Single-Minded Case in Combinatorial Auctions

Akhoury Shauryam

Algorithmic Game Theory

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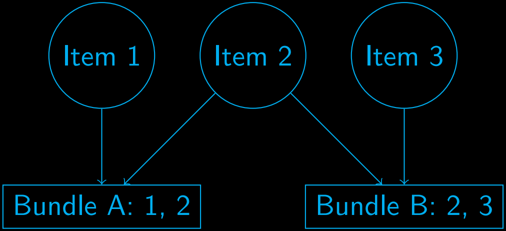
  - Strategic Behavior: Mechanisms must incentivize truthfulness.

Diagram



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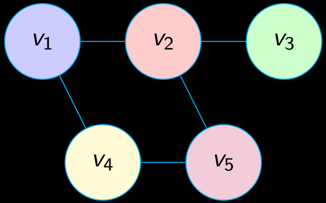

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- Size of the independent set:

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- Winning bidders pay the infimum v' needed to win, ensuring:

$$u = v^* - p \quad \text{(utility is maximized when bidding truthfully).}$$

- Misreporting risks:
  - Overreporting v^*: No effect on payment.
  - Underreporting v^*: Risk of losing allocation.

Continued

Sufficiency: Monotonicity and critical payment ensure truthful reporting is a dominant strategy.

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Necessity: Without these conditions, bidders can improve utility by misreporting.

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Output: An allocation W that approximates the optimal social welfare.

Approximation Guarantee: Setup and Bounding Contributions

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Goal: Prove.

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- For each $i \in W$, define its conflict-set OPT_i as:

$$OPT_i = \{j \in OPT \mid S_j^* \cap S_i^* \neq \emptyset\}.$$

Continued

Bounding the Contribution of OPT:

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- The total value of *OPT* can be rewritten as:

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Continued

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- Each $j \in OPT_i$ was not selected by the greedy algorithm, so:

$$v_j^* \leq v_i^* \cdot \frac{\sqrt{|S_j^*|}}{\sqrt{|S_i^*|}}.$$

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Applying Cauchy-Schwarz Inequality:

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Bounding the Contribution of OPT_i:

- Substituting this bound into the earlier inequality:

$$\sum_{j \in OPT_i} v_j^* \leq v_i^* \cdot \frac{\sqrt{|S_i^*|} \cdot \sqrt{m}}{\sqrt{|S_i^*|}} = v_i^* \cdot \sqrt{m}.$$

Continued

Final Bound for OPT:

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- Summing over all $i \in W$, the total contribution of OPT:

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$$\sum_{j \in OPT} v_j^* \leq \sqrt{m} \cdot \sum_{i \in W} v_i^*.$$

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- Therefore:

$$\frac{\text{Optimal Welfare}}{\text{Greedy Welfare}} \leq \sqrt{m}.$$

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- Greedy mechanism offers a $\sqrt{m}$-approximation with efficient computation.
- Future work could include exploring tighter approximation bounds and scalable mechanisms.

Thank You

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Questions are welcome!