Network robustness

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Outline

1. Motivation
2. Percolation processes
3. Errors in real networks
4. Attack Tolerance
5. Cascading Failures
6. Modeling cascading failure

Complex networks - what we have learn so far

- Real complex networks can be found everywhere: social networks, biological networks, technological networks, transportation networks, etc.

- Real complex networks are:
  - heterogeneous - degree distribution, hubs, centralities
  - small world - average shortest path grows logarithmically with network size
  - correlated - assortative or disassortative
  - clustered - high value of clustering coefficient

- Structure of real complex networks is strongly related to their function and evolution

Designed vs. self-organized

- Human designed systems - components are assembled according to set goal
- Real complex systems are self-organized and self-assembled with survival as the only goal
- Evolution plays an important role in robustness of complex networks

Robustness: what

- Human designed systems fail if one component stops working: car stops working due to failure of component, computer stops working due to chip error, etc
- Complex systems (networks) function despite frequent occurence of errors: protein folding fails offten, missing members of organizations, members of social groups change constantly
- Real complex network are robust

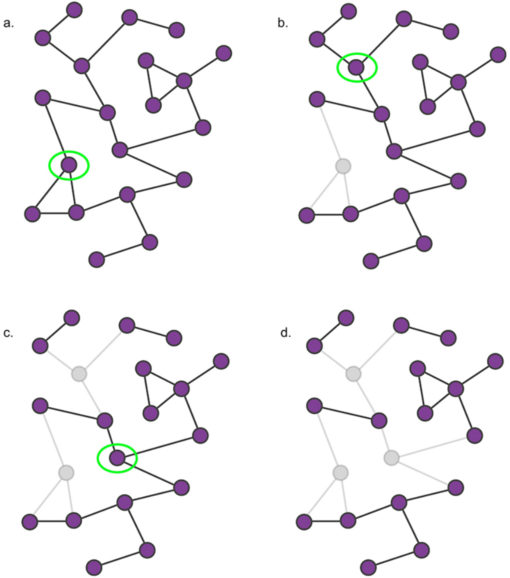
Robustness: why

- Understanding the origins of this robustness is important
    - origin of diseases
    - stability of societies and economies
    - ecosystem robustness and failure
    - smart design
- Complex networks and their structure play a key role in the robustness of biological, social and technological systems
- Reading material: chapter 8 in Network science book, http://networksciencebook.com/chapter/8

What we will cover

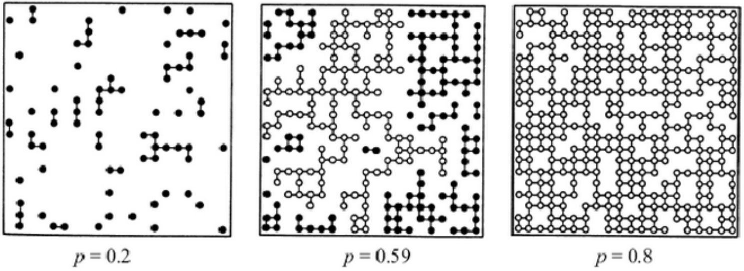
- Percolation processes
- Robustness of scale free networks
- Attack tollerance
- Cascading failures

Percolation example



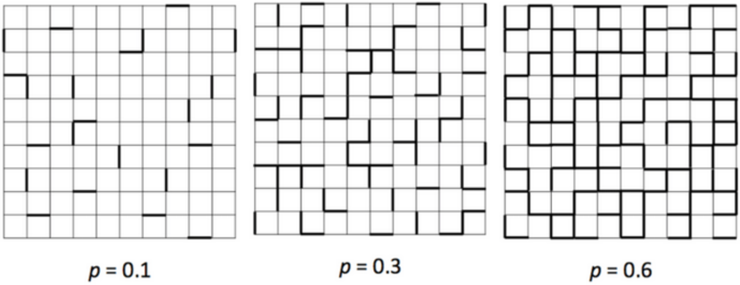
- Removal of one node does not influence the network connectivity
- How many nodes need to be removed to disconnect network
- Same can be asked for edges
- Source: Network science book

Site percolation



Source: Bunde et al., Anomalous transport and diffusion in percolation systems (2007)

Edge(bond) percolation



Source: Ghanbarian et al., Agricultural Water Management, 210, 208-216 (2018)

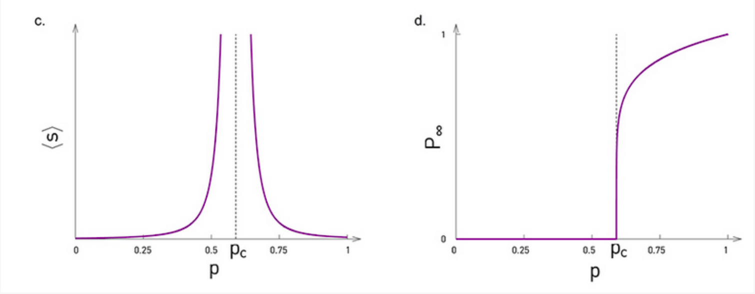
Percolation theory

- Subfield of statistical physics and mathematics
- It is one parameter process - probability p
- Questions:
  - What is the expected size of the largest cluster for a given p?
  - What is the average cluster size for a given p?
  - For what value of p we have spanning(infinite) cluster?
- Answers depend on the type of process (site vs. bond), probability p, and topology

Measured quantities

- Average size of finite cluster: $\langle s \rangle \sim |p - p_c|^{-\gamma_p}$
- Order parameter P_∞ - probability that randomly chosen site belongs to largest cluster: for $p > p_c$ $P_\infty \sim (p - p_c)^{\beta_p}$, $p \leq p_c$, $P_\infty = 0$
- Correlation length ξ - the mean distance between two sites belonging to the same cluster: $\xi \sim |p - p_c|^{-\nu}$

Site percolation on square lattice

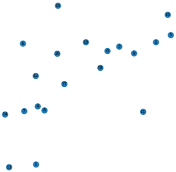


$p_c = 0.593$; $\gamma_p = \frac{43}{18}$, $\beta_p = \frac{5}{36}$, and $\nu = \frac{4}{3}$

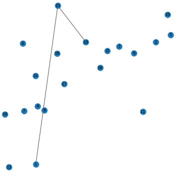
p_c depends on lattice type; γ_p, β_p and ν depend on lattice dimension

Source: Network science book

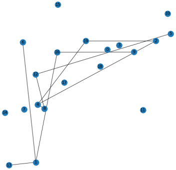
Erdos-Renyi graphs as percolation process



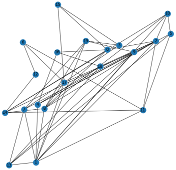
$p = 0$



$p = 0.01$



$p = 0.05$



$p = 0.3$

For $p > p_c = \frac{1}{N}$ giant component emerges

Inverse percolation process

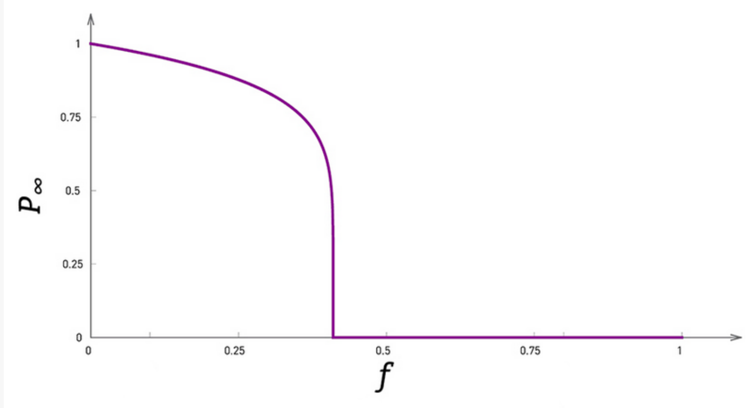
- Influence of node or edge failure on network integrity - robustness
- Inverse percolation process can be used to explain and understand this process
- How many nodes we need to remove to *destroy* giant component?

Square lattice

$$f = 1 - p$$

$f = 0.1$ $\quad$ $f = f_c = 1 - p_c$ $\quad$ $f = 0.8$

Square lattice - order parameter



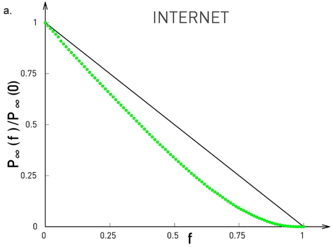
f_c depends on the type of lattice, exponents depend on lattice dimension

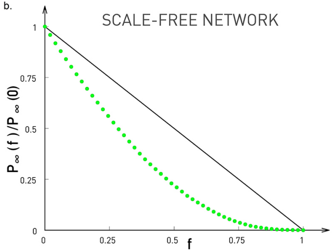
Error failure in ER graphs

- Error failure in ER graphs can be described by inverse percolation
- We remove f fraction of nodes in ER graph and observe how different properties change
- ER graphs have infinite dimension; $\gamma_p = 1$, $\beta_p = 1$ and $\nu = \frac{1}{2}$

Robustness of internet network

Real complex networks are robust to errors!





We need to remove almost all nodes ($fc = 1$) to distroy giant component

Source: Network science book

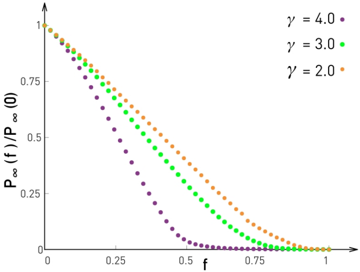
Estimate of f_c

- For a giant component to exist most nodes that belong to it must be connected to at least two other nodes
- Molloy-Reed Criterion $\kappa = \frac{\langle k^2 \rangle}{\langle k \rangle} > 2$; $\langle k^2 \rangle = \sum_k P(k) k^2$
- It is valid for any $P(k)$
- Critical threshold for networks: $f_c = 1 - \frac{1}{\frac{\langle k^2 \rangle}{\langle k \rangle} - 1}$

f_c for ER graphs

- ER graphs: $\langle k^2 \rangle = \langle k \rangle (\langle k \rangle + 1)$, $f_c = 1 - \frac{1}{\langle k \rangle}$
- For ER graphs f_c is finit; f_c is directly related to network density

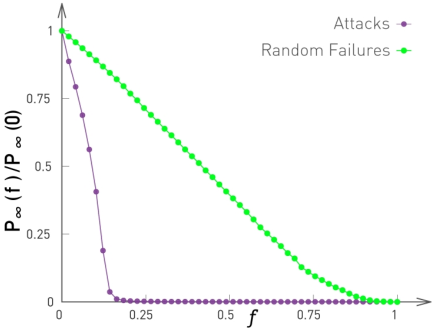
f_c for scale free networks



- For networks with $2 < \gamma < 3$ $(P(k) \sim k^{-\gamma})$: $f_c = 1 - \frac{1}{\frac{\gamma-2}{3-\gamma} k_{min}^{\gamma-2} k_{max}^{3-\gamma} - 1}$; for $N \to \infty$ $k_{max} \to \infty$ and $f_c = 1$
- For networks with $\gamma > 3$ $(P(k) \sim k^{-\gamma})$: $f_c = 1 - \frac{1}{\frac{\gamma-2}{\gamma-3} kmin - 1}$ and it is independent of the network size N

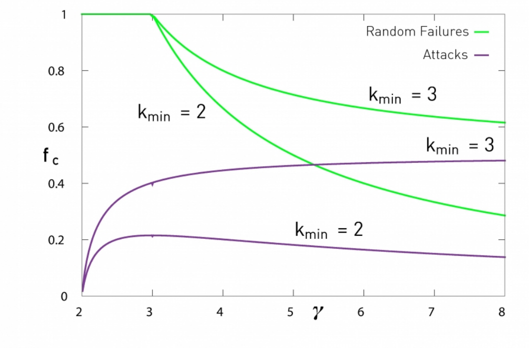
f_c for finit networks

- Real network have a finit size!
- Internet network $\kappa = 37.91$ and $f_c = 0.0972$
- Real networks are robust!
- They have enhanced robustness $f_c > f_c^{ER}$
- Similar is observed for link removal!



- We remove nodes according to their degree, starting from the node with the highest degree
- We need to remove less than 20% of nodes to decompose a network
- Real networks are less tolerant to attacks

f_c for attacks



$$f_c^{\frac{2-\gamma}{1-\gamma}} = 2 + \frac{2-\gamma}{3-\gamma} k_{min} \left(f_c^{\frac{3-\gamma}{1-\gamma}} - 1\right)$$

Real networks are vulnerable to deliberate attacks

f_c for attacks

Network	Random Failures (Real Network)	Random Failures (Randomized Network)	Attack (Real Network)
Internet	0.92	0.84	0.16
WWW	0.88	0.85	0.12
Power Grid	0.61	0.63	0.20
Mobile Phone Calls	0.78	0.68	0.20
Email	0.92	0.69	0.04
Science Collaboration	0.92	0.88	0.27
Actor Network	0.98	0.99	0.55
Citation Network	0.96	0.95	0.76
E. Coli Metabolism	0.96	0.90	0.49
Protein Interactions	0.88	0.66	0.06

Source: Network science book

More details in *Network science* Chapter 8

Cascading failures

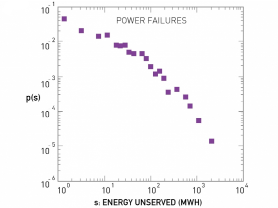
- Failures are not independent - blackouts (Power Grid network), financial crisis (financial networks)
- Attacks are not always independent - Denial of Service Attacks (Internet)
- Domino effect - failure of one node leads to incresed preassure on connected nodes and their failure
- Connected failures - cascading failures

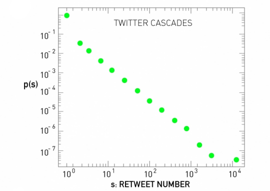
Avalanches

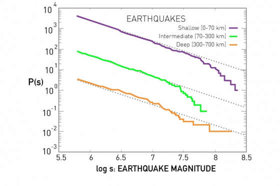
Avalanche size s equals the number of grains that fall of the platform

Distribution of avalanche sizes $P \sim s^{-\alpha}$ Source: Hesse et al., Frontiers in systems neuroscience 8, 166 (2014).

Avalanche sizes in different systems







Many other systems exhibit cascading failure: speacies, collective emotions, supply chains, neural systems etc.

$$1 < \alpha \leq 2$$

Most of the avalanches are small but large ones are still probable

Cascading failure

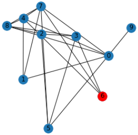
- Emergence of cascade:
  - Network properties
  - Propagation process
  - Breakdown criteries
- The distribution of sizes is universal
- Universality - the process does not depend on details
- Models that capture power-law size distribution

Failure Propagation Model

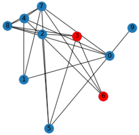
- Network of arbitrary structure
- Each agent can be in two states: active 0 and inactive 1
- Each agent has a breakdown treshold $\phi_i = \phi$
- At $t = 0$ all agents are active except one agent that is in state 1

Failure Propagation Model

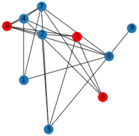
Dynamics rules



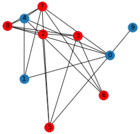
$t = 0$



$t = 5$



$t = 10$

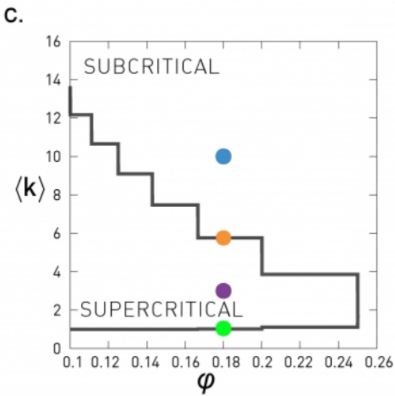


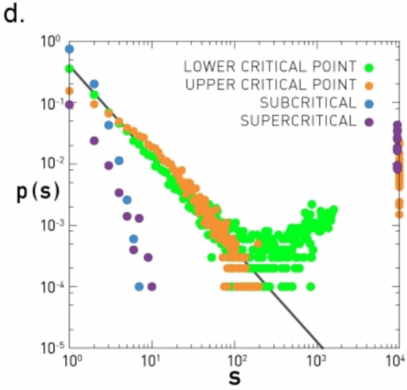
$t = 15$

At each time step we select a random agent i:

- If agent i is in state 0 we calculate the fraction of its k_i neighbours that are in the state 1 - ξ_i; if $\xi > \phi_i$ agent i becomes inactive
- If agent i is in a state 1 then nothing happens

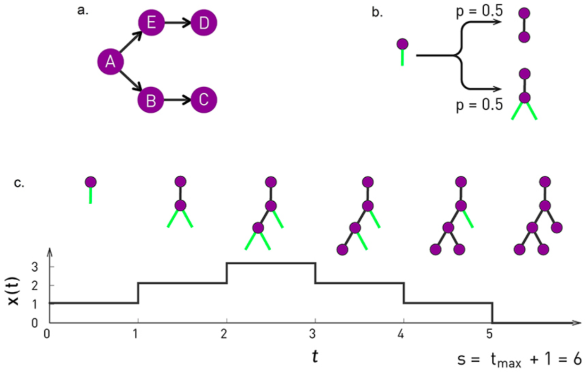
Failure Propagation Model - results





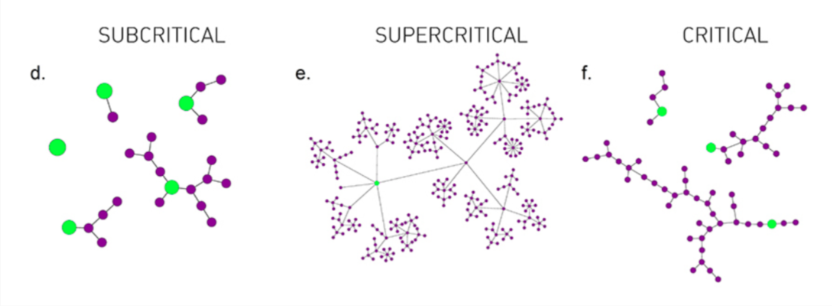
$\phi = 0.18$: $\langle k \rangle = 1.05$ (lower critical point), $\langle k \rangle = 3.0$ (supercritical), $\langle k \rangle = 5.76$ (upper critical point),$\langle k \rangle = 10.0$ (supercritical); $\alpha = 32$

Branching model



We begin with one node; in each step a new generation of nodes create a next generation of nodes taken from $P(k)$ distribution; the process stops when $k = 0$

Branching model



Source: Network science book