Finding transformation between two point clouds: (similarity transform)

- In case of Lidar data, if we know the actual point cloud of the map (A) and if we have the point cloud from the robot's lidar (R), we can estimate a transformation between A and R (Translation, Rotation and scale).

- Knowing this transformation tells us how off the robot's measurements are and correct them. So, we can improve the estimated poses using the surrounding data.

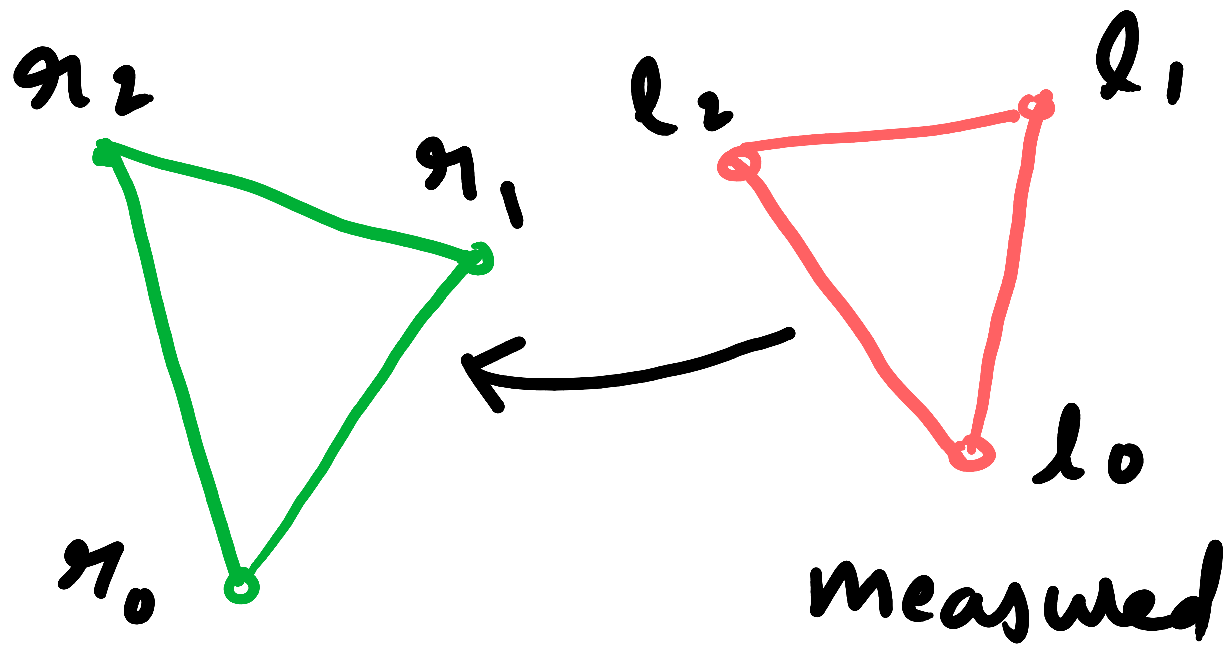
- This can be done in two ways:

a) **Feature based approach**: (Depends on features / landmarks)

- This is possible if we know specific objects in the surroundings that can be used as landmarks.

b) **Feature-less approach**:

- Here, we try to match the entire point cloud from the LiDar's sensor.
- Due to that, this approach is more accurate but slow.



Actual landmark locations from the robot's estimated position (r_i)

measured landmark locations (l_i)

Procedure :

- We need to find the transformation between the l_i and r_i to improve the robot's pose estimation.
- To do this, we get the similarity transform between l_i and r_i points

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} l_{ix} \\ l_{iy} \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix} = \begin{bmatrix} r_{ix} \\ r_{iy} \end{bmatrix}$$

&

$$\boxed{\lambda R l_i + t = r_i}$$

where; λ – Scale factor $\in \mathbb{R}$
α – Angle $\in [0, 2\pi]$
t – translation vector $\in \mathbb{R}^2$

* This is only for 2D transformation

To find the best λ, α and t, we minimize the squared distance between l_i and r_i:

$$\min \sum_i \| \lambda R l_i + t - r_i \|^2$$

But, this is a non-linear problem. Because, R consists of $\cos\alpha$ and $\sin\alpha$. So, we have to linearize and iterate until we find a global minimum.

- However, we have to remember that we might not find a global minimum.

→ So, another approach is proposed:

1) Compute the center of mass

$$\bar{l} = \frac{1}{m} \sum_i l_i \qquad \bar{r} = \frac{1}{m} \sum_i r_i$$

2) Get the reduced co-odinates

$$l_i' = l_i - \bar{l} \qquad r_i' = r_i - \bar{r} \qquad \text{so, } \sum_i l_i' = 0$$

$$\sum_i r_i' = 0$$

3) Now, replace the equation to be minimized with l_i' and r_i':

$$\min \sum_i \| \lambda R l_i - r_i + t \|^2$$

$$\min \sum_i \| \lambda R (l_i' + \bar{l}) - (r_i' + \bar{r}) + t \|^2$$

$$\min \sum_i \| \lambda R l_i' - r_i' + \underbrace{\lambda R \bar{l} - \bar{r} + t}_{t'} \|^2$$

$$\boxed{\min \sum_i \| \lambda R l_i' - r_i' + t' \|^2}$$

$$\sum_i \|\lambda R l_i' - r_i' + t'\|^2 \quad \left(\text{Use } (a+b)^2 = a^2 + 2ab + b^2\right)$$

$$= \sum_i \|\lambda R l_i' - r_i'\|^2 + 2t'^T \sum_i \lambda R l_i' - r_i' + \sum_i \|t'\|^2$$

$$= \sum_i \|\lambda R l_i' - r_i'\|^2 + 2t'^T \Big(\lambda R \underbrace{\sum_i l_i'}_{0} - \underbrace{\sum_i r_i'}_{0}\Big) + m\|t'\|^2$$

$$= \underbrace{\sum_i \|\lambda R l_i' - r_i'\|^2}_{\geq 0} \; ① + \underbrace{m\|t'\|^2}_{\geq 0} \; ②$$

→ So, now, we can minimize both of these terms individually

② ⇒ $m\|t'\|^2 = 0$

$$t' = 0$$

$$\lambda R \bar{l} - \bar{r} + t = 0$$

$$\boxed{t = \bar{r} - \lambda R \bar{l}}$$

① ⇒ $\min \sum_i \|\lambda R l_i' - r_i'\|^2$

Now, the equation is changed a bit by assuming the l_i point cloud is scaled by $\sqrt{\lambda}$ and r_i point cloud is scaled by $\frac{1}{\sqrt{\lambda}}$. Now, the new equation to minimize is

$$\min \sum_i \left\|\sqrt{\lambda} R l_i' - \frac{1}{\sqrt{\lambda}} r_i'\right\|^2$$

$$\min \lambda \underbrace{\sum \|R l_i'\|^2}_{a} - 2\underbrace{\sum r_i'^T R l_i'}_{b} + \frac{1}{\lambda} \underbrace{\sum \|r_i'\|^2}_{c}$$

$$\min \|\lambda a + b + \frac{1}{\lambda} c\| \quad \text{— (3)}$$

differentiating λ

$$a + \frac{(-1)}{\lambda^2} c = 0$$

$$\frac{c}{\lambda^2} = a$$

$$\lambda^2 = \frac{c}{a}$$

$$\lambda^2 = \frac{\Sigma \|r_i'\|^2}{\Sigma \|R l_i'\|^2}$$

We know that rotation R does not change the length

$$\boxed{\lambda = \sqrt{\frac{\Sigma \|r_i'\|^2}{\Sigma \| l_i'\|^2}}}$$

λ is independent of rotation and translation.

And, since b wasn't utilized and (3) had to be minimized, we minimize b

$$\min -2 \Sigma r_i^T R l_i'$$

$$\Rightarrow \max \Sigma r_i^T R l_i'$$

$$\max \sum \begin{bmatrix} r'_x & r'_y \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} l'_x \\ l'_y \end{bmatrix}$$

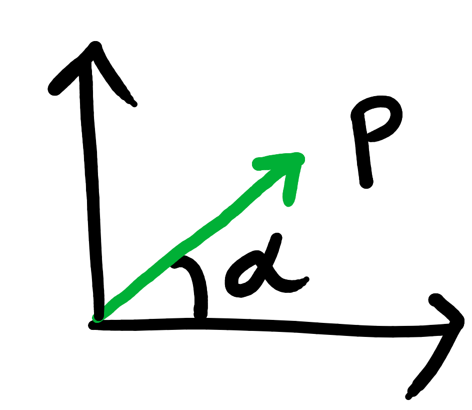
$$\max \sum \begin{bmatrix} r'_x & r'_y \end{bmatrix} \begin{bmatrix} l'_x \cos\alpha - l'_y \sin\alpha \\ l'_x \sin\alpha + l'_y \cos\alpha \end{bmatrix}$$

$$\max \sum r'_x l'_x \cos\alpha - r'_x l'_y \sin\alpha + r'_y l'_x \sin\alpha + r'_y l'_y \cos\alpha$$

$$\max \sum \cos\alpha \left(r'_x l'_x + r'_y l'_y \right) + \sin\alpha \left(-r'_x l'_y + r'_y l'_x \right)$$

$$\max \cos\alpha \sum r'_x l_x' + r'_y l_y' + \sin\alpha \sum -r'_x l_y' + r'_y l_x'$$

$$\max \begin{bmatrix} \cos\alpha & \sin\alpha \end{bmatrix} \underbrace{\begin{bmatrix} \sum r'_x l'_x + r'_y l'_y \\ \sum -r'_x l'_y + r'_y l'_x \end{bmatrix}}_{P}$$



From the equation, it can be seen that α is the angle made by the normalized vector P.

$$\boxed{\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} = \begin{bmatrix} \dfrac{P_x}{\sqrt{P_x^2 + P_y^2}} \\ \dfrac{P_y}{\sqrt{P_x + P_y^2}} \end{bmatrix}}$$

* For rigid body motion, we ignore the λ from the similarity transformation. We set $\lambda = 1$.