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離散型雙邊界選擇權之定價
Pricing Discrete Double Barrier Options

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論文提要內容:

界限選擇權是現在市場上相當受歡迎的新奇選擇權之一, 與一般選擇權的不同特點在, 它是一種路徑相依的選擇權, 界限選擇權的收益是在某個具體或非具體的資產價格 (標的物 or 利率) 碰觸到了某個特定的水準時, 依此來當作是選擇權失效的依據。大多數用於評價界限選擇權的模型都是假設在連續的觀測時間; 在此假設下, 選擇權大多可表示成封閉解。

許多現實中有關界限選擇權的合約觀測都是離散型的時間觀測點; 現實中現在還沒有公式可以定價這些離散型的選擇權, 儘管是利用 Monte carlo method 直接去跑模型, 可能數值收斂的速度不快。參考的文獻中發現離散型界限選擇權利用連續型公式做修正所得到公式可得到相當卓越的精準度。

本篇論文中首先說明利用標準的 Black-Scholes Model 和 diffusion-type 隨機微分方程來做爲其數學模型來架構並描述股價的動態狀況 ,一樣利用測度轉換將其作拆解, 然後特別去針對某一個機率做推導, 最後求得離散型的雙界限選擇權價錢的公式, 再去利用 Simpson 積分法則 , 求得數值解。

關鍵字: 界限選擇權、路徑相依選擇權、布朗運動在有限區間機率　、路徑跨越機率。

ABSTRACT

Pricing Discrete Double Barrier Options

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Barrier option is one of the Exotic options which are very popular in the market. Differ from other simple options, barrier option is continuous time and path depend. The payoff of a barrier option depends on whether or not a specified asset price, index, or rate reaches a specified level during the life of the option.

Most models for pricing barrier options assume continuous ,monitoring of the barrier; under this assumption, the option can often be priced in closed form. Many(if not most) real contracts with barrier provisions specify discrete monitoring constant; there are essentially no formulas for pricing these options, and even numerical pricing to run the model is probabily converge slowly.

In this thesis, first introduce the standard Black-Scholes Model and diffusion-type Stochastic differential equation as mathmatical model , and use it to describe the stock dynamical system . Then pricing the discrete type double barrier option heuristically. Compared to the Monte Carlo Algorithm, our discrete formula is cost not so much time.

Key word: barrier option, path-dependent options, Karatzas's brownian motion in finite interval, level crossing probabilities.

Contents

1 Preface **1**
1.1 introduction 1

2 The Discrete Price **4**
2.1 Pricing Double-barrier option 4

3 Main Steps of The Prove **7**
3.1 Approximation of the discrete probability 7
3.1.1 Discrete crossing probability 7
3.2 Discrete Price 12

4 Simulation **13**
4.1 Numerical solution 13
4.1.1 Case 1 13
4.1.2 Case 2 14
4.1.3 Case 3 14
4.1.4 Case 4 15
4.1.5 Case 5 15
4.2 Compare to other method 16
4.3 Conclusion 17

References **18**

Appendix A **19**

Chapter 1
Preface

1.1 introduction

A barrier option is activated(knocked in) or extinguished(knocked out) when a specified asset price, index, or rate reaches a specified level. The simplest such options are otherwise standard calls and puts that knocked in or knocked out by the underlying asset itself. Some variants tie the barrier crossing to one variable and the payoff to another; others specify ”binary” payoffs substitute the usual payoffs for calls and puts. Take together, these are among the most popular options with path-dependent payoffs. The knock-in and knock-out features lower the price of an option. Therefore, barrier options are prevailed in risk controll by matching a hedger's risk or speculator's view.

Most models of barrier option assume continuous monitoring of the barrier: a knock-in or knock-out is presumed to occur if the barrier is breached at any instant in the life of the option. Under this assumption, Merton(1973) obtained a formula for pricing a knock-out call. Subsequent work on pricing continuously monitored barrier options includes plenty of savants. However, a sizable protion of real contracts with barrier features specify fixed time for monitoring of the barrier-typically, daily closing. Numerical examples, indicate that there can be substantial price differences between discrete and continuous barrier option. Unfortunately, the exact pricing results available for continuous barriers do not extend to the discrete case.

Broadie, Glasserman, and Kou(1997), introduced the continuity correction of option with one barrier. In this paper, I justify a simple approximation of pricing discrete double barrier options. The method uses formulas for the prices of continuous barrier options but use random walk to approximate the crossing probability. The discrete formula is determined by the method of continuous type. Compared to continuous price, the discrete type value is close to continous type.

Here introduce the diffusion-type S.D.E(stochastic differential equation) as follow:

$$dX(t) = \mu(X(t), t)dt + \sigma(X(t), t)dB(t)$$

and we might have strong solution or weak solution. All the solutions of S.D.E have Markov property.

The Analysis is based on Black-Scholes market assumption. The asset price $\{S(t), t \geq 0\}$ follows the S.D.E. Let S(t) denote the price of stock and assume that it's an $It\hat{o}$ process.

$$dR(t) = \frac{dS(t)}{S(t)}$$

is process of the return on stock. And the stock price is the stochastic exponential of the return, and denoted by $\xi(R)$. In Black-scholes model it's assumed that the returns over non-overlapping time intervals are independent and have finite variance. This assumption leads to model for the return process R(t).

$$R(t) = rt + \sigma W_t$$

$$
\begin{aligned}
S(t) &= S(0)\xi(R)(t) \\
&= S(0)e^{R(t)-R(0)-\frac{1}{2}[R,R](t)} \\
&= S(0)e^{(r-\frac{1}{2}\sigma^2)+\sigma W_t}
\end{aligned}
$$

is the solution of Stochastic exponential. W is Brownian motion, r and σ(positive) are constants, and $S(0)$ is fixed. And let r denote the risk-free interest rate. The price of a claim contingent on S is the expected present value of its cash flows under the equivalent martingale measure.

Let U and L denote the upper barrier and lower barrier. In particular, always assume $S(0)$ is not equal to U and L. For double barrier, our hitting time defined as

$$\tau = inf\{t > 0 : S(t) \notin (L, U)\}$$

; A knock-out call option with maturity T and strike K pays $(S(T) - K)^+$ at time T if $\tau \geq T$ and zero otherwise. Its price is thus

$$e^{-rT}E[(S(T) - K)^+; \tau \geq T]$$

, the expectation taken with respect to the equivalent martingale measure.

Now suppose the barrier is monitored only at times iΔ, $i = 0, 1, ..., m$, where $\Delta = \frac{T}{m}$. Let us write S_i for $S_{i\Delta}$, so that $\{S_i, i = 0, 1, ...\}$ is asset price at monitoring instants. Define

$$\tilde{\tau} = inf\{n > 0 : S_n \notin (L, U)\}$$

The price of a discrete knock-out call option is given by

$$e^{-rT}E[(S_m - K)^+; \tilde{\tau} > m];$$

In general, there are no easily computed closed form expressions for the prices of these discrete barrier options. A consequence of our analysis is the obvious conclusion that the discrete price converges to the continuous price as the monitoring frequency increases. The following result shows how to adjust the continuous formula to obtain a far better approximation to the discrete price.

Chapter 2
The Discrete Price

2.1 Pricing Double-barrier option

The purpose of this section is to derive expressions for price V_m. I detail the case of the knock-out call option, the other cases following with only minor modification. Thus

$$\tilde{\tau} = inf\{m \geq 0 : S_n \notin (L, U)\}$$

$$V_m = e^{-rT} E[(S_m - K)^+; \tilde{\tau} \geq m]$$

is a knock-out call option. And define some notation as below:

$$S_n = S_0 e^{\bar{\mu} n\Delta - \sigma W_{n\Delta}}$$

where $\Delta = \frac{T}{m}$ and $W_{n\Delta} = \sqrt{\Delta} W_n$, W_n is sum of standard normal variables. $\bar{\mu} = r - \frac{\sigma^2}{2}$

$$dS_n = rS_n dn\Delta + \sigma S_n dW^P_{n\Delta}$$

is S.D.E that define in a measurable space (Ω, F, P).

$$
\begin{aligned}
V_m &= e^{-rT}E^P[(S_m - K)^+; \tilde{\tau} > m] \\
&= e^{-rT}E^P[(S_0 e^{\bar{\mu}m\Delta + \sigma W^P_{m\Delta}} - K)^+; \tilde{\tau} > m] \\
&= e^{-rT}E^P[(S_0 e^{\bar{\mu}m\Delta + \sigma W^P_{m\Delta}} - K) I_{\{S_m > K, L < min_n S_n < max_n S_n < U\}}] \\
&= e^{-rT}E^P[S_0 e^{\bar{\mu}m\Delta + \sigma W^P_{m\Delta}} I_{\{S_m > K, L < min_n S_n < max_n S_n < U\}}] \\
&- Ke^{-rT}E^P[I_{\{S_m > K, L < min_n S_n < max_n S_n < U\}}] \\
&= (1) - (2)
\end{aligned}
$$

First we solve (2), let $X_n = \dfrac{\ln \frac{S_n}{S_0}}{\sigma}$,$a = \dfrac{\ln \frac{L}{S_0}}{\sigma}$, $b = \dfrac{\ln \frac{U}{S_0}}{\sigma}$, $k = \dfrac{\ln \frac{K}{S_0}}{\sigma}$.
Under the P measure $dX_n = \dfrac{\bar{\mu}}{\sigma} dn\Delta + dW^P_{n\Delta}$
For removing drift then we use measure changing, $W^Q_{n\Delta} = \frac{\bar{\mu}}{\sigma} n\Delta + W^P_{n\Delta}$, under Q measure $dX_n = dW^Q_{n\Delta}$ we have the radon-Nikodyn derivative as below.

$$\frac{dP}{dQ} = e^{\frac{\bar{\mu}}{\sigma} W^Q_{m\Delta} - \frac{1}{2}(\frac{\bar{\mu}}{\sigma})^2 m\Delta}$$

$$
\begin{aligned}
&E^P[I_{\{X_m > k, a < min_n X(n) < max_n X(n) < b\}}] = E^Q[\frac{dP}{dQ} I_{\{X_m > k, a < min_n X_n < max_n X_n < b\}}] \\
&= E^Q[e^{\frac{\bar{\mu}}{\sigma} W^Q_{m\Delta} - \frac{1}{2}(\frac{\bar{\mu}}{\sigma})^2 m\Delta} I_{\{X_m > k, a < m^X_n < M^X_n < b\}}] \\
&= e^{-\frac{1}{2}(\frac{\bar{\mu}}{\sigma})^2 m\Delta} E^Q[e^{\frac{\bar{\mu}}{\sigma} W^Q_{m\Delta}} I_{\{X_m > k, a < m^X_n < M^X_n < b\}}]
\end{aligned}
$$

Noticed that $m^X_n = min_n X_n$ and $M^X_n = max_n X_n$

$$
\begin{aligned}
(2) &= Ke^{-rT - \frac{1}{2}(\frac{\bar{\mu}}{\sigma})^2 m\Delta} E^Q[e^{\frac{\bar{\mu}}{\sigma} X_m} I_{\{X_m > k, a < m^X_n < M^X_n < b\}}] \\
&= Ke^{-rT - \frac{1}{2}(\frac{\bar{\mu}}{\sigma})^2 m\Delta} \int_a^b e^{\frac{\bar{\mu}}{\sigma} x} Q(W^Q_{n\Delta} > k, a < m^X_n < M^X_n < b, W^Q_{n\Delta} \in dx) \\
&= Ke^{-rT - \frac{1}{2}(\frac{\bar{\mu}}{\sigma})^2 m\Delta} \int_k^b e^{\frac{\bar{\mu}}{\sigma} x} Q(a < m^X_n < M^X_n < b, W^Q_{n\Delta} \in dx)
\end{aligned}
$$

To derive a formula for (1),we need to apply two changes measure, first let $W^P_{n\Delta} = W^R_{n\Delta} + \sigma n\Delta$, then the radon-Nikodyn derivative is

$$\frac{dR}{dP} = e^{\frac{-\sigma^2}{2} m\Delta + \sigma W^P_{m\Delta}}$$

Under R measure $X_n = \frac{1}{\sigma}(r + \frac{\sigma^2}{2})n\Delta + W^R_{n\Delta}$ then let $W^{\bar{Q}}_{n\Delta} - \frac{1}{\sigma}(r + \frac{\sigma^2}{2})n\Delta = W^R_{n\Delta}$ the radon-Nikodyn derivative is

$$\frac{dR}{d\bar{Q}} = e^{\frac{1}{\sigma}(r+\frac{\sigma^2}{2})W^{\bar{Q}}_{m\Delta} - \frac{1}{2\sigma^2}(r+\frac{\sigma^2}{2})^2 m\Delta}$$

$$\begin{aligned}
(1) &= S_0 e^{-rT} E^P[e^{\bar{\mu}m\Delta + \sigma W^P_{m\Delta}} I_{\{S_m > K, L < min_n S_n < max_n S_n < U\}}] \\
&= S_0 E^R[I_{\{X_m > k, a < min_n X_n < max_n X_n < b\}}] \\
&= S_0 E^{\bar{Q}}(\frac{dR}{d\bar{Q}} I_{\{X_m > k, a < min_n X_n < max_n X_n < b\}}) \\
&= S_0 E^{\bar{Q}}[e^{\frac{1}{\sigma}(r+\frac{\sigma^2}{2})W^{\bar{Q}}_{m\Delta} - \frac{1}{2\sigma^2}(r+\frac{\sigma^2}{2})^2 m\Delta} I_{\{X_m > k, a < m^X_n < M^X_n < b\}}] \\
&= S_0 e^{-\frac{1}{2\sigma^2}(r+\frac{\sigma^2}{2})^2 m\Delta} E^{\bar{Q}}[e^{\frac{1}{\sigma}(r+\frac{\sigma^2}{2})W^{\bar{Q}}_{m\Delta}} I_{\{X_m > k, a < m^X_n < M^X_n < b\}}] \\
&= S_0 e^{-\frac{1}{2\sigma^2}(r+\frac{\sigma^2}{2})^2 m\Delta} \int_k^b e^{\frac{1}{\sigma}(r+\frac{\sigma^2}{2})x} \bar{Q}(a < m^X_n < M^X_n < b)
\end{aligned}$$

So the price of V_m, we express as follow:

Lemma 1. *With the notation as above, here has two measure, Q and $\bar{Q}$ $k = \frac{\ln(\frac{K}{S_0})}{\sigma}$, $a = \frac{\ln(\frac{L}{S_0})}{\sigma}$, $b = \frac{\ln(\frac{U}{S_0})}{\sigma}$, $W^Q_{m\Delta}$ is under Q measure, $W^{\bar{Q}}_{m\Delta}$ is under $\bar{Q}$ measure.*

$$\begin{aligned}
V_m = {} & S_0 e^{-\frac{1}{2\sigma^2}(r+\frac{\sigma^2}{2})^2 T} \int_k^b e^{\frac{1}{\sigma}(r+\frac{\sigma^2}{2})x} \bar{Q}(a < m^X_n < M^X_n < b, W^{\bar{Q}}_{m\Delta} \in dx) \\
& - K e^{-rT - \frac{1}{2}(\frac{\bar{\mu}}{\sigma})^2 T} \int_k^b e^{\frac{\bar{\mu}}{\sigma}x} Q(a < m^X_n < M^X_n < b, W^Q_{m\Delta} \in dx)
\end{aligned}$$

Chapter 3
Main Steps of The Prove

In this section, we give the main steps of the proof for the discrete type price. Most of the technical details of intermediate steps are relegated to appendices.

3.1 Approximation of the discrete probability

It is part of the derivation is heuristic, the complete prove still not finish.
To approximate the probability in Lemma.1
$Q(a < m_n^X < M_n^X < b, W_{m\Delta} \in dy) = P(\lambda_b \bigwedge \psi_a > m, W_{m\Delta} \in dy)$
The stopping time are defined as below:

$$\lambda_b = inf\{m \geq 0, W_{m\Delta} > b\}$$

$$\psi_a = inf\{m \geq 0, W_{m\Delta} < a\}$$

3.1.1 Discrete crossing probability

Use the techniques of Dynkin and Yushkevich (1969), in this section the upper bound and lower bound are a and 0.
Set $\sigma_0 = 0$, $\kappa_0 = T_0 = inf\{m > 0 : W_{m\Delta} \leq 0\}$. and define recursively

$$\sigma_n = inf\{m > \kappa_{n-1} : W_{m\Delta} \geq a\}$$

$$\kappa_n = inf\{m > \sigma_n : W_{m\Delta} \leq 0\}$$

$$\lim_{n\to\infty} P^x(\kappa_n \le m) = 0,$$

where m is finite.

Let $U_i = W_{\sigma_i\Delta} - a$ and $L_i = 0 - W_{\kappa_i\Delta}$ are overshoot and undershoot.

For any $y \in (0,\infty)$, the symmetry of Brownian motion that

$P^x(W_{m\Delta} \ge y \mid F_{\kappa_n}) = P^x(W_{m\Delta} \le -2L_n - y \mid F_{\kappa_n})$ on $\{\kappa_n \le m\}$

and so for any integer n ≥ 0, we have

$$\begin{aligned} &P^x(W_{m\Delta} \ge y, \kappa_n \le m) = P^x(W_{m\Delta} \le -2L_n - y, \kappa_n \le m) \\ &= P^x(W_{m\Delta} \le -2L_n - y, \sigma_n \le m) \end{aligned} \tag{3.1}$$

Similarly, for $y \in (-\infty, a)$

$P^x(W_{m\Delta} \le y \mid F_{\sigma_n}) = P^x(W_{m\Delta} \ge 2(U_n + a) - y \mid F_{\sigma_n})$ on $\{\sigma_n \le m\}$

Whence

$$\begin{aligned} &P^x(W_{m\Delta} \le y, \sigma_n \le m) = P^x(W_{m\Delta} \ge 2(U_n + a) - y, \sigma_n \le m) \\ &= P^x(W_{m\Delta} \ge 2(U_n + a) - y, \kappa_{n-1} \le m); n \ge 1 \end{aligned} \tag{3.2}$$

Apply (3.1) and (3.2) alternately and repeatedly to conclude, for $0 < y < a$, $n \ge 0:$

$$P^x(W_{m\Delta} \ge y, \kappa_n \le m) = P^x(W_{m\Delta} \le -y - 2na - 2(\sum_{i=0}^{n} L_i + \sum_{i=1}^{n} U_i)) \tag{3.3}$$

$$P^x(W_{m\Delta} \le y, \sigma_n \le m) = P^x(W_{m\Delta} \le y - 2na - 2(\sum_{i=1}^{n} U_i + \sum_{i=0}^{n-1} L_i)) \tag{3.4}$$

Now set $\pi_0 = 0$, $\rho_0 = T_a = inf\{m > 0 : W_{m\Delta} \ge a\}$ and define recursively

$$\pi_n = inf\{m > \rho_{n-1} : W_{m\Delta} \le 0\}$$

$$\rho_n = inf\{m > \pi_n : W_{m\Delta} \ge a\}$$

Proceed as previously to obtain the formulas, where m is finite,too.

$$\lim_{n\to\infty} P^x(\rho_n \le m) = 0; \tag{3.5}$$

And we have the following:

$$\begin{aligned} &P^x(W_{m\Delta} \le y, \rho_n \le m) \\ &= P^x(W_{m\Delta} \ge -y + 2(n+1)a + 2(\sum_{i=1}^{n} L_i + \sum_{i=0}^{n} U_i)) \end{aligned} \tag{3.6}$$

$$P^x(W_{m\Delta} \ge y, \pi_n \le m) = P^x(W_{m\Delta} \ge y + 2na + 2(\sum_{i=0}^{n-1} U_i + \sum_{i=1}^{n} L_i)) \tag{3.7}$$

Our goal is to solve the probabilities of (3.3) (3.4) (3.6) (3.7). First of all,Let $R_i^L = \dfrac{L_i}{\sqrt{\Delta_t}}$, $R_i^U = \dfrac{U_i}{\sqrt{\Delta_t}}$ for $i = 1, ..., n$ we analysis (3.3), and it can write as

$$\begin{aligned} &= P^x(W_{m\Delta} \ge y, \kappa_n \le m) = P^x(W_{m\Delta} \le -y - 2na - 2(\sum_{i=0}^{n} L_i + \sum_{i=1}^{n} U_i) \\ &= E(\Phi(\frac{(-y-2na-x)}{\sqrt{T}} - 2(\sum_{i=0}^{n} \frac{R_i^L}{\sqrt{m}} + \sum_{i=1}^{n} \frac{R_i^U}{\sqrt{m}}))) \end{aligned}$$

Using Taylor formula expand the function inside expectation by $\frac{1}{\sqrt{m}} = 0$

$$\begin{aligned} &\approx E(\Phi(\frac{-y-2na-x}{\sqrt{T}}) + \varphi(\frac{-y-2na-x}{\sqrt{T}})\frac{-2}{\sqrt{m}}(\sum_{i=0}^{n} R_i^L + \sum_{i=1}^{n} R_i^U) + o(\frac{1}{\sqrt{m}})) \\ &= \Phi(\frac{-y-2na-x}{\sqrt{T}}) + \varphi(\frac{-y-2na-x}{\sqrt{T}})\frac{-2}{\sqrt{m}}(2n+1)\beta + o(\frac{1}{\sqrt{m}}) \end{aligned}$$

Denote p(T;x;y) to be the transition probability function of Brownian motion

$$p(T;x;y) = \frac{1}{\sqrt{2\pi T}} e^{-\frac{(y-x)^2}{2T}}$$

From the Broadie and Glasserman and Kou(1997) "A continuity correction for discrete barrier option"

Therorem 1. *Convergence of all moments of the overshoot.* $E[(R_i^U)^q]$ *is qth moment of overshoot,* B_R *is the overshoot's distribution function and* Y *is the random variable for overshoot.*

$$\lim_{m\to\infty} E[(R_i^U)^q] = \int y^q dB_R(y)$$

for all $q > 0$ *, In particular,* $E[(R_i^U)] \to \beta$ *and* $\beta = -\frac{\zeta(\frac{1}{2})}{\sqrt{2\pi}} \approx 0.5826$ *with* ζ *the Riemann zeta function ,same way to obtain the undershoot's moment.*

Proof. See Appendix A.

Here assume the overshoots and undershoots are identically independent distributed, so denote the moment of overshoot and undershoot as β.
And now differential with respect to y:

$$P^x(W_{m\Delta} \in dy, \kappa_n \leq m)$$
$$= p(T;0;y+2na+x) - \frac{2(2n+1)\beta(y+2na+x)}{\sqrt{mT}} p(T;0;y+2na+x)$$

Same procced can obtain the following:

$$P^x(W_{m\Delta} \in dy, \sigma_n \leq m)$$
$$= p(T;0;y-2na-x) + \frac{2(2n)\beta(y-2na-x)}{\sqrt{mT}} p(T;0;y-2na-x)$$
$$P^x(W_{m\Delta} \in dy, \rho_n \leq m)$$
$$= p(T;0;y-2(n+1)a+x) + \frac{2(2n+1)\beta(y-2(n+1)a+x)}{\sqrt{mT}} p(T;0;y-2(n+1)a+x)$$
$$P^x(W_{m\Delta} \in dy, \pi_n \leq m)$$
$$= p(T;0;y+2na-x) - \frac{2(2n)\beta(y+2na-x)}{\sqrt{mT}} p(T;0;y+2na-x)$$

We can found the equivalent relation as following
$\kappa_{n-1} \bigvee \rho_{n-1} = \sigma_n \bigwedge \pi_n$ and $\sigma_n \bigvee \pi_n = \kappa_n \bigwedge \rho_n$

$$\begin{aligned}
&P^x(W_{m\Delta} \in dy, \rho_0 \bigwedge \kappa_0 \leq m) \\
&= \sum_{n=1}^{w} \{P^x(W_{m\Delta} \in dy, \kappa_{n-1} \leq m) + P^x(W_{m\Delta} \in dy, \rho_{n-1}) \\
&- P^x(W_{m\Delta} \in dy, \sigma_n \leq m) - P^x(W_{m\Delta} \in dy, \pi_n \leq m) \\
&+ P^x(W_{m\Delta} \in dy, \kappa_k \bigwedge \rho_k \leq m)\}
\end{aligned}$$

Due to the quantity of walk steps is finite, w is at most $\frac{m}{2}$ then the last term will be vanished.

$$\begin{aligned}
&P^x(W_{m\Delta} \in dy, \rho_0 \bigwedge \kappa_0 > m) \\
&= \sum_{n=-w}^{w} p(T; 0; y + 2na - x) - \sum_{n=-w+1}^{w} p(T; 0; y - 2na + x) \\
&- \sum_{n=1}^{w} \frac{2\beta}{\sqrt{mT}} \{(2n-1)p(T; 0; y - 2na + x)(y - 2na + x) \\
&- (2n-1)p(T; 0; y + 2(n-1)a + x)(y + 2(n-1)a + x) \\
&+ 2n((y + 2na - x)p(T; 0; y + 2na - x) - (y - 2na - x)p(T; 0; y - 2na - x))\}
\end{aligned}$$

And do some variable change,
Let $W'_{m\Delta} = W_{m\Delta} - a$
and $W_{0\Delta} = x - a$, with $y' = y - a$ and $x = 0$

$$\lambda'_q = inf\{m \geq 0, W'_{m\Delta} > q\}$$

$$\psi'_q = inf\{m \geq 0, W'_{m\Delta} < q\}$$

$$
\begin{aligned}
&P^0(W_{m\Delta} \in dy, \lambda_b \bigwedge \psi_a > m) \\
&= P^{-a}(W'_{m\Delta} \in dy', \lambda'_{b-a} \bigwedge \psi'_0 > m) \\
&= \sum_{n=-w}^{w} p(T;0;y+2n(b-a)) - \sum_{n=-w+1}^{w} p(T;0;y-2n(b-a)-2a) \\
&- \sum_{n=1}^{w} \frac{2\beta}{\sqrt{mT}} \{(2n-1)p(T;0;y-2n(b-a)-2a)(y-2n(b-a)-2a) \\
&- (2n-1)p(T;0;y+2(n-1)(b-a)-2a)(y-2(n-1)(b-a)-2a) \\
&+ 2n((y+2n(b-a))p(T;0;y+2n(b-a)) - (y-2n(b-a))p(T;0;y-2n(b-a)))\}
\end{aligned}
$$

Hence we have the probability for $Q(W_{m\Delta} \in dy, a < m_n^X < M_n^X < b)$

3.2 Discrete Price

By the Lemma.1

$$
\begin{aligned}
V_m &= S_0 e^{-\frac{1}{2\sigma^2}(r+\frac{\sigma^2}{2})^2 T} \int_k^b e^{\frac{1}{\sigma}(r+\frac{\sigma^2}{2})x} \bar{Q}(a < m_n^X < M_n^X < b, W_{m\Delta} \in dx) \\
&- K e^{-rT-\frac{1}{2}(\frac{\bar{\mu}}{\sigma})^2 T} \int_k^b e^{\frac{\bar{\mu}}{\sigma}x} Q(a < m_n^X < M_n^X < b, W_{m\Delta} \in dx)
\end{aligned}
$$

And replace the probability by the formula we obtain above.

Chapter 4
Simulation

4.1 Numerical solution

In this section, introduce how to use numerical method to compute the formula that we obtain as above. Frist we divide the formula to be two parts.
(1)$S_0 e^{-\frac{1}{2\sigma^2}(r+\frac{\sigma^2}{2})^2 T} \int_k^b e^{\frac{1}{\sigma}(r+\frac{\sigma^2}{2})x} \bar{Q}(a < m_n^X < M_n^X < b, W_{m\Delta} \in dx)$
(2)$Ke^{-rT-\frac{1}{2}(\frac{\bar{\mu}}{\sigma})^2 T} \int_k^b e^{\frac{\bar{\mu}}{\sigma}x} Q(a < m_n^X < M_n^X < b, W_{m\Delta} \in dx)$
Second use the simpson's rule which is default in Matlab Code.
Default setting of the double barrier option:
(MC I:use Monte carlo run first time with 100000 times)
(MC II:use Monte carlo run second time with 100000 times)
(MC III:use Monte carlo run thrid time with 100000 times)
P.S The table shows the M.C price and standard error

4.1.1 Case 1

- $S(0) = 90$ initial price, $\sigma = 0.1$ volatility, $T = 2$ long maturity
- $K = 88$ strike price,$r = 0.1$ risk-free interest rate, $m = 50$ random walk size

Table 4.1: Case 1

U	L	MCI	MCII	MCIII	Discrete
110	70	3.3660(5.2063)	3.3324(5.2140)	3.3269(5.1969)	3.3699
110	80	3.1405(5.1537)	3.1410(5.1389)	3.1546(5.1468)	3.1990
100	80	0.3986(1.5097)	0.3915(1.4894)	0.3908(1.4931)	0.3837

4.1.2 Case 2

- $S(0) = 90$ initial price, $U = 110$ upper barrier,$L = 70$ lower barrier, $\sigma = 0.1$
- $K = 88$ strike price,$r = 0.1$ risk-free interest rate, $m = 50$ random walk size

Table 4.2: Case 2

T	MCI	MCII	MCIII	Discrete
0.2	4.0907(3.4891)	4.0913(3.4805)	4.1278(3.4919)	4.1027
2	3.3425(5.2161)	3.3246(5.2040)	3.3390(5.1964)	3.3699
4	0.7322(2.5445)	0.7396(2.5631)	0.7305(2.5350)	0.7314

4.1.3 Case 3

- $S(0) = 90$ initial price, $U = 130$ upper barrier,$L = 60$ lower barrier, $T = 2$ long maturity
- $K = 88$ strike price,$r = 0.1$ risk-free interest rate, $m = 50$ random walk size

Table 4.3: Case 3

σ	MCI	MCII	MCIII	Discrete
0.1	13.0071(9.9440)	12.9938(9.9574)	12.9619(9.9526)	13.1196
0.3	2.4713(6.3403)	2.4732(6.3516)	2.5000(6.3940)	2.4853
0.5	0.3440(2.5280)	0.3516(2.5545)	0.3451(2.5398)	0.2779

4.1.4 Case 4

- $S(0) = 90$ initial price, $U = 110$ upper barrier,$L = 70$ lower barrier, $\sigma = 0.05$ volatility
- $K = 88$ strike price,$r = 0.02$ risk-free interest rate, $T = 2$ long maturity

Table 4.4: Case 4

m	MCI	MCII	MCIII	Discrete
10	5.7923	5.8254	5.8037	5.8588
100	5.7221	5.7760	5.7519	5.7445
500	5.7346	5.6870	5.7097	5.7152

4.1.5 Case 5

- $S(0) = 100$ initial price, $U = 130$ upper barrier,$L = 60$ lower barrier, $\sigma = 0.6$
- $K = 100$ strike price,$r = 0.1$ risk-free interest rate, $m = 50$ random walk size

Table 4.5: Case 5

T	MCI	MCII	MCIII	Discrete
0.2	2.3321(5.5696)	2.3371(5.5933)	2.3360(5.5754)	2.3388
2	0.0442(0.8047)	0.0465(0.8137)	0.0429(0.7716)	0.0209
3	0.0058(0.2704)	0.0088(0.3524)	0.0068(0.2871)	0.0015

4.2 Compare to other method

Now we show our method in a comparison to the corrected method and the continuous method. From the Broadie, Kou and Glasserman(1997), they show the down-and-out call option. Let the b $\rightarrow \infty$, then we can take knock-out option with double barrier as a down-and-out call

Table 4.6: Case 6

Barrier	Continuous barrier	Correrted barrier	True	Discrete barrier
85	6.308	6.322	6.322	6.3261
86	6.283	6.306	6.306	6.3123
87	6.244	6.281	6.281	6.2898
88	6.185	6.242	6.242	6.2543
89	6.099	6.184	6.184	6.2004
90	5.977	6.098	6.098	6.1206
91	5.808	5.977	5.977	6.0061
92	5.579	5.810	5.810	5.8466
93	5.277	5.584	5.584	5.6297
94	4.888	5.288	5.288	5.3423
95	4.398	4.907	4.907	4.9706
96	3.792	4.428	4.427	4.5002

To be continue

Table 4.6: Case 6

Barrier	Continuous barrier	Corrercted barrier	True	Discrete barrier
97	3.060	3.836	3.834	3.9176
98	2.189	3.121	3.126	3.2104
99	1.171	2.271	2.337	2.3675

Option parameters:

- $S_0 = 100$,$K = 100$, $\sigma = 0.3$, $r = 0.1$, $m = 50$, and $T = 0.2$

The true value is determined from a trinomial procedure modified in several ways to specifically handle discrete barriers and is fully descreibed in Broadie et al.(1996)

4.3 Conclusion

The Monte Carlo method, repeatly simulated 100000 times, the value still can't converge easily. Noticed that the sequense is decreasing as exponential decay in discrete formula. Observe inside the matlab work , we can found the terms n=-4 to 4 is converging almost. So compared to Monte Carlo Algorithm, it cost not so much time, and relative accuracy except in extreme cases with volatility and long maturity is big.

And compared to continuous method,corrected method and true value. We can found that our method won't affect by the barrier close to the initial value.

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Appendix A

Suppose b $\to \infty$ and $\mu \to 0$ in such way that for some $-\infty < \xi < \infty, \mu b \to \xi$. Then for all $0 \leq$ t, x $\leq \infty$, and $r > 0$, $\tau_w = \tau_w(b) = inf\{t : W(t) \geq b\}$, $G(t;\mu,b) = P_\mu(\tau_w(b) \leq t)$

$$\lim P_\mu(\tau(b) \leq b^2 t, S_{\tau_b} - b \leq x) = G(t;\xi,1)H(x)$$

and

$$\lim E_\mu[(S_{\tau(b)} - b)^r; \tau < \infty] = \frac{E_0 S_{\tau_+}^{r+1}}{(r+1)E_0(S_{\tau_+})}$$

and

$$H(x) = \frac{\int_{(0,x)} P_0(S_{\tau_+} \geq y)dy}{E_0 S_{\tau_+}}$$

Proof. Let $m = b^2 t$.

$$\begin{aligned}
&P_\mu(\tau(b) \leq m, S_{\tau(b)} - b \leq x) \\
&= E_0[e^{\theta S_r - \tau\psi(\theta)}; \tau \leq m, S_\tau - b \leq x] \\
&= e^{\theta b} E_0[e^{\theta(S_r - b) - \tau\psi(\theta)}; \tau \leq m, S_\tau - b \leq x]
\end{aligned}$$

it follows that $\theta b \to \xi$ and $\psi(\theta) \sim \frac{\theta^2}{2} \sim \frac{\xi^2}{2b^2}$. Hence at least for all finite x

$$\begin{aligned}
&e^\xi E_0[e^{\frac{\xi^2 \tau_w(1)}{2}}; \tau_w(1) \leq t]H(x) \\
&= E_0[e^{\xi W(\tau_w(1)) - \frac{\xi^2 \tau_w(1)}{2}}; \tau_w(1) \leq t]H(x) \\
&= P_\xi(\tau_w(1) \leq t)H(x) = G(t;\xi,1)H(x)
\end{aligned}$$

That this calculation is also valid when $x = \infty$ follows from the renewal theorem.
The indicated convergence of $E_\mu[(S_\tau - b)^r]$ follow by a similar calculation.

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