N M L4

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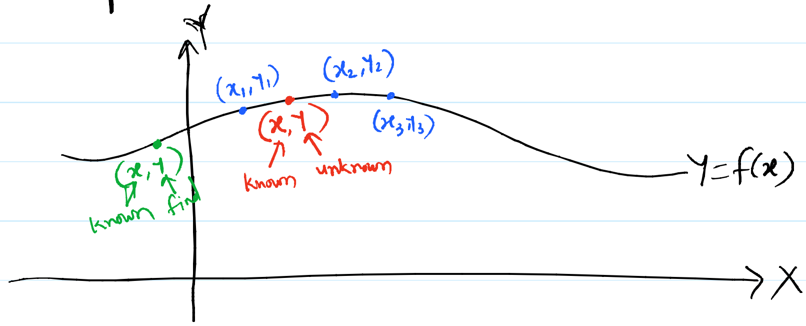
* Interpolation :



Evaluation of y at $x \notin (x_1, x_3)$ is called as Extrapolation

Evaluation of y at $x \in (x_1, x_3)$ is called as Interpolation.

I) Lagrange's Interpolation

$y=f(x)$ passes through the points.
(x_i, y_i) , $i = 1, 2, \cdots m$

$$y \text{ at } x = y(x) = y_1 L_1(x) + y_2 L_2(x) + y_3 L_3(x) + \cdots + y_m L_m(x)$$

where

$$L_1(x) = \frac{(x-x_2)(x-x_3)\cdots(x-x_m)}{(x_1-x_2)(x_1-x_3)\cdots(x_1-x_m)}$$

$$L_2(x) = \frac{(x-x_1)(x-x_3)\cdots(x-x_m)}{(x_2-x_1)(x_2-x_3)\cdots(x_2-x_m)}$$

& so on

$$L_i(x) = \frac{(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_m)}{(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_m)}$$

(x_i, y_i), $i = 1, 2, 3$

$$y(x) = y_1 L_1(x) + y_2 L_2(x) + y_3 L_3(x) \quad \text{——} (1)$$

$$L_1(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}$$

$$L_2(x) = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}$$

$$L_3(x) = (x-x_1)(x-x_2) / [(x_3-x_1)(x_3-x_2)]$$

$$y(x) = y_1 L_1 + y_2 L_2 + \cdots + y_m L_m = \sum_{i=1}^{m} y_i L_i$$

where
$$L_i(x) = L_i = \frac{\prod_{j=1}^{m} (x - x_j)}{\prod_{j=1}^{m} (x_i - x_j)} \qquad \text{if } i \neq j$$

* Examples

1) Find Lagrange's interpolating polynomial for

x	0	1	2
y	2	3	6

Also find y at $x = 1.5$.

→ Given $x_1 = 0$, $x_2 = 1$, $x_3 = 2$
$y_1 = 2$, $y_2 = 3$, $y_3 = 6$

∴ Using Lagrange's interpolⁿ formula

$$y(x) = y_1 L_1(x) + y_2 L_2(x) + y_3 L_3(x)$$

$$y(x) = 2L_1 + 3L_2 + 6L_3 \quad \text{——} \quad (1)$$

Now to evaluate L_1, L_2 & L_3

$$L_1(x) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} = \frac{(x-1)(x-2)}{(0-1)(0-2)} = \frac{x^2 - 3x + 2}{2}$$

$$L_2(x) = \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} = \frac{(x-0)(x-2)}{(1-0)(1-2)} = \frac{x^2 - 2x}{-1} = 2x - x^2$$

$$L_2(x) = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} = \frac{(x-0)(x-2)}{(1-0)(1-2)} = \frac{x^2-2x}{-1} = 2x-x^2$$

$$L_3(x) = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} = \frac{(x-0)(x-1)}{(2-0)(2-1)} = \frac{x^2-x}{2}$$

Substitute the values of $L_1(x)$, $L_2(x)$, $L_3(x)$ in (1)

$$Y(x) = \cancel{2}\left(\frac{x^2-3x+2}{\cancel{2}}\right) + 3\left(2x-x^2\right) + \cancel{6}^{3}\left(\frac{x^2-x}{\cancel{2}}\right)$$

$$= x^2-3x+2+6x-3x^2+3x^2-3x$$

$\therefore$ $\boxed{Y(x) = x^2+2}$ interpolating poly

$\therefore$ $y(1.5) = (1.5)^2+2$

$$\boxed{y(1.5) = 4.25}$$

2) Find the value of y at $x=0.4$ using Lagrange's interpolation formula for the points

x	0.1	0.3	0.6	0.8
y	0.72	1.81	2.73	3.47

→ Given $x_1 = 0.1$ $\quad x_2 = 0.3$ $\quad x_3 = 0.6$ $\quad x_4 = 0.8$

$y_1 = 0.72$ $\quad y_2 = 1.81$ $\quad y_3 = 2.73$ $\quad y_4 = 3.47$

$\therefore$ Using Lagrange's interpolation formula

$$Y(x) = y_1 L_1(x) + y_2 L_2(x) + y_3 L_3(x) + y_4 L_4(x)$$

$\therefore Y(0.4) = 0.72\, L_1(0.4) + 1.81\, L_2(0.4) + 2.73\, L_3(0.4) + 3.47\, L_4(0.4)$ —— ①

$$L_1(x) = \frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)}$$

$$L_1(0.4) = \frac{(0.4-0.3)(0.4-0.6)(0.4-0.8)}{(0.1-0.3)(0.1-0.6)(0.1-0.8)} = -0.1143 \text{ —— ②}$$

$$L_2(x) = \frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)}, \quad L_2(0.4) = 0.8 \text{ —— ③ check}$$

Similarly

$L_3(0.4) = 0.4$ —— ④ check

$L_4(0.4) = -0.0857$ —— ⑤ check

$\therefore$ from eqn ①, ②, ③, ④, ⑤

$$\boxed{Y(0.4) = 2.1603}$$

Q. Find Lagrange's interpolating polynomial passing through the set of points $(0,2)$, $(2,-2)$ and $(3,-1)$. Use it to find $\frac{dy}{dx}$ at $x = 2$.

2) Find y at $x = 1.07$ using Lagrange's interpolation formula for the points

x	1	1.2	1.3	1.5
y	1	1.728	2.197	3.375