EN1020 - Circuits, Signals, and Systems

1. Signals and Systems

Definition 1. *Signals*

Signals are represented as functions of independent varaible(s). We generally refer to the independent variable as time.

There are two basic types of signals;

1. *Continuous time signals $x(t)$: independent variable is continuous, that is the signal is defined for a continuum of values of the independent variable.*

2. *Discrete time signals $x[n]$: independent variables takes only a discrete set of values, that is the signal is defined only at a discrete set of values (i.e., $n \in \mathbb{Z}$ therefore $x[n]$ is a discrete time sequence).*

Remark: we can get a discrete time signal by sampling a continuous time signal.

Definition 2. *Signal Energy and Power*

For a continuous time signal x(t):

- The total energy over a finite time period $t_1 \leqslant t \leqslant t_2$ is defined as $\int_{t_1}^{t_2} |x(t)|^2 \, dt$.

- The average power over a finite time period $t_1 \leqslant t \leqslant t_2$ is defined as $\frac{\int_{t_1}^{t_2} |x(t)|^2 \, dt}{t_2 - t_1}$.

- For an infinite time period the total energy is defined as $E_\infty := \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$.

- For an infinite time period the average power is defined as $P_\infty := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$.

For a discrete time signal x[n]:

- The total energy over a finite time period $n_1 \leqslant n \leqslant n_2$ is defined as $\sum_{n=n_1}^{n_2} |x[n]|^2$.

- The average power over a finite time period $n_1 \leqslant n \leqslant n_2$ is defined as $\frac{\sum_{n=n_1}^{n_2} |x[n]|^2}{n_2 - n_1 + 1}$.

- For an infinite time period the total energy is defined as $E_\infty := \lim_{N \to \infty} \sum_{n=-N}^{N} |x[n]|^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2$.

- For an infinite time period the average power is defined as $P_\infty := \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 \, dt$.

Remark:

If $E_\infty < \infty$, that particular signal has fintite energy. And if it doesn't converge, it has infinite energy.

- Signals with finite total energy must have zero average power. [energy signals]

- Signals with finite average power must have inifinite total energy. [power signals]

- There are also signals with both infinite total energy and infinte average power.

Note 3. Transformations

Consider the signals $x(t)$ and $x\left(\alpha\left[t + \frac{\beta}{\alpha}\right]\right)$,

- Time shift: If $\beta < 0$, the resulting signal is delayed by $|\beta|$ (i.e., waveform shifts to the right by $|\beta|$). If $\beta > 0$, the signal is advanced β (i.e., waveform shifts to the left by β).

- Time reversal: If $\text{sgn}(\alpha) = -1$, the signal is reflected around $t + \frac{\beta}{\alpha} = 0$.

- Time scaling: If $|\alpha| > 1$, the resulting signal is stretched linearly by a factor of $|\alpha|$. Conversely it will be linearly compressed by a factor of an $|\alpha|$, if $|\alpha| < 1$.

Remark: these properties hold for discrete time signals as well.

Definition 4. *Periodic Signals - invariance under time shift*

For a continuous time signal x(t):

- A signal $x(t)$ is periodic if $\exists T > 0$, s.t., $x(t) = x(t+T)$. (i.e., unchanged by time shift of T). In that case we say that $x(t)$ is periodic with the period T. Otherwise we call it aperiodic.

- The fundamental period T_0 of $x(t)$ is the smallest $T > 0$, for which $x(t) = x(t+T)$. (this definition works works if $x(t)$ is not constant, in that case T_0 is undefined since $x(t)$ is periodic for any T).

- We can also deduce that $x(t) = x(t+mT)$, for $m \in \mathbb{Z}$.

For a discrete time signal x[n]:

- A signal $x[n]$ is periodic if $\exists N \in \mathbb{Z}^+$, s.t., $x[n] = x[n+N]$. In that case we say that $x[n]$ is periodic with the period N. Otherwise we call it aperiodic.

- The fundamental period N_0 of $x[n]$ is the smallest $N \in \mathbb{Z}^+$, for which $x[n] = x[n+N]$.

Definition 5. *Even and Odd Signals - symmetry under time reversal*

- A continunous time signal $x(t)$ is odd iff $x(-t) = -x(t)$ and even iff $x(-t) = x(t)$.

- A discrete time signal $x[n]$ is odd iff $x[-n] = -x[n]$ and even iff $x[-n] = x[n]$.

Any signal can be broken into a sum of two signal, one of which is even and one of which is odd.

i.e., $x(t) = \text{Ev}\{x(t)\} + \text{Od}\{x(t)\}$ *where,*

- $\text{Ev}\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$, *and*

- $\text{Od}\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]$.

Example 6.

1. An odd signal, $x(t)$ must be 0 at $t = 0$.

Since the signal is odd $x(-0) = x(0) = -x(0) \Rightarrow x(0) = 0$.

2. $\text{Ev}\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$ is an even signal.

$\text{Ev}\{x(-t)\} = \frac{1}{2}[(x(-t)) + (x(-(-t)))] = \frac{1}{2}[(x(-t)) + (x(t))] = \frac{1}{2}[x(t) + x(-t)] = \text{Ev}\{x(t)\}$, that is the even part of a signal is infact even.

3. $\text{Od}\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]$ is an odd signal.

$\text{Od}\{x(-t)\} = \frac{1}{2}[(x(-t)) - (x(-(-t)))] = \frac{1}{2}[(x(-t)) - (x(t))] = -\frac{1}{2}[x(t) - x(-t)] = -\text{Od}\{x(t)\}$, that is the odd part of a signal is infact odd.

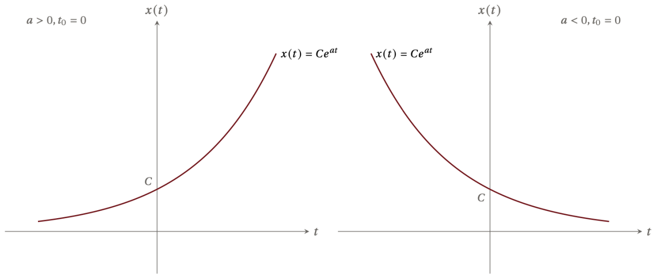
Note 7. Continuous Time Complex Exponential and Sinusodial Signals

- The continuous time complex exponential signal is of the form; $x(t) = C' e^{a(t+t_0)} = C' e^{at_0} e^{at} = Ce^{at}$, where $C, a \in \mathbb{C}$.

- Real exponential signals: if $C, a \in \mathbb{R}$ $x(t)$ is called a real exponential signal.

  $\rightarrow$ If $a > 0$, then $x(t)$ grows exponentially with t.

$\rightarrow$ If $a<0$, then $x(t)$ decays exponentially with t.



- Imaginary exponential signals/ Sinusodial signals: $x(t)=e^{j\omega_0 t}$.

$\rightarrow$ $x(t)$ is periodic iff $e^{j\omega_0 T}=1$. $[e^{j\omega_0 t}=e^{j\omega_0(t+T)}=e^{j\omega_0 t}\,e^{j\omega_0 T}]$.

$\rightarrow$ If $\omega_0=0$, then $x(t)$ is periodic for any T (so T is not defined, but $\omega_0=0$ is defined), if $\omega_0\neq 0$ then the fundamental period T_0 is the smallest positive value of T for which $T_0=\frac{2\pi}{|\omega_0|}$. [Thus $e^{j\omega_0 t}$, and $e^{-j\omega_0 t}$ both have the fundamental period]. $x(t)$ can be periodic for any ω_0.

$\rightarrow$ Larger the ω_0 the faster the signal oscillates, ω_0 is called the fundamental frequency.

$\rightarrow$ The kth harmonic of $x(t)$ is given by, $\phi_k(t)=e^{jk\omega_0 t}$, for $k\in\mathbb{Z}$. For $k=0$, $\phi_k(t)$ is constant, and for any other k, $\phi_k(t)$ is periodic with $\frac{T_0}{|k|}$ fundametal period. (fundamental period is $|k|\omega_0$).

- Sinusodial signals are closely related with periodic complex exponentials.

$\rightarrow$ We can write a complex exponential in terms of sinusoidals of the same funamental frequency. $[e^{j\omega_0 t}=\cos\omega_0 t+j\sin\omega_0 t]$.

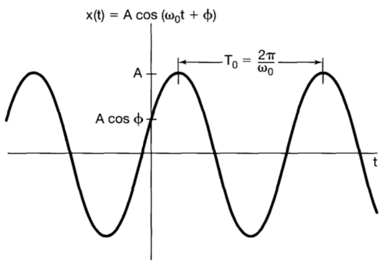
$\rightarrow$ Similarly we can write a sinusoidial in terms of periodic complex exponentials with the same fundamental frequency. $\Big[A\cos(\omega_0 t+\phi)=\left(\frac{A}{2}\,e^{j\phi}\right)e^{j\omega_0 t}+\left(\frac{A}{2}\,e^{-j\phi}\right)e^{-j\omega_0 t}\rightarrow$ note that the amplitudes are complex$\Big]$, or $[A\cos(\omega_0 t+\phi)=A\,\mathrm{Re}\{e^{j(\omega_0 t+\phi)}\}$, or $A\sin(\omega_0 t+\phi)=A\,\mathrm{Im}\{e^{j(\omega_0 t+\phi)}\}]$.

- Sinusodials and complex exponential signals are used to describe characteristics of many physical processes, in particular when the energy is consereved.

$\rightarrow$ These are power signals. (but not energy signals). $[E_{\text{period}}=\int_0^{T_0}|e^{j\omega_0 t}|^2\,dt=\int_0^{T_0}1\,dt=T_0]$.

- $E_{\text{period}}=\int_0^{T_0}|e^{j\omega_0 t}|^2\,dt=\int_0^{T_0}1\,dt=T_0$, $P_{\text{period}}=\frac{1}{T_0}\,E_{\text{period}}=1$.

- $E_\infty\rightarrow\infty$, $P_\infty=\lim_{T\rightarrow\infty}\frac{1}{2T}\int_{-T}^{T}|e^{j\omega_0 t}|^2\,dt=1$.



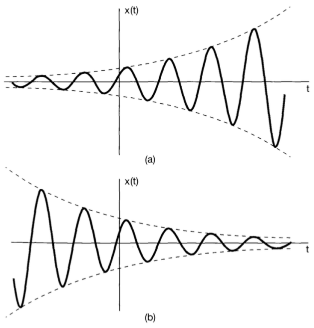
- General complex exponential signal: $x(t)=Ce^{at}$, where $C=|C|\,e^{j\theta}$, and $a=r+j\omega_0$.

$\rightarrow$ $x(t)=Ce^{at}=|C|\,e^{j\theta}\,e^{(r+j\omega_0)t}=|C|e^{rt}e^{j(\omega_0 t+\theta)}$.

$\rightarrow$ $x(t)=Ce^{at}=|C|e^{rt}\cos(\omega_0 t+\theta)+j\,|C|e^{rt}\sin(\omega_0 t+\theta)$.

$\rightarrow$ If $r=0$, then the real and imaginary parts are sinusodial.

→ If $r > 0$, then the real and imaginary parts are growing sinusodial. Simlilarly if $r < 0$, then the real and imaginary parts are decaying sinusodial, called damped sinusodials.



(a) Growing sinusoidal signal $x(t) = Ce^{rt}\cos(\omega_0 t + \theta)$, $r > 0$; (b) decaying sinusoid $x(t) = Ce^{rt}\cos(\omega_0 t + \theta)$, $r < 0$.

Note 8. Discrete Time Complex Exponential and Sinusodial Signals

- The discrete time complex exponential signal/sequence is of the form; $x[n] = C\alpha^n = Ce^{\beta n}$, where $C, \alpha \in \mathbb{C}$ and $\alpha = e^{\beta}$.

- Real exponential signals: if $C, \alpha \in \mathbb{R}$ then, $x(t)$ is called a real exponential signal.

→ if $|\alpha| > 1$, then $|x[n]|$ grows exponentially with n.

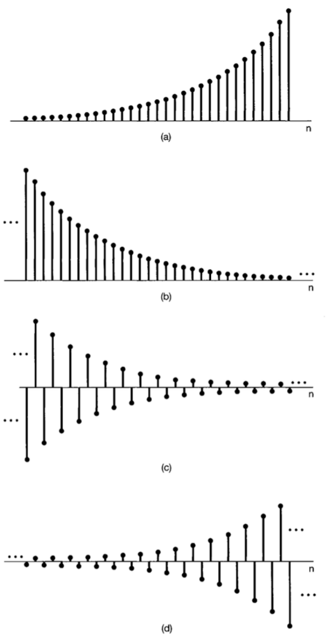
→ if $|\alpha| < 1$, then $|x[n]|$ decays exponentially with n.

→ if $\alpha > 0$, then $x[n]$ doesn't change its sign.

→ if $\alpha < 0$, then $x[n]$ alternates its sign.

→ if $\alpha = 1$, then $x[n]$ is constant.

→ if $\alpha = -1$, then $x[n]$ alternates between $+C$, $-C$ in every step.



The real exponential signal $x[n] = C\alpha^n$: (a) $\alpha > 1$; (b) $0 < \alpha < 1$; (c) $-1 < \alpha < 0$; (d) $\alpha < -1$.

- Imaginary exponential signals/ Sinusodial signals: $x[n] = e^{j\omega_0 n}$. ($|\alpha| = 1$).

→ Unlike in the continuous case (in that case the signal is distinct for distinct values of ω_0) we see that signal is same at ω_0, and $\omega_0 \pm 2m\pi n$, where $m \in \mathbb{Z}$. $[e^{j(\omega_0+2\pi)n} = e^{j2\pi n} e^{j\omega_0 n} = e^{j\omega_0 n}]$. Thus to study the signal we only have to consider frequency interval of 2π (usally we choose $0 \leqslant \omega_0 < 2\pi$ or $-\pi \leqslant \omega_0 \leqslant \pi$).

$\rightarrow$ So the increase rate of oscillation increase as we increase ω_0 from 0 to π, and decreases from π to 2π. (at 0 and 2π the rate of oscillation is same). Thus the slowly varying signals have ω_0 values near even multiples of $\pm\pi$, and rapidly varying signals have ω_0 values near odd multiples of $\pm\pi$ (change the sign at each point in time, as $e^{j\pi n} = (e^{j\pi})^n = (-1)^n$).

$\rightarrow$ The signal is periodic iff $e^{j\omega_0 n(n+N)} = e^{j\omega_0 n} \Rightarrow e^{j\omega_0 N} = 1 \Rightarrow \omega_0 N = 2\pi m \Rightarrow \frac{\omega_0}{2\pi} = \frac{m}{N} \in \mathbb{Q}$. Thus a discrete time exponential signal or a sinusodial to be periodic $\frac{\omega_0}{2\pi}$ must be rational. ($N > 0$, m, $N \in \mathbb{Z}$).

$\rightarrow$ The fundamental frequency ω_0 (or ω_0/m) of a signal with fundametal period N is given as, $\omega_0 = m \cdot \frac{2\pi}{N}$, where $m \in \mathbb{Z}$, s.t., $\omega_0 \in \mathbb{Z}^+$. ($N$ is not defined for $\omega_0 = 0$).

$\rightarrow$ In the continuous case all harmonically related complex signals are distinct, but in discrete time it not the case. (as $\phi_{k+N}[n] = e^{j(k+N)\left(\frac{2\pi}{N}\right)n} = e^{jk\left(\frac{2\pi}{N}\right)n} e^{j2\pi n} = \phi_k[n]$). Thus the harmonically related periodic exponential $\phi_k[n] = e^{jk\left(\frac{2\pi}{N}\right)n}$, where $k \in \mathbb{Z}$, only have N distinct periodic exponentials. ($k = 0, 1, \ldots, N-1$ are distinct of each other and $\{k = N, k = 0\}, \{k = N-1, k = -1\}, \ldots$ are identical).

- Sinusodial signals are closely related with periodic complex exponentials.

$\rightarrow$ $x[n] = e^{j\omega_0 n} = A\cos(\omega_0 n + \phi)$.

$\rightarrow$ We can write a complex exponential in terms of sinusoidals of the same fundamental frequency. $[e^{j\omega_0 n} = \cos\omega_0 n + j\sin\omega_0 n]$.

$\rightarrow$ Similalry we can write a sinusoidial in terms of periodic complex exponentials with the same fundamental frequency. $\left[A\cos(\omega_0 n + \phi) = \left(\frac{A}{2}e^{j\phi}\right)e^{j\omega_0 n} + \left(\frac{A}{2}e^{-j\phi}\right)e^{-j\omega_0 n}\right]$.

- Sinusodials and complex exponential signals are used to describe characteristics of many physical processes, in particular when the energy is consereved.

$\rightarrow$ These are power signals (but not energy signals).

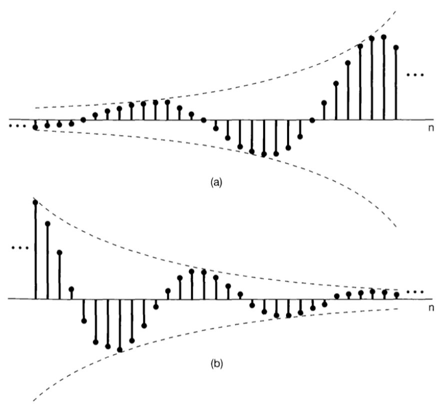
- Since $|e^{j\omega_0 n}|^2 = 1$, every sample contributes 1 to the energy. Thus the total energy is infinite.

- General complex exponential signal: $x(t) = C\alpha^n$, where $C = |C|\,e^{j\theta}$, and $\alpha = |\alpha|e^{j\theta}$.

$\rightarrow$ $x(t) = C\alpha^n = |C||\alpha|^n\cos(\omega_0 n + \theta) + j|C||\alpha|^n\sin(\omega_0 n + \theta)$.

$\rightarrow$ if $|\alpha| = 1$, then the real and imaginary parts are sinusodial.

$\rightarrow$ if $|\alpha| > 1$, then the real and imaginary parts are growing sinusodial. Simlilarly if $|\alpha| < 1$, then the real and imaginary parts are decaying sinusodial.

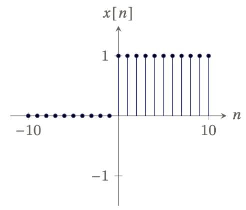


(a) Growing discrete-time sinusoidal signals; (b) decaying discrete-time sinusoid.

Definition 9.

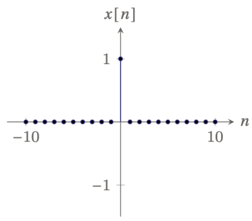
1. The discrete time unit step sequence, $u[n]$

$u[n] := \begin{cases} 0, & n < 0 \\ 1, & n \geqslant 0 \end{cases}.$



2. The discrete time unit impulse sequence, $\delta[n]$

$\delta[n] := \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}.$



- The discrete time unit impulse is the first difference of the discrete time step; $\delta[n] = u[n] - u[n-1]$.

Conversly, the discrete time unit step is the runnning sum of the unit impulse; $u[n] = \sum_{k=-\infty}^{n} \delta[k]$.

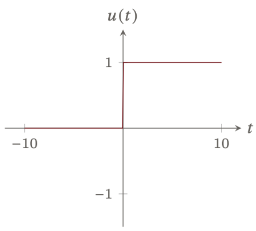
By changing the summation variable, $u[n] = \sum_{k=-\infty}^{0} \delta[n-k] = \sum_{k=0}^{\infty} \delta[n-k]$. *i.e., The discrete unit step function is a superposition of delayed discrete unit impulse.*

- The unit impulse sequence can be used to sample the value of a signal at $n = 0$. *In particular since* $\delta[n]$ *is nonzero (and 1) only for* $n = 0$, *it follows that;* $x[n]\delta[n] = x[0]\delta[n]$.

In generally, if we consider a unit impulse $\delta[n - n_0]$ *at* $n = n_0$, *then;* $x[n]\delta[n - n_0] = x[n_0]\delta[n - n_0]$.

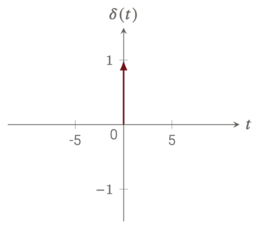
3. The continuous time unit step function, $u(t)$

$u(t) := \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}.$

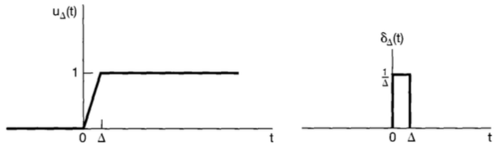


4. The continuous time unit impulse function, $\delta(t)$

$\delta(t) := \frac{du(t)}{dt}.$



Since $u(t)$ is discontinuous at $t=0$, it is not differentiable. Thus, consider the derivative of u_Δ and take the limit as $\Delta \to 0$, for $\delta(t)$; $\delta_\Delta(t) = \frac{du_\Delta(t)}{dt} \Rightarrow \delta(t) = \lim_{\Delta\to 0}\left(\frac{du_\Delta(t)}{dt}\right)$.



- The continuous time unit step is the running integral of the unit impulse; $u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$.

By changing the variable of integration, $u(t) = \int_{\infty}^{0} \delta(t-\sigma)\,(-d\sigma) = \int_{0}^{\infty} \delta(t-\sigma)\, d\sigma$.

The height of the arrow at $t=0$ being 1, indicates that the area of the impulse is 1. In general the impulse $k\delta(t)$ will have an area of k; $\int_{-\infty}^{t} k\delta(\tau)\, d\tau = ku(t)$.

- The continuous time impulse has a very important sampling property. We see that $x(t)\delta_\Delta(t)$ is zero outside the interval $0 \leqslant t \leqslant \Delta$. For small Δ, $x(t)$ approximately constant over this interval; $x(t)\,\delta_\Delta(t) \approx x(0)\delta_\Delta(t)$. Since $\delta_\Delta(t) \to \delta(t)$, as $\Delta \to 0$ it follows that, $x(t)\delta(t) = x(0)\delta(t)$.

In generally, for an impulse concentrated at an arbitary point t_0, $x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$.

2. Fourier Series

Note 10. Fourier Series

The Fourier series provides a means to represent a periodic signal as a sum of complex exponentials. These sinusodis have frequencies that are integer multiples of the fundamental frequency. ($\omega_0 = \frac{2\pi}{T}$, where T is the fundamental period). Periodic signals exhibit a line spectrum, which we refer to as the Fourier series representation of the signal. We here focus only on the Fourier series for continuous time signals.

Theorem 11. *Convergence of Fourier Series*

Not all functions have Fourier series that converge. But the convergence of Fourier seires is beyond the scope of this course.

Theorem 12. *Fourier Series for Continuous Time Periodic Signals*

1) A periodic signal is a linear combination if harmonically related complex exponentials. (i.e., whose frequencies are integer multiplications of fundamental frequency).

$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k\, e^{jk(2\pi/T)t}$. *[synthesis equation]*

2) Assuming a signal can be expressed as above the Fourier coefficients (or the spectral coefficients) are given by;

$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T}\int_T x(t)\, e^{-jk(2\pi/T)t}\, dt$. *[analysis equation]*

$a_0 = \frac{1}{T}\int_T x(t)\, dt$.

We use the notion $x(t) \xleftrightarrow{FS} a_k$ to denote this.

Remarks:

- The term for $k=0$, is a constant. The coefficient a_0 is the dc or constant component of $x(t)$ and it is the average value of $x(t)$ over a period.

- The components for $k=\pm N$, are refered to as the Nth harmonic components. ($k=\pm 1$, both have the fundamental frequency, and collectively called fundamental compoeenents or the first harmonic components).

Proof.

2) $x(t)=\sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t} \Rightarrow x(t)\, e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t} e^{-jn\omega_0 t} \Rightarrow \int_T x(t)\, e^{-jn\omega_0 t}\, dt = \sum_{k=-\infty}^{\infty} a_k [\int_T e^{j(k-n)\omega_0 t}\, dt] = \sum_{k=-\infty}^{\infty} a_k [\int_T \cos(k-n)\,\omega_0\, t\, dt + j\int_T \sin(k-n)\,\omega_0\, t\, dt] = \sum_{k=-\infty}^{\infty} a_k \begin{cases} T, & k=n \\ 0, & k\neq n \end{cases} = a_n T$. *(Hint: think about the area under the curve, alternaitvely we can use the orthogonality of complex exponentials).* □

Theorem 13. *Alternative Forms of Fourier Series*

1) $x(t)=a_0+\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t+\theta_k)$.

2) $x(t)=a_0+2\sum_{k=1}^{\infty} [B_k \cos k\omega_0\, t - C_k \sin k\omega_0\, t]$.

Proof. $x(t)=\sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t} = a_0+\sum_{k=1}^{\infty} [\, a_k\, e^{jk\omega_0 t} + a_{-k}\, e^{-jk\omega_0 t}] = a_0+\sum_{k=1}^{\infty} [\, a_k\, e^{jk\omega_0 t} + a_k^*\, e^{-jk\omega_0 t}] = a_0+\sum_{k=1}^{\infty} 2\,\mathrm{Re}\,\{a_k\, e^{jk\omega_0 t}\}$.

1) If a_k is expressed in polar form; $a_k=A_k\, e^{j\theta_k}$, we get 1st form.

2) If a_k is expressed in cartesian form; $a_k=B_k+jC_k$, we get 2nd form. □

Proposition 14. *Properties of Continuous Time Fourier Series*

1) Linearity:

Let $x(t), y(t)$ have the same fundamental period T, with $x(t) \xleftrightarrow{FS} a_k, y(t) \xleftrightarrow{FS} b_k$. And suppose that $z(t)=Ax(t)+By(t)$, where A, B are constants. Then, $z(t) \xleftrightarrow{FS} c_k = Aa_k + Bb_k$.

2) Time shifting:

Let $x(t)$ be a periodic signal, and $y(t)=x(t-t_0)$ with $x(t) \xleftrightarrow{FS} a_k$. Then, $y(t) \xleftrightarrow{FS} b_k = e^{-jk\omega_0 t_0}\, a_k$. (But $|a_k|=|b_k|$).

3) Frequency shifting:

Let $x(t)$, have a fundamental frequency of ω_0, and $y(t)=x(t)\, e^{jMk\omega_0 t}$ with $x(t) \xleftrightarrow{FS} a_k$. Then, $y(t) \xleftrightarrow{FS} b_k = a_{k-M}$.

4) Time scaling:

Let $x(t)$ be a periodic signal with period T, with $x(t) \xleftrightarrow{FS} a_k$. Then $y(t)=x(\alpha t)$, $\alpha>0$ will be periodic with a period T/α (fundametal frequency $\alpha\omega_0$), and $y(t)=\sum_{k=-\infty}^{\infty} a_k\, e^{jk(\alpha\omega_0)t}$.

5) Time reversal:

Let $x(t)$ be a periodic signal, and $y(t)=x(-t)$ with $x(t) \xleftrightarrow{FS} a_k$. Then, $y(t) \xleftrightarrow{FS} b_k = a_{-k}$.

$\rightarrow$ If $x(t)$ is even then a_k is also even. (i.e., $(x(-t)=x(t) \Rightarrow a_{-k}=a_k)$. Similarly, if $x(t)$ is odd then a_k is also odd. (i.e., $(x(-t)=-x(t) \Rightarrow a_{-k}=-a_k)$.

6) Let $x(t), y(t)$ have the same fundamental period T, with $x(t) \xleftrightarrow{FS} a_k, y(t) \xleftrightarrow{FS} b_k$. Then, $x(t)y(t)$ is also periodic and, $x(t)y(t) \xleftrightarrow{FS} h_k = \sum_{l=-\infty}^{\infty} a_l\, b_{k-l}$.

7) Conjugation and Conjugate Symmetry:

Let $x(t)$ be a periodic signal, and $y(t)=x^*(t)$ with $x(t) \xleftrightarrow{FS} a_k$. Then, $y(t) \xleftrightarrow{FS} b_k = a_{-k}^*(t)$.

→ *If* $x(t)$ *is real then,* a_0 *is real and* $|a_k| = |a_{-k}|$.

→ *If* $x(t)$ *is real then,* $\angle a_k = -\angle a_{-k}$.

→ *If* $x(t)$ *is real then,* $\mathrm{Re}\{a_k\} = \mathrm{Re}\{a_{-k}\}$*, and* $\mathrm{Im}\{a_k\} = -\mathrm{Im}\{a_{-k}\}$.

→ *If* $x(t)$ *is real then,* $x_e(t) \xleftrightarrow{FS} \mathrm{Re}\{a_k\}$*, and* $x_o(t) \xleftrightarrow{FS} j\,\mathrm{Im}\{a_k\}$.

→ *Moreover if* $x(t)$ *is real and even,* $a_k = a_{-k} = a_k^* \Rightarrow a_k = a_k^*$*, i.e., if* $x(t)$ *is real and even, then so are its Fourier series coefficients. (Thus* $a_0 = 0$ *if* $x(t)$ *is real and odd).*

→ *Similarly if* $x(t)$ *is real and odd, then its Fourier coefficients are purely imaginary and odd.*

8) Parseval's Relation:

$\frac{1}{T}\int_T |x(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |a_k|^2$.

9) Differentiation:

Let $x(t)$ *be a periodic signal, and* $y(t) = \frac{d}{dt}x(t)$ *with* $x(t) \xleftrightarrow{FS} a_k$*. Then,* $y(t) \xleftrightarrow{FS} b_k = jk\omega_0\,a_k$.

10) Integration:

Let $x(t)$ *be a periodic signal, and* $y(t) = \int_{-\infty}^{t} x(t)\,dt$ *(finite valued and periodic only if* $a_0 = 0$*) with* $x(t) \xleftrightarrow{FS} a_k$*. Then,* $y(t) \xleftrightarrow{FS} b_k = \frac{1}{jk\omega_0}\,a_k$.

Proof.

1) $c_k = \frac{1}{T}\int_T [Ax(t) + By(t)]e^{-j\omega_0 kt}\,dt = A\big[\frac{1}{T}\int_T x(t)\,e^{-j\omega_0 kt}\,dt\big] + B\big[\frac{1}{T}\int_T y(t)\,e^{-j\omega_0 kt}\,dt\big] = Aa_k + Bb_k$.

2) $b_k = \frac{1}{T}\int_T y(t)\ e^{-j\omega_0 kt}\ dt = \frac{1}{T}\int_T x(t - t_0)\ e^{-j\omega_0 kt}\ dt = \frac{1}{T}\int_T x(\tau)\ e^{-j\omega_0 k(\tau+t_0)}\ d(\tau + t_0) = e^{-jk\omega_0 t_0}\big[\frac{1}{T}\int_T x(\tau)\,e^{-j\omega_0 k\tau}\,d\tau\big] = e^{-jk\omega_0 t_0}\,a_k$.

3) $b_k = \frac{1}{T}\int_T [x(t)e^{jMw_0 t}]\,e^{-j\omega_0 kt}\,dt = \frac{1}{T}\int_T x(t)\,e^{-j\omega_0(k-M)t}\,dt = a_{k-M}$.

4) The Fourier coefficients remains same, the Fourier representation changes because of the change in the fundamental frequency.

5) $b_k = \frac{1}{T}\int_T y(t)\ e^{-j\omega_0 kt}\ dt = \frac{1}{T}\int_T x(-t)\ e^{-j\omega_0 kt}\ dt = \frac{1}{T}\int_T x(\tau)\ e^{-j\omega_0 k(-\tau)}\ d\ (-\tau) = \frac{1}{T}\int_0^{-T} -x(\tau)\,e^{-j\omega_0(-k)\tau}\,d\tau = \frac{1}{T}\int_T^0 x(\tau)\,e^{-j\omega_0(-k)k\tau}\,d\tau = a_{-k}$.

6) We obtain this multiplying the Fourier series representations of $x(t), y(t)$ *and noting that the kth harmonic component in the product will have a coefficient given by the sum of terms of the form* $a_l\,b_{k-l}$*. (Observe that this sum can be interpreted as the discrete time convolution of the sequence representation the Fourier coefficients of* $x(t)$ *and the sequence representing the Fourier coefficients of* $y(t)$*).*

7) $x(t) = \sum_{m=-\infty}^{\infty}\ a_m\ e^{jm\omega_0 t} \Rightarrow x^*(t) = \sum_{m=-\infty}^{\infty}\ a_m^*(\ e^{jm\omega_0 t})^* = \sum_{m=-\infty}^{\infty}\ a_m^* e^{j(-m)\omega_0 t} = \sum_{-k=-\infty}^{\infty} a_{-k}^* e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} b_k\,e^{jk\omega_0 t} = y(t)$.

8) $\frac{1}{T}\int_T |a_k\,e^{jk\omega_0 t}|^2\,dt = \frac{1}{T}\int_T |a_k|^2\,dt = |a_k|^2$*, i.e., the* $|a_k|^2$ *is the average power in kth harmonic component. Thus the average power of* $x(t)$*,* $\frac{1}{T}\int_T |x(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |a_k|^2$.

9) $y(t) = \frac{d}{dt}x(t) = \frac{d}{dt}\,[\sum_{k=-\infty}^{\infty}\ a_k\ e^{jk\omega_0 t}] = \sum_{k=-\infty}^{\infty}\ a_k\frac{d}{dt}(\ e^{jk\omega_0 t}) = \sum_{k=-\infty}^{\infty}\ (jk\omega_0\,a_k)\ e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} b_k\,e^{jk\omega_0 t}$*. Hence,* $b_k = jk\omega_0\,a_k$.

10) Similarly integrate $x(t) = \sum_{k=-\infty}^{\infty} a_k\,e^{jk\omega_0 t}$*, to get the desired result.* □

3. Fourier Transform

Note 15. Fourier Transform

We use Fourier transform for signals that are non periodic. As we will see a rather a large class of signals including all signals with finite energy can be represented through a linear combination of complex exponentials. Where as for periodic signals the complex exponential building blocks are harmonically related for aperiodic signals they are infintesimeally close in frequency, and the representation of linear combination takes the form of an integral rather than a sum. The resulting spectrum of coefficients is called the Fourier transform, and the synthesis integral itself which uses these coefficients to represent the signal as a linear combination of complex exponential is called the inverse Fourier transform.

An aperiodic signal can be viewed as a periodic signal with an infinite period. In the Fourier series representation of a periodic signal as the period increases the fundamental frequency decreases and the harmonically related components become closer in frequency. As the period becomes inifinite the frequency components form a continumm and the Fourier series sum becomes an integral.

Theorem 16. *Fourier Transform Pairs*

Fourier transform/ Fourier integral of $x(t)$*;* $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega.$

Synthesis equation/spectrum of $x(t)$*;* $X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, d\omega.$

Remarks:

The synthesis equation provides the informations needed to describe the signal as a linear combination of sinusodials, and the Fourier transform expresses the signal as a linear combination.