No.
Date : :

§ Propositional Logic (命題邏輯)
Statement Calculus

Def. A statement may be either true or false, but not both. → Simple

Ex : 2 is even. → Compound ← (combine 2 or 2↗ simple statement by connectives)

• Statement Connectives

Let P、Q symbolize statements

1. Conjunction : P and Q $\triangleq P \wedge Q$

$P \wedge Q$ is $\begin{cases} \text{True when both P、Q are True} \\ \text{False } \overline{\text{ow}} \end{cases}$

(Truth Table : 真值表)

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

2. Disjunction : P or Q $\triangleq P \vee Q$

$P \vee Q$ is $\begin{cases} \text{False when both P、Q are False} \\ \text{True } \overline{\text{ow}} \end{cases}$

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

3. Negation (逆命題) : Not P $\triangleq \sim P$

Ex : P : the car is red

$\sim P$: the car is not red

3* Negation of Negation : $\sim(\sim P) = P$

1+3. Negation of Conjunction : $\sim(P \wedge Q) = \sim P \vee \sim Q$

2+3. Negation of Disjunction : $\sim(P \vee Q) = \sim P \wedge \sim Q$

(De Morgan's Laws)

Chuyu culture

No.
Date

• Conditional Statement

(Antecedent) Hypothesis — (Consquent) Conclusion

Let P、Q symbolize statements

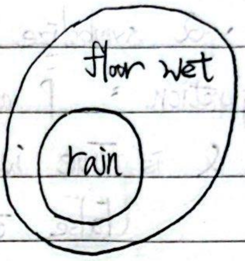
1. Conditional : If P then Q $\triangleq$ $P \to Q$

$P \to Q$ is False when and only when P is True and Q is False

若P則Q

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

vacuously true (空泛為真)



Remark. Q if P

P only if Q (P唯若Q)

P is suff. for Q (P為Q的充分)

P implies Q

Q is necessary for P (Q為P的必要)

Q whenever P

2. Contrapositive : $\sim Q \to \sim P$
(逆否命題)

3. Converse : $Q \to P$
(逆命題)

4. Biconditional : P if and only if Q $\triangleq$ $P \longleftrightarrow Q$ (P若且唯若Q)

$(P \to Q) \land (Q \to P)$

$P \leftrightarrow Q$ is True whenever the true value of P, Q are the same

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Remark. P iff Q

P is equivalent to Q

P is necessary and suff for Q

Def. An open sentence is a sentence contains variables. $\begin{cases} P(x) \\ P(x_1, \dots, x_n) \end{cases}$

Remark. Truth set (使P(x)為真的所有 x)

Def. With a universe Ω specified, $P(x)$ and $Q(x)$ are equivalent if they have same truth set of all $x \in \Omega$.

Ex. $P(x): 2+3x=20$

$Q(x): 2x-7=5$

$P(x)$ and $Q(x)$ equivalent

★ Two basic quantifiers $\begin{cases} \text{for all} \triangleq \forall \\ \text{there exist} \triangleq \exists \end{cases}$

$\exists x, P(x) \Leftrightarrow$ there exist x such that $P(x)$

$\forall x, P(x) \Leftrightarrow$ for all x, $P(x)$

Remark. In general,

element set

$$\forall x \in A, P(x)$$
$$\exists x \in A, P(x)$$

★ The negation of 「there exist」 statement is a 「for all」 statement

$$\forall x, P(x) \Leftrightarrow \sim \exists x, \sim P(x)$$
$$\exists x, Q(x) \Leftrightarrow \sim \forall x, \sim Q(x)$$

★ $\forall$, $\exists$ do not commute 交換率

★ A 「for all」/「there exist」 statement $\Leftrightarrow$ conjunction/disjunction of a very large number of simple statement.

No.
Date : :

Def. symbol $\exists!$, unique existential quantifier

$\exists! x.\ P(x) \Leftrightarrow$ there is a unique x such that $P(x)$

§ Method of Proof.

claim : $P \Rightarrow Q$

(I) Direct proof.

Ex. Sum of two odd number is even

<pf>

Suppose p, q are odd, then $p = 2r+1$, $q = 2s+1$ for some $r, s \in \mathbb{Z}$

Then $p+q = 2(r+s)+2 = 2(r+s+1)$

thus $p+q$ is even.

(II) Contraposition ($\sim Q \Rightarrow \sim P$)

Ex: $m \in \mathbb{Z}$, if m^2 is even, then m is even.

<pf>

Suppose m is odd, then $m = 2k+1$ for some $k \in \mathbb{Z}$

We shall show that m^2 is odd.

Then $m^2 = (2k+1)^2 = 2(2k^2+2k)+1$

thus m^2 is odd.

(III) Contradiction

Ex: Show $\sqrt{2} \in \mathbb{Q}^c$

<pf>

suppose $\sqrt{2} \in \mathbb{Q}$, then $\sqrt{2} = \frac{q}{p}$ for some $p, q \in \mathbb{Z}^+$ and $(p,q) = 1$

q^2 is even since $q^2 = 2p^2$

by previous ex $\Rightarrow$ q is even then $q = 2k$ for some $k \in \mathbb{Z}$

then $p^2 = \frac{(2k)^2}{2} = 2k^2$

by previous ex again $\Rightarrow$ p is even

$(p,q) \neq 1$ $\rightarrow\!\leftarrow$

Hence, $\sqrt{2} \in \mathbb{Q}^c$

Def. A set is a collection of distinct objs.

Remark. $\{1,2,3\}$ is set but $\underbrace{\{1,1,3\}}$ is not.

multiset 多重集

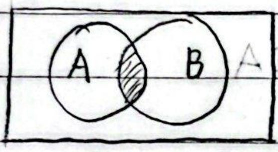
$\{x_1, \dots, x_N\}$ complete list

$\{x : P(x)\}$ truth set of $P(x)$. Ex. set of even integers $\{2n : n \in \mathbb{Z}\}$

Def. The empty set is a set containing no objs.
Write $\{\}$ or $\emptyset$

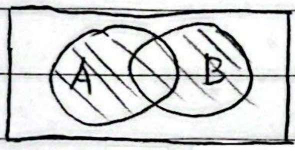
☆ Set Operation

Intersection : $A \cap B = \{x : x \in A \text{ and } x \in B\}$



Ex : $A = \{1,2,3,5\}$
$B = \{2,3,4,5\}$
$A \cap B = \{2,3,5\}$

Union : $A \cup B = \{x : x \in A \text{ or } x \in B\}$



$A \cup B = \{1,2,3,4,5\}$

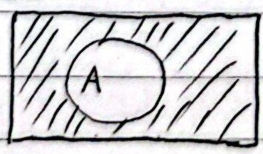
Remark. A and B are disjoint if $A \cap B = \emptyset$

Def. The universal set, at least for a given collection of set theoretic computations, is the set of all possible objs.

universal set

Complementation : $A^c = \{x : x \in \underset{\sim}{\Omega} \text{ or } x \notin A\}$
$= \{x : x \notin A\} = \Omega \setminus A$

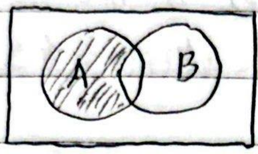
Ω



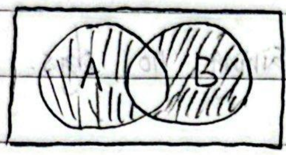
Ex. Let $\Omega = \{1,2,3,4,5,6\}$
$A^c = \{4,6\}$

Set Difference : $A \setminus B = A \cap B^c = \{x : x \in A \text{ and } x \notin B\}$

Ex. $A \setminus B = \{1\}$



Symmetric Difference : $A \triangle B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$



Ex. $A \triangle B$

$= (A \cup B) \setminus (A \cap B)$

$= \{1,2,3,4,5\} \setminus \{2,3,5\}$

$= \{1,4\}$

Def. A is subset of B , $A \subseteq B$

$\Leftrightarrow \; x \in A \Rightarrow x \in B$

Def. $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$

Thm. For any three sets $A \cdot B \cdot C$ defined on universal set Ω.

a. Commutative : $A \cup B = B \cup A$

$A \cap B = B \cap A$

b. Associative : $A \cup (B \cup C) = (A \cup B) \cup C$

$A \cap (B \cap C) = (A \cap B) \cap C$

c. Distributive : $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

d. Demorgan's law : $(A \cup B)^c = A^c \cap B^c$

$(A \cap B)^c = A^c \cup B^c$

Remark. $(A \cup B) \cap C \neq A \cup (B \cap C)$

$A \subseteq B \Leftrightarrow B^c \subseteq A^c$

$A \cap B = \emptyset \Leftrightarrow A \subseteq B^c$

§ Number system

Zero $= \{0\}$

$\mathbb{N} = \{1, 2, 3, \ldots\}$

$= \{x \in \mathbb{Z} \mid x > 0\}$

$= \mathbb{Z}^+$

$\mathbb{N} \to \mathbb{P}$: prime 質數

$\mathbb{N} \to$ composite 合數

$\mathbb{Z} = \{\cdots, -1, 0, 1, \cdots\}$

$\mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0\}$

$\mathbb{Q} \to \mathbb{Z}$

$\mathbb{Q} \to$ Fraction

$\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$

$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$ $\quad i = \sqrt{-1}$ imaginary unit

$$\begin{cases} i = i \\ i^2 = -1 \\ i^3 = -i \\ i^4 = 1 \end{cases}$$
$\Rightarrow$ power of i is cyclic

$a + bi$

imaginary part $\mathrm{Im}(a+bi) = b$

real part $\mathrm{Re}(a+bi) = a$

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

代數數 $\mathbb{A}$

Def. An algebraic number is a number that is root of non-zero polynomial in single variable w/ integer coef.

Ex $\frac{1+\sqrt{5}}{2}$ is a root of $x^2 - x - 1$

$1 + i$ is a root of $x^4 + 4$

Remark $\mathbb{Q} \subset \mathbb{A}$

超越數 $\mathbb{T}$

Def. A transcendental number is a number that is not algebraic.

Ex π, e

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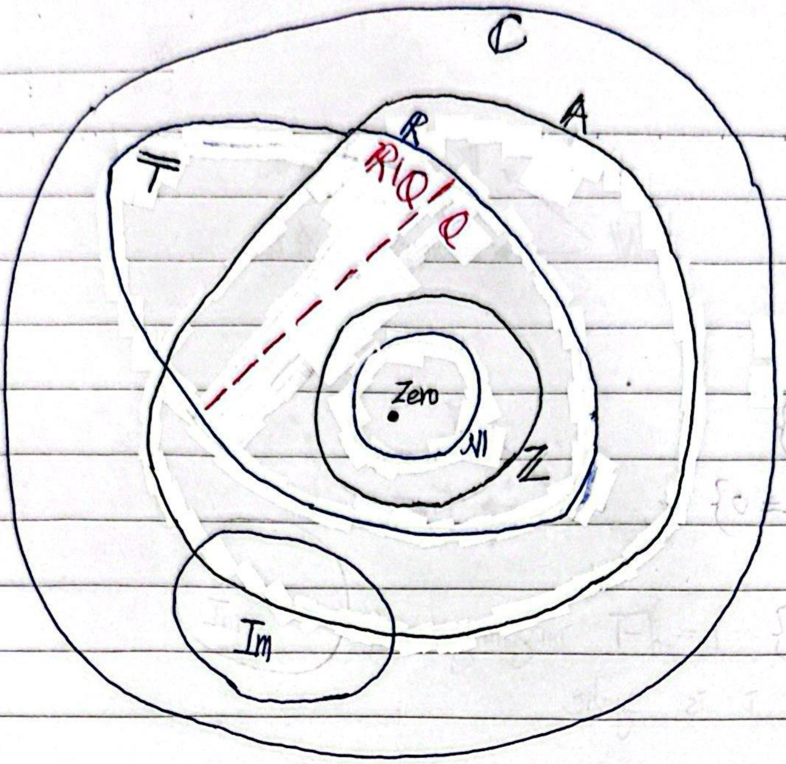

Thm. $\mathbb{Q}$ is dense in $\mathbb{R}$. (稠密性)

Remark. Given any two real numbers there is a rational number between them
(In fact, there are infinitely many such rationals)

Ex. $\frac{a}{b} < \frac{c}{d}$, $b, d > 0$

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

Thm (The Completeness of $\mathbb{R}$) (完備性)
Every nonempty set of $\mathbb{R}$ that has an upper bdd also has a supremum in $\mathbb{R}$

binary operation on $\mathcal{F}$ ($\mathcal{F} \times \mathcal{F} \longrightarrow \mathcal{F}$)

Def A field ($\mathcal{F}$, $+$, $\cdot$) together w/ ordering $\leq$ is an ordered field.

Thm. $\mathbb{R}$ is a complete ordered field.