Stochastic Process

Introduction

Definition. A stochastic process $\{X(t), t \in T\}$ is a collection of random variables. That is, for each $t \in T$, $X(t)$ is a random variable. The index t is often interpreted as time and, as a result, we refer to $X(t)$ as the state of the process at time t.

- The set T is called the index set of the process.

Examples

- Total number of customers that have entered a bank by time t
- The number phone calls occurring in a certain period of time t
- The total amount of sales that have been recorded in the market by time t

1.1 Discrete-time stochastic process

When T is a countable set, the stochastic process is said to be a discrete-time stochastic process.
eg: $\{X_n,\ n = 0, 1, ...\}$ is a discrete-time stochastic process indexed by the nonnegative integers.

Examples

1. The number of occupied channels in a telephone link at the arrival time of the n^{th} customer, $n = 1, 2, ...$
2. The number of packets in the buffer of a statistical multiplexer at the arrival time of the n^{th} customer, $n = 1, 2, ...$

1.2 Continuous-time stochastic process

If T is an interval of the real line, the stochastic process is said to be a continuous-time process.
eg: $\{X(t),\ t \geq 0\}$ is a continuous-time stochastic process indexed by the nonnegative real numbers.
Examples:

1. The number of occupied channels in a telephone link at time $t > 0$
2. The number of packets in the buffer of a statistical multiplexer at time $t > 0$

1.3 The state space of a stochastic process

The state space of a stochastic process is defined as the **set of all possible values** that the random variables $X(t)$ can assume.

- Thus, a stochastic process is a family of random variables that describes the evolution through time of some (physical) process.

1.4 Markov Chains

- Let $\{X_n,\ n = 0, 1, 2, ...,\}$ be a stochastic process that takes on a finite or countable number of possible values.
- Unless otherwise mentioned, this set of possible values of the process will be denoted by the set of nonnegative integers $\{0,\ 1,\ 2,\ .\ .\ .\}$.
- If $X_n = i$, then the process is said to be in state i at time n.
- We suppose that whenever the process is in state i, there is a fixed probability P_{ij} that it will next be in state j. That is, we suppose that

$$P\{X_{n+1} = j | X_n = i,\ X_{n-1} = i_{n-1}, ...,\ X_1 = i_1,\ X_0 = i_0\} = P_{ij} \tag{1.4.1}$$

for all states $i_0,\ i_1,\ .\ .\ .\ ,\ i_{n-1},\ i,\ j$ and all $n \geq 0$. Such a stochastic process is known as a Markov chain.

- For a Markov chain, the conditional distribution of any future state X_{n+1}, given the past states $X_0, X_1,\ .\ .\ .\ , X_{n-1}$ and the present state X_n, is independent of the past states and depends only on the present state, X_n.
- The value P_{ij} represents the probability that the process will, when in state i, next make a transition into state j.
- Since probabilities are nonnegative and since the process must make a transition into some state, we have

$$P_{ij} \geq 0,\ \ i, j \geq 0; \qquad \sum_{j=0}^{\infty} P_{ij} = 1,\ \ i = 0, 1, ...$$

Examples Markov Chains

- A game of snakes and ladders or any other game whose moves are determined entirely by dice is a Markov chain.
- The algorithm Google uses to determine the order of search results, called **PageRank**, is a type of Markov chain.

Let $\boldsymbol{P}$ denote the matrix of one-step transition probabilities P_{ij}, so that

$$\boldsymbol{P} = \begin{Vmatrix} P_{00} & P_{01} & P_{02} & \cdot & \cdot & \cdot \\ P_{10} & P_{11} & P_{12} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & & & \\ P_{i0} & P_{i1} & P_{i2} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & & & \end{Vmatrix}$$

Example 1. (Forecasting the Weather)
Suppose that the chance of rain tomorrow depends on previous weather conditions only through whether or not it is raining today and not on past weather conditions. Suppose also that if it rains today, then it will rain tomorrow with probability α; and if it does not rain today, then it will rain tomorrow with probability β.

Example 2. On any given day Gary is either cheerful (C), so-so (S), or glum (G). If he is cheerful today, then he will be C, S, or G tomorrow with respective probabilities 0.5, 0.4, 0.1. If he is feeling so-so today, then he will be C, S, or G tomorrow with probabilities 0.3, 0.4, 0.3. If he is glum today, then he will be C, S, or G tomorrow with probabilities 0.2, 0.3, 0.5.

Example 3. (Transforming a Process into a Markov Chain)
Suppose that whether or not it rains today depends on previous weather conditions through the last two days. Specifically, suppose that if it has rained for the past two days, then it will rain tomorrow with probability 0.7; if it rained today but not yesterday, then it will rain tomorrow with probability 0.5; if it rained yesterday but not today, then it will rain tomorrow with probability 0.4; if it has not rained in the past two days, then it will rain tomorrow with probability 0.2.

- If we let the state at time n depend only on whether or not it is raining at time n, then the preceding model is not a Markov chain. **(why not?)**
- However, we can transform this model into a Markov chain by saying that the state at any time is determined by the weather conditions during both that day and the previous day.

1.4.1 A Random Walk Model

A Markov chain whose state space is given by the integers $i = 0, \pm 1, \pm 2, ...$ is said to be a random walk if, for some number $0 < p < 1$,

$$P_{i,i+1} = p = 1 - P_{i,i-1}; \quad i = 0, \pm 1, \pm 2, \; ...$$

The preceding Markov chain is called a random walk for we may think of it as being a model for an individual walking on a straight line who at each point of time either takes one step to the right with probability p or one step to the left with probability $1 - p$.
Examples:

1. The wanderings of a drunken man as he walks along a straight line.
2. The winnings of a gambler who on each play of the game either wins or loses one dollar.

1.5 Chapman–Kolmogorov Equations

Define the n-step transition probabilities P_{ij}^n to be the probability that a process in state i will be in state j after n additional transitions. That is,

$$P_{ij}^n = P\left\{X_{n+k} = j | X_k = i\right\}, \quad n \geq 0, \; i, j \geq 0 \tag{1.5.1}$$

$$P_{ij}^1 = P_{ij}$$

The Chapman–Kolmogorov equations provide a method for computing these n-step transition probabilities. These equations are

$$P_{ij}^{n+m} = \sum_{k=0}^{\infty} P_{ik}^n P_{kj}^m \; \text{ for all } n, \; m \geq 0, \;\; \text{all } i, j \tag{1.5.2}$$

$P_{ik}^n P_{kj}^m$ represents the probability that starting in i the process will go to state j in $n + m$ transitions through a path which takes it into state k at the n^{th} transition.
If we let $\boldsymbol{P}^{(n)}$ denote the matrix of n-step transition probabilities P_{ij}^n, then Equation (1.5.1) asserts that

$$\boldsymbol{P^{(n+m)}} = \boldsymbol{P}^{(n)}.\boldsymbol{P}^{(m)}$$

Example 4. $\boldsymbol{P^{(2)}} = \boldsymbol{P}^{(1+1)} = \boldsymbol{P}.\boldsymbol{P} = \boldsymbol{P}^2$
$\boldsymbol{P^{(n)}} = \boldsymbol{P}^{(n-1+1)} = \boldsymbol{P}^{n-1}.\boldsymbol{P} = \boldsymbol{P}^n$

That is, the n-step transition matrix may be obtained by multiplying the matrix P by itself n times.

Example 5. Consider the previous Example in which the weather is considered as a two-state Markov chain. If $\alpha = 0.7$ and $\beta = 0.4$, then calculate the probability that it will rain four days from today given that it is raining today.

Example 6. Consider Example "Transforming a Process into a Markov Chain". Given that it rained on Monday and Tuesday, what is the probability that it will rain on Thursday?

Example 7. A Markov chain $\{X_n, \, n \geq 0\}$ with states 0, 1, 2, has the transition probability matrix $\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$. If $P\left\{X_0 = 0\right\} = P\left\{X_0 = 1\right\} = \frac{1}{4}$. Find $E(X_3)$.

Example 8. Suppose that coin 1 has probability 0.7 of coming up heads, and coin 2 has probability 0.6 of coming up heads. If the coin flipped today comes up heads, then we select coin 1 to flip tomorrow, and if it comes up tails, then we select coin 2 to flip tomorrow.

1. If the coin initially flipped is equally likely to be coin 1 or coin 2, then what is the probability that the coin flipped on the third day after the initial flip is coin 1?
2. Suppose that the coin flipped on Monday comes up heads. What is the probability that the coin flipped on Friday of the same week also comes up heads?

1.6 Classification of States

Accessible State

State j is said to be accessible from state i if $P_{ij}^{n}> 0$ for some $n \geq 0$.

- Note that this implies that state j is accessible from state i if and only if, starting in i, it is possible that the process will ever enter state j.
- This is true since if j is not accessible from i, then
  P{ever enter j| start in i} $= 0$

Communicate State

Two states i and j that are accessible to each other are said to communicate, and we write $i \leftrightarrow j$.
Note that any state communicates with itself since, by definition,

$$P_{ii}^{o} = P\left\{X_0 = i|X_0 = i\right\} = 1$$

The relation of communication satisfies the following three properties:

1. State i communicates with state i, all $i \geq 0$.
2. If state i communicates with state j, then state j communicates with state i.
3. If state i communicates with state j, and state j communicates with state k, then state i communicates with state k.

- Two states that communicate are said to be in the **same class**.
- The concept of communication divides the state space up into a number of separate classes.

Irreducible State

The Markov chain is said to be irreducible if there is only one class, that is, if all states communicate with each other.

Absorbing State

A state i is called absorbing if it is impossible to leave this state. Therefore, the state i is absorbing if and only if

$$P_{ii} = 1 \text{and } P_{ij} = 0 \text{ for } \quad i \neq j$$

If every state can reach an absorbing state, then the Markov chain is an absorbing Markov chain.

Example 9. Consider the Markov chain consisting of the three states 0, 1, 2 and having transition probability matrix

$$\boldsymbol{P} = \begin{Vmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{Vmatrix}$$

Is this Markov chain is irreducible?

Example 10. Consider a Markov chain consisting of the four states 0, 1, 2, 3 and having transition probability matrix

$$\boldsymbol{P} = \begin{Vmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 1 \end{Vmatrix}$$

Determine the classes of this Markov chain and verify whether it is irreducible or not.

Recurrent and Transient States

For any state i we let f_i denote the probability that, starting in state i, the process will ever reenter state i. State i is said to be **recurrent** if $f_i = 1$ and **transient** if $f_i < 1$.

- Suppose that the process starts in state i and i is recurrent. Hence, with probability 1, the process will eventually reenter state i.
- However, by the definition of a Markov chain, it follows that the process will be starting over again when it reenters state i and, therefore, state i will eventually be visited again. Continual repetition of this argument leads to the conclusion that if state i is recurrent then, starting in state i, the process will reenter state i again and again and again—in fact, infinitely often.
- On the other hand, suppose that state i is transient. Hence, each time the process enters state i there will be a positive probability, namely, $1-f_i$, that it will never again enter that state. Therefore, starting in state i, the probability that the process will be in state i for exactly n time periods equals $f_i^{n-1}(1-f_i)$, $n \geq 1$.
- In other words, if state i is transient then, starting in state i, the number of time periods that the process will be in state i has a geometric distribution with finite mean $1/(1-f_i)$.

Corollary. *If state i is recurrent, and state i communicates with state j, then state j is recurrent.*

Example 11. Let the Markov chain consisting of the states 0, 1, 2, 3 have the transition probability matrix

$$\boldsymbol{P} = \begin{Vmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{Vmatrix}$$

Determine which states are transient and which are recurrent.

Example 12. Consider the Markov chain having states 0, 1, 2, 3, 4 and

$$\boldsymbol{P} = \begin{Vmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{2} \end{Vmatrix}$$

Determine the recurrent state.

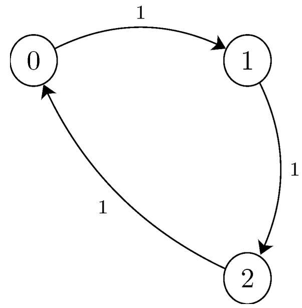
Example 13. Specify the classes of the following Markov chains, and determine whether they are transient or recurrent.

$$(1) \quad \begin{Vmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{Vmatrix} \qquad (2) \quad \begin{Vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{Vmatrix}$$

$$(3) \quad \begin{Vmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{Vmatrix} \qquad (4) \quad \begin{Vmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 1 & 0 & 0 & 0 & 0 \end{Vmatrix}$$

Periodicity

Consider the Markov chain shown in Figure.



- There is a periodic pattern in this chain.
- Starting from state 0, we only return to 0 at times $n = 3, 6, \ldots$.
- In other words, $P_{00}^{(n)} = 0$, if n is not divisible by 3. Such a state is called a periodic state with period $d(0) = 3$.

The period of a state i is the largest integer d satisfying the following property: $P_{ii}^{(n)} = 0$, whenever n is not divisible by d. The period of i is shown by $d(i)$. If $P_{ii}^{(n)} = 0$, for all $n > 0$, then we let $d(i) = \infty$.

- If $d(i) > 1$, we say that state i is periodic.
- If $d(i) = 1$, we say that state i is aperiodic.

We can show that all states in the same communicating class have the same period.

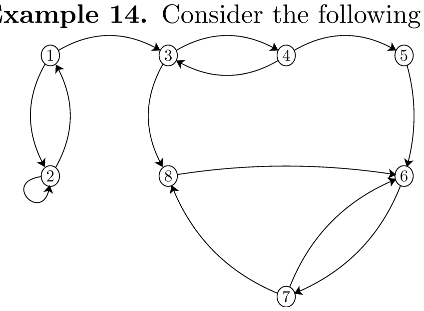
- A class is said to be periodic if its states are periodic.
- Similarly, a class is said to be aperiodic if its states are aperiodic. Finally, a Markov chain is said to be aperiodic if all of its states are aperiodic.

$$\text{If } i \leftrightarrow j, \text{then} \quad d(i) = d(j).$$

Consider a finite irreducible Markov chain X_n:

- If there is a self-transition in the chain ($p_{ii} > 0$ for some i), then the chain is aperiodic.
- Suppose that you can go from state i to state i in l steps, i.e., $P_{ii}^{(l)} > 0$. Also suppose that $P_{ii}^{(m)} > 0$. If $\gcd(l, m)$=1, then state i is aperiodic.
- The chain is aperiodic if and only if there exists a positive integer n such that all elements of the matrix P^n are strictly positive, i.e., $P_{ij}^{(n)} > 0$, for all $i,\ j \in S$.

Example 14. Consider the following Markov chain.



1. Is Class 1=$\{state\,1,\ state\,2\}$ aperiodic?
2. Is Class 2= $\{state\,3,\ state\,4\}$ aperiodic?
3. Is Class 4=$\{state\,6,\ state\,7,\ state\,8\}$ aperiodic?

Positive Recurrent State

If state i is recurrent, then it is said to be positive recurrent if, starting in i, the expected time until the process returns to state i is finite.

- It can be shown that in a finite-state Markov chain, all recurrent states are positive recurrent.

Ergodic State

Positive recurrent, aperiodic states are called ergodic.

Exercise. Consider a Markov chain on Ω= $\{1, 2, 3, 4, 5, 6\}$ specified by the transition probability matrix

$$\boldsymbol{P} = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

1. What are the (communicating) classes of this Markov chain?
2. Is the Markov chain irreducible ?
3. Which states are transient and which are recurrent? Justify your answer.
4. Let X_0 be the initial state with distribution $\pi_0 = (0, \frac{1}{4}, \frac{3}{4}, 0, 0, 0)^T$ corresponding to the probability of being in states 1, 2, 3, 4, 5, 6 respectively. Let X_0, X_1, X_2, . . . be the Markov chain constructed using $\boldsymbol{P}$ above. What is $\mathrm{E}(X_1)$?
5. What is $\mathrm{Var}(X_1)$?