Pb:1 Consider the real vector space $M_{2\times 3}(\mathbb{R})$ with standard addition and standard scalar multiplication. Let

$$W_1 = \left\{ \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \in M_{2\times 3}(\mathbb{R}) \;\middle|\; b = a+c \right\}$$

$$W_2 = \left\{ \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \in M_{2\times 3}(\mathbb{R}) \;\middle|\; a = 2c+1 \right\}$$

Does W_1, W_2 forms a vector subspace of $M_{2\times 3}(\mathbb{R})$?

Ans:- For W_1:- Let $\begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \end{bmatrix} \in W_1$, $\begin{bmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \end{bmatrix} \in W_1$

Then $b_1 = a_1 + c_1$; $b_2 = a_2 + c_2$. Now,

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & b_1+b_2 & c_1+c_2 \\ d_1+d_2 & e_1+e_2 & f_1+f_2 \end{bmatrix}$$

Moreover $b_1 + b_2 = (a_1+c_1) + (a_2+c_2) = (a_1+a_2) + (c_1+c_2)$

Thus $\begin{bmatrix} a_1+a_2 & b_1+b_2 & c_1+c_2 \\ e_1+e_2 & d_1+d_2 & f_1+f_2 \end{bmatrix} \in W_1$

Let $k \in \mathbb{R}$, $k\begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \end{bmatrix} = \begin{bmatrix} ka_1 & kb_1 & kc_1 \\ kd_1 & ke_1 & kf_1 \end{bmatrix}$.

We also have $kb_1 = k(a_1+c_1) = ka_1 + kc_1$

$\therefore \begin{bmatrix} ka_1 & kb_1 & kc_1 \\ kd_1 & ke_1 & kf_1 \end{bmatrix} \in W_1$

clearly $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in W_1$ $\therefore$ W_1 is a subspace of $M_{2\times 3}(\mathbb{R})$

For W_2:- W_2 is not a subspace because $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \notin W_2$

Pb: 2 Consider the real vector space $M_{n\times n}(\mathbb{R})$ with respect to standard addition and standard scalar multiplication. Let

$$W = \{A \in M_{n\times n}(\mathbb{R}) \mid A = A^T\} \quad (A^T = \text{transpose of } A)$$

= collection of all symmetric matrices.

T.S. W a vector subspace of $M_{n\times n}(\mathbb{R})$

Proof:- Let $A, B \in W$. then $A = A^T$, $B = B^T$

we know $(A+B)^T = A^T + B^T = A + B$

$\therefore A+B \in W$

Let $k \in \mathbb{R}$, $(kA)^T = kA^T = kA$ $\therefore kA \in W$

clearly Zero matrix is a symmetric matrix

Thus W forms a subspace.

Check:- collection of all skew matrices forms a subspace of $M_{n\times n}(\mathbb{R})$.

Linear combination Let V be a real vector space. A vector y in V is a linear combination of vectors $u_1, u_2, \dots u_n$ in V if there exists scalars $k_1, k_2, \dots, k_n$ in $\mathbb{R}$ such that

$$y = k_1 u_1 + k_2 u_2 + \dots + k_n u_n.$$

Alternative definition:- Let V be a real vector space. A vector y in V is a linear combination of vectors $u_1, u_2, \dots u_n$ in V if the vector equation

$$y = k_1 u_1 + k_2 u_2 + \dots + k_n u_n$$

has a solution.

Pb: 1 Express the following vectors as a linear combination of vectors $(1,0)$, $(0,1)$ in $\mathbb{R}^2$

(i) $(5,-4)$ (ii) $\left(\frac{7}{2}, \frac{5}{3}\right)$ (iii) (x,y)

Soln:-

(i) Take $y = (5,-4)$ and $u_1 = (1,0)$, $u_2 = (0,1)$

Form the vector equation

$$(5,-4) = k_1(1,0) + k_2(0,1) \longrightarrow (1)$$

$$\Rightarrow (5,-4) = (k_1, k_2)$$

$$\therefore k_1 = 5; \quad k_2 = -4$$

Thus (1) has a solution.

$$\therefore (5,-4) = 5(1,0) + (-4)(0,1)$$

(ii) $$\left(\frac{7}{2}, \frac{5}{3}\right) = \frac{7}{2}(1,0) + \frac{5}{3}(0,1)$$

(iii) $$(x,y) = x(1,0) + y(0,1).$$

Pb:2 Does the vector $(3,7,-4)$ in $\mathbb{R}^3$ can be written as linear combination of $(1,2,3)$, $(2,3,7)$, $(3,5,6)$?

Soln:-

Take $y=(3,7,-4)$, $u_1=(1,2,3)$, $u_2=(2,3,7)$, $u_3=(3,5,6)$

Form the vector equation $y = k_1u_1+k_2u_2+k_3u_3 \longrightarrow (1)$

If (1) has a solution then y is a linear combination of u_1, u_2, u_3

$$(3,7,-4) = k_1(1,2,3)+k_2(2,3,7)+k_3(3,5,6)$$
$$= (k_1+2k_2+3k_3,\ 2k_1+3k_2+5k_3,\ 3k_1+7k_2+6k_3)$$

$$\Rightarrow \begin{aligned} k_1+2k_2+3k_3 &= 3\\ 2k_1+3k_2+5k_3 &= 7\\ 3k_1+7k_2+6k_3 &= -4 \end{aligned}$$

We solve this by Gauss elimination

$$\begin{array}{c} \text{1st pivot} \\ \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3\\ 2 & 3 & 5 & 7\\ 3 & 7 & 6 & -4 \end{array}\right] \end{array} \xrightarrow[R_3 \to R_3-3R_1]{R_2\to R_2-2R_1} \left[\begin{array}{cccc} 1 & 2 & 3 & 3\\ 0 & -1 & -1 & 1\\ 0 & 1 & -3 & -5 \end{array}\right] \xrightarrow{R_2 \to (-1)R_2} \left[\begin{array}{cccc} 1 & 2 & 3 & 3\\ 0 & 1 & 1 & -1\\ 0 & 1 & -3 & -5 \end{array}\right]$$

2nd pivot

$$\xrightarrow{R_3\to R_3-R_2} \left[\begin{array}{cccc} 1 & 2 & 3 & 3\\ 0 & 1 & 1 & -1\\ 0 & 0 & -4 & -4 \end{array}\right]$$

$\therefore$ Reduced system

$$\begin{aligned} k_1+2k_2+3k_3 &= 3\\ k_2+k_3 &= -1\\ -4k_3 &= -4 \end{aligned}$$

$$\Rightarrow \boxed{k_1=4;\ k_2=-2;\ k_3=1}$$

$\therefore$ (1) has a solution and so y is a linear combination of u_1, u_2, u_3

and $\boxed{y = 4u_1 - 2u_2 + u_3}$

Pb. 3 Determine whether $p(t) = t^2 + t + 2$ is a linear combination of $p_1 = t^2 + 2t + 1$; $p_2 = t^2 + 3$ and $p_3 = t - 1$

Soln:- The vector equation is

$$p = k_1 p_1 + k_2 p_2 + k_3 p_3 \longrightarrow (1)$$

If (1) has a solution then p is the linear combination of p_1, p_2, p_3.

$$(t^2 + t + 2) = k_1(t^2 + 2t + 1) + k_2(t^2 + 3) + k_3(t - 1)$$

$$t^2 + t + 2 = (k_1 + k_2)t^2 + (2k_1 + k_3)t + (k_1 + 3k_2 - k_3)$$

$$\Rightarrow \begin{aligned} k_1 + k_2 &= 1 \\ 2k_1 + k_3 &= 1 \\ k_1 + 3k_2 - k_3 &= 2 \end{aligned}$$

We solve this by Gauss elimination.

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 \\ 1 & 3 & -1 & 2 \end{bmatrix} \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -2 & 1 & -1 \\ 0 & 2 & -1 & 1 \end{bmatrix} \; R_3 \to R_3 + R_2 . \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The reduced system is

$$\left.\begin{aligned} k_1 + k_2 &= 1 \\ -2k_1 + k_3 &= -1 \end{aligned}\right\} \Rightarrow \text{Infinite solutions are possible.}$$

$\therefore$ p can be written as a linear combination of p_1, p_2, p_3.