Signale und Systeme

Einträge der Arbeits-Kästen im Vorlesungsskript [1]

Studiengang Technische Informatik

(Bachelor, 3.Semester)

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[1] Gleichungsnummern (auf der rechten Seite von Gleichungen) bitte dem "echten" Vorlesungsskript entnehmen bzw. dort stehen lassen, da hier nur die Gleichungen in den Arbeits-Kästen konsekutiv durchnummeriert sind!

$\boxed{1}$

$$\underline{U}_1 = U_1\angle\varphi_1 = \frac{30}{\sqrt{2}}\angle\frac{\pi}{3} \approx 21.2\angle\frac{\pi}{3}$$

$$\underline{Z}_R = R = 1000$$

$$\underline{Z}_C = \frac{1}{j\omega C} = \frac{1}{sC} = \frac{1000}{s}$$

(und falls auch L im Netzwerk vorkäme: $\underline{Z}_L = j\omega L = sL$)

$\boxed{2}$

$$H(s) := \frac{\underline{U}_2}{\underline{U}_1} = \frac{\underline{Z}_C}{\underline{Z}_R + \underline{Z}_C} = \frac{1}{sC(R+1/sC)} = \frac{1}{1+sRC} = \frac{1}{1+s}$$

$\boxed{3}$

$$|H(j\omega)| = \left|\frac{1}{1+j\omega RC}\right| = \frac{1}{|1+j\omega RC|} = \frac{1}{\sqrt{1+(\omega RC)^2}} = \frac{1}{\sqrt{1+\omega^2}}$$

$$\sphericalangle H(j\omega) = \sphericalangle\frac{1\cdot(1-j\omega RC)}{(1+j\omega RC)(1-j\omega RC)} = \sphericalangle\frac{1-j\omega RC}{1+(RC)^2} = \arctan\frac{-\omega RC/(1+(RC)^2)}{1/(1+(RC)^2)}$$

$$= -\arctan(\omega RC) = -\arctan(\omega)$$

$$\underline{U}_2 = H(s)\cdot\underline{U}_1 = |H(j\omega)|\cdot|\underline{U}_1|\angle(\varphi_1 + \sphericalangle H(j\omega))$$

$$= \frac{1}{\sqrt{1+\omega^2}}\cdot|\underline{U}_1|\angle(\frac{\pi}{3} - \arctan(\omega)) = \ldots...$$

$\boxed{4}$

$$\text{a)}\underline{U}_2 = \frac{1}{\sqrt{1+(2\pi\cdot 0.5)^2}}\cdot 21.2\angle(\frac{\pi}{3} - \arctan(2\pi\cdot 0.5)) \approx 6.43\angle-0.21$$

$$\Rightarrow \quad u_2(t) = 6.43 * \sqrt{2}\,V\angle-0.21 \approx 9.09\,V\sin(\pi t - 0.21)$$

$$\text{b)}\underline{U}_2 = \frac{1}{\sqrt{1+(2\pi 5)^2}}\cdot 21.2\angle(\frac{\pi}{3} - \arctan(2\pi\cdot 5)) \approx 0.67\angle-0.49$$

$$\Rightarrow \quad u_2(t) = 0.95\,V\sin(10\pi t - 0.49)$$

$$\text{c)}\underline{U}_2 = \frac{1}{\sqrt{1+(2\pi 500)^2}}\cdot 21.2\angle(\frac{\pi}{3} - \arctan(2\pi\cdot 500)) \approx 0.00675\angle-0.523$$

$$\Rightarrow \quad u_2(t) = 6.75mV\sin(1000\pi t - 0.523)$$

$\boxed{5}$

Für $x(t) = 30\,V\sin(\pi t + \pi/3)$ ist $\mathcal{H}\{x(t)\} = 9.09\,V\sin(\pi t - 0.21)$.

Für $y(t) = 30\,V\sin(10\pi t + \pi/3)$ ist $\mathcal{H}\{y(t)\} = 0.95\,V\sin(\pi t - 0.21)$.

$$u_1(t) = 15\,V\,\sin(\pi t + \pi/3) + 60\,V\,\sin(10\pi t + \pi/3) = 0.5x(t) + 2y(t)$$

und damit $a = 0.5,\; b = 2$ und

$$u_2(t) := \mathcal{H}\{u_1(t)\} = \mathcal{H}\{0.5x(t) + 2y(t)\} \overset{(\textbf{??})}{=} 0.5\mathcal{H}\{x(t)\} + 2\mathcal{H}\{y(t)\}$$
$$= 4.55\,V \cdot \sin(\pi t - 0.21) + 1.9\,V \cdot \sin(\pi t - 0.21)\ .$$

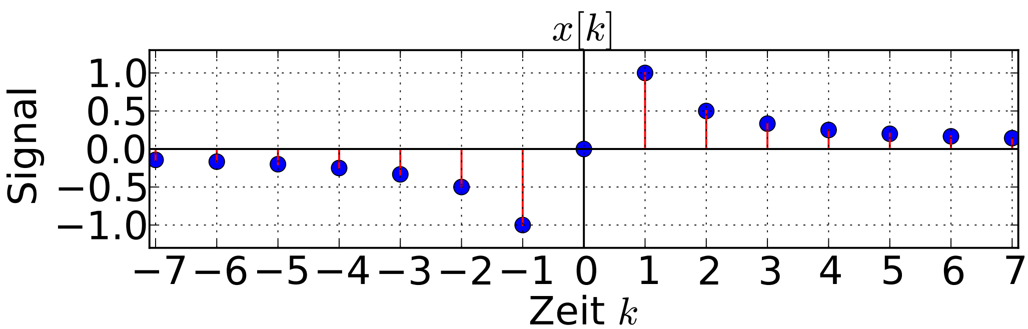
6

$$\mathcal{H}\{ax(t) + by(t) + cz(t)\} = \mathcal{H}\{ax(t) + 1 \cdot (by(t) + cz(t))\}$$
$$\overset{(\textbf{??})}{=} a\mathcal{H}\{x(t)\} + 1 \cdot \mathcal{H}\{by(t)) + cz(t)\} \overset{(\textbf{??})}{=} a\mathcal{H}\{x(t)\} + b\mathcal{H}\{y(t)\} + c\mathcal{H}\{z(t)\}\ .$$

7

$$x[-\infty], \ldots, x[-3], x[-2], x[-1], x[0], x[1], x[2], x[3], \ldots, x[\infty]$$

8



9

- $x[-k]$ die Spiegelung von $x[k]$ an der Signalpegel-Achse;
- $x[k + k_0]$ die Verschiebung von $x[k]$ um k_0 nach links.
- $x[k - k_0]$ die Verschiebung von $x[k]$ um k_0 nach rechts.

10

11 Hierbei gilt:

(b)
$$x[-k] = \begin{cases} -\frac{1}{k} & , \ k \neq 0 \\ 0 & , \ k = 0 \end{cases}$$

(c)
$$x[k+k_0] = x[k+3] = \begin{cases} \frac{1}{k+3} & , \ k \neq -3 \\ 0 & , \ k = -3 \end{cases}$$

(d)
$$x[k-k_0] = x[k-3] = \begin{cases} \frac{1}{k-3} & , \ k \neq 3 \\ 0 & , \ k = 3 \end{cases}.$$

12
$$x[k_0 - k] = x[-(k - k_0)]$$

13
$$\text{mit} \quad x[k_0 - k] = x[3-k] = \begin{cases} \frac{1}{3-k} & , \ k \neq 3 \\ 0 & , \ k = 3 \end{cases}.$$

14

- $x[k]$ heißt gerades Signal, falls $x[k] = x[-k] \ \forall k \in \mathbb{Z}$ gilt.
- $x[k]$ heißt ungerades Signal, falls $x[k] = -x[-k] \ \forall k \in \mathbb{Z}$ gilt.

15
$$x[-k] = \begin{cases} \frac{1}{-k} & , \ k \neq 0 \\ 0 & , \ k = 0 \end{cases} = \begin{cases} -\frac{1}{k} & , \ k \neq 0 \\ 0 & , \ k = 0 \end{cases} = -x[k]$$

16
$$y[-k] = \begin{cases} \frac{1}{(-k)^2} & , \ k \neq 0 \\ 0 & , \ k = 0 \end{cases} = \begin{cases} \frac{1}{k^2} & , \ k \neq 0 \\ 0 & , \ k = 0 \end{cases} = y[k]$$

17

- $x[k]$ heißt kausales Signal, falls gilt: $x[k] = 0 \ \forall k < 0$.
- $x[k]$ heißt nicht-kausales Signal falls es nicht kausal ist, d.h. falls gilt: $\exists k < 0 : x[k] \neq 0$.
- $x[k]$ heißt anti-kausales Signal falls $x[-k-1]$ kausal ist, d.h. falls gilt: $x[k] = 0 \ \forall k \geq 0$.

Lösung (siehe auch Skizze): 18

- $x[k]$ ist weder kausal noch anti-kausal, da es bei Null weder anfängt noch aufhört, d.h. da es von Null verschiedene Signalwerte sowohl für $k \geq 0$ als auch für $k < 0$ gibt. Insbesondere ist $x[k]$ also nicht kausal.
- $u[k]$ ist kausal, da es erst bei 0 anfängt, d.h. da $u[k] = 0$ für alle $k < 0$.
- $v[k]$ ist anti-kausal, da es bei 0 aufhört, d.h. da $v[k] = 0$ für alle $k \geq 0$.

19

$$\delta[k] := \begin{cases} 1 & , \ k = 0 \\ 0 & , \ k \neq 0 \end{cases}$$

20

$$\epsilon[k] := \begin{cases} 1 & , \ k \geq 0 \\ 0 & , \ k < 0 \end{cases}$$

21

$$\delta[k - k_0] = \begin{cases} 1 & , \ k = k_0 \\ 0 & , \ k \neq k_0 \end{cases}$$

bzw.

$$\delta[k + k_0] = \begin{cases} 1 & , \ k = -k_0 \\ 0 & , \ k \neq -k_0 \end{cases}$$

22

$$\begin{aligned} x[k] \cdot \delta[k - i] &= \begin{cases} x[i] & , k = i \\ 0 & , k \neq i \end{cases} \\ &= x[i] \cdot \delta[k - i] \end{aligned} \tag{1}$$

(Siebeigenschaft)

23

$$x[k] \;=\; \sum_{i=-\infty}^{\infty} x[i]\cdot\delta[k-i] \quad \text{für alle } k\in\mathbb{Z}$$

24

$$x[k] = \sum_{i=-K}^{K} x[i]\cdot\delta[k-i]$$

25

Lösung:

$$\begin{aligned} u[k] &= \delta[k+2]+\delta[k+1]+\delta[k]+\delta[k-1] \\ v[k] &= 2\delta[k+3]+\delta[k+1]-\delta[k-1]-2\delta[k-3] \end{aligned}$$

26

a) Signum-Folge: $\operatorname{sgn}[k] := \epsilon[k]-\epsilon[-k] = \begin{cases} 1 & , \; k>0 \\ 0 & , \; k=0 \\ -1 & , \; k<0 \end{cases}.$

27

b) Kamm-Folge oder Scha-Folge: $\text{Ш}[k] := \epsilon[k]+\epsilon[-k-1] = 1$ für alle $k\in\mathbb{Z}$.

28

c) Rechteck-Folge: $\operatorname{rect}_{k_1,k_2}[k] := \epsilon[k-k_1]-\epsilon[k-k_2-1] = \begin{cases} 1 & , \; k_1\le k\le k_2 \\ 0 & , \; \text{sonst} \end{cases}.$

29

d) Kausale Exponential-Folge: $x[k] = a^k\cdot\epsilon[k]$.

30

$$x[k]: 0,\ldots,0, x[0]=1, x[1]=0.7, x[2]=0.49, x[3]=0.343,\ldots$$

31

$$x[k]: 0,\ldots,0, x[0]=1, x[1]=-0.8, x[2]=0.64, x[3]=-0.512,\ldots$$

32

$$\begin{aligned} x[k]+y[k] &: x[-\infty]+y[-\infty],\ldots,x[0]+y[0],x[1]+y[1],\ldots,x[\infty]+y[\infty] \\ x[k]\cdot y[k] &: x[-\infty]\cdot y[-\infty],\ldots,x[0]\cdot y[0],x[1]\cdot y[1],\ldots,x[\infty]\cdot y[\infty] \\ c\cdot x[k] &: c\cdot x[-\infty],\ldots,c\cdot x[-1],c\cdot x[0],c\cdot x[1],\ldots,c\cdot x[\infty] \end{aligned}$$

$$\mathcal{S}_{k_1,k_2} := \{\vec{x} \in \mathcal{S} | x[k] = 0 \ \forall k < k_1 \text{ oder } k > k_2\}$$ 33

$$\begin{aligned}
\vec{x} &= (0\ 3\ 2\ 5\ 0\ 0) \\
\vec{y} &= (0\ 0\ 2\ \ -3\ 0\ 2) \\
\vec{x} + \vec{y} &= (0\ 3\ 4\ 2\ 0\ 2) \\
\vec{x} - \vec{y} &= (0\ 3\ 0\ 8\ 0\ \ -2) \\
\vec{x} \cdot \vec{y} &= (0\ 0\ 4\ \ -15\ 0\ 0) \\
c \cdot \vec{x} &= (0\ 15\ 10\ 25\ 0\ 0)
\end{aligned}$$ 34

$$(x * y)[k] := \sum_{i=-\infty}^{\infty} x[i] \cdot y[k-i]$$ 35

$$x[i] = (\ \underset{\uparrow\, i=0}{3}\ \ 2\ \ 1\ ), \quad y[i] = (\ \underset{\uparrow\, i=0}{1}\ \ -1\ \ 2\ ) \text{ bzw. } \quad y[0-i] = (\ 2\ \ -1\ \ \underset{\uparrow\, i=0}{1}\ )$$ 36

$$\begin{array}{c|cccccccc|cc}
 & x[i] = & & & \overset{i=0}{\downarrow} 3 & 2 & 1 & & & \sum x[i]y[k-i] \quad = & (x*y)[k] \\
\hline
k=0 & y[k-i] = & 2 & -1 & 1 & & & & & 3 \cdot 1 & = 3 \\
k=1 & & & 2 & -1 & 1 & & & & 3 \cdot (-1) + 2 \cdot 1 & = -1 \\
k=2 & & & & 2 & -1 & 1 & & & 3 \cdot 2 + 2 \cdot (-1) + 1 \cdot 1 & = 5 \\
k=3 & & & & & 2 & -1 & 1 & & 2 \cdot 2 + 1 \cdot (-1) & = 3 \\
k=4 & & & & & & 2 & -1 & 1 & 1 \cdot 2 & = 2
\end{array}$$ 37

$$x[k] * y[k] = 3\delta[k] - \delta[k-1] + 5\delta[k-2] + 3\delta[k-3] + 2\delta[k-4] \ .$$ 38

39

$$x[i] = (\overset{\overset{i=-43}{\downarrow}}{-1} \quad 3 \quad -2), \qquad \text{und}$$

$$y[i] = (\overset{\overset{i=19}{\downarrow}}{1} \quad -2 \quad 4 \quad -1) \quad \text{bzw.} \qquad y[-i] = (-1 \quad 4 \quad -2 \quad \overset{\overset{i=-19}{\downarrow}}{1})$$

40

k	$x[i] =$				$\overset{\overset{i=-43}{\downarrow}}{-1}$	3	-2				$(x*y)[k]$
-24	$y[k-i] =$	-1	4	-2	1						-1
-23			-1	4	-2	1					$2+3=5$
-22				-1	4	-2	1				$-4-6-2=-12$
-21					-1	4	-2	1			$1+12+4=17$
-20						-1	4	-2	1		$-3-8=-11$
-19							-1	4	-2	1	$-2=2$

41

$$(x*y)[k] = -\,\delta[k+24] + 5\delta[k+23] - 12\delta[k-22] + 17\delta[k+21] \\ -\,11\delta[k+20] + 2\delta[k+19]\ .$$

42

$$x[k] * y[k] \ \in \mathcal{S}_{a+c,b+d} \ \ \text{und hat Länge } n+m-1\ .$$

43

I) Kommutativität : $\quad x * y = y * x$

II) Assoziativität : $\quad w*(x*y) = (w*x)*y \quad$ und

$\quad c \cdot (x*y) = (c\cdot x)*y$

III) Distributivität : $\quad w*(x+y) = w*x + w*y$

IV) Neutrales Element : $\quad x * \delta = x$

V) Verschiebung : $\quad x[k] * \delta[k-k_0] = x[k-k_0]$

VI) Zeitinvarianz : $\quad x[k] * y[k-k_0] = (x[k]*y[k])[k-k_0]$

VII) Linearität : $\quad (c\cdot x + d\cdot y)*w = c\cdot(x*w) + d\cdot(y*w)$

$$p(z) := a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots + a_n z^n$$ 44

$$x[k] = a_0\delta[k] + a_1\delta[k-1] + a_2\delta[k-2] + a_3\delta[k-3] + \ldots + a_n\delta[k-n]$$ 45

$$p(z) \cdot q(z) = c_0 + c_1 z + \ldots + c_{2n} z^{2n} \quad \text{mit Koeffizienten } c_k = (x * y)[k] \; .$$ 46

$$p(z) = 3 + 2z + z^2 \qquad \text{und} \qquad q(z) = 1 - z + 2z^2$$ 47

$$\begin{aligned} &p(z) \cdot q(z) = (3 + 2z + z^2) \cdot (2z^2 - z + 1) \\ &= 3 \cdot 1 + z(3 \cdot (-1) + 2 \cdot 1) + z^2(3 \cdot 2 + 2 \cdot (-1) + 1 \cdot 1) + z^3(2 \cdot 2 + 1 \cdot (-1)) + z^4(1 \cdot 2) \\ &= 3 - z + 5z^2 + 3z^3 + 2z^4 \; . \end{aligned}$$ 48

$$E_x := \sum_{i=-\infty}^{\infty} |x[i]|^2$$ 49

$$P_x := \lim_{K \to \infty} \frac{1}{2K+1} \sum_{i=-K}^{K} |x[i]|^2$$ 50

$$\langle x[k], y[k] \rangle_E \;:=\; \sum_{k=-\infty}^{\infty} x^*[k] \cdot y[k] \; .$$ 51

$$\langle x[k], y[k] \rangle_P \;:=\; \lim_{K \to \infty} \frac{1}{2K+1} \sum_{k=-K}^{K} x^*[k] \cdot y[k] \; .$$ 52

53
$$
\begin{aligned}
||x[k]||_E &:= \sqrt{\langle x[k], x[k]\rangle_E} = \sqrt{E_x} \quad \text{bzw.}\\
||x[k]||_P &:= \sqrt{\langle x[k], x[k]\rangle_P} = \sqrt{P_x}\ .
\end{aligned}
$$

54
$$
\cos\Phi = \frac{\langle x[k], y[k]\rangle}{||x[k]|| \cdot ||y[k]||}\ .
$$

55
$$
\varphi_{xy}[\kappa] := \langle x[k], y[k+\kappa]\rangle
$$

56
$$
\varphi_{xx}[\kappa] := \langle x[k], x[k+\kappa]\rangle
$$

.

57
$$
\varphi_{xy}^{E}[\kappa] = x^*[-\kappa] * y[\kappa] \qquad \text{bzw.} \qquad \varphi_{xy}^{P}[\kappa] = \lim_{K\to\infty} \frac{1}{2K+1} x_K^*[-\kappa] * y_K[\kappa]
$$

58
$$
y[k] = \mathcal{H}\{x[k]\}
$$

59 **Beispiel**: Wir betrachten das System “Sparbuch” bei einer Bank. Wir nehmen an, das Sparbuch sei gebührenfrei und Einzahlungen werden mit Zinssatz p verzinst. Als Eingabe-Signal $x[k]$ definieren wir die Einzahlungen (z.B. zu Beginn) des Jahres k, und als Ausgabe-Signal $y[k]$ entsprechend das Guthaben des Sparbuchs.

a) Berechnen Sie für eine einmalige Einzahlung x_0 im Jahr $k = 0$ das Guthaben nach i Jahren!

b) Wie berechnet man für allgemeine Einzahlungen $x[k]$ das Guthaben $y[k]$?

Lösung: a) Für das Eingabe-Signal (bzw. die einmalige Einzahlung)

$$
x[k] = x_0 \cdot \delta[k] = \begin{cases} x_0 & , \ k = 0 \\ 0 & , \ k \neq 0 \end{cases}
$$

entwickelt sich das Guthaben des Sparbuchs wie folgt:

zu Beginn:	$y[0] = x_0$
nach 1 Jahr:	$y[1] = x_0 + p \cdot x_0 = (1+p)x_0$
nach 2 Jahren:	$y[2] = (1+p)x_0 + p \cdot (1+p)x_0 = (1+p)^2 x_0$
nach 3 Jahren:	$y[3] = (1+p)^2 x_0 + p \cdot (1+p)^2 x_0 = (1+p)^3 x_0$
$\vdots$	
nach i Jahren:	$y[i] = (1+p)^i x_0$.

D.h. das Ausgangssignal ist die kausale Exponentialfolge $y[k] = x_0 \cdot (1+p)^k \cdot \epsilon[k]$.

b) Allgemein errechnet sich das Guthaben im Jahr $k+1$ durch folgende Rekursion: 60

$$y[k+1] = y[k] \cdot (1+p) + x[k+1] \qquad (2)$$

D.h. $y[k+1]$ ergibt sich aus dem verzinsten Guthaben $y[k]$ des vorigen Jahres und den neuen Einzahlungen $x[k+1]$. .

$$\mathcal{H}\{c \cdot x_1[k] + d \cdot x_2[k]\} \;=\; c\mathcal{H}\{x_1[k]\} + d\mathcal{H}\{x_2[k]\} \;.$$ 61

$$y[0] = x[0] = c \cdot x_1[0] + dx_2[0] = cy_1[0] + dy_2[0] \;.$$ 62

63

$$\begin{aligned} y[k+1] &\overset{(??)}{=} y[k] \cdot (1+p) + x[k+1] = y[k] \cdot (1+p) + cx_1[k+1] + dx_2[k+1] \\ &\overset{(\mathrm{I.V.})}{=} (cy_1[k] + dy_2[k]) \cdot (1+p) + cx_1[k+1] + dx_2[k+1] \\ &= c(y_1[k](1+p) + x_1[k+1]) + d(y_2[k](1+p) + x_2[k+1]) \\ &\overset{(??)}{=} cy_1[k+1] + dy_2[k+1] \end{aligned}$$

64 $$y[k-k_0] = \mathcal{H}\{x[k-k_0]\}.$$

65 **I.A. ($k = k_0$):** Für die erste Einzahlung (o.B.d.A. bei $k = 0$) gilt wieder $y[0] = x[0]$ und $y[k] = 0$ für $k < 0$, und deshalb

$$z[k_0] = x[k_0 - k_0] = x[0] = y[0] = y[k_0 - k_0] \qquad \text{und } z[k] = 0 = y[k-k_0] \text{ für } k < k_0 \ .$$

66 **I.S. ($k \to k+1$):** Aus der I.V. $z[k] = y[k-k_0]$ und (**??**) folgt

$$z[k+1] \overset{(??)}{=} z[k]\cdot(1+p) + x[k+1-k_0] \overset{(\text{I.V.})}{=} y[k-k_0](1+p) + x[k-k_0+1]$$
$$\overset{(??)}{=} y[k+1-k_0] \ .$$

$\square$

67 Ein System $\mathcal{H}$ heißt kausal, wenn der Ausgabewert $y[k_0]$ zur Zeit k_0 nur von früheren Eingabewerten $x[k]$, $k \leq k_0$ abhängig ist.

68 $$|x[k]| < C \ \forall k \quad \Rightarrow \quad |y[k]| < D \ \forall k \ .$$

69 $$y[k] = x_0 \cdot (1+p)^k \cdot \epsilon[k] \to \infty \quad \text{für } k \to \infty \ .$$

70 Ein System heißt gedächtnislos, wenn der Ausgang $y[k]$ zur Zeit k nur vom Eingang $x[k]$ zur Zeit k abhängt.

71 Dagegen hat ein System ein Gedächtnis der Länge L, falls $y[k]$ nur von $x[\kappa]$ für $|\kappa - k| \leq L$ abhängt.

$$h[k] := \mathcal{H}\{\delta[k]\}$$ 72

73

$$\begin{aligned} y[k] &= \mathcal{H}\{x[k]\} = \mathcal{H}\{\sum_{i=-\infty}^{\infty} x[i]\cdot\delta[k-i]\} && \text{(wegen Satz \textbf{??})} \\ &= \sum_{i=-\infty}^{\infty} x[i]\cdot\mathcal{H}\{\delta[k-i]\} && \text{(wegen Linearität von } \mathcal{H}\text{)} \\ &= \sum_{i=-\infty}^{\infty} x[i]\cdot h[k-i] && \text{(wg. } h[k] := \mathcal{H}\{\delta[k]\} \text{ u. Zeitinv. } \mathcal{H}\text{)} \\ &= x[k] * h[k] && \text{(Def. Faltung, siehe Seite \textbf{??})} \end{aligned}$$

$$y[k] \;=\; x[k] * h[k] \quad \text{für alle } x[k] \in \mathcal{S}\ .$$ 74

$$h[k] := \mathcal{H}\{\delta[k]\} = (1+p)^k \cdot \epsilon[k] \qquad (3)$$ 75

$$y[k] \;=\; h[k] * x[k] \;=\; \sum_{i=-\infty}^{\infty} (1+p)^i \cdot \epsilon[i] \cdot x[k-i] \;=\; \sum_{i=0}^{\infty} (1+p)^i \cdot x[k-i]\ .$$ 76

$$y[k] \;=\; \sum_{i=0}^{k} (1+p)^i \cdot x[k-i]\ .$$ 77

$$\sum_{i=-\infty}^{\infty} |h[i]| < \infty\ .$$ 78

79

$$|y[k]| = |h[k]*x[k]| = |\sum_{i=-\infty}^{\infty} h[i]\cdot x[k-i]| \overset{\text{(DUG)}}{\leq} \sum_{i=-\infty}^{\infty} |h[i]\cdot x[k-i]|$$
$$= \sum_{i=-\infty}^{\infty} |h[i]|\cdot|x[k-i]| < M \sum_{i=-\infty}^{\infty} |h[i]| < M\cdot C < \infty \ ,$$

80

$$x[k] := \operatorname{sgn}(h[-k]) = \begin{cases} 1 & , \ h[-k] > 0 \\ 0 & , \ h[-k] = 0 \\ -1 & , \ h[-k] < 0 \end{cases}$$

81

$$x[k]\cdot h[-k] = \operatorname{sgn}(h[-k])\cdot h[-k] = |h[-k]| \geq 0 \ .$$

82

$$|y[0]| = |(x*h)[0]| = |\sum_{i=-\infty}^{\infty} x[i]\cdot h[-i]| = \sum_{i=-\infty}^{\infty} |h[-i]| = \sum_{i=-\infty}^{\infty} |h[i]| = \infty,$$

83

$$\sum_{i=-\infty}^{\infty} |h[i]| = \sum_{i=-\infty}^{\infty} (1+p)^i\cdot\epsilon[i] = \sum_{i=0}^{\infty}(1+p)^i$$

84

$|1+p| < 1$ bzw. äquivalent für $-2 < p < 0$.

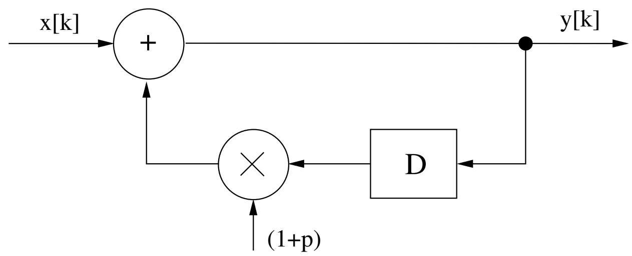
85

$$h[k] = 0 \ \forall k < 0 \ .$$

86

$$y[k] = h[k]*x[k] = \sum_{i=-\infty}^{\infty} h[i]x[k-i] = \sum_{i=0}^{\infty} h[i]x[k-i] \ ,$$

Lösung: Folgendes Blockschaltbild realisiert die Rekursion $y[k+1] = y[k] \cdot (1+p) + x[k+1]$ von (**??**) auf Seite **??**: 87



88

$$y[k] = h[k] * x[k] = \sum_{i=0}^{N} \beta_i x[k-i] \tag{4}$$

89

$$y[k] = \frac{1}{\alpha_0} \left(\sum_{i=0}^{N} \beta_i x[k-i] - \sum_{i=1}^{N} \alpha_i y[k-i] \right) \ . \tag{5}$$

90

$$\begin{aligned} \vec{u}[k+1] &= f_u(\vec{u}[k], \vec{x}[k]) \\ \vec{y}[k] &= f_y(\vec{u}[k], \vec{x}[k]) \end{aligned}$$

91

I) $\vec{y}[k_0] = f_y(\vec{u}[k_0], \vec{x}[k_0])$

II) $\vec{u}[k_0+1] = f_u(\vec{u}[k_0], \vec{x}[k_0])$

III) $\vec{y}[k_0+1] = f_y(\vec{u}[k_0+1], \vec{x}[k_0+1])$

IV) $\vec{u}[k_0+2] = f_u(\vec{u}[k_0+1], \vec{x}[k_0+1])$

V) $\vec{y}[k_0+2] = f_y(\vec{u}[k_0+2], \vec{x}[k_0+2])$

VI) $\vec{u}[k_0+3] = f_u(\vec{u}[k_0+2], \vec{x}[k_0+2])$

...

92

$$\vec{u}[k] := (x[k-1] \ x[k-2] \ \ldots \ x[k-L])^T$$

93

$$\begin{aligned} \vec{u}[k+1] &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \vec{u}[k] + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} x[k] \\ y[k] &= \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \end{pmatrix} \vec{u}[k] + a_0 x[k] \end{aligned}$$

94

$$X(z) \;:=\; \mathcal{Z}\{x[k]\} \;:=\; \sum_{k=-\infty}^{\infty} x[k]z^{-k}$$

95

$$\begin{aligned} a \;&:=\; \limsup_{k\to\infty} \sqrt[k]{|x[k]|} \\ b \;&:=\; \frac{1}{\limsup_{k\to\infty} \sqrt[k]{|x[-k]|}} \end{aligned}$$

96

$$X^{+}(z) \;:=\; \mathcal{Z}^{+}\{x[k]\} \;:=\; \sum_{k=0}^{\infty} x[k]z^{-k}\ .$$

97

a) $$\mathcal{Z}\{\delta[k]\} = \sum_{k=-\infty}^{\infty} \delta[k]z^{-k} = 1 \quad \text{für } z \in \mathbb{C}$$

b) $$\mathcal{Z}\{\delta[k-i]\} = \sum_{k=-\infty}^{\infty} \delta[k-i]z^{-k} = z^{-i} \quad \text{für } 0 < |z| < \infty$$

c) $$\begin{aligned} \mathcal{Z}\{\epsilon[k]\} &= \sum_{k=-\infty}^{\infty} \epsilon[k]z^{-k} = \sum_{k=0}^{\infty}(1/z)^{k} = \frac{1}{1-1/z} \\ &= \frac{z}{z-1} \quad \text{für } |1/z| < 1 \quad \Leftrightarrow \quad |z| > 1 \end{aligned}$$

d) $$\begin{aligned} \mathcal{Z}\{a^{k}\epsilon[k]\} &= \sum_{k=-\infty}^{\infty} a^{k}\epsilon[k]z^{-k} = \sum_{k=0}^{\infty}(a/z)^{k} = \frac{1}{1-a/z} \\ &= \frac{z}{z-a} \quad \text{für } |a/z| < 1 \quad \Leftrightarrow \quad |z| > |a| \end{aligned}$$

e) $$\begin{aligned} \mathcal{Z}\{-a^{k}\epsilon[-k-1]\} &= \sum_{k=-\infty}^{\infty} -a^{k}\epsilon[-k-1]z^{-k} = -\sum_{k=-\infty}^{-1} a^{k}z^{-k} = -\sum_{k=1}^{\infty} a^{-k}z^{k} \\ &= -\sum_{k=1}^{\infty}\left(\frac{z}{a}\right)^{k} = -\frac{z}{a}\sum_{k=0}^{\infty}\left(\frac{z}{a}\right)^{k} = -\frac{z}{a}\cdot\frac{1}{1-z/a} \\ &= \frac{z}{z-a} \quad \text{für } |z/a| < 1 \quad \Leftrightarrow \quad |z| < |a| \end{aligned}$$

$$\begin{aligned}\mathcal{Z}\{\alpha x[k]+\beta y[k]\} &= \sum_{k=-\infty}^{\infty}\left(\alpha x[k]+\beta y[k]\right)z^{-k}\\ &= \alpha\left(\sum_{k=-\infty}^{\infty}x[k]z^{-k}\right)+\beta\left(\sum_{k=-\infty}^{\infty}y[k]z^{-k}\right)\\ &= \alpha X(z)+\beta Y(z)\ .\end{aligned}$$

98

$$\begin{aligned}\mathcal{Z}\{x[k+k_0]\} = \sum_{k=-\infty}^{\infty}x[k+k_0]z^{-k} = z^{k_0}\sum_{k=-\infty}^{\infty}x[k+k_0]z^{-(k+k_0)}\\ = z^{k_0}\sum_{k'=-\infty}^{\infty}x[k']z^{-z'} = z^{k_0}X(z)\ .\end{aligned}$$

99

$$\mathcal{Z}\{\alpha^k\cdot x[k]\} = \sum_{k=-\infty}^{\infty}\alpha^k x[k]z^{-k} = \sum_{k=-\infty}^{\infty}x[k]\left(\frac{z}{\alpha}\right)^{-k} = X\left(\frac{z}{\alpha}\right)\ .$$

100

$$\mathcal{Z}\{x[-k]\} = \sum_{k=-\infty}^{\infty}x[-k]z^{-k} = \sum_{k'=-\infty}^{\infty}x[k']z^{k'} = \sum_{k'=-\infty}^{\infty}x[k']\left(\frac{1}{z}\right)^{-k'} = X\left(\frac{1}{z}\right)$$

101

$$x[0] = \lim_{z\to\infty}X(z)\ , \tag{6}$$

102

$$\lim_{k\to\infty}x[k] = \lim_{z\to 1}(z-1)X(z) \tag{7}$$

103

$$\mathcal{Z}\{\alpha^{k-1}\cdot\epsilon[k-1]\} = z^{-1}\mathcal{Z}\{\alpha^k\cdot\epsilon[k]\} = z^{-1}\cdot\frac{z}{z-\alpha} = \frac{1}{z-\alpha}$$

104

$$\mathcal{Z}\{A\alpha^{k-k_0}\cdot\epsilon[k-k_0]\} = Az^{-k_0}\mathcal{Z}\{\alpha^k\cdot\epsilon[k]\} = Az^{-k_0}\cdot\frac{z}{z-\alpha} = \frac{Az^{-(k_0-1)}}{z-\alpha}$$

105

106
$$\mathcal{Z}\{k\alpha^k\epsilon[k]\} = -z\frac{d}{dz}\mathcal{Z}\{\alpha^k\epsilon[k]\} = -z\frac{d}{dz}\frac{z}{z-\alpha} = -z\cdot\frac{1\cdot(z-\alpha)-z\cdot 1}{(z-\alpha)^2} = \frac{\alpha z}{(z-\alpha)^2}$$

107
$$Y(z) \;=\; H(z)\cdot X(z)$$

108
$$H(z) \;=\; \frac{Y(z)}{X(z)}$$

109
$$h[k] = (1+p)^k\epsilon[k]$$
$$\circ\!\!-\!\!\bullet$$
$$H(z) = \frac{z}{z-(1+p)}$$

110
$$y[k+1] = y[k]\cdot(1+p) + x[k+1]$$
$$\circ\!\!-\!\!\bullet$$
$$zY(z) = Y(z)\cdot(1+p) + zX(z) \qquad \Leftrightarrow \qquad Y(z)(z-(1+p)) = zX(z)$$

111
$$H(z) := \frac{Y(z)}{X(z)} = \frac{z}{z-(1+p)}\ . \tag{8}$$

112
$$x[k] = x_0\epsilon[k]$$
$$\circ\!\!-\!\!\bullet$$
$$X(z) = x_0\cdot\frac{z}{z-1}\ .$$

$$Y(z) = H(z)\cdot X(z) = \frac{z}{z-(1+p)}\cdot x_0 \cdot \frac{z}{z-1} = x_0 z^2 \frac{1}{(z-(1+p))\cdot(z-1)}$$ 113

$$\frac{1}{(z-(1+p))\cdot(z-1)} = \frac{p^{-1}}{z-(1+p)} - \frac{p^{-1}}{z-1} \qquad (9)$$ 114

$$\left(= \frac{p^{-1}(z-1-z-(1+p)}{\mathrm{HN}} = \frac{1}{\mathrm{HN}}\right) .$$

$$\frac{p^{-1}}{z-(1+p)} \;\bullet\!\!-\!\!\!-\!\!\circ\; p^{-1}\cdot(1+p)^{k-1}\epsilon[k-1] \qquad \text{und} \qquad -\frac{p^{-1}}{z-1} \;\bullet\!\!-\!\!\!-\!\!\circ\; -p^{-1}\cdot\epsilon[k-1]$$ 115

$$Y(z) = x_0\cdot z^2\cdot\left(\frac{p^{-1}}{z-(1+p)} - \frac{p^{-1}}{z-1}\right)$$ 116

$$\bullet\!\!-\!\!\!-\!\!\circ$$

$$y[k] = x_0\cdot\left(p^{-1}\cdot(1+p)^{k+1}\epsilon[k+1] - p^{-1}\cdot\epsilon[k+1]\right) = \frac{x_0}{p}\left((1+p)^{k+1}-1\right)\epsilon[k+1]$$

$$= \frac{x_0}{p}\left((1+p)^{k+1}-1\right)\epsilon[k+1]$$

$$\sum_{i=0}^{N}\alpha_i y[k-i] = \sum_{i=N-M}^{N}\beta_i x[k-i] \qquad (\alpha_0 \neq 0 \text{ und } \beta_{N-M}\neq 0)$$ 117

$$\circ\!\!-\!\!\!-\!\!\bullet$$

$$\sum_{i=0}^{N}\alpha_i z^{-i}Y(z) = \sum_{i=N-M}^{N}\beta_i z^{-i}X(z) \qquad \Leftrightarrow \qquad Y(z)\sum_{i=0}^{N}\alpha_i z^{-i} = X(z)\sum_{i=N-M}^{N}\beta_i z^{-i}$$

$$H(z) := \frac{Y(z)}{X(z)} = \frac{\sum_{i=N-M}^{N}\beta_i z^{-i}}{\sum_{i=0}^{N}\alpha_i z^{-i}} = \frac{\beta_{N-M}z^{M-N} + \beta_{N-M+1}z^{M-N-1} + \ldots + \beta_N z^{-N}}{\alpha_0 + \alpha_1 z^{-1} + \ldots + \alpha_N z^{-N}}$$ 118

$$= \frac{\beta_{N-M}z^{M} + \beta_{N-M+1}z^{M-1} + \ldots + \beta_N z^{0}}{\alpha_0 z^N + \alpha_1 z^{N-1} + \ldots + \alpha_N z^0}.$$

119

$$y[k] = \frac{1}{\alpha_0}\left(\sum_{i=N-M}^{N} \beta_i x[k-i] - \sum_{i=1}^{M} \alpha_i y[k-i]\right)$$

120

$$h[k] = 3 \cdot \left(\frac{1}{5}\right)^k \cdot \epsilon[k] + 2 \cdot \left(\frac{1}{2}\right)^k \cdot \epsilon[k]$$

$$\circ\!\!-\!\!\bullet$$

$$\begin{aligned} H(z) &= 3\frac{z}{z-\frac{1}{5}} + 2\frac{z}{z-\frac{1}{2}} \;=\; \frac{15z}{5z-1} + \frac{4z}{2z-1} \\ &= \frac{15z(2z-1)+4z(5z-1)}{(5z-1)(2z-1)} = \frac{30z^2-15z+20z^2-4z}{10z^2-7z+1} \\ &= \frac{50z^2-19z}{10z^2-7z+1} \;. \end{aligned}$$

121

$$\alpha_0 = 10, \alpha_1 = -7, \alpha_2 = 1 \quad \text{und} \quad \beta_0 = 50, \beta_1 = -19, \beta_2 = 0,$$

122

$$10y[k] - 7y[k-1] + y[k-2] = 50x[k] - 19x[k-1] \qquad \text{bzw. äquivalent}$$

$$y[k] = \frac{1}{10} \cdot (50x[k] - 19x[k-1] + 7y[k-1] - y[k-2])$$

123

$$h[k] = 3 \cdot 5^{-(k+2)}\epsilon[k+2] + 2 \cdot 2^{-k}\epsilon[k]$$

$$\circ\!\!-\!\!\bullet$$

$$\begin{aligned} H(z) &= 3z^2\frac{z}{z-\frac{1}{5}} + 2\frac{z}{z-\frac{1}{2}} \;=\; \frac{15z^3}{5z-1} + \frac{4z}{2z-1} \\ &= \frac{15z^3(2z-1)+4z(5z-1)}{(5z-1)(2z-1)} = \frac{30z^4-15z^3+20z^2-4z}{10z^2-7z+1} \;. \end{aligned}$$

$$\alpha_0 = 10, \alpha_1 = -7, \alpha_2 = 1 \quad \text{und} \quad \beta_{-2} = 30, \beta_{-1} = -15, \beta_0 = 20, \beta_1 = -4, \beta_2 = 0$$ 124

$$\begin{aligned} 10y[k] - 7y[k-1] + y[k-2] &= 30x[k+2] - 15x[k+1] + 20x[k] - 4x[k-1] \text{ bzw.} \\ y[k] &= \frac{30x[k+2] - 15x[k+1] + 20x[k] - 4x[k-1] + 7y[k-1] - y[k-2]}{10} \end{aligned}$$ 125

$$y[k-2] = \frac{30x[k] - 15x[k-1] + 20x[k-2] - 4x[k-3] + 7y[k-3] - y[k-4]}{10} .$$ 126

$$x[k] = \mathcal{Z}^{-1}\{X(z)\} = \mathcal{Z}^{-1}\{\sum_{k=-\infty}^{\infty} x[k]z^{-k}\}$$ 127

$$Y(z) = 1 \cdot z^{-1} - \frac{1}{2}z^{-2} + \frac{1}{3}z^{-3} - \frac{1}{4}z^{-4} + \ldots + \frac{(-1)^{k+1}}{k}z^{-k} + \ldots$$ 128

$$y[k] = \begin{cases} 0 & ,k \leq 0 \\ (-1)^{k+1} \cdot \frac{1}{k} & ,k > 0 \end{cases} .$$ 129

$$Y(z) \stackrel{!}{=} \frac{A_1}{z-\lambda_1} + \frac{A_2}{z-\lambda_2} + \ldots + \frac{A_N}{z-\lambda_N} \qquad (10)$$ 130

$$Y(z) \stackrel{!}{=} \frac{A_1(z-\lambda_2)\cdots(z-\lambda_N) + \ldots + A_N(z-\lambda_1)\cdots(z-\lambda_{N-1})}{\text{HN}} .$$ 131

$$\begin{aligned} Y(z) &= \frac{1}{(z-(1+p))\cdot(z-1)} = \frac{A}{z-(1+p)} + \frac{B}{z-1} \\ &= \frac{A(z-1) + B(z-(1+p))}{\text{HN}} = \frac{-A - B(1+p) + (A+B)z}{\text{HN}} \end{aligned}$$ 132

133

$$\begin{array}{rrcll} -A & -(1+p)B & = & 1 & (1) \\ A & +B & = & 0 & (2) \end{array}$$

134

$$(-1-p+1)B = 1 \qquad \text{bzw.} \qquad B = -\frac{1}{p}$$

135

$$A = -B = \frac{1}{p}$$

136

$$Y(z) = \frac{p^{-1}}{z-(1+p)} - \frac{p^{-1}}{z-1}$$

$$\bullet\!\!-\!\!\circ$$

$$\begin{aligned} y[k] &= p^{-1}(1+p)^{k-1} \cdot \epsilon[k-1] - p^{-1}\epsilon[k-1] \\ &= p^{-1}\left((1+p)^{k-1} - 1\right)\epsilon[k-1] \end{aligned}$$

137

$$\begin{aligned} Y(z) = \frac{A}{z-5} + \frac{B}{z-3} + \frac{C}{(z-3)^2} &= \frac{A(z-3)^2 + B(z-5)(z-3) + C(z-5)}{\text{HN}} \\ &= \frac{A(z^2-6z+9) + B(z^2-8z+15) + C(z-5)}{\text{HN}} \\ &= \frac{z^2(A+B) + z(-6A-8B+C) + 9A + 15B - 5C}{\text{HN}} \end{aligned}$$

138

$$\begin{array}{rrrrcll} & A & +B & & = & 2 & (1) \\ & -6A & -8B & +C & = & -9 & (2) \\ & 9A & +15B & -5C & = & 3 & (3) \\ \hline 5\cdot(2)+(3): & -21A & -25B & & = & -42 & (4) \\ 21\cdot(1)+(4): & & -4B & & = & 0 & (5) \end{array}$$

$$Y(z) = \frac{2}{z-5} + \frac{3}{(z-3)^2}$$ 139

$$\begin{array}{c}\bullet\\ |\\ \circ\end{array}$$

$$y[k] = 2 \cdot 5^{k-1} \cdot \epsilon[k-1] + \frac{3(k-1)}{3} \cdot 3^{k-1}\epsilon[k-1] \tag{11}$$

$$Y(z) = s(z) + \frac{r(z)}{q(z)}$$ 140

$$s(z) = s_0 + s_1 z + \ldots + s_{M-N} z^{M-N}$$ 141

$$\begin{array}{c}\bullet\\ |\\ \circ\end{array}$$

$$s[k] = s_0\delta[k] + s_1\delta[k+1] + \ldots + s_{M-N}\delta[k+M-N] \; .$$

$$Y(z) = 3z^2 - 2z + 1 + \frac{2z^2 - 9z + 3}{(z-5)(z-3)^2}$$ 142

$$Y(z) = 3z^2 - 2z + 1 + \frac{2z^2 - 9z + 3}{(z-5)(z-3)^2}$$ 143

$$\begin{array}{c}\bullet\\ |\\ \circ\end{array}$$

$$y[k] = 3\delta[k+2] - 2\delta[k+1] + \delta[k] + 2 \cdot 5^{k-1} \cdot \epsilon[k-1] + (k-1) \cdot 3^{k-1}\epsilon[k-1]$$

$$Y(z) = \sum_{i=0}^{M-N} s_i z^i + \sum_{i=1}^{Q} \sum_{v=1}^{n_i} \frac{A_{i,v}}{(z-\lambda_i)^v}$$ 144

$$\begin{array}{c}\bullet\\ |\\ \circ\end{array}$$

$$y[k] = \sum_{i=0}^{M-N} s_i \delta[k+i] + \sum_{i=1}^{Q} \sum_{v=1}^{n_i} A_{i,v} \binom{k-1}{v-1} \lambda_i^{k-v} \epsilon[k-1] \tag{12}$$

$$Y(z) = \frac{z}{(z-1)(z+1)} = \frac{A}{z-1} + \frac{B}{z+1}$$ 145

146
$$A = \frac{1}{0!}\left(\frac{d}{dz}\right)^0 \left[(z-1)\frac{z}{(z-1)(z+1)}\right]\Bigg|_{z=1} = \frac{z}{z+1}\bigg|_{z=1} = \frac{1}{2}$$
$$B = \frac{1}{0!}\left(\frac{d}{dz}\right)^0 \left[(z+1)\frac{z}{(z-1)(z+1)}\right]\Bigg|_{z=-1} = \frac{z}{z-1}\bigg|_{z=-1} = \frac{1}{2}\;.$$

147
$$\begin{aligned}\frac{A}{z-\alpha} + \frac{B}{z-\beta} &= \frac{\frac{a\cdot\alpha+b}{\alpha-\beta}(z-\beta) + \frac{a\cdot\beta+b}{\beta-\alpha}(z-\alpha)}{(z-\alpha)(z-\beta)} \\ &= \frac{z\cdot((a\cdot\alpha+b)-(a\cdot\beta+b)) - \beta(a\cdot\alpha+b) + \alpha(a\cdot\beta+b)}{(\alpha-\beta)(z-\alpha)(z-\beta)} \\ &= \frac{za(\alpha-\beta)+b(\alpha-\beta)}{(\alpha-\beta)(z-\alpha)(z-\beta)} = \frac{az+b}{(z-\alpha)(z-\beta)}\end{aligned}$$

148
$$A = \frac{a\cdot\alpha+b}{\alpha-\beta} = \frac{1\cdot 1+0}{1-(-1)} = \frac{1}{2}$$
$$B = \frac{a\cdot\beta+b}{\beta-\alpha} = \frac{1\cdot(-1)+0}{-1-1} = \frac{1}{2}$$

149
$$Y(z) = \frac{A}{z-\alpha} + \frac{B}{z-\beta} = \frac{0.5}{z-1} + \frac{0.5}{z+1}\;.$$

150
$$|\lambda_i| < 1 \;\forall i = 1, \ldots, N$$

151
$$y[k] = y[k-1] + y[k-2] + x[k] \tag{13}$$
$$\circ\!\!-\!\!\bullet$$
$$Y(z) = z^{-1}Y(z) + z^{-2}Y(z) + X(z) \qquad \Leftrightarrow \qquad Y(z)(1-z^{-1}-z^{-2}) = X(z)$$

152
$$H(z) := \frac{Y(z)}{X(z)} = \frac{1}{1-z^{-1}-z^{-2}} = \frac{z^2}{z^2-z-1}$$

153
$$\lambda_{1/2} = \frac{1\pm\sqrt{1+4}}{2} = \frac{1\pm\sqrt{5}}{2}\;,$$

$$Y(z) = H(z) \cdot X(z) = \frac{z}{z^2 - z - 1} = \frac{z}{(z-\lambda_1)(z-\lambda_2)}$$ 154

$$Y(z) = \frac{z}{(z-\lambda_1)(z-\lambda_2)} = \frac{A}{z-\lambda_1} + \frac{B}{z-\lambda_2}$$ 155

$$\begin{aligned} A &= \frac{a\cdot\alpha + b}{\alpha - \beta} = \frac{\lambda_1}{\lambda_1 - \lambda_2} = \frac{(1+\sqrt{5})/2}{(1+\sqrt{5})/2 - (1-\sqrt{5})/2} = \frac{1+\sqrt{5}}{2\sqrt{5}} \\ B &= \frac{a\cdot\beta + b}{\beta - \alpha} = \frac{\lambda_2}{\lambda_2 - \lambda_1} = \frac{(1-\sqrt{5})/2}{(1-\sqrt{5})/2 - (1+\sqrt{5})/2} = \frac{1-\sqrt{5}}{-2\sqrt{5}} \ . \end{aligned}$$ 156

$$Y(z) = \frac{A}{z-\lambda_1} + \frac{B}{z-\lambda_2}$$

$$\begin{array}{c}\bullet\\ |\\ \circ\end{array}$$

$$\begin{aligned} y[k] &= A\lambda_1^{k-1}\epsilon[k-1] + B\lambda_2^{k-1}\epsilon[k-1] \\ &= \frac{1+\sqrt{5}}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\epsilon[k-1] - \frac{1-\sqrt{5}}{2\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\epsilon[k-1] \\ &= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\right)\epsilon[k-1] \ . \qquad (14) \end{aligned}$$ 157

$$\begin{aligned} &U(z) = H_1(z)\cdot X(z) \qquad \text{und} \\ &Y(z) = H_2(z)\cdot U(z) = H_2(z)\cdot H_1(z)\cdot X(z) \end{aligned}$$ 158

$$H(z) := \frac{Y(z)}{X(z)} = H_1(z)\cdot H_2(z)$$

$$\begin{array}{c}\bullet\\ |\\ \circ\end{array}$$

$$h[k] = h_1[k] * h_2[k] \ .$$ 159

160

$$V(z) = H_1(z) \cdot X(z) \qquad \text{und} \qquad W(z) = H_2(z) \cdot X(z) \qquad \text{und also}$$
$$Y(z) = V(z) + W(z) = H_1(z) \cdot X(z) + H_2(z) \cdot X(z) = (H_1(z) + H_2(z)) \cdot X(z)$$

161

$$H(z) := \frac{Y(z)}{X(z)} = H_1(z) + H_2(z)$$
$$\bullet\!\!-\!\!\circ$$
$$h[k] = h_1[k] + h_2[k]\ .$$

162

$$\begin{aligned} V(z) = {}& X(z) - W(z) \qquad \text{und} \qquad W(z) = H_2(z) \cdot Y(z) \qquad\qquad \text{und} \\ Y(z) = {}& H_1(z) \cdot V(z) = H_1(z) \cdot (X(z) - H_2(z) \cdot Y(z)) \\ = {}& H_1(z) \cdot X(z) - H_1(z) \cdot H_2(z) \cdot Y(z) \qquad \text{bzw.} \\ \Leftrightarrow\ Y(z) \cdot (1 + H_1(z) \cdot H_2(z)) \ = {}&\ H_1(z) \cdot X(z) \end{aligned}$$

163

$$H(z) := \frac{Y(z)}{X(z)} = \frac{H_1(z)}{1 + H_1(z) \cdot H_2(z)}\ .$$

164

$$H(z) = H_1(z) \cdot H_2(z) = \frac{(z+3)(z+1)}{(z^2-1)(z+3)(z-1))} = \frac{1}{(z-1)^2} = \frac{1}{z^2-2z+1}\ .$$

165

$$y[k] - 2y[k-1] + y[k-2] = x[k-2]$$

166

$$y[k] = 2y[k-1] - y[k-2] + x[k-2]\ .$$

167

$$H(z) = H_1(z) + H_2(z) = \frac{z+3}{z^2-1} + \frac{z+1}{(z+3)(z-1)} = \frac{(z+3)^2 + (z+1)^2}{(z^2-1)(z+3)}$$
$$= \frac{2z^2 + 8z + 10}{z^3 + 3z^2 - z - 3} \qquad \text{(Systemfunktion)}$$
$$y[k] + 3y[k-1] - y[k-2] - 3y[k-3] = 2x[k-1] + 8x[k-2] + 10x[k-3]\ \text{(DGL)}$$
$$y[k] = 2x[k-1] + 8x[k-2] + 10x[k-3] - 3y[k-1] + y[k-2] + 3y[k-3]\ \text{(Alg.)}\ .$$

$$H(z) = \frac{H_1(z)}{1 + H_1(z)H_2(z)} = \frac{\frac{z+3}{z^2-1}}{1 + \frac{(z+3)(z+1)}{(z^2-1)(z+3)(z-1)}} = \frac{\frac{z+3}{(z+1)(z-1)}}{\frac{(z-1)^2+1}{(z-1)^2}} = \frac{\frac{z+3}{z+1}}{\frac{z^2-2z+2}{z-1}}$$
$$= \frac{(z+3)(z-1)}{(z+1)(z^2-2z+2)} = \frac{z^2+2z-3}{z^3-z^2+2} \qquad \text{(Systemfunktion)}$$
$$y[k] - y[k-1] + 2y[k-3] = x[k-1] + 2x[k-2] - 3x[k-3] \qquad \text{(Diff-Gl.)}$$
$$y[k] = x[k-1] + 2x[k-2] - 3x[k-3] + y[k-1] - 2y[k-3] \qquad \text{(rek. Alg.) .}$$

168

$$\int_a^b f(t)dt \; := \; \lim_{n\to\infty} \sum_{i=0}^{n-1} f(a + i\Delta t_n) \cdot \Delta t_n \qquad (15)$$

169

$$\Delta t_n := \frac{b-a}{n}$$

170

$$\int_{-\infty}^{t} \epsilon(\tau)d\tau = \begin{cases} 0 & , \; t \le 0 \\ \int_0^t 1 \; d\tau = [\tau]_0^t & , \; t > 0 \end{cases} = t\epsilon(t) =: \text{ramp}(t) \; . \qquad (16)$$

171

$$\int_{-\infty}^{t} \delta(\tau)d\tau = \epsilon(t) \; . \qquad (17)$$

172

$$\delta(t) := \frac{d}{dt}\epsilon(t) \; = \; \lim_{T\to 0} \frac{1}{T} \,\text{rect}\left(\frac{t}{T}\right)$$
$$= \; \lim_{T\to 0} \frac{1}{T} \,\text{tri}\left(\frac{t}{T}\right) \; = \; \lim_{T\to 0} \frac{2}{T} \,\text{si}\left(2\pi\frac{t}{T}\right) \; .$$

173

I) $x(\tau) \cdot \delta(\tau - t) = x(t) \cdot \delta(\tau - t)$

II) $\displaystyle\int_{-\infty}^{\infty} \delta(\tau - t)d\tau = 1$

III) $x(t) \; = \; \displaystyle\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$ für alle $t \in \mathbb{R}$

174

175
$$\int_{-\infty}^{\infty} \delta(\tau - t)\,d\tau = \int_{-\infty}^{\infty} \delta(\lambda)\,d\lambda = \epsilon(\infty) = 1\ .$$

176
$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau = \int_{-\infty}^{\infty} x(\tau)\delta(\tau - t)\,d\tau \overset{(I)}{=} \int_{-\infty}^{\infty} x(t)\delta(\tau - t)\,d\tau$$
$$= x(t)\int_{-\infty}^{\infty} \delta(\tau - t)\,d\tau \overset{(II)}{=} x(t)\ .$$

177
$$(x * y)(t) := \int_{-\infty}^{\infty} x(\tau)\cdot y(t-\tau)\ d\tau$$

178
$$\text{a)}\,\epsilon(t) * \epsilon(t) = \int_{-\infty}^{\infty} \epsilon(\tau)\epsilon(t-\tau)\,d\tau = \int_{0}^{\infty} \epsilon(t-\tau)\,d\tau \overset{\lambda:=t-\tau}{=} -\int_{t}^{-\infty} \epsilon(\lambda)\,d\lambda$$
$$= \int_{-\infty}^{t} \epsilon(\lambda)\,d\lambda \overset{(\mathbf{??})}{=} [\lambda\epsilon(\lambda)]_{-\infty}^{t} = t\epsilon(t) - 0 = t\epsilon(t) = \mathrm{ramp}(t)$$
$$\text{b)}\,\mathrm{rect}(t) * \epsilon(t) = \int_{-\infty}^{\infty} \mathrm{rect}(\tau)\epsilon(t-\tau)\,d\tau = \int_{-0.5}^{0.5} \epsilon(t-\tau)\,d\tau \overset{\lambda:=t-\tau}{=} -\int_{t-(-0.5)}^{t-0.5} \epsilon(\lambda)\,d\lambda$$
$$= \int_{t-0.5}^{t+0.5} \epsilon(\lambda)\,d\lambda \overset{(\mathbf{??})}{=} [\lambda\epsilon(\lambda)]_{t-0.5}^{t+0.5} = \mathrm{ramp}(t+0.5) - \mathrm{ramp}(t-0.5)$$
$$= \begin{cases} 0 & ,\ t \le -0.5 \\ t+0.5 & ,\ -0.5 \le t \le 0.5 \\ 1 & ,\, t \ge 0.5 \end{cases} \qquad =: \mathrm{sramp}(t) \quad \text{(saturierte Rampe; } \to \text{Skizze)}$$

179
$$\text{c)}\,\mathrm{rect}(t) * \epsilon(t) = (\epsilon(t+0.5) - \epsilon(t-0.5)) * \epsilon(t) \overset{\text{(III)}}{=} \epsilon(t+0.5) * \epsilon(t) - \epsilon(t-0.5) * \epsilon(t)$$
$$\overset{\text{(VI,a)}}{=} (t+0.5)\epsilon(t+0.5) - (t-0.5)\epsilon(t-0.5) = \mathrm{sramp}(t)$$

$$\cos\Phi = \frac{\langle x(t), y(t)\rangle}{||x(t)|| \cdot ||y(t)||}\ .$$ 180

$$E_{\delta(t-\tau)} = ||\delta(t-\tau)||^2 = \int_{-\infty}^{\infty} \delta^2(t-\tau)\,dt = \lim_{T\to 0}\int_{\tau-T/2}^{\tau+T/2} \frac{1}{T^2}\,dt = \lim_{T\to 0}\frac{1}{T} = \infty$$

$$\cos\Phi = \frac{\langle \delta(t-\tau_1), \delta(t-\tau_2)\rangle}{||\delta(t-\tau_1)|| \cdot ||\delta(t-\tau_2)||} = \frac{\int_{-\infty}^{\infty} \delta(t-\tau_1)\delta(t-\tau_2)\,dt}{1\cdot 1} = \int_{-\infty}^{\infty} 0\ \, dt = 0$$

$$\Rightarrow \Phi = \frac{\pi}{2} = 90^\circ$$ 181

Lösung: 182

$$||\,\mathrm{rect}(t-\tau)||^2 = \int_{-\infty}^{\infty} \mathrm{rect}^2(t-\tau)\,d\tau \overset{(\lambda=t-\tau)}{=} \int_{-\infty}^{\infty} \mathrm{rect}(\lambda)\,d\lambda \overset{(\text{Def.rect})}{=} 1$$

$$\cos\Phi = \frac{\langle \mathrm{rect}(t-\tau_1), \mathrm{rect}(t-\tau_2)\rangle}{||\,\mathrm{rect}(t-\tau_1)|| \cdot ||\,\mathrm{rect}(t-\tau_2)||} \overset{(\lambda=t-\tau_1)}{=} \frac{\int_{-\infty}^{\infty} \mathrm{rect}(\lambda)\,\mathrm{rect}(\lambda-(\tau_2-\tau_1))\,d\lambda}{1\cdot 1}$$

$$\overset{(\text{Symm.rect})}{=} \int_{-\infty}^{\infty} \mathrm{rect}(\lambda)\,\mathrm{rect}((\tau_2-\tau_1)-\lambda)\,d\lambda$$

$$\overset{(\text{Def.Faltung})}{=} (\mathrm{rect} * \mathrm{rect})(\tau_2-\tau_1) \overset{(\text{S. \textbf{??},Bsp.d})}{=} \mathrm{tri}(\tau_2-\tau_1)$$

$$\Rightarrow \Phi = \arccos \mathrm{tri}(\tau_2-\tau_1)$$

Lösung: 183

$$\langle \sin(n\omega_0 t), \sin(m\omega_0 t)\rangle_T = \int_{-T/2}^{T/2} (\frac{1}{2j}(e^{jn\omega_0 t} - e^{-jn\omega_0 t}))(\frac{1}{2j}(e^{jm\omega_0 t} - e^{-jm\omega_0 t}))\,dt$$

$$= -\frac{1}{4}\int_{-T/2}^{T/2} e^{j(n+m)\omega_0 t} - e^{j(m-n)\omega_0 t} - e^{j(n-m)\omega_0 t} - e^{-j(n+m)\omega_0 t}\,dt$$

$$\overset{(\text{??})}{=} -\frac{T}{4}(\delta[n+m] - \delta[m-n] - \delta[n-m] + \delta[-(n+m)])$$

$$= \frac{T}{4}(2\delta[n-m] - 2\delta[n+m]) = \frac{T}{2}\delta[n-m] = \begin{cases} 0 & , n \neq m \\ T/2 & , n = m \end{cases} \tag{18}$$

184
$$\vec{k} = \mathbf{B}^{-1}\vec{v} \tag{19}$$

185
$\vec{b}_1 \cdot \vec{b}_2 = 2 - 4 + 2 = 0, \qquad \vec{b}_1 \cdot \vec{b}_3 = 2 + 2 - 4 = 0, \qquad \vec{b}_2 \cdot \vec{b}_3 = 4 - 2 - 2 = 0$
mit gleichen quadrierten Normen
$||\vec{b}_1||^2 = 1 + 4 + 4 = 9, \qquad ||\vec{b}_2||^2 = 4 + 4 + 1 = 9, \qquad ||\vec{b}_3||^2 = 4 + 1 + 4 = 9$.

186
$$\mathbf{B}^{-1} = \overline{\begin{pmatrix} \frac{\vec{b}_1}{||\vec{b}_1||^2} & \frac{\vec{b}_2}{||\vec{b}_2||^2} & \frac{\vec{b}_3}{||\vec{b}_3||^2} \end{pmatrix}}^T = \frac{1}{9}\begin{pmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{pmatrix} , \tag{20}$$

187
$$\vec{k} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \mathbf{B}^{-1}\vec{v} = \frac{1}{9}\begin{pmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{pmatrix} \vec{v} . \tag{21}$$

188
$$\vec{k} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \end{pmatrix} \vec{v} . \tag{22}$$

189
$$\mathbf{B} = \begin{pmatrix} \frac{1}{2}(t) & \cos(\omega_0 t) & \sin(\omega_0 t) & \cos(2\omega_0 t) & \sin(2\omega_0 t) & \cdots \end{pmatrix}$$

190
$$\mathbf{B}^{-1} = \begin{pmatrix} \frac{1}{T/4} \cdot \frac{1}{2}(t)^T \\ \frac{1}{T/2} \cdot \cos(\omega_0 t)^T \\ \frac{1}{T/2} \cdot \sin(\omega_0 t)^T \\ \frac{1}{T/2} \cdot \cos(2\omega_0 t)^T \\ \frac{1}{T/2} \sin(2\omega_0 t)^T \\ \vdots \end{pmatrix} = \frac{2}{T}\begin{pmatrix} 1(t)^T \\ \cos(\omega_0 t)^T \\ \sin(\omega_0 t)^T \\ \cos(2\omega_0 t)^T \\ \sin(2\omega_0 t)^T \\ \vdots \end{pmatrix}$$

$$\begin{pmatrix} a_0 \\ a_1 \\ b_1 \\ a_2 \\ b_2 \\ \vdots \end{pmatrix} = \mathbf{B}^{-1} \cdot (x_F(t)) = \frac{2}{T} \begin{pmatrix} 1(t)^T \\ \cos(\omega_0 t)^T \\ \sin(\omega_0 t)^T \\ \cos(2\omega_0 t)^T \\ \sin(2\omega_0 t)^T \\ \vdots \end{pmatrix} \cdot \big(x(t)\big) = \frac{2}{T} \begin{pmatrix} \langle 1(t), x(t)\rangle \\ \langle \cos(\omega_0 t), x(t)\rangle \\ \langle \sin(\omega_0 t), x(t)\rangle \\ \langle \cos(2\omega_0 t), x(t)\rangle \\ \langle \sin(2\omega_0 t), x(t)\rangle \\ \vdots \end{pmatrix} ,$$ 191

$$\begin{aligned} a_0 &= \frac{2}{T}\int_{-T/2}^{T/2} x(t)\,dt \\ a_1 &= \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(\omega_0 t)\,dt & b_1 &= \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(\omega_0 t)\,dt \\ a_2 &= \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(2\omega_0 t)\,dt & b_2 &= \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(2\omega_0 t)\,dt \\ &\vdots & &\vdots \end{aligned}$$ 192

$$\begin{aligned} a_n &= \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(n\omega_0 t)\,dt = \frac{2}{T}\int_{-T_i/2}^{T_i/2} \hat{x}\cos(n\omega_0 t)\,dt = \frac{2\hat{x}}{T}\left[\frac{\sin(n\omega_0 t)}{n\omega_0}\right]_{-T_i/2}^{T_i/2} \\ &= \frac{2\hat{x}}{Tn\omega_0}\cdot(\sin(n\omega_0 T_i/2) - \sin(-n\omega_0 T_i/2)) = \frac{2\hat{x}}{Tn\omega_0}\cdot 2\sin(n\omega_0 T_i/2) \\ &= \frac{4\hat{x}}{T\cdot 2/T_i}\cdot\frac{\sin(n\omega_0 T_i/2)}{n\omega_0 T_i/2} = 2\hat{x}\frac{T_i}{T}\cdot \mathrm{si}(n(2\pi/T)\,T_i/2) = 2\hat{x}\frac{T_i}{T}\cdot \mathrm{si}\left(n\pi\frac{T_i}{T}\right) \end{aligned}$$ 193

$$\begin{aligned} b_n &= \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(n\omega_0 t)\,dt = \frac{2}{T}\int_{-T_i/2}^{T_i/2} \hat{x}\sin(n\omega_0 t)\,dt = \frac{2\hat{x}}{T}\left[\frac{-\cos(n\omega_0 t)}{n\omega_0}\right]_{-T_i/2}^{T_i/2} \\ &= \frac{2\hat{x}}{Tn\omega_0}\cdot(-\cos(n\omega_0 T_i/2) + \cos(-n\omega_0 T_i/2)) = 0 \end{aligned}$$ 194

195

$$a_0 = 2\hat{x}\frac{T_i}{T}\ , \qquad a_n = 2\hat{x}\frac{T_i}{T}\cdot \mathrm{si}\left(n\pi\frac{T_i}{T}\right)\ , \qquad b_n = 0 \tag{23}$$

196

$$\begin{aligned} x_F(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) = \hat{x}\frac{T_i}{T} + \sum_{n=0}^{\infty} 2\hat{x}\frac{T_i}{T}\cdot \mathrm{si}\left(n\pi\frac{T_i}{T}\right)\cdot\cos(n\omega_0 t) \\ &= 2\hat{x}\frac{T_i}{T}\left(\frac{1}{2} + \mathrm{si}\left(\pi\frac{T_i}{T}\right)\cos(\omega_0 t) + \mathrm{si}\left(2\pi\frac{T_i}{T}\right)\cos(2\omega_0 t) + \right. \\ &\qquad \left. \mathrm{si}\left(3\pi\frac{T_i}{T}\right)\cos(3\omega_0 t) + \dots\right) \end{aligned} \tag{24}$$

197

$$\begin{aligned} x_F(t) \approx \frac{0.5}{2} &+ 0.45\cos(\omega_0 t) + 0.32\cos(2\omega_0 t) + 0.15\cos(3\omega_0 t) \\ &+ 0\cdot\cos(4\omega_0 t) - 0.09\cdot\cos(5\omega_0 t)\ . \end{aligned}$$

198

$$\begin{aligned} r_n\cos(n\omega_0 t + \varphi_n) &= r_n\cos(\varphi_n)\cos(n\omega_0 t) - r_n\sin(\varphi_n)\sin(n\omega_0 t) \\ &= a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t) \end{aligned}$$

199

$$r_n = \sqrt{a_n^2 + b_n^2} \quad \text{und} \quad \varphi_n = \tilde{\mathrm{sgn}}(b_n)\cdot\arccos\frac{a_n}{r_n}$$

$$\left(\text{bzw. } \varphi_n = \arctan\left(\frac{b_n}{a_n}\right) + \pi(1-\epsilon(a_n))\tilde{\mathrm{sgn}}(b_n)\right)$$

200

$$\begin{aligned} r_0 &= \frac{a_0}{2} = \hat{x}\frac{T_i}{T}\delta(f) \\ r_n &= \sqrt{a_n^2 + b_n^2} = |a_n| = 2\hat{x}\frac{T_i}{T}\cdot\left|\mathrm{si}\left(n\pi\frac{T_i}{T}\right)\right| \end{aligned}$$

$$X(f) = \sum_{n=0}^{\infty} r_n \delta(f - nf_0)$$
$$= \hat{x}\frac{T_i}{T}\delta(f) + \sum_{n=1}^{\infty} 2\hat{x}\frac{T_i}{T} \cdot \mathrm{si}\left(n\pi\frac{T_i}{T}\right) \cdot \delta(f - nf_0)\ .$$
201

$$X_H(f) = 2\hat{x}\frac{T_i}{T} \cdot \mathrm{si}\left(n\pi\frac{T_i}{T}\right) = 2\hat{x}\frac{T_i}{T} \cdot \mathrm{si}(\pi\frac{T_i}{T}\frac{f}{f_0}) \tag{25}$$
202

$$f \in \{4f_0, 8f_0, 12f_0, \ldots\}$$
203

$$c_0 := \frac{a_0}{2}\ , \quad c_n := \frac{1}{2}(a_n - jb_n)\ , \quad c_{-n} := \frac{1}{2}(a_n + jb_n) = c_n^*$$
204

$$a_0 = 2c_0\ , \quad a_n = c_n + c_{-n} = 2\,\mathrm{Re}(c_n)\ , \quad b_n = j(c_n - c_{-n}) = -2\,\mathrm{Im}(c_n)$$
205

$$c_n := \frac{1}{2}(a_n - jb_n) = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\cos(n\omega_0 t)\,dt - j\frac{1}{T}\int_{-T/2}^{T/2} x(t)\sin(n\omega_0 t)\,dt$$
$$= \frac{1}{T}\int_{-T/2}^{T/2} x(t)(\cos(-n\omega_0 t) + j\sin(-n\omega_0 t))\,dt = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jn\omega_0 t}\,dt$$
$$c_{-n} := (c_n)^* = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{jn\omega_0 t}\,dt$$
206

$$c_0 = \frac{a_0}{2} = \hat{x}\frac{T_i}{T}$$
$$c_n = \frac{1}{2}(a_n - jb_n) = \frac{a_n}{2} = \hat{x}\frac{T_i}{T} \cdot \mathrm{si}\left(n\pi\frac{T_i}{T}\right)$$
$$c_{-n} = (c_n)^* = c_n$$
207

208

$$c_k = \hat{x}\frac{T_i}{T} \cdot \mathrm{si}\left(k\pi\frac{T_i}{T}\right) \; . \tag{26}$$

209

$$x_F(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \text{für} \qquad c_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t}\, dt \; .$$

210

$$\begin{aligned} x_F(t) &= \sum_{k=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} \int_{-T/2}^{T/2} x(t') e^{-j\omega_k t'}\, dt' \cdot e^{j\omega_k t} \\ &= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \left[\int_{-T/2}^{T/2} x(t') e^{-j\omega_k t'}\, dt'\right] \cdot e^{j\omega_k t} \cdot \Delta\omega \; . \end{aligned}$$

211

$$x_F(t) \overset{(T\to\infty)}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(t') e^{-j\omega t'}\, dt'\right] \cdot e^{j\omega t}\, d\omega \quad = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega$$

212

$$x(t) = \delta(t) \circ\!\!-\!\!\bullet\; X(f) = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi f t}\, dt = e^0 = 1 \tag{27}$$

213

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}\, d\omega = \int_{-\infty}^{\infty} e^{j2\pi f t}\, df \tag{28}$$

214

$$\begin{aligned} x(t) \circ\!\!-\!\!\bullet\; X(\omega) &= \int_{-\infty}^{\infty} \hat{x}\,\mathrm{rect}(t/T_i) e^{-j\omega t}\, dt = \hat{x}\int_{-T_i/2}^{T_i/2} e^{-j\omega t}\, dt = \hat{x}\left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-T_i/2}^{T_i/2} \\ &= \frac{\hat{x}}{-j\omega}(e^{-j\omega T_i/2} - e^{j\omega T_i/2}) = \frac{\hat{x}}{-j\omega} 2j\,\mathrm{Im}(e^{-j\omega T_i/2}) \\ &= \frac{2\hat{x}}{-\omega}\sin(-\omega T_i/2) \overset{(\omega=2\pi f)}{=} \frac{\hat{x}}{\pi f}\sin(\pi f T_i) = \hat{x}\, T_i \frac{\sin(\pi f T_i)}{\pi f T_i} \\ &= \hat{x}\, T_i\,\mathrm{si}(\pi f T_i) \; . \end{aligned} \tag{29}$$

$$X(f) = \delta(f - f_0) \;\bullet\!\!-\!\!\!-\!\!\circ\; x(t) \overset{(??)}{=} \int_{-\infty}^{\infty} \delta(f - f_0)\, e^{j2\pi ft}\, df \overset{(\text{Satz } \mathbf{??})}{=} e^{j2\pi f_0 t}\ . \qquad (30)$$ 215

$$c_1 x_1(t) + c_2 x_2(t) \;\circ\!\!-\!\!\!-\!\!\bullet\; c_1 X_1(\omega) + c_2 X_2(\omega)$$ 216

$$\mathrm{rect}(\frac{t}{2T}) \;\circ\!\!-\!\!\!-\!\!\bullet\; 2T\,\mathrm{si}(T\omega) \qquad \text{und} \qquad \mathrm{rect}(\frac{t}{4T}) \;\circ\!\!-\!\!\!-\!\!\bullet\; 4T\,\mathrm{si}(2T\omega)$$ 217

$$\begin{aligned} x(t) &= 2\,\mathrm{rect}(\frac{t}{2T}) + 0.5\,\mathrm{rect}(\frac{t}{4T}) \\ &\;\;\circ\!\!-\!\!\!-\!\!\bullet \\ X(\omega) &= 2 \cdot 2T\,\mathrm{si}(T\omega) + 0.5 \cdot 4T\,\mathrm{si}(2T\omega) = 4T\,\mathrm{si}(T\omega) + 2T\,\mathrm{si}(2T\omega) \\ &= 4T\,\mathrm{si}(\pi 2Tf) + 2T\,\mathrm{si}(\pi 4Tf)\ \big|_{f=\omega/2\pi}\ . \end{aligned}$$ 218

$$\cos(2\pi f_0 t) = \frac{1}{2} e^{j2\pi f_0 t} + \frac{1}{2} e^{-j2\pi f_0 t} \;\circ\!\!-\!\!\!-\!\!\bullet\; \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0) \quad \text{und} \qquad (31)$$ 219

$$\begin{aligned} \sin(2\pi f_0 t) = \frac{1}{2j} e^{j2\pi f_0 t} - \frac{1}{2j} e^{-j2\pi f_0 t} \;\circ\!\!-\!\!\!-\!\!\bullet\; &\frac{1}{2j}\delta(f - f_0) - \frac{1}{2j}\delta(f + f_0) \\ &= \frac{j}{2}\delta(f + f_0) - \frac{j}{2}\delta(f - f_0)\ . \qquad (32) \end{aligned}$$

$$x(t - t_0) \;\circ\!\!-\!\!\!-\!\!\bullet\; e^{-j\omega t_0} X(\omega)\ .$$ 220

$$x(t) = \mathrm{rect}(\frac{t - t_0}{T}) \;\circ\!\!-\!\!\!-\!\!\bullet\; X(\omega) = e^{-j\omega t_0}\, T\,\mathrm{si}(\pi f T)\ .$$ 221

$$x(at) \;\circ\!\!-\!\!\!-\!\!\bullet\; \frac{1}{|a|} X\left(\frac{\omega}{a}\right)\ ,$$ 222

insbesondere also $\quad x(-t) \;\circ\!\!-\!\!\!-\!\!\bullet\; X(-\omega)\ .$

223
$$x_2(t) = \mathrm{rect}(0.5t) \circ\!\!-\!\!\bullet\; X_2(f) := 2\,\mathrm{si}(\pi \frac{f}{0.5}) \qquad \text{und}$$
$$x_3(t) = \mathrm{rect}(2t) \circ\!\!-\!\!\bullet\; X_3(f) := 2\,\mathrm{si}(\pi \frac{f}{2})$$

224
$$\delta(at) \circ\!\!-\!\!\bullet\; \frac{1}{|a|} \bullet\!\!-\!\!\circ\; \frac{1}{|a|}\delta(t), \quad \text{d.h.} \quad \delta(at) = \frac{1}{|a|}\delta(t) \;\Box \tag{33}$$

225
I) Falls $x(t) \circ\!\!-\!\!\bullet\; X_\omega(\omega)$ gilt, dann gilt auch $X_\omega(t) \circ\!\!-\!\!\bullet\; 2\pi x(-\omega)$.

II) Falls $x(t) \circ\!\!-\!\!\bullet\; X_f(f)$ gilt, dann gilt auch $X_f(t) \circ\!\!-\!\!\bullet\; x(-f)$.

226
$$\begin{aligned} X(t) = T'\,\mathrm{si}(\pi T' t) &\circ\!\!-\!\!\bullet\; \mathrm{rect}(-f/T') = \mathrm{rect}(f/T') \\ \overset{(T'=1/T)}{\Leftrightarrow} \frac{1}{T}\,\mathrm{si}(\pi \frac{t}{T}) &\circ\!\!-\!\!\bullet\; \mathrm{rect}(Tf) \\ \overset{(\mathrm{Lin.})}{\Leftrightarrow} \mathrm{si}(\pi \frac{t}{T}) &\circ\!\!-\!\!\bullet\; T\,\mathrm{rect}(Tf) \end{aligned} \tag{34}$$

227
$$\frac{d}{dt}x(t) \circ\!\!-\!\!\bullet\; j\omega X(\omega)\;.$$

228
$$\begin{array}{c} x''(t) + 3x'(t) + x(t) = \mathrm{rect}(t) \\ \circ \\ | \\ \bullet \\ (j\omega)^2 X(\omega) + 3j\omega X(\omega) + X(\omega) = \mathrm{si}(\omega/2) \end{array}$$

229
$$X(\omega) = \frac{\mathrm{si}(\omega/2)}{(j\omega)^2 + 3j\omega + 1}$$

230
$$\mathrm{sgn}'(t) = \epsilon'(t) - \epsilon'(-t) = \delta(t) - \delta(-t)\cdot(-1) = 2\delta(t)$$
$$\begin{array}{c} \circ \\ | \\ \bullet \end{array}$$
$$j\omega\,Y(\omega) = 2\cdot 1 \qquad \Leftrightarrow \qquad Y(\omega) = \frac{2}{j\omega} \overset{(\omega=2\pi f)}{=} \frac{1}{j\pi f}$$

$$\epsilon(t) = \frac{1}{2} + \frac{1}{2}\,\mathrm{sgn}(t)$$

$$\circ\!\!-\!\!\bullet$$

$$X(f) = \frac{1}{2}\delta(f) + \frac{1}{j2\pi f} \stackrel{(??)}{=} \pi\delta(\omega) + \frac{1}{j\omega}$$

231

$$e^{at}\epsilon(t) \circ\!\!-\!\!\bullet \int_{-\infty}^{\infty} e^{at}\epsilon(t)e^{-j\omega t}\,dt = \int_{0}^{\infty} e^{(a-j\omega)t}\,dt = \left[\frac{e^{(a-j\omega)t}}{a-j\omega}\right]_0^\infty \stackrel{(a<1)}{=} 0 - \frac{1}{a-j\omega}$$

$$= \frac{1}{j\omega - a} = \frac{1}{j2\pi f - a}$$

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I) Faltungstheorem: $x(t) * y(t) \circ\!\!-\!\!\bullet\ X(\omega)\cdot Y(\omega)$

(bzw. $\circ\!\!-\!\!\bullet_f\ X_f(f)\cdot Y_f(f)$)

II) Multiplikationstheorem: $x(t)\cdot y(t) \circ\!\!-\!\!\bullet\ \frac{1}{2\pi}X(\omega) * Y(\omega)$

(bzw. $\circ\!\!-\!\!\bullet_f\ X_f(f) * Y_f(f)$)

233

$$Y(f) = X(f)\cdot\mathrm{rect}(\frac{f}{2f_g}) = \begin{cases} X(f) & ,|f| < f_g \\ 0 & ,|f| > f_g \end{cases}$$

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$$Y(f) = X(f)\cdot\mathrm{rect}(\frac{f}{2f_g})$$

$$\bullet\!\!-\!\!\circ$$

$$y(t) = x(t) * 2f_g\,\mathrm{si}(2\pi f_g t)\ , \qquad (35)$$

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$$\int_{-\infty}^{t} x(\tau)\,d\tau \multimap \frac{1}{j\omega}X(\omega) + \pi X(0)\cdot\delta(\omega)$$
$$= \frac{1}{j2\pi f}X_f(f) + \frac{1}{2}X_f(0)\cdot\delta(f)$$

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$$e^{j\omega_0 t}\cdot x(t) \multimap X(\omega-\omega_0) \qquad (\,= X_f(f-f_0)\,)$$

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$$X(s) \;:=\; \mathcal{L}\{x(t)\} \;:=\; \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

239
$$\mathcal{L}\{x(t)\} \;=\; \int_{-\infty}^{\infty} x(t)e^{-(\alpha+j\omega)t}\,dt = \int_{-\infty}^{\infty} x(t)e^{-\alpha t}\cdot e^{-j\omega t}\,dt = \mathcal{F}\{x(t)\cdot e^{-\alpha t}\}$$

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$$X_{\mathcal{F}}(\omega) = X(s)\,|_{s=j\omega}\;\;. \tag{36}$$