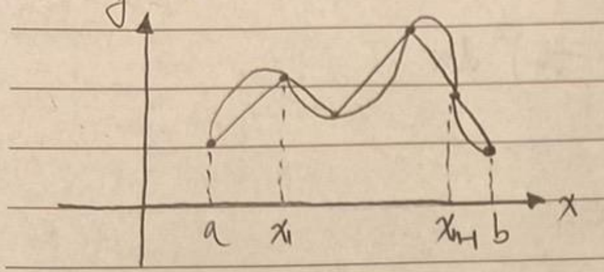
§ Arc Length of a curve



Let $L_k = \|(x_k, y_k) - (x_{k-1}, y_{k-1})\|_2 = (\Delta x_k^2 + \Delta y_k^2)^{\frac{1}{2}}$

and $\Delta y_k = y_k - y_{k-1} = f(x_k) - f(x_{k-1})$

by MVT $\exists\ \xi_k \in [x_{k-1}, x_k]$ s.t $f'(\xi_k)\Delta x_k = \Delta y_k$

$$L_k = \Delta x_k \left(1 + \frac{\Delta y_k^2}{\Delta x_k^2}\right)^{\frac{1}{2}} = \left(1 + f'(\xi_k)^2\right)^{\frac{1}{2}} \Delta x_k$$

Consider the sum

$$\sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \left(1 + f'(\xi_k)^2\right)^{\frac{1}{2}} \Delta x_k$$

$$\Big\downarrow \ \|P\| \to 0$$

$$L = \int_a^b \left(1 + f'(x)^2\right)^{\frac{1}{2}} dx$$

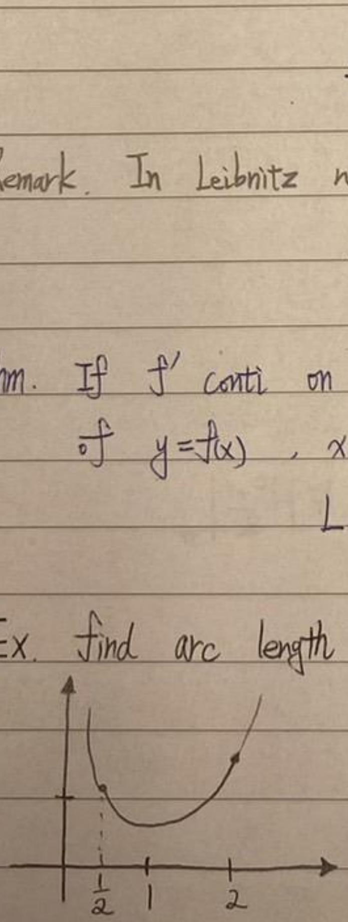
Remark. In Leibnitz notation

$$L = \int_a^b \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{1}{2}} dx$$

thm. If f' conti on $[a, b]$ then the length of $y = f(x)$, $x \in [a, b]$ is

$$L = \int_a^b \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx$$

Ex. find arc length of $y = \frac{x^3}{6} + \frac{1}{2x}$ on $[\frac{1}{2}, 2]$



$$\frac{dy}{dx} = \frac{x^2}{2} - \frac{1}{2x^2} = \frac{1}{2}\left(x^2 - \frac{1}{x^2}\right)$$

$$L = \int_{\frac{1}{2}}^{2} \left[1 + \frac{1}{4}\left(x^4 - 2 + \frac{1}{x^4}\right)\right]^{\frac{1}{2}} dx$$

$$= \frac{1}{2}\int_{\frac{1}{2}}^{2} \left(x^2 + \frac{1}{x^2}\right) dx = \frac{1}{2}\left(\frac{x^3}{3} - \frac{1}{x}\Big|_{\frac{1}{2}}^{2}\right)$$

$$= \frac{1}{2}\left(\left[\frac{8}{3} - \frac{1}{2}\right] - \left[\frac{1}{24} - 2\right]\right) = \frac{1}{2}\left(\frac{13}{6} + \frac{47}{24}\right) = \frac{33}{16}$$

Ex. arc length of $y=\ln x$ on $[1,2]$

$$L=\int_1^2\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}dx=\int_1^2\left(1+\frac{1}{x^2}\right)^{\frac{1}{2}}dx$$

$$=\int_1^2\sqrt{\frac{x^2+1}{x^2}}\,dx=\int_1^2\frac{x^2+1}{x\sqrt{x^2+1}}\,dx$$

$$=\int_1^2\frac{x}{\sqrt{x^2+1}}\,dx+\int_1^2\frac{1}{x\sqrt{x^2+1}}\,dx$$

$$\frac{1}{2}\int_1^2(x^2+1)^{-\frac{1}{2}}\,d\,x^2+1=(x^2+1)^{\frac{1}{2}}\Big|_1^2$$

$$=\sqrt{5}-\sqrt{2}$$

(M1) Hyperbolic function

$$\int_1^2\frac{-1}{\sqrt{1+\left(\frac{1}{x}\right)^2}}\,d\frac{1}{x}$$

$$=-\left(\sinh^{-1}\left(\frac{1}{x}\right)\Big|_1^2\right)$$

$$=\sinh^{-1}(1)-\sinh^{-1}\left(\frac{1}{2}\right)$$

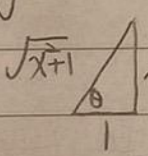
(M2)

let $x=\tan(\theta)$

$dx=\sec^2(\theta)\,d\theta$

$$\int\frac{1}{\tan(\theta)\sqrt{1+\tan^2(\theta)}}\sec^2(\theta)\,d\theta=\int\frac{\sec(\theta)}{\tan(\theta)}\,d\theta$$

$$=\int\csc(\theta)\,d\theta=-\ln|\csc(\theta)+\cot(\theta)|+C$$



$$=-\ln\left|\frac{\sqrt{x^2+1}+1}{x}\right|+C$$

$$\Rightarrow\int_1^2\frac{1}{x\sqrt{x^2+1}}\,dx=-\left(\ln\left|\frac{\sqrt{x^2+1}+1}{x}\right|\Big|_1^2\right)=\ln|\sqrt{2}+1|-\ln\left|\frac{\sqrt{5}+1}{2}\right| ※$$

• Arc length function

$$L(x)=\int_a^x\left(1+[f'(t)]^2\right)^{\frac{1}{2}}dt$$

$$\frac{dL(x)}{dx}=\left(1+[f'(x)]^2\right)^{\frac{1}{2}}$$

Ex. find arc length fn for $y = x^2 - \frac{1}{8}\ln x$

w/ init pt = (1, 1)

$\frac{dy}{dx} = 2x - \frac{1}{8x}$

$1 + \left[\frac{dy}{dx}\right]^2 = 1 + 4x^2 - \frac{1}{2} + \frac{1}{64x^2} = \left(2x + \frac{1}{8x}\right)^2$

$L(x) = \int_1^x 2t + \frac{1}{8t}\, dt = t^2 + \frac{\ln t}{8} \Big|_1^x$

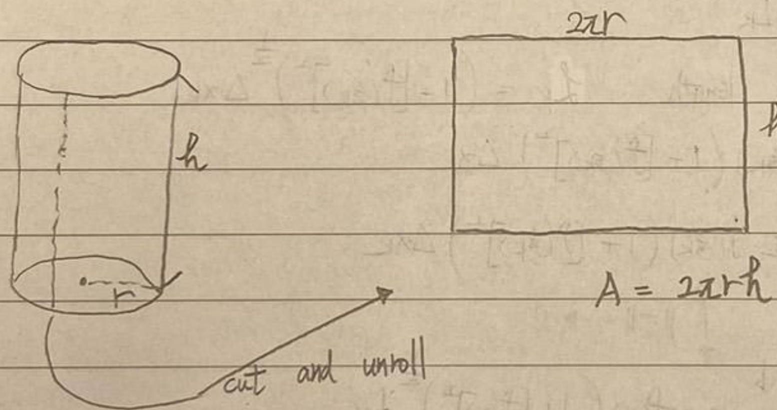
$= x^2 + \frac{\ln x}{8} - 1$

find arc length along $x=1$ to $x=3$

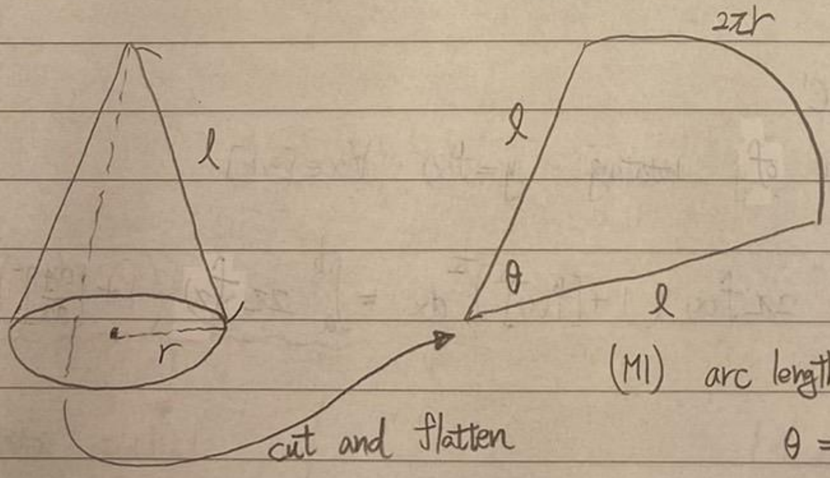
$L(3) = 8 - \frac{\ln 3}{8}$ #

§ Area of surface revolution

• A curve rotated about a line



$A = 2\pi r h$



(M1) arc length $= 2\pi r = l\theta$

$\theta = \frac{2\pi r}{l}$

$A = \frac{1}{2} l^2 \theta = \pi r l$

(M2) $A = \pi l^2 \left(\frac{2\pi r}{2\pi l}\right) = \pi r l$

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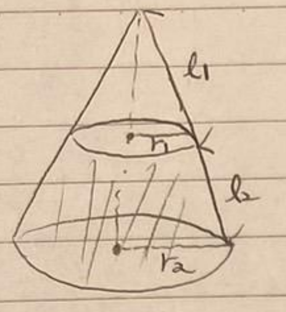
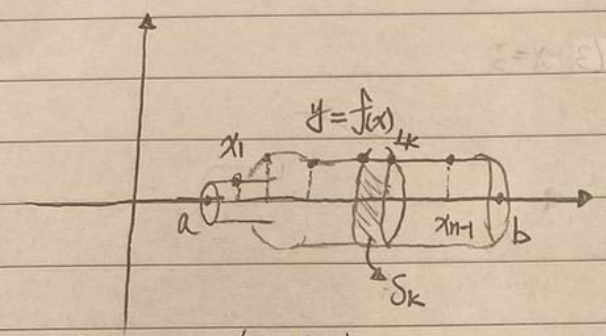


$$S = \pi r_2 (l_1 + l_2) - \pi r_1 l_1$$
$$= \pi (r_2 - r_1) l_1 + \pi r_2 l_2 \quad \cdots\cdots (\Delta)$$

$$\frac{l_1}{r_1} = \frac{l_1 + l_2}{r_2} \Rightarrow \begin{aligned} l_1 r_2 &= r_1 l_1 + r_1 l_2 \\ (r_2 - r_1) l_1 &= r_1 l_2 \end{aligned}$$

$$(\Delta) = \pi (r_1 + r_2) l_2 = 2\pi \bar{r} l_2 \ ※$$



$$S_k = 2\pi \left(\frac{y_k + y_{k-1}}{2} \right) L_k$$

apply thm of arc length, $L_k = \left(1 + [f'(\xi_k)]^2\right)^{\frac{1}{2}} \Delta x_k$

$$\Rightarrow \quad S_k \approx 2\pi f(\xi_k) \left(1 + [f'(\xi_k)]^2\right) \Delta x_k$$

$$\sum_{k=1}^{n} S_k \approx \sum_{k=1}^{n} 2\pi f(\xi_k) \left(1 + [f'(\xi_k)]^2\right) \Delta x_k$$

$$\Big\downarrow \ \|P\| \longrightarrow 0$$

$$S = \int_a^b 2\pi f(x) \left(1 + [f'(x)]^2\right)^{\frac{1}{2}} dx$$

thm. $f \geq 0$ and $f \in C^1$

define surface area by rotating $y = f(x)$ $\forall x \in [a, b]$

about x-axis

$$S = \int_a^b 2\pi f(x) \left(1 + [f'(x)]^2\right)^{\frac{1}{2}} dx = \underbrace{\int_a^b 2\pi y \left(1 + \left[\frac{dy}{dx}\right]^2\right)^{\frac{1}{2}} dx}_{\text{Leibniz notation}}$$

Remark $C^0 = \{\text{all conti fn}\}$

$C^1 = \{\text{all diff fn whose derivative is conti}\}$

$C^k \Leftrightarrow f', f'', \cdots, f^{(k)}$ exist and conti

Smooth fn $= C^\infty$

Notes. recall arc length fn

$$L(x) = \int_a^x \left(1 + [f'(t)]^2\right)^{\frac{1}{2}} dt$$

$$\frac{dL}{dx} = \left(1 + [f'(x)]^2\right)^{\frac{1}{2}} = \left(1 + \left[\frac{dy}{dx}\right]^2\right)^{\frac{1}{2}}$$

$$(dL)^2 = (dx)^2 + (dy)^2$$

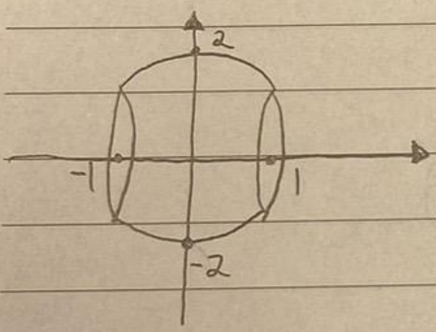
$$dL = \left(1 + \left[\frac{dx}{dy}\right]^2\right)^{\frac{1}{2}} dy = \left(1 + \left[\frac{dy}{dx}\right]^2\right)^{\frac{1}{2}} dx$$

rotate x-axis $\Rightarrow \int 2\pi y \, dL$

y-axis $\Rightarrow \int 2\pi x \, dL$

Ex. $y = \sqrt{2^2 - x^2}$, $x \in [-1, 1]$

Find area of surface by rotating about x-axis



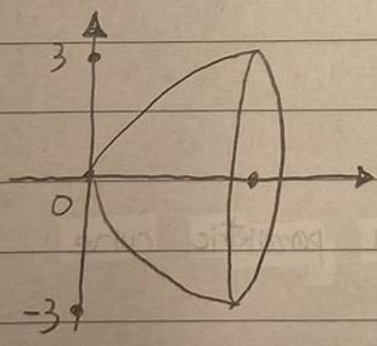
$$\frac{dy}{dx} = \frac{1}{2}(4-x^2)^{-\frac{1}{2}}(-2x) = \frac{-x}{\sqrt{4-x^2}}$$

$$S = \int_{-1}^{1} 2\pi\sqrt{4-x^2}\left(1 + \frac{x^2}{4-x^2}\right)^{\frac{1}{2}} dx$$

$$= \int_{-1}^{1} 2\pi\sqrt{4-x^2}\,\frac{2}{\sqrt{4-x^2}}\,dx = 8\pi$$

Ex. $x = \frac{2}{3}y^{\frac{3}{2}}$ between $y=0$ to $y=3$

rotated about x-axis



$$\frac{dx}{dy} = y^{\frac{1}{2}}$$

$$S = \int_0^3 2\pi y\,(1+y)^{\frac{1}{2}}\,dy = 2\pi \int_0^3 y\,\frac{2}{3}\,d(1+y)^{\frac{3}{2}}$$

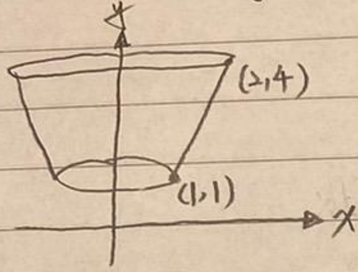
$$= \frac{4\pi}{3}\left(y(1+y)^{\frac{3}{2}}\Big|_0^3 - \int_0^3 (1+y)^{\frac{3}{2}} dy \right)$$

$$= \frac{4\pi}{3}\left(24 - \left(\frac{2}{5}(1+y)^{\frac{5}{2}}\Big|_0^3\right)\right) = \frac{4\pi}{3}\left(24 - \left(\frac{2^6-2}{5}\right)\right)$$

$$= \frac{4\pi}{3}\left(\frac{120-64+2}{5}\right) = \frac{232\pi}{15} \#$$

Ex $y = x^2$, $(1,1)$ to $(2,4)$

rotated about y-axis



$$x = \sqrt{y}$$

$$\frac{dx}{dy} = \frac{1}{2\sqrt{y}}$$

$$S = \int_1^4 2\pi\sqrt{y}\left(1 + \frac{1}{4y}\right)^{\frac{1}{2}} dy$$

$$= 2\pi \int_1^4 \left(y + \frac{1}{4}\right)^{\frac{1}{2}} dy$$

$$= 2\pi \left(\frac{2}{3}\left(y + \frac{1}{4}\right)^{\frac{3}{2}} \Big|_1^4 \right)$$

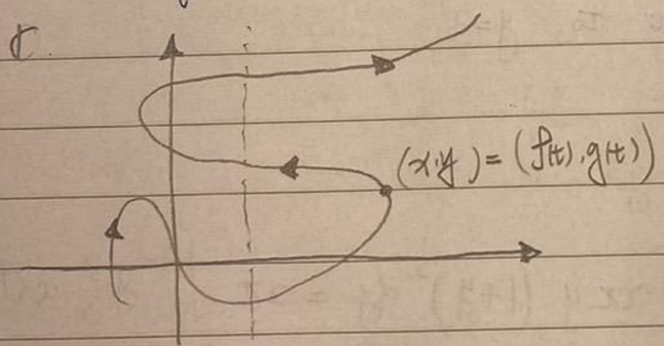
$$= \frac{4\pi}{3}\left(\left(4 + \frac{1}{4}\right)^{\frac{3}{2}} - \left(1 + \frac{1}{4}\right)^{\frac{3}{2}} \right)$$

$$S = \int_1^2 2\pi x \left(1 + (2x)^2\right)^{\frac{1}{2}} dx$$

$$= \pi \int_1^2 (1 + 4x^2)^{\frac{1}{2}} dx^2$$

$$= \frac{\pi}{4}\left[\frac{2}{3}(1 + 4x^2)^{\frac{3}{2}} \Big|_1^2 \right] = \frac{\pi}{6}\left(17^{\frac{3}{2}} - 5^{\frac{3}{2}}\right)$$

§ Parametric Equation



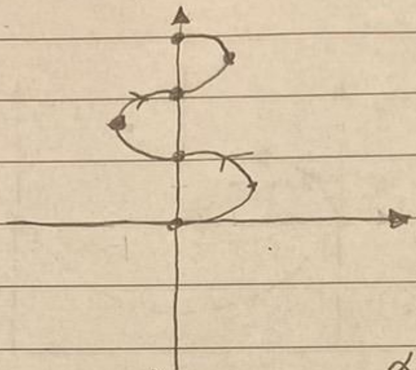
$C = \{ (x,y) \in \mathbb{R}^2 \mid x = f(t),\ y = g(t),\ t \in I \}$

is a parametric curve on I

Curve C fail Vertical Line tests

Ex Sketch the curve

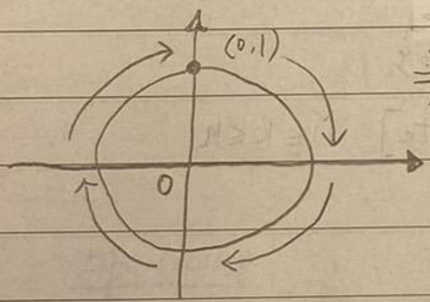
$$\begin{cases} x = \sin\left(\frac{\pi t}{2}\right) \\ y = t \end{cases}, \quad 0 \le t \le 6$$



(Remark : $y = f(x)$ $\Longleftrightarrow$ $\begin{matrix} x = t \\ y = f(t) \end{matrix}$)
natural parametrization

Ex Sketch the curve

$$\begin{cases} x = \sin(2t) \\ y = \cos(2t) \end{cases}, \quad 0 \le t \le 2\pi \qquad \Longleftrightarrow \quad x^2 + y^2 = 1$$



$\Rightarrow$ move twice around the circle in the clockwise direction.
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Eliminating parameter

Parametric equation $\Rightarrow$ Solve for t in one equation $\Rightarrow$ Substitude into another equation $\Rightarrow$ Rectangular equation

Ex. $\begin{cases} x = t^2 - 4 \\ y = \frac{t}{2} \end{cases}$ $\quad t = 2y \quad x = (2y)^2 - 4 \quad x = 4y^2 - 4$

Ex $\begin{cases} x = t^2 - 2t \\ y = t + 1 \end{cases}$ $\quad t = y - 1 \quad x = (y-1)^2 - 2(y-1) \quad x = y^2 - 4y + 3$

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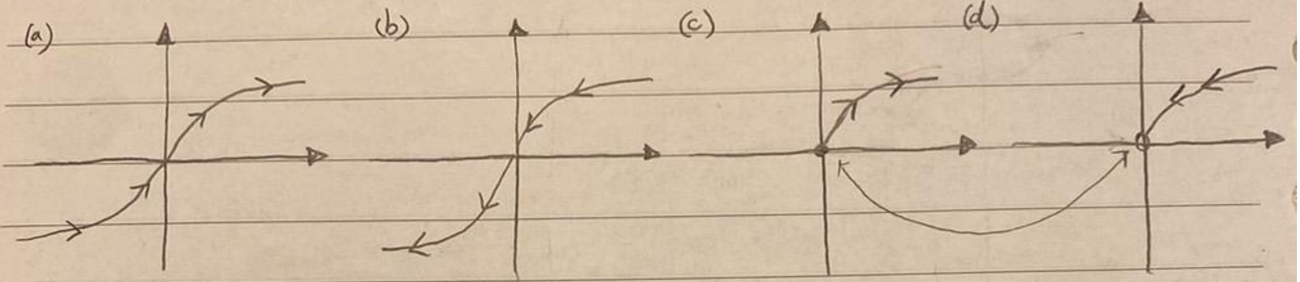
Ex (a) $x = t^3$, $y = t$
(b) $x = -t^3$, $y = -t$
(c) $x = t^{\frac{3}{2}}$, $y = \sqrt{t}$
(d) $x = e^{-3t}$, $y = e^{-t}$

$\Rightarrow$ Eliminating parameter gives $y^3 = x$



Def. Curve $\gamma : I \longrightarrow \mathbb{R}^2$

$t \longmapsto \gamma(t) = (f(t), g(t))$

Smooth Curve

$\gamma \in C^1(I)$ and $\gamma'(t) \neq 0$ on $\underline{\mathrm{int}(I)}$ ← interior pts.

Curve piecewise smooth

$\gamma : I \rightarrow \mathbb{R}^2$

$\exists\ P = \{a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b\}$

s.t γ is smooth on each $[t_{k-1}, t_k]$ $1 \le k \le n$

§ Parametric Equation and Calculus

Thm. If a smooth cure γ is given by

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$

then the slope of γ at (x,y) is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad \frac{dx}{dt} \neq 0$$

$$\frac{dy}{dt} \cdot \frac{dt}{dx}$$

chain rule

$$\frac{\Delta y}{\Delta x} = \frac{g(t+\Delta t) - g(t)}{f(t+\Delta t) - f(t)}$$

$$\downarrow \qquad \qquad \downarrow$$

$$\frac{dy}{dx} \leftarrow \frac{g'(t)}{f'(t)}$$