A Generalized Online Mirror Descent with Applications to Regression

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Definitions

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  $\forall u, v : f(v) \geq f(u) + \langle \nabla f(u), v - u \rangle + \frac{\alpha}{2}\|u - v\|^2$
- A function is β-smooth for a norm $\|\cdot\|$ if
  $\forall u, v : f(v) \leq f(u) + \langle \nabla f(u), v - u \rangle + \frac{\beta}{2}\|u - v\|^2$

Fenchel Conjugate

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- A vector x is a subgradient of f at v if $\forall u : f(u) - f(v) \geq \langle u - v, x\rangle$, the set of such x is denoted as $\partial f(v)$
- The Fenchel-Young inequality states that if $x \in \partial f(v)$ then $f(v) + f^*(x) = \langle v, x\rangle$

Fenchel Conjugate Properties

- The Fenchel conjugate f^* of an α-strongly convex function f is everywhere differentiable and $\frac{1}{\alpha}$-strongly smooth. This means that, for all $u, v \in X$,

$$f^*(v) \leq f^*(u) + \langle \nabla f^*(u), v - u \rangle + \frac{1}{2\alpha} \|u - v\|^2$$

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- $\nabla f^*(u) = \arg\max_{v \in S} \langle v, u \rangle - f(v)$

Online Convex Optimization

In the online convex optimization protocol, an algorithm sequentially chooses elements from a convex set $S \in X$, each time incurring a certain loss. At each step $t = 1, 2, \ldots$ the algorithm chooses $w_t \in S$ and then observes a convex loss function $l_t : S \Rightarrow R$. The value $l_t(w_t)$ is the loss of the learner at step t, and the goal is to control the regret.

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Regret here is defined as:

$$R_T(u) = \sum_{t=1}^{T} l_t(w_t) - \sum_{t=1}^{T} l_t(u)$$

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So we try to fine tune our algorithm to select w_t which minimises the Regret over all u.

Online Mirror Descent

The standard Online Mirror Descent looks like:

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Algorithm OMD

Input: parameter $\eta > 0$, regularization function $R(x)$.

Let y_1 be such that $\nabla R(y_1) = 0$ and $x_1 = \text{argmin } B_R(x \| y_1)$.

for $t = 1$ to T **do**

  Play x_t.

  Observe the loss function f_t and let $\nabla_t = \nabla f_t(x_t)$.

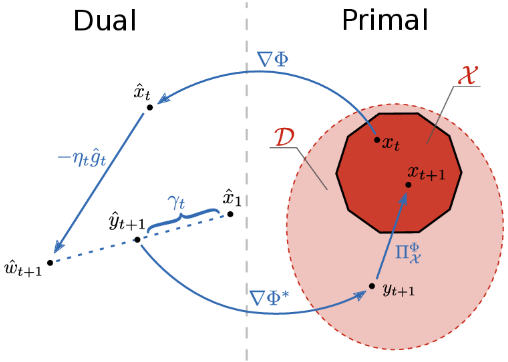
  Update y_{t+1} according to the rule:

Lazy: $\nabla R(y_{t+1}) = \nabla R(y_t) - \eta \nabla_t$

Agile: $\nabla R(y_{t+1}) = \nabla R(x_t) - \eta \nabla_t$

  Project: $x_{t+1} = \text{argmin } B_R(x \| y_{t+1})$

Online Mirror Descent



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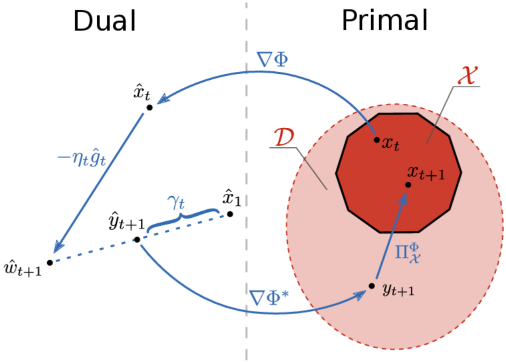

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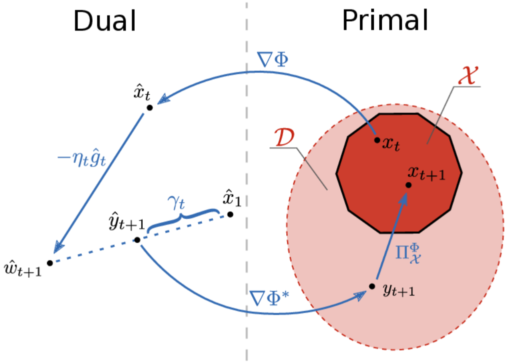

- In OMD the gradient update is carried out in a "dual" space on the choice of the regularization.
- This transformation enables better bounds depending on what space we begin with.

Generalized Online Mirror Descent

The OMD algorithm is generalized into:

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Algorithm General Online Mirror Descent

1: **Parameters:** A sequence of strongly convex functions $f_1, f_2, \ldots$ defined on a common convex domain $S \subseteq X$.
2: **Initialize:** $\theta_1 = 0 \in X$
3: **for** $t = 1, 2, \ldots$ **do**
4: Choose $w_t = \nabla f_t^*(\theta_t)$
5: Observe $z_t \in X$
6: Update $\theta_{t+1} = \theta_t + z_t$

Identities for GOMD

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And each f_t is α_t-strongly convex with respect to the norm $\|\cdot\|_t$.
Let $\|\cdot\|_t^*$ be the dual norm of $\|\cdot\|_t$, for $t = 1, 2, \ldots, T$. Then, for any $u \in S$

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Assume OMD is run with functions $f_1, f_2, \ldots, f_T$ defined on a common convex domain $S \subseteq X$
And each f_t is α_t-strongly convex with respect to the norm $\|\cdot\|_t$. Let $\|\cdot\|_t^*$ be the dual norm of $\|\cdot\|_t$, for $t = 1, 2, \ldots, T$. Then, for any $u \in S$

$$\sum_{t=1}^{T} \langle z_t, u - w_t \rangle \leq f_T(u) + \sum_{t=1}^{T} \frac{\|z_t\|_{t,*}^2}{2\alpha_t} + f_t^*(\theta_t) - f_{t-1}^*(\theta_t)$$

Where we set $f_0^*(0) = 0$.

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Where we set $f_0^*(0) = 0$. Moreover, for all $t \geq 1$, we have

$$f_t^*(\theta_t) - f_{t-1}^*(\theta_t) \leq f_{t-1}(w_t) - f_t(w_t)$$

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$$= f_t^*(\theta_t) - f_{t-1}^*(\theta_t) + \langle w_t, z_t \rangle + \frac{1}{2\alpha_t} \|z_t\|_{t,*}^2.$$

Proof for Lemma 1

The Fenchel-Young inequality implies

$$\sum_{t=1}^{T} \Delta_t = f_T^*(\theta_{T+1}) \geq \langle u, \theta_{T+1} \rangle - f_T(u) = \sum_{t=1}^{T} \langle u, z_t \rangle - f_T(u).$$

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Summing the last 2 inequalities, we get:

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$$\sum_{t=1}^{T} \langle u, z_t \rangle - f_T(u) \leq \sum_{t=1}^{T} \Delta_t$$

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$$\sum_{t=1}^{T}\langle u, z_t\rangle - f_T(u) \leq \sum_{t=1}^{T}\left(f_t^*(\theta_t) - f_{t-1}^*(\theta_t) + \langle w_t, z_t\rangle + \frac{1}{2\beta_t}\|z_t\|_t^2\right).$$

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So combining the two sum inequalities, we get:

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Rearranging this we get the proof for Lemma 1

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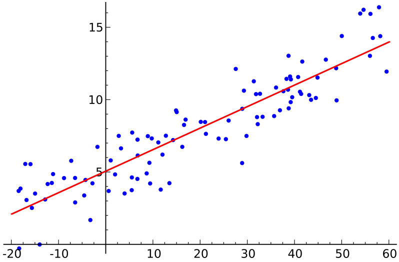
Combining the two, we get
$f_t^*(\theta_t) - f_{t-1}^*(\theta_t) \leq f_{t-1}(w_t) - f_t(w_t)$, as desired.

Linear Regression



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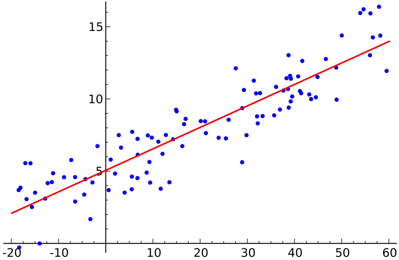

- Linear regression is a method that utilizes a linear framework to model the predictive association between a single response variable and one or more explanatory variables.
- We are given pairs of x_t, y_t where $y_t = u^\top x_t + \nu_t$ where ν_t is a random noise and our goal is to recover u.

Online Regression

At time step $t = 1, 2, .., T$, we recieve (x_t, y_t).

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Here $x_t \in \mathbb{R}^d$ and $y_t \in \mathbb{R}$

Vovk-Azoury-Warmuth Algorithm for Online Regression

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$$= \arg\min_w \left(\tfrac{a}{2}\|w\|^2 + \tfrac{1}{2}\textstyle\sum_{s=1}^{t-1}(w^\top x_s)^2 - \sum_{s=1}^{t-1} y_{s'} w^\top x_s \right)$$

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$$= \arg\min_w \left(\tfrac{1}{2} w^\top \left(aI + \textstyle\sum_{i=1}^{t-1} x_s x_s^\top \right) w - \sum_{s=1}^{t-1} y_s w^\top x_s \right)$$

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$$= \left(aI + \textstyle\sum_{s=1}^{t-1} x_s x_s^\top \right)^{-1} \sum_{i=1}^{t-1} y_i \cdot x_i$$

Fitting VAW to GOMD

- Now, by letting $A_0 = aI$, $A_t = A_{t-1} + x_t x_t^\top$ for $t \geq 1$, and $z_s = y_s \cdot x_s$, we obtain the OMD update $w_t = A_t^{-1}\theta_t = \nabla f_t^*(\theta_t)$

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Hence, the regret $R_T(u)$ is controlled as follows:

Regret for GOMD-VAW

$$R_T(u) = \frac{1}{2}\sum_{t=1}^{T}(y_t - w_t^\top x_t)^2 - \frac{1}{2}\sum_{t=1}^{T}(y_t - u^\top x_t)^2$$

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$$= \sum_{t=1}^{T}(y_t u^\top x_t - y_t w_t^\top x_t) - f_T(u) + \frac{a}{2}\|u\|_2^2 + \frac{1}{2}\sum_{t=1}^{T}(w_t^\top x_t)^2$$

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$$\leq f_T(u)+\sum_{t=1}^{T} y_t^2\|x_t\|_{2t,*}^2+f_t^*(\theta_t)-f_{t-1}^*(\theta_t)-f_T(u)+\frac{a}{2}\|u\|_2^2+\frac{1}{2}\sum_{t=1}^{T}(w_t^\top x_t)^2$$

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$$\leq f_T(u) + \sum_{t=1}^{T} y_t^2 \|x_t\|_{2t,*}^2 + f_t^*(\theta_t) - f_{t-1}^*(\theta_t) - f_T(u) + \frac{a}{2}\|u\|_2^2 + \frac{1}{2}\sum_{t=1}^{T}(w_t^\top x_t)^2$$

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Regret for GOMD-VAW

$$R_T(u) = \frac{1}{2}\sum_{t=1}^{T}(y_t - w_t^\top x_t)^2 - \frac{1}{2}\sum_{t=1}^{T}(y_t - u^\top x_t)^2$$

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$$\text{since } f_t^*(\theta_t) - f_{t-1}^*(\theta_t) \leq f_{t-1}(w_t) - f_t(w_t) = -\frac{1}{2}(w_t^\top x_t)^2$$

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Where we used the X_t^2-strong convexity of f_t and the fact that $f_t \geq f_{t-1}$.

AF Regret Bounding

Combining previous inequalities and the bound for $R_T(u)$, we get:

$$R_T^{AF}(u) \leq 2X_T^2 f(u) + \sum_{t=1}^{T} (y_t - u^\top x_t)^2$$

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- They further expanded upon this to work for Classification models using GOMD.

Thank You!