CSCI-GA.1170-001 Homework 2

Ankit Sati

TOTAL POINTS

27 / 36

QUESTION 1

1 Different Methods for Recurrences 8 / 14

✓ **+ 1 pts** ***Part a)** Correct values for $$c_1$$ and $$n$$ (for Big-O)*

✓ **+ 1 pts** ***Part a)** Correct values for $$c_2$$ and $$n$$ (for $$\Omega$$)*

✓ **+ 1 pts** ***Part c)** Correct Level 0*

✓ **+ 1 pts** ***Part c)** Correct Level $$1$$*

✓ **+ 0.5 pts** ***Part c)** Correct Level $$d-1$$*

✓ **+ 1 pts** ***Part c)** Correct Level $$d$$*

✓ **+ 0.5 pts** ***Part c)** Correct depth*

+ 2 *Point adjustment*

Part a is wrong.

1 On what assumption n is being replaced with nlogn ?

2 Incorrect T(n) and asymptotic bound

Incorrect/missing steps for part c. Final answer missing/incorrect

QUESTION 2

2 Fun with Recurrences 14 / 15

✓ **+ 1 pts** *(c) correct substitution*

✓ **+ 1 pts** *(c) correct relation for step 2*

✓ **+ 1 pts** *(c) correct relation for step 3*

✓ **+ 1 pts** *(a) Extra credit for correct constant*

✓ **+ 5 pts** *(B) All Correct*

✓ **+ 5 pts** *(a) all correct*

QUESTION 3

3 Expected Running Time 5 / 7

✓ **- 0 pts** *Correct*

✓ **- 1 pts** *Incorrect/Missing recurrence relation for n > 5 in part (a)*

✓ **- 1 pts** *Incorrect/Missing base case for n <= 5 in part (a)*

✓ **- 0.5 pts** *Incorrect/Missing T(n) in part (a)*

✓ **+ 0.5 pts** *Correct final answer in part (c)*

3 Missing justification for why Bla(Foo) is constant time

4 Missing base case (n <= 1)

Problem 2-1.

a) $T(n) = 2T\left(\frac{n}{2}+1\right) + n-2$

given $T(3) = 0$

To prove. $T(n) = \Theta(n-2)\log(n-2)$ ∴ Tight bounds.

→ Prove by master theorem for Cheat value.

$T(n) = aT\left(\frac{n}{b}\right) + 1.$

$f(n) = \Theta(n^k \cdot \log^p n)$

$\log_b a = 1$; $k = 1$; $P = 0$.

Case 2 $f(n) = \Theta(n^k \cdot \log^{p+1} n)$ → Cheat Value by Masters Theor.

$\boxed{= \Theta(n \cdot \log n)}$ → best guess.

Guess then Verify

$T(n) = 2T\left(\frac{n+2}{2}\right) + n - 2.$

Guessing the value

$= 2c\left(\frac{n\log n + 2}{2}\right) + n\log n - 2.$

$= c(n\log n + 2) + n\log n - 2.$

$\simeq \Theta(n\log n).$

Since $T(3) = 0$.

∴ $(n-2)\log(n-2)$ ∀ $n \geq 3$.

which means for every number $(n \geq 3)$ the below expression holds true

$\Omega c_1 n\log n \leq \Theta(n-2)\log(n-2) \leq O c_2 n \log n$

where c_1 & $c_2 > 0$

1

$\therefore c_1\, n \log n \leq (n-2) \log(n-2) \leq c_2\, n \log n$

which means that our function is a tight bound.

$\forall\, c_1, c_2 > 0 \Rightarrow f(n) = \Theta\, g(n)$

$$\boxed{= \Theta \cdot (n-2) \log(n-2)}$$ ← Proven by guess then verify.

b) Domain Range substitution for a Reducing Recurrence.

$$T(n) = 2T\left(\frac{n}{2}+1\right) + (n-2)$$

Assuming that the function is reducing, we will solve towards $T(n) = 1$.

$$T_n = \begin{cases} 0 & ; n = 3 \\ 2T\left(\frac{n}{2}+1\right) + n-2 & ; n > 1 \end{cases}$$

$$T(n) = 2T\left(\frac{n}{2}+1\right) + n-2 \quad \text{--- (1)}$$

Solving for $T\left(\frac{n}{2}\right)$

$$\cancel{T\left(\frac{n}{2}\right) = 2\left[2T\left(\frac{n+1}{2^2}\right) + \left(\frac{n}{2}-2\right)\right]}$$

$$T\left(\frac{n}{2}\right) = 2\left[2T\left(\frac{n+1}{2^2}\right) + \frac{n}{2} - 2\right] + n-2$$

$$2^2 T\, \frac{n+1}{2^2} + n - 2 + n - 2 \quad \text{--- (2)}$$

⋮

Relation to function in reduced form: ↗ Total work for full credit (Reduced form)

$$f(n) = \boxed{T_n = 2^k \cdot T\left(\frac{n+1}{2^k}\right) + k(n-2)} \longrightarrow \text{Main equation}$$

$\therefore f(3) = 0$. min value $(n-2)$

$$T\left(\frac{n-2}{2^k}\right) = T(1). \qquad \frac{n-2}{2^k} = 1$$

$$k = \log(n-2)$$

Cont. 2

Since

$$T(n) = 2^K T\left(\frac{n-2}{2^K}\right) + K(n-2)$$

& $K = \log(n-2)$ —— (4)

Substituting the values

$$\boxed{\begin{aligned} T(n) &= 2^K \cdot T(3) + K(n-2) \\ T(n) &= K(n-2) \\ T(n) &= (n-2)\cdot \log(n-2) \end{aligned}}$$

← Answer. * Approach 1.

Alternative Approach:

$$T(n) = \Omega((n-2)\log(n-2))$$

$$\Rightarrow T(n) \geq c\,((n-2)\log(n-2))$$

Solution for sub problems with size $\left(\frac{n}{2}+1\right)$

here. $T\left(\frac{n}{2}+1\right) \geq c\left(\left(\frac{n}{2}+1\right)-2\right)\log\left(\frac{n}{2}+1-2\right)$

$$\geq c\left(\frac{n}{2}-1\right)\log\left(\frac{n}{2}-1\right)$$

$$T(n) = 2T\left(\frac{n}{2}+1\right) + (n-2)$$

$$T(n) \geq 2\left[c\left(\frac{n}{2}-1\right)\log\left(\frac{n}{2}-1\right)\right] + (n-2)$$

$$T(n) \geq c(n-2)\log\left(\frac{n-2}{2}\right) + n-2$$

$$T(n) \geq c(n-2)\left[\log(n-2) - \log 2\right] + (n-2)$$

$$T(n) \geq c(n-2)\log(n-2) + (n-2)\left[1 - c\log 2\right]$$

$$T(n) \geq c(n-2)\log(n-2) \quad \forall \boxed{c \geq \frac{1}{\log 2}}$$

$\therefore$ $T(n) = \Omega((n-2)\log(n-2))$

Since $T(n) = O((n-2)\log(n-2))$ → Approach 2.

$$\boxed{\therefore T(n) = \Theta((n-2)\log(n-2)) \quad \forall\, c_1, c_2 > 0.}$$ (3)

2-1(c) Recursion Tree Method

$T(0) = 1.$

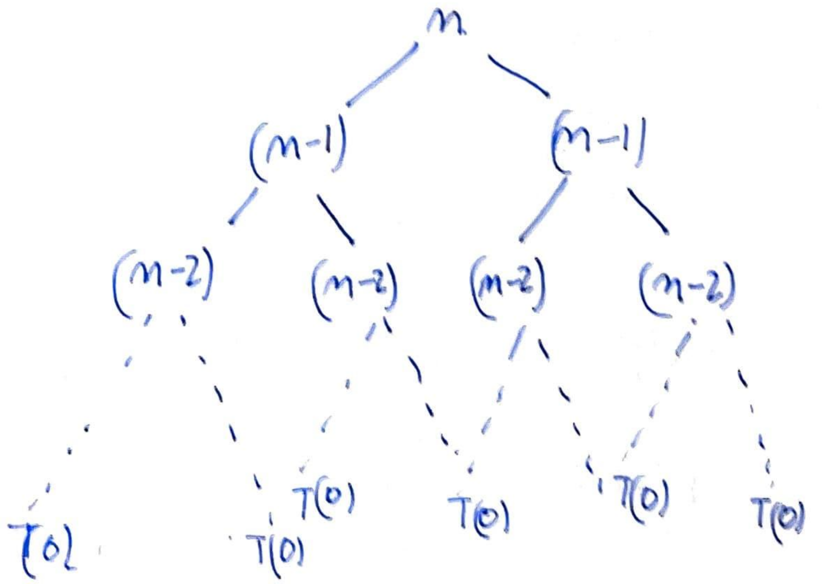
$T(n) = 2T(n-1) + n, \quad T(0) = 1.$

$T(n) = aT\left(\frac{n}{b}\right) + f(n)$

$\cancel{f(n) = \Theta(n^d)}$

$f(n) = \Theta(n^k \cdot \log^p n)$

Recursion Tree



$T(0) = 1.$

Total Cost

n

$2(n-1)$

$2^2(n-2)$

$2^3(n-3)$

$\vdots$

$2^k \cdot (n-k)$

For the exact value of $T(n)$

$$T(n) = n + 2(n-1) + 2^2(n-2) + \dots 2^{(n-1)}(n-(n-1)) + 2^n \cdot T(0)$$

$$T(n) = n + 2n + 2^2 n + 2^3 n \dots + 2^{(n-1)} n - (1\cdot 2 + 2\cdot 2^2 + 3\cdot 2^3 + \dots (n-1)2^{(n-1)})$$

$$T(n) = n\left[1 + 2 + 2^2 + \dots 2^{n-1}\right] - \sum_{i=1}^{n-1} i\, 2^i + 2^n$$ As Per hint

$$= \frac{n\cdot(2^n - 1)}{2-1} + \boxed{\frac{(n-1)2^{n+1} - 2^n \cdot n + 2 + 2^n}{(2-1)^2}}$$

$$= \cancel{n\cdot 2^n} - n + n\cdot 2^{n+1} - 2^{n+1} - \cancel{n\cdot 2^n} + 2 + 2^n$$

5

Cont.

$= 2 \cdot m 2^m - 2^m - m + 2.$

$$\boxed{T(m) = 2 \cdot m 2^m - 2^m - m + 2.} \rightarrow$$ Correct answer as per the question ~~Assign~~

$$\boxed{T(m) = \Theta(m \cdot 2^m)} \rightarrow$$ Asymptotic Notation.

Table.

Level	Size of pr.	No of Pr.	Non recursive cost	Total cost
0	m	1	m	m
1	$(m-1)$	2	$2(m-1)$	$2(m-1)$
$d-1$	$(m-(d-1))$	2^{d-1}	$2^{d-1}(m-(d-1))$	$2^{d-1}(m-(d-1))$
d	$(m-d)$	2^d	~~1(d(a(a)))~~ 1	$2^d \cdot 1$

Last level d

$m - d = 0$

$\therefore m = d$. (as mentioned above)

Cost at level $d = 2^m$.

cost at level $d-1 = 2^{m-1}(m-(m-1))$

$= 2^{m-1} \cdot 1$ Ans.

~~Conf~~ Asymptotically equal.

$m \rightarrow$ CEO

$2 \rightarrow$ V.P.

~~Martin Hoffman~~ ~~Ro~~

1 Different Methods for Recurrences 8 / 14

✓ **+ 1 pts** ***Part a)** Correct values for $$c_1$$ and $$n$$ (for Big-O)*

✓ **+ 1 pts** ***Part a)** Correct values for $$c_2$$ and $$n$$ (for $$\Omega$$)*

✓ **+ 1 pts** ***Part c)** Correct Level 0*

✓ **+ 1 pts** ***Part c)** Correct Level $$1$$*

✓ **+ 0.5 pts** ***Part c)** Correct Level $$d-1$$*

✓ **+ 1 pts** ***Part c)** Correct Level $$d$$*

✓ **+ 0.5 pts** ***Part c)** Correct depth*

+ 2 *Point adjustment*

Part a is wrong.

1 On what assumption n is being replaced with nlogn ?

2 Incorrect T(n) and asymptotic bound

Problem 2-2. (Fun with Recurrences)

(a). $T(n,1) = 3n$

$T(1,n) = 3n$.

$T(n,n) = 3n + T\left(\frac{n}{3}, \frac{n}{3}\right)$

$$T(n,n^2) = T\left(\frac{n}{3}, \frac{n^2}{3^2}\right) + 3n.$$

$$= T\left(\frac{n}{3^2}, \frac{n^2}{9}\right) + 3n + 3 \cdot \frac{n}{3}$$

$$= T\left(\frac{n}{3^3}, \frac{n^2}{3^3}\right) + 3n + 3\frac{n}{3} + 3\frac{n}{3^2} \ldots$$

$$= \cancel{T\left(\frac{n}{3^4}\right.}$$

$$T\left(\frac{n}{3^k}, \frac{n^2}{3^k}\right) + 3n\left[1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3} + \cdots \frac{1}{3^{k-1}}\right]$$

Put $\frac{n}{3^k} = 1. \Rightarrow 3^k = n \Rightarrow k = \log_3 n.$

$$\Rightarrow T(n,n^2) = T\left(1, \frac{n^2}{n}\right) + 3n\left[\frac{1 \cdot (1-1/3)^k}{(1-1/3)}\right]$$

$$= T(1,n) + 3n \times \frac{3}{2}\left(1 - \frac{1}{3^k}\right)$$

$$= 3n + \frac{9n}{2}\left(1 - \frac{1}{n}\right)$$

$$= 3n + \frac{9n}{2} - \frac{9}{2}.$$

Answer.

$$= \frac{15n}{2} + \frac{9}{2}. = O(n)$$

$$\boxed{\therefore\ T(n,n^2) = O(n)}$$

Extra Credit. $T(n) = 15\frac{n}{2} + \frac{9}{2}$ $\rightarrow O(1)$.

$$\boxed{= 15\frac{n}{2} + O(1) \Rightarrow 15\frac{n}{2} + O(1)}$$

8

2-2 Recurrence Tree.

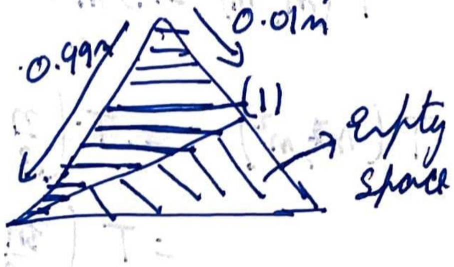
Unbalanced

$T(n) = T(0.01n) + T(0.99n) + cn$

$\hookrightarrow$ Shortest branch VP. $\hookrightarrow$ Longest branch $\rightarrow$ VP. $cn \hookrightarrow$ CEO

$$T(n) = T\left(\frac{n}{100}\right) + T\left(\frac{99n}{100}\right) + cn.$$

Reference diagram



Recursion tree

Shortest Bra. $\rightarrow$ cn $\rightarrow$ Longest branch — $c(n)$

$\frac{n}{100}$ $\quad$ $\frac{99n}{100}$ — $c(n)$

$\left(\frac{n}{100}\right)^2$ $\quad$ $\frac{99n^2}{(100)^2}$ $\quad$ $c\frac{.99n^2}{(100)^2}$ $\quad$ $\left(\frac{99}{100}\right)^2 n^2$ — $c(n)$

$\rightarrow$ Shorter $< cn.$

$< cn.$

(1) $\quad$ (1) $\quad$ $T\left(\left(\frac{99}{100}\right)^k n\right)$

depth of the smallest branch $\rightarrow$ $\boxed{m = \log_{100} n}$

$$\frac{n}{100^m} = 1.$$

$$100^m = n.$$

$$m = \log_{100} n.$$

0

depth of the largest branch. : $n \cdot \left(\frac{99}{100}\right)^K = 1$.

$$n \approx \left(\frac{100}{99}\right)^K.$$

$$K = \log_{\frac{100}{99}} n.$$

$T(n) < Cn + Cn + Cn \ldots$ upto $\log_{\frac{100}{99}} n$ levels.

$$T < Cn \cdot \log_{\frac{100}{99}} n.$$

$$\boxed{T(n) = O\left(n \log_{\frac{100}{99}} n\right)} \text{ — I}$$

& $T(n) > Cn + Cn + Cn \ldots$ upto $\log_{100} n$ level

$$T(n) > c\,n \log_{100} n$$

$$\boxed{T(n) = \Omega\left(n \log_{100} n\right)} \text{ — II.}$$

From equation I & II. we can conclude Asymtotically

$$T(n) = \Theta(n \log n)$$

2-2.

(c) Domain Range Substitution

$$Tm = 9T\left(\frac{m}{3} + \frac{m^2}{\log m}\right)$$

let $m = 3^K$. $\Rightarrow k = \log_3 m$.

$\Rightarrow m^2 = (3)^{2K} = 9^K$.

$$T(3^K) = 9T\left(\frac{3^K}{3}\right) + \frac{9^K}{K}.$$

$$T(3k = 9T(3^{K-1}) + 9^{(K-1)}$$

$f(k) = T(3^k)$

$f(k-1) = T(3^{K-1})$

$$f(k) = 9f(k-1) + \frac{9^k}{k}.$$

$$= 9\left[9f(k-2) + \frac{9^{K-1}}{(K-1)}\right] + \frac{9^k}{k}.$$

$$= 9^2\left[9f(k-3) + \frac{9^{k-2}}{(k-2)}\right] + \frac{9^k}{K-1} + \frac{9^k}{k}$$

$$f(k) = 9^k \cdot f(0) + \frac{9^k}{1} + \frac{9^k}{2} \cdots + \frac{9}{K-1} + \frac{9}{K}.$$

$$= 9^k \cdot \theta(1) + 9^k\left[1 + \frac{1}{2} + \cdots \frac{1}{K-1} + \frac{1}{K}\right].$$

$$T(k) = 9^k \cdot \theta(1) + 9^k \log k.$$

$$T(3^k) = 9^k \cdot \theta(1) + 9^k \log k.$$

Since $3^k = m \Rightarrow 9^k = m^2$.

$K = \log_3 m$.

cont.

2 Fun with Recurrences 14 / 15

✓ **+ 1 pts** *(c) correct substitution*

✓ **+ 1 pts** *(c) correct relation for step 2*

✓ **+ 1 pts** *(c) correct relation for step 3*

✓ **+ 1 pts** *(a) Extra credit for correct constant*

✓ **+ 5 pts** *(B) All Correct*

✓ **+ 5 pts** *(a) all correct*

Incorrect/missing steps for part c. Final answer missing/incorrect

$$T(n) = n^2 : \theta(1) + n^2 \log\log n.$$

$$= \theta(n^2 \log\log n) \quad \text{Tightly bound.}$$

2-3 Solution

$$T(n) = T\left(\frac{n}{3^{3/4}}\right) + \theta(1).$$

$$T(n) = T\left(\frac{n}{3^{3/4}}\right) + C.$$

Compare with $T(n) = a\,T\left(\frac{n}{b}\right) + f(n).$

$a=1,\ b=3,\ f(n)=C.$

$n^{\log_b a} = n^{\log_{3^{3/4}} 1} = n^0 = 1 \quad f(n) = \underline{\underline{C}}.$

by Master theorem Case 2.

$$T(n) = \theta(1 \cdot \log_{3^{3/4}} n)$$

$$= \theta(\log_{3^{3/4}} n)$$

(b) The function BLA returns Value $\underline{\underline{1}}$.

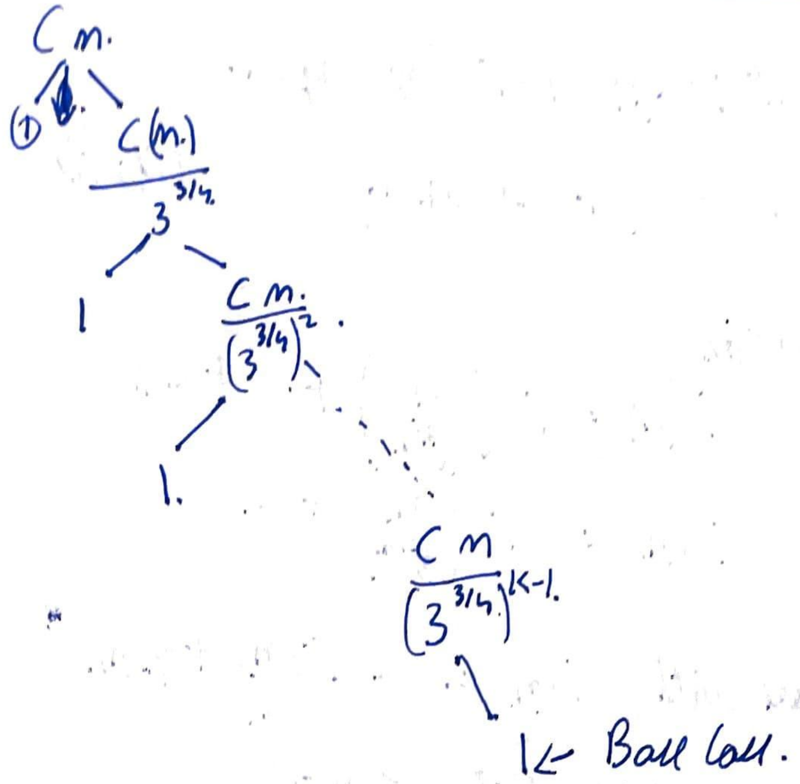
(c) Expecting running time of Foo.

Deriving the Recurrence.

$$S(n) = S\left(\frac{n}{3^{3/4}}\right) + \theta(n)$$

$$= S\left(\frac{n}{3^{3/4}}\right) + C(n)$$

Cont.



$$S(n) = Cn + C\frac{n}{3^{3/4}} + \frac{Cn}{(3^{3/4})^2} + \frac{Cn}{(3^{3/4})^{k-1}} + S(1)$$

$$= Cn\left[1 + \frac{1}{3^{3/4}} + \frac{1}{(3^{3/4})^2} \cdots \frac{1}{3^{3/4(k-1)}}\right] + c'$$

$$= C(n)\,\frac{1 - \frac{1}{(3^{3/4})^k}}{\left(1 - \frac{1}{3^{3/4}}\right)} \qquad \therefore (3^{3/4})^k = n \text{ (base case.}$$

$$k = \log_{3^{3/4}} n.$$

$$Sn = Cn\left(\frac{3^{3/4}}{(3^{3/4}-1)}\right)\cdot\left(1 - \frac{1}{n}\right)$$

$$Sn = \frac{3^{3/4}}{(3^{3/4}-1)}\cdot(Cn - C)$$

$$Sn = \Theta(n)$$

3 Expected Running Time 5 / 7

✓ **- 0 pts** *Correct*

✓ **- 1 pts** *Incorrect/Missing recurrence relation for n > 5 in part (a)*

✓ **- 1 pts** *Incorrect/Missing base case for n <= 5 in part (a)*

✓ **- 0.5 pts** *Incorrect/Missing T(n) in part (a)*

✓ **+ 0.5 pts** *Correct final answer in part (c)*

3 Missing justification for why Bla(Foo) is constant time

4 Missing base case (n <= 1)