Probability & Statistics

1. The number of flaws in a 1-inch length of copper wire manufactured by a certain process varies from wire to wire. Overall, 48% of the wires produced have no flaws, 39% have one flaw, 12% have two flaws, and 1% have three flaws. Let X be the number of flaws in a randomly selected piece of wire.

   (a) Identify the pmf of X.

   (b) Compute F (0), F (1), F(2) and F(3).

   (c) Compute F(1.25) and F(3.5).

2. A discrete random variable X has a probability function, as shown in the table below, where a and b are constants. given that $E(x) = 1.5$.

X	0	1	2	3
$P(X = x)$	0.3	a	b	0.2

   (a) Find the value of a and the value of b.

   (b) Find,

      i. P($0 < X < 1.5$)

      ii. E (3X-3)

      iii. Show that $Var(X) = 1.25$

      iv. Var(3X-3)

3. A company manufactures electronic components, and each component is tested for defects. Let X be a discrete random variable representing the quality rating of a randomly selected component based on a testing process. The probability mass function of X is given by:

$$P(X = x) = \begin{cases} k(2-x), & x = 0, 1, 2 \\ k(x-2), & x = 3 \\ 0, & \text{otherwise} \end{cases}$$

   Where k is a positive constant.

   (a) Determine the value of k.

   (b) Compute the variance of a modified rating system, where the final adjusted score is given by $Y = 3X - 2$.
   Two independent components, X_1 and X_2, are randomly selected from the production line for quality testing.

   (c) Prove that the probability of their total quality score being 5 is 0.

   (d) Find the probability that the total quality score of two randomly selected components falls between 1.3 and 3.2.

4. Measurements of scientific systems are always subject to variation, some more than others. There are many structures for measurement error, and statisticians spend a great deal of time modeling these errors. Suppose the measurement error X of a certain physical quantity is decided by the density function.

$$f(x) = \begin{cases} k(3 - x^2), & \text{-1} \leq x \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

   (a) Determine k that renders f(x) a valid density function.
   (b) Find the probability that a random error in the measurement is less than $\frac{1}{2}$.
   (c) For this particular measurement, it is undesirable if the magnitude of the error (i.e., $|x|$) exceeds 0.8. What is the probability that this occurs?

5. Suppose that three batteries are randomly chosen from a group of 3 new, 4 used but still working, and 5 defective batteries. If we let X and Y, denote the number of new and used but still working batteries that are chosen, respectively, find

   (a) The joint probability mass distribution of X and Y.
   (b) The probability of at least 2 defective batteries was chosen.
   (c) The marginal probability distribution of X.

6. Let X and Y have the joint probability density function given by

$$f(x, y) = \begin{cases} kxy, & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

   (a) Find the value of k that makes a probability density function.
   (b) Find $P(X \leq \frac{1}{2}, Y \leq \frac{3}{4})$

7. If the joint probability density of X and Y is given by

$$f(x, y) = \begin{cases} 2e^{-x}e^{-2y}, & 0 < x, 0 < y \\ 0, & \text{elsewhere} \end{cases}$$

   Find,

   (a) The marginal distribution functions of X and Y.
   (b) The conditional densities, $f_{(X|Y)}(x|y)$ and $f_{(Y|X)}(y|x)$.

8. A hospital diagnostic center has two sequential procedures for patients. Let X and Y denote the time(hour) taken to complete patient registration and the total time (hour) a patient spends in the diagnostic process. The joint probability density function of X and Y is given by,

$$f(x,y) = \begin{cases} 6(1-y), & 0 \le x \le y \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

   (a) Verify that is a valid joint probability density function.
   (b) Find the marginal density function of the registration time.
   (c) Compute the probability that a patient completes the diagnostic process within 0.5 hours, given that registration took at most 0.75 hours.
   (d) Find the marginal probability density function of total diagnostic time given the registration time.
   (e) Find the probability that the diagnostic process takes at least 0.75 hours, given that the registration took at least 0.5 hours.

Multivariate Calulus & PDE

1. Consider the function

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

   (a) Is f continuous at $(0,0)$?
   (b) Find $f_x(0,0)$ and $f_y(0,0)$.
   (c) Is f differentiable at the origin?

2. Consider the function

$$f(x,y) = \begin{cases} \frac{2x^3y}{x^2+2y^2}\cos(xy) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

   (a) Find the partial derivative of f at the point $(0,0)$
   (b) Prove that f is continuous on all $\mathbb{R}^2$. (Hint: Note that for $(x,y) \neq (0,0)$, we have that

$$\frac{1}{x^2+2y^2} \le \frac{1}{x^2+y^2}$$

   (c) Is f differentiable at $(0,0)$?

3. Prove that if the partial derivatives f_x and f_y exist in some neighborhood of the point $(a,b) \in \mathbb{R}^2$ and they are continuous at (a,b), then f is differentiable at (a,b).

4. Discuss the differentiability of the following functions.
   (a) $f(x, y) = \sqrt{x^2 + y^2}$
   (b) $g(x, y) = \log(1 - x^2 - y^2)$
5. Show that $f(x, y) = 5 + x \ln(xy - 5)$ is differentiable at (2,3).
6. Find the cartesian equation of the tangent plane to the surface

$$z = 1 - x^2 - y^2$$

   at the point $(1, 2, -4)$.