Computing Exercise 3: Using the Euler Method to Approximate Free Fall

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Introduction

The overall objective of this exercise is to estimate Felix Baumgartner's record breaking free fall from a Helium balloon.

The equation detailing the fall is a second order ordinary differential equation due to the fact that the acceleration is changing throughout the fall. This can be simplified by reducing it to two separate first order differential equations, one for the velocity and one for the position. These differential equations can be determined using the Euler method.

The Euler method is a way of estimating first order differential equations by summing up the velocity and height series equations:

$$v_{y,n+1} = v_{y,n} - \Delta t\left(g + \frac{k}{m}|v_{y,n}|v_{y,n}\right) \tag{1}$$

$$y_{n+1} = y_n + \Delta t.v_{y,n} \tag{2}$$

$$t_{n+1} = t_n + \Delta t \tag{3}$$

where v_y is the velocity, y is the height, t is the time, Δt is the increment of time taken in the series equations, g is the the acceleration due to gravity, m is the mass of the free falling body and k is a constant determined by:

$$k = \frac{C_d \rho_0 A}{2} \tag{4}$$

where C_d is the drag coefficient (in this exercise $C_d = 1.15$ was used for a sky diver), A is the cross sectional area (a value of $0.7m^2$ was used in this exercise) and ρ_0 is the air density (for which a density of $1.2kgms^{-1}$ was used for ambient pressure and temperature).

The Euler method works by calculating the value of the gradient at the start of each small step size between the points on a curve and then summing them.

Part a) The Euler Method

The first part of the exercise was to write a program to calculate the height and velocity as they changed with time using the Euler method.

This was achieved by creating separate empty arrays for time, height and velocity. Then equations 1 and 2 should be run, appending each new value of height, velocity and time to the relevant empty array before increasing the time by Δt to keep the series running.

A condition must be put on the height in order to stop the simulation from running for infinite time therefore a start height was set (initially set to be 1km) and a while loop was created such that when y became zero (the falling body reached the ground) the simulation ended.

The arrays were then used to create two graphs, one for the height against time and one for the velocity against time.

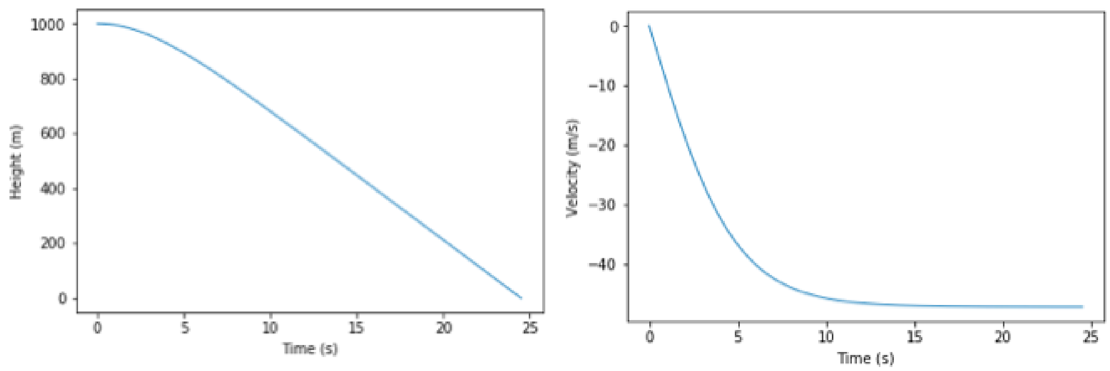

Figure 1: A graph showing how the height and velocity of the falling body changes with time for the Euler method with constant air density. A set Δt of 0.05 and a start height of 1km were used.

These graphs follow the general pattern of how the height and velocity should vary during a free fall, including how the velocity reaches a maximum (negative) terminal velocity and then remains at this value for the remainder of the simulation.

Part b) Analytical prediction

The second order differential equation for the free fall of a body can also be approximated using an analytical prediction:

$$y = y_0 - \frac{m}{2k} log_e \left[cosh^2 \left(\sqrt{\frac{kg}{m}}.t \right) \right] \tag{5}$$

$$v_y = -\sqrt{\frac{mg}{k}} tanh \left(\sqrt{\frac{kg}{m}}.t \right) \tag{6}$$

where y_0 is the starting height.

This can be achieved exactly the same as was done in part a but instead of using equations 1 and 2, equations 5 and 6 should be used.

The graphs created for height against time and velocity against time are almost identical to the graphs produced in part a, such that the difference between the two methods is very small relative to the scale.

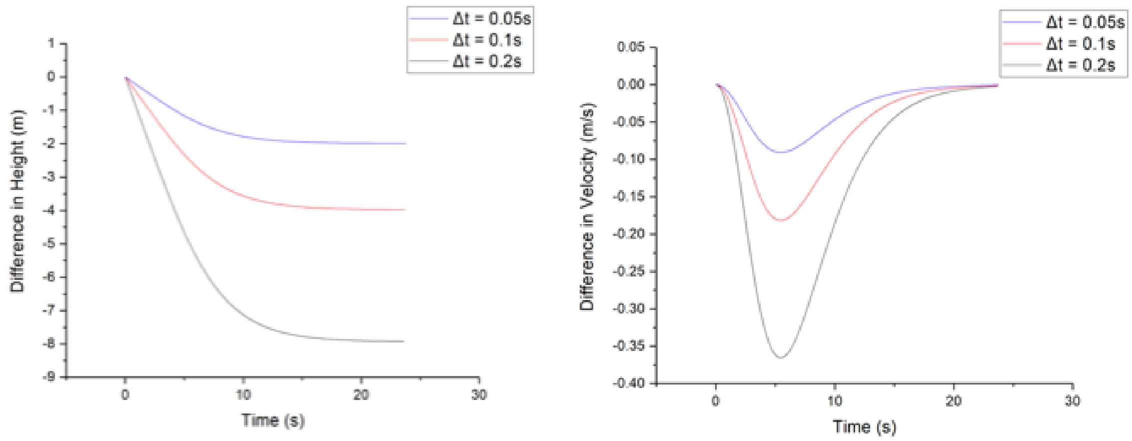

Figure 2: A graph showing how the difference between the two methods (Euler method and analytical method) of height or velocity changes with time for two different Δts. A set start height of 1km was used.

From this comparative graph, it can be seen that as you increase Δt, the difference between the Euler method and the analytical method gets bigger and more prominent compared to the scales of the axis.

At comparatively large values of Δt there are much fewer points on the Euler method plot, therefore the simulation begins to visibly vary from the analytical prediction.

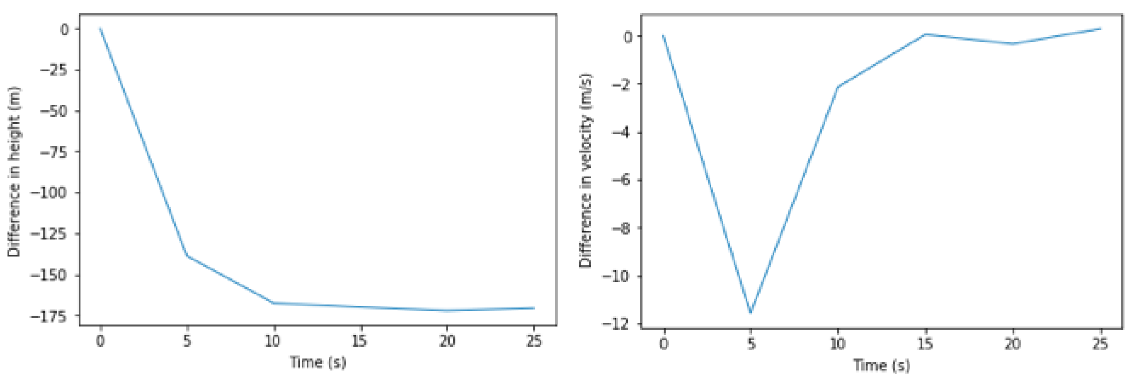

Figure 3: A graph showing how the difference between the two methods (Euler method and analytical method) of height or velocity changes with time for a Δt of 5. A set start height of 1km was used.

Part c) Modified Euler Method

Due to the fact that the Euler method calculates the gradient at the start of each time increment and the gradient of this time step is assumed to be linear its duration, the Euler method can very quickly vary from the actual curve.

One way to reduce this overshoot is to use the modified Euler method which calculates the gradient at the midpoint of the time increment as well as at the beginning.

This involves adding an extra velocity equation to incorporate the midpoint and then adding this to the velocity series equation.

The difference between the Euler method and the modified Euler method can be calculated for each point on the graph; then this can be plotted against time to show how the difference changes.

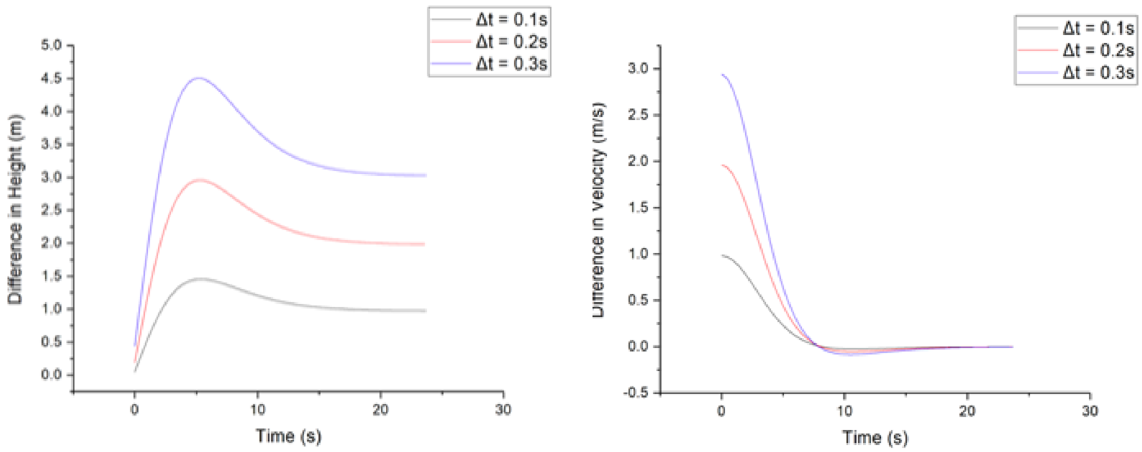

Figure 4: A graph showing how the difference between the two methods (Euler method and modified Euler method) of height or velocity changes with time for two different Δts. A set start height of 1km was used.

As shown in the figure, the larger the Δt, the larger the difference between the two methods is. This follows the theoretical understanding as the larger step size results in less points on the graph, reducing the accuracy of the estimation.

Part d) Estimating Baumgartner's Jump

Previously the starting height of the free fall has been set at 1km, meaning the air density was roughly the same for the entire fall, however Felix Baumgartner actually jumped from a height of 39045m. Therefore the air density is much lower at such a high altitude due to weaker gravity and less air molecules pressing down, meaning the air molecules can spread out more.

Therefore the air density must be adapted to be a function of the height.

$$\rho(y) = \rho_0 e^{-\frac{y}{h}} \tag{7}$$

where ρ_0 is the air density at ambient pressure and temperature and h is the scale height of the atmosphere (assumed to be 7.64km in this exercise).

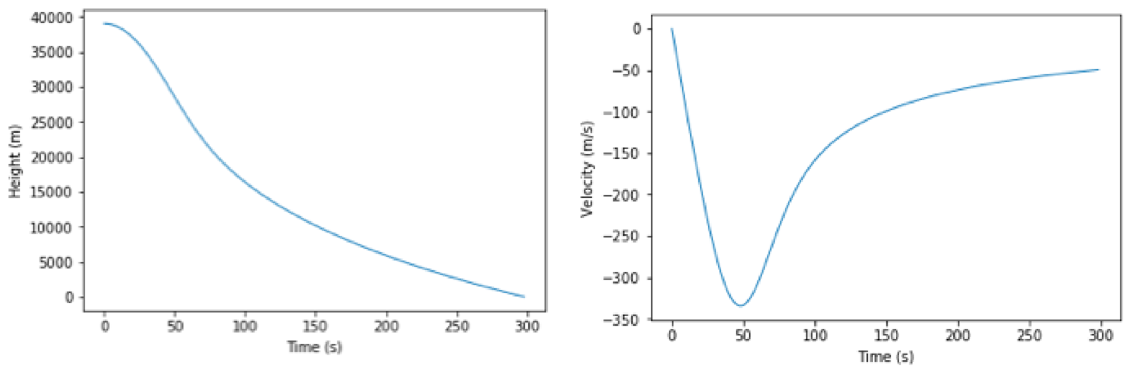

Figure 5: A graph showing how the height and velocity of the falling body changes with time for the modified Euler method with varying air density. A set Δt of 0.05 and a start height of 39045m were used.

The velocity graph shows an interesting shape of a maximum velocity before the velocity decreases at a much slower rate.

The maximum velocity is reached due to the balancing of the gravitational and air resistance forces, the same as when terminal velocity is reached for a normal free fall. This maximum velocity is however much greater than the terminal velocity reached during a fall in constant air density.

The velocity then decreases due to the fact that the air density increases as the height decreases therefore the drag force upwards constantly increases throughout the free fall. Ultimately this is because the terminal velocity reached is dependant on the altitude.

According to my simulation with the variables I estimated, the maximum speed reached was $333.999081152 ms^{-1}$. This is lower than the speed needed to break the sound barrier ($343ms^{-1}$) and is also lower that the maximum speed reached by Felix Baumgartner during his free fall (recorded to be $373ms^{-1}$). This is likely due to the fact that the Euler method is only an approximation and is not accurate enough to calculate the exact parameters of the free fall.

Part e) Varying the Parameters

Varying the start height of the free fall or the ratio $C_d A/m$ has an effect on the maximum velocity reached as shown in the following comparison graphs and tables:

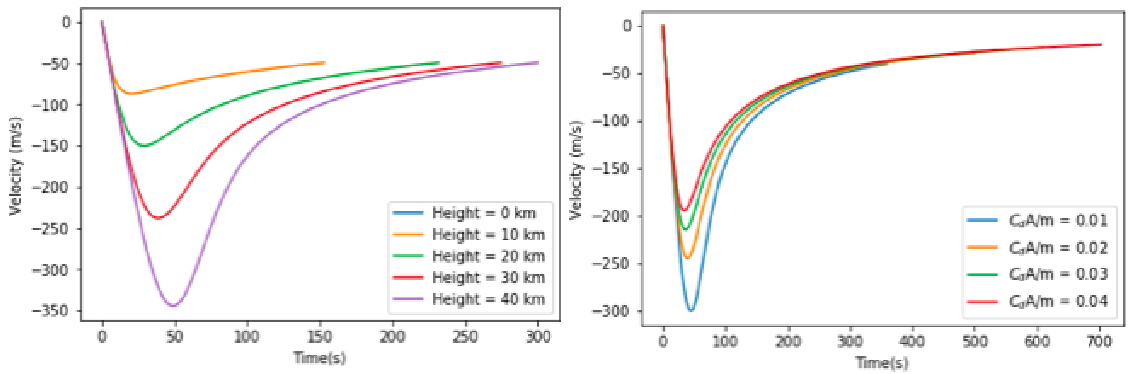

Figure 6: A graph showing how the velocity of the falling body changes with time for the modified Euler method with varying air density. On the left for various start heights and on the right for various values of C_d A/m. A set Δt of 0.05 was used.

Start Height (m)	Maximum Velocity (m/s)	C_dA/m	Maximum Velocity (m/s)
10	87.1997970641	0.01	300.576353016
20	150.309095689	0.02	245.343425864
30	238.387231522	0.03	215.233557977
40	344.639106825	0.04	195.067577609

TABLE 1: A table showing the how the maximum velocity varies for different start heights and different values of C_dA/m.

This shows that as the start height gets higher, a higher maximum velocity can be reached. This agrees with the theory because from a higher start height there is a lower initial air density, meaning larger initial terminal velocities can be reached.

Also outlined in the comparison graphs is that as the value for C_dA/m increases, the maximum velocity reached decreases. This is due to the fact that if there is more surface area to mass, the more air resistance has an effect on the body.

Discussion

There is a local truncation error in effect during the Euler method due to the error in each increment and can be determined as the difference between the solution after each increment. There is also a global truncation error occurring that is due to the overall combination of the local truncation errors.

Additionally a rounding error also affects the Euler method due to the fact that the program has to round each series solution once calculated so once the solutions are summed, the rounding error begins to make a difference. This can be reduced by using a Kahan summation algorithm to add a sequence of floating point numbers with finite precision.

An alternate to the Euler method is the backwards Euler method, in which the function is calculated at the end of each time increment instead of at the start by finding the implicit equation. This method is however more complicated that the Euler method.

The main improvement on the Euler method is the fourth order Runge-Kutta approach. This is very similar to the Euler and modified Euler methods, in that it calculates the differential in each increment but instead of just calculating it once for each increment, the gradient is calculated four times in each increment.

The solution found by the Runge-Kutta approach can be expressed as

$$y_{n+1} = y_n + \left(k_1 + 2k_2 + 2k_3 + k_4 \right) \tag{8}$$

where k_1, k_2, k_3 and k_4 are determined by:

$$k_1 = f(x_n, y_n) \tag{9}$$

$$k_2 = f\big(x_n + \frac{h}{2}, y_n + \frac{hk_1}{2}\big) \tag{10}$$

$$k_3 = f\big(x_n + \frac{h}{2}, y_n + \frac{hk_2}{2}\big) \tag{11}$$

$$k_4 = f\big(x_n + h, y_n + hk_3\big) \tag{12}$$

In the simulation for Baumgartner's jump, various approximations were made, such as the gravity not changing throughout the duration of the fall whereas in reality it would vary a tiny amount. The assumption was also made that the surface area of the body would not change whereas it could change if he moved his body to a different shape with a slightly different surface area.

These approximations would have a very small effect on the estimate of Baumgartner's free fall, therefore the most likely reason for the program returning slightly different results is the approximation of the Euler method.