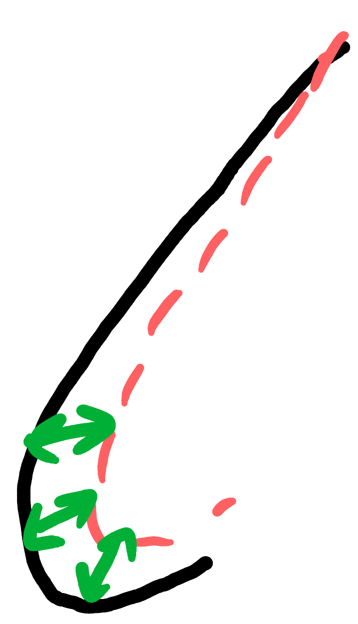
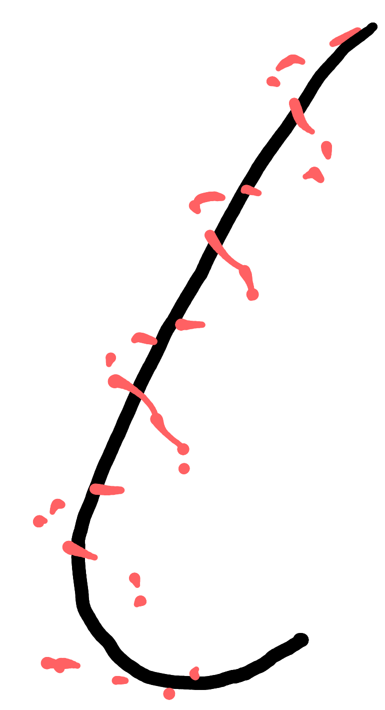
Trajectory Errors :

- **Systematic Error :**

This is the error occured when the estimated trajectory has a drift from the actual trajectory. It is likely caused by an incorrect parameter in the motion model.



Systematic Error



Random Error

- **Random Error :**

This type of error is difficult to pin point. It could be caused by various issues : sensor noise , environmental error , etc

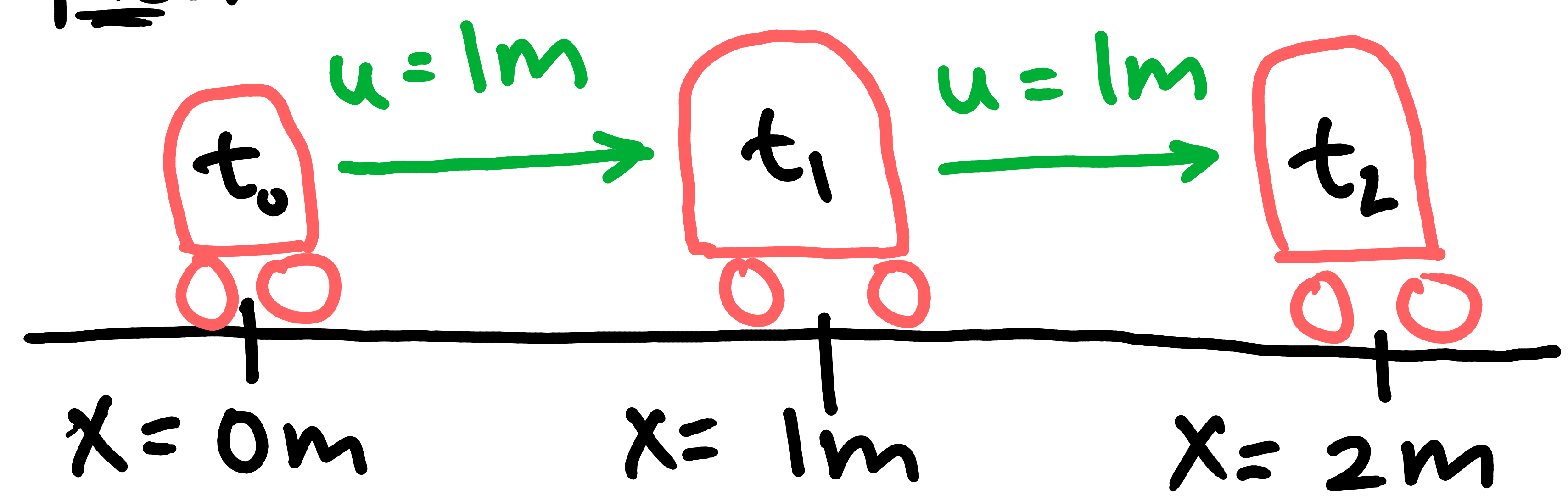
- So, all the real world properties that can be captured accurately categorize systematic errors while everything else is considered a random error

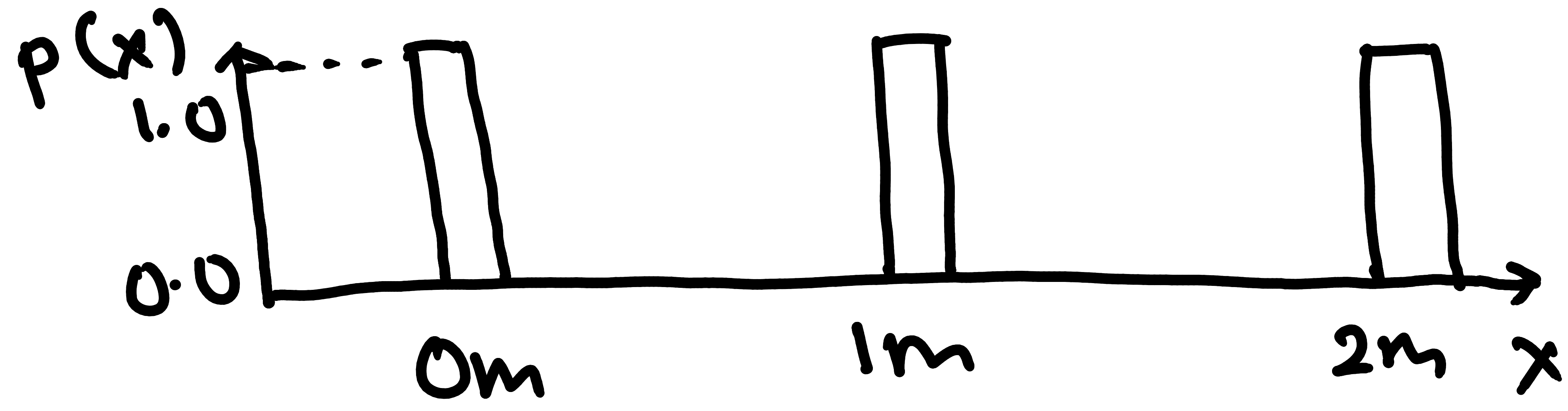
Probability Modelling : (Modelling uncertainity)

1D space:

Ideal Case :

(No Errors)





- Assuming, giving an input (u) to move 1m move the state (x) by exactly 1m.
- So, the probability that the robot is at x = 1m at t_1 is 100%. (1.0)

- But, if we say that the probability of the robot being exactly at the given position is 0.5 and the probability of the robot being ahead or behind the given position is 0.25. Then, we get a probability distribution. And, since its on discrete time, we have a discrete probability distribution.

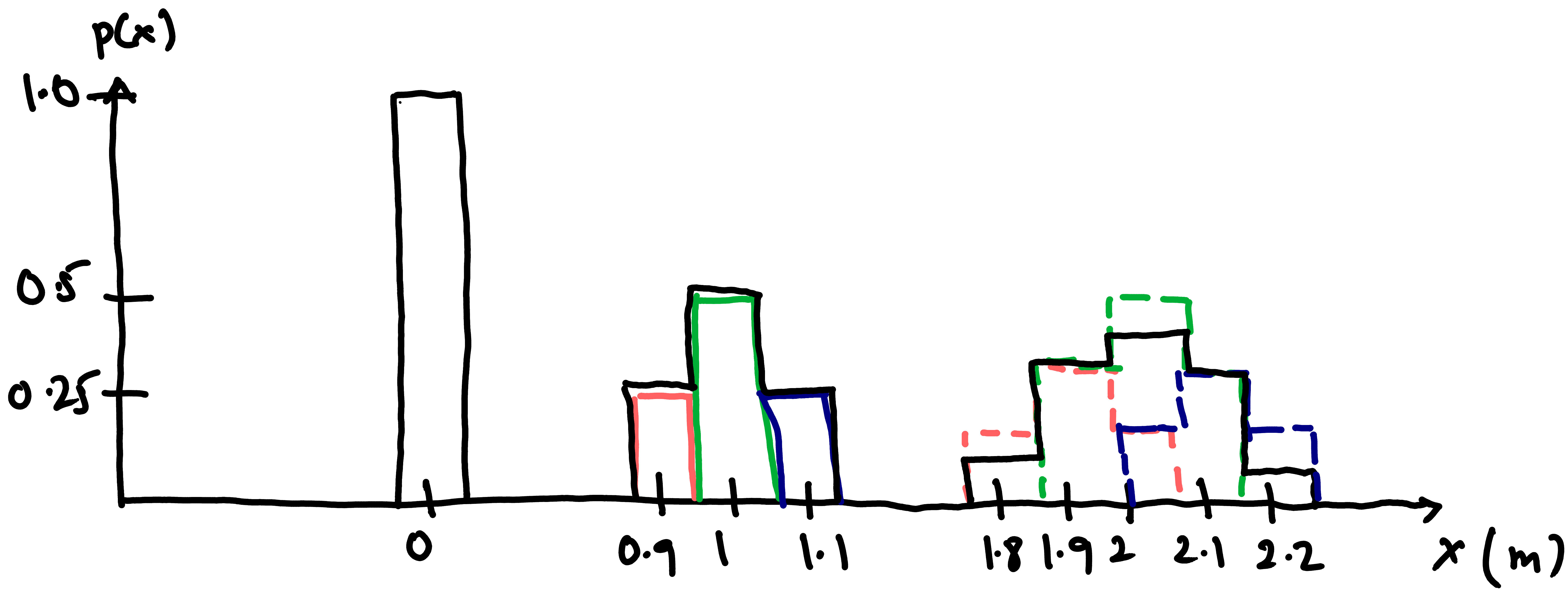
$$0 \leq p(x_i) \leq 1.0$$

$$\sum_{i=-\infty}^{\infty} p(x_i) = 1.0$$

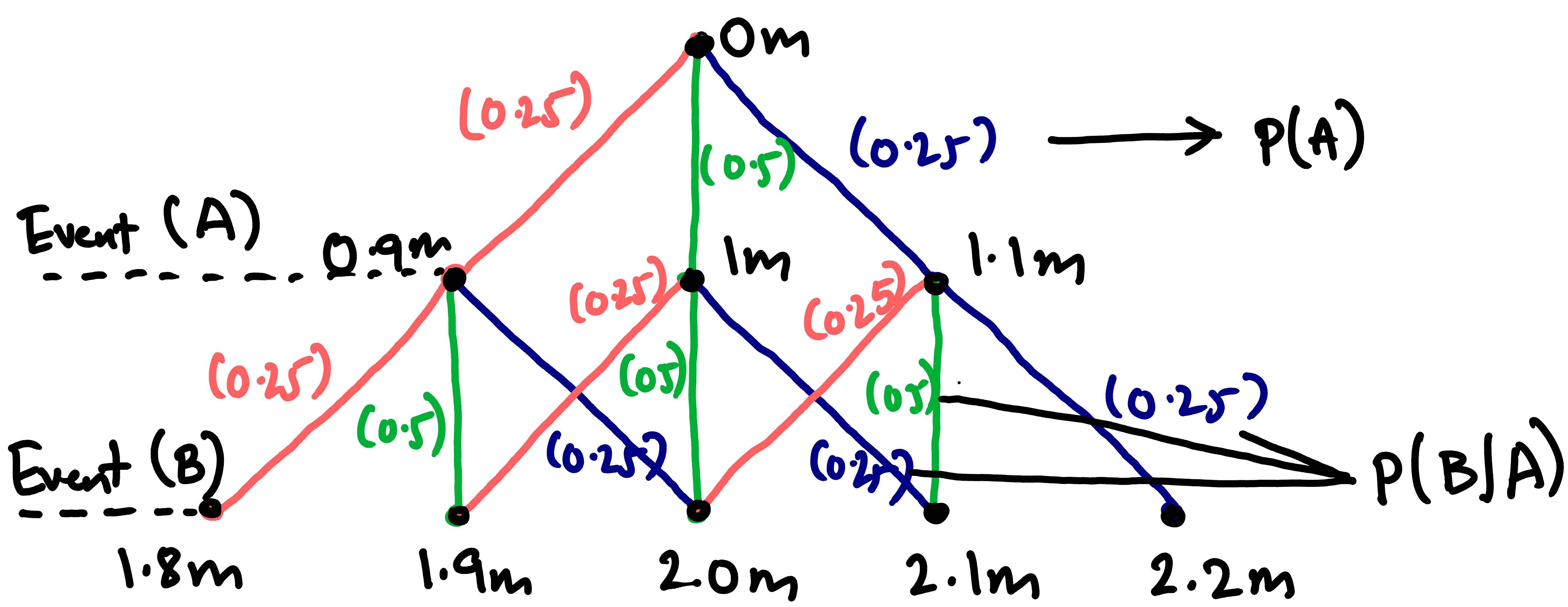
- But, this distribution has the error only in its initial position and the same error is propogated through actions. So, no errors are caused due to actions which is still not close to the real world scenarios.

- So, lets say initially, the robot has no error but when the robot is moved, we get the same triangle distribution as before.

(Remember, this error is from the control)



To get the resulting distribution, we convolve the input and action distributions.



The probability of both events A and B occuring

Using conditioned probability, $P(B \cap A) = P(B|A)\,P(A)$

1.8m	1.9m	2.0m	2.1m	2.2m
(0.25)(0.25)	(0.25)(0.5)	(0.5)(0.5)	(0.25)(0.5)	(0.25)(0.25)
	(0.5)(0.25)	(0.25)(0.25)	(0.5)(0.25)	
		(0.25)(0.25)		

The final probability at every position = sum of all the conditional probabilities

	1.8m	1.9m	2.0m	2.1m	2.2m
⇒	0.0625	0.125	0.25	0.125	0.0625
		+ 0.125	+ 0.0625	+ 0.125	
			+ 0.0625		
	0.0625	0.25	0.375	0.25	0.0625

→ So, assuming we have a prior distribution of position $p(x)$ and a measurement distribution $p(z/x)$ from the sensor;

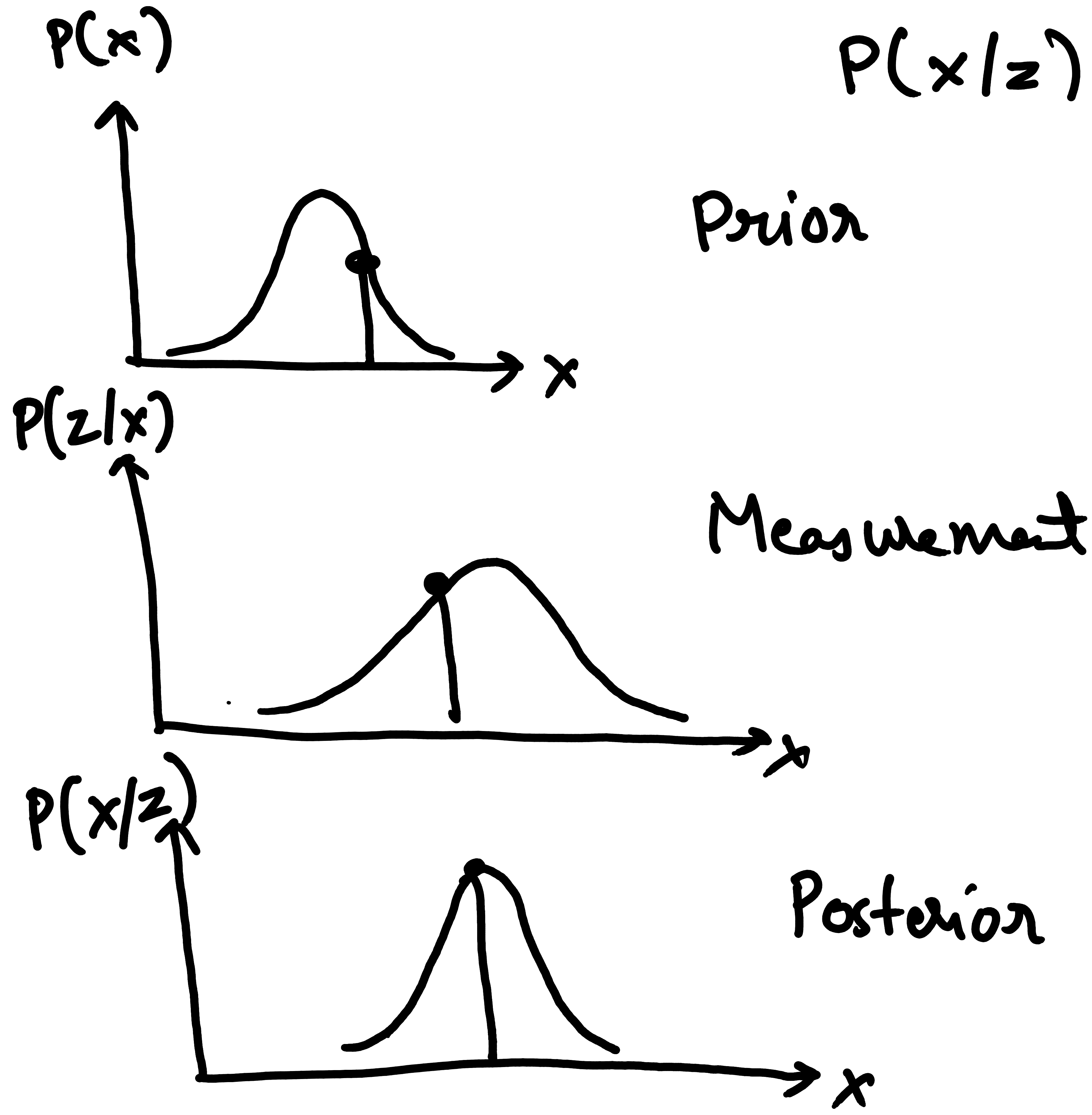
posterior distribution $P(x/z) = \dfrac{P(z|x).P(x)}{P(z)}$ [Using Bayes' theorem]

$$P(x|z) = \frac{P(z/x).P(x)}{\sum_{x'} P(z/x').P(x')}$$

$$P(x/z) = \underbrace{\alpha}_{\text{Normlization constant}} . P(z/x)\, P(x)$$

[$P(z \cap x)$ and $P(z)$ is the sum of $P(z \cap x)$ for all the possible values of x]

- So, $p(z)$ is the total probability of both z and x events happening. $\left(\sum_{x'} P(z \cap x')\right)$



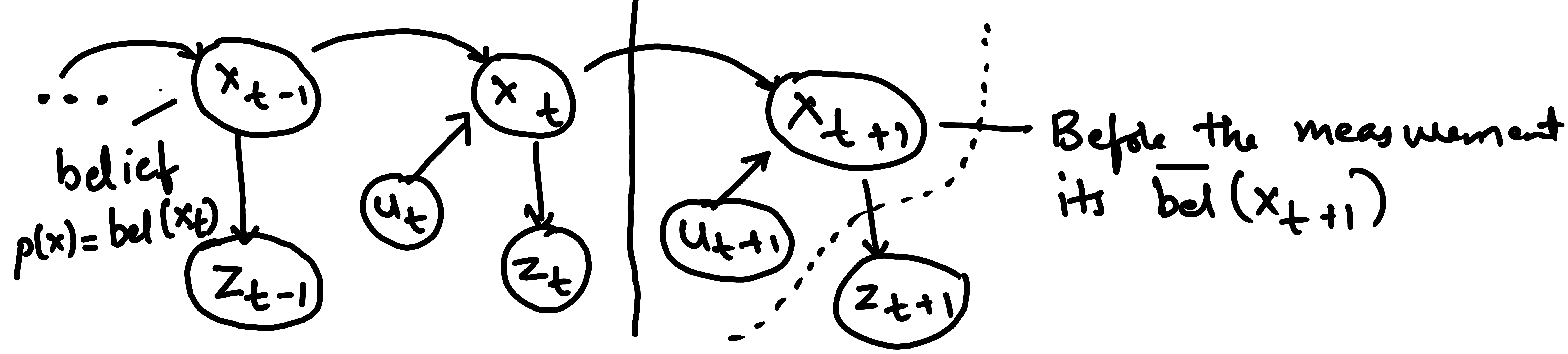
→ Therefore, the steps for pose estimation involving uncertainity are:

① Motion - Convolution:

$$P(x) = \sum_y P(x/y).P(y)$$ [y - previous position]

② Measurement - Multiplication:

$$P(x/z) = \alpha . p(z/x).p(x)$$



Bayes' Filter ($bel(x_{t-1})$, u_t, z_t):

For all x_t:

$$\overline{bel}(x_t) = \sum_{x_{t-1}} p(x_t \mid x_{t-1}, u_t) \cdot bel(x_{t-1})$$

$$bel(x_t) = \alpha \cdot p(z_t \mid x_t) \cdot \overline{bel}(x_t)$$

return $bel(x_t)$

This is the basis for histogram filter, kalman filter and the particle filter.
↳ discrete bayes' filter

<u>Note</u>:

- The accuracy of the estimated position with a measurement update decreases overtime.
- The estimated position using a bayes filter with the measurement step is either the same or better than the position estimated without the measurement step.

Normal distribution density: (Gaussian distribution)

$$\mathcal{N}(x;\mu;\sigma^2) = \frac{1}{\sqrt{2\pi}\cdot\sigma}\cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

σ - standard deviation = $\sqrt{\text{variance}}$

- The density is designed in such a way that the integral over all possible x is equal to one.

<u>Kalman</u> <u>Filter</u>:

- Instead of using a triangle distribution, use a normal distribution for both control and measurement.

→ Modifying the measurement step to use normal distributions:

$$\underbrace{bel(x)}_{\sim\mathcal{N}(x;\mu,\sigma^2)} = \alpha\cdot\underbrace{p(z|x)}_{\sim\mathcal{N}(z;cx,\sigma_Q^2)}\cdot\underbrace{\overline{bel}(x)}_{\sim\mathcal{N}(x;\bar{\mu},\bar{\sigma}^2)}$$

c: transformation factor (Eg: input is in m but, measurement is in cm)

$$\Rightarrow bel(x) = \alpha'\cdot e^{-\frac{1}{2}\left(\frac{z-cx}{\sigma_Q^2}\right)}\cdot e^{-\frac{1}{2}\left(\frac{x-\bar{\mu}}{\bar{\sigma}^2}\right)}$$

$$\Rightarrow \boxed{bel(x) = \alpha'\cdot e^{\underbrace{-\frac{1}{2}\left(\frac{z-cx}{\sigma_Q^2}\right)^2-\frac{1}{2}\left(\frac{x-\bar{\mu}}{\bar{\sigma}^2}\right)^2}_{\text{Quadratic in } x}}} \quad \text{—— (1)}$$

We know that any quadratic equation can be converted to the form $-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$.
Therefore, we can prove that the resulting distribution is a gaussian

$x\sim\mathcal{N}(x;\mu,\sigma^2)$; bel(x) is a normal distribution.

Assuming, $g(x) = \frac{1}{2\sigma_\varphi^2}(z - cx)^2 + \frac{1}{2\bar{\sigma}^2}(x - \bar{\mu})^2$ [from (1)]

Since $g(x)$ is a quadratic term, we can assume the following form.

$$\boxed{g(x) = \frac{1}{2}A(x-B)^2 + C} \quad \text{— (2)}$$

$$g'(x) = \frac{dg(x)}{dx} = A(x-B) \stackrel{!}{=} 0 \qquad (\text{Assuming } g'(x) = 0)$$

$$\Rightarrow x = B$$

$$g''(x) = \frac{d^2g(x)}{dx} = A$$

So, we can get B from $g'(x)$ and A from $g''(x)$

$$\therefore g'(x) = \frac{1}{\sigma_\varphi^2}(z - cx)(-c) + \frac{1}{\bar{\sigma}^2}(x - \bar{\mu}) \stackrel{!}{=} 0$$

$$\Rightarrow \frac{-zc}{\sigma_\varphi^2} + \frac{c^2x}{\sigma_\varphi^2} + \frac{x}{\bar{\sigma}^2} - \frac{\bar{\mu}}{\bar{\sigma}^2} = 0$$

$$\Rightarrow x\left[\frac{c^2}{\sigma_\varphi^2} + \frac{1}{\bar{\sigma}^2}\right] = \frac{zc}{\sigma_\varphi^2} + \frac{\bar{\mu}}{\bar{\sigma}^2}$$

$$\Rightarrow \boxed{x = \left[\frac{zc}{\sigma_\varphi^2} + \frac{\bar{\mu}}{\bar{\sigma}^2} \Big/ \frac{c^2}{\sigma_\varphi^2} + \frac{1}{\bar{\sigma}^2}\right] = B} \quad \text{— (3)}$$

$$\boxed{g''(x) = \left[\frac{c^2}{\sigma_\varphi^2} + \frac{1}{\bar{\sigma}^2}\right] = A} \quad \text{— (4)}$$

From 1 and 2,

$$bel(x) = \alpha' \cdot e^{-\frac{1}{2}A(x-B)^2 + C} = \alpha' \cdot e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

$$A = \frac{1}{\sigma^2} \; ; \; B = \mu$$

From 3, $$\sigma^2 = \frac{1}{\frac{c^2}{\sigma_Q^2} + \frac{1}{\bar{\sigma}^2}} \quad \text{— 5}$$

From 4, $$\mu = \frac{\frac{zc}{\sigma_Q^2} + \frac{\bar{\mu}}{\bar{\sigma}^2}}{\frac{c^2}{\sigma_Q^2} + \frac{1}{\bar{\sigma}^2}} = \sigma^2\left[\frac{zc}{\sigma_Q^2} + \frac{\bar{\mu}}{\bar{\sigma}^2}\right]$$

$$\Rightarrow \mu = \sigma^2\left[\frac{c}{\sigma_Q^2}(z + c\bar{\mu} - c\bar{\mu}) + \frac{\bar{\mu}}{\bar{\sigma}^2}\right]$$

$$\Rightarrow \mu = \sigma^2\left[\frac{c}{\sigma_Q^2}(z - c\bar{\mu}) + \bar{\mu}\left[\frac{c^2}{\sigma_Q^2} + \frac{1}{\bar{\sigma}^2}\right]\right]$$

$$\Rightarrow \mu = \sigma^2\left[\frac{c}{\sigma_Q^2}(z - c\bar{\mu}) + \frac{\bar{\mu}}{\sigma^2}\right]$$

$$\Rightarrow \mu = \underbrace{\boxed{\frac{\sigma^2}{\sigma_Q^2} \cdot c}}_{\text{Kalman Gain — 6}}(z - c\bar{\mu}) + \bar{\mu}$$

$$\boxed{\mu = \bar{\mu} + K(z - c\bar{\mu})} \quad \text{— 7}$$

→ Correction step of the Kalman Filter:

$$\boxed{\mu = \underbrace{\bar{\mu}}_{\text{predicted state}} + \underbrace{K}_{\text{Kalman Gain}} \underbrace{\Big(\underbrace{z}_{\text{actual measurement}} - \underbrace{c\,\underbrace{\bar{\mu}}_{\text{predicted state}}}_{\text{predicted measurement}} \Big)}_{\text{innovation}}} \qquad \text{(from ⑦)}$$

So, if $k = 0$; $\mu = \bar{\mu} + 0 = \bar{\mu}$
if $k = 1$; $\mu = \bar{\mu} + z - \bar{\mu} = z$

$$\Rightarrow K = \frac{\sigma^2}{\sigma_\varphi^2} \cdot c \qquad \text{(from ⑥)}$$

$$\Rightarrow K = \frac{c}{\sigma_\varphi^2 \left[\frac{c^2}{\sigma_\varphi^2} + \frac{1}{\bar{\sigma}^2} \right]} = \frac{c}{c^2 + \frac{\sigma_\varphi^2}{\bar{\sigma}^2}}$$

$$\therefore \boxed{K = \frac{c \cdot \bar{\sigma}^2}{c^2 \cdot \bar{\sigma}^2 + \sigma_\varphi^2}} \quad \text{⑧}$$

$\bar{\sigma}^2$ – (predicted) prior variance

σ_φ^2 – measurement variance

From ⑤, $\sigma^2 = \dfrac{1}{\left[\frac{c^2}{\sigma_\varphi^2} + \frac{1}{\bar{\sigma}^2} \right]} = \dfrac{\sigma_\varphi^2 \cdot \bar{\sigma}^2}{c^2 \cdot \bar{\sigma}^2 + \sigma_\varphi^2} - \bar{\sigma}^2 + \bar{\sigma}^2$

$$\Rightarrow \sigma^2 = \Bigg[1 - c\underbrace{\frac{c\,\bar{\sigma}^2}{c^2 \bar{\sigma}^2 + \sigma_\varphi^2}}_{K} \Bigg] \bar{\sigma}^2 \Rightarrow \boxed{\sigma^2 = (1 - Kc)\,\bar{\sigma}^2} \quad \text{⑨}$$

→ So, instead of calculating and storing all the values in a distribution, we just calculate K, μ, and σ^2.

From (7), (8), and (9):

$$\boxed{\begin{aligned} &(1)\ K = \frac{c\,\bar{\sigma}^2}{c^2\bar{\sigma}^2 + \sigma_Q^2} \\ &(2)\ \mu = \bar{\mu} + K(z - c\bar{\mu}) \\ &(3)\ \sigma^2 = (1 - K.C)\,\bar{\sigma}^2 \end{aligned}}$$

→ Prediction step of the Kalman Filter:

From bayes filter,

$$\overline{bel}(x_t) = \int \underbrace{p(x_t \mid x_{t-1}, u_t)}_{\sim \mathcal{N}(x_t;\ \underbrace{a x_{t-1} + u_t}_{\text{affine transformation}},\ \underbrace{\sigma_R^2}_{\text{system noise}})} \cdot \underbrace{bel(x_{t-1})}_{\sim \mathcal{N}(x_{t-1};\ \mu_{t-1},\ \sigma_{t-1}^2)} \cdot dx_{t-1}$$

$$\Rightarrow \overline{bel}(x_t) = \int e^{-\frac{1}{2}\left(\frac{x_t - (a x_{t-1} + u_t)}{\sigma_R}\right)^2} \cdot e^{-\frac{1}{2}\left(\frac{x_{t-1} - \mu_{t-1}}{\sigma_{t-1}}\right)^2} \cdot dx_{t-1}$$

- So, is the integration of a normal distribution still a normal distribution?
  - Yes. But, the proof is tedious so, it is not included.

(For the proof, check out "Probabilistic Robotics" by Thrun, Fox and Burgard)

Result:

$$\boxed{\begin{aligned} &(1)\quad \bar{\mu}_t = a \cdot \mu_{t-1} + u_t \\ &(2)\quad \bar{\sigma}_t^2 = a^2 \cdot \sigma_{t-1}^2 + \sigma_R^2 \end{aligned}} \qquad (a x_{t-1} + u_t)$$

Kalman Filter $((\mu_{t-1}, \sigma_{t-1}^2), (u_t, \sigma_R^2), (z_t, \sigma_Q^2))$:

→ Prediction:

- $\bar{\mu}_t = a \cdot \mu_{t-1} + u_t$
- $\bar{\sigma}_t^2 = a^2 \sigma_{t-1}^2 + \sigma_R^2$

→ Correction:

- $K = \dfrac{c\bar{\sigma}^2}{c^2\bar{\sigma}^2 + \sigma_Q^2}$
- $\mu_t = \bar{\mu}_t + K(z_t - c\bar{\mu}_t)$
- $\sigma_t^2 = (1 - Kc)\bar{\sigma}_t^2$

return μ_t, σ_t^2