NLA EXAM 1

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Part I: Fill in the Blank

1. Let $\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & 0 & -4 \\ -2 & 1 & 4 \end{bmatrix}$

   Then $\|\mathbf{A}\|_F =$ ____________________

   and $\|\mathbf{A}\|_1 =$ ____________________

2. Let $\mathbf{A} = \mathbf{QR}$ be the full QR factorization of $\mathbf{A}$, where $\mathbf{A}$ is $m \times n$, $m \geq n$ and is full rank. Then the first n columns of $\mathbf{Q}$ span ______________________________. The last $m - n$ columnes of $\mathbf{Q}$ are in the ______________________________ of $\mathbf{A}^T$

3. Suppose $\mathbf{P}$ is an orthonormal proctor onto a subspace of $\mathbb{C}^m$ of dimension $k < m$. What is $\|P\|_F$ ______________________________.

   What is the dimension of the space onto which $(\mathbf{I} - \mathbf{P})$ projects? ________________________

4. Give a (non-trivial) example of 2×2 matrices $\mathbf{A}, \mathbf{B}$ such that $\|\mathbf{A} + \mathbf{B}\|_\infty = \|\mathbf{A}\|_\infty + \|\mathbf{B}\|_\infty$. Your $\mathbf{A}$ and $\mathbf{B}$ must have all nonzero entries.

   ____

5. Assume the largest two singular values of $\mathbf{A}$ are 10 and 3. Let $\mathbf{x} = -2\mathbf{v_1} + 3\mathbf{v_2}$ where $\mathbf{v_1}, \mathbf{v_2}$ are the first two right singular vectors. What is $\|\mathbf{Ax}\|_2^2$ (Note the square on the norm term). ______________________________

6. Let $\mathbf{A} = \mathbf{QR}$ be the full QR for $m \times n$ matrix $\mathbf{A}$, and $\mathbf{b} = \mathbf{Ax}$ for where $\|\mathbf{x}\|_2 = 5$. Suppose $\|\mathbf{R}\|_2 \leq 10$. Give an upper bound on $\|\mathbf{b}\|_2$ $\mathbf{A}$. ____________________

7. True or False: If $\mathbf{A}$ is Hermitian, the eigenvalues of $\mathbf{A}$ are equal to the singular values of $\mathbf{A}$. _______________

8. True or False: The full QR factorization of an $m \times n$ matrix $\mathbf{A}$ always exists, even if the matrix is rank deficient. ____________________

9. True or False: If the singular values of the matrix $\mathbf{A}$ are distinct and non-zero, then the SVD is unique. ____________________

Part II: Short Answer

1. Suppose $\text{col}(A) = \text{span}\left\{ \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \right\}$

   Find an orthonormal basis for the column space of $\mathbf{A}$.

2. Let $W = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Does $\|\mathbf{x}\| := \sqrt{\mathbf{x}^T W^T W \mathbf{x}}$ define a norm on $\mathbb{R}^2$? If yes, prove it. If no, explain why not.

3. Give a formula for the inverse of $\mathbf{I} - \mathbf{u}\mathbf{e}_k^T$, where $\mathbf{e}_k^T$ is the standard unit vector (i.e kth column of the identity matrix) and $\mathbf{u}$ has non-zeros *only* on the rows $k+1$ to n. Here $k < n$. Verify your formula is indeed an inverse.

4. Let $\mathbf{q_1}, \mathbf{q_2}$, be an orthonormal basis for a subspace $\mathcal{S}$ in $\mathbb{C}^m$. Let $\mathbf{P}$ be the orthogonal projector onto $\mathcal{S}$, and let $\mathbf{P}^\perp$ denotre the orthogonal projector onto the complementary space $\mathcal{S}^\perp$. Answer the following.

   (a) Give a formula for $\mathbf{P}$

   (b) Give a formula for an orthogonal projector (call it $\mathbf{P_1}$) onto span$\{\mathbf{q_1}\}$ and find a formula for an orthogonal projector onto span$\{\mathbf{q_2}\}$ (call it $\mathbf{P_2}$)

   (c) Prove that $\mathbf{P}^\perp = (\mathbf{I} - \mathbf{P_1})(\mathbf{I} - \mathbf{P_2})$

5. Let $\mathbf{P}$ be an orthogonal projector. Show that $(\mathbf{I} - 2\mathbf{P})$ is a unitary matrix.

6. Let $\mathbf{P}$ be an orthogonal projector. Show, *using the definition of the 2-norm as an induced matrix norm*, that $\|\mathbf{P}\|_2 = 1$

7. Let $\mathbf{A}$ be a block diagonal matrix $\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 \end{bmatrix}$ where $\mathbf{A}_i$ are each $n \times n$ invertible. Assume $\mathbf{A} = \mathbf{U_i S_i V_i^*}$, $i = 1, 2$ be the SVD's of the $\mathbf{A_i}$.

   Find the SVD of $\mathbf{A}$.

8. Let $\mathbf{A} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{\sqrt{2}} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and use it to answer the following (show/verify all work). You do NOT need to compute A

   (a) Verify that the given factorization for $\mathbf{A}$ is the SVD of $\mathbf{A}$

   (b) Find $\|\mathbf{A}\|_2$.

   (c) Find $\|\mathbf{A}\|_F$.

   (d) Find the **economy size** SVD of $\mathbf{A}$.

   (e) Find an orthogonal projection matrix $\mathbf{P}$, onto $col(\mathbf{A})$.

   (f) Find an orthogonal projection matrix $\mathbf{W}$, onto $null(\mathbf{A})$.

   (g) Find the best rank-1 matrix approximation, $\mathbf{A_1}$ of $\mathbf{A}$ in the matrix 2-norm.

   (h) Give the error $\|\mathbf{A} - \mathbf{A_1}\|_2$

9. You have a square matrix $\mathbf{A}$, and are given $\mathbf{A} = \mathbf{QR}$, the QR factorization of $\mathbf{A}$. But you ned to solve $\mathbf{A}^*\mathbf{z} = \mathbf{c}$ for $\mathbf{z}$. Explain how to do this without explicitly computing the inverse of any matrices.

NLA EXAM 2

1. Suppose $\mathbf{A}$ is $m \times m$ invertible. Give the total number of floating point operations (big-Oh or asymptotic is fine) for computing the solution to $\mathbf{Ax} = \mathbf{b}$, including the time to reduce $\mathbf{A}$ to upper triangular form using Householder reflectors. (Break down the cost per step).

2. On homework for the case $m = 2$, we gained intuition on the Kahan-Gastinel theorem, which says for an $m \times m$ invertible matrix $\mathbf{A}$, $\frac{1}{\text{cond}(\mathbf{A})}$ gives the distance from the set of all singular matrices; thus, the larger the condition number, the smaller the distance to singularity. (Here, $\text{cond}(\mathbf{A})$ is with respect to any of the p-norms.) But we also know that if A is not invertible, $\det(\mathbf{A}) = 0$. So is it true that a small determinant gives information on the distance of $\mathbf{A}$ from singularity? Let's check.

   (a) Let $\mathbf{A}$ be any invertible matrix, and let c be a positive number. Show $\text{cond}(\mathbf{A}) = \text{cond}(c\mathbf{A})$.

   (b) Give the relationship between $\det(\mathbf{A})$ and $\det(c\mathbf{A})$.

   (c) What do you conclude about the use of the determinant to gauge the distance of a matrix from singularity, and why?

3. Give the formula for the relative condition number associated with evaluating the function $f(x) = e^x$ for $x > 1$. What is happening to the condition number as a function of x, and, knowing this measures relative sensitivity of output to small relative changes in input, why does this make sense, given the graph of the function?

4. A base-10, normalized floating point number system has 2 digits of precision, the exponent range is $-3 \leq e \leq 3$. Give the upper bound on the relative distance between any pair of adjacent (on the number line) floating point numbers (excluding 0).

5. Let $\mathbf{A} = \mathbf{U} \begin{bmatrix} 1 & 0 \\ 0 & 10^{-3} \end{bmatrix} \mathbf{V}^*$ with $\mathbf{U}, \mathbf{V}$ orthogonal. The columns of $\mathbf{U}$ are $\mathbf{u_1}, \mathbf{u_2}$. Consider the two systems $\mathbf{Ax} = \mathbf{b}, \mathbf{A\tilde{x}} = \tilde{\mathbf{b}}$, where $\mathbf{b} = 5\mathbf{u_1}$, and $\delta\mathbf{b} := \mathbf{b} - \tilde{\mathbf{b}} = 10^{-2}\mathbf{u_2}$. Show that

$$\frac{\|\mathbf{x} - \tilde{\mathbf{x}}\|_2}{\|\mathbf{x}\|_2} = 2$$

6. Let c, x_1, x_2 be real numbers, assume none are 0 . Show that the algorithm to compute the scalar-vector product $c \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ on a computer using normalized floating point arithmetic and satisfying the Fundamental Theorem of Floating Point Arithmetic is backward stable.

7. Let $\mathbf{A} = \begin{bmatrix} 8 & 3 & 4 \\ 3 & 5 & -2 \\ 4 & -2 & -1 \end{bmatrix}$ (note $\mathbf{A}$ is symmetric). Making use of Householder reflectors, find an orthogonal matrix Q so that $\mathbf{T} := \mathbf{QAQ}^T$ is tridiagonal (i.e. $T_{3,1} = 0 = T_{1,3}$).

8. Let $a < b$ be real numbers (but not necessarily positive). We can compute the midpoint of the interval $[a, b]$ using either

   (a) $\frac{a+b}{2}$ or

   (b) $a + \frac{b-a}{2}$.

   Give at least 2 scenarios in which, under floating point arithmetic, one formula might be preferred to the other.

9. Let $\hat{\mathbf{x}}$ be the solution to the linear least squares problem (i.e. $\hat{\mathbf{x}}$ minimizes $\|\mathbf{b} - \mathbf{A}\mathbf{x}\|_2$ over all vectors in $\mathbb{R}^2$) where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

   Let the residual vector be $\mathbf{r} = \mathbf{b} - \mathbf{A}\hat{\mathbf{x}}$. Without computing $\hat{\mathbf{x}}$, which of the following three vectors is a possible value for the residual vector, and why?

$$\text{(a)} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \text{(b)} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \quad \text{(c)} \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

10. Let $\mathbf{q}_1 = \begin{bmatrix} .5 \\ .5 \\ 0 \\ .5 \\ .5 \\ 0 \end{bmatrix}$, $\mathbf{q}_2 = \begin{bmatrix} .5 \\ -.5 \\ .5 \\ 0 \\ 0 \\ .5 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$. Compute the orthogonal projection, $\hat{\mathbf{b}}$, of $\mathbf{b}$ onto span $\{\mathbf{q}_1, \mathbf{q}_2\}$. Then, compute vector $\mathbf{w}$ so that $\mathbf{b} = \hat{\mathbf{b}} + \mathbf{w}$ and verify $\mathbf{w}$ is orthogonal to $\hat{\mathbf{b}}$.

11. Let $\mathbf{Q}_1 = [\mathbf{q}_1, \mathbf{q}_2]$ be the 6×2 matrix formed from the $\mathbf{q}_1, \mathbf{q}_2$ in the previous problem. Define $\mathbf{R}_1 = \begin{bmatrix} -4 & 2 \\ 0 & 1 \end{bmatrix}$. Define $\mathbf{A} = \mathbf{Q}_1\mathbf{R}_1$, so that $\mathbf{Q}_1\mathbf{R}_1$ is the economy QR factorization of $\mathbf{A}$. Use this fact to find the least squares solution $\hat{\mathbf{x}}$ that minimizes $\|\mathbf{b} - \mathbf{A}\mathbf{x}\|_2^2$, where $\mathbf{b}$ is as defined in the previous problem.

12. Now suppose $\mathbf{q}_3$ has unit length and is orthogonal to $\mathbf{q}_1, \mathbf{q}_2$, so that $\mathbf{Q}_1 = [\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3]$ let $\mathbf{R}_1 = \begin{bmatrix} -4 & -2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Define $\mathbf{A} = \mathbf{Q}_1\mathbf{R}_1$, so $\mathbf{A}$ is now 6×3 but $\mathbf{R}_1$ is rank deficient.

    (a) Give the dimension of Null($\mathbf{A}$) and give a basis for Null($\mathbf{A}$).

    (b) Express the (set of) least squares solution(s) to $\min_{\mathbf{x}} \|\mathbf{b} - \mathbf{A}\mathbf{x}\|_2^2$ using the information given above.

    (c) Give the unique minimum norm least squares solution.

NLA FINAL

Part I: Short Answer

1. Suppose LU decomposition with partial pivoting is used to factor $\mathbf{A}$ as $\mathbf{PA} = \mathbf{LU}$. Give the steps needed (Not the details of the factorization algorithm!) to compute the solution to $\mathbf{Ax} = \mathbf{b}$. Give the (big-Oh) flop count for each step. (Your answer should not be more than about 4 lines)
2. What does it mean for an eigenvalue λ, of an $m \times m$ matrix $\mathbf{A}$ to be defective?
3. If partial piviting is used to factor $\mathbf{PA} = \mathbf{LU}$ for $m \times m$ matrix $\mathbf{A}$, we know the $|L_{ij}| \leq$ ______
4. Let $\mathbb{F}$ be a normalized floating point number system using base 10, 4 digits of precision. Then the machine unit roundoff, $\epsilon_{mach} =$ ______. This means that if $0 < \epsilon < \epsilon_{mach}$, $fl(1 + \epsilon) =$ ______
5. Let A be $m \times m$ invertible matrix. The p-norm condition number of $\mathbf{A}$ is defined as (do not assume p =2): ______
6. We wise to solve $\mathbf{Ax} = \mathbf{b}$ for invertible $\mathbf{A}$. We have an algorithm that returns $\hat{\mathbf{x}}$, with residual $\mathbf{R} = \mathbf{b} - \mathbf{A}\hat{\mathbf{x}}$. Then $\frac{\|\mathbf{x}-\hat{\mathbf{x}}\|_p}{\|\mathbf{x}\|_p} \leq$ ______
7. Suppose we compute a least-squares solution $\hat{x}$ to $\min_x \|\mathbf{Ax} - \mathbf{b}\|_2$ for $m \times n$ full rank matrix $\mathbf{A}$. Then the residual $\mathbf{r} = \mathbf{b} - \mathbf{A}\hat{\mathbf{x}}$ lives in what subspace of $\mathbb{C}^m$ ______
8. Let $m \times m$ matrix $\mathbf{B}$ be defined as $\mathbf{B} := \mathbf{I} - \mathbf{uv}^*$. Explain how to compute the product $\mathbf{Bx}$ in $\mathcal{O}(m)$ flops.
9. Give the steps in the shifted QR iteration for computing eigenvalues. Then, show that 2 successive iterations are similar matrices.
10. Suppose you have computed 3 eigenvectors v_1, v_2, v_3 for an $m \times m$ symmetric matrix $\mathbf{A}$. Explain how to compute a starting guess x_0 for the Rayleigh Quotient iteration so that the iteration will converge to one of the eigenpairs you haven't already found.

Part II: Short Answer

1. Let $A = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$
   (a) What are the Gershgorian disks for the matrix.
   (b) Perform 1 iteration of the (unshifted) QR iteration on the matrix
2. For the matrix in the previous problem, the eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = -1$.
   (a) To which of the eigenvalues will the inverse iteration with shift of $\mu = 1$ converge (explain)?
   (b) How fast will it converge?
3. Suppose $\mathbf{A}$ is $m \times m$ full rank matrix but is not symmetric. We want to solve $\mathbf{Ax} = \mathbf{b}$ and clearly there is a unique solution for which we could use pivoted LU or QR to solve. However, if we multiply both sides by $\mathbf{A}^*$, the solution to $\mathbf{A}^*\mathbf{Ax} = \mathbf{A}^*\mathbf{b}$ is the same as the original but now we have an HPD matrix for which we can compute a Cholesky decomposition instead and use it to solve the second system.
   (a) In big-Oh, compare the flops required of the three approaches (The three approaches being using $\mathbf{PA} = \mathbf{LU}$ on $\mathbf{Ax} = \mathbf{b}$, using $\mathbf{A} = \mathbf{QR}$ on $\mathbf{Ax} = \mathbf{b}$, or using Cholesky on $\mathbf{A}^*\mathbf{Ax} = \mathbf{b}$)
   (b) Which approach(es) would be preferred in practice, and why?
4. Let $\mathbf{A}$ be a (fixed) symmetric positive definite matrix in $\mathbb{R}^{m \times m}$. Use the *eigendecomposition* of $\mathbf{A}$ to prove that $\|v\|_{\mathbf{A}} := \sqrt{\mathbf{v^T A v}}$ defines a valid norm on $\mathbb{C}^m$

5. Let $\mathbf{A} \in \mathbb{R}^{m \times m}$ be (fixed)symmetric and positive definite matrix. Two vectors $\mathbf{w}, \mathbf{v} \in \mathbb{R}^m$ are called A-conjugate (or A-orthogonal) if $\mathbf{w^T A v} = 0$. Because $\mathbf{A} = \mathbf{R^T R}$ for non-singular $\mathbf{R}$, we can show that $\mathbf{w^T A v} = \mathbf{w^T R^T R v}$ defines a valid inner product on $\mathbb{R}^m$ (a fact you do not need to prove).

   Let $\mathcal{S} = \text{span}\{\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}\}$, with $\mathbf{x_i} \in \mathbb{R}^m$ and $\dim(\mathcal{S}) = 3$. Give a Gram-Schmidt-like algorithm to compute an A-conjugate basis for $\mathcal{S}) = 3$ (they only have to be A-conjugate, there's no 'normalization' necessary)

6. An *invariant subspace* of a matrix $\mathbf{A} \in \mathbb{C}^{m \times m}$ is a subspace of $\mathcal{S}$ such that $\mathbf{Ax} \in \mathcal{S}$ for every $\mathbf{x} \in \mathcal{S}$.

   (a) Let $\mathbf{A} = \mathbf{QTQ}^*$ be the Schur decomposition of $\mathbf{A}$, which implies $\mathbf{AQ} = \mathbf{QT}$. Suppose that the first 3 diagonal elements of $\mathbf{T}$ are non-zero. Partition $\mathbf{Q}, \mathbf{T}$ so that:

   $$\mathbf{A}\begin{bmatrix} Q_1 & Q_2 \end{bmatrix} = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} = \end{bmatrix}$$

   So that $\mathbf{T}_{11}$ is 3×3. This relationship shows that $\text{span}\{\mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3}\}$ is an invariant subspace of $\mathbf{A}$. Why?

   (b) Since $\mathbf{T}$ was triangular, $\mathbf{T_{11}}$ is triangular. And $\mathbf{Q_1}$ has orthonormal columns. Use these facts to find the least squares solution $\mathbf{y}$, to the problem $\min_{y \in \mathbb{C}^m} \|(\mathbf{AQ_1})\mathbf{y} - \mathbf{b}\|_2^2$

7. The so-called "generalized eigenvalue problem" $\mathbf{Ax} = \lambda \mathbf{Mx}$ occurs in applications involving mass-spring systems, where $\mathbf{A}$ is called the stiffness matrix, $\mathbf{M}$ is called the mass matrix. If $\mathbf{M}$ is SPD, then $\mathbf{M}$ is invertible, so this can (in theory!) be converted into an eigenvalue problem $\mathbf{M^{-1}Ax} = \lambda \mathbf{x}$. However, if $\mathbf{A}$ is also symmetric, this unfortunately doesn't preserve symmetry as the matrix $\mathbf{M^{-1}A}$ is now not symmetric. Fortunately, $\mathbf{M}$ is SPD and can be factored with a Cholesky factorization $\mathbf{M} = \mathbf{R^T R}$.

   (a) Use this factorization to convert $(\mathbf{M^{-}1A})\mathbf{x} = \lambda \mathbf{x}$ into an eigenvalue/eigenvector problem[1], $\mathbf{R^{-T}AR^{-1}y} = \lambda \mathbf{y}$ with the same eigenvalues. What is the relationship between the eigenvectors $\mathbf{x}$ and $\mathbf{y}$ of each of the systems?

   (b) If $\mathbf{A}$ is symmetric positive definite, show that $\mathbf{R^{-T}AR^{-1}}$ is also symmetric positive definite.

   (c) Suppose we want to apply power iteration to find a dominant eigenpair of $\mathbf{R^{-T}AR^{-1}}$ (assuming one exists). Explain how, in practice, you can compute the necessary matrix vector product $\mathbf{z^{(k+1)}} = \mathbf{R^{-T}AR^{-1}y^{(k)}}$ **without computing the inverse of the Cholesky factor!**

[1] The notation $\mathbf{R^{-T}}$ is equivalent to $(\mathbf{R^{-1}})^T = (\mathbf{R^T})^{-1}$.

Matrix Analysis

Final Exam

1. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ Hermitian matrices. Assume that $\mathbf{B}$ has rank at most r. Prove that $\lambda_{k+r}(\mathbf{A}) \geq \lambda_k(\mathbf{A}+\mathbf{B})$ for $k = 1, 2, ..., n-2r$.

2. Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$. Let $\mathbf{I}$ denote the $n \times n$ identity matrix.

   (a) Show that $(\mathbf{I} \bigotimes \mathbf{A})^k = \mathbf{I} \bigotimes \mathbf{A}^k$ and$(\mathbf{B} \bigotimes \mathbf{I})^k = \mathbf{B}^k \bigotimes \mathbf{I}$ for all integers k.

   (b) show that $e^{\mathbf{I} \bigotimes \mathbf{A}} = \mathbf{I} \bigotimes e^{\mathbf{A}}$ and $e^{\mathbf{B} \bigotimes \mathbf{I}} = e^{\mathbf{B}} \bigotimes \mathbf{I}$

   (c) Show that the matrices $\mathbf{I} \bigotimes \mathbf{A}$ and $\mathbf{B} \bigotimes \mathbf{I}$ commute.

   (d) show that $e^{\mathbf{B} \bigotimes \mathbf{B}} = e^{(\mathbf{I} \bigotimes \mathbf{A})+(\mathbf{B} \bigotimes \mathbf{I})} = e^{\mathbf{B}} \bigotimes e^{\mathbf{A}}$

3. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{pmatrix}$$

   The eigenvalues of $\mathbf{A}$ are $\lambda_1 = 1, \lambda_2 = 1$ and $\lambda_3 = 2$

   (a) Show that $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is a generalized eigenvector of order 2 corresponding to $\lambda = 1$

   (b) Find an eigenvector corresponding to $\lambda = 2$

   (c) Express $\mathbf{A}$ in its Jordan canonical form i.e. $\mathbf{A} = \mathbf{QJQ^{-1}}$. You only need to specify $\mathbf{Q}$ and $\mathbf{J}$.

4. Let $\mathbf{A}$ be an $n \times n$ matrix. Recall the definition of the i-th Gershgorian radius:

$$R_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} |A_{i,j}|$$

   If $|A_{i,j}| > R_i$ for k different values i, prove that $k \leq rank(\mathbf{A})$

   [**Hint:** Consider a certain principal submatrix $\mathbf{A}$]

5. This problem concerns matrix functions and partial order.

   (a) Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$. Consider the function $f(\mathbf{A} = trace(\mathbf{B^T A B})$. Find the gradient of f with respect to $\mathbf{A}$ i.e., $\nabla f_{\mathbf{A}}$. note that $\mathbf{B}$ is a fixed matrix in the definition of $f(\mathbf{A})$.

   (b) Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$. Assume that $\mathbf{A} \preceq \mathbf{B}$. Prove that $\lambda_i(\mathbf{A}) \leq \lambda_i(\mathbf{B})$ for $1 \leq i \leq n$ i.e., prove that the i-th eigenvalue of $\mathbf{A}$ is less than te i-th eigenvalue of $\mathbf{B}$.

6. Show that adding a row to a matrix cannot decrease its largest singular value.

Numerical Analysis

Midterm I

1. Consider the following function $g : \mathbb{R}^2 \to \mathbb{R}^2$ defined as:

$$g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} \frac{1}{4}x^2 + \frac{1}{16}y + \frac{23}{32} \\ x + \frac{1}{2}y^2 - \frac{5}{8} \end{bmatrix}$$

.

   (a) Show that $\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$ is fixed point of g.

   (b) Determine whether the fixed point iteration defined by g is locally convergent to $\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$

2. Find a linear polynomial that is the best least square fit to the function $f(x) = e^x$ on the interval $[0, 1]$. [Note: the underlying inner product is defined $\langle f, g\rangle = \int_0^1 f(x)g(x)\, dx$.]

3. (a) Compute the Lagrange interpolating polynomial that passes through the points $(0, 1), (2, 3)$ and $(3, 0)$. [**Note:** you just have to write the polynomial. You do not have to simplify it.]

   (b) Given $f(x) = e^x$, let P denote the Lagrange interpolating polynomial that interpolates f at the points $-1, -.05, 0, 0.5, 1$. Bound the error $|f(1/4) - P(1/4)|$. [**Remark:** it is not necessary to construct P]

   (c) Given $f(x) = e^x$, let Q denote the degree 4 Chebyshev interpolating polynomial on $[-1, 1]$. Bound the error $|f(x) - Q(x)|$ for any $x \in [-1, 1]$ i.e. provide the explicit worst-case error bound. [**Remark:** it is not necessary to construct Q]

4. Let $f \in C^2([a, b])$. We divide $[a, b]$ into n sub-intervals and set $h = \frac{b-a}{n}$. Define the nodes $x_1, x_2, ..., x_{n+1}$ as follows:

$$x_1 = 0 \text{ and } x_i = x_{i-1} + h \text{ for } i \in [2, n+1].$$

   Given $(x_1, f(x_1)), (x_2, f(x_2)), ..., (x_{n+1}, f(x_{n+1}))$, let $g(x)$ be the interpolating linear spline to f. Formally $g(x)$ is continuous on $[x_1, x_{n+1}]$, is a straight line on each interval $[x_i, x_{i+1}]$ for $i = 1, 2, ..., n$ and it interpolates $f(x)$ at the nodes. Prove that

$$|f(x) - g(x)| \leq \frac{1}{8}h^2\|f"\|_\infty \quad \text{for any } x \in [a, b]$$

   where $\|f"\|_\infty = \max_{a \leq x \leq b} |f"(x)|$

5. Let $f(x)$ be a continuous function on $[a, b]$. prove that its minimax polynomial is unique. [Recall the minimax polynomial is the polynomial of degree at most n that is the closet to f in L^∞ norm]. [**Hint:** Use the Chebyshev equioscillation theorem.]