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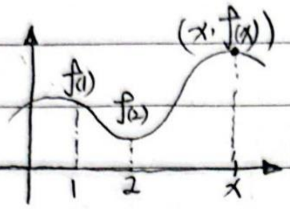
§ Relation and Function

Def. Let D and E be sets. A relation on D and E is a subset of $\underbrace{D \times E}$

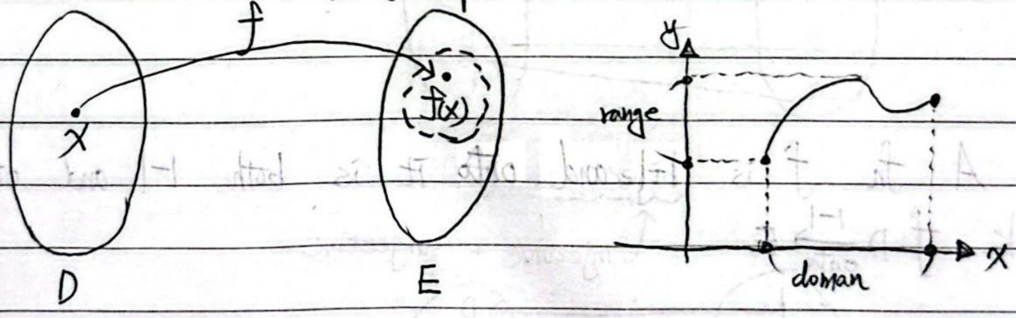
(A set of order pairs (x,y), $x \in D$, $y \in E$)



If f is a relation write $(x,y) \in f$ or $x f y$
(Indicate x is related y)

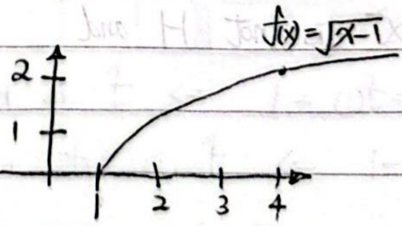
Def. A fn maps D into E is a relation f from D to E such that
(i) $\forall x \in D$, $\exists y \in E$ s.t. $(x,y) \in f$
(ii) If $(x,y) \in f$ and $(x,y^*) \in f$ then $y = y^*$

Remark. Write $f: D \to E$
or $\underbrace{f: x \to x^2}$
fn f maps x to x^2



$D = D(f)$ is the domain of f
E is co-domain of f
$R(f) = \{ f(x) : x \in D(f) \}$ is range of f, $R(f) \subseteq E$
$G(f) = \{ (x, f(x)) : x \in D(f) \}$ is graph of f.

Ex. find domain and range.
(a) $f(x) = \sqrt{x-1}$
(b) $g(x) = \sqrt{4 - x^2}$



A: (a) $x - 1 \geq 0 \Rightarrow x \geq 1$
$R(f) = [0, \infty)$

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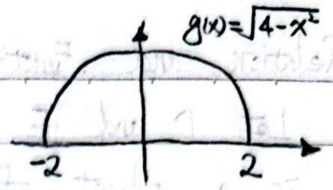
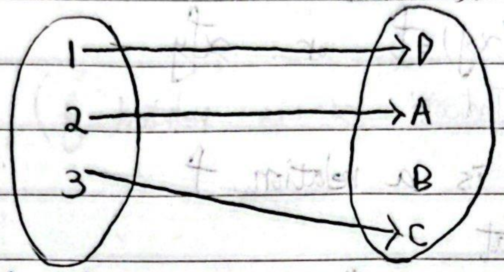


(b) $4-x^2 \geq 0 \Rightarrow 4 \geq x^2 \Rightarrow -2 \leq x \leq 2$

$R(f) = [0, 2]$

Def. A fn f is $\underbrace{1\text{-}1}_{}$ if $f(x) = f(y) \Rightarrow x = y$ → injective

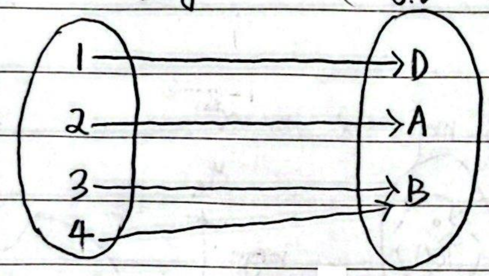
Remark. $f: D \xrightarrow{1-1} E$ $\left(\begin{array}{l} (x,y) \in f \text{ and } (x^*,y) \in f \\ \text{then } x = x^* \end{array} \right)$.



Def. A fn f is onto if $R(f) = E$

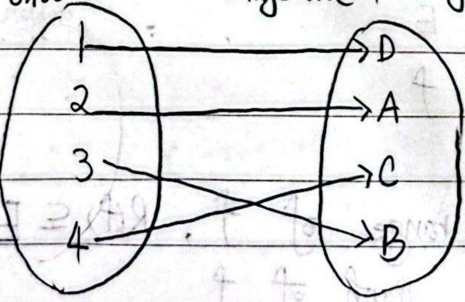
Remark. $f: D \xrightarrow{onto} E$ $\left(\begin{array}{l} \forall y \in E,\ \exists x \in D \\ \text{s.t } x = f(y) \end{array} \right)$

surjective



Def. A fn f is bijective if it is both 1-1 and onto.

Remark $f: D \xrightarrow[onto]{1-1} E$ injective + surjective



Ex $f(x) = x^2$

$f(-1) = f(1) = 1 \Rightarrow f$ is not 1-1

$x^2 \neq -1 \Rightarrow f$ is not onto

≷ Transformation of Functions

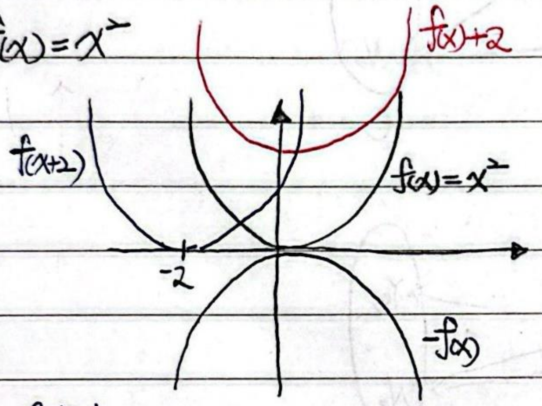
$y = f(x)$

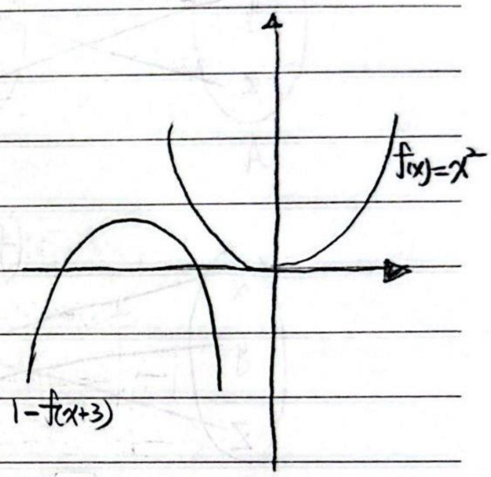
(1) Horizontal shift $\left|\begin{matrix} +c \\ -c \end{matrix}\right|$ units to $\left|\begin{matrix} \text{right} \\ \text{left} \end{matrix}\right|$: $y = \left|\begin{matrix} f(x+c) \\ f(x-c) \end{matrix}\right|$

(2) Vertical shift $\left|\begin{matrix} +c \\ -c \end{matrix}\right|$ unit $\left|\begin{matrix} \text{upward} \\ \text{downward} \end{matrix}\right|$: $y = \left|\begin{matrix} f(x)+c \\ f(x)-c \end{matrix}\right|$

(3) Reflection about $\left|\begin{matrix} x\text{-axis} \\ y\text{-axis} \\ (0,0) \end{matrix}\right|$: $y = \left|\begin{matrix} -f(x) \\ f(-x) \\ -f(-x) \end{matrix}\right|$

Ex $f(x) = x^2$



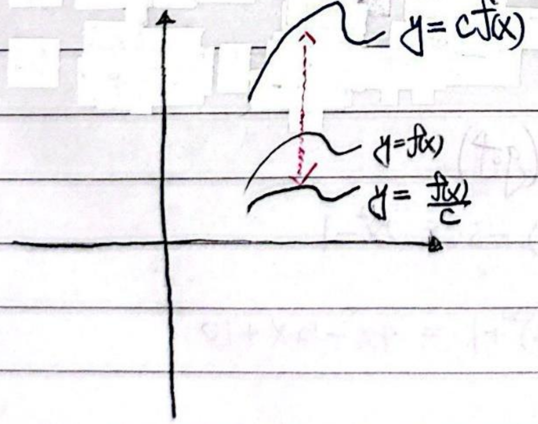


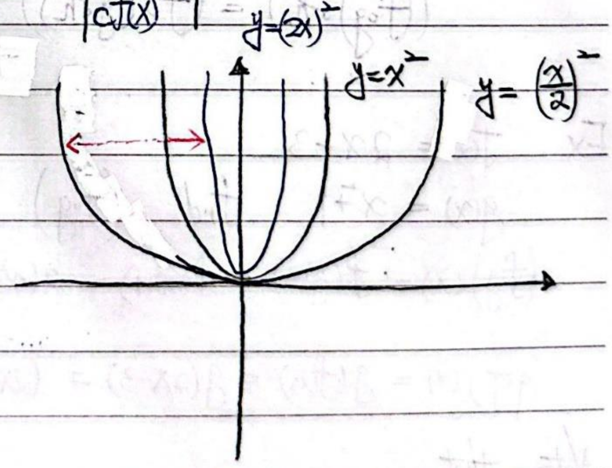
Suppose $c > 1$,

(4) Horizontally $\left|\begin{matrix} \text{shrink} \\ \text{stretch} \end{matrix}\right|$ by c : $y = \left|\begin{matrix} f(cx) \\ f(\frac{x}{c}) \end{matrix}\right|$

(5) Vertically $\left|\begin{matrix} \text{shrink} \\ \text{stretch} \end{matrix}\right|$ by c : $y = \left|\begin{matrix} \frac{f(x)}{c} \\ cf(x) \end{matrix}\right|$

Ex.





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§ Composition of Functions.

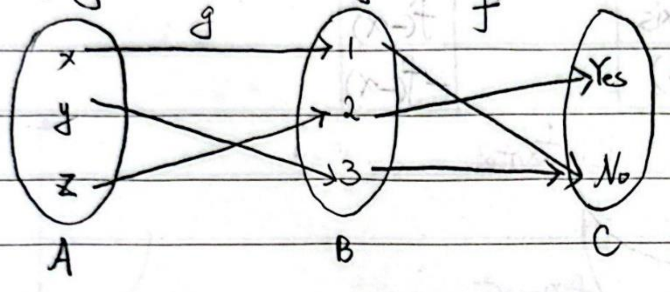
Def The composition of two fn

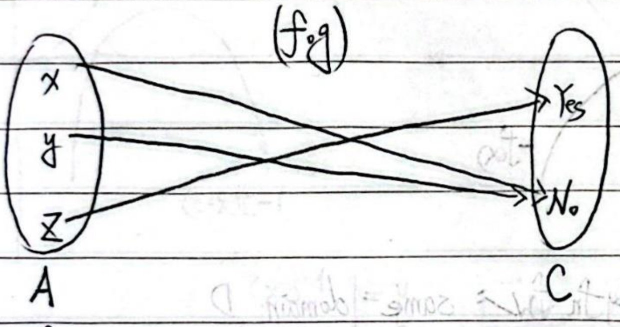
$$f : B \to C$$

$$g : A \to B$$

is the fn $(f \circ g) : A \to C$

Remark. $(f \circ g) : x \longmapsto f(g(x))$





$(g \circ f)$ is not possible unless $A = C$

★ Composition is associative

$$((f \circ g) \circ h) = (f \circ (g \circ h)) \rightarrow \text{結合律}$$

Ex. $f(x) = 2x - 3$

$g(x) = x^2 + 1$ find $(f \circ g)$, $(g \circ f)$

$(f \circ g)(x) = f(g(x)) = f(x^2+1) = 2(x^2+1) - 3 = 2x^2 - 1$

$(g \circ f)(x) = g(f(x)) = g(2x-3) = (2x-3)^2 + 1 = 4x^2 - 12x + 10$

Note that

$(f \circ g)(x) \neq (g \circ f)(x) \Rightarrow$ Not Commute → 交換率

§ Arithmetic Operation of Functions

Def. If f and g are fn,

for $x \in D(f) \cap D(g)$

(1) $(f \pm g)(x) = f(x) \pm g(x)$

(2) $(fg)(x) = f(x)g(x)$

(3) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, $g(x) \neq 0$

Ex. $f(x) = \sqrt{x}$; $D(f) = [0, \infty)$

$g(x) = \sqrt{1-x}$, $D(g) = (-\infty, 1]$

$(f \pm g)(x) = \sqrt{x} \pm \sqrt{1-x}$ w/ domain $= [0,1]$

$(fg)(x) = \sqrt{x - x^2}$ w/ domain $= [0,1]$

$\left(\frac{f}{g}\right)(x) = \sqrt{\frac{x}{1-x}}$ w/ domain $= [0,1)$

$y = f(x) \iff x = u(y)$

y: depend var / output

x: independent var / input

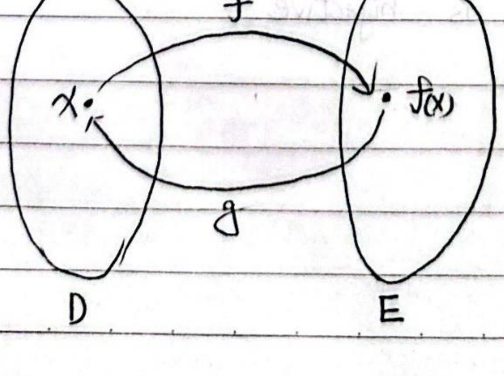
Inverse of relation f^{-1} is the relation $(x,y) \in f \iff (y,x) \in f^{-1}$

$\left(x f y \iff y f^{-1} x\right)$

Def. (Left inverse of fn)

$g: E \to D$ is left inverse of $f: D \to E$

if $g(f(x))$ $\forall x \in D$

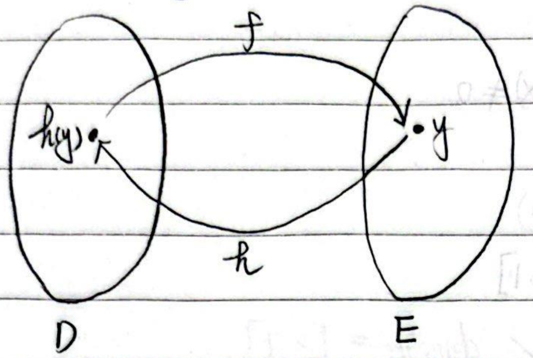


$\Rightarrow$ Left inverse tells how to go back when you started

Def. (Right inverse of fn)

$h: E \to D$ is right inverse of f $f: D \to E$

if $f(h(y)) = y \quad \forall y \in E$



$\Rightarrow$ Right inverse tells a possible place to start

Thm. 1. A fn is $\left|\begin{matrix}\text{1-1} \\ \text{Onto}\end{matrix}\right| \iff$ it has $\left|\begin{matrix}\text{left} \\ \text{right}\end{matrix}\right|$ inverse

☆ If a fn has both left and right inverse then two inverse are identical, and this common inverse is unique.

Remark called two-sided inverse or inverse f^{-1} of fn f

Thm. A fn is $\underbrace{\text{bijective}}$ $\iff$ it has two-sided inverse

1-1 + onto

Remark $f(f^{-1}(x)) = (f \circ f^{-1})(x) = x$

$f^{-1}(f(x)) = (f^{-1} \circ f)(x) = x$

$y = f(x) \iff x = f^{-1}(y)$

Ex $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 3x+5$ is bijective

$(f \circ f^{-1})(x) = f(f^{-1}(x)) = x$

$3f^{-1}(x) + 5 = x$

$f^{-1}(x) = \frac{x-5}{3}$

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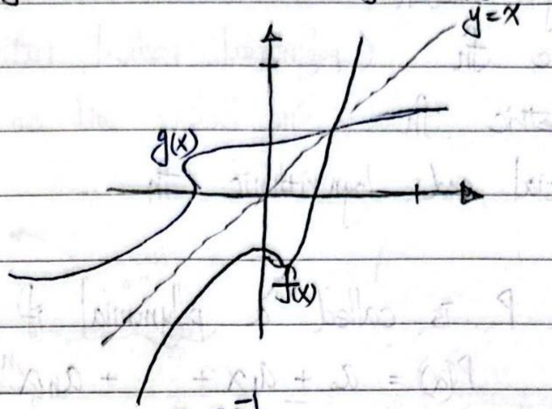
Ex. $f(x) = 2x^3 - 1$

$g(x) = \left(\frac{x+1}{2}\right)^{\frac{1}{3}}$, show f, g are inverse fn. of each other.

$f(g(x)) = 2\left(\frac{x+1}{2}\right) - 1 = x$

$g(f(x)) = \left(\frac{2x^3 - 1 + 1}{2}\right)^{\frac{1}{3}} = x$



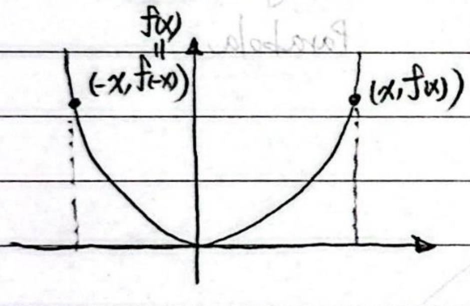
☆ Reflective

Thm. $G(f)$ contains pt (x,y) $\Leftrightarrow$ $G(f^{-1})$ contains pt. (y,x)

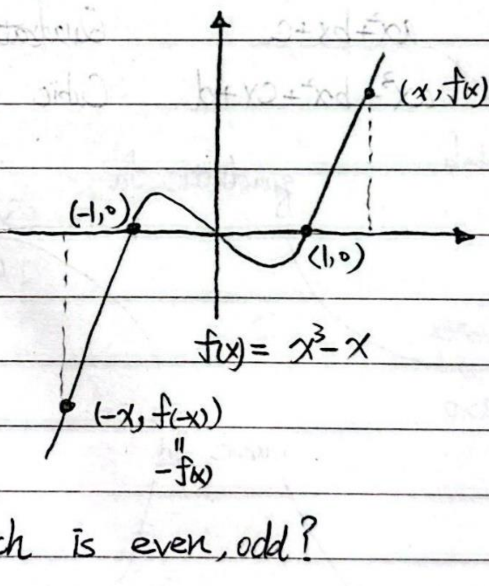
§ Even and Odd function

Def. A fn f is said to be $\left|\begin{matrix}\text{even}\\ \text{odd}\end{matrix}\right|$ if $f(-x) = \left|\begin{matrix} f(x) \\ -f(x)\end{matrix}\right|$ $\forall x \in D(f)$

Remark. $\left|\begin{matrix}\text{Even}\\ \text{Odd}\end{matrix}\right|$ is symmetric about $\left|\begin{matrix}\text{y-axis}\\ \text{origin}\end{matrix}\right|$



$f(x) = x^2$



$f(x) = x^3 - x$

Ex. $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ Determine each is even, odd?

$g(x) = |x|$

$h(x) = \sin(x)$

$f(-x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(-x)^2}{2}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = f(x) \Rightarrow$ even

$g(-x) = |-x| = |x| = g(x) \Rightarrow$ even

$h(-x) = \sin(-x) = -\sin(x) = -h(x) \Rightarrow$ odd

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Elementary Function

(i) Algebraic fn (polynomial, radical, rational)
(ii) Trigonometric fn (sine, cosine and so on)
(iii) Exponential and logarithmic fn

多项式

Def. A fn P is called a polynomial if

$$P(x) = a_0 + a_1 x + \cdots + a_n x^n$$

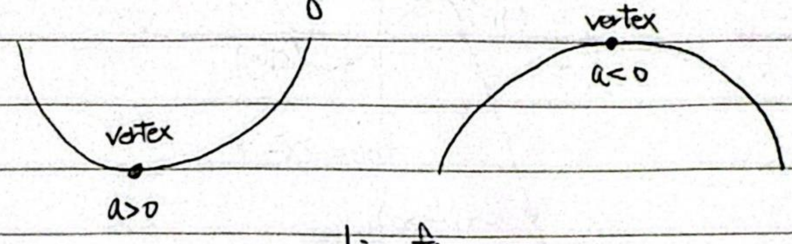
$n \in \mathbb{Z}^+$ and $a_0, a_1, \ldots, a_n$ are constants

$D(P) = \mathbb{R}$.

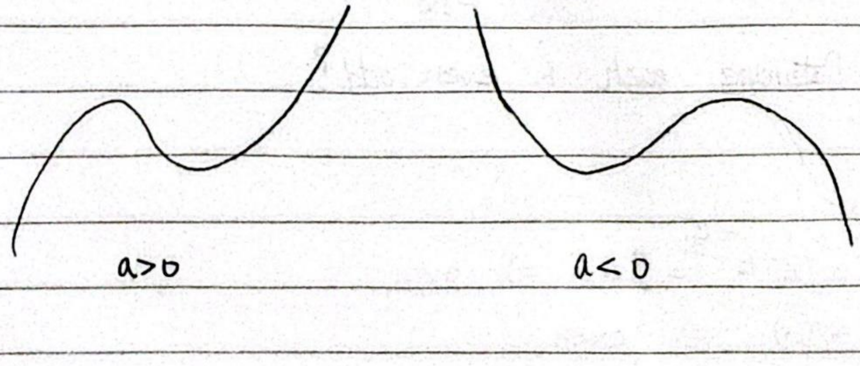
Remark If $a_n \neq 0$ then $\underbrace{Deg}_{\text{degree}}(P) = n$

Deg (P)	f(x)	
0	a	Constant fn
1	$ax+b$ (straight line)	Linear fn
2	ax^2+bx+c (parabola)	Quadratic fn
3	ax^3+bx^2+cx+d	Cubic fn

quadratic fn



cubic fn



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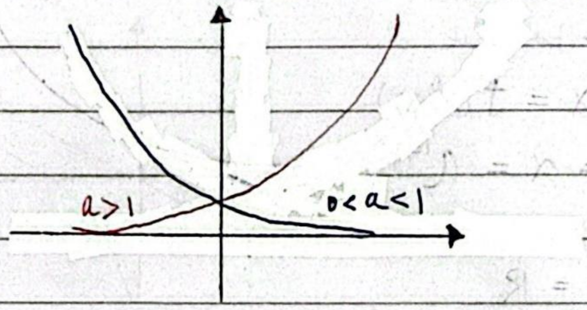
§ Exponential Function

$$f(x) = \underset{\text{base}}{a}^{\overset{\text{exponent}}{x}}, \quad a > 0$$

if $x = n \in \mathbb{Z}^+ \Rightarrow a^n = \underbrace{a \cdots a}_{n \text{ factors}} \quad a^{-n} = \frac{1}{a^n}$

if $x = 0 \Rightarrow a^0 = 1$

if $x = \frac{m}{n} \in \mathbb{Q} \Rightarrow a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$



if $\left| \begin{matrix} 0 < a < 1 \\ a > 1 \end{matrix} \right| \Rightarrow f$ is exponential $\left| \begin{matrix} \text{decay} \\ \text{growth} \end{matrix} \right|$

★ Laws of exponent

(1) $a^{x+y} = a^x a^y$

(2) $a^{x-y} = \frac{a^x}{a^y}$

(3) $(a^x)^y = a^{xy}$

(4) $(ab)^x = a^x b^x$

★ Speial base e

$e \approx 2.718\ldots\ldots$

$\lim_{n \to \infty} \left(1 + \frac{r}{n}\right)^n = e^r$

compound interest

$e^r = 1 + r + \frac{r^2}{2} + \frac{r^3}{6} + \cdots = \sum_{n=0}^{\infty} \frac{r^n}{n!}$

$f(x) = e^x$ is natural exponential fn.

Choyo culture

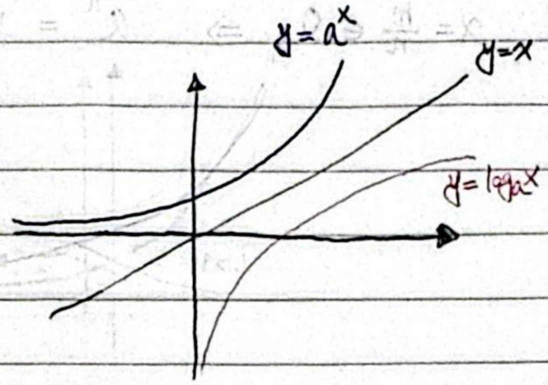
§ Logarithmic Function

Recall the inverse of fn

$$y = f(x) \iff x = f^{-1}(y)$$

then

$$\underbrace{y = \log_a x}_{\substack{\text{logarithmic fn} \\ \text{w/ base } a}} \iff x = a^y$$



since $f(f^{-1}(x)) = x = f^{-1}(f(x))$

then $\log_a(a^x) = x = a^{\log_a x}$

$D(f) = (0, \infty)$, $R(f) = \mathbb{R}$

★ Laws of logarithms

1. $\log_a(xy) = \log_a x + \log_a y$
2. $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
3. $\log_a(x^b) = b \log_a(x)$

Ex. expand $\ln\left(\frac{x^2\sqrt{x^2+2}}{3x+1}\right)$

$$= \ln(x^2) + \ln\sqrt{x^2+2} - \ln(3x+1)$$

$$= 2\ln x + \frac{1}{2}\ln(x^2+2) - \ln(3x+1)$$

★ If base $a = e \Rightarrow \log_e x = \ln x$ is natural logarithm

Then the inverse relation

$$y = e^x \iff x = \ln y$$

and

$$\ln(e^x) = x = e^{\ln x}$$

★ Change of Base

$$\log_a x = \frac{\ln x}{\ln a} \quad \text{Ex: } \log_8 7 = \frac{\ln 7}{\ln 8}$$

Ex. Solve $e^{5-3x} = 10$

$$\ln(e^{5-3x}) = 5 - 3x = \ln 10 \Rightarrow x = \frac{5}{3} - \frac{\ln 10}{3}$$

$$3x = 5 - \ln 10$$

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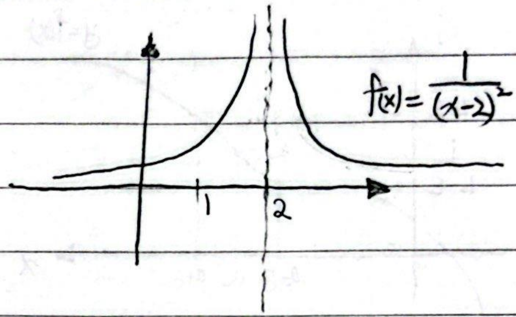
Def. A rational fn f is of the form
$$f(x) = \frac{P(x)}{Q(x)}$$
where P, Q are polynomials. $D(f) = \{Q(x) \neq 0\}$

Remark (1) Every polynomial is rational fn $\frac{P(x)}{1}$

(2) If denominator Q is zero at $x=a$
numerator P is not zero at $x=a$
then graph of f tends to vertical as
$x \to a$, line $x=a$ is called vertical asymptote.
→ 水平渐近线

Ex: $f(x) = \dfrac{1}{x^2-4x+4} = \dfrac{1}{(x-2)^2}$



§ Power function

Def. A fn of the form
$$f(x) = x^a$$
(i) $a = n$, $n \in \mathbb{Z}^+$

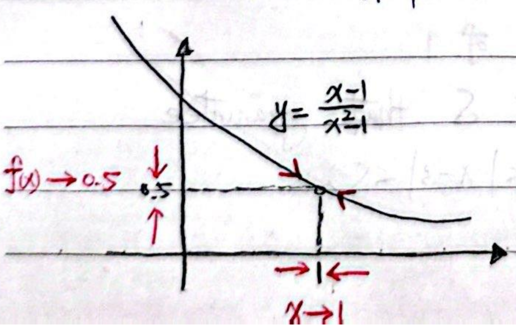
(ii) $a = \frac{1}{n}$, $n \in \mathbb{Z}^+$, root fn. Ex: $n=2$, $f(x) = x^{\frac{1}{2}} = \sqrt{x}$ square root fn

$n=3$, $f(x) = x^{\frac{1}{3}} = \sqrt[3]{x}$ cubic root fn

(iii) $a = -1$, reciprocal fn

§ Limit

Consider $f(x) = \dfrac{x-1}{x^2-1}$ for value of x close to 1



x	$f(x)$
0.5	0.667
0.9	0.526
0.99	0.502
1.01	0.497
1.1	0.476

$\Rightarrow \lim_{x \to 1} \dfrac{x-1}{x^2-1} = 0.5$

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