- EE 387 - Signal Processing
- Assignment 1

Problem 1. Determine the fundamental period of the following signals

(a) $x(t) = 3\cos(10t+1) - \sin(4t-1)$

$x(t) = 3\cos(\omega_1 t + \phi_1) - \sin(\omega_2 t + \phi)$

Fundamental period of $3\cos(10t+1)$

$= T_1$

$= \frac{2\pi}{\omega_1}$

$= \frac{2\pi}{10}$

$= \pi/5$

Fundamental period of $\sin(4t-1)$

$= T_2$

$= \frac{2\pi}{\omega_2}$

$= \frac{2\pi}{4}$

$= \pi/2$

So, Fundamental period of $x(t) = T_0$

$= LCM(T_1, T_2)$

$= LCM(\pi/5, \pi/2)$

$= \pi$

$T_0 = \frac{2\pi k}{\omega_0}$

if $k = 1$

$T_0 = \frac{2\pi}{\omega_0}$

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(b) $x[n] = 1 + e^{j4\pi n/7} - e^{j2\pi n/5}$

$x[n] = 1 + e^{j(4\pi/7)n} - e^{j(2\pi/5)n}$

$x[n] = 1 + e^{j\omega_1 n} - e^{j\omega_2 n}$

$N_0 = \frac{2\pi}{\omega_0}k$

Fundamental Period of $e^{j\left(\frac{4\pi}{7}\right)n}$

N_1

$= \frac{2\pi}{\omega_1}k$

$= \frac{2\pi}{4\pi/7}k$

$= \frac{7}{2}k$

for $k = 2 \ni N_1 = 7 \ (\in \mathbb{Z})$

Fundamental period of $e^{j\left(\frac{2\pi}{5}\right)n}$

$= N_2$

$= \frac{2\pi}{\omega_2}k$

$= \frac{2\pi}{2\pi/5}k$

$= 5h$

for $h = 1 \ni N_2 = 5 \ (\in \mathbb{Z})$

So, Fundamental period of $x[n] = N_0$

$= LCM(N_1, N_2)$

$= LCM(7, 5) = 35$

Problem 2. Determine whether the following signals are Periodic or aperiodic? If periodic also find the period.

(a) $x(t) = 2\cos(4t + \pi/3)$

$x(t) = A\cos(\omega_0 t + \phi)$

$\omega = 4$

$2\pi f = 4$

$f = \frac{2}{\pi}$

$T = \pi/2$

if periodic $x(t) = x(t+T)$

$x(t+T) = 2\cos\left(4\left(t+\pi/2\right)+\pi/3\right)$

$x(t+T) = 2\cos\left(2\pi + \left(4t+\pi/3\right)\right)$

$x(t+T) = 2\cos\left(4t+\pi/3\right) = x(t) \quad (\because \cos(2\pi+\theta) = \cos\theta)$

So, signal $2\cos(4t+\pi/3)$ is periodic with fundamental period $= T_0 = \pi/2$

(b) $x(t) = \left[\sin\left(2t - \pi/4\right)\right]^2$

$\frac{1-\cos 2\theta}{2} = \sin^2\theta$

$x(t) = \sin^2\left(2t - \pi/4\right)$

$x(t) = \frac{1 - \cos 2\left(2t - \pi/4\right)}{2} = \frac{1}{2} - \frac{1}{2}\cos\left(4t - \pi/2\right)$

(b) $x(t) = \frac{1}{2} - \frac{1}{2}\cos\left(4t - \frac{\pi}{2}\right)$

$$\omega = 4 = 2\pi f$$

$$f = \frac{4}{2\pi} = \frac{2}{\pi}$$

$$T = \frac{1}{f} = \frac{\pi}{2}$$

$x(t+T)$

$$= \frac{1}{2} - \frac{1}{2}\cos\left(4\left(t+\frac{\pi}{2}\right) - \frac{\pi}{2}\right)$$

$$= \frac{1}{2} - \frac{1}{2}\cos\left(4t - \frac{\pi}{2} + 2\pi\right)$$

$$= \frac{1}{2} - \frac{1}{2}\cos\left(2\pi + \left(4t - \frac{\pi}{2}\right)\right) \quad \left(\because \cos(2\pi+\theta) = \cos\theta\right)$$

$$= \frac{1}{2} - \frac{1}{2}\cos\left(4t - \frac{\pi}{2}\right)$$

$$= x(t)$$

So, signal $x(t) = \left[\sin\left(2t - \frac{\pi}{4}\right)\right]^2$ is periodic with fundamental period $T_0 = \frac{\pi}{2}$

(c) $x[n] = \sin\left(\frac{6\pi}{7}n + 1\right)$

$$\omega = \frac{6\pi}{7} = 2\pi f$$

$$f = \frac{3}{7} = \frac{k}{N} \text{ is Rational}$$

$x(n+N)$

$$= \sin\left(\frac{6\pi}{7}(n+N)+1\right)$$

$$= \sin\left(\frac{6\pi}{7}n + \frac{6\pi}{N} + 1\right)$$

So, $x[n] = \sin\left(\frac{6\pi}{7}n + 1\right)$ is periodic with fundamental period $N = 7$ "

(d) $x[n] = \cos\left(\frac{\pi}{8} n^2\right)$

$$x[n+N] = \cos\left(\frac{\pi}{8}(n+N)^2\right)$$

$$x[n+N] = \cos\left(\frac{\pi}{8} n^2 + \frac{\pi}{8} N^2 + \frac{\pi}{8}(2nN)\right)$$

If signal $x[n]$ to be periodic $x[n+N] = x[n]$

i.e $\cos\left(\frac{\pi}{8} n^2 + \frac{\pi}{8} N^2 + \frac{\pi}{8}(2nN)\right) = \cos\left(\frac{\pi}{8} n^2\right)$

This is only possible if

$$\frac{\pi}{8} N = 2\pi m \text{ and } \frac{\pi}{8} N^2 = 2\pi k$$

$(\because \cos(2\pi + \theta) = \cos\theta)$

$\Rightarrow m = 1, \; k = 4 \quad N = 8$

Hence,

$x[n+N]$

$= \cos\left(\frac{\pi}{8} n^2 + \frac{\pi}{8} 8^2 + \frac{\pi}{8}(16n)\right)$

$= \cos\left(8\pi + \frac{\pi}{8} n^2 + 2\pi n\right)$

$= \cos\left(2\pi n + \pi/8 \, n^2\right) \quad (\because n \in \mathbb{Z}, \cos(2n\pi + \theta) = \cos\theta)$

$= \cos\left(\frac{\pi}{8} n^2\right)$

$= x[n]$

Then, signal $x(t) = \cos\left(\frac{\pi}{8} n^2\right)$ is periodic with a fundamental period of $N = 8$

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(e) $x(t) = \sin\left(\frac{\pi}{8} t^2\right)$

$$x(t+T) = \sin\left(\frac{\pi}{8}(t+T)^2\right)$$

$$x(t+T) = \sin\left(\frac{\pi}{8}t^2 + \frac{\pi}{8}T^2 + \frac{\pi}{8}(2tT)\right)$$

For above equation, For signal $x(t)$ to be periodic $x(t+T)$ should be equal to $x(t)$

i.e

$$\sin\left(\frac{\pi}{8}t^2 + \frac{\pi}{8}T^2 + \frac{\pi}{8}(2tT)\right) = \sin\left(\frac{\pi}{8}t^2\right)$$

This is only possible if

$$T = \frac{2\pi}{\pi/8} = 16$$

$T = 16 \Rightarrow \sin\left(\frac{\pi}{8}t^2 + \frac{\pi}{8}\cdot 16^2 + \frac{\pi}{8}(2 \times t \times 16)\right)$

$$= \sin\left(\frac{\pi}{8}t^2 + 32\pi + 4\pi t\right)$$

$$= \sin\left(\frac{\pi}{8}t^2\right) \qquad (\sin(2\pi n + \theta) = \sin\theta)$$

So, $x(t) = \sin\left(\frac{\pi}{8}t^2\right)$ is periodic with fundamental period $T = 16$.

(f) $x[n] = \cos\left(\frac{\pi}{2}n\right)\cos\left(\frac{\pi}{4}n\right)$

$x[n+N] = \cos\left(\frac{\pi}{2}(n+N)\right)\cos\left(\frac{\pi}{4}(n+N)\right)$

$x[n+N] = \cos\left(\frac{\pi}{2}n+\frac{\pi}{2}N\right)\cos\left(\frac{\pi}{4}n+\frac{\pi}{4}N\right)$

From the above equation for the signal to be periodic $\cos\left(\frac{\pi}{2}n+\frac{\pi}{2}N\right)\cos\left(\frac{\pi}{4}n+\frac{\pi}{4}N\right) = \cos\left(\frac{\pi}{2}n\right)\cos\left(\frac{\pi}{4}n\right)$

That is $x[n+N] = x[n]$

This is only possible if,

$\frac{\pi}{2}N_1 = 2\pi m_1$, $\frac{\pi}{4}N_2 = 2\pi m_2$ $(\because \cos(2\pi+\theta)=\cos\theta)$

Where N_1 and N_2 are the periods of $\cos\left(\frac{\pi}{2}n\right)$ and $\cos\left(\frac{\pi}{4}n\right)$ respectively. and $m_1, m_2 \in \mathbb{Z}^+$

$\Rightarrow N_1 = 4$, $m_1 = 1$, $N_2 = 8$, $m_2 = 1$

Let $x_1[n] = \cos\left(\frac{\pi}{2}n\right)$ and $x_2[n] = \cos\left(\frac{\pi}{4}n\right)$ then

$x_1[n+N_1] = \cos\left(\frac{\pi}{2}n+\frac{\pi}{2}N_1\right) = \cos\left(2\pi+\frac{\pi}{2}n\right) = \cos\left(\frac{\pi}{2}n\right) = x_1[n]$

$x_2[n+N_2] = \cos\left(\frac{\pi}{4}n+\frac{\pi}{4}N_2\right) = \cos\left(2\pi+\frac{\pi}{4}n\right) = \cos\left(\frac{\pi}{4}n\right) = x_2[n]$

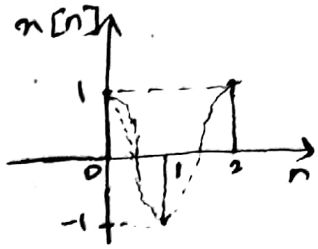
Hence, the fundamental period of the signal $x[n]$ will be L.C.M (N_1, N_2) that is $LCM(4,8) = 8$

Therefore, The given signal $x[n] = \cos\left(\frac{\pi}{2}n\right)\cos\left(\frac{\pi}{4}n\right)$ is periodic with a fundamental period of $N=8$

Problem 3. Classify each of the signals below as a power signal or an energy signal. In addition, find the power or the energy of the signal.

(a)

$$x[n] = \begin{cases} \cos(\pi n) & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



$$E_\infty = \lim_{n\to\infty} \sum_{-N}^{N} |x[n]|^2 = \sum_{-\infty}^{\infty} |x[n]|^2$$

$2\cos^2\theta = 1+\cos 2\theta$

$$E_\infty = \sum_{-\infty}^{\infty} |\underbrace{\cos(\pi n)}_{x[n]}|^2$$

$$E_\infty = \sum_{-\infty}^{\infty} \underbrace{\cos^2(\pi n)}_{x[n]} = \sum_{-\infty}^{-1} \underbrace{0}_{0} + \sum_{0}^{\infty} (\cos^2 \pi n)$$

$$E_\infty = \sum_{0}^{\infty} \frac{1+\cos(2\pi n)}{2}$$

$$E_\infty = \frac{1}{2} \sum_{0}^{\infty} (1+\cos 2\pi n)$$

$$E_\infty = \frac{1}{2} \sum_{0}^{\infty} 1 + \frac{1}{2} \sum_{0}^{\infty} \underbrace{\cos(2\pi n)}_{0}$$

$2\cos^2\theta = 1+\cos 2\theta$

$$E_\infty = \infty \ (\text{infinite})$$

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} [\cos^2(\pi n)]$$

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} \left(\frac{1+\cos 2\pi n}{2}\right)$$

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} \left(\frac{1}{2} + \frac{\cos 2\pi n}{2}\right)$$

$$P_\infty = \frac{1}{2N+1} \lim_{N\to\infty} \left(\sum_{-N}^{N} \frac{1}{2} + \sum_{-N}^{N} \underbrace{\frac{\cos 2\pi n}{2}}_{0}\right)$$

$$P_\infty = \frac{1}{2N+1} \lim_{N\to\infty} \left(\frac{1}{2} \sum_{-N}^{N} 1\right) \qquad \left(\because \sum_{-N}^{N} 1^n = N_2 - N_1 + 1 = 2N+1\right)$$

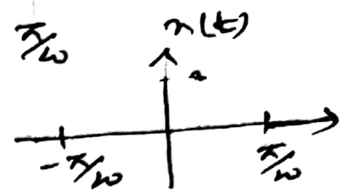
$$P_\infty = \frac{1}{2N+1} \times \frac{1}{2} \times \lim_{N\to\infty} (2N+1)$$

$$P_\infty = \frac{1}{2} \ (\text{finite})$$

$\therefore$ Above $x[n]$ is a power signal with finite average power $\frac{1}{2}$ and infinite total energy.

(b)

$$x(t) = \begin{cases} \frac{1}{2}(\cos(\omega t)+1) & -\pi/\omega < t \le \pi/\omega \\ 0 & \text{otherwise} \end{cases}$$



Continuous time sinusoidal signals are always periodic. So, This $x(t) = \frac{1}{2}(\cos(\omega t)+1)$ is periodic.

$$E_\infty = \lim_{t\to\infty} \int_{-T}^{T} |x(t)|^2 \cdot dt = \int_{-\infty}^{\infty} |x(t)|^2 \cdot dt$$

⇓ periodic

$$E_\infty = \int_{-\pi/\omega}^{\pi/\omega} x\left[\frac{1}{2}(\cos(\omega t)+1)\right]^2 \cdot dt.$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$E_\infty = \frac{1}{4}\int_{-\pi/\omega}^{\pi/\omega} (\cos^2(\omega t) + 2\cos\omega t + 1) \cdot dt$$

$$E_\infty = \frac{1}{4}\int_{-\pi/\omega}^{\pi/\omega} \left(\frac{1+\cos(2\omega t)}{2} + 2\cos\omega t + 1\right) \cdot dt$$

$$E_\infty = \frac{1}{8}\int_{-\pi/\omega}^{\pi/\omega} (\cos(2\omega t) + 4\cos\omega t + 3) \cdot dt$$

$$E_\infty = \frac{1}{8}\left[\underbrace{\int_{-\pi/\omega}^{\pi/\omega} \cos(2\omega t)\cdot dt}_{0} + 4\underbrace{\int_{-\pi/\omega}^{\pi/\omega} \cos(\omega t)\cdot dt}_{0} + \int_{-\pi/\omega}^{\pi/\omega} 3 \cdot dt\right]$$

$$E_\infty = \frac{1}{8} \times 3[t]_{-\pi/\omega}^{\pi/\omega} = \frac{3\pi}{4\omega}\ J < \infty \text{ (finite)}$$

This is an energy signal with finite total energy $\frac{3\pi}{4\omega}$ J and average power $P_\infty = 0$ "

(area under the curve)

$$\left(\because P = \lim_{T\to\infty} \frac{E}{T}\right)$$

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Problem 4. Determine the Fourier series representation for the following signal, $x(t)$, with period 4:

$$x(t) = \begin{cases} \sin(\pi t) & 0 \le t \le 2 \\ 0 & 2 < t \le 4 \end{cases}$$

$T_0 = 4$ $\qquad \omega_0 = 2\pi \times \frac{1}{4}$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{\pi}{2}$$

where $a_k = \dfrac{1}{T_0} \displaystyle\int_{T_0} x(t) \cdot e^{-jk\omega_0 t} \cdot dt$

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t)$$

$$a_0 = \frac{1}{4}\int_0^2 0 \cdot dt = 0$$

$$\sin(\pi t) = \frac{e^{j\pi t} - e^{-j\pi t}}{2j}$$

$$a_k = \frac{1}{4}\int_0^2 \sin \pi t \cdot e^{-jk\frac{\pi}{2}t}$$

$$= \frac{1}{4}\int_0^2 \left(\frac{e^{j\pi t} - e^{-j\pi t}}{2j}\right) \cdot e^{-j\frac{\pi}{2}kt} \cdot dt$$

$$= \frac{1}{4j} \times \frac{1}{2} \int_0^2 \left(e^{jt(\pi - \frac{\pi}{2}k)} - e^{-jt(\pi + \frac{\pi}{2}k)}\right) \cdot dt$$

$$= \frac{1}{8j}\left[\int_0^2 e^{jt(\pi - \frac{\pi}{2}k)} \cdot dt - \int_0^2 e^{-jt(\pi + \frac{\pi}{2}k)} \cdot dt\right]$$

$$= \frac{1}{8j}\left[\int_0^2 e^{j\pi(1 - k/2)t} \cdot dt - \int_0^2 e^{-j\pi(1 + k/2)t} \cdot dt\right]$$

$$= \frac{1}{8j}\left[\left.\frac{e^{j\pi(1-k/2)t}}{j\pi(1-k/2)}\right|_0^2 + (-)\left.\frac{e^{-j\pi(1+k/2)t}}{(-)j\pi(1+k/2)}\right|_0^2\right]$$

$$= \frac{1}{8j}\left[\frac{e^{j\pi(2-k)} - 1}{j\pi(1-k/2)} + (-)\frac{e^{-j\pi(2+k)} - 1}{j\pi(1+k/2)}\right]$$

Problem 4

a_k

$$= \frac{1}{8j}\left[\frac{e^{j\pi(2-k)}}{j\pi(1-k/2)} - \frac{1}{j\pi(1-k/2)} - \left(\frac{e^{-j\pi(2+k)}}{-j\pi(1+k/2)} - \frac{1}{-j\pi(1+k/2)}\right)\right]$$

$$= \frac{1}{8j}\left[\frac{e^{2\pi j}\, e^{-j\pi k}}{j\pi(1-k/2)} - \frac{1}{j\pi(1-k/2)} + \frac{e^{-2\pi j}\cdot e^{-j\pi k}}{j\pi(1+k/2)} - \frac{1}{j\pi(1+k/2)}\right]$$

$$= \frac{1}{8j(j\pi)}\left[\frac{(-1)\cdot e^{-j\pi k}}{(1-k/2)} - \frac{1}{\cancel{j\pi}(1-k/2)} + \frac{(-1)\cdot e^{-j\pi k}}{(1+k/2)} - \frac{1}{(1+k/2)}\right]$$

$$= \frac{1}{-8\pi}\left[-e^{j\pi k}\left(\frac{1}{(1-k/2)} + \frac{1}{(k/2+1)}\right) - \left(\frac{1}{(1-k/2)} + \frac{1}{(1+k/2)}\right)\right]$$

$$= \frac{-1}{8\pi}\left[e^{-j\pi k}\left(\frac{1+\cancel{k/2}+1-\cancel{k/2}}{(1-k^2/4)}\right) - \left(\frac{1+\cancel{k/2}+1-\cancel{k/2}}{(1-k^2/4)}\right)\right]$$

$$= \frac{-1}{\cancel{8}\pi}\left[e^{-j\pi k}\,\frac{\cancel{8}}{(4-k^2)} - \frac{\cancel{8}}{(4-k^2)}\right]$$

$$= \frac{1}{\pi}\left[\frac{e^{-j\pi k}}{(k^2-4)} - \frac{1}{(k^2-4)}\right]$$

$$= \frac{1}{\pi(k^2-4)}\left(e^{-j\pi k}-1\right)$$

$$= \frac{e^{-j\pi k}-1}{\pi(k^2-4)}$$,,

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Problem 4

$x(t)$

$$= \sum_{k=-\infty}^{\infty} a_k e^{jk(\pi/2)t}$$

$$= \sum_{k=-\infty}^{\infty} \left(\frac{-1 + e^{-j\pi k}}{\pi(k^2-4)}\right) e^{jk(\pi/2)t} \quad \text{where } k \neq 0 \; (\because a_0 = 0)$$

Problem 5 Consider the following three continuous-time signals with a fundamental period of $T = 1/2$

$$x(t) = \cos(4\pi t)$$

$$y(t) = \sin(4\pi t)$$

$$z(t) = x(t) \cdot y(t)$$

(a) Determine the Fourier series coefficients of $x(t)$

$$x(t) = \cos(4\pi t)$$

$$\cos(4\pi t) = \frac{e^{4\pi t} + e^{-4\pi t}}{2}$$

$$\cos(4\pi t) = \frac{1}{2} e^{4\pi t} + \frac{1}{2} e^{-4\pi t}$$

$$\cos(4\pi t) = a_1 e^{4\pi t} + a_{-1} e^{-4\pi t}$$

$$a_1 = a_{-1} = 1/2 \, , \quad a_0 = 0$$

$$a_k = 0 \; ; \; k \neq \pm 1$$

(b) Determine the Fourier series coefficients of $y(t)$

$$y(t) = \sin(4\pi t) = \frac{e^{4\pi t} - e^{-4\pi t}}{2j}$$

$$y(t) = \frac{1}{2j} e^{4\pi t} + \left(-\frac{1}{2j}\right) e^{-4\pi t} = b_1 e^{4\pi t} + b_{-1} e^{-4\pi t}$$

$$b_1 = \frac{1}{2j} \, , \quad b_{-1} = -\frac{1}{2j}$$

$$b_k = 0 \; ; \; k \neq \pm 1$$

Problem 5

(c) Use the results of parts (a) and (b) along with the multiplication property of the continuous-time Fourier series, to determine the Fourier series coefficients of $z(t) = x(t)y(t)$.

$$x(t)\,y(t) \xleftrightarrow{FS} c_k \text{ with period } T \text{ where}$$

$$x(t) \xleftrightarrow{FS} a_k \text{ with period } T$$

$$y(t) \xleftrightarrow{FS} b_k \text{ with period } T$$

$$c_k = \sum_{l=-\infty}^{\infty} a_l\, b_{k-l} \quad (\text{* Discrete Convolution})$$

$$c_k = \sum_{l-\infty}^{\infty} a_l\, b_{k-l} = 0 \,;\; k \neq \pm 2$$

$$\cancel{k=\pm 2} \Rightarrow c_k = a_k * b_k = \sum_{l=-\infty}^{\infty} a_l\, b_{k-l} = \frac{1}{4j}\delta[k-2] - \frac{1}{4j}\delta[k+2]$$

$$c_2 = \frac{1}{4j},\; c_{-2} = -\frac{1}{4j} \quad [k \neq \pm 2 \Rightarrow c_k = 0]$$

$$(\because c_2 = a_1 b_{2-1},\; c_{-2} = a_{-1} b_{-2-(-1)})$$

(d) $z(t) = x(t) \cdot y(t)$, Determine the Fourier series coefficients of $z(t)$ through direct expansion of $z(t)$ in trigonometric form, and compare your result with that of part c.

$$z(t) = \cos(4\pi t) \cdot \sin(4\pi t)$$

$$z(t) = \frac{1}{2}\left(2\sin(4\pi t) \cdot \cos(4\pi t)\right)$$

$$z(t) = \frac{1}{2}\sin(8\pi t) = \frac{1}{2}\left(\frac{e^{8\pi t} - e^{-8\pi t}}{2j}\right)$$

$$z(t) = \frac{1}{4j}e^{8\pi t} - \frac{1}{4j}e^{-8\pi t}$$

$\therefore$ The same result as part (c) is obtained.

$$c_2 = \frac{1}{4j},\; c_{-2} = -\frac{1}{4j} \;;\; c_k = 0 \text{ for } k \neq \pm 2$$

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Problem 7. Given that $x(t)$ has the Fourier transform $X(j\omega)$, express the Fourier transforms of the signals listed below in terms of $X(j\omega)$. You may use the Fourier transform properties.

(a) $x_1(t) = x(1-t) + x(-1-t)$

(b) $x_2(t) = x(3t-6)$

(c) $x_3(t) = \frac{d^2 x}{dt^2}(t-1)$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot dt = F\{x(t)\}$$

where $$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t} \cdot d\omega = F^{-1}\{X(j\omega)\}$$

$$x(t) \xleftrightarrow{F.T} X(j\omega)$$

$$x(-t) \xleftrightarrow{F.T} X(-j\omega) \quad (\because \text{reversal property})$$

$$x(-t+1) \xleftrightarrow{F.T} e^{-j\omega_0(-1)} X(-j\omega)$$

$$x(-t+1) \xleftrightarrow{F.T} e^{j\omega_0} X(-j\omega) \text{ — ①} \quad (\because \text{time shifting property})$$

$$x(-t-1) \xleftrightarrow{F.T} e^{j\omega_0(-1)} \cdot X(-j\omega)$$

$$x(-t-1) \xleftrightarrow{F.T} e^{-j\omega_0} \cdot X(-j\omega) \text{ — ②}$$

Applying linearity property.

$$① + ② \Rightarrow x(-t+1) + x(-t-1) = e^{j\omega_0} X(-j\omega) + e^{-j\omega_0} X(-j\omega)$$

$$x(1-t) + x(-1-t) = \left(e^{j\omega_0} + e^{-j\omega_0}\right) X(-j\omega)$$

$$x_1(t) = 2\cos\omega\, X(-j\omega)$$

Problem 7

(b) $x(t) \xleftrightarrow{F.T} \cancel{x(t-2)}\, X(j\omega)$

$x(t-2) \xleftrightarrow{F.T} e^{-j\omega_0 (2)}$ (time shifting property)

$x(t-2) \xleftrightarrow{F.T} e^{-2j\omega}$

$x(at) \xleftrightarrow{F.T} \frac{1}{a} e^{-2j\omega} X\left(\frac{j\omega}{a}\right)$ ($\because$ Time scaling property)

$x\,3(t-2) \xleftrightarrow{F.T} \frac{1}{3} e^{-2j\omega} X\left(\frac{j\omega}{3}\right)$

$x(3t-6) \xleftrightarrow{F.T} \frac{1}{3} e^{-2j\omega} X\left(\frac{j\omega}{3}\right)$

(c) $x(t) \xleftrightarrow{F.T} X(j\omega)$

Applying differentiation property:

$$\frac{dx(t)}{dt} \xleftrightarrow{F.T} j\omega X(j\omega)$$

First using time shifting property

$$x(t-1) \xleftrightarrow{F.T} e^{-j\omega(1)} . X(j\omega)$$

$$x(t-1) \xleftrightarrow{F.T} e^{-j\omega} X(j\omega)$$

$$\frac{dx(t-1)}{dt} \xleftrightarrow{F.T} e^{-j\omega} . j\omega X(j\omega)$$

$$\frac{d}{dt}\left(\frac{dx(t-1)}{dt}\right) \xleftrightarrow{F.T} e^{-j\omega} . j\omega . (j\omega) X(j\omega)$$

$$\frac{d^2 x(t-1)}{dt^2} \xleftrightarrow{F.T} e^{-j\omega} (j^2\omega^2) X(j\omega)$$

$$\frac{d^2 x(t-1)}{dt^2} \xleftrightarrow{F.T} -e^{-j\omega} . \omega^2 X(j\omega)$$

Problem 8. Consider the Fourier transform pair

$$e^{-|t|} \xleftrightarrow{F.T} \frac{2}{1+\omega^2}$$

(a) Use the appropriate Fourier transform properties to find the Fourier transform of $t \cdot e^{-|t|}$.

Applying differentiation property in frequency of F.T

$$t\, e^{-|t|} \xleftrightarrow{F.T} j \cdot \frac{d}{d\omega}\left[\frac{2}{1+\omega^2}\right]$$

$$t\, e^{-|t|} \xleftrightarrow{F.T} j \cdot \left[\frac{-2}{(1+\omega^2)^2} \cdot 2\omega\right]$$

$$t \cdot e^{-|t|} \xleftrightarrow{F.T} \frac{-4j\omega}{(1+\omega^2)^2}$$

(b) Use the above result, along with the duality property, to determine the Fourier transform of

$$\frac{4t}{(1+t^2)^2}$$

$$x(t) = t \cdot e^{-|t|} \xleftrightarrow{F.T} X(j\omega) = \frac{-4j\omega}{(1+\omega^2)^2}$$

$$t \cdot e^{-|t|} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-4j\omega}{(1+\omega^2)^2}\, e^{j\omega t}\, d\omega \quad \text{— ①}$$

$$① \times 2\pi \;\&\; t \to \omega \Rightarrow 2\pi\omega e^{-|\omega|} = \int_{-\infty}^{\infty} \frac{-4jt}{(1+t^2)^2}\, e^{j\omega t}\, dt$$

$$t \to (-t) \Rightarrow 2\pi\omega e^{-|\omega|} = \int_{-\infty}^{\infty} \frac{-4j(-t)}{(1+(-t)^2)^2}\, e^{j\omega(-t)}\, d(-t)$$

$$2\pi\omega e^{-|\omega|} = \int_{-\infty}^{\infty} \frac{-4jt}{(1+t^2)^2}\, e^{-j\omega t}\, dt \quad (\because j^2 = -1)$$

$\times j$

$$j2\pi\omega e^{-|\omega|} = \int_{-\infty}^{\infty} \frac{4t}{(1+t^2)^2}\, e^{-j\omega t}\, dt$$

So, F.T of $\frac{4t}{1+t^2}$ is $j2\pi\omega e^{-|\omega|}$

Problem 9. Let $X(j\omega)$ denote the Fourier transform of the signal $x(t)$ depicted in Figure 1.

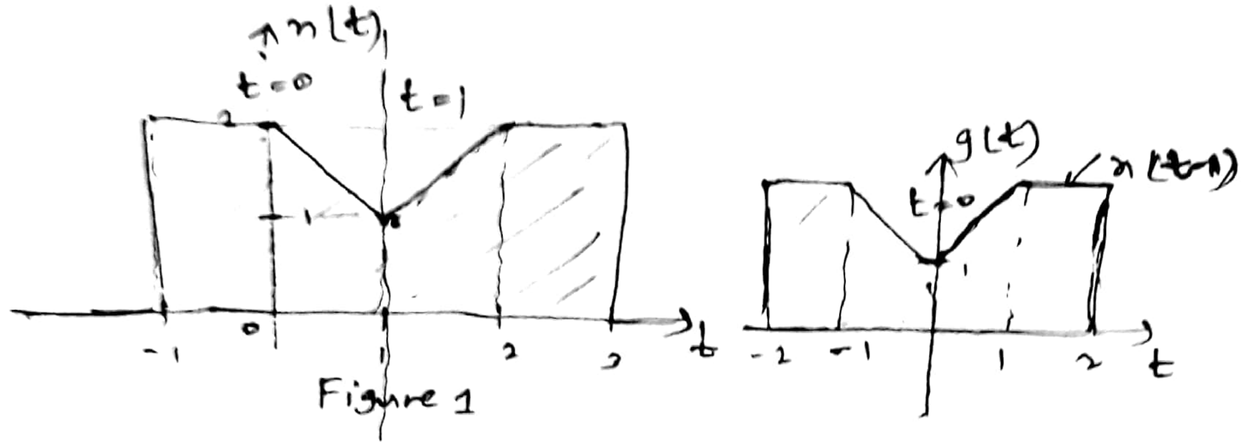


Figure 1

(a) Find $\angle X(j\omega)$.

$$X(j\omega) = A(j\omega)e^{j\theta(j\omega)} \quad ; A(j\omega) \text{ and } \theta(j\omega) \in \text{Real}$$

$$X(j\omega) = |X(j\omega)|e^{j\angle X(j\omega)}$$

$x(t)$ is symmetric about $t=1$

a signal $g(t)$ is symmetric about $t=0$

$$g(t) = x(t+1)$$

$$x(t) = g(t-1)$$

$$X(j\omega) = G(j\omega)e^{-j\omega(1)} = A(j\omega)e^{j\theta(j\omega)}$$

First computing $\angle G(j\omega)$ ($\because G(j\omega)$ is real.)

We are supposed to solve the problem without explicitly evaluating any Fourier Transform.

Here $\angle G(j\omega) = \pm\pi$ ($\because$ magnitude > 0)

$A(j\omega)$ is ~~to be~~ real $\Rightarrow X(j\omega) = G(j\omega)e^{-j\omega(1)}$

$$X(j\omega) = G(j\omega)e^{-j\omega(1)} = A(j\omega)e^{j\theta(\omega)}$$

L.H.S to be equal to R.H.S $\Rightarrow$

$$A(j\omega) = G(j\omega) \text{ and } e^{j\theta(\omega)} = e^{-j\omega(1)} = e^{j(-\omega)}$$

$$\theta(\omega) = (-\omega)$$

$$\angle X(j\omega) = (-\omega).$$

Problem 9

(b) Find $X(j0)$ $\qquad X(jw) = \int_{-\infty}^{\infty} x(t) e^{-jwt} \cdot dt$

$$w=0 \Rightarrow X(j0) = \int_{-\infty}^{\infty} x(t) e^{-jt(0)} \cdot dt = \int_{-\infty}^{\infty} x(t) \cdot 1 \cdot dt$$

$$\int_{-\infty}^{\infty} x(t) \cdot dt = \text{total area under the curve}$$

$$X(j0) = [3-(-1)] \times 2 - \left(\frac{1}{2} \times 2 \times 1\right) = 7$$

$$X(j0) = 7$$

(c) $\int_{-\infty}^{\infty} X(jw) \cdot dw$ $\qquad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(jw) \cdot dw$

$$\because x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(jw) \, dw$$

$$\int_{-\infty}^{\infty} X(jw) \cdot dw = 2\pi \, x(0)$$

$$\int_{-\infty}^{\infty} X(jw) \cdot dw = 2\pi (2) = 4\pi$$

(d) Evaluate $\int_{-\infty}^{\infty} |X(jw)|^2 \cdot dw$

Using Parseval's theorem;

$$\int_{-\infty}^{\infty} |x(t)|^2 \cdot dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(jw)|^2 \cdot dw$$

$$\int_{-\infty}^{\infty} |x(jw)|^2 \cdot dw$$

$$= 2\pi \int_{-\infty}^{\infty} |x(t)|^2 \cdot dt$$

$$= 2\pi \left[\int_{-1}^{0} (2^2) \cdot dt + \int_{0}^{1} (2-t)^2 \cdot dt + \int_{1}^{2} t^2 \, dt + \int_{2}^{3} (2^2) \cdot dt \right]$$

$$= 2\pi \left[4 + \frac{(2-t)^3}{-3} \Big|_0^1 + \frac{t^3}{3} \Big|_1^2 + 4 \right]$$

Problem 9

(d) $\int_{-\infty}^{\infty} |X(j\omega)|^2 \cdot d\omega$

$= 2\pi\left[4 - \left(\frac{1}{3} - \frac{8}{3}\right) + \left(\frac{8}{3} - \frac{1}{3}\right) + 4\right]$

$= 2\pi \times \frac{38}{3}$

$= \frac{76\pi}{3}$

(e) Sketch the inverse Fourier transform of $Re\{X(j\omega)\}$.

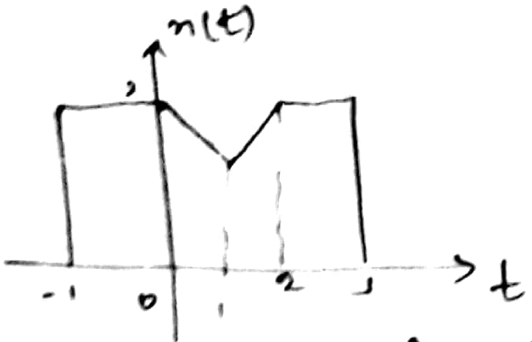
Even part of the signal is the $Re\{X(j\omega)\}$

$Ev\{x(t)\} \Rightarrow Re\{X(j\omega)\}$

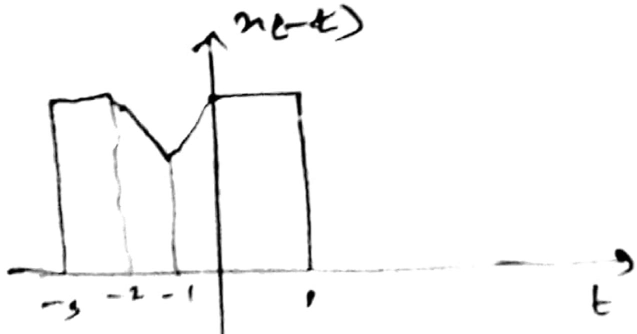
$Re\{(X(j\omega))\}$

$= Ev\{x(t)\}$

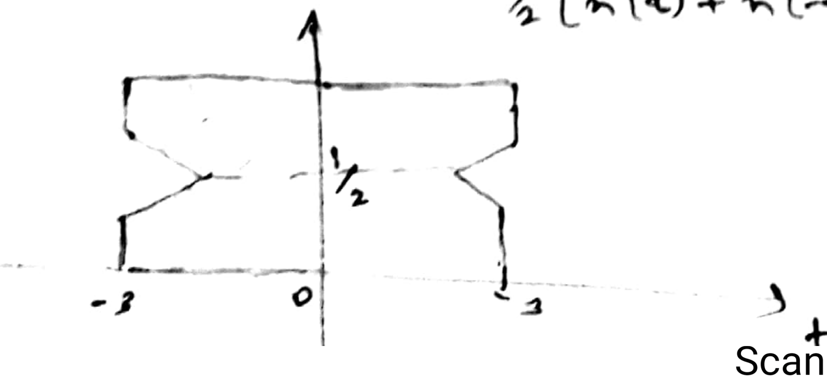
$= \frac{1}{2}[x(t) + x(-t)]$



$\oplus$



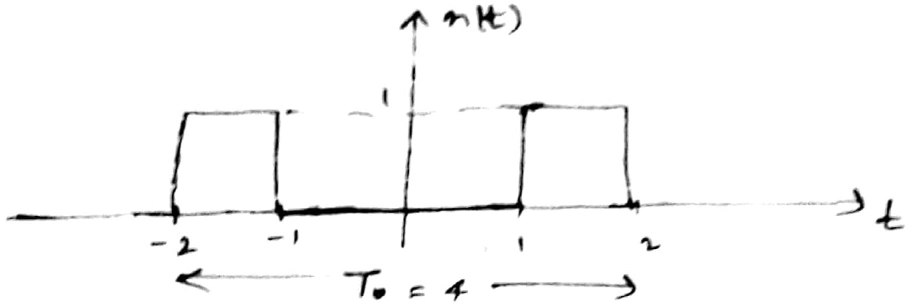
$Ev\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$



Problem 6. Consider the following periodic square wave pulse (symmetric) with a period of 4.

$$x(t) = \begin{cases} 0, & |t| < 1 \quad ; \ -1 < t < 1 \\ 1, & 1 < |t| < 2 \quad ; \ \begin{matrix} 1 < t < 2 \\ -2 < t < -1 \end{matrix} \end{cases}$$

(a) Determine the Fourier series coefficients of $x(t)$.



$$\omega_0 = \frac{2\pi}{T_0}$$

$$\omega_0 = \frac{2\pi}{4} = \pi/2$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

where $a_k = \frac{1}{T_0}\int_{T_0} x(t) \cdot e^{-jk\omega_0 t} \cdot dt$

$$a_k = \frac{1}{4}\left[\int_{-2}^{-1} 1 \cdot e^{-jk(\pi/2)\cdot t}\, dt + \int_{-1}^{1} 0 \cdot e^{-jk(\pi/2)t} \cdot dt + \int_{1}^{2} 1 \cdot e^{-jk(\pi/2)\cdot t} \cdot dt\right]$$

$$a_k = \frac{1}{4}\left[\int_{-2}^{-1} e^{-jk(\pi/2)t} \cdot dt + \int_{1}^{2} e^{-jk(\pi/2)\cdot t} \cdot dt\right]$$

$$a_k = \frac{1}{4}\left[\left.\frac{e^{-jk(\pi/2)\cdot t}}{-jk(\pi/2)}\right|_{-2}^{-1} + \left.\frac{e^{-jk(\pi/2)t}}{-jk(\pi/2)}\right|_{1}^{2}\right]$$

$$a_k = \frac{1}{2\pi jk}\left[e^{\frac{jk\pi}{2}} - e^{jk\pi} + \left(e^{-jk\pi} - e^{-\frac{jk\pi}{2}}\right)\right]$$

$$a_k = \frac{1}{\pi k}\left[\frac{e^{\frac{jk\pi}{2}} - e^{-\frac{jk\pi}{2}}}{2j} + \left(\frac{e^{-jk\pi} - e^{jk\pi}}{2j}\right)\right]$$

$$a_k = \frac{1}{\pi k}\left(\sin\left(\frac{\pi k}{2}\right) - \sin(\pi k)\right) \text{, for all } k$$

$k = 0$, $a_0 = 0$