Multiplex and temporal networks

M. Mitrović Dankulov and A. Alorić

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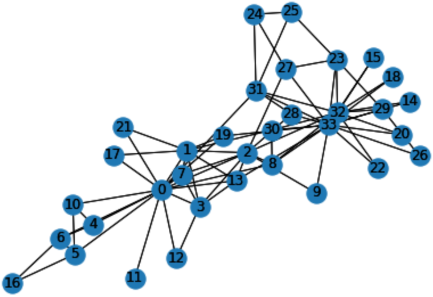
Outline

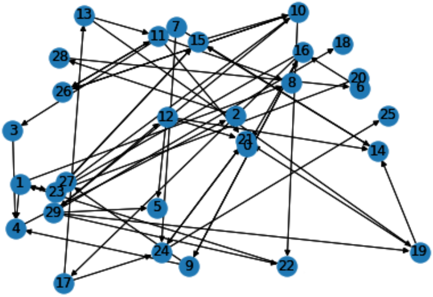
Types of networks we covered so far

Undirected, unweighted network



$A_{ij} = 1$ or $A_{ij} = 0$; $A_{ij} = A_{ji}$

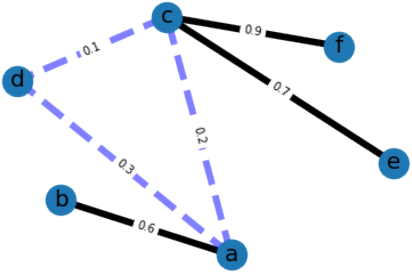
Directed, unweighted network



$A_{ij} = 1$ or $A_{ij} = 0$; $A_{ij} \neq A_{ji}$

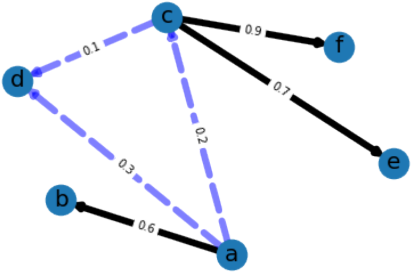
Types of networks we covered so far

Undirected, weighted network



$A_{ij} \geq 0$; $A_{ij} = A_{ji}$

Directed, weighted network



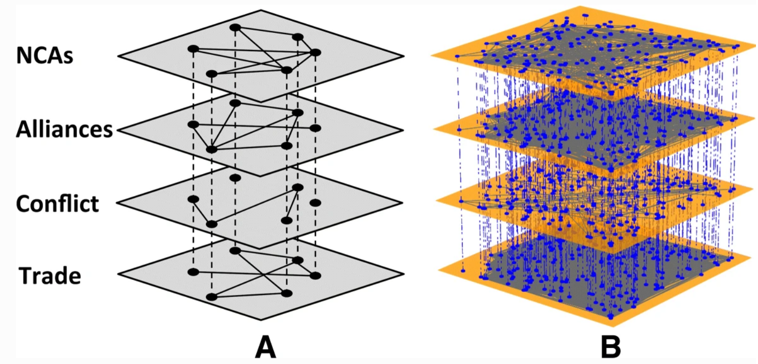
$A_{ij} \geq 0$; $A_{ij} \neq A_{ji}$

Types of networks we covered so far

- Single layer or monoplex - all edges are of the same type
  - collaboration between scientists, predator-pray relation, friendship network, links between airports - any company flight
- Edges are static - they constant from the moment they occur in the network
  - predator-pray relation, friendship network, family links

Different types of links

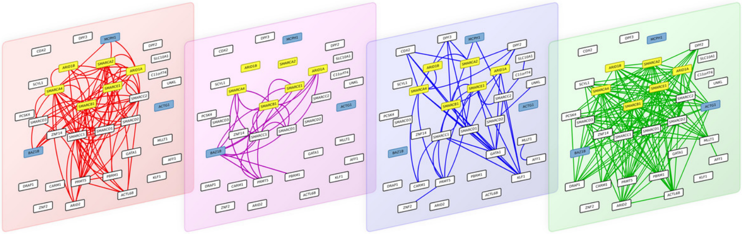
Nuclear power states



Source: Goldblum et al., Applied Network Science 4, 36(2019)

Different types of edges

Gene interaction network

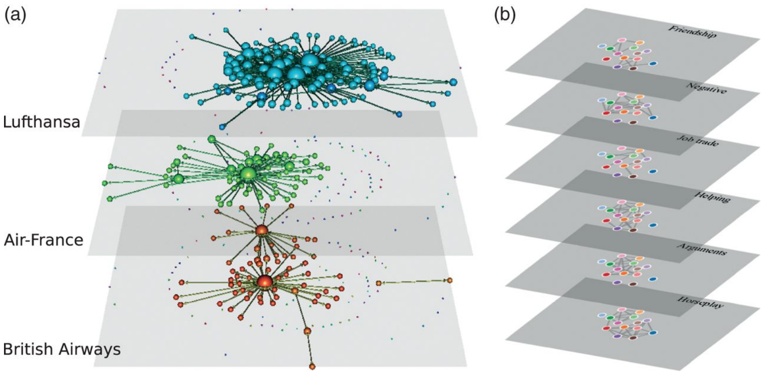


pathways, co-expression, PPIs and complexes networks

Source: Didier et al., PeerJ 2015, 1525 (2015)

Multiplex networks: example

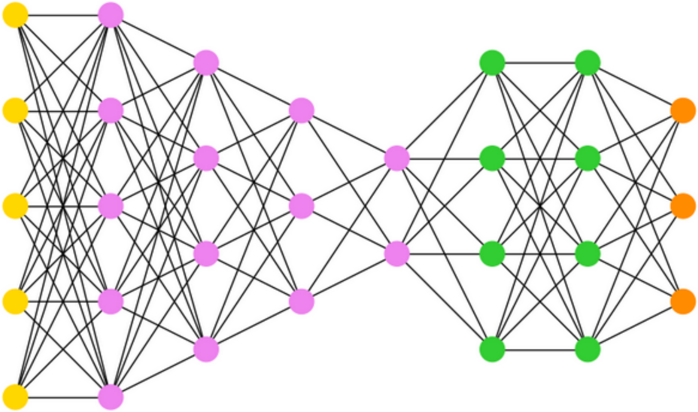
Each layer has the same set of nodes



Kiela et. al. Multilayer Networks, arXiv:1309.7233, March 2014

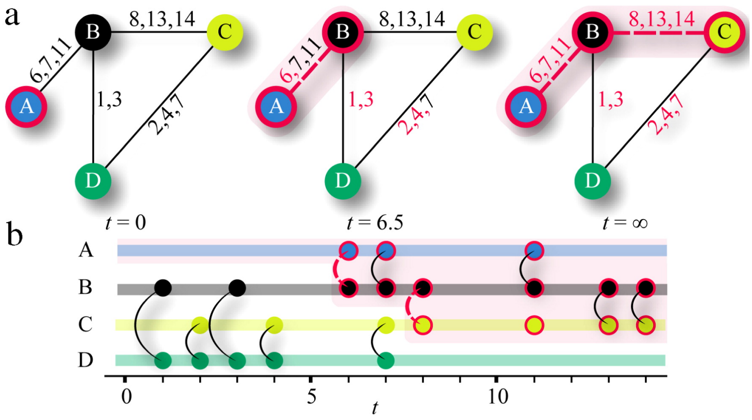
Multilayer networks: example

Deep learning neural networks



Temporal links

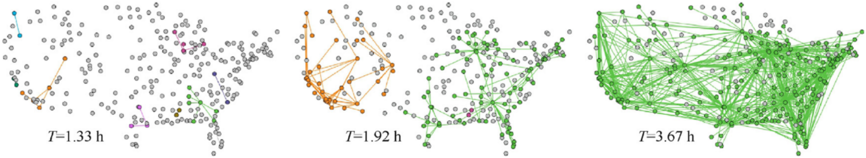
Contact networks



Source: Holme et al., Physics Reports 519, 97–125 (2012)

Temporal links

Airport network



(a) Temporal airport network with time window of 1.33 h
(b) Temporal airport network with time window of 1.92 h
(c) Temporal airport network with time window of 3.67 h

Source: Liu et al., Chinese Journal of Aeronautics Volume 33, 2019-226 (2020)

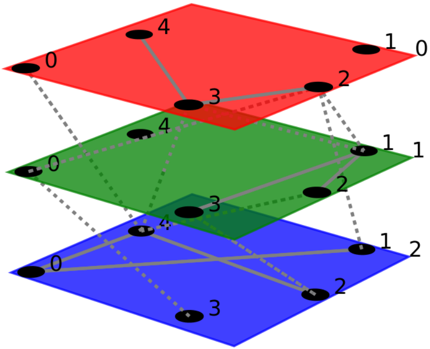
Multiplex networks

- There can be more than one relation between two nodes
- There can be different groups of nodes in each layer
- Multiplex or multilayer networks:
  - Each layer corresponds to one type of interactions
  - Layers can be independent, but offten are not
  - Nodes in different layers can be coupled

Multiplex networks: measures and questions

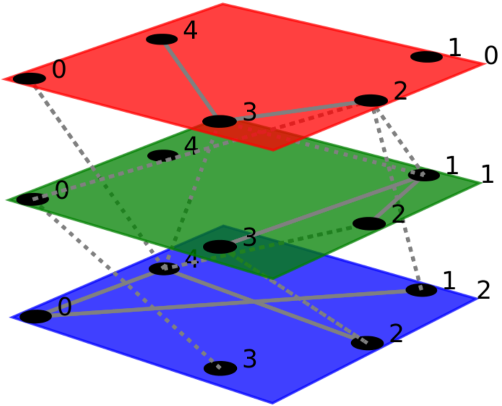
- Need of generalization of usual measures:
  - Degree, Neighbourhood, Centralities, Paths and distances, Clustering coefficient
- New layer-oriented questions to answer:
  - Which layers determine the centrality of a user; Which layers are relevant to measure the similarity of two nodes; How one layer influence the evolution of another; ...

Multiplex networks:definition



- α - layer index
- For each α (V^{α}, E^{α}): V^{α} - set of nodes and E^{α} - set of edges in layer α
- $C = \{E_{\alpha,\beta}\}$ - set of edges between nodes in different layer

Multiplex networks:definition



- Sets of nodes:
  $V^0 = \{0, 1, 2, 3, 4\}, V^1 = \{0, 1, 2, 3, 4\}, V^2 = \{0, 1, 2, 3, 4$
- Sets of intra-layer edges:
  $E^0 = \{(0, 1), (4, 2)\}$,
  $E^1 = \{(1, 4), (0, 3)\}$,
  $E^2 = \{(0, 2), (1, 2), (3, 4)\}$
- Sets of inter-layer edges:
  $E^{0,1} = \{(2, 0), (2, 1), (3, 1)\}$,
  $E^{0,2} = \{(0, 4), (2, 1), (3, 4)\}$,
  $E^{1,2} = \{(0, 3), (2, 4), (3, 2)\}$

Multiplex networks: basics

- Can be directed, weighted or k-partite

- Coupling between layers:
  - Ordinal: inter-layer links among consecutive layers
  - Categorical: inter-layer links between all pairs of layers
  - Generalized coupling: decay function, etc.

- The most simple case is categorical coupling: each layer has the same set of nodes coupled with each other only types of edges differe: airport network with companies, communication network with different types of communications (email, chats, cell phones, etc.), Gene interaction network

Multiplex networks: notations

- Adjacency matrix of layer α: $A_{ij}^{\alpha} = 1$ if $(i^{\alpha}, j^{\alpha}) \in E^{\alpha}$, 0 otherwise
- Adjacency matrix between layers α and β: $A_{ij}^{\alpha,\beta} = 1$ if $(i^{\alpha}, j^{\beta}) \in E^{\alpha,\beta}$, 0 otherwise
- L^{α} - number of links in layer α, $L^{\alpha,\beta}$ number of links between layers α and β
- Neighbours of node i in layer α: $\mathcal{N}^{\alpha}(i) = j \in V^{\alpha}, (i^{\alpha}, j^{\alpha}) \in E^{\alpha}$
- Total neighbours of node i: $\mathcal{N}^{tot}(i) = \cup_{\alpha} \mathcal{N}^{\alpha}(i)$

Multiplex networks: notations

- Degree of node i in layer α: $q_i^\alpha = |\mathcal{N}^\alpha(i)|$
- Total degree of node i: $q_i^{tot} = ||\mathcal{N}^{tot}(i)||$
- Centrality measures can be defined for each frame
- We use many terms for multiplex networks: multilayer, multimodal, network of networks, coupled networks, etc.
- According to Boccaletti et al., Physics Reports 544, 1-122 (2014), multiplex networks are special case of multilayer network where $V^1 = V^2 = \ldots = V^M$ and the only type of inter-layer connections are those in which a given node is only connected to its counterpart nodes in the rest of layers

Distances and paths

- Intra-layer paths and distances are the same as in monoplec networks
- Paths in multilayer network also include information about the layer
- Inter-layer connections can make a difference:
  - disconnected components in the layer may become connected
  - shortest path between nodes in the same layer maybe through nodes in other layers

Multiplex networks: comparison of layers

- Inter-layer links are trivial
- We can create a monopartite network aggregate - weighted network: $W_{ij} = \sum_\alpha A_{ij}^\alpha$
- We can create a projection network $\overline{A_{ij}} = 1$ if $(i^\alpha, j^\alpha) \in E^\alpha$ for any α
- We can consider each layer separately and study their mutual relations and influence
- Similarity between connectivity of different layers: $Overlap^{\alpha,\beta} = \sum_{i,j} A_{ij}^\alpha A_{ij}^\beta$

Multiplex networks: clustering

- Node clustering is a M dimensional vector: $(c_i^1, c_i^2, \ldots, c_i^M)$, c_i^α is clustering of node i in layer α

- We can use projection networks: $\overline{c_i} = \frac{\sum_{j,k} \overline{A_{ij}}\ \overline{A_{ik}}}{\sum_i \overline{A_{ij}}(\sum_i \overline{A_{ij}} - 1)}$

- We can also define a global clustering: $\langle c_i \rangle = \frac{2\sum_\alpha \overline{E^\alpha}}{\sum_\alpha |\mathcal{N}^\alpha(i)|(|\mathcal{N}^\alpha(i)| - 1)}$, $\overline{E_\alpha} = \{(i,j) \in E^\alpha; i,j \in \mathcal{N}^\alpha(i)\}$

- $\frac{1}{M-\theta_i}\overline{c_i} \leq \langle c_i \rangle \leq \overline{c_i}$, θ_i number of layers in which i has less than two neighbours

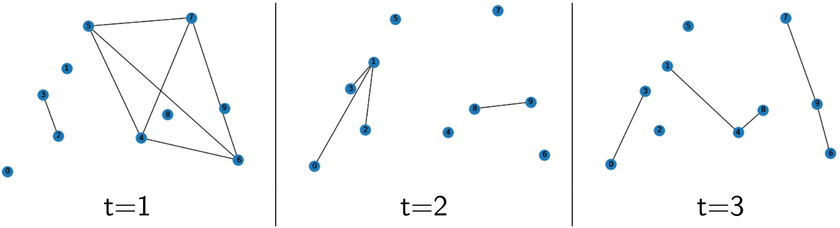
Degree-degree correlations

- Degree-degree correaltion distribution $P(k^\alpha, k^\beta) = \frac{N(k^\alpha, k^\beta)}{N}$
- Average degree of a node in layer α conditioned to the degree of the same node in layer β: $\langle k^\alpha \rangle (k^\beta) = \frac{\sum_{k^\alpha} k^\alpha P(k^\alpha, k^\beta)}{\sum_{k^\alpha} P(k^\alpha, k^\beta)}$
- If $\langle k^\alpha \rangle (k^\beta)$ is independent of k^β, then degree of nodes in layer α and β are uncorrelated
- Pearson correlation coefficient $r_{\alpha,\beta} = \frac{\langle k^\alpha k^\beta \rangle - \langle k^\alpha \rangle \langle k^\beta \rangle}{\sigma_\alpha \sigma_\beta}$, $\sigma_\alpha = \sqrt{\langle k^\alpha k^\alpha \rangle - \langle k^\alpha \rangle^2}$

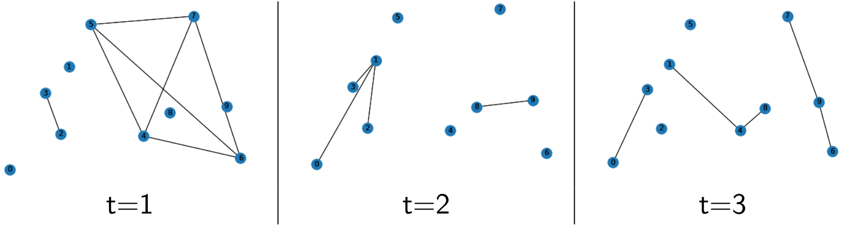
Time in networks

- Timestamps: Facebook or Twitter: friends (followers) added and removed over time, flight or bus ride time tables
- Duration: spending times with friend, duration of flight or ride
- Frequency: how offten you talk to your friends, number of flights

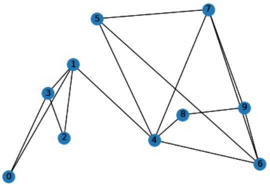
Temporal networks



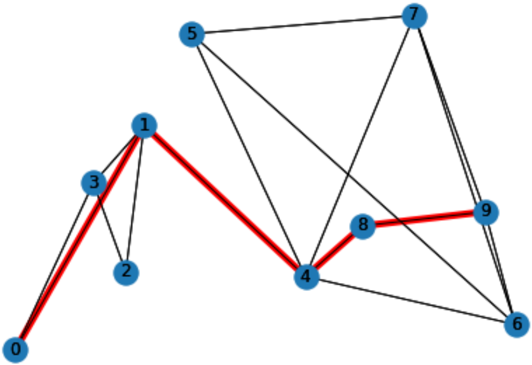
Temporal networks



Aggregated network

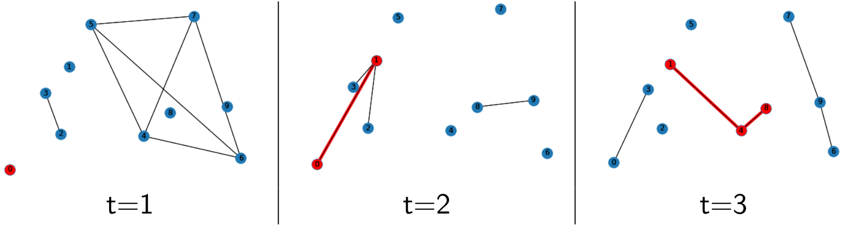


Paths in static networks



Shortest path from 0 to 9 is $(0, 1, 4, 8, 9)$, length 4

Paths in temporal networks

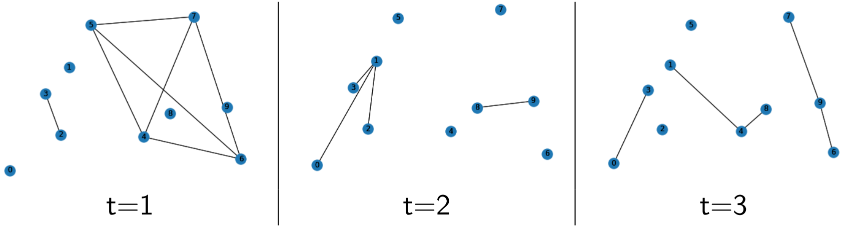


Node 9 is not reacheable from node 0

Temporal networks: paths

- Paths need to follow temporal order of edges
- Paths are temporal, and begin and end at certain points in time
- Their length is time dependent
- Observation period $t \in [t_0, T]$; $R(i)$ is a set of nodes that can be reached from node i during this interval; $R(i)$ is a set of influence of node i

Temporal networks

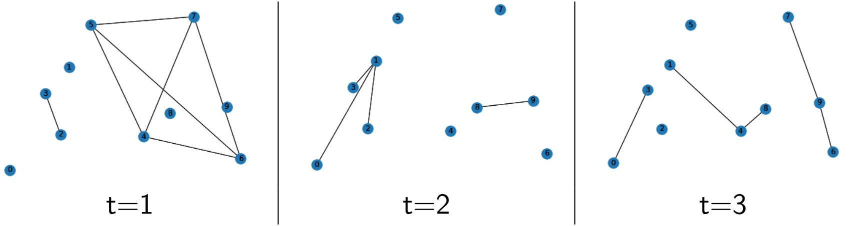


$R(1) = \{1, 2, 3, 4, 8\}$; $R(2) = \{0, 1, 3, 4, 8\}$; $R(3) = \{0, 1, 2, 3, 4, 8\}$;
$R(4) = \{5, 6, 7, 8, 1\}$

Distances,latencies, and fastest paths

- Duration - temporal path length
- Fastest time-respecting path(s) between two nodes
- The shortest time within which i can reach j is called their latency
- Distance - number of links; Duration and latency for measuring time

Temporal networks



$\lambda_{ij}(t) = 1$: $(3,2)$, $(4,5)$, $(4,6)$, $(4,7)$, $(5,6)$, $(5,7)$, $(6,7)$

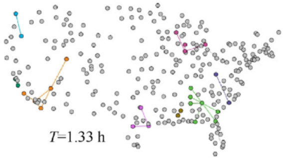
$\lambda_{ij}(t) = 2$: $(0,1)$, $(0,2)$, $(0,3)$, $(1,2)$, $(1,3)$, $(8,9)$

$\lambda_{ij}(t) = 3$: $(0,4)$, $(0,8)$, $(1,4)$, $(1,8)$, $(2,4)$, $(2,8)$, $(3,4)$, $(3,8)$, $(4,8)$, $(5,8)$, $(6,8)$, $(7,8)$

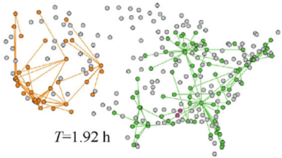
Centrality measures in temporal networks

- Static closeness and betweenness based on static shortest path
- Closeness and betweenness in temporal networks: duration, time-order, frequency
- Betweenness in temporal networks is generalized based on one for static by adding a dependence on time and counting the fraction of shortest or fastest time-respecting paths that pass through node i
- Closeness centrality $q_i(t) = \frac{N-1}{\sum_{i \neq j} \lambda_{i,t}(t)}$, $\lambda_{i,t}(t)$ is latency between i and j

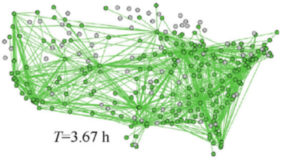
Observation time window



(a) Temporal airport network with time window of 1.33 h



(b) Temporal airport network with time window of 1.92 h



(c) Temporal airport network with time window of 3.67 h

Temporal scale