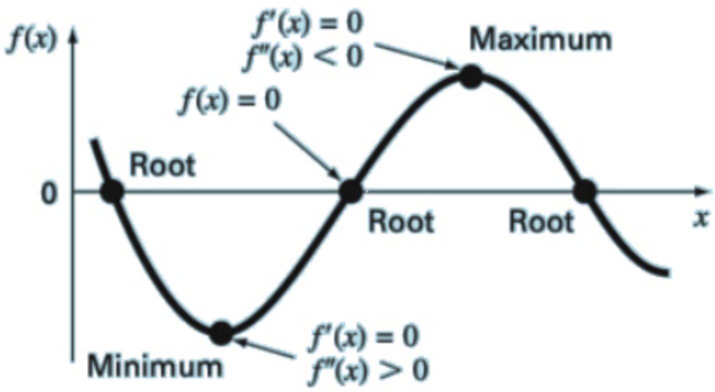
Chapter IV

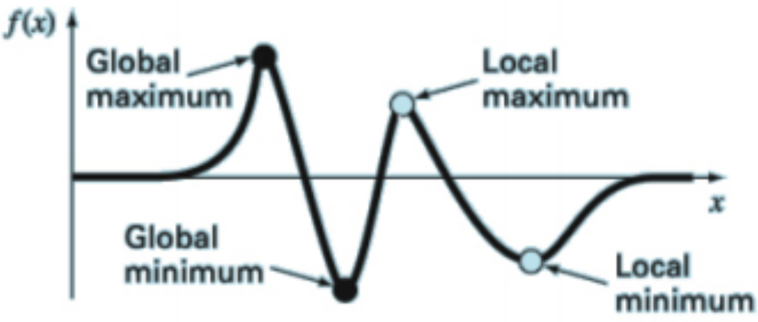
Optimization

4.1 Introduction

- Optimization is the term often used for minimizing or maximizing a function.
- Geometrically, the maximum or minimum occurs at the turning point or at the end points of a function.
- Mathematically, the derivative of function is zero at the turning point. Moreover, the second derivative, $f'(x)$, indicates whether the optimum is a minimum or a maximum: if $f''(x) < 0$, the point is a maximum; if $f''(x) > 0$, the point is a minimum.



- In one dimensional optimization problem, we need to find x that corresponds the derivative $f'(x)$ is equal to zero.
- In engineering, the quantity that we wish to optimize, $f(x)$, called the merit function or objective function, and the quantities that we are free to adjust, x, known as the design variables.
- There are two main optimizations:
  - Constrained optimization : restrictions or constraints are placed on the design variables
  - Unconstrained optimization : no restrictions are placed on the design variables
- Sometimes both local and global optima can occur in optimization as shown in figure. Such cases are called multimodal. If function has single optimum (i.e. maximum or minumum), then it is unimodal.
- In almost all instances, we will be interested in finding the absolute highest or lowest value of a function. Thus, we must take care that we do not mistake a local result for the global optimum.



4.1.1 Multidimensional Unconstrained Optimization

- Consider a two-variable function $f(x, y)$.
- As in one-dimensional case, in multidimensional case, maximum or minimum occurs where the partial derivatives equal to zero.
- If $f(x, y)$ has a maximum or minimum at (a, b), and the first order partial derivatives of $f(x, y)$ exist at (a, b), then

$$\left.\frac{\partial f}{\partial x}\right|_{(a,b)} = 0 \ , \ \left.\frac{\partial f}{\partial y}\right|_{(a,b)} = 0. \tag{5.25}$$

- Further, maximum or minimum occurs involves the second partial with respect to x and y.
- Assuming that the partial derivatives are continuous at and near the point (a, b), the following quantity can be computed:

$$\Delta = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \tag{5.26}$$

  - If $\Delta > 0$ and $\left.\frac{\partial^2 f}{\partial x^2}\right|_{(a,b)} > 0$, then $f(a, b)$ is a minimum.
  - If $\Delta > 0$ and $\left.\frac{\partial^2 f}{\partial x^2}\right|_{(a,b)} < 0$, then $f(a, b)$ is a maximum.
  - If $\Delta < 0$, then $f(a, b)$ is a saddle point.

Example 5.3 *$f(x,y) = 1 - x^2 - y^2$ has a maximum at $(0,0)$ since $\dfrac{\partial f}{\partial x}(0,0) = 0$, $\dfrac{\partial f}{\partial y}(0,0) = 0$ and*

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(0,0)} = -2\ ,\ \left.\frac{\partial^2 f}{\partial y^2}\right|_{(0,0)} = -2\ ,\ \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(0,0)} = 0 \implies \Delta > 0, \frac{\partial^2 f}{\partial x^2} < 0$$

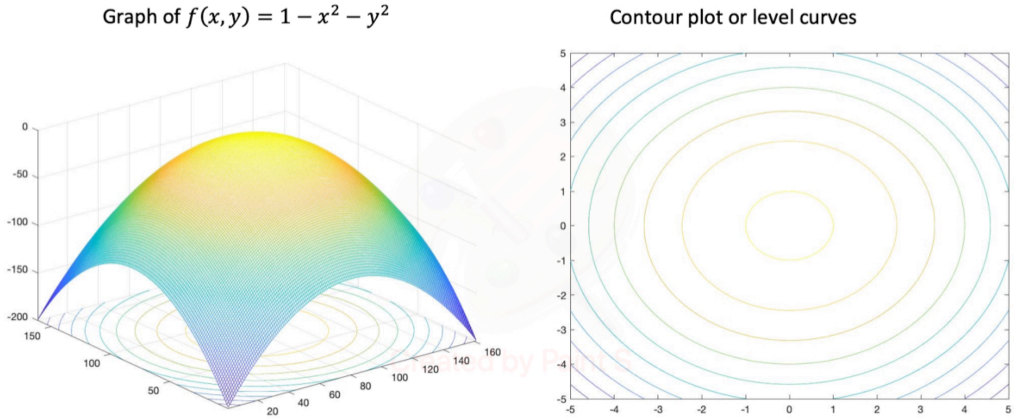


Figure 5.7

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- The derivative of function $f(x,y)$ is maximized in the direction of gradient and it is minimized in the opposite direction of gradient.
- The gradient of $f(x,y)$ is
$$\nabla f(x,y) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} \tag{5.27}$$
- An obvious strategy for climbing a hill would be to determine the maximum slope at your starting position and then start walking in that direction.
- Since the slope is changed point to point you could walk a short distance along the gradient direction, stop and reevaluate the gradient and walk another short distance. By repeating the process you would eventually get to the top of the hill.
- Using this stratergy, the following method was developed.

5.3.1 Steepest Ascent Method

To determine the maximum of $f(x,y)$:

- Start from (x_0, y_0) evaluate gradient at (x_0, y_0).
$$\nabla_{h_0} = \frac{\partial f}{\partial x}(x_0, y_0)\vec{i} + \frac{\partial f}{\partial y}(x_0, y_0)\vec{j} \tag{5.28}$$
- Search along the direction of the gradient, h_0, until we find a maximum.
  - Start from (x_0, y_0), the direction of the gradient can be expressed as
$$x = x_0 + h_0.\frac{\partial f}{\partial x}(x_0, y_0)\ \ ,\ \ y = y_0 + h_0.\frac{\partial f}{\partial y}(x_0, y_0) \tag{5.29}$$

- Evaluate $f(x, y)$ in direction h_0.

$$g(h_0) = f\Big(x_0 + h_0.\frac{\partial f}{\partial x}(x_0, y_0), y_0 + h_0.\frac{\partial f}{\partial y}(x_0, y_0)\Big) \tag{5.30}$$

- Set $g'(h_0) = 0$ and find h_0.

- The process is then repeated.

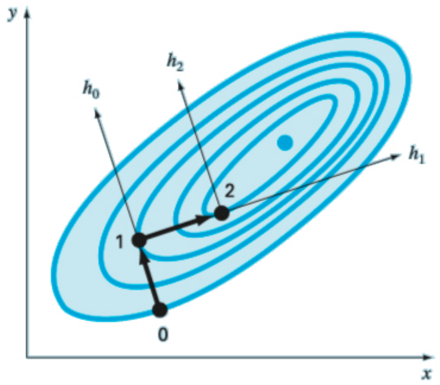


Figure 5.8

- Note that to find the minimum of $f(x, y)$, follow the above steps using $-\nabla f(x, y)$. This method is called as steepest descent method.

Example 5.4 *Maximize the following function:*

$$f(x, y) = 2xy + 2x - x^2 - 2y^2 \tag{5.31}$$

using initial guesses, $x = -1$ and $y = 1$.

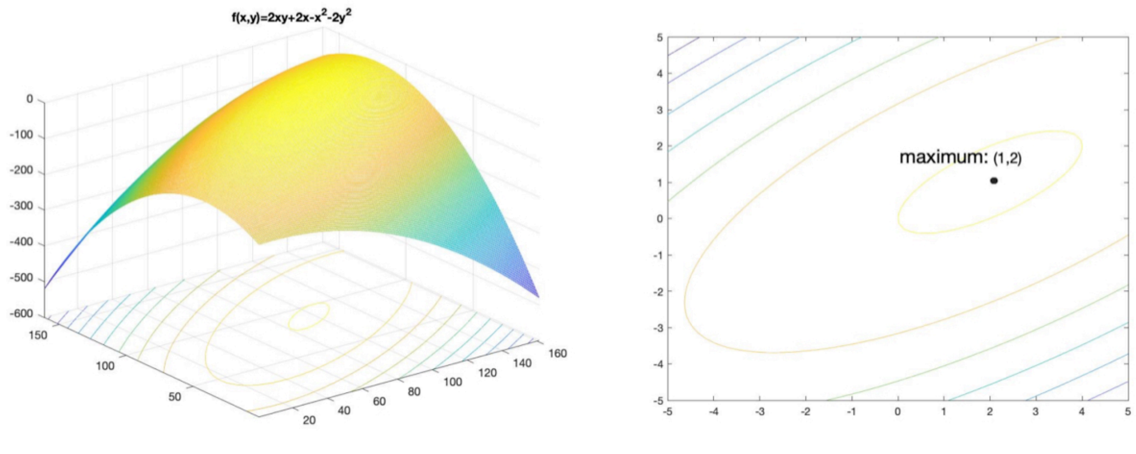


Figure 5.9

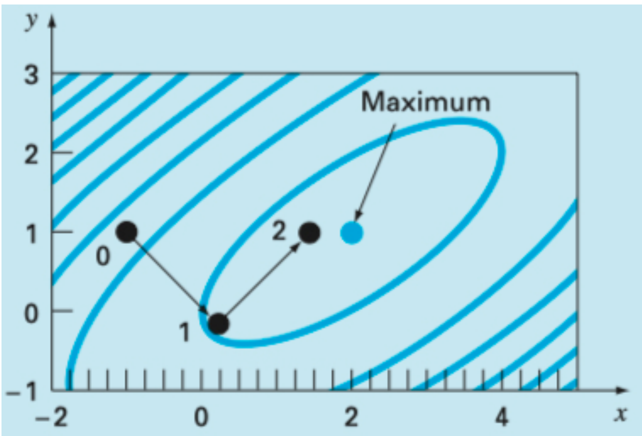
SOLUTION:

Now let us implement steepest ascent.

Note that

$$\frac{\partial f}{\partial x} = 2y + 2 - 2x \ , \ \frac{\partial f}{\partial y} = 2x - 4y \tag{5.32}$$

- *Initial approximation:* $(x_0, y_0) = (-1, 1)$
- *1st Iteration:*
  - $\left.\dfrac{\partial f}{\partial x}\right|_{(-1,1)} = 6 \ , \ \left.\dfrac{\partial f}{\partial y}\right|_{(-1,1)} = -6 \implies \nabla f = 6\vec{i} - 6\vec{j}$
  - $g(h_0) = f(x_0 + 6h_0, y_0 - 6h_0) = f(-1 + 6h_0, 1 - 6h_0) = -180h_0^2 + 72h_0 - 7$
  - $g'(h_0) = -360h_0 + 72 = 0 \implies h_0 = 0.2$
  - $x_1 = x_0 + 6h_0 = -1 + 6 \times 0.2 = 0.2,\ y_1 = y_0 - 6h_0 = 1 - 6 \times 0.2 = -0.2$
- *2nd Iteration:*
  - $\left.\dfrac{\partial f}{\partial x}\right|_{(0.2,-0.2)} = 1.2 \ , \ \left.\dfrac{\partial f}{\partial y}\right|_{(0.2,-0.2)} = 1.2 \implies \nabla f = 1.2\vec{i} + 1.2\vec{j}$
  - $g(h_1) = f(x_1 + 1.2h_1, y_1 + 1.2h_1) = f(0.2 + 1.2h_1, -0.2 + 1.2h_1) = -1.44h_1^2 + 2.88h_1 + 0.2$
  - $g'(h_1) = -2.88h_1 + 2.88 = 0 \implies h_1 = 1$
  - $x_2 = x_1 + 1.2h_1 = 0.2 + 1.2 = 1.4,$
    $y_2 = y_1 + 1.2h_1 = -0.2 + 1.2 = 1$
- *3rd Iteration:*
  - $\left.\dfrac{\partial f}{\partial x}\right|_{(1.4,1)} = 1.2 \ , \ \left.\dfrac{\partial f}{\partial y}\right|_{(1.4,1)} = -1.2 \implies \nabla f = 1.2\vec{i} - 1.2\vec{j}$
  - $g(h_2) = f(x_2 + 1.2h_2, y_2 - 1.2h_2) = f(1.4 + 1.2h_2, 1 - 1.2h_2) = -7.2h_2^2 + 2.88h_2 + 1.64$
  - $g'(h_2) = -14.2h_2 + 2.88 = 0 \implies h_2 = 0.2028$
  - $x_3 = x_2 + 1.2h_2 = 1.4 + 1.2 \times 0.2028 = 1.64336,$
    $y_3 = y_2 - 1.2h_2 = 1 - 1.2 \times 0.2028 = 0.75664$



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5.4 Constrained Optimization

- Constraint optimization is the process of optimizing an objective function with respect to some variables in the presence of constraints on those variables.

- General constrained minimization problem may be written as follows:

$$\begin{array}{rlll} \min & f(\mathbf{x}) & & \\ \text{subject to} & g_i(\mathbf{x}) = c_i & \text{for } i = 1, \ldots, n & \text{Equality constraints} \\ & h_j(\mathbf{x}) \geq d_j & \text{for } j = 1, \ldots, m & \text{Inequality constraints} \end{array} \tag{5.33}$$

5.4.1 Linear Programming

- The basic linear programming problem consists of two major parts:

  - a linear objective function

$$z = f(x) = a_1x_1 + a_2x_2 + \cdots + a_nx_n \tag{5.34}$$

  - a set of constraints (linear ineaqualities)

$$a_{i1}x_1 + a_{i2} + \cdots + a_{in}x_n \leq b_i \tag{5.35}$$

or

$$a_{i1}x_1 + a_{i2} + \cdots + a_{in}x_n \geq b_i \tag{5.36}$$

Example 5.5 *Energy Savers, Inc., produces heaters of types S and L. The wholesale price is \$40 per heater for S and \$88 for L. Two time constraints result from the use of two machines M_1 and M_2. On M_1 one needs 2 min for an S heater and 8 min for an L heater. On M_2 one needs 5 min for an S heater and 2 min for an L heater. Determine production figures x_1 and x_2 for S and L, respectively (number of heaters produced per hour), so that the hourly revenue*

$$z = f(x) = 40x_1 + 88x_2 \tag{5.37}$$

is maximum.

SOLUTION:

- *The objective function (to be maximized)*

$$z = f(x) = 40x_1 + 88x_2 \tag{5.38}$$

- *Four constraints*

$$\begin{aligned} 2x_1 + 8x_2 &\leq 60 \quad \textit{(time on machine } M_1\textit{)} && (5.39) \\ 5x_1 + 2x_2 &\leq 60 \quad \textit{(time on machine } M_2\textit{)} && (5.40) \\ x_1 \qquad &\geq 0 && (5.41) \\ x_2 &\geq 0 && (5.42) \end{aligned}$$

- *Since this problem has limited number of varaibles, x_1, x_2, the solution can be approximated by graphically.*

Q7) S.A.M to locate maximum of

$$f(x,y) = 3.5x + 2y + x^2 - x^4 - 2xy - y^2$$

with initial condition $(0,0) = (x_0, y_0)$

3 iter.

all iter.

=

$\boxed{2.35}$

$\Rightarrow \nabla f|_{x_0 y_0}$

$x = x_0 + \frac{\partial f}{\partial x} h_0$

$x, y \Rightarrow y = y_0 + \frac{\partial f}{\partial y} h_0$

$f(x, y) = g(h_0)$

$g' = 0 \rightsquigarrow h_0$

Quiz:

$$f(x,y) = 2.25xy + 1.75 - 1.5x^2 - 2y^2$$

start $x_0 = 1, \quad y_0 = 1$

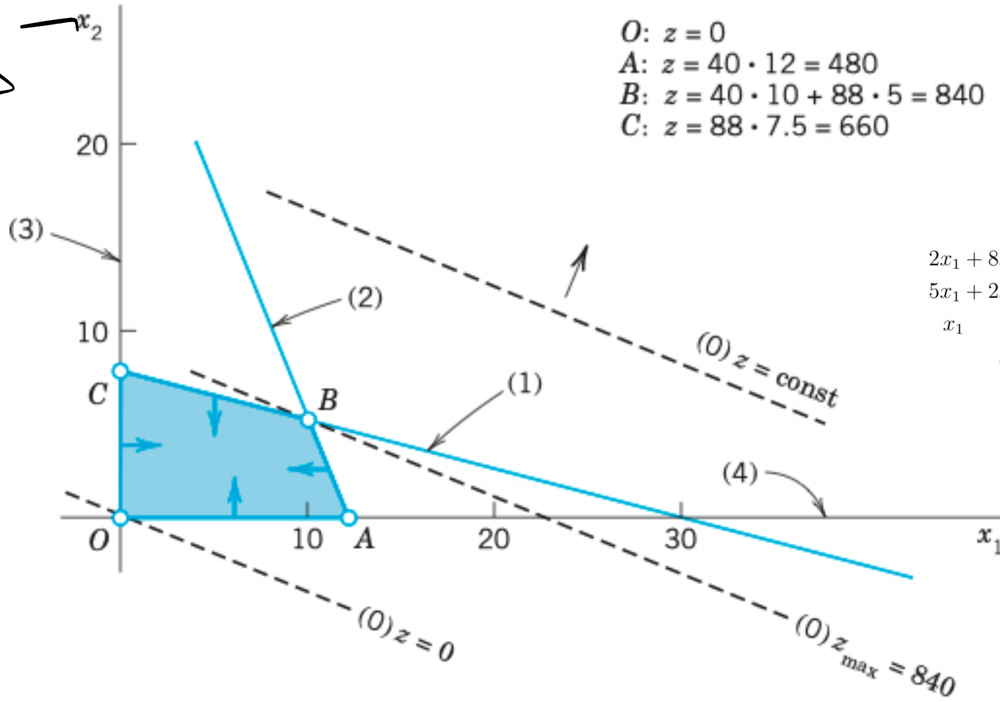
appl 3 application of Steepest ascent method to find optimum point.

✓ ∇

✓ x

y

✓ $g(h) = 0$



$z = f(x) = 40x_1 + 88x_2$

$$\begin{aligned} 2x_1 + 8x_2 &\le 60 \quad \textit{(time on machine } M_1\textit{)} \\ 5x_1 + 2x_2 &\le 60 \quad \textit{(time on machine } M_2\textit{)} \\ x_1 &\ge 0 \\ x_2 &\ge 0 \end{aligned}$$

Figure 5.11 Graphical solution of a linear programming problem

- *The optimal solution is obtained by moving the line of constant revenue* (0) *up as much as possible without leaving the feasibility region completely.*
- *Obviously, this optimum is reached when that line passes through B, the intersection* (10, 5) *of (1) and (2). We see that the optimal revenue*

$$z_{max} = (40)(10) + (88)(5) = \$840 \tag{5.43}$$

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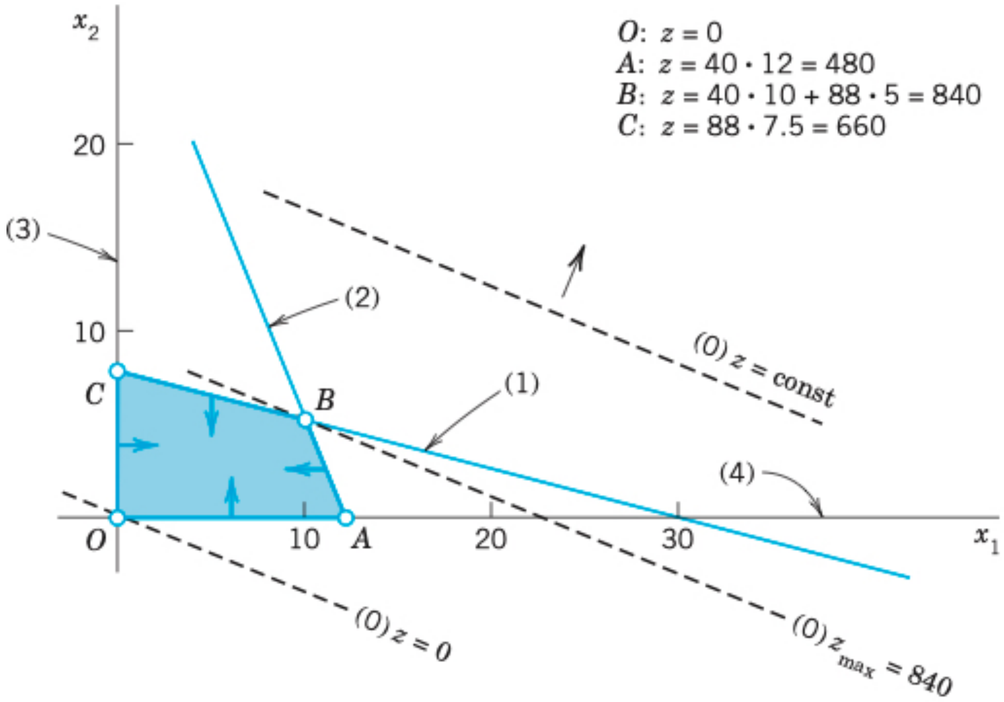
[1]**References:**

(i) *Numerical Methods for Engineers*, Steven C. Chapra, Raymond P. Canale



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