Problem 2-1.

a) $T(n) = 2T\left(\frac{n}{2}+1\right) + n-2$

given $T(3) = 0$

To prove. $T(n) = \theta(n-2)\log(n-2)$ ∴ Tight bound.

→ Prove by master theorem for cheat value.

$T(n) = aT\left(\frac{n}{b}\right) + 1.$

$f(n) = \theta(n^k . \log^p n)$

$\log_b a = 1$; $k = 1$; $P = 0$.

Case 2 $f(n) = \theta(n^k . \log^{P+1} n)$ → Cheat Value by Masters Theo.

$\boxed{= \theta(n . \log n)}$ → best guess.

Guess then Verify

$T(n) = 2T\left(\frac{n+2}{2}\right) + n-2.$

Expressing the value

$= 2.c\left(\frac{n\log n}{2}+2\right) + n\log n - 2.$

$= c(n\log n + 2) + n\log n - 2.$

$\simeq \theta(n\log n).$

Since $T(3) = 0$.

∴ $(n-2)\log(n-2)$ ∀ $n \geq 3$.

which means for every number ($n \geq 3$) the below expression holds true

$\Omega c_1 n \log n \leq \theta(n-2)\log(n-2) \leq O c_2 n \log n$

where c_1 & $c_2 > 0$

1

$\therefore c_1 \underbrace{n \log n}_{g} \leq (n-2)\log(n-2) \leq c_2\, n \log n$

which means that our function is a tight bound.

$\forall\, c_1, c_2 > 0 \Rightarrow f(n) = \theta\, g(n)$

$\boxed{= \theta \cdot (n-2)\log(n-2)}$ ← Proven by guess then verify.

b) Domain Range substitution for a Reducing Recurrence.

$T(n) = 2T\left(\frac{n}{2}+1\right) + (n-2)$

Assuming that the function is reducing, we will solve towards $T(n) = 1$.

$$T_n = \begin{cases} 0 & ; n = 3 \\ 2T\left(\frac{n}{2}+1\right) + n - 2 & ; n > 1 \end{cases}$$

$T(n) = 2T\left(\frac{n}{2}+1\right) + n - 2$ —— 1

Solving for. $C\left(\frac{n}{2}\right)$

$\cancel{T\left(\frac{n}{2}\right) = 2\left[2T\left(\frac{n+1}{2^2}\right) + \left(\frac{n}{2}-2\right)\right]}$

$T\left(\frac{n}{2}\right) = 2\left[2T\left(\frac{n+1}{2^2}\right) + \frac{n}{2} - 2\right] + n - 2.$

$2^2 T\frac{n+1}{2^2} + n - 2 + n - 2$ —— 2

⋮

Relation to function in reduces form. → Total work for full credit (Reduced form)

$f(n) = \boxed{T_n = 2^k \cdot T\left(\frac{n+1}{2^k}\right) + k(n-2)}$ → Main equation

$T\left(\frac{n-2}{2^k}\right) = T(1).$ $\quad \frac{n-2}{2^k} = 1$ $\quad \therefore f(3) = 0.$ min value $(n-2)$

$k = \log(n-2)$

Cont. 2

Since

$$T(n) = 2^K \frac{T(n-2)}{2^K} + K(n-2)$$

& $K = \log(n-2)$ —— (4)

Substituting the values.

$$\boxed{\begin{aligned} T(n) &= 2^K \cdot \overset{\nearrow 0}{T(3)} + K(n-2) \\ T(n) &= K(n-2) \\ T(n) &= (n-2) \cdot \log(n-2) \end{aligned}}$$ ← Answer. * Approch 1.

Alternative Approach:

$$T(n) = \Omega((n-2)\log(n-2))$$

$$\Rightarrow T(n) \geq c((n-2)\log(n-2))$$

Solution for sub problem with size $\left(\frac{n}{2}+1\right)$

here. $T\left(\frac{n}{2}+1\right) \geq c\left(\left(\frac{n}{2}+1\right)-2\right)\log\left(\frac{n}{2}+1-2\right)$

$$\geq c\left(\frac{n}{2}-1\right)\log\left(\frac{n}{2}-1\right)$$

$$T(n) = 2T\left(\frac{n}{2}+1\right) + (n-2)$$

$$T(n) \geq 2\left[c\left(\frac{n}{2}-1\right)\log\left(\frac{n}{2}-1\right)\right] + (n-2)$$

$$T(n) \geq c(n-2)\log\left(\frac{n-2}{2}\right) + n-2$$

$$T(n) \geq c(n-2)\left[\log(n-2) - \log 2\right] + (n-2)$$

$$T(n) \geq c(n-2)\log(n-2) + (n-2)[1 - c\log 2]$$

$$T(n) \geq c(n-2)\log(n-2) \quad \forall \boxed{c \geq \frac{1}{\log 2}}$$

$\therefore T(n) = \Omega(n-2)\log(n-2)$

Since $T(n) = O(n-2)\log(n-2)$ ↗ Approch 2.

$$\boxed{\therefore T(n) = \Theta((n-2)\log(n-2)) \ \forall \ c_1, c_2 > 0.}$$ (3)

2-4(c) Recursion Tree Method

$T(0) = 1.$

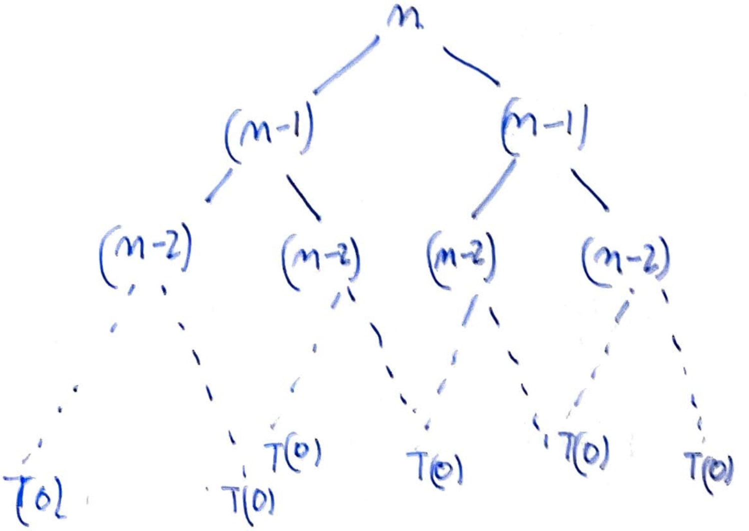
$T(n) = 2T(n-1) + n, \quad T(0) = 1.$

$T(n) = \cancel{aT}\ aT\left(\frac{n}{b}\right) + f(n)$

~~$f(n) = \theta(n^d)$~~

$f(n) = \theta(n^k \cdot \log^p n)$

Recursion Tree



$T(0) = 1.$

Total Cost.

n

$2(n-1)$

$2^2(n-2)$

$2^3(n-3)$

$\vdots$

$2^k \cdot (n-k)$

For the Exact value of $T(n)$

$T(n) = n + 2(n-1) + 2^2(n-2) + \dots 2^{(n-1)}(n-(n-1)) + 2^n \cdot T(0)$

$T(n) = n + 2n + 2^2 n + 2^3 n \dots + 2^{(n-1)} n - (1.2 + 2.2^2 + 3.2^3 + \dots (n-1)2^{(n-1)}) + \dots$

$T(n) = n\left[1 + 2 + 2^2 + \dots 2^{n-1}\right] - \sum_{i=1}^{n-1} i 2^i + 2^n$ As Per hint

$= \frac{n \cdot (2^n - 1)}{2-1} + \boxed{\frac{(n-1)2^{n+1} - 2^n \cdot n + 2 + 2^n}{(2-1)^2}}$

$= \cancel{n \cdot 2^n} - n + n \cdot 2^{n+1} - 2^{n+1} - \cancel{n \cdot 2^n} + 2 + 2^n$

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Cont.

$= 2 \cdot m2^m - 2^m - m + 2.$

$\boxed{T(m) = 2 \cdot m2^m - 2^m - m + 2.}$ → ~~Asym~~ Correct answer as per the questn

$\boxed{T(m) = \Theta(m \cdot 2^m)}$ → Asymptatic Notation.

Table.

Level	Size of pr.	No of Pr.	Non recursion cost	Total cost
0	n	1	m	n
1	$(n-1)$	2	$2(n-1)$	$2(n-1)$
$d-1$	$(n-(d-1))$	2^{d-1}	$2^{d-1}(n-(d-1))$	$2^{d-1} \cdot (n-(d-1))$
d	$(n-d)$	2^d	~~1~~	$2^d \cdot 1$

Last level d

$n-d=0$

$\therefore n = d.$ (as mentioned above) Cost at level $d = 2^n$.

cost at level $d-1 = 2^{n-1}(n-(n-1))$

$= 2^{n-1} \cdot 1$ Ans.

~~Grow~~ Asymptotically equal.

$n \to$ CEO

$2 \to$ V.P.

~~[illegible]~~

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Problem 2-2. (Fun with Recurrences)

(a). $T(m,1) = 3m$

$T(1,m) = 3m.$

$T(m,m) = 3m + T\left(\frac{m}{3}, \frac{m}{3}\right)$

$$T(m^2, m^2) = T\left(\frac{m}{3}, \frac{m^2}{3^2}\right) + 3m.$$

$$= T\left(\frac{3m}{3^2}, \frac{m^2}{9}\right) + 3m + 3\cdot\frac{m}{3}$$

$$= T\left(\frac{m}{3^3}, \frac{m^2}{3^3}\right) + 3m + 3\frac{m}{3} + 3\frac{m}{3^2} \ldots$$

$$= \cancel{T\left(\frac{m}{3^3}\right)}$$

$$T\left(\frac{m}{3^k}, \frac{m^2}{3^k}\right) + 3m\left[1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3} + \cdots \frac{1}{3^{k-1}}\right]$$

Put $\frac{m}{3^k} = 1. \Rightarrow 3^k = m \Rightarrow k = \log_3 m.$

$$\Rightarrow T(m, m^2) = T\left(1, \frac{m^2}{m}\right) + 3m\left[\frac{1\cdot(1-1/3)^k}{(1-1/3)}\right]$$

$$= T(1, m) + 3m \times \frac{3}{2}\left(1 - \frac{1}{3^k}\right)$$

$$= 3m + \frac{9m}{2}\left(1 - \frac{1}{m}\right)$$

$$= 3m + \frac{9m}{2} - \frac{9}{2}.$$

$$= \frac{15m}{2} + \frac{9}{2}. = O(m)$$

Answer.

$$\boxed{\therefore T(m, m^2) = O(m)}$$

Extra Credit. $T(m) = 15\frac{m}{2} + \underbrace{\frac{9}{2}}_{O(1).}$

$$\boxed{= 15\frac{m}{2} + O(1) \Rightarrow 15\frac{m}{2} + O(1)}$$

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2-2 Recurrence Tree.

Unbalanced

$T(n) = T(0.01n) + T(0.99n) + Cn$

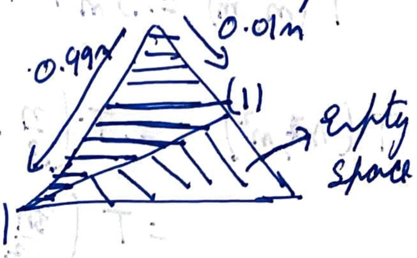
Cn → CEO

→ longest branch → VP.

→ Shortest branch VP.

$T(n) = T\left(\frac{n}{100}\right) + T\left(\frac{99n}{100}\right) + Cn.$

Reference diagram



Recursion tree

Shortest Bra..

Longest branch

$Cn.$ — $C(n)$

$\frac{n}{100}$ $\quad \frac{99n}{100}$ — $C(n)$

$\left(\frac{n}{100}\right)^2 \quad \frac{99n^2}{(100)^2} \quad C\frac{.99n^2}{(100)^2} \quad \left(\frac{99}{100}\right)^2 n^2$ — $C(n)$

→ Shorter

$< Cn.$

$< Cn.$

(1) $\qquad$ (1) $\qquad T\left(\left(\frac{99}{100}\right)^k n\right)$

depth of the Smallest branch → $\boxed{m = \log_{100} n.}$

$\frac{n}{100^m} = 1.$

$100^m = n.$

$m = \log_{100} n.$

depth of the longest branch. : $n \cdot \left(\frac{99}{100}\right)^k = 1.$

$$n \approx \left(\frac{100}{99}\right)^k.$$

$$k = \log_{\frac{100}{99}} n.$$

$T(n) < Cn + Cn + Cn \ldots$ upto $\log_{\frac{100}{99}} n$ levels.

$$T < Cn \cdot \log_{\frac{100}{99}} n.$$

$$\boxed{T(n) = O\left(n \log_{\frac{100}{99}} n\right)} \text{ --- I}$$

2. $T(n) > Cn + Cn + Cn \ldots$ upto $\log_{100} n$ level

$$T(n) > C n \log_{100} n$$

$$\boxed{T(n) \; \Omega\left(n \log_{100} n\right)} \text{ --- II.}$$

From equation I & II. we can conclude
Asymptotically
$$T(n) = \Theta(n \log n)$$

2-2.

(c) Domain Range Subsitution

$$T m = 9T\left(\frac{m}{3} + \frac{m^2}{\log m}\right)$$

let $m = 3^K \Rightarrow k = \log_3 m$

$$\Rightarrow m^2 = (3)^{2K} = 9^K$$

$$T(3^K) = 9T\left(\frac{3^K}{3}\right) + \frac{9^K}{K}$$

$$T(3^k) = 9T(3^{k-1}) + 9^{(k-1)}$$

$$f(k) = T(3^k)$$

$$f(k-1) = T(3^{k-1})$$

$$f(k) = 9f(k-1) + \frac{9^k}{k}$$

$$= 9\left[9f(k-2) + \frac{9^{k-1}}{(k-1)}\right] + \frac{9^k}{k}$$

$$= 9^2\left[9f(k-3) + \frac{9^{k-2}}{(k-2)}\right] + \frac{9^k}{k-1} + \frac{9^k}{k}$$

$$\vdots$$

$$f(k) = 9^k f(0) + \frac{9^k}{1} + \frac{9^k}{2} \dots + \frac{9^k}{k-1} + \frac{9^k}{k}$$

$$= 9^k \cdot \theta(1) + 9^k\left[1 + \frac{1}{2} + \dots \frac{1}{k-1} + \frac{1}{k}\right]$$

$$T(k) = 9^k \cdot \theta(1) + 9^k \log k$$

$$T(3^k) = 9^k \cdot \theta(1) + 9^k \log k$$

$$\boxed{\begin{array}{l}\text{Since } 3^k = m \Rightarrow 9^k = m^2 \\ K = \log_3 m\end{array}}$$

cont.

$T(n) = n^2 \cdot \theta(1) + n^2 \log \log n.$

$= \theta(n^2 \log \log n)$ Tightly bound.

2-3 Solution

$T(n) = T\left(\frac{n}{3^{3/4}}\right) + \theta(1)$

$T(n) = T\left(\frac{n}{3^{3/4}}\right) + C.$

Compare with $T(n) = a\,T\left(\frac{n}{b}\right) + f(n).$

$a=1,\ b=3,\ f(n)=C.$

$n^{\log_b a} = n^{\log_{3^{3/4}} 1} = n^0 = 1.\ \ f(n) = \underline{\underline{C}}.$

by Master theorem Case 2.

$T(n) = \theta\left(1 \cdot \log_{3^{3/4}} n\right)$

$= \theta\left(\log_{3^{3/4}} n\right)$

(b) The function BLA returns Value $\underline{\underline{1}}$.

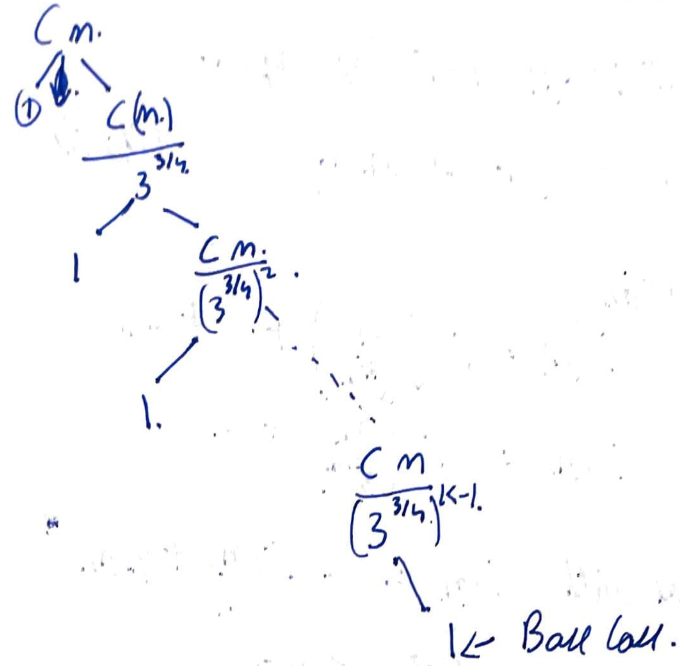
(c) Expecting running time of Foo.

Deriving the Recurrence.

$S(n) = S\left(\frac{n}{3^{3/4}}\right) + \theta(n)$

$= S\left(\frac{n}{3^{3/4}}\right) + E(n)$

Cont.



$$S(n) = Cn + C\frac{n}{3^{3/4}} + \frac{Cn}{(3^{3/4})^2} + \frac{Cn}{(3^{3/4})^{k-1}} + S(1)$$

$$= Cn\left[1 + \frac{1}{3^{3/4}} + \frac{1}{(3^{3/4})^2} \cdots\cdots \frac{1}{3^{3/4\,(k-1)}}\right] + C'$$

$$= C(n)\, \frac{1 - \frac{1}{(3^{3/4})^k}}{\left(1 - \frac{1}{3^{3/4}}\right)} \qquad \therefore (3^{3/4})^k = n \text{ (base case.}$$

$$k = \log_{3^{3/4}} n.$$

$$Sn = Cn\left(\frac{3^{3/4}}{(3^{3/4}-1)}\right) \cdot \left(1 - \frac{1}{n}\right)$$

$$Sn = \frac{3^{3/4}}{(3^{3/4}-1)} \cdot (Cn - C)$$

$$Sn = \Theta(n)$$