GAMMA AND BETA DISTRIBUTIONS

Beta Function

Definition: If m > 0, n > 0, the integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ is called a beta function and is denoted by β(m, n) e.g.

i) $\int_0^1 \sqrt{x}(1-x)^2\,dx = \int_0^1 x^{\frac{3}{2}-1}(1-x)^{3-1}\,dx = \beta\left(\frac{3}{2},3\right)$

or $\int_0^1 \sqrt{x}(1-x)^2\,dx = \beta\left(\frac{1}{2}+1,2+1\right) = \beta\left(\frac{3}{2},3\right)$

ii) $\int_0^1 x^{-\frac{1}{3}}(1-x)^{-\frac{1}{3}}\,dx = \beta\left(-\frac{1}{3}+1,-\frac{1}{3}+1\right) = \beta\left(\frac{2}{3},\frac{2}{3}\right)$

Properties of Beta Function

1. Beta function is symmetric function i.e. β(m, n) = β(n, m)
2. There are some other forms also of Beta function. One of these forms, which will be helpful in defining beta distribution of second kind, is

$$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

3. (i) $\dfrac{\beta(p,q+1)}{q} = \dfrac{\beta(p+1,q)}{p}$

   (ii) β(p, q) = β(p + q, q) × β(p, q + 1)

On the basis of the above discussion, you can try the following exercise.

E1) Express the following as a beta function:

i) $\int_0^1 x^{-\frac{1}{3}}(1-x)^{\frac{1}{2}}\,dx$

ii) $\int_0^1 x^{-2}(1-x)^5\,dx$

iii) $\int_0^\infty \frac{x^2}{(1+x)^5}\,dx$

iv) $\int_0^\infty \frac{x^{-\frac{1}{2}}}{(1+x)^2}\,dx$

Gamma Function

we are now defining it with more properties, examples and exercises to make you clearly understand this special function.

Definition: If n > 0, the integral $\int_0^\infty x^{n-1}e^{-x}dx$ is called a gamma function and is denoted by $\Gamma(n)$

e.g.

(i) $\int_0^\infty x^2e^{-x}dx = \Gamma(2+1) = \Gamma(3)$

(ii) $\int_0^\infty \sqrt{x}e^{-x}dx = \Gamma\left(\frac{1}{2}+1\right) = \Gamma\left(\frac{3}{2}\right)$

Some Important Results on Gamma Function

1. If n > 1, $\Gamma(n) = (n-1)\Gamma(n-1)$
2. If n is a positive integer, $\Gamma n = (n-1)!$
3. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Relationship between Beta and Gamma Functions

If m > 0, n > 0, then $\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

You can now try the following exercise.

E2) Evaluate:

(i) $\int_0^\infty e^{-x}x^{\frac{5}{2}}dx$

(ii) $\int_0^\infty (1-x)^{10}dx$

(iii) $\int_0^\infty x^{-\frac{1}{2}}e^{-x}dx$

GAMMA DISTRIBUTION

Gamma distribution is a generalisation of exponential distribution. Both the distributions are good models for waiting times. For exponential distribution, the length of time interval between successive happenings is considered i.e. the time is considered till one happening occurs whereas for gamma distribution, the length of time between 0 and the instant when r^{th} happening occurs is considered. So, if r = 1, then the situation becomes the exponential situation. Let us now define gamma distribution:

Definition: A random variable X is said to follow gamma distribution with parameters r > 0 and λ > 0 if its probability density function is given by

$$f(x) = \begin{cases} \dfrac{\lambda^r e^{-\lambda x} x^{r-1}}{\Gamma(r)}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

Remark 1:

(i) It can be verified that

$$\int_0^\infty f(x)\,dx = 1$$

Verification:

$$\int_0^\infty (x)dx = \int_0^\infty \frac{\lambda^r e^{-\lambda x} x^{r-1}}{\Gamma(r)}dx$$

$$= \int_0^\infty \frac{\lambda e^{-\lambda x} (\lambda x)^{r-1}}{\Gamma(r)}dx$$

Putting $\lambda x = y \Rightarrow \lambda dx = dy$

Also, when $x = 0, y = 0$ and when $x \to \infty, y \to \infty$

$$= \frac{1}{\Gamma(r)} \int_0^\infty e^{-y} y^{r-1} dx$$

$$= \frac{1}{\Gamma(r)} \Gamma(r) \qquad \text{[Using gamma function defined]}$$

$= 1$

(ii) If X is a gamma variate with two parameters r > 0 and λ > 0, it is expressed as X ~ γ(λ, r).

(iii) If we put r = 1, we have

$$f(x) = \frac{\lambda e^{-\lambda x} x^0}{\Gamma(1)}, x > 0$$

$$= \lambda e^{-\lambda x}, x > 0$$

which is probability density function of exponential distribution.

Hence, exponential distribution is a particular case of gamma distribution.

(iv) If we put λ = 1, we have

$$f(x) = \frac{e^{-x}.x^{r-1}}{\Gamma r}, x > 0, r > 0$$

It is known as gamma distribution with single parameter r. This form of the gamma distribution is also widely used. If X follows gamma distribution with single parameter r > 0, it is expressed as $X \sim \gamma(r)$.

Mean and Variance of Gamma Distribution

If X has a gamma distribution with parameters r > 0 and $\lambda > 0$, then its

Mean = $\frac{r}{\lambda}$, Variance = $\frac{r}{\lambda^2}$.

If X has a gamma distribution with single parameter r > 0, then its

Mean = Variance = r.

Additive Property of Gamma Distribution

1. If $X_1, X_2, \ldots, X_k$ are independent gamma variates with parameters $(\lambda, r_1), (\lambda, r_2), \ldots, (\lambda, r_k)$ respectively, then $X_1 + X_2 + \ldots + X_k$ is also a gamma variate with parameter $(\lambda, r_1 + r_2 + \ldots + r_k)$.
2. If $X_1, X_2, \ldots, X_k$ are independent gamma variates with single parameters $r_1, r_2, \ldots, r_k$ respectively, then $X_1 + X_2 + \ldots, + X_k$ is also a gamma variate with parameter $r_1 + r_2 + \ldots + r_k$.

Example 1: Suppose that on an average 1 customer per minute arrive at a shop. What is the probability that the shopkeeper will wait more than 5 minutes before

(i) both of the first two customers arrive, and

(ii) the first customer arrive?

Assume that waiting times follows gamma distribution.

Solution:

i) Let X denotes the waiting time in minutes until the second customer arrives, then X has gamma distribution with r = 2 (as the waiting time is to be considered up to 2nd customer)

λ = 1 customer per minute.

$$\therefore\ P[X > 5] = \int_5^\infty f(x)dx = \int_5^\infty \frac{\lambda^r e^{-\lambda x} x^{r-1}}{\Gamma(r)} dx$$

$$= \int_5^\infty \frac{(1)^2 e^{-x} x^{2-1}}{\Gamma(2)} dx = \int_5^\infty \frac{e^{-x} x^1}{\lfloor 1} dx = \int_5^\infty x^1 e^{-x} dx$$

$$= \left[\left\{ x \frac{e^{-x}}{-1} \right\}_5^\infty - \int_5^\infty (1) \frac{e^{-x}}{-1} dx \right] \qquad \text{[Integrating by parts]}$$

$$= \left(0 + 5e^{-5}\right) + \int_5^\infty e^{-x} dx = 5e^{-5} + \left[\frac{e^{-x}}{-1}\right]_5^\infty$$

$$= 5e^{-5} - \left(0 - e^{-5}\right)$$

$$= 6e^{-5}$$

$$= 6 \times 0.0070$$

$$= 0.042$$

ii) In this case r = 1, λ = 1 and hence

$$P\left[X > 5\right] = \int_5^{\infty} \frac{\lambda^r e^{-\lambda x} . x^{r-1}}{\overline{|}(r)} dx$$

$$= \int_5^{\infty} \frac{(1)^1 e^{-x} x^0}{\overline{|}(1)} dx = \int_5^{\infty} e^{-x} dx = \left[\frac{e^{-x}}{-1}\right]_5^{\infty} = 0 + e^{-5} = 0.0070$$

Alternatively,

As r = 1, so it is a case of exponential distribution for which

$$f\left(x\right) = \lambda e^{-\lambda x}, x > 0$$

$$\therefore P\left[X > 5\right] = \int_5^{\infty} \lambda e^{-\lambda x} dx = \int_5^{\infty} (1) e^{-x} dx = \left[\frac{e^{-x}}{-1}\right]_5^{\infty} = 0 + e^{-5} = 0.0070$$

Here is an exercise for you.

E3) Telephone calls arrive at a switchboard at an average rate of 2 per minute. Let X denotes the waiting time in minutes until the 4^{th} call arrives and follows gamma distribution. Write the probability density function of X. Also find its mean and variance.

Let us now discuss the beta distributions in the next two sections:

BETA DISTRIBUTION OF FIRST KIND

You have studied that beta function is related to gamma function in the following manner:

$$\beta\left(m, n\right) = \frac{\overline{|}(m) \overline{|}(n)}{\overline{|}(m+n)}$$

Now, we are in a position to define beta distribution which is defined with the help of beta function. There are two kinds of beta distribution – beta distribution of first kind and beta distribution of second kind. Beta distribution of second kind is defined in next section of the unit whereas beta distribution of first kind is defined as follows:

Definition: A random variable X is said to follow beta distribution of first kind with parameters m > 0 and n > 0, if its probability density function is given by

$$f(x) = \begin{cases} \dfrac{1}{\beta\left(m, n\right)} x^{m-1} \left(1 - x\right)^{n-1}, 0 < x < 1 \\ 0, \qquad \text{otherwise} \end{cases}$$

The random variable X is known as beta variate of first kind and can be expressed as $X \sim \beta_1(m, n)$

Remark 5: If m = 1 and n = 1, then the beta distribution reduces to

$$f(x) = \frac{1}{\beta(1,1)} x^{1-1}(1-x)^{1-1},\ 0 < x < 1$$

$$= \frac{x^0(1-x)^0}{\beta(1,1)},\ 0 < x < 1$$

$$= \frac{1}{\beta(1,1)}, 0 < x < 1$$

But $\beta(1,1) = \frac{\Gamma 1 \Gamma 1}{\Gamma 2} = \frac{\lfloor 0 \lfloor 0}{\lfloor 1}$

Therefore, $f(x) = \frac{(1)(1)}{(1)} = 1$

$\therefore f(x) = 1,\ 0 < x < 1$

$$= \frac{1}{1-0}, 0 < x < 1$$

which is uniform distribution on (0, 1).

[$\because$ p.d.f. of uniform distribution on (a, b) is $f(x) = \frac{1}{b-a}$, $a < x < b$]

So, continuous uniform distribution is a particular case of beta distribution.

Mean and variance of Beta Distribution of First Kind

Mean and Variance of this distribution are given as

Mean = $\frac{m}{m+n}$

Variance = $\frac{mn}{(m+n)^2(m+n+1)}$

Example 4: Determine the constant C such that the function $f(x) = Cx^3(1-x)^6, 0 < x < 1$ is a beta distribution of first kind. Also, find its mean and variance.

Solution: As $f(x)$ is a beta distribution of first kind.

$$\therefore \int_0^1 f(x) = 1$$

$$\Rightarrow \int_0^1 Cx^3(1-x)^6\,dx = 1$$

$$\Rightarrow C\int_0^1 x^3(1-x)^6\,dx = 1$$

$\Rightarrow \; C\beta(3+1, 6+1) = 1$ [By definition of Beta distribution of first kind]

$$\Rightarrow \; C = \frac{1}{\beta(4,7)}$$

$$= \frac{\Gamma(4+7)}{\Gamma(4)\Gamma(7)} \qquad \left[\because \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right]$$

$$= \frac{\Gamma(11)}{\Gamma(4)\Gamma(7)} = \frac{\lfloor 10}{\lfloor 3 \lfloor 6}$$

$$= \frac{10 \times 9 \times 8 \times 7 \times \lfloor 6}{3 \times 2 \times \lfloor 6} = 840$$

Thus, $f(x) = 840x^3(1-x)^6$

$$= 840x^{4-1}(1-x)^{7-1}$$

$$= \frac{x^{4-1}(1-x)^{7-1}}{\beta(4,7)}$$

$\left[\because \frac{1}{\beta(4,7)} = 840\right.$ just obtained above in this example]

$\therefore \quad m = 4, n = 7$

$\Rightarrow$ Mean $= \dfrac{m}{m+n} = \dfrac{4}{4+7} = \dfrac{4}{11}$,

and Variance $= \dfrac{mn}{(m+n)^2(m+n+1)}$

$$= \frac{4 \times 7}{(4+7)^2(4+7+1)}$$

$$= \frac{28}{(121)(12)} = \frac{7}{(121)(3)} = \frac{7}{363}$$

Now, you can try the following exercises.

E4) Using beta function, prove that

$$\int_0^1 60x^2(1-x)^3\,dx = 1$$

E5) Determine the constant k such that the function

$f(x) = kx^{-\frac{1}{2}}(1-x)^{\frac{1}{2}}, 0 < x < 1,$ is a beta distribution of first kind. Also find its mean and variance.

BETA DISTRIBUTION OF SECOND KIND

Let us now define beta distribution of second kind.

Definition: A random variable X is said to follow beta distribution of second kind with parameters m > 0, n > 0 if its probability density function is given by

$$f(x) = \begin{cases} \dfrac{x^{m-1}}{\beta(m,n)(1+x)^{m+n}}, & 0 < x < \infty \\ 0, & \text{elsewhere} \end{cases}$$

Remark 6: It can be verified that $\int_0^\infty \frac{x^{m-1}}{\beta(m,n)(1+x)^{m+n}} dx = 1$

Verification:

$$\int_0^\infty \frac{x^{m-1}}{\beta(m,n)(1+x)^{m+n}} dx = \frac{1}{\beta(m,n)} \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

$$= \frac{1}{\beta(m,n)} \beta(m,n) \quad \left[\because \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx \text{ is another form of beta function. (see Sec. 16.2 of this Unit)} \right]$$

$= 1$

Remark 7: If X is a beta variate of second kind with parameters m > 0, n > 0, then it is expressed as $X \sim \beta_2(m, n)$

Mean and Variance of beta Distribution of second kind

Mean = $\dfrac{m}{n-1}, n > 1;$

Variance = $\dfrac{m(m+n-1)}{(n-1)^2(n-2)}, n > 2$

Example 5: Determine the constant k such that the function

$$f(x) = \frac{kx^3}{(1+x)^7}, 0 < x < \infty,$$

is the p.d.f of beta distribution of second kind. Also find its mean and variance.

Solution: As $f(x)$ is a beta distribution of second kind,

$$\therefore \int_0^\infty f(x) dx = 1$$

$$\Rightarrow \int_0^\infty \frac{kx^3}{(1+x)^7}dx = 1$$

$$\Rightarrow k\int_0^\infty \frac{x^{4-1}}{(1+x)^{4+3}}dx = 1$$

$$\Rightarrow k\beta(4,3) = 1$$

$$\Rightarrow k = \frac{1}{\beta(4,3)} = \frac{\overline{|7}}{\overline{|4}\,\overline{|3}} = \frac{\underline{|6}}{\underline{|3}\,\underline{|2}} = \frac{6\times5\times4}{2} = 60$$

Here m = 4, n = 3

$$\therefore \text{ Mean } = \frac{m}{n-1} = \frac{4}{3-1} = \frac{4}{2} = 2$$

$$\text{Variance } = \frac{m(m+n-1)}{(n-1)^2(n-2)} = \frac{4(4+3-1)}{(3-1)^2(3-2)} = \frac{4\times6}{4\times1} = 6$$

Now, you can try the following exercises.

E6) Using beta function, prove that

$$\int_0^\infty \frac{x^3}{(1+x)^{\frac{13}{2}}}dx = \frac{64}{15015}$$

E7) Obtain mean and variance for the beta distribution whose density is given by

$$f(x) = \frac{60x^2}{(1+x)^7}, 0 < x < \infty$$

SUMMARY

The following main points have been covered in this unit:

1) A random variable X is said to follow **gamma distribution with parameters $r > 0$ and $\lambda > 0$** if its probability density function is given by

$$f(x) = \begin{cases} \dfrac{\lambda^r e^{-\lambda x} x^{r-1}}{\overline{|(r)}}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

2) **Gamma distribution** of random variable X **with single parameter** $r > 0$ is defined as $f(x) = \frac{e^{-x}x^{r-1}}{\overline{|r}}, x > 0, r > 0$

3) For **gamma distribution** with two parameters λ and r, **Mean** = $\frac{r}{\lambda}$ and **Variance** = $\frac{r}{\lambda^2}$.

4) A random variable X is said to follow **beta distribution of first kind** with parameters m > 0 and n > 0, if its probability density function is given by

$$f(x) = \begin{cases} \dfrac{1}{\beta(m,n)} x^{m-1}(1-x)^{n-1}, 0 < x < 1 \\ 0, \qquad \text{otherwise} \end{cases}$$

Its **mean and variance** are $\dfrac{m}{m+n}$ and $\dfrac{mn}{(m+n)^2(m+n+1)}$, respectively.

5) A random variable X is said to follow **beta distribution of second kind** with parameters m > 0, n > 0 if its probability density function is given by:

$$f(x) = \begin{cases} \dfrac{x^{m-1}}{\beta(m,n)(1+x)^{m+n}}, 0 < x < \infty \\ 0, \qquad \text{elsewhere} \end{cases}$$

Its **Mean and Variance** are $\dfrac{m}{n-1}, n > 1;$ and $\dfrac{m(m+n-1)}{(n-1)^2(n-2)}, n > 2$

respectively.

6) Exponential distribution is a particular case of gamma distribution and continuous uniform distribution is a particular case of beta distribution.