$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$Ax = \lambda x$$

$$\Rightarrow Ax - \lambda I x = 0 \Rightarrow (A - \lambda I)x = 0$$

$$\det(A - \lambda I) = 0$$

$$\left| \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0 \Rightarrow \left| \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \left| \begin{bmatrix} 2-\lambda & 1-0 \\ 1-0 & 2-\lambda \end{bmatrix} \right| = 0 \Rightarrow \left| \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow (2-\lambda)^2 - 1 = 0 \Rightarrow \lambda^2 - 4\lambda + 3 = 0$$

$$\Rightarrow 4 - 4\lambda + \lambda^2 - 1 = 0$$

$$\Rightarrow \lambda^2 - (3+1)\lambda + 3 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda - \lambda + 3 = 0$$

$$\Rightarrow \lambda(\lambda - 3) - 1(\lambda - 3) = 0$$

$$\Rightarrow (\lambda - 1)(\lambda - 3) = 0 \Rightarrow \boxed{\begin{matrix} \lambda = 1 \\ \lambda = 3 \end{matrix}}$$

$\underline{\underline{\lambda = 1}}$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2x_1 + x_2 \\ x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow 2x_1 + x_2 = x_1 \Rightarrow x_1 = -x_2$$

$$\Rightarrow x_1 + 2x_2 = x_2 \Rightarrow x_1 = -x_2$$

$$x_2 = t \Rightarrow x_1 = -t \Rightarrow \begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad (t \neq 0)$$

$\underline{\underline{\lambda = 3}}$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2x_1 + x_2 \\ x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix}$$

$$\Rightarrow 2x_1 + x_2 = 3x_1 \Rightarrow -x_1 = -x_2 \Rightarrow x_1 = x_2$$

$$\Rightarrow x_1 + 2x_2 = 3x_2 \Rightarrow x_1 = x_2$$

$$x_2 = s \Rightarrow x_1 = s \Rightarrow \begin{bmatrix} s \\ s \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (s \neq 0)$$

$\therefore$ eigen values = 1, 3

eigen vectors = $\begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow$ Orthogonal

$A x_1 = \lambda x_1$ $\qquad$ $A x_2 = \lambda x_2$

$A \frac{x_1}{|x_1|} = \lambda \frac{x_1}{|x_1|}$ $\qquad$ $A \frac{x_2}{|x_2|} = \lambda \frac{x_2}{|x_2|}$

$\Rightarrow \boxed{A \hat{x}_1 = \lambda \hat{x}_1}$ $\qquad$ $\boxed{A \hat{x}_2 = \lambda \hat{x}_2}$

$$\left\{ \begin{matrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \\ \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \end{matrix} \right\} \Rightarrow \text{(Orthonormal)}$$

Let A be a $n \times n$ symmetric matrix

Consider $\lambda_1, \lambda_2, \dots \lambda_n$ are eigenvalues of A.

& correspondingly $v_1, v_2, \dots v_n$ " eigenvectors of A.

(Orthonormal)

$$A = S \Lambda S^{-1} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \lambda_2 & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \cdots & v_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}^{-1}$$

I_p

$\Rightarrow A = P D P^T$ (P = Orthogonal matrix)

$P^{-1} = P^T$

$\Rightarrow \boxed{A = P D P^T}$

$$A = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} v_1 \rightarrow \\ v_2 \rightarrow \\ \vdots \\ v_n \rightarrow \end{bmatrix}$$

$$A = \lambda_1 v_1 v_1^T + v_2 \lambda_2 v_2^T + \dots + \lambda_n v_n v_n^T$$

$$\Rightarrow \boxed{A = \sum_{i=1}^{n} \lambda_i v_i v_i^T}$$

For example

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \lambda = 1,\ x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow x_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\lambda = 3,\ x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow x_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$A = PDP^T$$

$$= \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T$$

$$\Rightarrow A = 1 \cdot \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}_{2\times 1} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}_{1\times 2} + 3 \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} + 3 \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 3/2 & 3/2 \\ 3/2 & 3/2 \end{bmatrix} + \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$