Pb:1 Consider the real vector space $\mathbb{R}^3$ with respect to standard addition and standard scalar multiplication. Let

$$W_1 = \{(x, y, z) \in \mathbb{R}^3 \mid x - 2y + 3z = 0\} =$$

$$W_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x \geq 0;\ y \geq 0\}$$

$$W_3 = \{(x, y, z) \in \mathbb{R}^3 \mid z = x + y\}$$

Is W_1, W_2, W_3 are vector subspaces of $\mathbb{R}^3$?

Soln:-

For W_1:- Geometrically W_1 represents a plane passing through origin.

clearly $(0, 0, 0) \in W_1$;

Let $(x_1, y_1, z_1) \in W_1$, $(x_2, y_2, z_2) \in W_2$ then

$$x_1 - 2y_1 + 3z_1 = 0 \quad \text{and} \quad x_2 - 2y_2 + 3z_2 = 0$$

Now, $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + y_1,\ x_2 + y_2,\ x_3 + y_3)$

we observe the following

$$(x_1 + y_1) - 2(x_2 + y_2) + 3(x_3 + y_3) = (x_1 - 2y_1 + 3z_1) + (x_2 - 2y_2 + 3z_2)$$
$$= 0 + 0 = 0$$

Thus $(x_1 + y_1,\ x_2 + y_2,\ x_3 + y_3) \in W_1$

Let $k \in \mathbb{R}$, $(x, y, z) \in W_1$ then $x - 2y + 3z = 0$

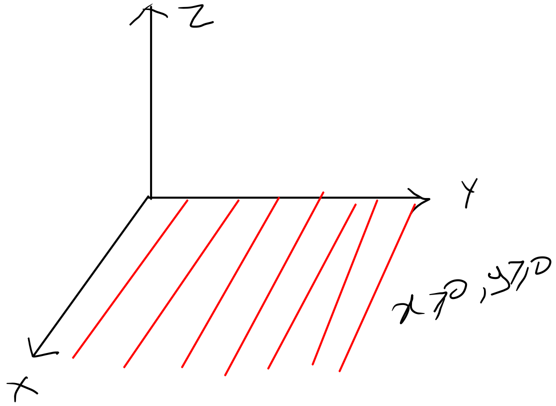
Now, $k(x, y, z) = (kx, ky, kz)$

We observe that $(kx) - 2(ky) + 3(kz) = k(x - 2y + 3z)$
$$= k \cdot 0 = 0$$

$\therefore k(x, y, z) \in W_1$

$\therefore W_1$ is a subspace.

For W_2:-



$W_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x \geq 0 ; y \geq 0\}$.

clearly $(0,0,0) \in W_2$

Let $(x_1, y_1, z_1) \in W_2$, $(x_2, y_2, z_2) \in W_2$. Then

$x_1 \geq 0 ; y_1 \geq 0 ;$ $\quad x_2 \geq 0 ; \; y_2 \geq 0 \quad \Rightarrow \quad x_1 + x_2 \geq 0 ; \; y_1 + y_2 \geq 0$

Now, $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

Since $x_1 + x_2 \geq 0$; $\quad y_1 + y_2 \geq 0$

$\therefore (x_1 + x_2, \; y_1 + y_2, \; z_1 + z_2) \in W_2$.

Consider the vector $(1,1,1) \in \mathbb{R}^3$ clearly $(1,1,1) \in W_2$

Take the scalar "-2"

$-2(1,1,1) = (-2,-2,-2) \notin W_2$

$\therefore$ $k(x, y, z)$ for some $k \in \mathbb{R}$, $(x, y, z) \in W_2$

need be an element of W_2.

$\therefore W_2$ is not a subspace.

check! W_3 is a subspace.

Remark:

1) The only subspaces of $\mathbb{R}^2$ are lines passing through origin, $\mathbb{R}^2$ and $\{(0,0)\}$

2) In $\mathbb{R}^3$ the only subspaces are $\{(0,0,0)\}$, lines passing through origin, planes passing through origin and $\mathbb{R}^3$

3) Consider the vector space $\mathbb{R}^n$ with standard addition and scalar multiplication. Suppose $a_1, a_2, \cdots a_n, b_0$ are real numbers, then the subset

$$W = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \mid a_1x_1 + a_2x_2 + \cdots + a_nx_n = b_0\}$$

is a subspace of $\mathbb{R}^n$ if $b_0 = 0$.

Q:- Consider the real vector space $\mathbb{R}^{50}$ with usual addition and scalar multiplication. Let

$$W_1 = \{(x_1, x_2, \cdots, x_{50}) \mid x_1 - x_2 + x_{39} + x_{50} = 0\}$$

$$W_2 = \{(x_1, x_2, \cdots, x_{50}) \mid 2x_9 - 7x_{14} + x_{49} = 2\}$$

Is W_1, W_2 a vector subspace of $\mathbb{R}^{50}$?

Ans:- By remark 3, W_1 is a subspace, W_2 is not a subspace.

Consider the real vector space $P_2(\mathbb{R})$ with standard addition and standard scalar multiplication. Let

$$E_1 = \{a_0 + a_1 t + a_2 t^2 \in P_2(\mathbb{R}) \mid a_0 = 0\}.$$

$$E_2 = \{a_0 + a_1 t + a_2 t^2 \in P_2(\mathbb{R}) \mid a_0 + a_1 + a_2 = 2\}$$

Does E_1, E_2 form a vector subspace of $P_2(\mathbb{R})$?

Ans:- For E_1:-

Let $p(t) = a_0 + a_1 t + a_2 t^2 \in E_1 \Rightarrow a_0 = 0$

$q(t) = b_0 + b_1 t + b_2 t^2 \in E_2 \Rightarrow b_0 = 0$

$\Rightarrow a_0 + b_0 = 0$

Now, $(p+q)(t) = p(t) + q(t) = (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2$

Since $a_0 + b_0 = 0$

$\Rightarrow (p+q)(t) \in E_1$

Let $k \in \mathbb{R}$, $(kp)(t) = ka_0 + ka_1 t + ka_2 t^2$

Since $a_0 = 0 \Rightarrow ka_0 = 0$

$\therefore (kp)(t) \in E_1$

clearly the zero polynomial in E_1

$\therefore E_1$ is a subspace.

E_2 is not a subspace because zero polynomial is not in E_2