[redacted] September 8, 2021

Problem Set 1

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Problem 1-1 (Asymptotic Comparisons) 10 Points

For each of the following pairs of functions $f(n)$ and $g(n)$, state whether f is $O(g)$; whether f is $o(g)$; whether f is $\Theta(g)$; whether f is $\Omega(g)$; and whether f is $\omega(g)$. (More than one of these can be true for a single pair or none of it!)

(a) $f(n) = (n^n)^3; \quad g(n) = n^{(n^3)}$.

(b) $f(n) = n!; \quad g(n) = (n+1)!$.

(c) $f(n) = n^{0.99} + 15(\log_2 n)^{100}; \quad g(n) = 4^{\log_{16} n^2}$.

(d) $f(n) = n^{0.000001}; \quad g(n) = (\log n)^{1000001}$.

(e) $f(n) = n^{5/\log_2 n}; \quad g(n) = 10000$.

Problem 1-2 (Order of growth) 12 points

(a) (8 points) Write the following functions in the Θ-notation in the following simple form $c^n \cdot n^d \cdot (\log n)^r$, where c, d, r are constants. E.g., $20n^2 + 5n + 7 \sim n^2$, you can omit "Θ" in front of n^2. Briefly justify your answers

$$\begin{aligned}
\frac{n^2+2}{1+n^3\cdot 2^{-n}} &\sim \\
\log(\sqrt[3]{n})\cdot\log(37n^3+45) &\sim \\
\log(n!) + 10^{205}n &\sim \\
\frac{6^n - 1000}{2^n+1} &\sim \\
n^{2/\log n} + 1000 &\sim \\
2^{n+\log n} &\sim \\
2^n + 10(\log n)^{(\log n)} &\sim \\
2^{3n} + 4^n &\sim
\end{aligned}$$

(b) (4 points) List the simple functions you derived in part (a) in the order of asymptotic growth, from the smallest to the largest. (E.g., if one of the "complicated" functions in part (a) was $20n^2 + 5n + 7$, use the equivalent "simple" function n^2 in your ordering for this problem).

Problem 1-3 (Bubble Sort) 21(+3) points

Consider sorting n numbers stored in an array A by iterating through the array and exchanging $A[i]$ and $A[i+1]$ if they are out of order. Repeat this until the array is in order.

(a) (4 points) Write (non-recursive) pseudocode for this algorithm, which is known as Bubble-Sort.

(b) (4 points) In the worst case, the outer loop of Bubble-Sort only needs to run $n-1$ times. Prove this by giving a loop invariant that the algorithm maintains. Give the best-case and worst-case running times of Bubble-Sort in Θ-notation.

(c) (4 points) Let $A[1,\ldots,n]$ be an array of n distinct numbers. If $i < j$ and $A[i] > A[j]$, then the pair (i,j) is called an *inversion* of A.

For example, the inversions of the array $\langle 7,5,2,9,6\rangle$ are (1,2),(1,3),(1,5),(2,3),(4,5).

What is the relationship between the number of exchanges in Bubble-Sort, and the number of inversions I in the input array? *Justify your answer.*

(d) (3 points) Give an example of an array where the *running time* of Bubble-Sort is $\Theta(n^2)$ but the number of inversions is $\Theta(n)$.

(e) (2 points) Give an example of an array for which Insertion-Sort runs in time $\Theta(n)$ but Bubble-Sort runs in time $\Theta(n^2)$.

(f) (4 points) Can you give an example of an array for which Bubble-Sort runs in time $\Theta(n)$ but Insertion-Sort runs in time $\Theta(n^2)$? *Justify your answer.*

(g) (3 points) [**Extra credit**] Let $A[1,\ldots,n]$ be a random permutation of $\{1,2,\ldots,n\}$. What is the expected number of inversions of A. What can you conclude about the average case running time of Bubble-Sort (where the average is taken over the choice ofl permutation A of size n)?

Hint: Recall the linearity of expectation, i.e., for any real a,b,c and any random variables X,Y,

$$E(aX+bY+c) = aE(X)+bE(Y)+c\ .$$