)-f(x)}{h} \xrightarrow{h\to 0} \text{slope of tangent line}$$

Δy: increment of y

Δx: increment of x

Def. A fn f is diff at x if

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \text{ exist}$$

It is called derivative of f at x

$$\left(\text{Alternative} \quad \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a) \right)$$

Remark. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$\underbrace{\frac{dy}{dx}}$, y', $\frac{d}{dx}\{f(x)\}$

$\frac{dy}{dx}$: derivative of y w.r.t x

Ex find tangent line to $y=x^2$ at pt $(1,1)$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{2xh+h^2}{h} = 2x$$

$m = f'(1) = 2$

$1 = 2\cdot 1 + b \Rightarrow b = -1$

the tangent line is $y = 2x-1$

or

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} x+1 = 2 = m$$

Ex find tangent line to $y=\sqrt{x}$ at $(1,1)$

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0} \frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$m = f'(1) = \frac{1}{2}$

$1 = \frac{1}{2}\cdot 1 + b \Rightarrow b = \frac{1}{2}$

the tangent line is $y = \frac{1}{2}x + \frac{1}{2}$

Ex $f(xy) = f(x) + f(y)$ $\forall x, y > 0$ (抽象)

find (1) $f(1)$

(2) f conti at 1, prove f conti on $\mathbb{R}^+$

(3) $f'(1) = 1$, find $f(x)$ $\forall x > 0$

<pf>

(1) $f(1) = f(1) + f(1) \Rightarrow f(1) = 0$

(2) f conti at 1 $\Rightarrow \lim_{x\to 1} f(x) = f(1)$

given $c > 0$, $\lim_{x\to c} f(x) = \lim_{x\to c} f(\frac{x}{c} \times c) = \lim_{x\to c} \{f(\frac{x}{c}) + f(c)\} = f(c)$

(3) $f'(1) = \lim_{x\to 1} \frac{f(x) - f(1)}{x-1} = \lim_{x\to 1} \frac{f(x)}{x-1} = 1$

$$f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x-c} = \lim_{x\to c} \frac{\{f(\frac{x}{c}) + f(c) - f(c)\}}{x-c} = \lim_{x\to c} \frac{f(\frac{x}{c})}{x-c}$$

$$= \frac{1}{c} \lim_{\frac{x}{c}\to 1} \frac{f(\frac{x}{c})}{\frac{x}{c} - 1} = \frac{1}{c} \lim_{y\to 1} \frac{f(y)}{y-1} = \frac{1}{c} \#$$

$$\begin{tikzcd} S(t) \arrow[r, bend left, "\text{diff}"] & S'(t) \arrow[r, bend left, "\text{diff}"] \arrow[l, dashed, bend left, "\text{integrate}"] & S''(t) \arrow[l, dashed, bend left, "\text{integrate}"] \end{tikzcd}$$

position　velocity　instantaneous velocity

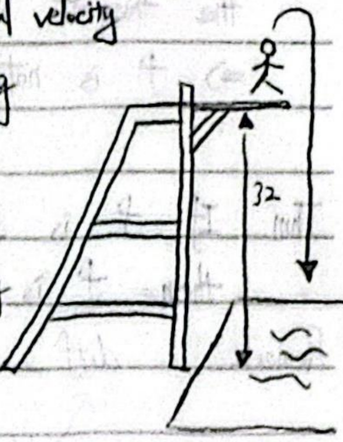
$S(t) = -\frac{1}{2}\boxed{g}t^2 + \boxed{V_0}t + \boxed{S_0}$

acceleration due to gravity

initial velocity

initial height

Ex. At time $t=0$, a diver jumps from platform diving board that is 32 feet above the water. The initial velocity of diver is 16 feet/s. When diver hit water? What is the diver's velocity at impact?



$g = 9.8\ m/s \approx 32$ feet/s

$S(t) = -\frac{1}{2}(32)t^2 + 16t + 32 = -16t^2 + 16t + 32$

$= -16(t^2 - t - 2) = -16(t-2)(t+1)$

$S(t) = 0 \Rightarrow t = 2$ (−1 不合)

$S'(t) = -32t + 16$

$S'(2) = -64 + 16 = -48$

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Def $\lim\limits_{h\to 0^{+}_{-}} \dfrac{f(a+h)-f(a)}{h} = \lim\limits_{x\to a^{+}_{-}} \dfrac{f(x)-f(a)}{x-a}$ is $\left|\begin{matrix}\text{right}\\ \text{left}\end{matrix}\right|$ hand derivative at a

$\underbrace{\phantom{\lim\limits_{h\to 0^{+}} \dfrac{f(a+h)-f(a)}{h}}}_{f'(a^{+}_{-})}$

Def. $f'(a)\ \exists \iff f'(a^{+}) = f'(a^{-})$

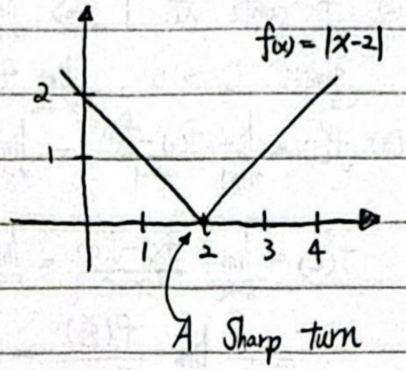
Ex. $f(x) = |x-2|$ is conti at $x=2$

$\lim\limits_{x\to 2^{-}} \dfrac{f(x)-f(2)}{x-2} = \lim\limits_{x\to 2^{-}} \dfrac{|x-2|}{x-2} = -1$

$\lim\limits_{x\to 2^{+}} \dfrac{f(x)-f(2)}{x-2} = 1$

$\Rightarrow$ f is not diff at $x=2$

$\Rightarrow$ f has no tangent line at $(2,0)$

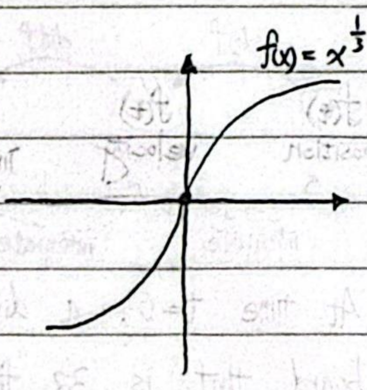


Ex. $f(x) = x^{\frac{1}{3}}$ is conti at $x=0$

$\lim\limits_{x\to 0} \dfrac{f(x)-f(0)}{x} = \lim\limits_{x\to 0} \dfrac{x^{\frac{1}{3}}}{x} = \lim\limits_{x\to 0} \dfrac{1}{x^{\frac{2}{3}}} = \nexists$

the tangent line is vertical at $x=0$

$\Rightarrow$ f is not diff at $x=0$



Thm. If f is diff at $x=a$

then f is conti at $x=a$

Remark. diff $\longrightarrow$ conti $\longrightarrow$ limit exist

(連續不一定可微)

Ex $f(x) = |x|$

$f'(0^{-}) = \lim\limits_{x\to 0^{-}} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0^{-}} \dfrac{|x|}{x} = -1$

$f'(0^{+}) = \lim\limits_{x\to 0^{+}} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0^{+}} \dfrac{|x|}{x} = 1$

$\Rightarrow$ $f'(0^{-}) \neq f'(0^{+})$

$f'(0)$ does not exist

but f conti at $x=0$

2nd-derivative, nth-derivative

$$f \longrightarrow f' \longrightarrow f'' \longrightarrow \cdots \longrightarrow f^{(n)}$$

$$y=f(x) \longrightarrow \frac{dy}{dx} \longrightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) \longrightarrow \cdots \longrightarrow \frac{d^ny}{dx^n} \quad \text{(Leibniz symbol)}$$

§ Differentiation Rules

1. Constant Rule $\frac{d}{dx}\{\underset{\text{constant}}{c}\} = 0$

2. Power Rule $\frac{d}{dx}\{x^n\} = nx^{n-1}$

pf) Binomial thm $(x+y)^n = \sum_{i=0}^{n}\binom{n}{i}x^i y^{n-i} = x^n + \binom{n}{n-1}x^{n-1}y + \cdots + \binom{n}{1}x^1y^{n-1} + y^n$

$$\lim_{h\to 0}\frac{(x+h)^n - x^n}{h} = \lim_{h\to 0}\frac{\binom{n}{n-1}x^{n-1}h + \cdots + \binom{n}{1}xh^{n-1} + h^n}{h} = nx^{n-1} \quad \#$$

Ex $\frac{d}{dx}\{x^3\} = 3x^2$

$\frac{d}{dx}\{x^{\sqrt{2}}\} = \sqrt{2}\,x^{\sqrt{2}-1}$

$\frac{d}{dx}\{x^{-\frac{4}{3}}\} = -\frac{4}{3}x^{-\frac{7}{3}}$

$\frac{d}{dx}\{\sqrt{x^{2+\pi}}\} = \frac{d}{dx}\{x^{1+\frac{\pi}{2}}\} = \left(1+\frac{\pi}{2}\right)x^{\frac{\pi}{2}}$

3 Constant Multiple Rule $\frac{d}{dx}\{cf(x)\} = c\frac{d}{dx}\{f(x)\}$

Let $g(x) = cf(x)$

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{c\{f(x+h)-f(x)\}}{h} = cf'(x)$$

$\Rightarrow (cf(x))' = cf'(x)$

Ex. $\frac{d}{dx}\{4x^6\} = 4(6x^5) = 24x^5$

$\frac{d}{dx}\{-3x^{-7}\} = -3(-7x^{-8}) = 21x^{-8}$

4. Sum and Difference Rule $\frac{d}{dx}\{f(x) \pm g(x)\} = \frac{d}{dx}\{f(x)\} \pm \frac{d}{dx}\{g(x)\}$

Let $A(x) = f(x) + g(x)$

$$A'(x) = \lim_{h\to 0}\frac{A(x+h)-A(x)}{h} = \lim_{h\to 0}\frac{\{f(x+h)-f(x)\} + \{g(x+h)-g(x)\}}{h} = f'(x) + g'(x)$$

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3 + 4 ⇒ Linearity

Ex. $\frac{d}{dx}\{x^5+5x^2\} = \frac{d}{dx}\{x^5\} + 5\frac{d}{dx}\{x^2\} = 5x^4 + 5(2x) = 5x^4+10x$

$\frac{d}{dx}\{\frac{3}{x^4} - 2x^2 + 6x - 7\} = 3\frac{d}{dx}\{\frac{1}{x^4}\} - 2\frac{d}{dx}\{x^2\} + 6\frac{d}{dx}\{x\} - \frac{d}{dx}\{7\}$

$= 3(-4)\frac{1}{x^5} - 2(2x) + 6$

$= \frac{-12}{x^5} - 4x + 6$

5. Product Rule $\frac{d}{dx}\{f(x)g(x)\} = \frac{d}{dx}\{f(x)\}g(x) + f(x)\frac{d}{dx}\{g(x)\}$

$\lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + g(x+h)f(x) - f(x)g(x)}{h}$

$= \lim_{h\to 0}\left\{\left(\frac{f(x+h)-f(x)}{h}\right)g(x+h) + f(x)\left(\frac{g(x+h)-g(x)}{h}\right)\right\} = f'(x)g(x) + f(x)g'(x)$

Ex $\frac{d}{dx}\{(3x-2x^2)(5+4x)\} = \frac{d}{dx}\{3x-2x^2\}(5+4x) + (3x-2x^2)\frac{d}{dx}\{5+4x\}$

$= (3-4x)(5+4x) + (3x-2x^2)4$

Ex $\frac{d}{dt}\{\sqrt{t}(a+bt)\} = \frac{d}{dt}\{\sqrt{t}\}(a+bt) + \sqrt{t}\frac{d}{dt}\{a+bt\}$

$= \frac{(a+bt)}{2\sqrt{t}} + \sqrt{t}b$

6. Quotient Rule $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{\frac{d}{dx}\{f(x)\}g(x) - f(x)\frac{d}{dx}\{g(x)\}}{[g(x)]^2}$

$\lim_{h\to 0} \frac{\left[\frac{f(x+h)}{g(x+h)}\right] - \left[\frac{f(x)}{g(x)}\right]}{h} = \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x)g(x+h)}$

$= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{hg(x)g(x+h)}$

$= \lim_{h\to 0} \frac{\left(\frac{f(x+h)-f(x)}{h}\right)g(x) - f(x)\left(\frac{g(x+h)-g(x)}{h}\right)}{g(x)g(x+h)}$

$= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

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Ex. $\frac{d}{dx}\left\{\frac{5x-2}{x^2+1}\right\} = \frac{\frac{d}{dx}\{5x-2\}(x^2+1) - (5x-2)\frac{d}{dx}\{x^2+1\}}{(x^2+1)^2}$

$= \frac{5(x^2+1) - (5x-2)(2x)}{(x^2+1)^2}$

Ex. $\frac{d}{dx}\left\{\frac{(x-1)(x^2-2x)}{x^4}\right\} = \frac{d}{dx}\left\{\frac{x^3-3x^2+2x}{x^4}\right\} = \frac{d}{dx}\left\{\frac{1}{x} - \frac{3}{x^2} + \frac{2}{x^3}\right\}$

$= \frac{-1}{x^2} + \frac{6}{x^3} - \frac{6}{x^4}$

7. $\frac{d}{dx}\{e^x\} = e^x$

$\lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} e^x\left(\frac{e^h - 1}{h}\right)$

since $e^h - 1 = \left(1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots\right) - 1$

then $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$

Hence $\frac{d}{dx}\{e^x\} = e^x$

§ Appendix : Rule of Trigonometric Function

$\frac{d}{dx}\{\sin(x)\} = \cos(x) \longrightarrow \left(\lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} = \lim_{h\to 0} \frac{2\sin\left(\frac{h}{2}\right)\cos\left(x+\frac{h}{2}\right)}{h} \right.$

$\left. = \lim_{h\to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \cos\left(x+\frac{h}{2}\right) = \cos(x) \right)$

$\frac{d}{dx}\{\cos(x)\} = -\sin(x)$

$\frac{d}{dx}\{\tan(x)\} = \sec^2(x)$

$\frac{d}{dx}\{\cot(x)\} = -\csc^2(x)$

$\frac{d}{dx}\{\sec(x)\} = \sec(x)\tan(x)$

$\frac{d}{dx}\{\csc(x)\} = -\csc(x)\cot(x)$

Ex $\frac{d}{dx}\left\{\frac{\sec(x)}{1+\tan(x)}\right\} = \frac{\frac{d}{dx}\{\sec(x)\}(1+\tan(x)) - \sec(x)\frac{d}{dx}\{1+\tan(x)\}}{[1+\tan(x)]^2}$

$= \frac{\sec(x)\tan(x)(1+\tan(x)) - \sec(x)\sec^2(x)}{[1+\tan(x)]^2}$

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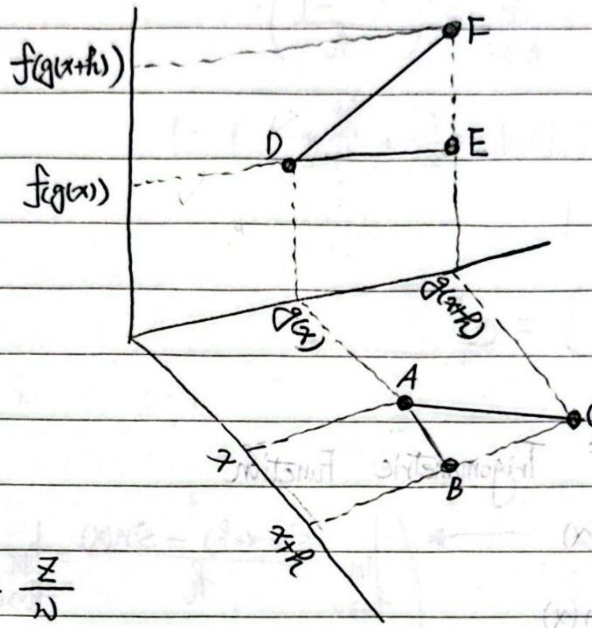
Ex find 27th derivative of $\cos(x)$

$$\left.\begin{aligned} f' &= -\sin(x) \\ f'' &= -\cos(x) \\ f''' &= \sin(x) \\ f^{(4)} &= \cos(x) \end{aligned}\right\} \text{ cycle length 4}$$

$27 \div 4 = 6 \cdots 3 \Rightarrow f^{(27)} = \sin(x)$

§ Chain Rule



$g'(x) \approx \dfrac{\overline{BC}}{\overline{AB}} = \dfrac{w}{h}$

$w \approx g'(x)h$

$f'(g(x)) \approx \dfrac{\overline{EF}}{\overline{DE}} = \dfrac{z}{w}$

$z \approx f'(g(x))\, w \approx f'(g(x))\, g'(x)\, h$

$$\big(f(g(x))\big)' = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} = \lim_{h\to 0} \frac{z}{h} = \lim_{h\to 0} \frac{f'(g(x))g'(x)h}{h}$$

$$= f'(g(x))g'(x)$$

Thm. If g diff at x and f diff at $g(x)$ then $f(g(x))$ is diff at x and

$$\frac{d}{dx}\{f(g(x))\} = f'(g(x))g'(x)$$

Remark $y = f(u)$, $u = g(x)$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

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Ex $\frac{d}{dx}\{\sqrt{x^2+1}\}$

$f(x) = \sqrt{x} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}}$

$g(x) = x^2+1 \quad g'(x) = 2x$

then $f'(g(x))\,g'(x) = \frac{1}{2\sqrt{x^2+1}}\,2x = \frac{x}{\sqrt{x^2+1}}$

Ex. $\frac{d}{dx}\{(x^3-1)^{100}\}$

$f(x) = x^{100} \Rightarrow f'(x) = 100x^{99}$

$g(x) = x^3-1 \quad g'(x) = 3x^2$

then $f'(g(x))\,g'(x) = 100(x^3-1)^{99}\,3x^2 = 300\,x^2(x^3-1)^{99}$.

$\frac{d}{dx}\{a^x\} = \frac{d}{dx}\{e^{x\ln a}\} = a^x \cdot \ln a$

by chain rule

Thm. (Derivative of Inverse fn)

Let f be diff on I. If f has inverse f^{-1}

then g is diff at any x which $f'(f^{-1}(x)) \neq 0$

Moreover,

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Remark. $f(f^{-1}(x)) = x$, by Chain Rule

$f'(f^{-1}(x))\,(f^{-1})'(x) = 1$

Cor. Let $f(x) = \log_a x$, $f^{-1}(x) = a^x$

then $(\log_a x)' = \frac{1}{a^{\log_a x}\cdot \ln a} = \frac{1}{x\ln a}$

In addition, $a=e$ $(\ln x)' = \frac{1}{x}$

Ex $\frac{d}{dx}\{e^{ax^2+bx+c}\} = \left(\frac{d\,e^{ax^2+bx+c}}{d\,ax^2+bx+c}\right)\left(\frac{d\,ax^2+bx+c}{dx}\right)$

$= e^{ax^2+bx+c}(2ax+b)$

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In general, given $f_1, \dots, f_n$ w/ composite fn

$$f_1 \circ (f_2 \circ \cdots (f_{n-1} \circ f_n))$$

If each f_i is diff at its immediate input then the composite fn is also diff. by the repeated application of Chain Rule

$$\frac{df_1}{dx} = \frac{df_1}{df_2}\frac{df_2}{df_3} \cdots \cdots \frac{df_n}{dx}$$

Ex $\frac{d}{dx}\left\{\sqrt{1+\sqrt{1+\sqrt{x}}}\right\}$

$$= \left(\frac{d\sqrt{1+\sqrt{1+\sqrt{x}}}}{d\,1+\sqrt{1+\sqrt{x}}}\right)\left(\frac{d\,1+\sqrt{1+\sqrt{x}}}{d\,1+\sqrt{x}}\right)\left(\frac{d\,1+\sqrt{x}}{dx}\right)$$

$$= \frac{1}{2\sqrt{1+\sqrt{1+\sqrt{x}}}} \quad \frac{1}{2\sqrt{1+\sqrt{x}}} \quad \frac{1}{2\sqrt{x}}$$

Ex. $\frac{d}{dx}\{x^x\} = \frac{d}{dx}\{e^{x\ln x}\}$

$$= \left(\frac{d\,e^{x\ln x}}{dx\ln x}\right)\left(\frac{dx\ln x}{dx}\right)$$

$$= x^x(\ln x + 1)$$

Ex $\frac{d}{dx}\{x^{x^x}\} = \frac{d}{dx}\{e^{x^x\ln x}\}$

$$= \left(\frac{d\,e^{x^x\ln x}}{dx^x\ln x}\right)\left(\frac{d\,x^x\ln x}{dx}\right)$$

$$= x^{x^x}\left\{x^x(\ln x+1)\ln x + x^{x-1}\right\}$$

§ Implicit Differentiation

$y = f(x)$ is explicit form

Some fn are defined implicity by a relation between x and y such as

$$x^2 + y^2 = 25$$

or $$x^3 + y^3 = 6xy$$

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Ex find $\frac{dy}{dx}$ give that $y^3+y^2-5y-x^2=-4$

$$3y^2\frac{dy}{dx}+2y\frac{dy}{dx}-5\frac{dy}{dx}-2x=0$$

$$\frac{dy}{dx}\left\{3y^2+2y-5\right\}=2x$$

$$\frac{dy}{dx}=\frac{2x}{3y^2+2y-5} \quad \#$$

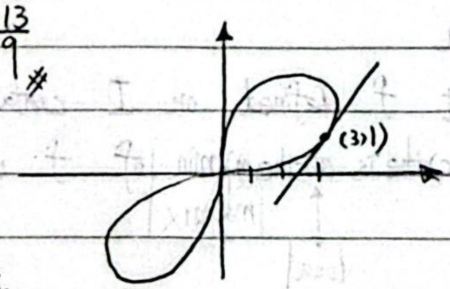
Ex find Slope of $3(x^2+y^2)^2=100xy$ at pt $(3,1)$

$$6(x^2+y^2)(2x+2y\frac{dy}{dx})=100(y+x\frac{dy}{dx})$$

$$12x(x^2+y^2)-100y=\frac{dy}{dx}\left\{100x-12y(x^2+y^2)\right\}$$

$$\frac{dy}{dx}=\frac{12x(x^2+y^2)-100y}{100x-12y(x^2+y^2)}=\frac{3x(x^2+y^2)-25y}{25x-3y(x^2+y^2)}$$

$$\left.\frac{dy}{dx}\right|_{(3,1)}=\frac{90-25}{75-30}=\frac{65}{45}=\frac{13}{9} \quad \#$$



Ex $x^3y^2=4$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

$$3x^2y^2+2x^3y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{3x^2y^2}{-2x^3y}=-\frac{3}{2}\frac{y}{x}$$

$$\frac{d}{dx}\left\{\frac{dy}{dx}\right\}=-\frac{3}{2}\left(\frac{\frac{dy}{dx}x-y}{x^2}\right)=-\frac{3}{2}\left(\frac{-\frac{3}{2}y-y}{x^2}\right)$$

$$=\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\frac{y}{x^2}=\frac{15}{4}\frac{y}{x^2} \quad \#$$

§ Derivative of Inverse Trigonometric Function

$$\frac{d}{dx}\left\{\sin^{-1}(x)\right\}=\frac{1}{\sqrt{1-x^2}}$$

(M1)

$$\sin(\sin^{-1}(x))=x$$

$$(\sin^{-1}(x))'=\frac{1}{\cos(\sin^{-1}(x))}=\frac{1}{\sqrt{1-[\sin(\sin^{-1}(x))]^2}}=\frac{1}{\sqrt{1-x^2}} \quad ※$$

(M2)

$$y=\sin^{-1}(x) \Rightarrow x=\sin(y) \Rightarrow \frac{dx}{dy}=\cos(y)=\sqrt{1-\sin^2(y)}=\sqrt{1-x^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}$$

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$$\frac{d}{dx}\{\tan^{-1}(x)\} = \frac{1}{1+x^2}$$

$y = \tan^{-1}(x) \Rightarrow x = \tan(y) \Rightarrow \frac{dx}{dy} = \sec^2(y) = 1 + \tan^2(y) = 1 + x^2$

$\frac{dy}{dx} = \frac{1}{1+x^2}$

$$\frac{d}{dx}\{\cos^{-1}(x)\} = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\{\cot^{-1}(x)\} = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}\{\sec^{-1}(x)\} = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\{\csc^{-1}(x)\} = \frac{-1}{x\sqrt{x^2-1}}$$

§ Extrema

Def. Let f defined on D containing c. then $f(c)$ is the abs. $\left|\begin{matrix}\text{min} \\ \text{max}\end{matrix}\right|$ of f on D if $f(c) \left|\begin{matrix}\le \\ \ge\end{matrix}\right| f(x)$ $\forall x \in D$

local $\qquad$ when x near c

Thm. (Extreme Value Theorem)

If f conti on closed interval $[a,b]$, then f has both a abs. $\begin{cases}\text{max} & f(c) \\ \text{min} & f(d)\end{cases}$ at some $\begin{cases} c \in [a,b] \\ d \in [a,b]\end{cases}$

Thm (Fermat's Theorem)

If f has local max or min at c and if $f'(c)$ exists, then $f'(c) = 0$

stationary pt.

Def. A critical pt of f is a $c \in D(f)$ s.t. $f'(c) = 0$ or $f'(c)$ does not exist

Remark. If f has local max or min at c then c is critical pt. of f.

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Ex. $f(x) = x^3$

$f'(x) = 3x^2$

$f'(0) = 0 \Rightarrow$ But f has no max or min at 0

It demonstrate that even when $f'(c) = 0$

there need not to be a max or min at c

Ex Find critical pt of f

(a) $f(x) = x^{\frac{1}{3}}(x-3)^{\frac{2}{3}} \quad \forall x \in \mathbb{R}$

$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}(x-3)^{\frac{2}{3}} + x^{\frac{1}{3}}\frac{2}{3}(x-3)^{-\frac{1}{3}}$

$= \dfrac{\frac{1}{3}(x-3) + \frac{2}{3}x}{x^{\frac{2}{3}}(x-3)^{\frac{1}{3}}} = \dfrac{x-1}{x^{\frac{2}{3}}(x-3)^{\frac{1}{3}}}$

$\Rightarrow f'(0) \nexists \qquad \Rightarrow x = 0 \vee 1 \vee 3$

$f'(1) = 0 \qquad$ are critical pt of f

$f'(3) \nexists$

(b) $f(x) = xe^x \quad x \in (0, \infty)$

$f'(x) = e^x + xe^x = e^x(1+x)$

$f'(-1) = 0$ but $-1 \notin (0,\infty) = D(f)$

(c) $f(x) = \dfrac{x^2}{x-2}$

$f'(x) = \dfrac{2x(x-2) - x^2}{(x-2)^2} = \dfrac{x(x-4)}{(x-2)^2}$

$\Rightarrow f'(0) = 0 \qquad \Rightarrow \quad x = 0 \vee 4$

$f'(4) = 0 \qquad$ are critical pt of f

$f'(2) \nexists$ but $f(2)$ undefined

★ Find Extrema on a closed interval $[a,b]$

1. Find critical pt of f
2. Evaluate f at each critical pt in (a,b)
   endpoint of $[a,b]$
3. The | least / greatest | of these values is | min / max |

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Ex. find abs $\left\{\begin{matrix}\min \\ \max\end{matrix}\right.$ of $f(x) = x^3 - 3x^2 + 1 \quad -\frac{1}{2} \le x \le 4$

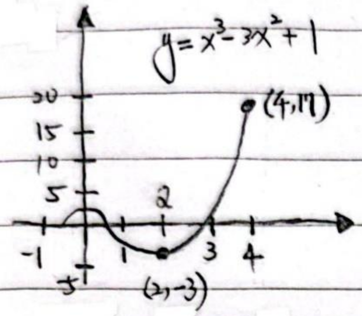
$f'(x) = 3x^2 - 6x = 3x(x-2) \Rightarrow x = 0 \vee 2.$

$f(0) = 1 \quad \Rightarrow \quad \max$ is $f(4)$

$f(2) = -3 \qquad \min$ is $f(2)$

$f(\frac{1}{2}) = \frac{1}{8}$

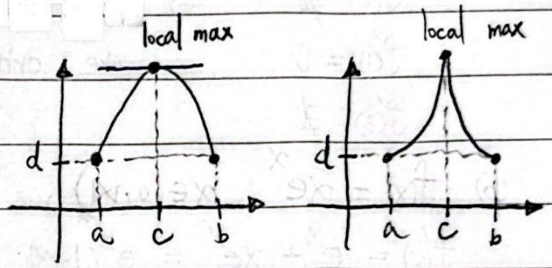
$f(4) = 17$



Thm. (Rolle's theorem)

1. f is conti on $[a,b]$
2. f is diff on (a,b)
3. $f(a) = f(b)$

then $\exists c \in (a,b)$ s.t. $f'(c) = 0$



(When diff requiment is dropped. f will still have critical pt $c \in (a,b)$, but it may not yield a horizontal tangent.)

Remark.

幾何意義 ⇒ 連續且平滑並且端點等高之函數圖形必至少有一水平切線

代數意義 ⇒ 連續且平滑之函數在兩等根間其一階導數必具有一零根

Ex. $s = f(t)$ (position function) If the obj is in the same place at 2 different instant $t=a$ and $t=b$, then $f(a) = f(b)$. Rolle's thm says that there is some instant of time $t = c \in (a,b)$ when $f'(c) = 0$ that is the velocity is 0.

Chryu culture