See discussions, stats, and author profiles for this publication at: https://www.researchgate.net/publication/227389342

A First Course in Optimization Theory

Book · Ja	anuary 1996			
DOI: 10.1017	//CBO9780511804526 · Source: RePEc			
CITATIONS		READS		
296		16,364		
		,		
1 author	:			
	Rangarajan K. Sundaram			
	New York University			
	72 PUBLICATIONS 5,761 CITATIONS			
	SEE PROFILE			
Sama of	Constitution of the control of the c			
Some of	Some of the authors of this publication are also working on these related projects:			



A FIRST COURSE IN OPTIMIZATION THEORY

RANGARAJAN K. SUNDARAM

New York University



Contents

Preface page			page xiii	
Ac	kno	wledgen	nents	xvii
1	Ma	thematic	cal Preliminaries	1
	1.1	Notatio	on and Preliminary Definitions	2
		1.1.1	Integers, Rationals, Reals, \mathbb{R}^n	2
		1.1.2	Inner Product, Norm, Metric	4
	1.2	Sets ar	nd Sequences in \mathbb{R}^n	7
		1.2.1	Sequences and Limits	7
		1.2.2	Subsequences and Limit Points	10
		1.2.3	Cauchy Sequences and Completeness	11
		1.2.4	Suprema, Infima, Maxima, Minima	14
		1.2.5	Monotone, Sequences in ℝ	17
		1.2.6	The Lim Sup and Lim Inf	18
		1.2.7	Open Balls, Open Sets, Closed Sets	22
		1.2.8	Bounded Sets and Compact Sets	23
		1.2.9	Convex Combinations and Convex Sets	23
		1.2.10	Unions, Intersections, and Other Binary Operations	. 24
	1.3	Matric	es	30
		1.3.1	Sum, Product, Transpose	30
		1.3.2	Some Important Classes of Matrices	32
			Rank of a Matrix	33
			The Determinant	35
		1.3.5	The Inverse	38
		1.3.6	Calculating the Determinant	39
	1.4	Functi		41
		1.4.1	Continuous Functions	41
		1.4.2	Differentiable and Continuously Differentiable Functions	43

viii Contents

		1.4.3 Partial Derivatives and Differentiability		46
		1.4.4 Directional Derivatives and Differentiability		48
		1.4.5 Higher Order Derivatives		49
	1.5	Quadratic Forms: Definite and Semidefinite Matrices		50
		1.5.1 Quadratic Forms and Definiteness		50
		1.5.2 Identifying Definiteness and Semidefiniteness		53
	1.6	Some Important Results		55
		1.6.1 Separation Theorems		56
		1.6.2 The Intermediate and Mean Value Theorems		60
		1.6.3 The Inverse and Implicit Function Theorems		65
	1.7	Exercises		66
2	Opt	imization in \mathbb{R}^n		74
	2.1	Optimization Problems in \mathbb{R}^n		74
	2.2	Optimization Problems in Parametric Form		77
	2.3	Optimization Problems: Some Examples		78
		2.3.1 Utility Maximization		78
		2.3.2 Expenditure Minimization		79
		2.3.3 Profit Maximization		80
		2.3.4 Cost Minimization		80
		2.3.5 Consumption-Leisure Choice		81
		2.3.6 Portfolio Choice		81
		2.3.7 Identifying Pareto Optima		82
		2,3.8 Optimal Provision of Public Goods		83
		2.3.9 Optimal Commodity Taxation		84
	2.4	Objectives of Optimization Theory		85
	2.5	A Roadmap		86
	2.6	Exercises		88
3	Exi	stence of Solutions: The Weierstrass Theorem		90
	3.1	The Weierstrass Theorem		90
	3.2	The Weierstrass Theorem in Applications		92
	3.3	A Proof of the Weierstrass Theorem	,	96
	3.4	Exercises		97
4	Unc	onstrained Optima		100
7	4.1 "Unconstrained" Optima		~.	100
		First-Order Conditions		101
		Second-Order Conditions		103
		Using the First- and Second-Order Conditions		104
		O		

	Contents	ix
4.5	A Proof of the First-Order Conditions	106
4.6	A Proof of the Second-Order Conditions	108
4.7	Exercises	110
Equ	ality Constraints and the Theorem of Lagrange	112
5.1	Constrained Optimization Problems	112
5.2	Equality Constraints and the Theorem of Lagrange	113
	5.2.1 Statement of the Theorem	114
	5.2.2 The Constraint Qualification	115
	5.2.3 The Lagrangean Multipliers	116
5.3	Second-Order Conditions	117
5.4	Using the Theorem of Lagrange	121
	5.4.1 A "Cookbook" Procedure	121
	5.4.2 Why the Procedure Usually Works	122
	5.4.3 When It Could Fail	123
	5.4.4 A Numerical Example	127
5.5	Two Examples from Economics	128
D	5.5.1 An Illustration from Consumer Theory	128
	5.5.2 An Illustration from Producer Theory	130
	5.5.3 Remarks	132
5.6	A Proof of the Theorem of Lagrange	135
	A Proof of the Second-Order Conditions	137
	Exercises	142
	quality Constraints and the Theorem of Kuhn and Tucker	145
6.1	The Theorem of Kuhn and Tucker	145
	6.1.1 Statement of the Theorem	145
	6.1.2 The Constraint Qualification	147
	6.1.3 The Kuhn-Tucker Multipliers	148
6.2	Using the Theorem of Kuhn and Tucker	150
	6.2.1 A "Cookbook" Procedure	150
	6.2.2 Why the Procedure Usually Works	151
	6.2.3 When It Could Fail	152
	6.2.4 A Numerical Example	155
6.3	Illustrations from Economics	157
	6.3.1 An Illustration from Consumer Theory	158
	6.3.2 An Illustration from Producer Theory	161
6.4	The General Case: Mixed Constraints	164
6.5	A Proof of the Theorem of Kuhn and Tucker	165
6.6	Exercises	168

x Contents

7 、	Cor	nvex Structures in Optimization Theory	172
	7.1	Convexity Defined	173
		7.1.1 Concave and Convex Functions	174
		7.1.2 Strictly Concave and Strictly Convex Functions	176
	7.2	Implications of Convexity	177
		7.2.1 Convexity and Continuity	177
		7.2.2 Convexity and Differentiability	179
		7.2.3 Convexity and the Properties of the Derivative	183
	7.3	Convexity and Optimization	185
		7.3.1 Some General Observations	185
		7.3.2 Convexity and Unconstrained Optimization	187
		7.3.3 Convexity and the Theorem of Kuhn and Tucker	187
	7.4	Using Convexity in Optimization	189
	7.5	A Proof of the First-Derivative Characterization of Convexity	190
	7.6	A Proof of the Second-Derivative Characterization of Convexity	191
	7.7	A Proof of the Theorem of Kuhn and Tucker under Convexity	194
	7.8	Exercises	198
8	Qua	asi-Convexity and Optimization	203
	8.1	Quasi-Concave and Quasi-Convex Functions	204
	8.2	Quasi-Convexity as a Generalization of Convexity	205
	8.3	Implications of Quasi-Convexity	209
,	8.4	Quasi-Convexity and Optimization	213
	8.5	Using Quasi-Convexity in Optimization Problems	215
	8.6	A Proof of the First-Derivative Characterization of Quasi-Convexity	216
	8.7	A Proof of the Second-Derivative Characterization of	
		Quasi-Convexity	217
	8.8	A Proof of the Theorem of Kuhn and Tucker under Quasi-Convexity	220
	8.9	Exercises	221
9	Parametric Continuity: The Maximum Theorem 22		
	9.1	Correspondences	225
		9.1.1 Upper- and Lower-Semicontinuous Correspondences	225
		9.1.2 Additional Definitions	228
		9.1.3 A Characterization of Semicontinuous Correspondences	229
		9.1.4 Semicontinuous Functions and Semicontinuous	
		Correspondences	233
	9.2	Parametric Continuity: The Maximum Theorem	235
		9.2.1 The Maximum Theorem	235
		9.2.2 The Maximum Theorem under Convexity	237

xi

	9.3	An Application to Consumer Theory	240
		9.3.1 Continuity of the Budget Correspondence	240
		9.3.2 The Indirect Utility Function and Demand	
		Correspondence	242
	9.4	An Application to Nash Equilibrium	243
		9.4.1 Normal-Form Games	243
		9.4.2 The Brouwer/Kakutani Fixed Point Theorem	244
		9.4.3 Existence of Nash Equilibrium	240
	9.5	Exercises	24
10	Supe	ermodularity and Parametric Monotonicity	253
	10.1	Lattices and Supermodularity	254
		10.1.1 Lattices	254
		10.1.2 Supermodularity and Increasing Differences	255
	10.2	Parametric Monotonicity	258
	10.3	An Application to Supermodular Games	262
		10.3.1 Supermodular Games	262
		10.3.2 The Tarski Fixed Point Theorem	263
		10.3.3 Existence of Nash Equilibrium	263
	10.4	A Proof of the Second-Derivative Characterization of	
		Supermodularity	264
	10.5	Exercises	260
11	Finite-Horizon Dynamic Programming		268
		Dynamic Programming Problems	268
		Finite-Horizon/Dynamic Programming	268
	11.3	Histories, Strategies, and the Value Function	269
		Markovian Strategies	
		5 Existence of an Optimal Strategy	
	11.6	6 An Example: The Consumption–Savings Problem	
	11.7	Exercises	278
12	Stationary Discounted Dynamic Programming		283
	12.1	Description of the Framework	283
	12.2	Histories, Strategies, and the Value Function	282
	12.3	The Bellman Equation	283
	12.4	A Technical Digression	286
		12.4.1 Complete Metric Spaces and Cauchy Sequences	286
		12.4.2 Contraction Mappings	28
		12.4.3 Uniform Convergence	289

xii Contents

Index

12.5 Existe	ence of an Optimal Strategy	291
	A Preliminary Result	292
12.5.2	2 Stationary Strategies	294
12.5.3	B Existence of an Optimal Strategy	295
	ample: The Optimal Growth Model	298
	The Model	299
12.6.2	2 Existence of Optimal Strategies	300
12.6.3	3 Characterization of Optimal Strategies	301
12.7 Exerci	ises	309
Appendix A Se	et Theory and Logic: An Introduction	315
A.1 Sets, U	Jnions, Intersections	315
A.2 Propos	sitions: Contrapositives and Converses	316
-	ifiers and Negation	318
A.4 Necess	sary and Sufficient Conditions	320
Appendix B T	he Real Line	323
B.1 Constr	ruction of the Real Line	323
B.2 Proper	rties of the Real Line	326
Appendix C St	tructures on Vector Spaces	330
C.1 Vector	Spaces	330
C.2 Inner l	Product Spaces	332
C.3 Norme	ed Spaces	333
C.4 / Metric		336
/ C.4.1	Definitions	336
C.4.2	Sets and Sequences in Metric Spaces	. 337
C.4.3	Continuous Functions on Metric Spaces	339
C.4.4	Separable Metric Spaces	340
C.4.5	Subspaces	341
C.5 Topolo	ogical Spaces	342
C.5.1	Definitions	342
C.5.2	Sets and Sequences in Topological Spaces	343
C.5.3	Continuous Functions on Topological Spaces	343
C.5.4	· · · · · · · · · · · · · · · · · · ·	343
C.6 Exerci	ises	345
Ribliography		240

351