Poisson Process

The Poisson process is one of the most widely-used counting processes. It is usually used in scenarios where we are counting the occurrences of certain events that appear to happen at a certain rate, but completely at random (without a certain structure). For example, suppose that from historical data, we know that earthquakes occur in a certain area with a rate of 2 per month. Other than this information, the timings of earthquakes seem to be completely random. Thus, we conclude that the Poisson process might be a good model for earthquakes. In practice, the Poisson process or its extensions have been used to model

– the number of car accidents at a site or in an area;
– the location of users in a wireless network;
– the requests for individual documents on a web server;
– the outbreak of wars;
– photons landing on a photodiode.

Poisson random variable: Here, we briefly review some properties of the Poisson random variable that you have discussed in the previous modules. Remember that a discrete random variable X is said to be a *Poisson* random variable with parameter μ, shown as $X \sim Poisson(\mu)$, if its range is $R_X = \{0, 1, 2, 3, \dots\}$, and its PMF is given by

$$P_X(k) = \begin{cases} \frac{e^{-\mu}\mu^k}{k!} & \text{for } k \in R_X \\ 0 & \text{otherwise} \end{cases}$$

Here are some useful facts that we have seen before:

1. If $X \sim Poisson(\mu)$, then $EX = \mu$, and $\mathrm{Var}(X) = \mu$.
2. If $X_i \sim Poisson(\mu_i)$, for $i = 1, 2, \cdots, n$, and the X_i's are independent, then

$$X_1 + X_2 + \cdots + X_n \sim Poisson(\mu_1 + \mu_2 + \cdots + \mu_n).$$

3. The Poisson distribution can be viewed as the limit of binomial distribution.

Definition of the Poisson Process:

The above construction can be made mathematically rigorous. The resulting random process is called a Poisson process with rate (or intensity) λ. Here is a formal definition of the Poisson process.

The Poisson Process

Let $\lambda > 0$ be fixed. The counting process $\{N(t), t \in [0, \infty)\}$ is called a **Poisson process** with **rate**s λ if all the following conditions hold:

1. $N(0) = 0$;
2. $N(t)$ has independent increments;
3. the number of arrivals in any interval of length $\tau > 0$ has $Poisson(\lambda\tau)$ distribution.

Note that from the above definition, we conclude that in a Poisson process, the distribution of the number of arrivals in any interval depends only on the length of the interval, and not on the exact location of the interval on the real line. Therefore the *Poisson process has stationary increments*.

Example 01

The number of customers arriving at a grocery store can be modeled by a Poisson process with intensity $\lambda = 10$ customers per hour.

1. Find the probability that there are 2 customers between 10:00 and 10:20.
2. Find the probability that there are 3 customers between 10:00 and 10:20 and 7 customers between 10:20 and 11.

Example 02

Let $\{N(t), t \in [0, \infty)\}$ be a Poisson process with rate $\lambda = 0.5$.

a. Find the probability of no arrivals in $(3, 5]$.
b. Find the probability that there is exactly one arrival in each of the following intervals: $(0, 1]$, $(1, 2]$, $(2, 3]$, and $(3, 4]$.