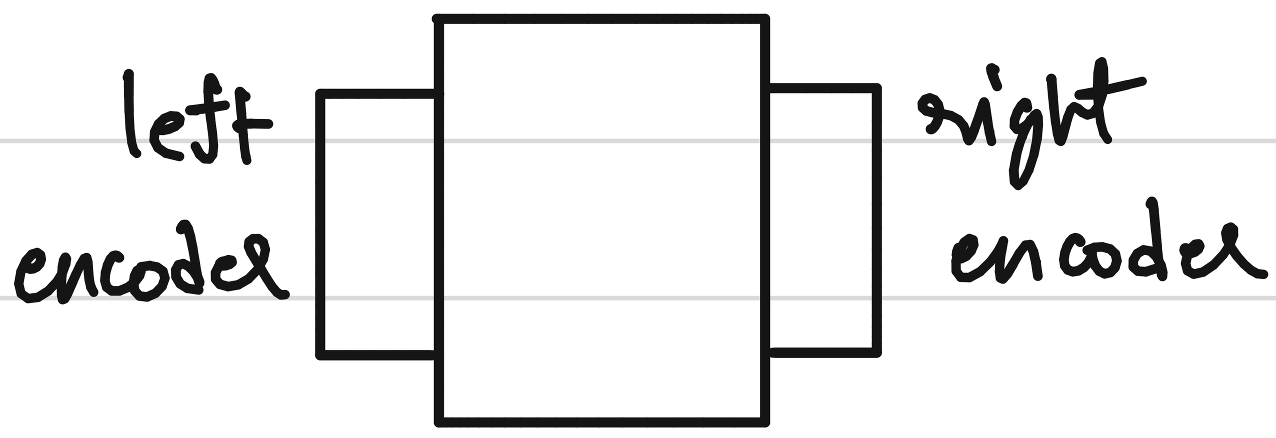
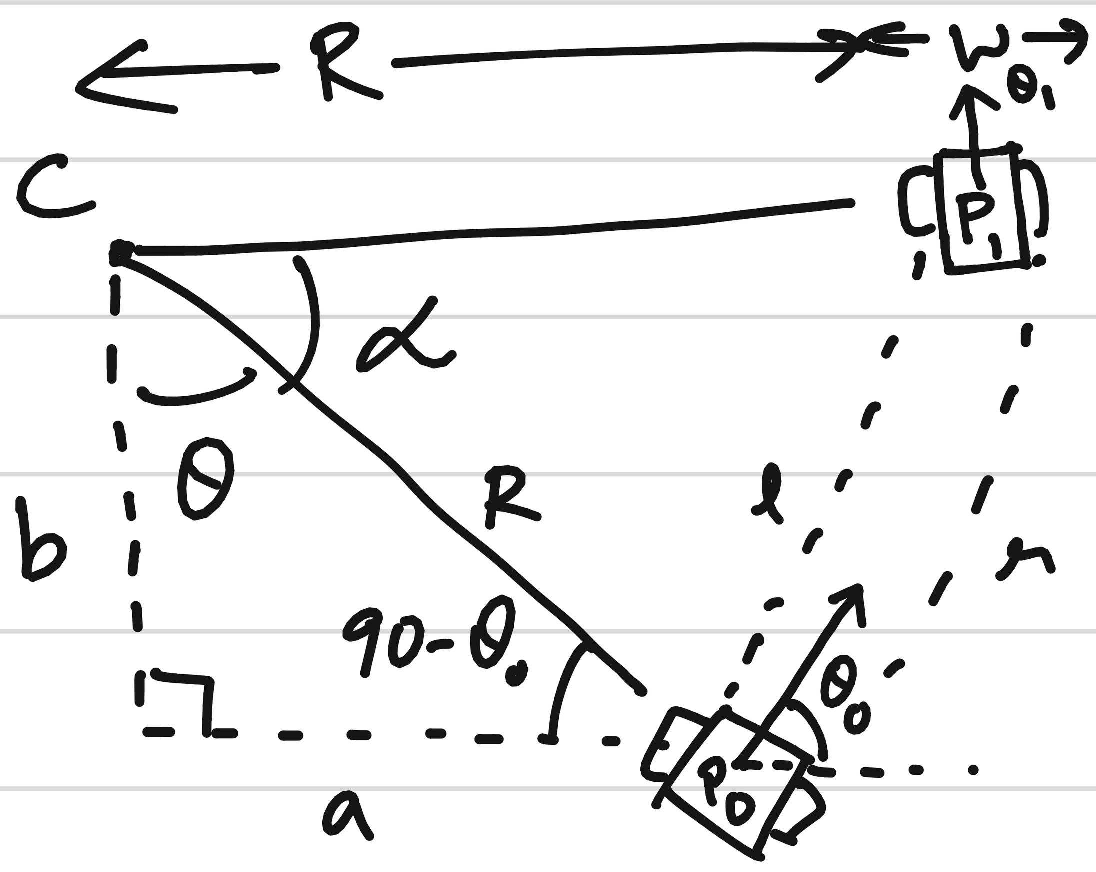
⇒ Problem statement :

Find all the equations necessary to perform filtering on the given robot.



$1 \text{ Tick} \simeq 0.349 \text{ mm}$

⇒ Motion Model :



→ $\dfrac{l}{\cancel{2\pi} R} = \dfrac{\alpha}{\cancel{2\pi}}$ (Equating angle and perimeter proportions)

$\boxed{l = \alpha R}$ — (1) $\Rightarrow r = \alpha(R+w)$

$r - l = \alpha(R+w) - \alpha R$

→ $\sin\theta = \dfrac{a}{(R+w/2)}$ $\boxed{\alpha = \dfrac{r-l}{w}}$

C – Center (Pivot) of rotation

α – Angle of rotation

l – distance travelled by left encoder

r – distance travelled by right encoder

w – width of the robot

P – Position of the robot

θ – Heading angle of the robot

R – Radius of rotation

$a = \left(R + \frac{w}{2}\right)\sin\theta$

$b = \left(R + \frac{w}{2}\right)\cos\theta$

→ $C = \begin{bmatrix} P_{0x} - a \\ P_{0y} + b \end{bmatrix}$

$C = P_0 - \begin{bmatrix} a \\ -b \end{bmatrix} = P_0 - \left(R + \frac{w}{2}\right)\begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}$ — (2)

→ When $\alpha \neq 0$ i.e., $l \neq r$, $\theta' = \theta + \alpha$

$$P_1 = C + \left(R + \frac{W}{2}\right)\begin{bmatrix}\sin(\theta+\alpha) \\ -\cos(\theta+\alpha)\end{bmatrix}$$

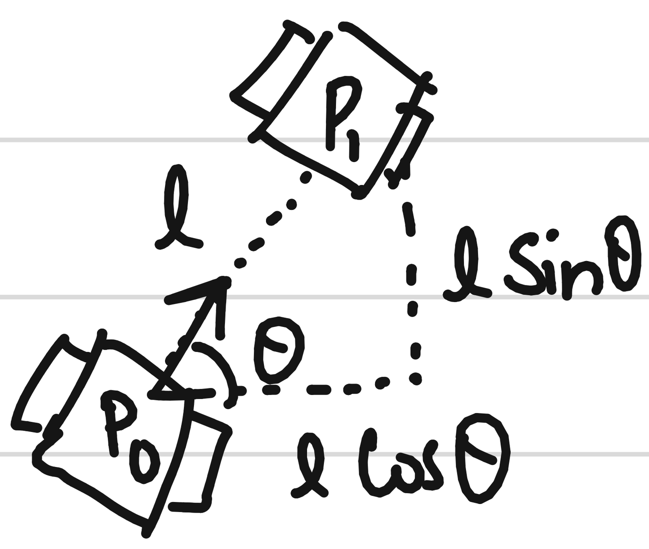
From (2)

$$P_1 = P_0 - \left(R+\frac{W}{2}\right)\begin{bmatrix}\sin\theta \\ -\cos\theta\end{bmatrix} + \left(R+\frac{W}{2}\right)\begin{bmatrix}\sin(\theta+\alpha) \\ -\cos(\theta+\alpha)\end{bmatrix}$$

$$P_1 = P_0 + \left(R+\frac{W}{2}\right)\begin{bmatrix}\sin(\theta+\alpha) - \sin\theta \\ \cos\theta - \cos(\theta+\alpha)\end{bmatrix}$$

→ When $\alpha = 0$ i.e., $l = r$, $\theta' = \theta$

$$P_1 = P_0 + l\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix}$$



Final Motion Model :

- When $\alpha_t \neq 0$ $(l_t \neq r_t)$:

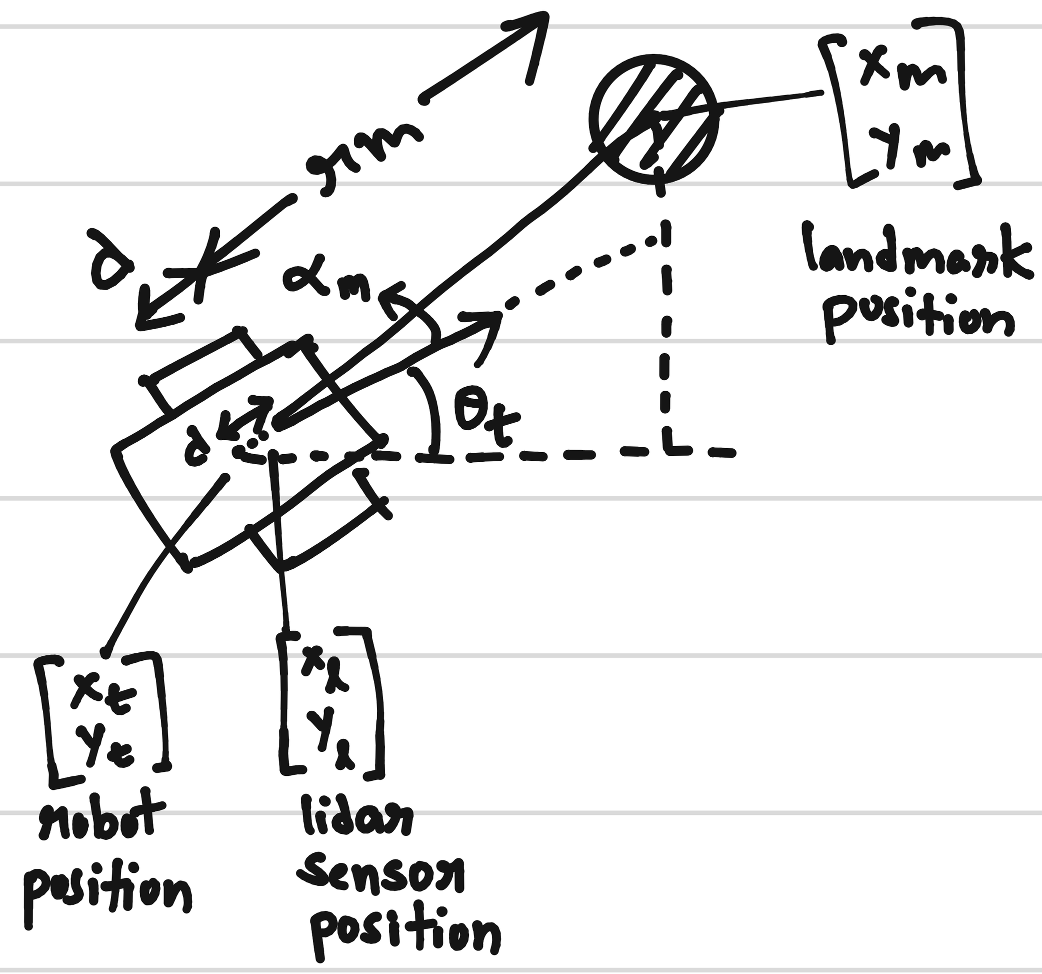
$$X_t = \begin{bmatrix}x_{t+1} \\ y_{t+1} \\ \theta_{t+1}\end{bmatrix} = \begin{bmatrix}x_t \\ y_t \\ \theta_t\end{bmatrix} + \begin{bmatrix}(R+W/2)(\sin(\theta_t+\alpha_t) - \sin\theta_t) \\ (R+W/2)(\cos\theta_t - \cos(\theta_t+\alpha_t)) \\ l_t/R\end{bmatrix}$$

- When $\alpha_t = 0$ $(l_t = r_t)$:

$$X_t = \begin{bmatrix}x_{t+1} \\ y_{t+1} \\ \theta_{t+1}\end{bmatrix} = \begin{bmatrix}x_t \\ y_t \\ \theta_t\end{bmatrix} + \begin{bmatrix}l_t\cos\theta_t \\ l_t\sin\theta_t \\ 0\end{bmatrix}$$

⇒ Measurement Model :

$$\begin{bmatrix} x_l \\ y_l \end{bmatrix} = \begin{bmatrix} x_t \\ y_t \end{bmatrix} + d \begin{bmatrix} \cos\theta_t \\ \sin\theta_t \end{bmatrix}$$



$$Z_t = \begin{bmatrix} r_m \\ \alpha_m \end{bmatrix} = \begin{bmatrix} \sqrt{(x_m - x_l)^2 + (y_m - y_l)^2} \\ \operatorname{atan}\left(\frac{y_m - y_l}{x_m - x_l}\right) - \theta_t \end{bmatrix}$$

↖ The laser scanner mounted on the robot provides a measurement angle α_m and a measurement distance r_m.

⇒ Kalman Filter :

Prediction :

$$x_t = A_t x_{t-1} + B_t u_t + \varepsilon_Q$$

- x – state of robot
- A – Transformation of state
- u – control of robot
- B – Transformation of control
- ε_Q – Robot process error
- $\overline{bel}(x)$ – Predicted Belief (x)
- Q – known robot noise
- μ – mean state of robot
- Σ – covariance of state of robot
- $\bar{\mu}$ – Predicted state of robot
- $\bar{\Sigma}$ – predicted covariance of state of robot

→ However, we can't model the motion perfectly so, we look at a distribution of state instead. Here, we use a Gaussian distribution.

$$\left.\begin{aligned} \bar{\mu}_t &= A_t \mu_{t-1} + B_t \mu_t \\ \bar{\Sigma}_t &= A_t \Sigma_{t-1} A_t^T + Q_t \end{aligned}\right\} \overline{bel}(x_t)$$

Correction :

$$Z_t = C_t \bar{X}_t + \varepsilon_R$$

$$K_t = \bar{\Sigma}_t C_t^T \left(C_t \bar{\Sigma}_t C_t^T + R_t\right)^{-1}$$

$$\left.\begin{aligned} \mu_t &= \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t) \\ \Sigma_t &= (I - K_t C_t)\bar{\Sigma}_t \end{aligned}\right\} bel(x_t)$$

$\bar{X}$ – Measured state (raw) of robot
C – Transformation of $\bar{X}$
ε_R – Measurement error
Z – Measured state of robot
$bel(X_t)$ – corrected belief (x)
K – Kalman Gain
R – Known measurement noise
μ_t – corrected mean of robot state
Σ_t – corrected variance of state of robot

→ However, the regular Kalman filter only works with a linear motion model and a linear measurement model.

→ The given robot has non-linear models because they include non-linear functions such as $\sin\theta$, $\cos\theta$, $\tan\theta$, and x^2.

→ For non-linear functions, we need to linearize the functions before using the Kalman filter. This is also called as the Extended Kalman Filter.

⇒ Extended Kalman Filter:

Prediction step:

$$x_t = g(x_{t-1}, u_t)$$

$$\left.\begin{aligned} \bar{\mu}_t &= g(x_{t-1}, u_t) \\ \bar{\Sigma}_t &= G \Sigma_{t-1} G^T + V \Sigma_{control} V^T \end{aligned}\right\} \overline{bel}(x_t) - \text{Predicted Belief of } X$$

g – non-linear function
G – Jacobian of g w.r.t state X
V – Jacobian of g w.r.t input u
$\Sigma_{control}$ – Covariance matrix of control noise.

Here are the prediction step equations for the given robot,

$$\Sigma_{\text{Control}} = \begin{bmatrix} \sigma_l^2 & 0 \\ 0 & \sigma_r^2 \end{bmatrix}$$

(Elements other than the diagonals are non-zero because left and right motor are independent).

→ How to get σ_l^2 and σ_r^2?

Lets say we know that the encoders have a 30% error and their is 60% slip error between left and right wheels when rotating.

$$\sigma_l^2 = (\gamma_1 l)^2 + (\gamma_2 (l-r))^2$$

$$\sigma_r^2 = (\gamma_1 r)^2 + (\gamma_2 (l-r))^2 \quad \text{where } \gamma_1 = 0.3 \text{ and } \gamma_2 = 0.6$$

We need to square each term because the LHS is quadratic.

→ Since the given state depends on (x, y, θ),

$$G = \begin{bmatrix} | & | & | \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial \theta} \\ | & | & | \end{bmatrix}_{3\times 3}$$

$$G = \begin{bmatrix} 1 & 0 & (R+W/2)(\cos(\theta_t+\alpha_t) - \cos\theta_t) \\ 0 & 1 & (R+W/2)(\sin(\theta_t+\alpha_t) - \sin\theta_t) \\ 0 & 0 & 1 \end{bmatrix} \quad \text{when } \alpha \neq 0, \; l \neq r$$

$$G = \begin{bmatrix} 1 & 0 & -l\sin\theta \\ 0 & 1 & l\cos\theta \\ 0 & 0 & 1 \end{bmatrix} \quad \text{when } \alpha = 0 \text{ i.e., } l = r$$

→ Since the given control depends on (l, R),

$$V = \begin{bmatrix} | & | \\ \frac{\partial g}{\partial l} & \frac{\partial g}{\partial r} \\ | & | \end{bmatrix}_{3\times 2}$$

- $\alpha \neq 0$ i.e., $l \neq r$

$$\frac{\partial g_{11}}{\partial l} = \frac{wr}{(r-l)^2}\left(\sin(\theta_t+\alpha_t) - \sin\theta_t\right) - \frac{(r+l)}{2(r-l)}\cos(\theta_t+\alpha_t)$$

$$\frac{\partial g_{21}}{\partial l} = \frac{wr}{(r-l)^2}\left(-\cos(\theta_t+\alpha_t) + \cos\theta_t\right) - \frac{(r+l)}{2(r-l)}\sin(\theta_t+\alpha_t)$$

$$\frac{\partial g_{31}}{\partial l} = -1/w$$

$$\frac{\partial g_{11}}{\partial l} = \frac{-wl}{(r-l)^2}\left(\sin(\theta_t+\alpha_t) - \sin\theta_t\right) + \frac{r+l}{2(r-l)}\cos(\theta_t+\alpha_t)$$

$$\frac{\partial g_{21}}{\partial r} = \frac{-wl}{(r-l)^2}\left(-\cos(\theta_t+\alpha_t) + \cos\theta_t\right) + \frac{r+l}{2(r-l)}\sin(\theta_t+\alpha_t)$$

$$\frac{\partial g_{31}}{\partial r} = 1/w$$

- When $\alpha = 0$ i.e., $l = r$,

$$V = \begin{bmatrix} \frac{1}{2}\left(\cos\theta + \frac{l}{w}\sin\theta\right) & \frac{1}{2}\left(-\frac{l}{w}\sin\theta + \cos\theta\right) \\ \frac{1}{2}\left(\sin\theta - \frac{l}{w}\cos\theta\right) & \frac{1}{2}\left(\frac{l}{w}\cos\theta + \sin\theta\right) \\ -1/w & 1/w \end{bmatrix}$$

Correction step :

$$Z_t = h(x_t)$$

$$K_t = \bar{\Sigma}_t H_t^T (H_t \bar{\Sigma}_t H_t^T + R)^{-1}$$

$$bel(x_t) \begin{cases} \mu_t = \bar{\mu}_t + K_t (z_t - h(\bar{\mu}_t)) \\ \Sigma_t = (I - K_t H_t) \bar{\Sigma}_t \end{cases}$$

h - non-linear function
$\bar{X}$ - Measured state (raw) of robot
H - Jacobian of h w.r.t state X
Z - Measured state of robot
$bel(x_t)$ - corrected belief (x)
K - Kalman Gain
R - known measurement noise
μ_t - corrected mean of robot state
Σ_t - corrected variance of state of robot

→ Here are the correction step equations of the given robot,

$$R = \begin{bmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\alpha^2 \end{bmatrix}$$

σ_r - measurement distance std dev
σ_α - measurement angle std dev

→ Since the given measurement depends on the state (x, y, θ),

Assume, $\Delta x = x_m - x_l$ $\Delta y = y_m - y_l$, $q = \Delta x^2 + \Delta y^2$

$$H = \begin{bmatrix} \frac{-\Delta x}{\sqrt{q}} & \frac{-\Delta y}{\sqrt{q}} & \frac{d(\Delta x \sin\theta - \Delta y \sin\theta)}{\sqrt{q}} \\ \frac{\Delta y}{q} & \frac{-\Delta x}{q} & \left[\frac{-d(\Delta x \cos\theta + \Delta y \sin\theta)}{q}\right] - 1 \end{bmatrix}_{2\times 3}$$