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Q1

Given (Ω, F, P) is a probability space
$$G = \{A \in F : P(A) = 0 \text{ or } 1\}$$

To show: G is σ-algebra on Ω

As (Ω, F, P) is a probability space

(1) $P(E) \geq 0 \;\forall E \in F$

(2) $P(\Omega) = 1$

(3) Let $F_1, F_2 \ldots \in F$ such that $F_i \cap F_j = \phi$ $\forall i, j \in (1, 2, \ldots, \infty)$

then $P(\bigcup_{i=1}^{\infty} F_i) = \sum_{i=1}^{\infty} P(F_i)$

By 2nd condition:
$$P(\Omega) = 1 \Rightarrow \Omega \in G$$

Now ~~by~~ let $B \in G$, then $P(B) = 0$ or 1.

If $P(B) = 0$ then $P(B^c) = 1 \Rightarrow B^c \in G$

$P(B) = 1$ then $P(B^c) = 0$ $B^c \in G$

Satkar

Now let $B_1, B_2, B_3 \in G$

Then $B_1, B_2 \ldots \in F$

Let's define $C_1 = B_1$, $C_2 = B_2 - B_1 = B_2 \cap B_1^c, \ldots$

$C_n = B_n - B_{n-1} \forall n = 2, 3 \ldots$

$C_i = B_i - B_{i-1} = B_i \cap B_{i-1}^c$

$B_i, B_{i-1} \in F, \Rightarrow B_i^c, B_{i-1}^c \in F$

$B_i \cap B_{i-1}^c = (B_i^c \cap B_{i-1})^c \in F$

$C_1, C_2 \ldots \in F$ (if $A, B \in F$ then $A \cup B \in F$, $(A \cup B)^c \in F$)

and $C_i \cap C_j = \emptyset$, $i, j \in 1, 2 \ldots$

$$P\left(\bigcup_{i=1}^{\infty} B_i\right) = P\left(\bigcup_{i=1}^{\infty} C_i\right) = \sum_{i=1}^{\infty} P(C_i).$$ (By 3rd axiom of prob.)

$$P\left(\bigcup_{i=1}^{\infty} B\right) = P(C_1) + P(C_2) - P(C_1) + P(C_3) - P(C_2) + \ldots$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} P(C_i) - P(C_{i-1}) = \lim_{n \to \infty} P(C_n)$$

Now $C_n = B_n - B_{n-1}$, $B_n, B_{n-1} \in G$

Solution

(3)

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$B_{n-1} \in G, \Rightarrow B_{n-1}^c \in G$

$P(B_n \cap B_{n-1}^c) = P(B_n) + P(B_{n-1}^c) - P(B_n \cup B_{n-1}^c)$

Since, $B_n, B_{n-1}^c \in G$, $P(B_n) = 0$ or 1 $P(B_{n-1}^c) = 0$ or 1

if both $P(B_n) = 0$, $P(B_{n-1}^c) = 0$ then,

$$P(B_n \cup B_{n-1}^c) = 0$$

$P(C_n) = 0$, if any one of them has zero probability

$P(B_n) + P(B_{n-1}^c) = 1$ & $P(B_n \cup B_{n-1}^c) = 1$

$P(C_n) = 0$

if both of them s.t $P(B_n) = P(B_{n-1}^c) = 1$

$$P(B_n \cup B_{n-1}^c) = 1$$

$P(C_n) = 1 + 1 - 1 = 1$

$P(C_n) = 1$ or 0

$\lim_{n \to \infty} P(C_n) = 1$ or 0

$\therefore P(\bigcup_{i=1}^{\infty} B) = 1$ or 0

$\bigcup_{i=1}^{\infty} B \in G$

Hence we proved that all three criteria for G to be σ algebra on Ω.

Sathoak

Q2

Ans 2 Let us consider the discreate random variable which is defined on the support $S_X = N$ with PMF given by:

$$f_X(x) = \begin{cases} \frac{a}{x^4} & \text{if } x \in N, \quad a = \left[\sum_{n=1}^{\infty} \left(\frac{1}{x^4} \right) \right]^{-1} \\ 0 & \text{o.w} \end{cases}$$

$$\sum_{x \in S_X} f_X(x) = \sum_{x \in N} \frac{a}{x^4} = a \sum_{n=1}^{\infty} \frac{1}{x^4} = 1$$

and $f_X(x) \geq 0$, $x \in \mathbb{R}$

Hence $f_X(x)$ is valid PMF.

$$E(X^2) = \sum_{x \in S_X} x^2 f_X(x)$$

$$= \sum_{x \in N} \left(x^2 \cdot \frac{a}{x^4} \right) = a \sum_{n=1}^{\infty} \frac{1}{x^2}$$

As, $\sum_{x \in S_X} |x^2| f_X(x) = E(x^2) = a \sum_{i=1}^{\infty} \frac{1}{x^2} < \infty$

Hence, $E(X^2)$ exist.

Now check for $E(X^3)$:

$$\sum_{x \in S_X} |x^3| f_X(x) = \sum_{x \in N} x^3 \frac{a}{x^4} = c \sum_{n=1}^{\infty} \frac{1}{\cancel{n}}$$

As $\sum_{n=1}^{\infty} \frac{1}{n} \to \infty$, As $n \to \infty$

Hence, $E(X^3)$ doesn't exist.

Q3
Ans 3 Using 1st order taylor series approximation:

$$E\left(\frac{X}{Y}\right) = \frac{E(X)}{E(Y)}$$

$$X = X_1 + X_2 + \cdots + X_{37}$$

$$Y = X_1 + \cdots X_{100}$$

$$E(X) = E(X_1 + X_2 + \cdots + X_{37})$$

$$= E(X_1) + E(X_2) + \cdots + E(X_{37})$$

$$= \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}$$

$$= \left(\frac{37}{2}\right)$$

$$E(Y) = E(X_1 + X_2 + \cdots + X_{100})$$

$$= E(X_1) + E(X_2) + \cdots + E(X_{100})$$

$$= \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}$$

$$= \frac{100}{2}$$

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$$E\left(\frac{x}{y}\right) = \frac{E(x)}{E(y)}$$

Here : $E(x) = \frac{37}{2}$

$$E(y) = \frac{100}{2}$$

$$E\left(\frac{x}{y}\right) = \frac{37/2}{100/2}$$

$$= \frac{37}{100}$$

Sarthak

Q4
Ans 4

$$P(\{x > t\} \cup \{x < -t\}) = P(|x| > t)$$

Let us consider the function:

$$g\ [0, \infty] \rightarrow [0, \infty)$$

$$g(x) = \frac{x^2}{1+x^2}$$

$$g(x) = 1 - \frac{1}{1+x^2}$$

$$g'(x) = \frac{2x}{(1+x^2)^2} \geq 0$$

so, g is non-decreasing

$E(g(x)) = E\left(\frac{x^2}{1+x^2}\right)$ exist.

Moment inequality:

$$P(|X| \geq t) \leq \frac{E(g(X))}{g(t)}$$

$$P(|X| \geq t) \leq \frac{E(X^2/1+X^2)}{g(t)}$$

$$P(|X| \geq t) \leq \frac{E(X^2/1+X^2)}{(t^2/1+t^2)}$$

$$P(|X| \geq t) \leq \frac{1+t^2}{t^2} E\left(\frac{X^2}{1+X^2}\right)$$

$$P(|X| < t) \leq P(|X| \leq t)$$

$$P(|X| < t) \leq \frac{1+t^2}{t^2} E\left(\frac{X^2}{1+X^2}\right)$$

Sarthak

Q5

Here Y will follow binomial distribution:

$Y \sim Bin(n, p)$

& $p \sim U(0, 0.25)$

Expectation of uniform variable:

$$E(p) = \int_0^b \left(\frac{p}{b-a}\right) dp$$

$$= \frac{b^2 - a^2}{2(b-a)} = \frac{b+a}{2}$$

$$= \frac{0 + 0.25}{2} = 0.125$$

$$E(Y) = E(E(Y|P)) = E(nP)$$

$$= n E(P)$$

$$= 10 \times 0.125$$
$$= 1.25$$

Variance of Y be will equal to:

$$V(Y) = E[V(Y|P)] + V[E(Y|P)]$$

$= E(np(1-p)) + V(np)$

$= n(E(p) - E(p^2)) + n^2 V(p)$

$= 10\left(0.125 - \left(\frac{(0.25)^2}{3}\right)\right) + 10^2 \times \frac{(0.25-0)^2}{12}$

$\left[\because V(p) = E(p^2) - (E(p))^2 \right.$

$E(p^2) = \int_0^b \frac{p^2}{(b-a)}\,dp$

$= \frac{b^3 - a^3}{3(b-a)}$

$= \frac{b^2 + ba + a^2}{3}$

$V(p) = \frac{b^2 + ab + a^2}{3} - \frac{(b+a)^2}{4}$

$= \frac{(b-a)^2}{12}$

$= 1.0417 + 0.521$

$= 1.563$

Aathira

Q6
Ans 6

$X_1, X_2, \ldots X_n$ be a random sample of size $n \geq 1$, from a cumulative distribution function F_θ, where $\theta \in \Theta$.

$T_n = T(X_1, \ldots X_n)$ is a MLE of θ.

To show:

T_n is consistent

$(P_\varepsilon(T_n = \theta) \to 1)$ as $n \to \infty$.

Let us first assume T_n is consistent:

$$\lim_{n \to \infty} P(|T_n - \theta| \leq \varepsilon) = 1 \quad \forall \varepsilon > 0 \ \& \ \theta \in \Theta$$

let us assume $\varepsilon = \frac{1}{n}$

$$P\left(T_n - \theta \leq \frac{1}{n}\right) = P\left(\frac{-1}{n} \leq (T_n - \theta) \leq \frac{1}{n}\right)$$

$$= F\left(\frac{1}{n}\right) - F\left(\frac{-1}{n}\right) \cdot \{\text{Here } F \text{ is CDF of } (T_n - \varepsilon)\}$$

$$= F\left(\frac{1}{n}\right) - \left[F\left(\frac{-1}{n}\right) - P\left(T_n - \theta = -\frac{1}{n}\right)\right]$$

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$$= F\left(\frac{1}{n}\right) - F\left(\frac{-1}{n}\right) + P\left(T_n - \theta = \frac{-1}{n}\right)$$

Taken $n \to \infty$

$$\lim_{n\to\infty} P\left(|T_n - \theta| \leq \frac{1}{n}\right) = \lim_{n\to\infty} F\left(\frac{1}{n}\right) - F\left(\frac{-1}{n}\right) + P\left(T_n - \theta = \frac{-1}{n}\right)$$

$$= \lim_{n\to\infty} \cancel{F(0) - F(0)} + \lim_{n\to\infty} \left(P(T_n - \theta = 0)\right)$$

$$= \lim_{n\to\infty} P(T_n = \theta)$$

Now as T_n is consistent $\lim_{n\to\infty} P\left(|T_n - \theta| \leq \frac{1}{n}\right) = 1$

$$\lim_{n\to\infty} P(T_n = \theta) = 1$$

Hence, $P(T_n = \theta) \to 1$ as $n \to \infty$

T_n is consistent $\Rightarrow P(T_n = \theta) \to 1$ as $n \to \infty$ — (1)

Now assume $P(T_n = \theta) \to 1$ as $n \to \infty$

Let $\varepsilon > 0$ be any real numbers

$\{T_n - \theta = 0\} \subset \{-\varepsilon \leq |T_n - \theta| \leq \varepsilon\}$

$P(T_n - \theta = 0) \leq P(-\varepsilon \leq |T_n - \theta| \leq \varepsilon)$

$P(T_n = \theta) \leq P(|T_n - \theta| \leq \varepsilon)$

As, $P(x) \leq 1$ for any x:

$P(T_n = \varepsilon) \leq P(|T_n - \varepsilon| \leq \varepsilon) \leq 1$

As $n \to \infty$:

$\lim_{n\to\infty} P(T_n = \theta) = 1$ and $\lim_{n\to\infty} 1 = 1$

By sandwich theo:

$\lim_{n\to\infty} P(|T_n - \theta| \leq \varepsilon) = 1$ $\forall\ \varepsilon > 0$ — (2)

Hence T_n is consistent.

$P(T_n = \theta) \to 1$ as $n \to \infty$

So by (1) & (2):

$P(T_n = \theta) \Leftrightarrow T_n$ is consistent.

Aathrav

Q8
Ans 8

As given $\frac{X_1}{X_2} \sim C(0,1)$

$$U_n = \sum_{i=1}^{n} \frac{X_{2i-1}}{X_{2i}}$$

M & N be independent cauchy :
$M \sim$ Cauchy $(0, \sigma)$
$N \sim$ Cauchy $(0, \tau)$

$$f_M(u) = \frac{1}{\pi\sigma} \frac{1}{1+\left(\frac{u}{\sigma}\right)^2}$$

$$f_N(v) = \frac{1}{\pi\tau} \frac{1}{1+\left(\frac{v}{\tau}\right)^2}$$

pdf of $U+V$:

$$\cancel{f_2(z) = \int_{-\infty}^{+\infty} \frac{1}{\pi\sigma}\left(1+\frac{u^2}{\sigma^2}\right)} \times \cancel{\frac{1}{\pi\tau}\left(1+\frac{v}{}\right)}$$

$$f_2(z) = \int_{-\infty}^{+\infty} \frac{1}{\pi\sigma} \frac{1}{\left(1+\left(\frac{w}{\sigma}\right)^2\right)} \left(\frac{1}{\pi\sigma}\right) \left(\frac{1}{1+((z-w)/\tau)^2}\right) dw$$

$$= \frac{1}{\pi(\sigma+\tau)} \frac{1}{1+\left(z/(\sigma+\tau)\right)^2}$$

Sum of two independent cauchy is again a cauchy, with scale parameters adding

$z_1 - z_n$ are i.i.d cauchy $(0,1)$

then $\sum z_i$ is cauchy $(0,n)$

so $U_n = \sum_{i=1}^{n} \frac{X_{2i-1}}{X_{2i}} \sim C(0,n)$ (1)

$V_n = \sum_{i=1}^{n} X_i^2$

$E(V_n) = E(X_n^2) = 1$ (give $N(0,1)$)

Using strong law of large numbers:

$V_n = \frac{X_1^2 + X_2^2 + \cdots X_n^2}{n} \to 1$ almost surely (2)

From (1) & (2):

$n\left(\frac{U_n}{V_n}\right) \to$ cauchy $(0,n)$ in distribution

Hence, $Z_n = \frac{U_n}{V_n} \to$ Cauchy$(0,1)$ in distribution.

Aathsay

Q10
Ans 10

$x_1, x_2 \ldots x_n$ be a RS from a population with:

$$f(x,\theta) = \begin{cases} Q(\theta)M(x) & \text{if } 0<x<\theta \\ 0 & o.w \end{cases}$$

Consider x_n = max$(x_1 \ldots x_n)$ and

x_1 = min$(x_1, \ldots x_n)$

likelihood maximization:

$$L(\theta, x) = \{ (Q(\theta)\, M(x))^n I_{(0,\infty)}(x_{(1)}) I_{(0,\theta)}(x_{(n)})$$

By Neyman Fisher:

$$L(\theta, x) = h(x)\, g_\theta(T(x))$$

$$h(x) = (M(x))^n I_{(0,\infty)}(x_n)$$

$$g_\theta(T(x)) = (Q(\theta))^n I_{(0,\theta)}(x_{(n)})$$

$h(x)$ is independent of θ & $g_\theta(T(x))$ depend on θ & x.

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$T(x) = x_{(n)}$

Hence, $T = X_n$ is a sufficient statistics for θ.

~~Q 8~~

~~Ans 8~~ $U_n = \sum_{i=1}^{n} \frac{X_{2i-1}}{X_{2i}}$

As given, $\frac{X_1}{X_2} \sim C(0,1)$

$U_n = \sum_{i=1}^{n} \frac{X_{2i-1}}{X_{2i}}$

Sattam

Q12
Ans 12

X_1 : lifetime of component manufactured by A

X_2 : lifetime of component manufactured by B

T be any random variable

$$T = \min(X_1, X_2)$$

T will denote time to which both component work.

	1	2	3	4	5
X_1	116	2	34	34	113
X_2	36	12	14	47	5
T	36	2	14	34	5

Distribution of T:

for $t \geq 0$, we know $T \geq t$, if and only if both $X_1 \geq t$ & $X_2 \geq t$ so $P(T \geq t)$

$$P(T \geq t) = P(X_1 \geq t, X_2 \geq t)$$

$$= P(X_1 \geq t)\, P(X_2 \geq t)$$

$$\therefore P(X \geq x) = 1 - P(X < x) = 1 - \int_0^x \lambda e^{-\lambda x} dx$$

$$= 1 - (1 - e^{-\lambda x})$$

$$= e^{-\lambda x}$$

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$$P(X_1 \geq t, X_2 \geq t) = P(X_1 \geq t)\, P(X_2 \geq t)$$
$$= e^{-(\lambda_1+\lambda_2)t}$$

$$F_T(t) = P(T \leq t) = 1 - e^{-(\lambda_1+\lambda_2)t}$$
$$f_T(t) = F_T'(t) = (\lambda_1+\lambda_2)\, e^{-(\lambda_1+\lambda_2)t}$$

T is an exponential random variable with mean $\left(\frac{1}{\lambda_1+\lambda_2}\right)$

Confidence interval for exponential distr.:

$\cancel{f(x/\lambda) = \lambda e^{-\lambda x}}$ $\quad [\because \lambda = (\lambda_1+\lambda_2)]$

~~density function~~ $y = 2\lambda T$ follows exponential distribution with parameter $1/2$ i.e χ^2_2

$$f(x/\lambda) = \lambda e^{-\lambda x}$$

density f^n: $\quad g(y) = \frac{1}{2} e^{-y/2}$ $\quad$ for $y > 0$

y: has exponential dist. with parameter $1/2$ i.e chi-square dist. with degree of freedom 2.

Sattar

Pivot:

$$h(T_1, \dots, T_n, \lambda) = 2\lambda \sum_{i=1}^{n} T_i \equiv \sum_{i=1}^{n} y_i$$

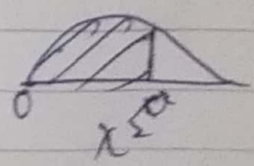
each $y_i = 2\lambda x_i$ follow χ^2_2 dist. and independent. so h follow χ^2_{2n} dist.

So,

$$P\left(\chi^2_{2n}(\alpha/2) < 2\lambda / \left(\sum_{i=1}^{n} T_i\right) \le \chi^2_n\left(1-\frac{\alpha}{2}\right)\right) = (1-\alpha)$$

$$P\left(\frac{\chi^2_{2n}(\alpha/2)}{2/\left(\sum_{i=1}^{n} T_i\right)} \le \lambda \le \frac{\chi^2_{2n}(1-\alpha/2)}{2\sum_{i=1}^{n} T_i}\right) = (1-\alpha)$$

$(2n = 10)$, $\alpha = 0.05$



$$\chi^2_{10}(0.975) = 20.5$$

$$\chi^2_{10}(0.025) = 3.25$$

$$\cancel{P\left(\frac{3.25}{2(36+2+14+34+5)} \le \lambda \le \frac{20.5}{2(36+2+14+34+21)}\right)}$$

A/ $\underline{3.25}$

Confidence interval for mean time $\left(\frac{1}{\lambda}\right)$ $\nu =$

$$P\left(\frac{2\sum_{i=1}^{n} T_i}{\chi^2_{2n}(1-\alpha/2)} \le \nu \le \frac{2\sum_{i=1}^{n} T_i}{\chi^2_{2n}\left(\frac{\alpha}{2}\right)}\right) = (1-\alpha)$$

$$\sum_{i=1} T_i = (36+2+14+34+5) = 91$$

$$P\left(\frac{2\times 91}{20.5} \le \mu \le \frac{2\times 91}{3.25}\right) = (1-0.05)$$

Interval t $\left(\frac{2\times 91}{20.5} \le \nu \le \frac{2\times 91}{3.25}\right)$

$(8.88 \le \nu \le 56)$

Confidence Int. : $(8.88, 56)$

Sattar

Q13

Ans 13 (a) $H_0 : \mu = 50$

$H_1 : \mu < 50$

- Here null hypothesis means that there is no change in the average number of blocked intrusion attempts.
- Here alternative hypothesis means that there is ~~sig~~ ~~chan~~ reduction in the average number of blocked intrusion attempts.

(b) $H_0 : \mu = 50$

$H_1 : \mu < 50$

S.A.I.T. $\bar{x} = \dfrac{53+21+32+ - - + 39+45}{20}$

$= 40.2$

Variance, $s^2 = \dfrac{1}{(N-1)} \sum_{i=1}^{N} (x_i - \bar{x})^2$

$= \dfrac{1}{19}\left((53-40.2)^2 + - + (45-40.2)^2\right)$

$= \dfrac{1203.2}{19} = 63.326$

$S = \sqrt{63.326}$

$S = 7.9577$

$\bar{x}$ & s^2 denotes sample mean & sample variance. $\bar{x}$ is the mle of μ and $[(n-1)/n]s^2$ is the mle of σ^2.

Random variable

$$T = \frac{\bar{x} - \mu}{\left(\frac{S}{\sqrt{n}}\right)}$$ has t-dist. with $(n-1)$ degree of freedom

$0 < \alpha < 1$,

$t_{\alpha/2, n-1}$ be the upper $\frac{\alpha}{2}$ critical point of a t-dist. with $(n-1)$ degree of freedom. i.e $\frac{\alpha}{2} = P(T > t_{\alpha/2, n-1})$

$$(1-\alpha) = P(-t_{\alpha/2, n-1} < T < t_{\alpha/2, n-1})$$

$$= P\left(-t_{\alpha/2, (n-1)} < \frac{\bar{x} - \mu}{S/\sqrt{n}} < t_{\alpha/2, n-1}\right)$$

$$= P\left(-t_{\alpha/2, n-1}\frac{S}{\sqrt{n}} < \bar{x} - \mu < t_{\alpha/2, n-1}\frac{S}{\sqrt{n}}\right)$$

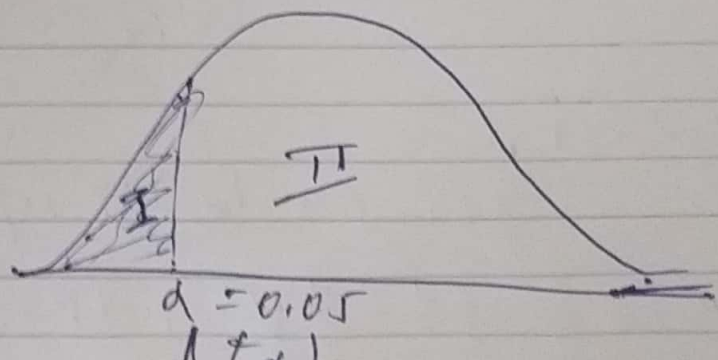
$$= P\left(\bar{x} - t_{\alpha/2, n-1}\frac{S}{\sqrt{n}} < \mu < \bar{x} + t_{\alpha/2, n-1}\frac{S}{\sqrt{n}}\right)$$

A $(1-\alpha)100\%$ confidence interval of μ is given by:

$$\left(\bar{x} - t_{\alpha/2, n-1}\frac{s}{\sqrt{n}}, \bar{x} + t_{\alpha/2, n-1}\frac{s}{\sqrt{n}}\right)$$

But by looking at sign of our alternative hypothesis we can say that it is left tail hypothesis)

So right side of ~~factor~~ value can go to infinity.



- If our t score $\left(\frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}}\right)$ is less than t_α then we will reject null hypothesis,

Means if our t-score is in region I we will reject our null hypothesis and if t-score is in region II we will accept our null hypothesis.

Salbelly

(c)

T-score calculations:

$$\frac{\bar{X} - 50}{\left(\frac{7.9577}{\sqrt{20}}\right)} \sim t_{(N-1)}$$

$$t = \frac{40.2 - 50}{1.7794}$$

$$= -5.507$$

for $\alpha = 0.05$

$$t_{19, \alpha} = -1.729$$

Hence our t-score fall in region 1, hence will reject null-hypothesis