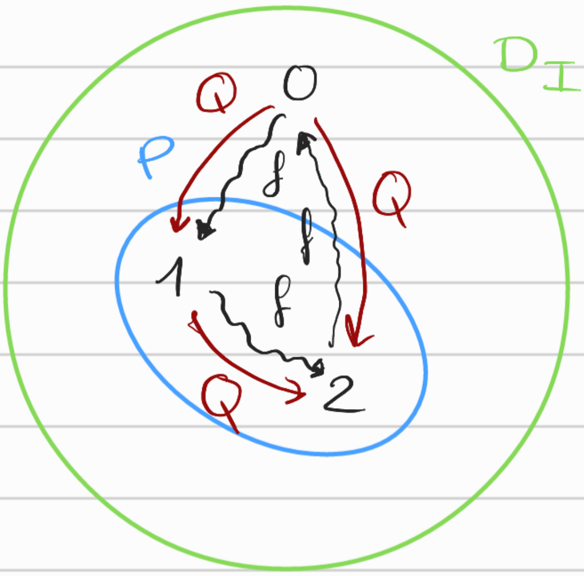
Exercice 1 :

① $S_p = \{P_1, Q_2\}$ $\quad S_f = \{f_1\}$ $\quad V = \{x, y\}$



$I(f) = \{(0,1),(1,2),(2,0)\}$

$f(0) = 1$
$f(1) = 2$
$f(2) = 0$

et

$I(P)(0) = T$
$I(P)(1) = T$
$I(P)(2) = F$

$$[\![\forall x . P(f(x))]\!]^I_\rho = \bigwedge_{d \in D_I} [\![P(f(x))]\!]^I_{\rho[d/x]}$$

$$= \bigwedge_{d \in D_I} I(P)(I(f))[\![x]\!]^I_{\rho[d/x]} = \bigwedge_{d \in D_I} I(P)(I(f)\,\rho[d/x](x))$$

$$= I(P)(I(f)(0)) \wedge_\beta I(P)(I(f)(1)) \wedge_\beta I(P)(I(f)(2))$$

$$= I(P)(1) \wedge_\beta I(P)(2) \wedge_\beta I(P)(0)$$

$$= T \wedge_\beta T \wedge_\beta F = F$$

③ $\forall x . \exists y . Q(f(x), y)$

$$= \bigwedge_{\substack{d \in D_I \\ 0,1,2}} \left(\bigvee_{\substack{d' \in D_I \\ 0,1,2}} \left(I(Q)(I(f)(\rho[d/x][d'/y]x)(\rho[d/x][d'/y]y) \right) \right)$$

$(I(Q)(I(f)(0), 0) \vee I(Q)(I(f)(0), 1) \vee I(Q)(I(f)(0), 2)) \wedge \ldots$

$$= \left(I(Q)(I(f)(0),0) \vee_\beta I(Q)(I(f)(0),1) \vee_\beta \underbrace{I(Q)(I(f)(0),2)}_{T} \right) \wedge_\beta$$

$$\left(I(Q)(I(f)(1),0) \vee_\beta I(Q)(I(f)(1),1) \vee_\beta I(Q)(I(f)(1),2) \right) \wedge_\beta$$

$$\left(I(Q)(I(f)(2),0) \vee_\beta \underbrace{I(Q)(I(f)(2),1)}_{T} \vee_\beta I(Q)(I(f)(2),2) \right)$$

$= F$

⑤ $\forall x . P(x) \Rightarrow \exists y . Q(f(y), x)$

$$= \bigwedge_{v \in D_I} \left(I(P)(v) \Rightarrow_\beta \bigvee_{v' \in D_I} P(Q)(I(f)(v', v)) \right)$$

$$= \left(I(P)(0) \Rightarrow_\beta \bigvee_{v' \in D_I} I(Q)(I(f)(v', 0)) \right) \wedge_\beta$$

On développe le OU en cherchant les valides

$$= \left(I(P)(1) \Rightarrow_\beta I(Q)(I(f)(2), 1) \vee_\beta \bigvee_{v' \in D_I \setminus \{0\}} I(Q)(I(f)(v'), 0) \right) \wedge_\beta$$

$$= \left(I(P)(2) \Rightarrow_\beta I(Q)(I(f)(0), 2) \vee_\beta \bigvee_{v' \in D_I \setminus \{0\}} I(Q)(I(f)(v'), 0) \right) \wedge_\beta$$

$= T$ Pour résumer :

$$= (F \Rightarrow \ldots) \wedge_\beta (T \Rightarrow T \vee_\beta \ldots) \wedge_\beta (T \Rightarrow_\beta T \vee_\beta \ldots) = T$$

Exercice 2 : Correction perso

On cherche une interprétation Fausse pour chercher une formule non-valide

On pose $P(x,y) = x \leq y$ avec $x, y \in D_I = \{0,1,2\}$

On pose $f(x) = \begin{cases} f(0) = 1 \\ f(1) = 2 \\ f(2) = 0 \end{cases}$

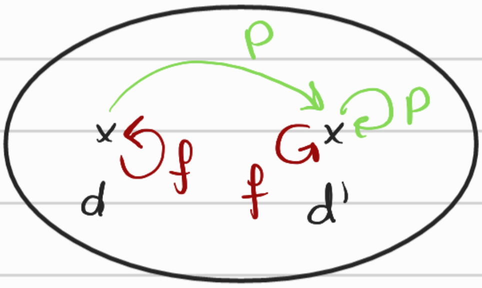
On fixe $x \leq f(x)$: $f(0)=1$ ok
$f(1)=2$ ok
$f(2)=0$ FAUX

Correction prof

$D_I \{d, d'\}$

$I(P) = \{(d,d',T),(d',d',T),(d',d,F),(d,d,F)\}$

$I(f) = \{(d,d),(d',d')\}$



$$[\![F_1 \Rightarrow F_2]\!]^I_\rho = \left(\bigwedge_{d \in D_I} \bigvee_{d' \in D_I} I(P)(d,d')\right) \Rightarrow_\beta \left(\bigwedge_{d'' \in D_I} I(P)(d'', I(f)(d''))\right)$$

$$= \left(I(P)(d,d') \vee_\beta \dots\right) \wedge_\beta \left(I(P)(d',d') \vee_\beta \dots\right) \Rightarrow_\beta \left(I(P)(d, I(f)(d))\right) \wedge_\beta \left(I(P)(d', I(f)(d'))\right)$$

$$= (T \vee_\beta \dots) \wedge_\beta (T \vee_\beta \dots) \Rightarrow_\beta (F \wedge_\beta T)$$

$$= T \Rightarrow_\beta F = F$$

Avec les séquents :

$$\vdash \forall x.\exists y.P(x,y) \Rightarrow \forall x.P(x,f(x))$$
$$\overline{\hspace{20em}} \Rightarrow_D$$
$$\forall x.\exists y.P(x,y) \vdash \forall x.P(x,f(x))$$
$$\overline{\hspace{20em}} \forall_D$$

$$\dfrac{\dfrac{\forall x.\exists y.P(x,y) \vdash (P(x,f(x)))_{[x'/x]}}{(\exists y.P(x,y))_{[x'/x]} \vdash P(x',f(x'))}\ \forall_G}{\exists y.P(x',y) \vdash P(x',f(x'))}$$

donc $F_1 \Rightarrow F_2$ pas démontrable avec séquents

$F_2 \Rightarrow F_1$ démontrable avec séquents $(\Rightarrow_D, \forall_D, \exists_D, \forall_G)$

$$[\![F_2 \Rightarrow F_1]\!]_\rho^I = [\![\forall x.P(x,f(x))]\!] \Rightarrow_\beta [\![\forall x.\exists y.P(x,y)]\!]$$

$$= \underbrace{\bigwedge_{v \in D_I} P(I)(v, I(f)(v))}_{[\![F_2]\!]_\rho^I} \Rightarrow_\beta \underbrace{\bigwedge_{v' \in D_I} \bigvee_{v'' \in D_I} I(P)(v',v'')}_{[\![F_1]\!]_\rho^I}$$

- Si $[\![F_2]\!]_\rho^I = F$, alors $[\![F_2 \Rightarrow F_1]\!]_\rho^I = F \Rightarrow_\beta [\![F_1]\!]_\rho^I = T$ ← Si F, alors T
- Sinon, $[\![F_2]\!]_\rho^I = \bigwedge_{v \in D_I} I(P)(v, I(f)(v))$ ← Si T, alors T pour chaque $v \in D_I$

$$= I(P)(v_1, I(f)(v_1)) \wedge_\beta \ldots = T$$

Pour n'importe quel élément $v \in D_I$, on a :

$$((d, I(f)(d)), T) \in I(P) \quad (i)$$

Montrons alors que $[\![F_1]\!]_\rho^I = T$ (comme ça on aura $[\![F_2 \Rightarrow F_1]\!] = T \Rightarrow_\beta T = T$)

$$[\![F_1]\!]_\rho^I = \bigwedge_{v' \in D_I} \bigvee_{v'' \in D_I} I(P)(v, v'')$$

• Pour que $[\![F]\!]_\rho^I$ il faut que, pour tout $v' \in D_I$, je trouve $v'' \in D_I$ tel que $I(P)(v', v'') = T$

$(T \lor \ \ ) \land (\ \ \lor T) \land \ldots$ on cherche 1 vrai dans chaque clause

Prenons un v' quelconque $\in D_I$, choisissons $v'' = I(f)(v')$ qui forcément existe car $v' \in D_I$ et que $I(f)$ doit être définie par D.

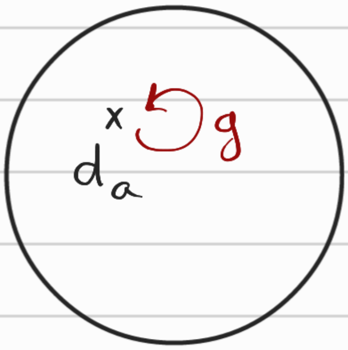
$$I(P)(v', v'') = I(P)(v', I(f)(v')) = T \text{ par la condition } (i)$$

Exercice 3 :

Soit I une interprétation qui rend vrai H_1, H_2, H_3 on sait donc que :

• $[\![g(a) \doteq a]\!]_\rho^I \underset{M}{=} I(\doteq)\big(I(g)(I(a)), I(a)\big)$
$\underset{M}{=} I(g)(I(a)) = I(a)$
$\underset{M}{=} \underbrace{I(g)(d_a) = d_a}_{(ii)\quad T} \underset{M}{=} T$

$S_P = \{P_2\}$ $\quad S_f = \{a_0, f_1, g_1\}$ $\quad V = \{x\}$



• $[\![\forall x . g(f(x)) = g(x)]\!]_\rho^I$

$\underset{M}{=} \bigwedge_{d \in D} I(g)(I(f)(d)) = I(g)(d)$

$\underset{M}{=} \underline{\big(I(g)(I(f)(d_a)) = I(g)(d_a)\big)} \land_\beta \Big(\bigwedge_{d \in D \setminus \{d_a\}} I(g)(I(f)(d)) = I(g)(d)\Big)$

$(i) \quad \top \quad d \in D \setminus \{d_a\}$

- $[\![\forall x . P(x,x)]\!]^I_\rho$

$$\underset{M}{=} \underbrace{I(P(d_a,d_a))}_{(iii)\ \top} \wedge_B \bigwedge_{d \in D_I \setminus \{d_a\}} I(P)(d,d) \underset{M}{=} \top$$

- $[\![P(a, g(f(a)))]\!]^I_\rho \underset{M}{=} I(P)(d_a, I(g)(I(f)(d_a)))$

$$\overset{(i)}{\underset{M}{=}} I(P)(d_a, I(g)(d_a)) \overset{(ii)}{=} I(P)(d_a, d_a) \overset{(iii)}{\underset{M}{=}} \top$$

Exercice 4 :

$\mathcal{E}q$

$$\frac{s \doteq t \in \mathcal{E}}{s \doteq t}\ ax \qquad \frac{}{s \doteq s}\ \text{reflexivité}$$

$$\frac{t \doteq s}{s \doteq t}\ \text{symmmetrie} \qquad \frac{s \doteq t \quad t \doteq u}{s \doteq u}\ \text{transitivité}$$

$$\frac{s \doteq t}{\sigma(s) = \sigma(t)}\ \text{substraction} \qquad \frac{s \doteq t}{\mathcal{U}[s]_p \doteq \mathcal{U}[t]_p}\ \text{contexte}$$

$$\frac{\dfrac{\dfrac{p(x,s(y)) \doteq s(p(x,y))}{}\ \left[s^2(0) \quad s(0)\right]}{p(s^2(0),s^2(0)) \doteq s(p(s^2(0),s(0)))}\ \text{subst} \qquad \dfrac{p(s^2(0),s(0)) \doteq s^3(0)\quad n=1}{s(p(s^2(0),s(0))) \doteq s^4(0)}\ \text{contexte}}{P(s(s(0)), s(s(0)))}\ \text{transitivité}$$

$$\left[\quad /x \;,\; /y\right] \qquad u = s(x)$$

$$\frac{}{\quad}\;\mathrm{AX} \qquad\qquad \frac{}{\quad}\;\text{transitivité}$$

$$p(s^2(0), s(0)) \doteq s(p(s^2(0), 0)) \qquad\qquad s(p(s^2(0), 0)) \doteq s^3(0)$$

$$\frac{}{}\;\text{subst}\;\left[{}^{s^2(0)}/_{x}\right]\left[{}^{0}/_{y}\right] \qquad\qquad \frac{}{}\;\text{contexte} \quad p = 1,\; u = s(x)$$

$$\frac{\dfrac{}{p(x, s(y)) \doteq s(p(x, y))}\;\mathrm{AX}}{p(s^2(0), s(0)) \doteq s(p(s^2(0), 0))}\;\text{subst}\left[{}^{s^2(0)}/_{x}\right]\left[{}^{0}/_{y}\right]$$

$$\frac{\dfrac{\dfrac{}{p(x, 0) \doteq x}\;\mathrm{AX}}{p(s^2(0), 0) \doteq s^2(0)}\;\text{subst}\left[{}^{s^2(0)}/_{x}\right]}{s(p(s^2(0), 0)) \doteq s^3(0)}\;\text{contexte}\; p = 1,\; u = s(x)$$