Computing Exercise 4: Investigating Rocket Orbits using the Runge Kutta Method

Saskia Jackson

sj16276

Introduction

The overall objective of this exercise is to simulate various motion of rockets in space. Firstly to determine circular and elliptical orbits around the Earth and then secondly to map the motion of a rocket travelling from Earth, around the moon and then back past Earth.

To do this the fourth-order Runge-Kutta method of differentiation is used. The Runge-Kutta method is an extension and improvement on the Euler method, developed around 1900 by C. Runge and M. W. Kutta, and works in a similar way; by taking the gradient of the curve being differentiated at various points in each time step.

The difference between the Euler Method and the Runge-Kutta method is that instead of the gradient being taken only at the beginning and end, the gradient is also taken at two middle points on the time step. This means that a more accurate estimation of the gradient can be calculated in the time step.

The four points in the Runge-Kutta calculation of gradient in one step size are determined as follows:

$$k_1 = f(x_n, y_n) \tag{1}$$

$$k_2 = f(x_n + \frac{h}{2}, y_n + \frac{hk_1}{2}) \tag{2}$$

$$k_3 = f(x_n + \frac{h}{2}, y_n + \frac{hk_2}{2}) \tag{3}$$

$$k_4 = f(x_n + h, y_n + hk_3) \tag{4}$$

These points are summed together in equation 5 to determine the differential.

$$y_{n+1} = y_n + \frac{h}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right] \tag{5}$$

where h is the step size.

The fourth-order Runge-Kutta method is used specifically in this exercise to find both the distance components and the velocity components as they change with time throughout an orbit.

The distance components are found using the fact that at any given point the differential of the distance in the x (or y) direction with respect to time is the velocity in that direction.

The velocity components were found by differentiating the below equations:

$$\frac{dv_x}{dt} = f_3(t, x, y, v_x, v_y) = \frac{-GMx}{(x^2 + y^2)^{\frac{3}{2}}} \tag{6}$$

$$\frac{dv_y}{dt} = f_3(t, x, y, v_x, v_y) = \frac{-GMy}{(x^2 + y^2)^{\frac{3}{2}}} \tag{7}$$

where dv_x and dv_y are the velocity components of the rocket, G is the gravitational constant ($6.67 \times 10^{-11} m^3 kg^{-1} s^{-2}$), M is the mass of the planet being orbited (in the case of the Earth being orbited $M = 5.9722 \times 10^24 kg$, m is the mass of the rocket and x and y are the position components.

Part a): Circular and Elliptical Orbits around a Planet

The first part of the exercise is designed to find various circular and elliptical orbits around the Earth.

This was achieved by writing a function for the equation describing the rate of change of velocity in both components, as shown in equations 6 and 7, plus a function to set the values of k in the Runge-Kutta method.

Then these k values were summed up and each value for x, y, v_x, v_y were appended to empty arrays. These arrays can then be used to plot various graphs.

The kinetic energy (E_k) was found using the velocity components in the following equation:

$$E_k = \frac{1}{2} m \left(\sqrt{(v_x^2 + v_y^2)} \right)^2 \tag{8}$$

The gravitational energy (GE) is calculated using equation 9.

$$GE = \frac{-GMm}{\sqrt{x^2 + y^2}} \tag{9}$$

The x component of the position can be plotted against the y component of the position to give a graph showing the path of the rocket's orbit and the three different energies can be plotted against time to show how they vary throughout the orbit.

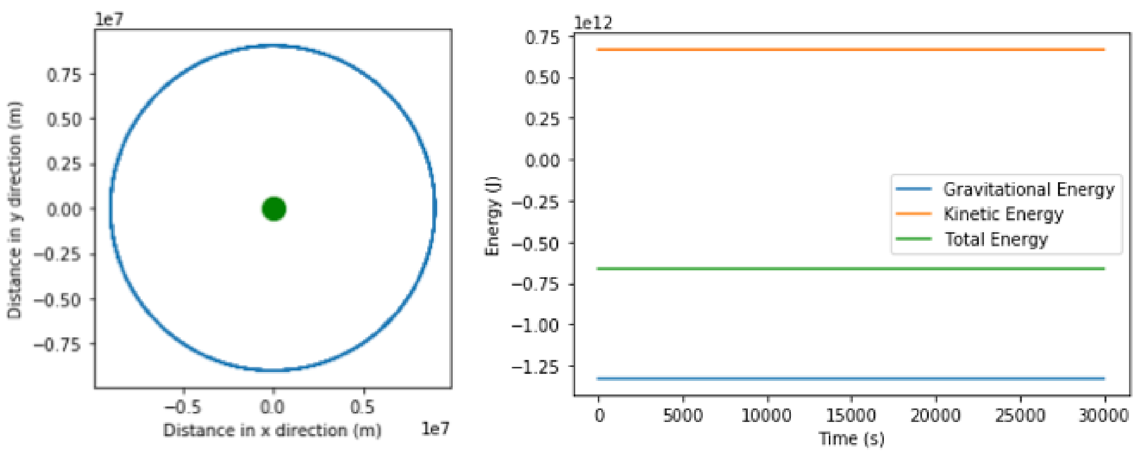

Figure 1: Graph of the position of the rocket as it orbits the Earth on the left and a graph of the total energy, kinetic energy and gravitational potential energy compared to time on the right, both for a circular orbit with a starting distance of $9000km$.

Figure 1 shows a circular orbit where the start velocity (v_s) is determined by the Vis Viva equation (as shown in equation 10).

$$v_s = \sqrt{\frac{GM}{x_s}} \tag{10}$$

where (x_s) is the start distance from the center of the Earth

If the start velocity is increased the orbit becomes elliptical.

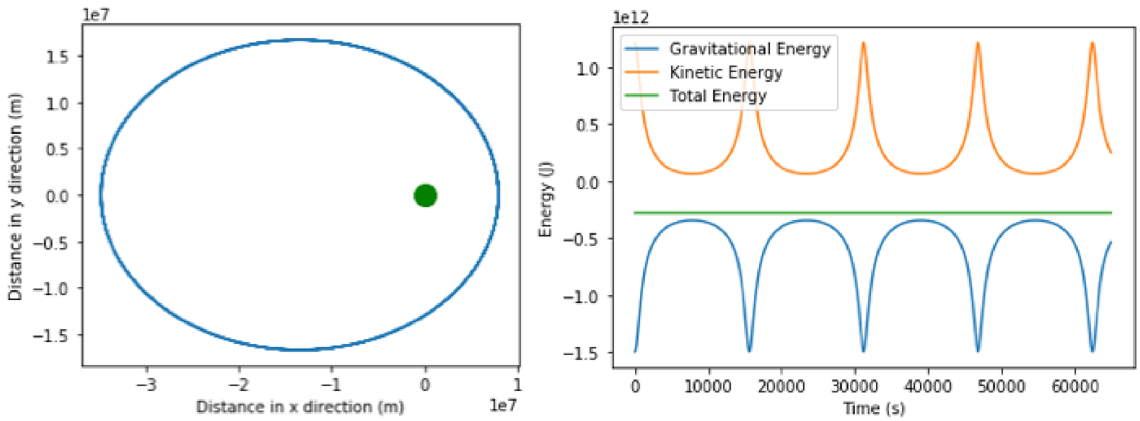

Figure 2: Graph of the position of the rocket as it orbits the Earth on the left and a graph of the total energy, kinetic energy and gravitational potential energy compared to time on the right, both for an elliptical orbit with a starting distance of $8000km$ and a starting velocity of $9000ms^{-1}$.

The time step used in the fourth order Runge-Kutta method doesn't have a noticeable effect on the position or energy at this large scale but on the small scale there are small deviations.

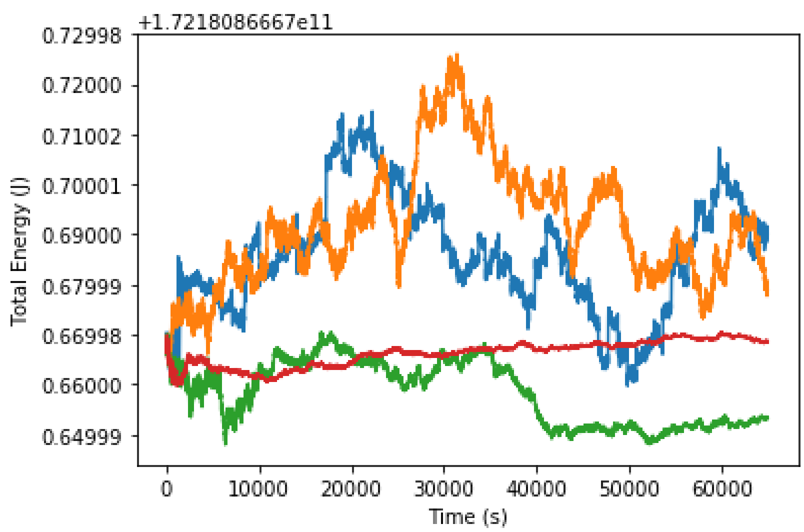

Figure 3: Graph comparing the total energy against time for various step sizes for an elliptical orbit with a starting distance of $9000km$ and a start velocity of $8000ms^{-1}$.

As is shown by figure 3, on the small scale comparison of the total energy against time, a different step size does have an effect on the position and velocity.

Part b): Rocket Orbiting the Moon before Returning Past Earth

The second part of the exercise aims to graphically show the motion of a rocket as it leaves the Earths atmosphere, passes close enough to the moon to take photographs and then returns back towards the Earth to send the information back.

In this part of the exercise the gravitational force is no longer just from the Earth, the Moon's gravitational field also has an effect on the rockets orbit.

The components of the velocity can no longer be determined using equations 6 and 7 so are now determined by differentiating equation 11.

$$m\ddot{r} = \frac{mM_E G}{|r - R_E|^3}(r - R_E) - \frac{mM_m G}{|r - R_M|^3}(r - R_M) \tag{11}$$

where $\ddot{r}$ is the acceleration, M_E is the mass of the Earth, R_E is the distance to the center of the earth (in this exercise the Earth is set to be at the origin so $R_E = 0$), M_m is the mass of the Moon and R_M is the distance to the centre of the moon.

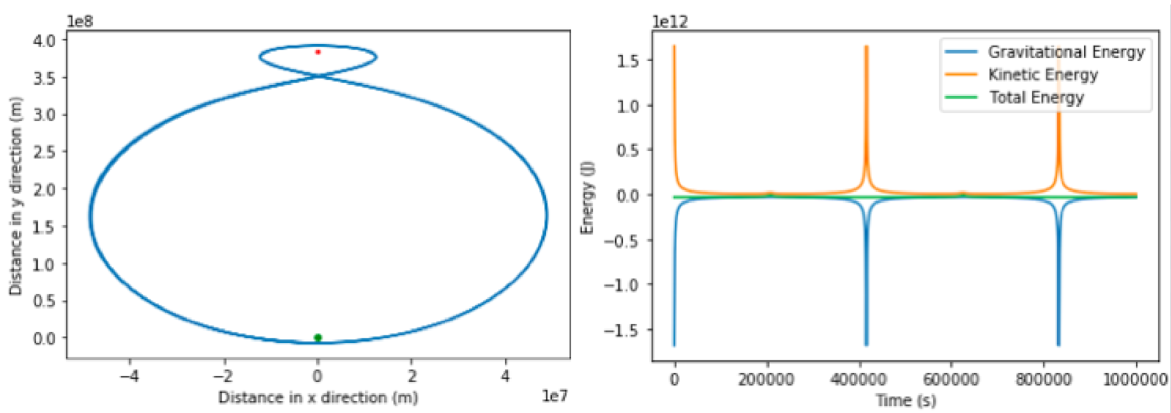

Figure 4: Graph to show the position of the rocket around the Earth and the Moon on the left and the kinetic energy, gravitational energy and total energy against time.

The start velocity used to get this orbit was $10.481500 ms^{-1}$ and the start position was $7100000m$. These starting conditions were obtained through trial and error.

A test should be put into place in the code to test whether the rocket crashes into either the Earth or the Moon. This can be done by setting a break function so that the for loop ends if the radius of the orbit becomes smaller than the radius of the Earth or the Moon.

The period of the orbit around the Earth and the Moon can be determined by setting up an if statement such that if the kinetic energy reaches a maximum then the time at that point will be printed. Due to the fact that the kinetic energy will be at its maximum at the start point and then again when the rocket reaches the same position again, the time taken between these points is equivalent to the period.

The period for the orbit calculated in this exercise was determined to be $409604s$ which is roughly equivalent to 4.7 days.

The closest distance of the rocket to the Moon can be calculated by finding the maximum y component of distance and subtracting the distance of the Moon from the Earth from this value. For the variables outlined above the closest distance the rocket got to the Moon was $7099851.82m$.

Discussion

As with any approximation for calculating differential equations, there are a range of quantization errors present in the calculation.

There will be a local truncation error in each increment due to the difference between the approximation and the numerical answer. This when added together creates a larger global truncation error in the entire approximation.

A rounding error is also present due to the summation of the differential terms and the difference due to the numerical answer due to rounding at each increment. One way of reducing the rounding error is to use the Kahan summation algorithm that adds a sequence of finite precision point numbers by keeping a running variable to account for small errors.

The Runge-Kutta approximation is one of the more accurate approximations for differential equations but can still be inproved upon. For instance, the variation on the Runge-Kutta approximation: the 3/8th rule could be used that has different coefficients to the original fourth-order Runge-Kutta method. This method reduces the error coefficients, causing a more accurate approximation.

In section b of the exercise, a huge approximation was made by stating that the Earth and Moon were stationary compared to each other when in fact that Moon is orbiting the Earth. This was not taken into account when the path of the rocket was calculated.

One way of achieving this would be to find the k values of the distance components using a function that takes into account to motion of the Moon around the Earth.