• Higher-order derivative

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left\{\frac{dy}{dx}\right\} = \frac{\frac{d}{dt}\left\{\frac{dy}{dx}\right\}}{\frac{dx}{dt}}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left\{\frac{d^2y}{dx^2}\right\}$$

Ex. $\begin{cases} x = \sqrt{t} \\ y = \frac{1}{4}(t^2-4) \quad t \geq 0 \end{cases}$ $\quad 3 = \frac{1}{4}(t^2-4) \Rightarrow t = 4$

find slope and concavity at pt $(2,3)$

$$\begin{cases} \frac{dy}{dt} = \frac{t}{2} \\ \frac{dx}{dt} = \frac{1}{2\sqrt{t}} \end{cases} \Rightarrow \frac{dy}{dx} = \frac{\frac{t}{2}}{\frac{1}{2\sqrt{t}}} = t^{\frac{3}{2}}\Big|_{t=4} = 8$$

$$\frac{d}{dt}\left\{\frac{dy}{dx}\right\} = \frac{3}{2}t^{\frac{1}{2}} \Rightarrow \frac{d^2y}{dx^2} = \frac{\frac{3}{2}t^{\frac{1}{2}}}{\frac{1}{2\sqrt{t}}} = 3t\Big|_{t=4} = 12 > 0$$

concave upward

Ex $\begin{cases} x = t^2 \\ y = t^3 - 3t \end{cases}$ (a) Show γ has two tangent at $(3,0)$
(b) Find pt on γ s.t tangent is horizontal or vertical
(c) concavity

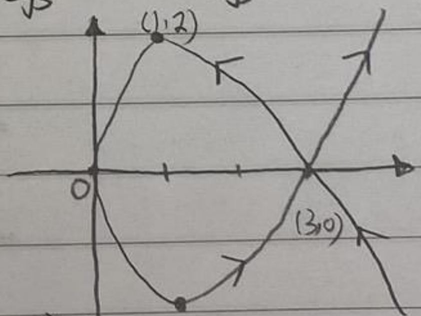
(a) $\begin{cases} \frac{dy}{dt} = 3t^2-3 \\ \frac{dx}{dt} = 2t \end{cases} \Rightarrow \frac{dy}{dx} = \frac{3t^2-3}{2t} \Rightarrow$ since $t = \sqrt{3}$ or $-\sqrt{3}$

$\frac{dy}{dx} = \frac{6}{2\sqrt{3}}$ or $-\frac{6}{2\sqrt{3}}$

(b)

$t = \pm 1 \Rightarrow (1,-2), (1,2)$ $\quad \left(\frac{dy}{dx} = 0\right)$

$t = 0 \Rightarrow (0,0)$ $\quad \left(\frac{dy}{dx}\text{ undefined}\right)$



(c) $$\frac{d}{dt}\left\{\frac{dy}{dx}\right\} = \frac{6t\cdot 2t - (3t^2-3)2}{(2t)^2} = \frac{-6t^2+6+12t^2}{4t^2} = \frac{3t^2+3}{2t^2}$$

$$\frac{d^2y}{dx^2} = \frac{3t^2+3}{4t^3} \begin{cases} t > 0 & \text{concave upward} \\ t < 0 & \text{〃 downward} \end{cases}$$

• Area

Recall that area under a curve $y = F(x) \geq 0$ from a to b is given by

$$A = \int_a^b F(x)\,dx$$

Suppose that

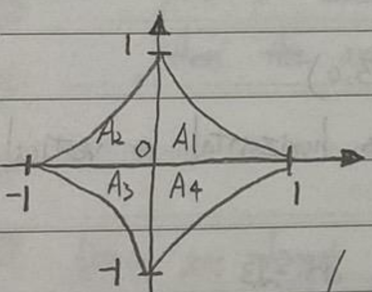
$$\mathcal{J} = \{ (x,y) \mid y = F(x) \geq 0,\ x \in [a,b] \}$$
$$= \{ (x,y) \mid x = f(t),\ y = g(t) \geq 0 \text{ and } f: [\alpha,\beta] \to [a,b] \text{ is 1-1 and onto} \}$$

then by substitution rule

$$A = \int_a^b y\,dx = \int_{\alpha \atop \beta}^{\beta \atop \alpha} g(t)\,f'(t)\,dt \quad \text{if } f'(t) \underset{<}{>} 0,\ t \in [\alpha,\beta]$$

Ex. Find area enclosed by the astroid

$$\begin{cases} x = \cos^3(t) \\ y = \sin^3(t) \end{cases} \qquad 0 \leq t \leq 2\pi$$



$$\int_{\frac{\pi}{2}}^{0} \sin^3(t)\, d(\cos^3(t))$$

$$A = 4A_1$$

$$A_1 = \int_0^1 y\,dx = \int_{\frac{\pi}{2}}^{0} \sin^3(t)\, 3\cos^2(t)\, [-\sin(t)]\,dt$$

$$= 3\int_0^{\frac{\pi}{2}} \sin^4(t)\cos^2(t)\,dt$$

$$\left(\begin{array}{l} p = \sin^2(t) \\ dp = 2\sin(t)\cos(t)\,dt \end{array} \right)$$

$$= \frac{3}{2}\int_0^{\frac{\pi}{2}} [\sin^2(t)]^{\frac{3}{2}} [\cos^2(t)]^{\frac{1}{2}}\, 2\sin(t)\cos(t)\,dt$$

$$= \frac{3}{2}\int_0^1 y^{\frac{5}{2}-1}(1-y)^{\frac{3}{2}-1}\,dy$$

$$= \frac{3}{2} B\left(\frac{5}{2}, \frac{3}{2}\right) = \frac{3}{2}\,\frac{\Gamma(\frac{5}{2})\Gamma(\frac{3}{2})}{\Gamma(4)}$$

$$= \frac{3}{2}\,\frac{(\frac{3}{2})(\frac{1}{2})\sqrt{\pi}\cdot(\frac{1}{2})\sqrt{\pi}}{3!} = \frac{3\pi}{2^5} = \frac{3\pi}{32}$$

$$A = 4\cdot\frac{3\pi}{32} = \frac{\pi}{8}$$ \#

• Arc Length

Recall that a curve $y = F(x)$ $x \in [a,b]$, F' is conti

$$L = \int_a^b \left(1 + \left[\frac{dy}{dx}\right]^2\right)^{\frac{1}{2}} dx$$

$\gamma = \{(x,y) \mid x = f(t),\ y = g(t),\ f : [\alpha,\beta] \to [a,b]$ 1-1, onto w/ $\frac{dx}{dt} > 0$ $\forall t \in [\alpha,\beta]\}$

then by substitution rule

$$L = \int_\alpha^\beta \left(1 + \left[\frac{dy/dt}{dx/dt}\right]^2\right)^{\frac{1}{2}} \left[\frac{dx}{dt}\right] dt$$

$$= \int_\alpha^\beta \left(\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2\right)^{\frac{1}{2}} dt$$

Thm. If a smooth curve γ is given by

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$

s.t γ does not intersect itself on $t \in [\alpha,\beta]$ (except possibly at the endpoints)

then the arc length of γ over $[\alpha,\beta]$ is

$$L = \int_\alpha^\beta \left(\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2\right)^{\frac{1}{2}} dt$$

Notes. arc length fn

$$L(t) = \int_\alpha^t \left([f'(u)]^2 + [g'(u)]^2\right)^{\frac{1}{2}} du$$

$$\frac{dL}{dt} = \left([f'(t)]^2 + [g'(t)]^2\right)^{\frac{1}{2}}$$

$$dL = \left(\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2\right)^{\frac{1}{2}} dt$$

Ex. Find length of the astroid

$$\begin{cases} x = \cos^3(t) \\ y = \sin^3(t) \end{cases} \quad 0 \le t \le 2\pi$$

$\frac{dx}{dt} = -3\cos^2(t)\sin(t)$

$\frac{dy}{dt} = 3\sin^2(t)\cos(t)$

$$\Rightarrow \left(\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2\right)^{\frac{1}{2}} = \left(9\left[\cos^4(t)\sin^2(t) + \sin^4(t)\cos^2(t)\right]\right)^{\frac{1}{2}}$$

$$= 3\cos(t)\sin(t)$$

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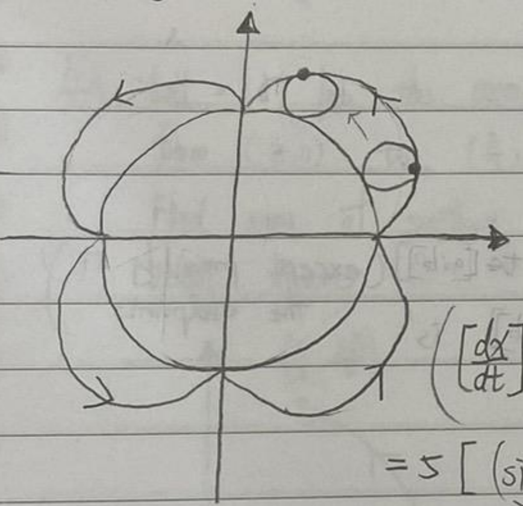
$$\int_0^{\frac{\pi}{2}} 3\cos(t)\sin(t)\,dt = \frac{3}{2}\int_0^{\frac{\pi}{2}} \sin(2t)\,dt$$
$$= \frac{3}{4}\left(-\cos(2t)\Big|_0^{\frac{\pi}{2}}\right)$$
$$= \frac{3}{2}$$

then arc length of astroid $4 \times \frac{3}{2} = 6$ #

Ex. find length of epicycloid 外擺線

$$\begin{cases} x = 5\cos(t) - \cos(5t) \\ y = 5\sin(t) - \sin(5t) \end{cases}$$



$$\frac{dx}{dt} = 5\sin(5t) - 5\sin(t)$$
$$= 5(\sin(5t) - \sin(t))$$
$$\frac{dy}{dt} = 5\cos(t) - 5\cos(5t)$$
$$= 5(\cos(t) - \cos(5t))$$

$$\left(\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2\right)^{\frac{1}{2}}$$
$$= 5\left[\underbrace{(\sin(5t) - \sin(t))}_{2\cos(3t)\sin(2t)}^2 + \underbrace{(\cos(t) - \cos(5t))}_{-2\sin(3t)\sin(2t)}^2\right]^{\frac{1}{2}}$$

(和差化積)

$$= 5\left(4\sin^2(2t)\left[\cos^2(3t) + \sin^2(3t)\right]\right)^{\frac{1}{2}}$$
$$= 5 \cdot 2 \cdot \sin(2t)$$

$$L = 4 \cdot 5 \cdot 2 \int_0^{\frac{\pi}{2}} \sin(2t)\,dt = 20\left(-\cos(2t)\Big|_0^{\frac{\pi}{2}}\right)$$
$$= 20 \times 2 = 40 \text{ #}$$

Notes. when $\begin{cases} x = t \\ y = f(t) \end{cases} \Rightarrow \left[\frac{dx}{dt}\right]^2 = 1$, $\left[\frac{dy}{dt}\right]^2 = f'(t)^2$ $\Rightarrow dL = \left(1 + [f'(t)]^2\right)^{\frac{1}{2}} dt$

- Area of surface of Revolution

thm. If a smooth curve γ given by

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$

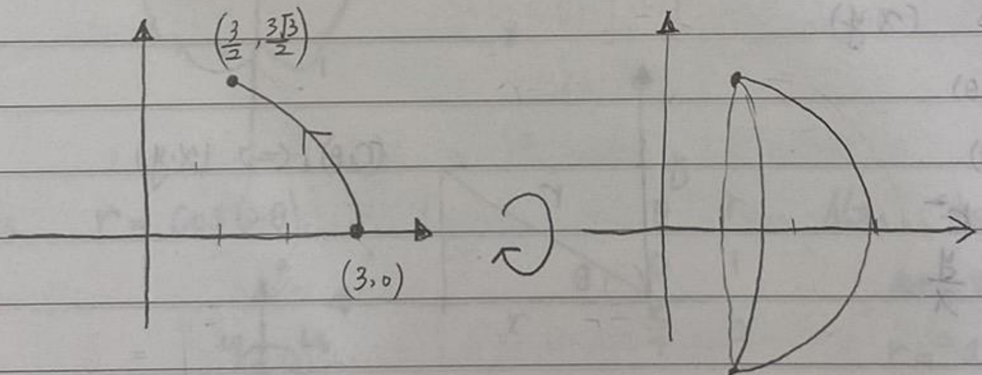
does not cross itself on $[\alpha, \beta]$, then the area S of the surface of revolution by revolving γ about

x-axis $\Rightarrow S = \int_\alpha^\beta 2\pi\, g(t)\, dL$

y-axis $\Rightarrow S = \int_\alpha^\beta 2\pi\, f(t)\, dL$

Ex. Let γ be the arc of $x^2+y^2=9$ from $(3,0)$ to $(\frac{3}{2}, \frac{3\sqrt{3}}{2})$

Find area of surface by revolving γ about x-axis.



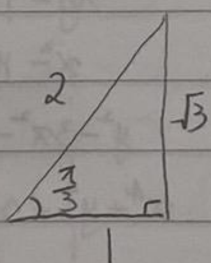
since $x^2+y^2=9 \Rightarrow$ Let $x = 3\cos(t)$, $y = 3\sin(t)$, $t \in [0, \frac{\pi}{3}]$

$$S = \int_0^{\frac{\pi}{3}} 2\pi\, 3\sin(t) \left([-3\sin(t)]^2 + [3\cos(t)]^2 \right)^{\frac{1}{2}} dt$$

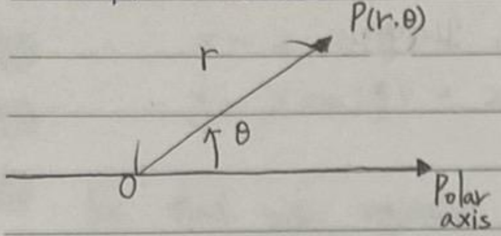
$$= 18\pi \int_0^{\frac{\pi}{3}} \sin(t)\, dt$$

$$= 18\pi \left(-\cos(t) \Big|_0^{\frac{\pi}{3}} \right)$$

$$= 18\pi (1 - \tfrac{1}{2}) = 9\pi$$



⋛ Polar Coordinate



r = directed distance from O to P

θ = directed angle, counterclockwise from polar axis to $\overline{OP}$ 逆時針

Notes. i) With Cartesian coordinates, each (x,y) is unique expression

This is not true w/ polar coordinate

(r,θ) and $(r, \theta+2\pi)$

ii) since r is a directed distance

(r,θ) and $(-r, \theta+\pi)$

In general, $\forall n \in \mathbb{Z}$

$$(r,\theta) = (r, \theta + 2n\pi) = (-r, \theta + (2n+1)\pi)$$

Polar $\qquad$ Cartesian

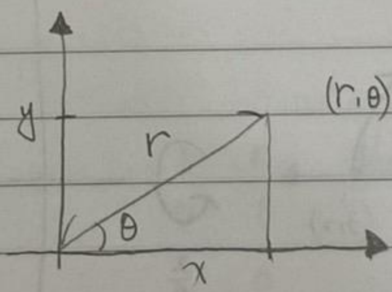
$(r,\theta) \iff (x,y)$

$x = r\cos(\theta)$

$y = r\sin(\theta)$

$x^2 + y^2 = r^2$

$\tan(\theta) = \frac{y}{x}$



$(r,\theta) \iff (x,y)$

Ex. Here are some plane curve expressed in terms of both coordinate systems.

$r\cos(\theta) = 2$	$x = 2$
$r^2\cos(\theta)\sin(\theta) = 4$	$xy = 4$
$r^2\cos^2(\theta) - r^2\sin^2(\theta) = 1$	$x^2 - y^2 = 1$
$r = 1 + 2r\cos(\theta)$	$y^2 - 3x^2 - 4x - 1 = 0$
$r = 1 - \cos(\theta)$	$x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$

Ex. Convert $(2, \frac{\pi}{3})$ to (x, y)

$$x = 2\cos(\frac{\pi}{3}) = 2 \cdot \frac{1}{2} = 1$$

$$y = 2\sin(\frac{\pi}{3}) = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3} \Rightarrow$$

$(2, \frac{\pi}{3}) \downarrow (1, \sqrt{3})$

Ex. find polar equation for $x^2 + (y-3)^2 = 9$

$$r^2\cos^2(\theta) + (r\sin(\theta) - 3)^2 = 9$$

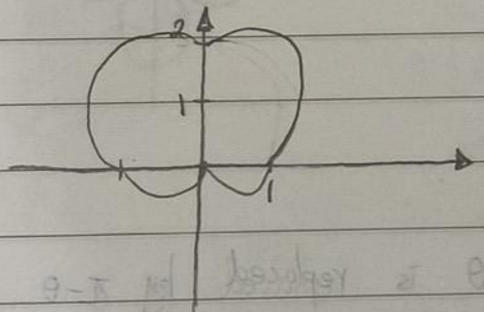
$$r^2\cos^2(\theta) + r^2\sin^2(\theta) - 6r\sin(\theta) + 9 = 9$$

$$r^2 = 6r\sin(\theta)$$

$$r = 6\sin(\theta) \#$$

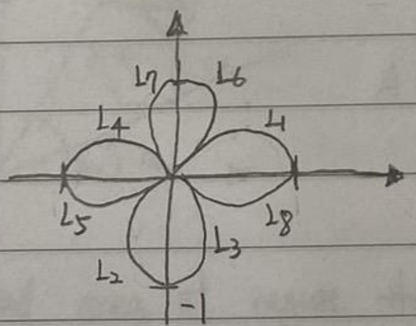
• Graph of Polar curves

Ex. $r = 1 + \sin(\theta)$



It's called cardioid (心臟線)

Ex. $r = \cos(2\theta)$



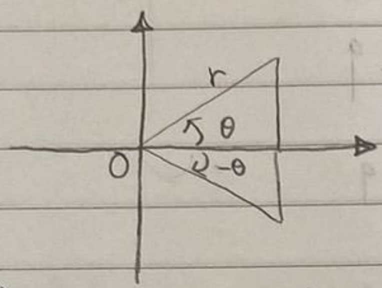
It's a four-leaved rose #

θ	r
0	1
$\frac{\pi}{4}$	0
$\frac{\pi}{2}$	-1
$\frac{3\pi}{4}$	0
π	1
$\frac{5\pi}{4}$	0
$\frac{6\pi}{4}$	-1
$\frac{7\pi}{4}$	0
2π	1

Notes. the rose curves are of the form $r = a\cos(n\theta)$ or $r = a\sin(n\theta)$

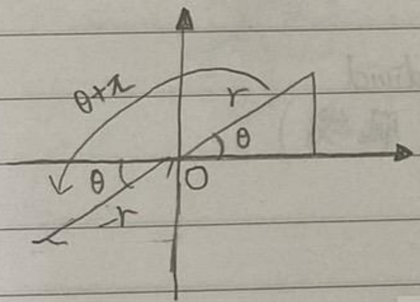
Symmetry

(i) If polar equation is unchanged, when θ is replaced by $-\theta$
the curve is symmetric about polar axis

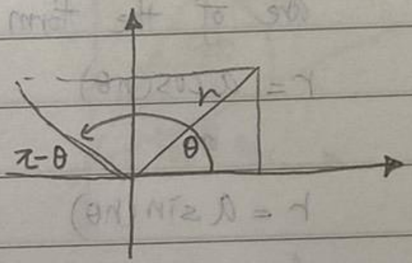


(ii) If equation is unchanged, when r is replaced by $-r$
or θ 〃 $\theta+\pi$

the curve is symmetric about the pole.

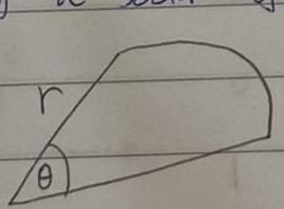


(iii) If equation is unchanged, when θ is replaced by $\pi-\theta$
the curve is symmetric about vertical line $\theta=\frac{\pi}{2}$

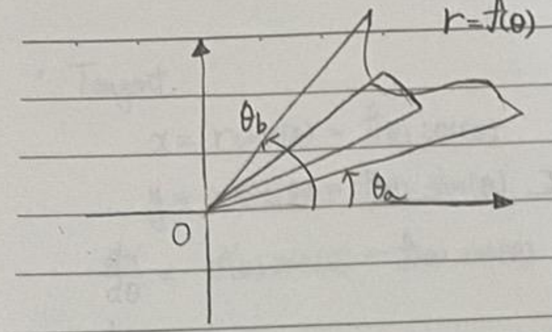


ξ Calculus in Polar Coordinate

Recall area of a sector of circle,



$$A = \frac{1}{2}r^2\theta = \pi r^2\left(\frac{\theta}{2\pi}\right)$$



$$A_k = \frac{1}{2}[f(\xi_k)]^2 \Delta\theta_k$$

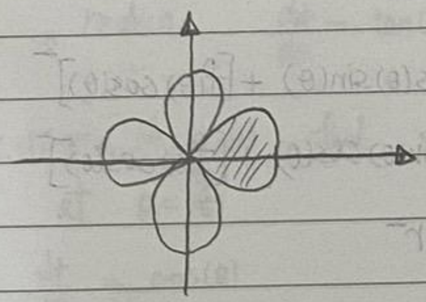
$$\Rightarrow \sum_{k=1}^{n} \frac{1}{2}[f(\xi_k)]^2 \Delta\theta_k$$

$$\downarrow \ \|\Pi\| \to 0$$

$$A = \int_{\theta_a}^{\theta_b} \frac{1}{2}[f(\theta)]^2 d\theta$$

$$= \int_{\theta_a}^{\theta_b} \frac{1}{2} r^2 d\theta$$

Ex. find area enclosed by one loop of $r=\cos(2\theta)$



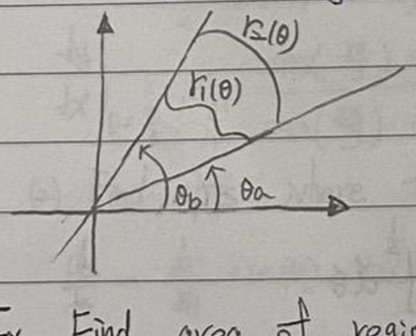
$$A = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}\cos^2(2\theta)\, d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1+\cos(4\theta)}{2} d\theta$$

$$= \frac{1}{4}\left(\frac{\pi}{2} + \frac{1}{4}\left(\sin(4\theta)\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\right)\right)$$

$$= \frac{\pi}{8}$$

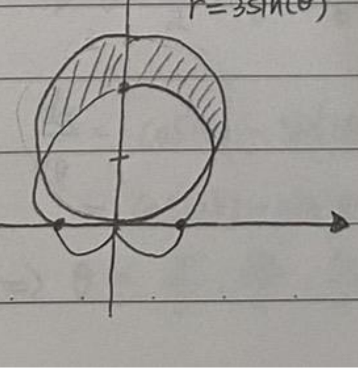
• Area of the Region, $0 \le r_1(\theta) \le r \le r_2(\theta)$



$$A = \int_{\theta_a}^{\theta_b} \frac{1}{2}\left([r_2(\theta)]^2 - [r_1(\theta)]^2\right) d\theta$$

Ex. Find area of region that lie inside $r=3\sin(\theta)$
outside $r=1+\sin(\theta)$

$3\sin(\theta) = 1+\sin(\theta)$
$2\sin(\theta) = 1$
$\sin(\theta) = \frac{1}{2}$
$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$



$$A = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} [3\sin(\theta)]^2 - [1+\sin(\theta)]^2 d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(8\sin^2(\theta) - 2\sin(\theta) - 1\right) d\theta$$

Chryuc

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$$A = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 4 - 4\cos(2\theta) - 2\sin(\theta) - 1 \, d\theta$$

$$= \frac{1}{2}\left(3\theta - 2\sin(2\theta) + 2\cos(\theta) \Big|_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \right) = \pi$$

• Arc Length

$$x = r\cos(\theta) = f(\theta)\cos(\theta)$$

$$y = r\sin(\theta) = f(\theta)\sin(\theta)$$

$$\frac{dx}{d\theta} = f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)$$

$$\frac{dy}{d\theta} = f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)$$

$$\left[\frac{dx}{d\theta}\right]^2 + \left[\frac{dy}{d\theta}\right]^2 = [f'(\theta)\cos(\theta)]^2 - 2f'(\theta)f(\theta)\cos(\theta)\sin(\theta) + [f(\theta)\sin(\theta)]^2$$

$$+ [f'(\theta)\sin(\theta)]^2 + 2f'(\theta)f(\theta)\sin(\theta)\cos(\theta) + [f(\theta)\cos(\theta)]^2$$

$$= [f'(\theta)]^2 + [f(\theta)]^2 = \left[\frac{dr}{d\theta}\right]^2 + r^2$$

$$L = \int_{\theta_a}^{\theta_b} \left(r^2 + \left[\frac{dr}{d\theta}\right]^2\right)^{\frac{1}{2}} d\theta$$

Ex. Find length of $r = 1 - \cos(\theta)$

$$\frac{dr}{d\theta} = \sin(\theta)$$

$$r^2 + \left[\frac{dr}{d\theta}\right]^2 = 1^2 - 2\cos(\theta) + \cos^2(\theta) + \sin^2(\theta)$$

$$= 2(1 - \cos(\theta))$$

$$L = \int_0^{2\pi} [2(1-\cos(\theta))]^{\frac{1}{2}} d\theta = \int_0^{2\pi} \left[4\sin^2\left(\frac{\theta}{2}\right)\right]^{\frac{1}{2}} d\theta$$

$$= 2\int_0^{2\pi} \sin\left(\frac{\theta}{2}\right) d\theta = 4\left(-\cos\left(\frac{\theta}{2}\right)\Big|_0^{2\pi}\right)$$

$$= 4(1+1) = 8$$

culture

- Tangent

$x = r\cos(\theta) = f(\theta)\cos(\theta)$

$y = r\sin(\theta) = f(\theta)\sin(\theta)$

$\frac{dx}{d\theta} = f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)$

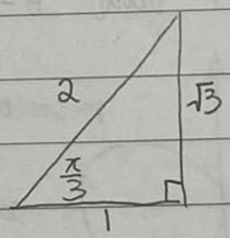
$\frac{dy}{d\theta} = f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)}{f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)} = \frac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)}$$

Notes tangent line at the pole, then $r=0$

reduce $\frac{dy}{dx} = \tan(\theta)$

Ex.(1) $r = 1+\sin(\theta)$, find slope of tangent line at $\theta = \frac{\pi}{3}$



$\frac{dr}{d\theta} = \cos(\theta)$

$$\frac{dy}{dx} = \frac{\cos(\theta)\sin(\theta) + [1+\sin(\theta)]\cos(\theta)}{\cos^2(\theta) - [1+\sin(\theta)]\sin(\theta)}$$

$$= \frac{2\cos(\theta)\sin(\theta) + \cos(\theta)}{\cos^2(\theta) - \sin^2(\theta) - \sin(\theta)} = \frac{\sin(2\theta) + \cos(\theta)}{\cos(2\theta) - \sin(\theta)}$$

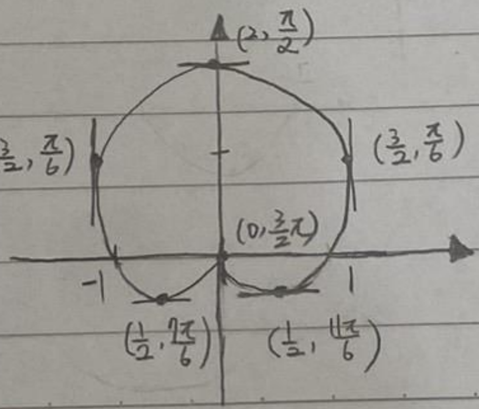
$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = \frac{\sin(\frac{2\pi}{3}) + \cos(\frac{\pi}{3})}{\cos(\frac{2\pi}{3}) - \sin(\frac{\pi}{3})} = \frac{\frac{1+\sqrt{3}}{2}}{-(\frac{1+\sqrt{3}}{2})} = -1$$ ※

(2) Find pts where tangent line is horizontal or vertical

$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin(\theta) + r\cos(\theta) = 2\cos(\theta)\sin(\theta) + \cos(\theta)$

$= \sin(2\theta) + \cos(\theta) = 0$

$\Rightarrow \theta = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$



$\frac{dx}{d\theta} = \cos^2(\theta) - \sin^2(\theta) - \sin(\theta)$

$= \cos(2\theta) - \sin(\theta) = 0$

$\Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{9\pi}{6}$

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- Area of a Surfcace of revolution

thm Let $f \in C^1([\theta_a, \theta_b])$

the area of surface by revolving $r = f(\theta)$

from θ_a to θ_b

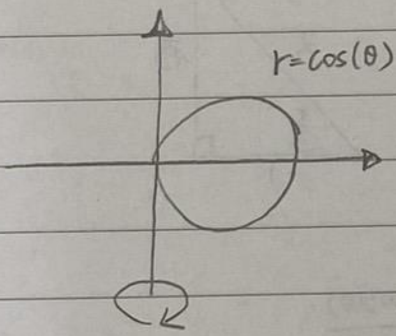
about polar axis $S = \int_{\theta_a}^{\theta_b} 2\pi y \, dL$

$$= \int_{\theta_a}^{\theta_b} 2\pi f(\theta) \sin(\theta) \left([f(\theta)]^2 + [f'(\theta)]^2 \right)^{\frac{1}{2}} d\theta$$

about $\theta = \frac{\pi}{2}$ $S = \int_{\theta_a}^{\theta_b} 2\pi x \, dL$

$$= \int_{\theta_a}^{\theta_b} 2\pi f(\theta) \cos(\theta) \left([f(\theta)]^2 + [f'(\theta)]^2 \right)^{\frac{1}{2}} d\theta$$

Ex. Find area of the surface by revolving the $r = \cos(\theta)$ about $\theta = \frac{\pi}{2}$



$$\frac{dr}{d\theta} = -\sin(\theta)$$

$$r^2 + \left[\frac{dr}{d\theta}\right]^2 = 1$$

$$S = \int_0^{\pi} 2\pi \cos^2(\theta) \, d\theta$$

$$= \pi \int_0^{\pi} 1 + \cos(2\theta) \, d\theta$$

$$= \pi^2 + \frac{\pi}{2} \left(\sin(2\theta) \Big|_0^{\pi} \right) = \pi^2$$