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Def. Let f be fn defined at least on an open interval except possibly at a itself

$$\lim_{x\to a} f(x) = L$$

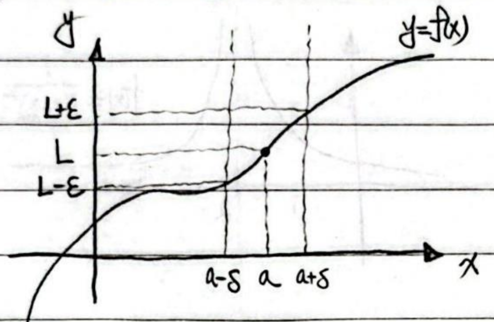
means that

$\forall \varepsilon > 0,\ \exists \delta > 0$ s.t. $0 < |x-a| < \delta$ (s.t.: such that; $0<|x-a|<\delta$: x-tolerance)

$\Rightarrow |f(x) - L| < \varepsilon$ ($\Rightarrow$: implies; $|f(x)-L|<\varepsilon$: y-tolerance)

Remark. $f(x)$ tends | approaches to L as x tends | approaches to a

$f(x)$ is near L whenever x is near a



Ex. Given $\lim_{x\to 3} 2x-5 = 1$

find δ s.t. $|2x-5-1| < 0.01$

whenever $0<|x-3|<\delta$

$\varepsilon = 0.01$

$|2x-5-1| = |2x-6| = 2|x-3| < 0.01$

choose $\delta = \frac{0.01}{2} = 0.005$

$0 < |x-3| < 0.005$

implies $|2x-5-1| = 2|x-3| < 2\delta = 0.01$

As result, x-tolerance within 0.005 of 3

, y-tolerance within 0.01 of 1

Note that 0.005 is the largest value of δ that guarantee

$|2x-5-1| < 0.01$ whenever $0<|x-3|<\delta$

Any smaller positive δ would also work.

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Ex $\lim_{x\to 2} x^2 = 4$

$$|x^2-4| = |(x+2)(x-2)| < C|x-2| < \varepsilon$$

by taking $\delta = \frac{\varepsilon}{C}$

assume $|x-2| < 1$ then $1 < x < 3$

so $3 < x+2 < 5$

thus we have $|x+2| < 5$

$C = 5$ is a suitable choice

but now we have $|x-2| < 1$ and $|x-2| < \frac{\varepsilon}{5}$

take $\delta = \min\{1, \frac{\varepsilon}{5}\}$

It follows that

if $|x-2| < \delta$ then $|x^2-4| \leq \varepsilon$

Thm. Suppose that α is a constant and

$$\lim_{x\to a} f(x) = L \quad \text{and} \quad \lim_{x\to a} g(x) = M$$

exist. Then

1. $\lim_{x\to a} \alpha f(x) = \alpha L$ (Scalar multiple)
2. $\lim_{x\to a} [f(x) \pm g(x)] = L \pm M$ (Sum and Difference)
3. $\lim_{x\to a} f(x)g(x) = LM$ (Product)
4. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$, $M \neq 0$ (Quotient)
5. $\lim_{x\to a} [f(x)]^n = L^n$ (Power)
6. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{L}$ (Root)

Ex $\lim_{x\to 2} 4x^2+3 = 4(\lim_{x\to 2} x^2) + 3$

$= 16 + 3 = 19$

$\lim_{x\to 1} \frac{x^2+x+2}{x+1} = \frac{1+1+2}{1+1} = 2$

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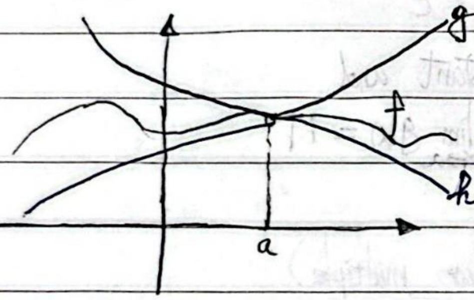
Ex Find $\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x+1)(x-1)}{x-1} = \lim_{x\to 1} x+1 = 2$
(Dividing Out Technique)

Ex. Find $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} = \lim_{x\to 0} \frac{x}{x(1+\sqrt{1+x})} = \frac{1}{1+\sqrt{1+x}} = \frac{1}{2}$
(Rationalizing Technique)

Thm. If $h(x) \le f(x) \le g(x)$ for all x in an open interval containing a, except possibly at a itself, and if

$$\lim_{x\to a} h(x) = L = \lim_{x\to a} g(x)$$

then $\lim_{x\to a} f(x) = L$



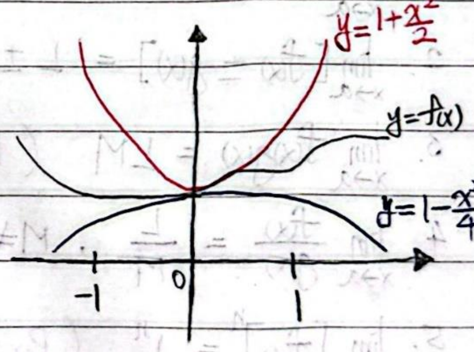
Ex $1-\frac{x^2}{4} \le f(x) \le 1+\frac{x^2}{2}$

find $\lim_{x\to 0} f(x)$

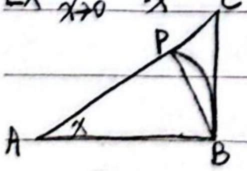
A. Since $\lim_{x\to 0} 1-\frac{x^2}{4} = 1$

$\lim_{x\to 0} 1+\frac{x^2}{2} = 1$

by Squeeze thm, $\lim_{x\to 0} f(x) = 1$



Ex $\lim_{x\to 0} \frac{\sin(x)}{x}$



$\triangle ABP \le \triangle ABP \le \triangle ABC$

$\frac{1}{2} 1 \cdot 1 \cdot \sin(x) \le \frac{1}{2} \cdot 1^2 \cdot x \le \frac{1}{2} \cdot 1 \cdot \frac{\sin(x)}{\cos(x)}$

$1 \le \frac{x}{\sin(x)} \le \frac{1}{\cos(x)}$

$1 \ge \frac{\sin(x)}{x} \ge \underset{\downarrow}{\cos(x)}$ by Squeeze thm $\Rightarrow \lim_{x\to 0} \frac{\sin(x)}{x} = 1$

$\quad 1$

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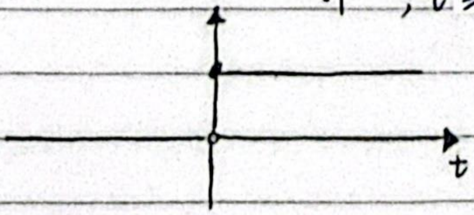
★ Heaviside fn $H(t) = \begin{cases} 0 & , t < 0 \\ 1 & , t \geq 0 \end{cases}$



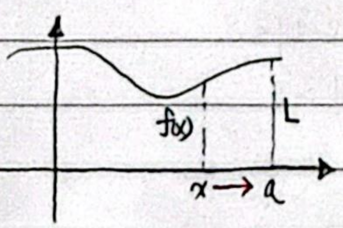
$\lim_{t\to 0} H(t)$ does not exist

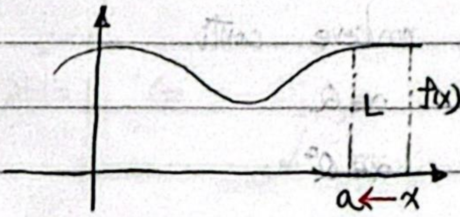
t approaches 0 from $\begin{vmatrix} \text{left} \\ \text{right} \end{vmatrix}$, $H(t)$ approaches $\begin{vmatrix} 0 \\ 1 \end{vmatrix}$

write $\left.\begin{aligned} \lim_{t\to 0^+} H(t) = 1 \\ \lim_{t\to 0^-} H(t) = 0 \end{aligned}\right\}$ One-Sided Limit

Def $\lim_{x\to a^-} f(x) = L \iff \forall \varepsilon > 0, \exists \delta > 0$ s.t. $a-\delta < x < a$ $\Rightarrow |f(x)-L| < \varepsilon$

a^+ $\quad a < x < a+\delta$





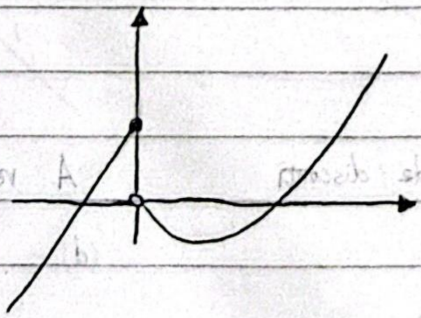
Ex $f(x) = \begin{cases} 2x+1 & , x \leq 0 \\ x^2-x & , x > 0 \end{cases}$

$\lim_{x\to 0} f(x)$ does not exist

<pf>

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 2x+1 = 1$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2-x = 0$



Thm $\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$

Remark 極限存在 $\iff$ 左極限 = 右極限

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Ex $\lim_{x\to 0} \frac{|x|}{x}$

$$\lim_{x\to 0^-} \frac{-x}{x} = -1 \neq 1 = \lim_{x\to 0^+} \frac{x}{x}$$

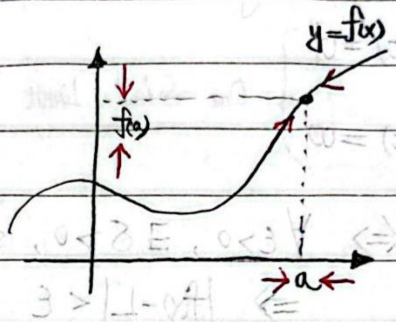
$\Rightarrow \lim_{x\to 0} \frac{|x|}{x}$ does not exist

§ Continuity

Def. A fn f is conti at a if $\lim_{x\to a} f(x) = f(a)$ $\left(\begin{array}{l} \forall \varepsilon > 0, \exists \delta > 0 \\ \text{s.t. } 0 < |x-a| < \delta \\ \Rightarrow |f(x) - f(a)| < \varepsilon \end{array} \right)$

Note that def require 3 things.

1. $f(a)$ is defined
2. $\lim_{x\to a} f(x)$ exist
3. $\lim_{x\to a} f(x) = f(a)$



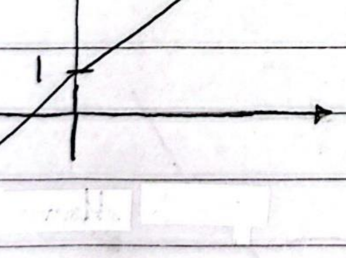
☆ Dirichlet fn is nowhere conti. $\varepsilon = \frac{1}{2}$, $|x-a| < \delta$

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & \overline{\mathbb{Q}} \end{cases} \Rightarrow 1 = |f(x) - f(a)| > \varepsilon$$

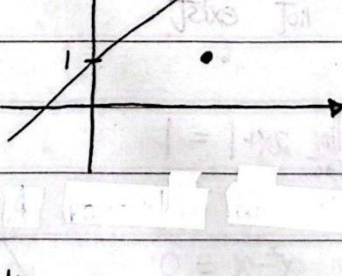
dense

• Discontinuities

(a)

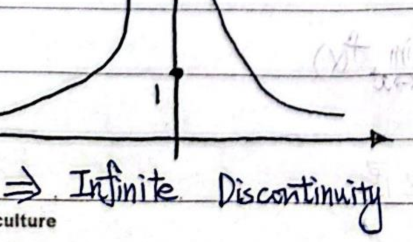


(b)



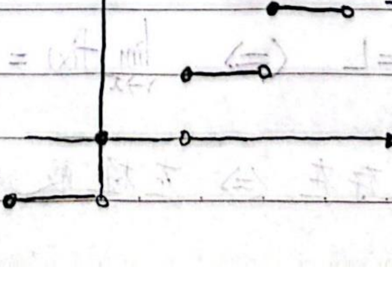
$\Rightarrow$ Removable Discontinuity

(c)



$\Rightarrow$ Infinite Discontinuity

(d)



$\Rightarrow$ Jump Discontinuity

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Ex. (a) $f(x) = \frac{x^2-x-2}{x-2}$ (b) $f(x) = \begin{cases} \frac{x^2-x-2}{x-2} & x \neq 2 \\ 1 & x = 2 \end{cases}$

(c) $f(x) = \begin{cases} \frac{1}{x^2} & x \neq 0 \\ 1 & x = 0 \end{cases}$ (d) $f(x) = \lfloor x \rfloor$ → { floor fn / greatest integer fn

A: (a) $f(2)$ not defined

(b) $f(2)$ is defined but $\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{(x-2)(x+1)}{(x-2)} = 3 \neq 1 = f(2)$

(c) $f(0)$ is defined but $\lim_{x\to 0} \frac{1}{x^2}$ does not exist

(d) $\lfloor x \rfloor$ has discontinuities at all of the integers

$\lim_{x\to n} \lfloor x \rfloor$ does not exist if $n \in \mathbb{Z}$

Def. A fn f is conti from $\left|\begin{matrix} \text{left} \\ \text{right} \end{matrix}\right|$ at a if $\lim_{x\to a^-} f(x) = f(a)$ (a^+)

Ex $f(x) = \lfloor x \rfloor$

At each $n \in \mathbb{Z}$, $\lim_{x\to n^-} f(x) = n-1 \neq f(n)$

$\lim_{x\to n^+} f(x) = n = f(n)$

is conti from right but disconti from left

Def. f is conti at $a \iff f(a) = \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$

Remark 連續 $\iff$ 左連續且右連續

Ex. $f(x) = \begin{cases} 1 & x = 1 \\ ax^2+b & 1 < x < 2 \\ \frac{6x^2}{x+1} & 2 \leq x \end{cases}$

f is conti on $\mathbb{R}$, find a and b

A: f conti at $x=1$, $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} ax^2+b = a+b = 1 = f(1)$

$x=2$, $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} ax^2+b = 4a+b = 8 = f(2)$

$\Rightarrow a = \frac{7}{3}$

$b = -\frac{4}{3}$

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Ex. $f(x) = 1-\sqrt{1-x^2}$

A: for $-1 < a < 1$

$$\lim_{x\to a} f(x) = 1-\sqrt{1-a^2} = f(a)$$

thus f is conti at a if $-1 < a < 1$

Similar $\lim_{x\to -1^+} f(x) = f(-1)$

$$\lim_{x\to 1^-} f(x) = f(1)$$

So f is conti from $\begin{cases} \text{right at } -1 \\ \text{left at } 1 \end{cases}$

Thm. If f and g conti at a then

(i) $f \pm g$

(ii) fg

(iii) $\frac{f}{g}$, $g(a) \neq 0$

(iv) αf, $\alpha \in \mathbb{R}$

are also conti at a

Remark. polynomials $P(x) = a_0 + a_1x_1 + \cdots + a_nx^n$

rational fn. $\frac{P(x)}{Q(x)}$, $Q(x) \neq 0$

not fn $\sqrt[n]{x}$

$\begin{cases} \text{trigonometric fn} & \sin(x), \cos(x) \\ \text{exponential fn} & e^x \\ \text{logarithmic fn} & \ln x \end{cases}$ 互為反函數

Ex. $F(x) = 3|x| + \frac{x^3-x}{x^2-5x+6} + 4$

let $f(x) = |x|$

$g(x) = x^3 - x$

$h(x) = x^2 - 5x + 6$

$f \cdot g \cdot h$ are everywhere conti

$\Rightarrow F$ is conti except at 2 and 3

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Def. f is conti on I (interval) if it is conti at each interior pt of I and one-sided conti at whatever endpoints the interval may contain.

Thm. If f is conti at b and $\lim_{x\to a} g(x) = b$
then $\lim_{x\to a} f(g(x)) = f(b) = f(\lim_{x\to a} g(x))$

order of these two symbols can be reversed

Ex. $\lim_{x\to \frac{\pi}{2}} \cos(2x + \sin(\frac{3\pi}{2}+x))$
$= \cos(\lim_{x\to\frac{\pi}{2}} 2x + \sin(\frac{3\pi}{2} + \lim_{x\to\frac{\pi}{2}} x))$
$= \cos(\pi + \sin(2\pi)) = -1$ ※

Thm. If g conti at a, f is conti at g(a)
then f(g(x)) is conti at a.
g is conti at a $\Rightarrow \lim_{x\to a} g(x) = g(a)$
f   g(a) $\Rightarrow \lim_{x\to g(a)} f(x) = f(g(a))$
$\Rightarrow \lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(g(a))$

Thm. (Bolzano's thm) 勘根定理
Let f be conti on [a,b], then if $f(a)f(b) < 0$ (opposite sign)
$\exists\, c \in (a,b)$ s.t. $f(c) = 0$
Remark. $x=c$ 為 f 的零根
Ex Prove $x^9 = 3x^6 - 1$ has sol in (0,1)
<pf> Set $f(x) = x^9 - 3x^6 + 1$, f is conti on [0,1]
and $f(0)f(1) = 1\cdot -1 < 0$
by Bolzano's thm, $\exists\, c \in (0,1)$ s.t. $f(c) = 0$
Hence $x^9 = 3x^6 - 1$ has sol in (0,1)

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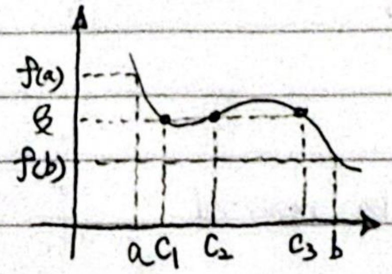
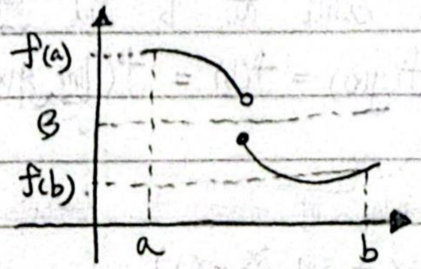
Thm. (Intermediate Value thm)

If f conti on $[a,b]$ and β is any value between $f(a)$ and $f(b)$, then there exist at least $c \in (a,b)$ s.t. $f(c) = \beta$





<pf> for any β between $f(a)$ and $f(b)$

set $F(x) = f(x) - \beta \quad \forall x \in [a,b]$

F is conti on $[a,b]$

$F(a)F(b) = (f(a) - \beta)(f(b) - \beta) < 0$

by Bolzano's thm $\exists\, c \in (a,b)$

$F(c) = f(c) - \beta = 0 \Rightarrow f(c) = \beta$

Ex. Show $\sqrt{2x+5} = 4 - x^2$ has a sol

<pf> write $\sqrt{2x+5} - 4 + x^2 = 0$

Set $f(x) = \sqrt{2x+5} - 4 + x^2$

f is conti on $[-\frac{2}{5}, \infty)$

we find $f(0) = \sqrt{5} - 4 \approx -1.76$

$f(2) = 3$

by Intermediate value thm

$\exists\, c$ s.t $f(c) = 0$

c solve the original equation.

§ Infinite Limits

Consider $f(x) = \frac{3}{x-2}$, $\lim_{x \to 2^-} \frac{3}{x-2} = -\infty$

$\lim_{x \to 2^+} \frac{3}{x-2} = \infty$

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Def. $\lim_{x\to a} f(x) = \left|\begin{matrix}\infty \\ -\infty\end{matrix}\right| \iff \forall \left|\begin{matrix}M^*<0 \\ M>0\end{matrix}\right| \ \exists\, \delta>0$ s.t $0<|x-a|<\delta$

$\Rightarrow f(x) \left|\begin{matrix}>M \\ <M^*\end{matrix}\right|$

Remark infinite limit from $\left|\begin{matrix}\text{left} \\ \text{right}\end{matrix}\right|$ replace $0<|x-a|<\delta$ by $\left|\begin{matrix}a-\delta<x<a \\ a<x<a+\delta\end{matrix}\right|$

Ex prove $\lim_{x\to 0}\frac{1}{x^2}=\infty$

<pf> Let $M>0$

claim: find δ s.t $0<|x-a|<\delta$ then $\frac{1}{x^2}>M$

$\frac{1}{x^2}>M \iff \frac{1}{M}>x^2$

$\iff \frac{1}{\sqrt{M}}>x$

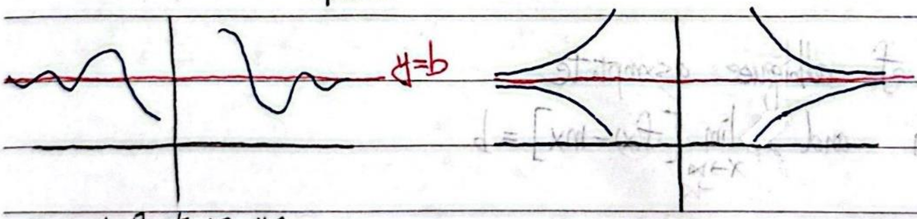
$\iff \frac{1}{\sqrt{M}}>|x|$

choose $\delta=\frac{1}{\sqrt{M}}$ and $0<|x|<\delta$ then $\frac{1}{x^2}>M$

☆ Asymptote

Def The line $\left|\begin{matrix}y=mx+b \\ y=b \\ x=a\end{matrix}\right|$ is $\left|\begin{matrix}\text{oblique} \\ \text{horizontal} \\ \text{vertical}\end{matrix}\right|$ asymptote of $f(x)$

if either $\left|\begin{matrix}\lim_{x\to a^+} f(x)=\pm\infty \text{ or } \lim_{x\to a^-} f(x)=\pm\infty \\ \lim_{x\to\infty} f(x)=b \text{ or } \lim_{x\to-\infty} f(x)=b \\ \lim_{x\to\infty}[f(x)-(mx+b)]=0 \text{ or } \lim_{x\to-\infty}[f(x)-(mx+b)]=0\end{matrix}\right|$



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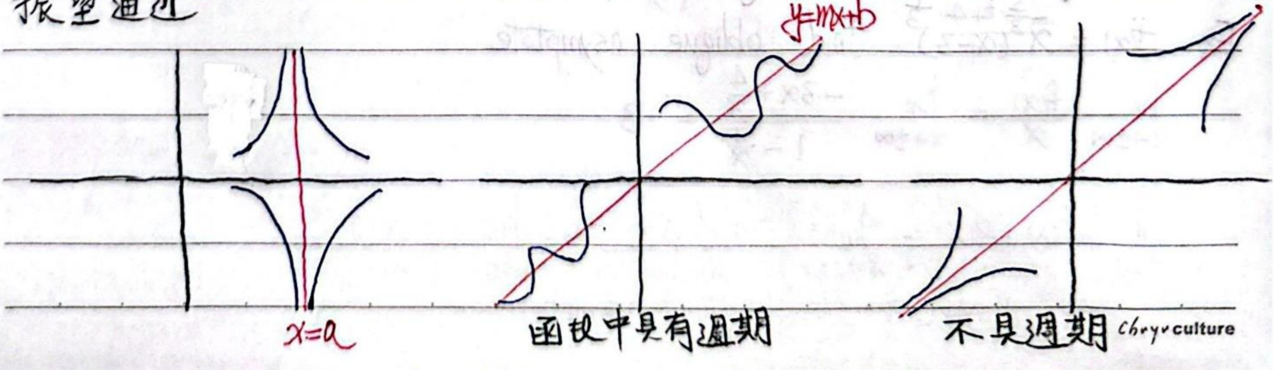
不具週期



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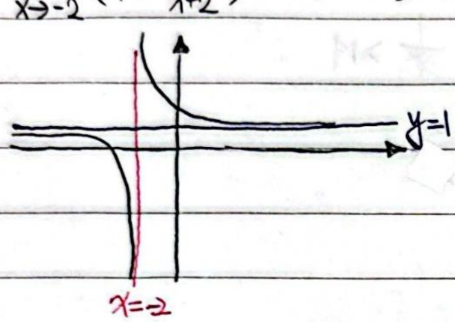
Ex Find the horizontal and vertical asymptote of

$$f(x) = \frac{x+3}{x+2}$$
$$= 1 + \frac{1}{x+2}$$

$\lim\limits_{x\to\infty} \left(1+\frac{1}{x+2}\right) = 1 \Rightarrow y=1$ horizontal asymptote

$\lim\limits_{x\to-\infty} \left(1+\frac{1}{x+2}\right) = 1$

$\lim\limits_{x\to-2} \left(1+\frac{1}{x+2}\right) = \infty \Rightarrow x=-2$ vertical asymptote



Ex find oblique asymptote of $f(x) = \frac{-3x^2+4}{x-1}$

$$\frac{-3x^2+4}{x-1} = \frac{(x-1)(-3x-3)+1}{x-1}$$
$$= (-3x-3) + \frac{1}{x-1}$$

$$\begin{array}{r} -3x-3 \\ x-1 \enclose{longdiv}{-3x^2+4} \\ \underline{-3x^2+3x} \\ -3x+4 \\ \underline{-3x+3} \\ 1 \end{array}$$

$\lim\limits_{x\to\infty} \frac{-3x^2+4}{x-1} - (-3x-3) = 0$

$\lim\limits_{x\to-\infty} \frac{-3x^2+4}{x-1} - (-3x-3) = 0$

$\Rightarrow y = -3x-3$ oblique asymptote

★ Alternative def of oblique asymptote

$\lim\limits_{\substack{x\to\infty \\ -\infty}} \frac{f(x)}{x} = m$ and $\lim\limits_{\substack{x\to\infty \\ -\infty}} [f(x) - mx] = b$

then $y = mx+b$ is oblique asymptote

Ex $f(x) = \frac{-3x^2+4}{x-1}$ find oblique asymptote

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{-3x+\frac{4}{x}}{1-\frac{1}{x}} = -3$$

$$\lim_{x\to\pm\infty} f(x)+3x = \lim_{x\to\pm\infty} \frac{-3x^2+4+3x^2-3x}{x-1} = \lim_{x\to\pm\infty} \frac{-3x+4}{x-1} = -3$$

$\Rightarrow y = -3x-3$ oblique asymptote.

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★ $$\lim_{x\to\pm\infty}\frac{a_0+a_1x+\cdots+a_nx^n}{b_0+b_1x+\cdots+b_mx^m}=\begin{cases}\pm\infty & n>m\\ \frac{a_n}{b_n} & n=m\\ 0 & n<m\end{cases}$$

<pf> $$\frac{a_0+a_1x+\cdots+a_nx^n}{b_0+b_1x+\cdots+b_mx^m}=\frac{x^n\left[\frac{a_0}{x^n}+\frac{a_1}{x^{n-1}}+\cdots+\frac{a_{n-1}}{x}+a_n\right]}{x^n\left[\frac{b_0}{x^n}+\frac{b_1}{x^{n-1}}+\cdots+\frac{b_{m-1}}{x^{n-m+1}}+\frac{b_m}{x^{n-m}}\right]}=(\Delta)$$

for $n>m$,
$$\lim_{x\to\pm\infty}(\Delta)=\pm\infty$$

for $n=m$
$$\lim_{x\to\pm\infty}(\Delta)=\lim_{x\to\pm\infty}\frac{\left[\frac{a_0}{x^n}+\frac{a_1}{x^{n-1}}+\cdots+\frac{a_{n-1}}{x}+a_n\right]}{\left[\frac{b_0}{x^n}+\frac{b_1}{x^{n-1}}+\cdots+\frac{b_{n-1}}{x}+b_n\right]}=\frac{a_n}{b_n}$$

for $n<m$
$$\text{rewrite }(\Delta)=\frac{x^m\left[\frac{a_0}{x^m}+\frac{a_1}{x^{m-1}}+\cdots+\frac{a_{n-1}}{x^{m-n+1}}+\frac{a_n}{x^{m-n}}\right]}{x^m\left[\frac{b_0}{x^m}+\frac{b_1}{x^{m-1}}+\cdots+\frac{b_{m-1}}{x}+b_m\right]}$$
$$\lim_{x\to\pm\infty}(\Delta)=0$$

Ex $\displaystyle\lim_{x\to\infty}\frac{3x^2-x-2}{5x^2+4x+1}=\frac{3}{5}$

Ex $\displaystyle\lim_{x\to\infty}\frac{\sqrt{2x^2+1}}{3x-5}=\lim_{x\to\infty}\frac{\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}}=\frac{\sqrt{2}}{3}$

★ Tangent Line Problem $\Longleftrightarrow$ differentiate 微分