☆ $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$ (the 1st n squares)

$$\sum_{i=1}^{4} i^2 = \sum_{i=1}^{4} i + \sum_{i=2}^{4} i + \sum_{i=3}^{4} i + \sum_{i=4}^{4} i$$

1×1

$(1+1) \times 2$

$(1+1+1) \times 3$

$(1+1+1+1) \times 4$

(1) $\left(\sum_{j=1}^{k} j\right) - \left(\sum_{j=1}^{i-1} j\right) = \frac{k(k+1)}{2} - \frac{(i-1)i}{2}$

(2) $i + (i+1) + \cdots + (i+k-i)$
$= i(k-i+1) + \frac{(k-i)(k-i+1)}{2}$
$= (k-i+1)\left\{\frac{2i+k-i}{2}\right\}$

$$S = \sum_{i=1}^{k} i^2 = \sum_{i=1}^{k}\left(\sum_{j=i}^{k} j\right) = \sum_{i=1}^{k} \frac{(k+i)(k-i+1)}{2} = \sum_{i=1}^{k} \frac{k^2 + \cancel{ki} - \cancel{ki} - i^2 + k + i}{2}$$

$$= \frac{1}{2}\sum_{i=1}^{k}\left\{k(k+1) - i(i-1)\right\} = \frac{1}{2}\left\{k^2(k+1) - S + \frac{k(k+1)}{2}\right\}$$

$$\frac{3}{2}S = \frac{2k^2(k+1) + k(k+1)}{4} = \frac{k(k+1)(2k+1)}{4}.$$

$$\Rightarrow S = \frac{k(k+1)(2k+1)}{6} \quad \#$$

$$(x+y)^n = \sum_{i=1}^{n} x^i y^{n-i} \binom{n}{i}$$

$$(i-1)^4 = \sum_{k=1}^{4} i^k (-1)^{4-i} \binom{4}{k}$$

☆ $\sum_{i=1}^{k} i^3 = \left[\frac{k(k+1)}{2}\right]^2$ (the 1st n cubes)

$$\sum_{i=1}^{k} i^4 - (i-1)^4 \left(\begin{aligned} &= \{k^4 - (k-1)^4\} + \{(k-1)^4 - (k-2)^4\} + \cdots + \{2^4 - 1\} + 1 \\ &= k^4 \end{aligned} \right)$$

$$= \sum_{i=1}^{k} i^4 - (i^4 - 4i^3 + 6i^2 - 4i + 1) = \sum_{i=1}^{k} 4i^3 - 6i^2 + 4i - 1$$

$$= 4S - k(k+1)(2k+1) + 2k(k+1) - k$$

then

$$4S = k^4 + k(k+1)(2k+1) - 2k(k+1) + k$$
$$= k(k^3 + (k+1)(2k+1) - 2(k+1) + 1)$$
$$= k(k^3 + 2k^2 + \cancel{2k} + k + \cancel{1} - \cancel{2k} - \cancel{2} + \cancel{1})$$
$$= k^2(k^2 + 2k + 1)$$
$$= k^2(k+1)^2$$

$$\Rightarrow S = \left[\frac{k(k+1)}{2}\right]^2 \quad \#$$

No.
Date : :

Def (Anti-derivative)

A fn F is an antiderivative of f on I if $F'(x) = f(x) \quad \forall x \in I$

Ex Let $f(x) = \frac{1}{N}$ for $x = 1, \dots, N$ (discrete uniform)

Ex: Roll a fair dice

$$\mathbb{E}(X) = \sum_{x=1}^{N} x f(x) = \sum_{x=1}^{N} \frac{x}{N} = \frac{1}{N} \frac{N(N+1)}{2} = \frac{N+1}{2}$$

$$\mathbb{E}(X^2) = \sum_{x=1}^{N} x^2 f(x) = \sum_{x=1}^{N} \frac{x^2}{N} = \frac{1}{N} \frac{N(2N+1)(N+1)}{6} = \frac{(N+1)(2N+1)}{6}$$

$$V(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = \frac{(N+1)(2N+1)}{6} - \frac{(N+1)^2}{4} = \frac{2(N+1)(2N+1) - 3(N+1)^2}{12}$$

$$= \frac{(N+1)\{4N+2-3N-3\}}{12} = \frac{N^2-1}{12}$$

Expectation
1st moment

Variance
2nd central moment

$$H(X) = -\sum_{x=1}^{N} \frac{1}{N} \log\left(\frac{1}{N}\right) = \ln N$$

entropy : 火商
average level of {information / uncertainty}

Thm. If F is an antiderivative of f on I
then G is an antiderivative of f on I
$\iff$ G is of the form

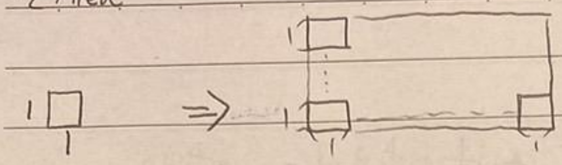
$$G(x) = F(x) + C \quad \forall x \in I$$

where C is a constant.

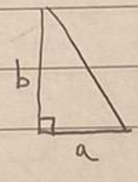
Chryu culture

(Calculus II)

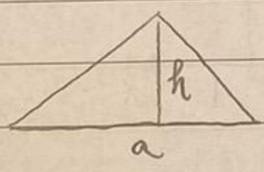
§ Area



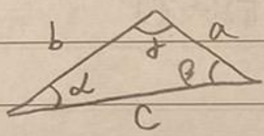
$A = ab$



$A = \frac{ab}{2}$,



$A = \frac{ah}{2}$



$s = \frac{a+b+c}{2}$

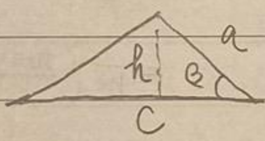
$A = [s(s-a)(s-b)(s-c)]^{\frac{1}{2}}$

海龍公式

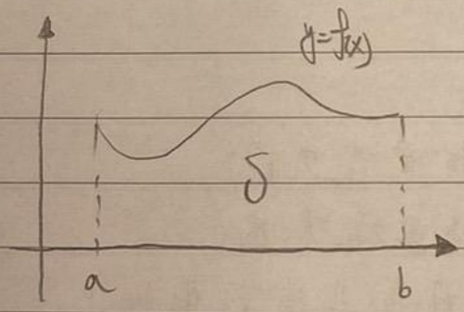
$A = \frac{1}{2} ab \sin(\gamma)$

$= \frac{1}{2} ac \sin(\beta)$

$= \frac{1}{2} cb \sin(\alpha)$

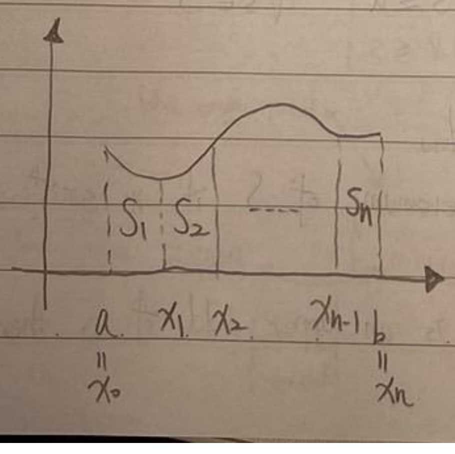


$\sin(\beta) = \frac{h}{a} = \frac{對}{斜}$



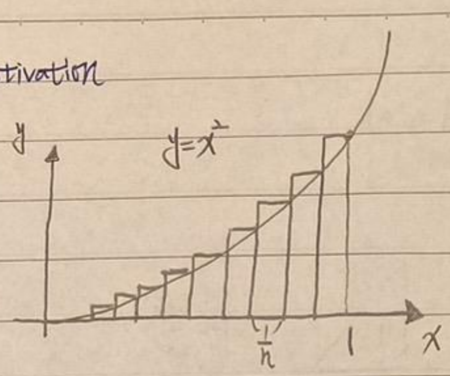
$S = \{(x,y) \mid a \le x \le b,\ 0 \le y \le f(x)\}$

$\Downarrow$



$S = S_1 + \cdots + S_n$

Ⓐ Motivation



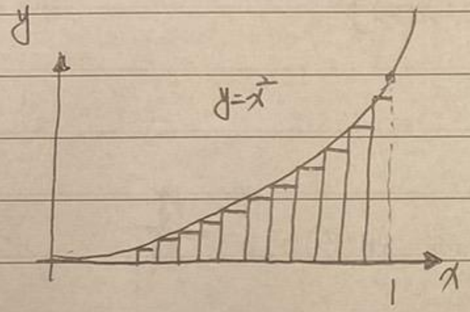
$$U_n = f\left(\frac{1}{n}\right)\cdot\frac{1}{n} + f\left(\frac{2}{n}\right)\frac{1}{n} + \cdots + f\left(\frac{n}{n}\right)\frac{1}{n}$$

$$= \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\frac{1}{n} = \sum_{k=1}^{n}\left(\frac{k}{n}\right)^2\frac{1}{n}$$

$$= \frac{1}{n^3}\left(\sum_{k=1}^{n} k^2\right) = \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6}$$

$$= \frac{(n+1)(2n+1)}{6n^2} = \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

$$\lim_{n\to\infty} U_n = \frac{2}{6} = \frac{1}{3} \quad \#$$



$$L_n = \sum_{k=1}^{n}\left(\frac{k-1}{n}\right)^2\frac{1}{n} = \frac{1}{n^3}\sum_{k=1}^{n}\left(k^2-2k+1\right)$$

$$= \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6} - n(n+1) + n\right)$$

$$\lim_{n\to\infty} L_n = \frac{1}{3} \quad \#$$

Hence

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} L_n = \frac{1}{3} \quad \#$$

Riemann 1826～1866，1854 提出（実数系建構未成）

Darboux 1842～1917，1870 提出（実数系建構完成，実数系具有完備性）

★ Completeness of $\mathbb{R}$ $\left(\begin{array}{l} u = \sup S \\ l = \inf S \end{array}\right)$

- S is bdd $\left|\begin{array}{l}\text{above}\\ \text{below}\end{array}\right|$ if $\exists$ $\left|\begin{array}{l}u\in\mathbb{R}\\ l\in\mathbb{R}\end{array}\right|$ s.t $\left|\begin{array}{l}s\le u\\ l\le s\end{array}\right|$ $\forall s\in S$
- S is bdd if both bdd above and below
- 若 S is bdd $\left|\begin{array}{l}\text{above}\\ \text{below}\end{array}\right|$, then $\left|\begin{array}{l}u\in\mathbb{R}\\ l\in\mathbb{R}\end{array}\right|$ is $\left|\begin{array}{l}\text{superemum}\\ \text{infimum}\end{array}\right|$ of S if it satisfies

least upper bdd

greatest lower bdd

(i) $\left|\begin{array}{l}u\\ l\end{array}\right|$ is $\left|\begin{array}{l}\text{upper}\\ \text{lower}\end{array}\right|$ bdd of S (ii) If $\left|\begin{array}{l}u'\\ l'\end{array}\right|$ is an $\left|\begin{array}{l}\text{upper}\\ \text{lower}\end{array}\right|$ bdd of S, then $\left|\begin{array}{l}u\le u'\\ l'\le l\end{array}\right|$

Def A partition P of $[a,b]$ is a finite set
$P = \{x_0, x_1, \cdots, x_n\}$ s.t $a = x_0 < x_1 < \cdots < x_n = b$
denote $\Delta x_k = x_k - x_{k-1}$ for $k = 1, 2, \cdots, n$
the norm of P is $\|P\| = \max_i \Delta x_i$
(the mesh of P)

$$[a,b]$$

$$[x_0, x_1] \cup [x_1, x_2] \cup \cdots \cup [x_{k-2}, x_{k-1}] \cup [x_{k-1}, x_k]$$

☆ Unequal subinterval

$x_0 = a$

$x_1 = a + (x_1 - x_0) = a + \Delta x_1$

$x_2 = a + (x_2 - x_1) + (x_1 - x_0) = a + \Delta x_2 + \Delta x_1$

$\vdots$

$x_n = a + \Delta x_n + \cdots + \Delta x_1 = a + (b-a) = b$

☆ Equal subinterval

$x_0 = a$

$x_1 = a + \Delta x$

$x_2 = a + 2\Delta x$

$\vdots$

$x_n = a + n \cdot \Delta x$

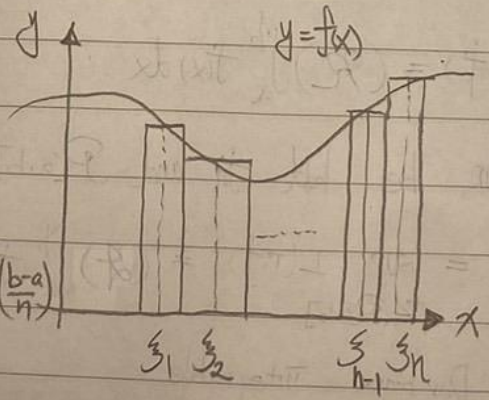
Def. Let $f : [a,b] \to \mathbb{R}$ and P is partition of $[a,b]$
$\xi_k \in [x_{k-1}, x_k]$

$$S(P,f) = \sum_{k=1}^{n} f(\xi_k) \Delta x_k$$

is called Riemann Sum

☆
If pick ξ_k to be right endpoint then w/ equal width subinterval

$$S(P,f) = \sum_{k=1}^{n} f\left(a + \left(\frac{b-a}{n}\right) k\right)\left(\frac{b-a}{n}\right)$$



Def Let $f:[a,b] \to \mathbb{R}$ be bdd fn and P be partition of $[a,b]$

define $M_k(f) = \sup\{f(x) : x \in [x_{k-1}, x_k]\}$

$m_k(f) = \inf\{f(x) : x \in [x_{k-1}, x_k]\}$

then $U(P,f) = \sum_{k=1}^{n} M_k(f)\Delta x_k$ is Darboux upper sum

$L(P,f) = \sum_{k=1}^{n} m_k(f)\Delta x_k$ " lower "

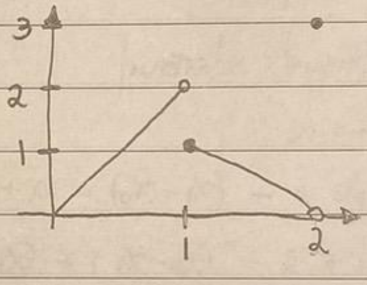
Remark: $L(P,f) \le U(P,f)$

P^* is refinement of P if $P \subset P^*$

$L(P,f) \le L(P^*,f) \le U(P^*,f) \le U(P,f)$

Ex let $f:[0,2] \to \mathbb{R}$

$$f(x) = \begin{cases} 2x & 0 \le x < 1 \\ 2-x & 1 \le x < 2 \\ 3 & x = 2 \end{cases}$$



let $P = \{0, 1, \frac{3}{2}, 2\}$ w/ $\xi = (\frac{1}{3}, 1, \frac{7}{4})$

$S(P,f) = 2\cdot\frac{1}{3}(1-0) + (2-1)\cdot(\frac{3}{2}-1) + (2-\frac{7}{4})(2-\frac{3}{2})$

$U(P,f) = 2(1-0) + 1(\frac{3}{2}-1) + 3(2-\frac{3}{2})$

$L(P,f) = 0(1-0) + (2-\frac{3}{2})(\frac{3}{2}-1) + 0(2-\frac{3}{2})$

Def. Let $f:[a,b] \to \mathbb{R}$, f is Riemann Integrable on $[a,b]$

if $\lim_{\|P\| \to 0} S(P,f) = (R)\int_a^b f(x)\,dx$

Def. Let $f:[a,b] \to \mathbb{R}$ be bdd fn and $\mathcal{P}[a,b]$ are all partition of $[a,b]$

if $\inf_{P \in \mathcal{P}[a,b]} U(P,f) = \sup_{P \in \mathcal{P}[a,b]} L(P,f) = (D)\int_a^b f(x)\,dx$

then f is Darboux Integrable

Integral of f from a to b

upper limit

★ Riemann Intergrability = Darboux Integrability

integrand

$$(\mathcal{R})\int_a^b f(x)\,dx = (\mathcal{D})\boxed{\int_{a}^{b} f(x)\,dx}$$

variable of Integration

lower limit

thm Each of following Conditions is suff. for the existence of Riemann Integral $\int_a^b f(x)\,dx$

a) f is conti on $[a,b]$

b) f is bdd variation on $[a,b]$

Riemann Integral is not good enough!

- does not handle fn w/ many discontinuities.
- 〃 unbounded fn

Ex.

let $f:[0,1]\to\mathbb{R}$, $f(x) = \begin{cases} 1 & x\in\mathbb{Q} \\ 0 & x\in\mathbb{Q}^c \end{cases}$

given any interval $[a,b]\subset[0,1]$

then $\inf_{[a,b]} f = 0$, $\sup_{[a,b]} f = 1$

$0 = L(P,f) \longrightarrow \sup_P L(P,f)$

$\neq$ $\Rightarrow$ f is not Riemann Integrable

$1 = U(P,f) \longrightarrow \inf_P U(P,f)$

Ex. (Unbounded fn)

let $f:[0,1]\to\mathbb{R}$, $f(x) = \begin{cases} \frac{1}{\sqrt{x}} & 0<x\le 1 \\ 0 & x=0 \end{cases}$

Consider a partition P, $\sup_{[x_0,x_1]} f = \infty \Rightarrow U(P,f) = \infty$

★ Integration is a linear functional

$$f \xmapsto{I(\cdot)} \int_a^b f(x)\,dx$$

$$I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$$

$$\left(\begin{array}{l} \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \\ \text{if } a=b \Rightarrow \int_a^a f(x)\,dx = 0 \end{array} \right)$$

★ Interval decomp. $[a,b] = [a,m] \cup [m,b]$

$$\int_a^m f(x)\,dx + \int_m^b f(x)\,dx = \int_a^b f(x)\,dx$$

If $[c,d] \subseteq [a,b]$ and $f(x) \geq 0$

$$\int_c^d f(x)\,dx \leq \int_a^b f(x)\,dx$$

★ Ineq.

If $f(x) \leq g(x) \quad \forall x \in [a,b]$

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

when $f = 0 \Rightarrow 0 \leq \int_a^b g(x)\,dx$

★ $$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx \leq \sup_{x\in[a,b]} |f(x)| \cdot (b-a)$$

Ex. $\int_0^1 (4+3x^2)\,dx$

$$= \int_0^1 4\,dx + \underbrace{3\int_0^1 x^2\,dx}$$

$\left(\text{In prev Ex.} \quad \int_0^1 x^2\,dx = \frac{1}{3} \right)$

$$= 4(1-0) + 3 \times \frac{1}{3} = 5$$

Thm (MVT, integral ver.)

Let $f: [a,b] \longrightarrow \mathbb{R}$ be conti fn

then $\exists\ c \in [a,b]$ s.t $\int_a^b f(x)\,dx = f(c)(b-a)$

<pf>

By Extreme value thm, $\exists\ x_1, x_2 \in [a,b]$

$m = f(x_1)$ ☆ min of f

$M = f(x_2)$ ☆ max of f

Since $m \le f(x) \le M$

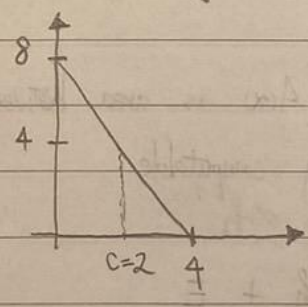
$\Rightarrow \underbrace{\int_a^b m\,dx}_{=m(b-a)} \le \int_a^b f(x)\,dx \le \int_a^b M\,dx = M(b-a)$

$\Rightarrow m \le \dfrac{\int_a^b f(x)\,dx}{b-a} \le M$

by intermediate value thm, $\exists\ c$ between x_1 and x_2

s.t $f(c) = \dfrac{\int_a^b f(x)\,dx}{b-a}$ #

Ex. Find average value of $f(x) = 8-2x$ over $[0,4]$ and find c s.t $f(c)$ equal to the average value of f over $[0,4]$



Ans:

$A = \frac{1}{4-0} \int_0^4 8-2x\,dx$

$= \frac{1}{4} \times \frac{4\times 8}{2} = 4$

$f(c) = 8-2c = 4 \Rightarrow c = 2$

Ex. Given $\int_0^3 x^2\,dx = 9$, find c s.t. $f(c)$ is equal to the average of $f(x) = x^2$ over $[0,3]$

Ans.

$f(c) = c^2 = \frac{9}{3} \Rightarrow c = \pm\sqrt{3}$

$(-\sqrt{3} \notin [0,3])$

No. P-8
Date : :

§ Fundamental theorem of Calculus (FTC)

$\hookrightarrow$ tangent problem $\qquad\qquad\qquad$ $\hookrightarrow$ area problem

differential calculus $\xleftrightarrow{\text{FTC}}$ integral calculus

☆ FTC give the precise inverse relationship between derivative and integral

Newton (1642 ~ 1727) 指控抄襲 Leibniz (1646 ~ 1716) $\left(\text{Notation } \dfrac{dy}{dx},\ \int_a^b f(x)\,dx\right)$

Thm. (FTC, part I)

Let $f : [a,b] \to \mathbb{R}$ be conti fn, and F defined by

$$F(x) = \int_a^x f(t)\,dt \quad \forall x \in [a,b]$$

then F is conti on $[a,b]$

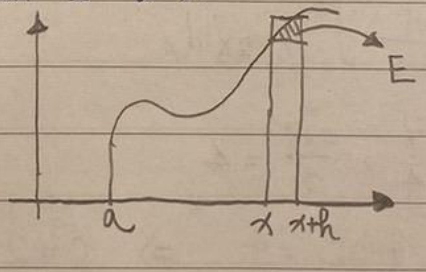
and diff on (a,b), and $F'(x) = f(x)$

(Remark. ☆ F is anti-derivative of f
(反導數)

☆ Conti fn on compact set is unif conti)

☆ Geometric meaning

$x \longmapsto A(x)$, $A(x)$ is area between a to x



$A(x)$ may not easily computable

$$\underbrace{A(x+h) - A(x)}_{\text{area between } x \text{ to } x+h} = f(x)\cdot h + E$$

$$f(x) = \frac{A(x+h) - A(x)}{h} - \frac{E}{h}$$

$$|E| \le h\,[\underbrace{f(x+h_1)}_{\max} - \underbrace{f(x+h_2)}_{\min}]$$

$$\left| f(x) - \frac{A(x+h) - A(x)}{h} \right| = \frac{|E|}{h} \le f(x+h_1) - f(x+h_2) \longrightarrow 0 \text{ as } h \to 0$$

$$f(x) = \lim_{h\to 0} \frac{A(x+h) - A(x)}{h} \triangleq A'(x)$$