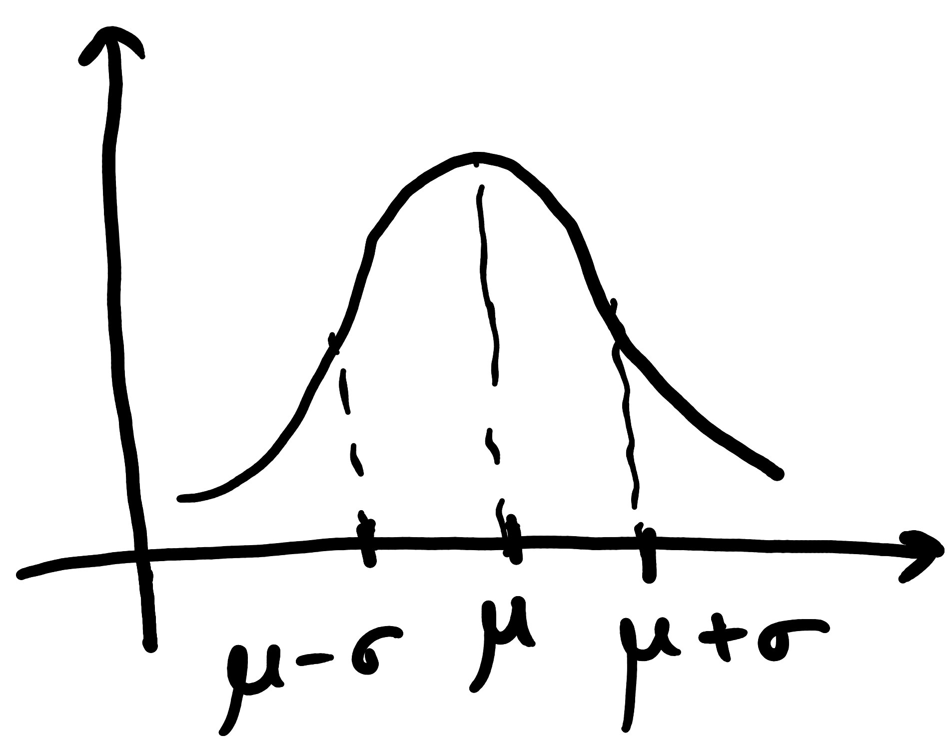
Kalman Filtering in multiple dimensions:

→ In a 1D normal distribution,

$$X \sim \mathcal{N}(x; \mu, \sigma^2) \qquad \Rightarrow \quad x = \mu \pm \sigma$$

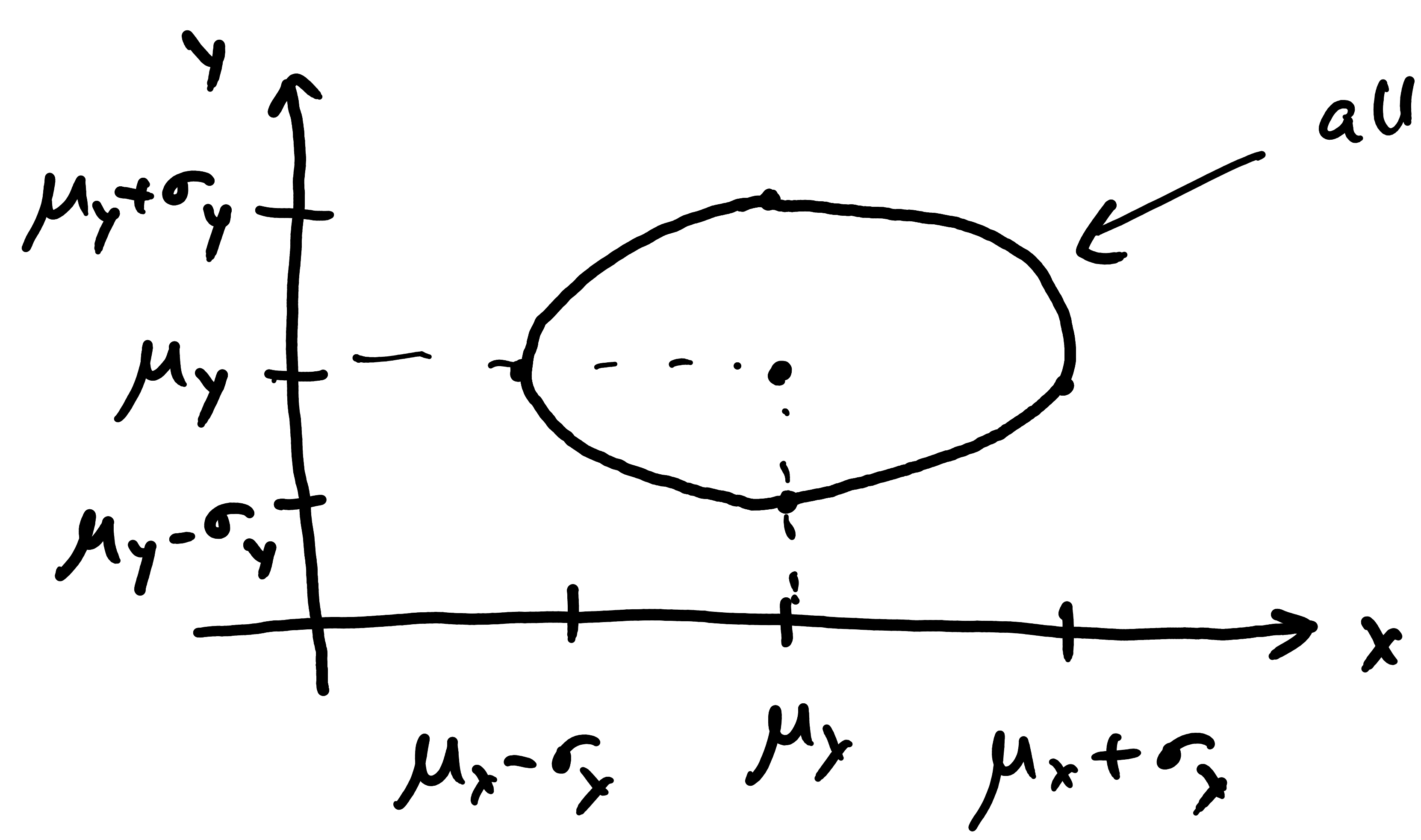
$$p(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad \text{when } \frac{(x-\mu)^2}{\sigma^2} = 1$$

where in the probability distribution function (PDF),



$\pm 1\sigma$: 68%
$\pm 2\sigma$: 95.5%
$\pm 3\sigma$: 99.7%

→ In 2D, the locus that satisfies : $\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} = 1$ is



all the points on the ellipse.

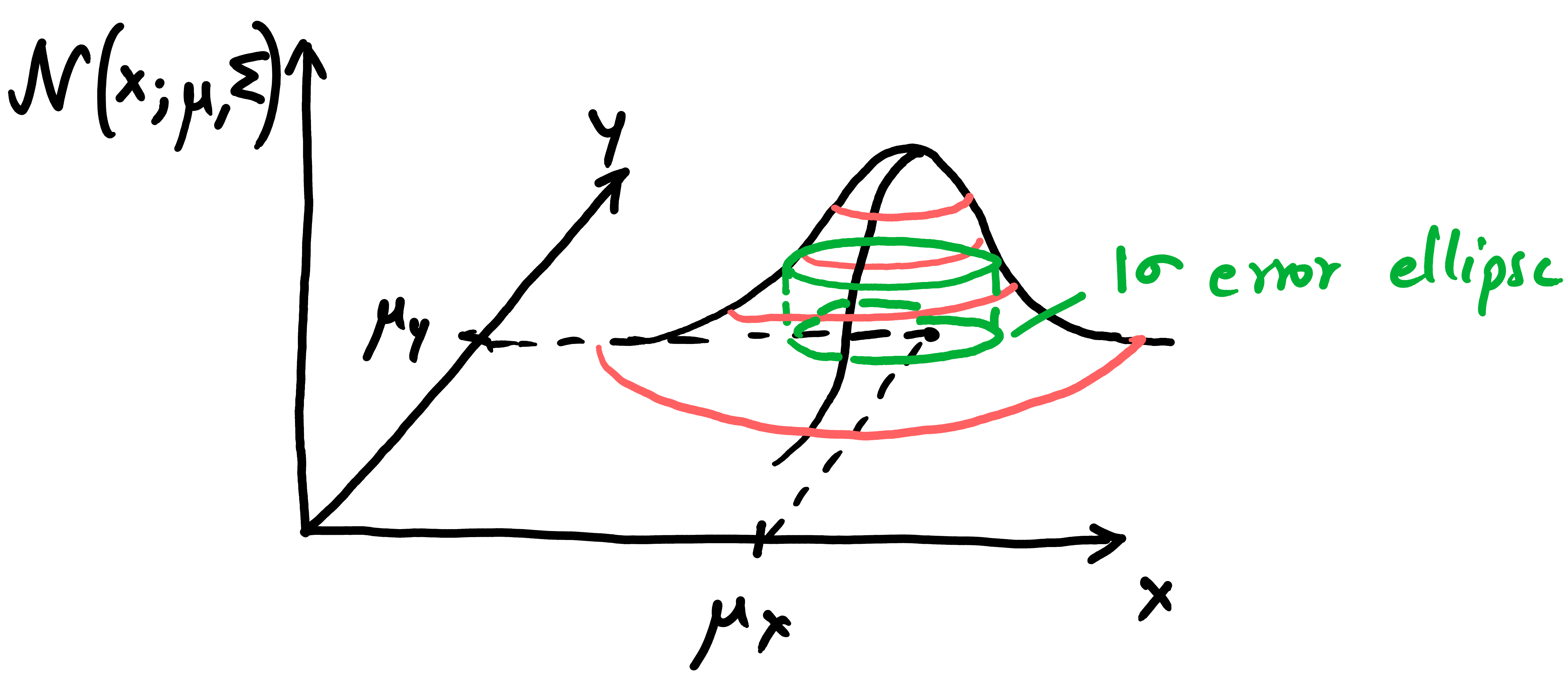
→ In matrix form, we can write $\frac{(x-\mu_x)^2}{\sigma_{x^2}} + \frac{(y-\mu_y)^2}{\sigma_{y^2}}$ as:

$$[x-\mu_x,\ y-\mu_y]\begin{bmatrix}1/\sigma_x^2 & 0\\ 0 & 1/\sigma_y^2\end{bmatrix}\begin{bmatrix}x-\mu_x\\ y-\mu_y\end{bmatrix} = (x-\mu)^T\Sigma^{-1}(x-\mu)$$

where $(x-\mu) = [x-\mu_x,\ y-\mu_x]$

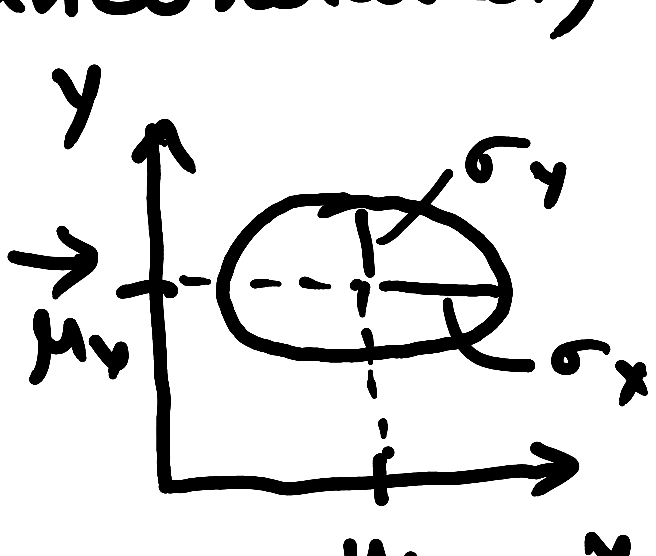
and Σ (Covariance matrix) $= \begin{bmatrix}\sigma_x^2 & 0\\ 0 & \sigma_y^2\end{bmatrix}$

→ Therefore, $\text{PDF} = \alpha \cdot e^{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)}$



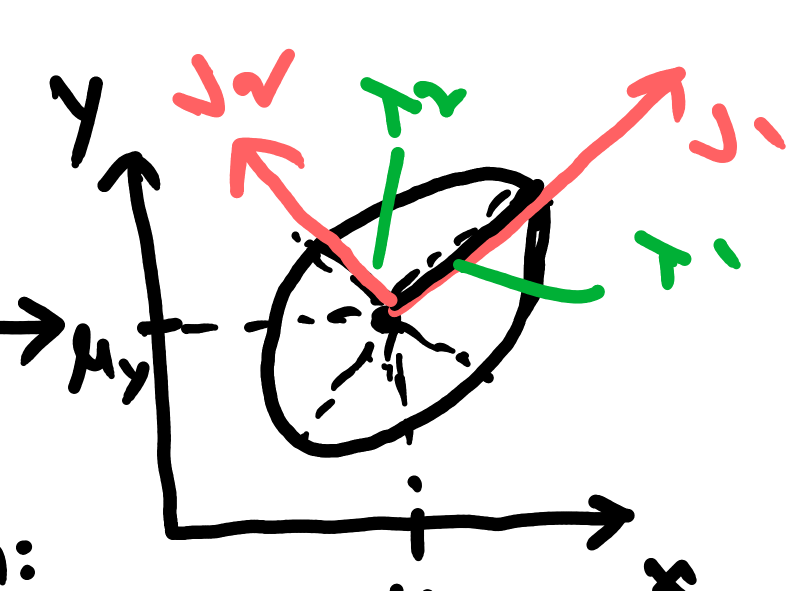
→ When X and Y are independent (X and Y are uncorrelated)

$$\Sigma = \begin{bmatrix}\sigma_x^2 & 0\\ 0 & \sigma_y^2\end{bmatrix} \rightarrow$$



→ When x and y are correlated

$$\Sigma = \begin{bmatrix}\sigma_x^2 & \sigma_{xy}\\ \sigma_{xy} & \sigma_{y^2}\end{bmatrix} \rightarrow$$



From eigen decomposition:

$$\Sigma = V\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}V^T \rightarrow \begin{bmatrix}| & |\\ V_1 & V_2\\ | & |\end{bmatrix}$$

Eigen values

Eigen vectors

→ In nD,

$$p(x) = \det(2\pi\Sigma)^{-\frac{1}{2}} \cdot e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)}$$

$$\hookrightarrow \mathcal{N}(\underset{n}{x} ; \underset{n}{\mu} ; \underset{n \times n}{\Sigma})$$

→ **Kalman Filter (1D):**

Prediction:

$$x_t = a_t x_{t-1} + b_t u_t + \varepsilon_R$$

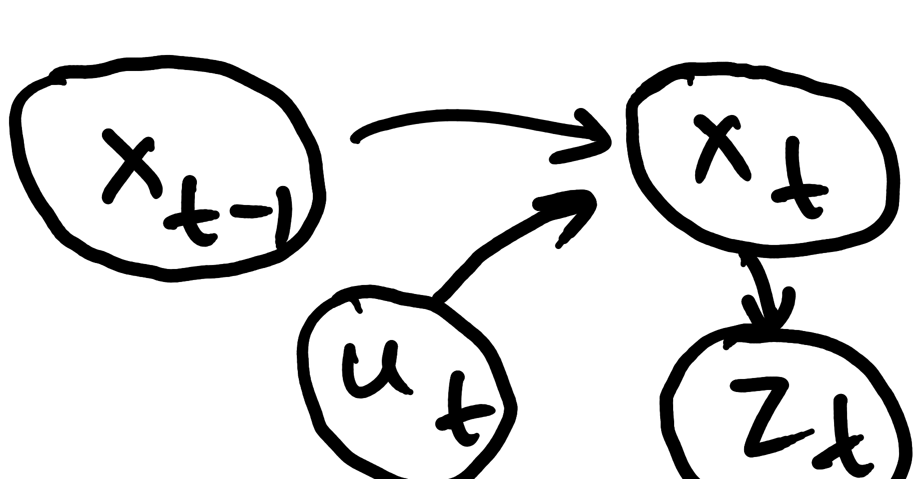
$$\overline{bel}(x_t)\begin{cases} \bar{\mu}_t = a_t \cdot \mu_{t-1} + b_t \cdot u_t \\ \bar{\sigma}_t^2 = a_t^2 \cdot \sigma_{t-1}^2 + \sigma_R^2 \end{cases}$$

Correction:

$$z_t = c_t \cdot \bar{x}_t + \varepsilon_Q$$

$$k_t = \frac{c_t \cdot \bar{\sigma}_t^2}{c_t^2 \bar{\sigma}_t^2 + \sigma_Q^2}$$

$$bel(x_t)\begin{cases} \mu_t = \bar{\mu}_t + k_t(z_t - c_t\bar{\mu}_t) \\ \sigma_t^2 = (1 - k_t c_t)\bar{\sigma}_t^2 \end{cases}$$



Kalman Filter (nD):

Prediction:

$$x_t = A_t x_{t-1} + B_t u_t + \varepsilon_R$$

$$\overline{bel}(x_t)\begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t \mu_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t \end{cases}$$

Correction:

$$z_t = C_t \bar{x}_t + \varepsilon_Q$$

$$K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$$

$$bel(x_t)\begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - C_t\bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t)\bar{\Sigma}_t \end{cases}$$

Proof: Probabilistic robotics by Thourn, Brogard, and Fox.

→ Extended Kalman Filter :-

- The kalman filter works only with linear motion models.
- To make it work with non-linear motion models, we linearize the model at every state and use the kalman filter on the linearized motion model and the linearized measurement model.
- This version of the kalman filter is called extended kalman filter (EKF)

Prediction :

$$x_t = g(x_{t-1}, u_t) \qquad g: \text{non-linear motion model}$$

$$\overline{bel}(x_t) \begin{cases} \bar{\mu}_t = g(\mu_{t-1}, u_t) \\ \bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + V_t \Sigma_{control} V_t^T \end{cases}$$

Correction :

$$Z_t = h(x_t)$$

$$K_t = \bar{\Sigma}_t H_t^T (H_t \bar{\Sigma}_t H_t^T + Q_t)^{-1}$$

$$bel(x_t) \begin{cases} \mu_t = \bar{\mu}_t + K_t \overbrace{(z_t - h(\bar{\mu}_t))}^{\text{innovation}} \\ \Sigma_t = (I - K_t H_t) \bar{\Sigma}_t \end{cases}$$

H: Jacobian of h w.r.t state

$$: \left[\frac{\partial h}{\partial \text{state}}\right]$$

G: Jacobian matrix of g w.r.t state

$$\Rightarrow G(\mu_{t-1}, u_t) : \left[\frac{\partial g}{\partial \text{state}}\right]$$

$$: \begin{bmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} & \frac{\partial g_1}{\partial \theta} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} & \frac{\partial g_2}{\partial \theta} \\ \frac{\partial g_3}{\partial x} & \frac{\partial g_3}{\partial y} & \frac{\partial g_3}{\partial \theta} \end{bmatrix}$$

⇒ V: Jacobian matrix of g w.r.t control

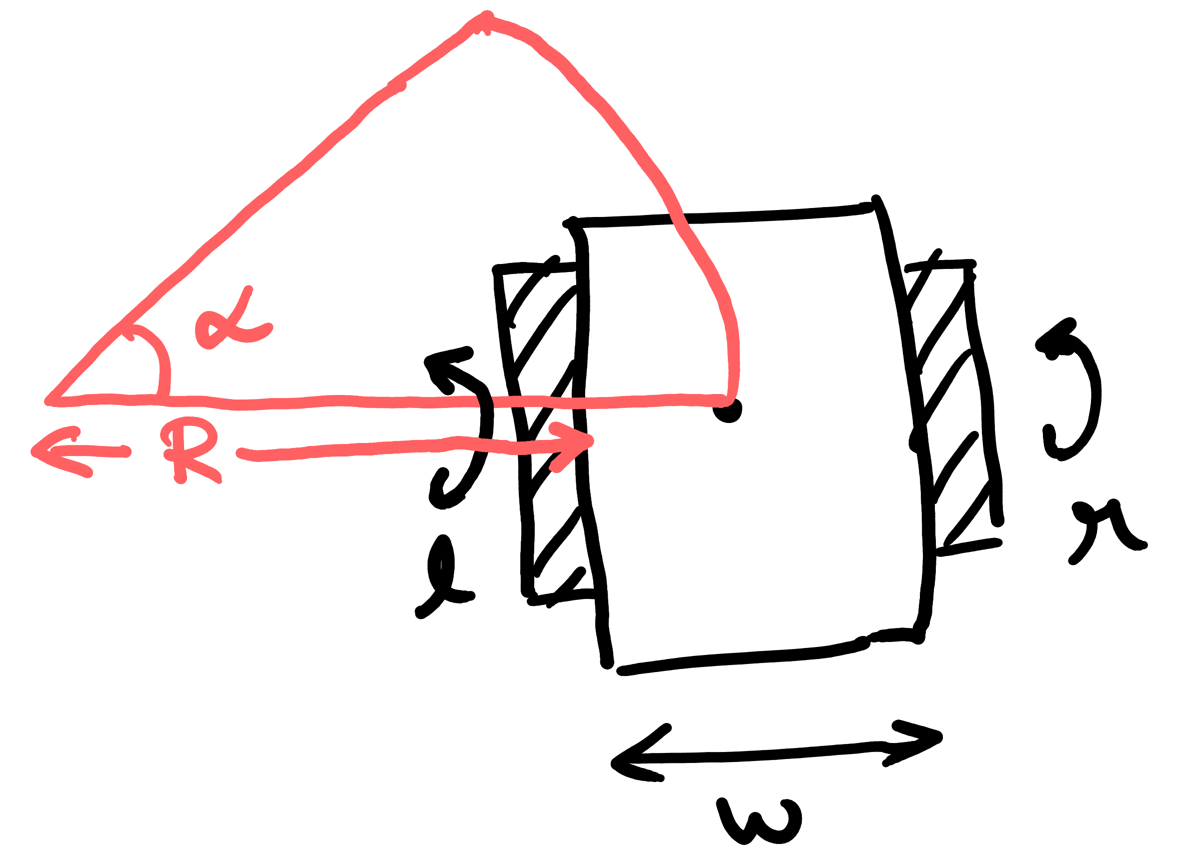
$$V(\mu_{t-1}, u_t) : \left[\frac{\partial g}{\partial \text{control}}\right]$$

Example:-

→ **Motion model**:

$$\alpha = \frac{r - l}{w}$$

$$R = \frac{l}{\alpha} \quad \text{when } r \neq l$$



w – width of the robot
θ – current heading
α – Change in heading
l, r – distance moved by each wheel

- $r \neq l$:

$$X_{t+1} = \begin{bmatrix} x_{t+1} \\ y_{t+1} \\ \theta_{t+1} \end{bmatrix} = \begin{bmatrix} x_t \\ y_t \\ \theta_t \end{bmatrix} + \begin{bmatrix} \left(R+\frac{w}{2}\right)\left(\sin(\theta_t+\alpha_t) - \sin\theta_t\right) \\ \left(R+\frac{w}{2}\right)\left(-\cos(\theta_t+\alpha_t) + \cos\theta_t\right) \\ \alpha_t \end{bmatrix} = g(\underbrace{x_t, y_t, \theta_t}_{\text{state}}, \underbrace{l_t, r_t}_{\text{control}})$$

$$= g(x_t, u_t)$$

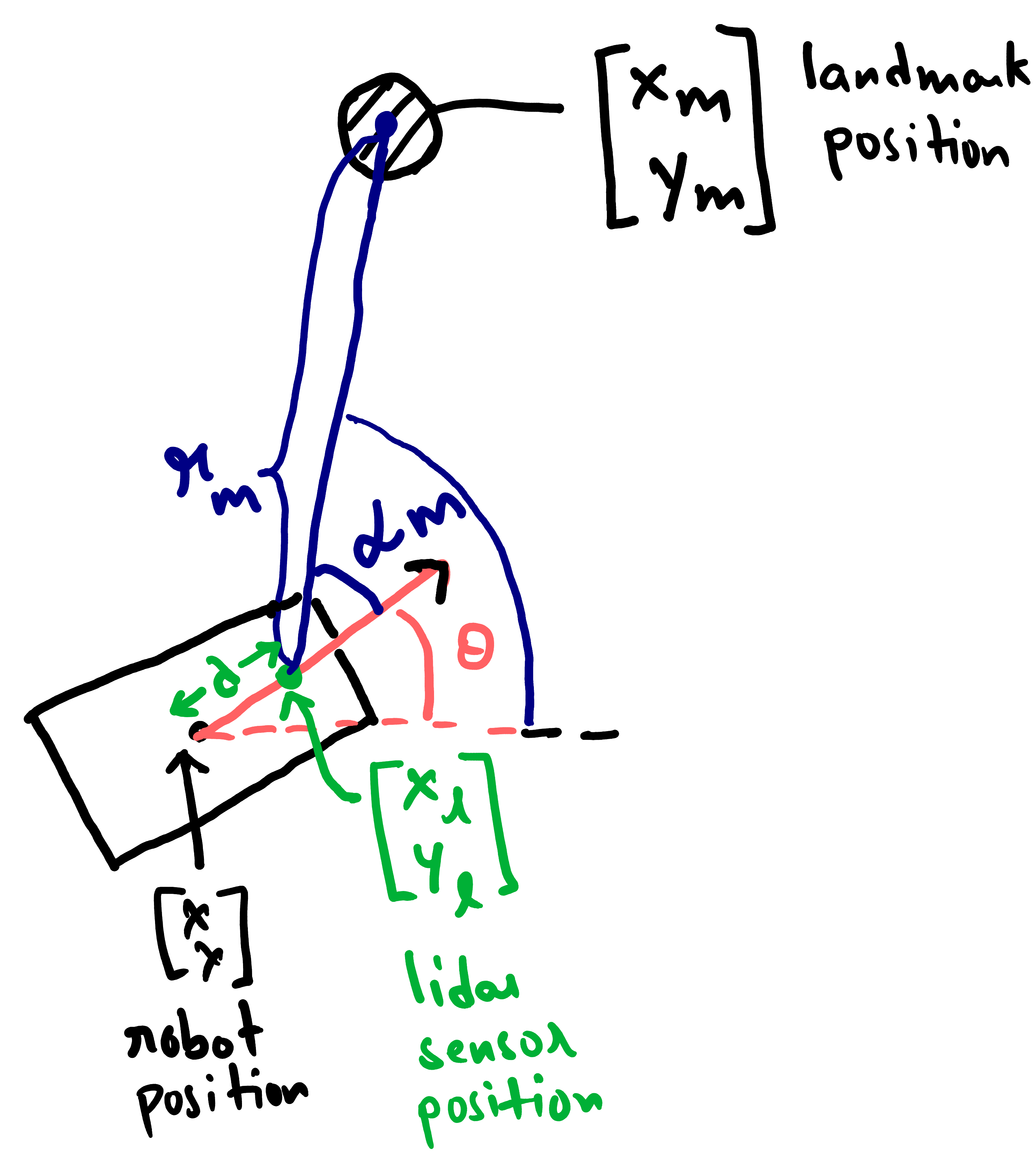
- $r = l$:

$$x_{t+1} = \begin{bmatrix} x_t \\ y_t \\ \theta_t \end{bmatrix} + \begin{bmatrix} l\cos\theta_t \\ l\sin\theta_t \\ 0 \end{bmatrix}$$

→ **Measurement Model**:

$$\begin{bmatrix} x_l \\ y_l \end{bmatrix} = \begin{bmatrix} x_t \\ y_t \end{bmatrix} + d\begin{bmatrix} \cos\theta_t \\ \sin\theta_t \end{bmatrix}$$

$$z_t = \begin{bmatrix} r_m \\ \alpha_m \end{bmatrix} \quad r_m = \sqrt{(x_m - x_l)^2 + (y_m - y_l)^2}, \quad \alpha_m = \operatorname{atan}\left(\frac{y_m - y_l}{x_m - x_l}\right) - \theta_t$$



→ Jacobean for motion model (G) w.r.t state :

$$X_{t+1} = g(x_t, u_t) = \begin{bmatrix} x_t \\ y_t \\ \theta_t \end{bmatrix} + \begin{bmatrix} (R+\frac{w}{2})(\sin(\theta_t+\alpha_t) - \sin\theta_t) \\ (R+\frac{w}{2})(-\cos(\theta_t+\alpha_t) + \cos\theta_t) \\ \alpha_t \end{bmatrix}$$

• $r \neq l$:

$$\frac{\partial g_{11}}{\partial x_t} = 1 \qquad \frac{\partial g_{11}}{\partial y_t} = 0 \qquad \frac{\partial g_{11}}{\partial \theta_t} = \left(R+\frac{w}{2}\right)\left(\cos(\theta_t+\alpha_t) - \cos\theta_t\right)$$

$$\frac{\partial g_{21}}{\partial x_t} = 0 \qquad \frac{\partial g_{21}}{\partial y_t} = 1 \qquad \frac{\partial g_{21}}{\partial \theta_t} = \left(R+\frac{w}{2}\right)\left(\sin(\theta_t+\alpha_t) - \sin\theta_t\right)$$

$$\frac{\partial g_{31}}{\partial x_t} = 0 \qquad \frac{\partial g_{31}}{\partial y_t} = 0 \qquad \frac{\partial g_{31}}{\partial \theta_t} = \alpha_t$$

• $r = l$:

When $r = l$, $\alpha \to 0$

$$\frac{\partial g_{11}}{\partial \theta_t} = R\left(\cos(\theta+\alpha) - \cos\theta\right) + \frac{w}{2}\left(\cos(\theta+\alpha) - \cos\theta\right)$$

$$\lim_{\alpha\to 0} \frac{\partial g_{11}}{\partial \theta_t} = \underbrace{\frac{l}{\alpha}}_{\to\infty}\underbrace{\left(\cos(\theta+\alpha) - \cos\theta\right)}_{\to 0} + \frac{w}{2}\underbrace{\left(\cos(\theta+\alpha) - \cos\theta\right)}_{\to 0}$$

Using l'Hopital rule, differentiate the numerator & denominator w.r.t α

$$= \lim_{\alpha\to 0} \frac{l \cdot -\sin(\theta+\alpha)}{1}$$

$$= -l\sin\theta$$

Finally,

$$G = \begin{bmatrix} 1 & 0 & -l\sin\theta \\ 0 & 1 & l\cos\theta \\ 0 & 0 & 0 \end{bmatrix}$$

→ Jacobian for motion model (V) w.r.t control:

- $r \neq l$:

$$X_{t+1} = g(x_t, u_t) = \begin{bmatrix} x_t \\ y_t \\ \theta_t \end{bmatrix} + \begin{bmatrix} \left(R + \frac{w}{2}\right)\left(\sin(\theta_t + \alpha_t) - \sin\theta_t\right) \\ \left(R + \frac{w}{2}\right)\left(-\cos(\theta_t + \alpha_t) + \cos\theta_t\right) \\ \alpha_t \end{bmatrix}$$

$$R = \frac{l}{\alpha} \; ; \; \alpha = \frac{r-l}{w} \Rightarrow \left(R + \frac{w}{2}\right) = \frac{lw}{r-l} + \frac{w}{2} = \frac{w}{2}\left(\frac{2l}{r-l} + 1\right) = \frac{w}{2}\left(\frac{r+l}{r-l}\right)$$

$$\frac{\partial g_{11}}{\partial l} = \frac{w}{2} \cdot \frac{1}{(r-l)} \cdot \left(\sin(\theta_t + \alpha_t) - \sin\theta_t\right) - \frac{w}{2}\frac{(r+l)(-1)}{(r-l)^2}\left(\sin(\theta_t + \alpha_t) - \sin\theta_t\right)$$
$$+ \frac{\cancel{w}}{2}\left(\frac{r+l}{r-l}\right)\left(\cos(\theta_t + \alpha_t)\left(-\frac{1}{\cancel{w}}\right)\right)$$

$$= \frac{w}{2} \cdot \frac{(r - \cancel{l}) + (r + \cancel{l})}{(r-l)^2} \cdot \left(\sin(\theta_t + \alpha_t) - \sin\theta_t\right)$$
$$- \frac{1}{2}\left(\frac{r+l}{r-l}\right) \cdot \cos(\theta_t + \alpha_t)$$

$$= \frac{wr}{(r-l)^2}\left(\sin(\theta_t + \alpha_t) - \sin\theta_t\right) - \frac{(r+l)}{2(r-l)}\cos(\theta_t + \alpha_t)$$

$$\frac{\partial g_{21}}{\partial l} = \frac{wr}{(r-l)^2}\left(-\cos(\theta_t + \alpha_t) + \cos\theta_t\right) + \frac{\cancel{w}}{2}\left(\frac{r+l}{r-l}\right)\left(\sin(\theta_t + \alpha_t)\left(-\frac{1}{\cancel{w}}\right)\right)$$

$$= \frac{wr}{(r-l)^2}\left(-\cos(\theta_t + \alpha_t) + \cos\theta_t\right) - \frac{(r+l)}{2(r-l)}\sin(\theta_t + \alpha_t)$$

$$\frac{\partial g_{31}}{\partial l} = \frac{-1}{w}$$

$$\frac{\partial g_{11}}{\partial r} = \frac{-\omega l}{(r-l)^2}\left(\sin(\theta_t+\alpha_t) - \sin\theta_t\right) + \frac{r+l}{2(r-l)}\cos(\theta_t+\alpha_t)$$

$$\frac{\partial g_{21}}{\partial r} = \frac{-\omega l}{(r-l)^2}\left(-\cos(\theta_t+\alpha_t) + \cos\theta_t\right) + \frac{r+l}{2(r-l)}\cdot\sin(\theta_t+\alpha_t)$$

$$\frac{\partial g_{31}}{\partial r} = \frac{1}{\omega}$$

- $\underline{r = l}$:

(Not sure how these were obtained)

$$\frac{\partial g_{11}}{\partial l} = \frac{1}{2}\left(\cos\theta + \frac{l}{\omega}\sin\theta\right)$$

$$\frac{\partial g_{4}}{\partial l} = \frac{1}{2}\left(\sin\theta - \frac{l}{\omega}\cos\theta\right)$$

$$\frac{\partial g_{31}}{\partial l} = \frac{-1}{\omega}$$

$$\frac{\partial g_{11}}{\partial r} = \frac{1}{2}\left(-\frac{l}{\omega}\sin\theta + \cos\theta\right)$$

$$\frac{\partial g_{21}}{\partial r} = \frac{1}{2}\left(\frac{l}{\omega}\cos\theta + \sin\theta\right)$$

$$\frac{\partial g_{31}}{\partial r} = \frac{1}{\omega}$$

→ Jacobian for measurement model (H) w.r.t state :

$$Z_t = \begin{bmatrix} r_{mt} \\ \alpha_{mt} \end{bmatrix} = \begin{bmatrix} \sqrt{(x_{m_t}-x_{l_t})^2 + (y_{m_t}-y_{l_t})^2} \\ \operatorname{atan}\left(\frac{y_{mt}-y_{lt}}{x_{mt}-x_{lt}}\right) - \theta_t \end{bmatrix} \; ; \; \begin{bmatrix} x_{lt} \\ y_{lt} \end{bmatrix} = \begin{bmatrix} x_t \\ y_t \end{bmatrix} + d \begin{bmatrix} \cos\theta_t \\ \sin\theta_t \end{bmatrix}$$

$$\frac{\partial h_{11}}{\partial x} = \frac{1}{\cancel{2}\sqrt{(x_{mt}-x_{lt})^2 + (y_{mt}-y_{lt})^2}} \cdot \cancel{2}(x_{mt}-x_{lt}) \cdot (-1)$$

$$= \frac{-(x_{m_t}-x_{l_t})}{\sqrt{\Delta x^2 + \Delta y^2}}$$

Assume $\Delta x = x_{m_t} - x_{lt}$

$\Delta y = y_{mt} - y_{lt}$

$$\frac{\partial h_{11}}{\partial y} = \frac{-(y_{m_t}-y_{lt})}{\sqrt{\Delta x^2 + \Delta y^2}}$$

$$\frac{\partial h_{11}}{\partial \theta} = \frac{d(\Delta x . \sin\theta - \Delta y . \cos\theta)}{\sqrt{\Delta x^2 + \Delta y^2}}$$

$$\frac{\partial h_{21}}{\partial x} = \frac{\Delta y}{\Delta x^2 + \Delta y^2}$$

$$\frac{\partial h_{21}}{\partial y} = \frac{-\Delta x}{\Delta x^2 + \Delta y^2}$$

$$\frac{\partial h_{21}}{\partial \theta} = \frac{-d(\Delta x \cos\theta + \Delta y \sin\theta)}{\Delta x^2 + \Delta y^2} - 1$$

→ EKF for the given example :

- Prediction Step :

$$\bar{\mu}_t = g(\mu_{t-1}, u_t)$$

$$\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + V_t \Sigma_{control} V_t^T$$

$$\Sigma_{control} = \begin{bmatrix} \sigma_l^2 & 0 \\ 0 & \sigma_r^2 \end{bmatrix}$$

where, $\sigma_l^2 = (\gamma_1 \cdot l)^2 + (\gamma_2 \cdot (l - r))^2$

$\sigma_r^2 = (\gamma_1 \cdot r)^2 + (\gamma_2 \cdot (l - r))^2$

where (control motion factor) γ_1 : Error in control (30%)

(control turn factor) γ_2 : Observed error in slip when rotating (60%)

Note : $\Sigma_{control}$ tries to model as many observed errors as possible.

- Correction Step :

$$K_t = \bar{\Sigma}_t H_t^T (H_t \bar{\Sigma}_t H_t^T + Q)^{-1}$$

$$\mu_t = \bar{\mu}_t + K_t (z_t - h(\bar{\mu}_t))$$

$$\Sigma_t = (I - K_t H_t) \bar{\Sigma}_t$$

Measurement Covariance matrix

$$Q = \begin{bmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\alpha^2 \end{bmatrix}$$

where σ_r : measurement distance std dev

σ_α : measurement angle std dev