Analogy Between Prime Sieve and Irrational Sieve

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Abstract

This paper explores a structural analogy between the sieve of Eratosthenes, used to isolate prime numbers, and an analogous operation on the real numbers that isolates irrational numbers by successively removing rationals with increasing denominators. Both can be interpreted as infinite filtering processes that remove finite or periodic structures, leaving behind elements that cannot be finitely represented.

1 Introduction

The sieve of Eratosthenes removes numbers that are multiples of smaller integers, leaving behind primes—numbers that cannot be expressed as products of smaller integers.

This paper proposes an analogous process on the real number line: starting with the continuum $\mathbb{R}$, we remove all rational numbers that can be expressed with denominators below a given bound, and let that bound tend to infinity. The remaining set, in the limit, consists of the irrational numbers—those that cannot be finitely expressed as ratios of integers.

2 The Sieve of Eratosthenes as a Filtering Process

Let $\mathbb{N} = \{2, 3, 4, \ldots\}$. For each integer $p \geq 2$, define the set of multiples:

$$M_p = \{n \in \mathbb{N} \mid p \text{ divides } n\}.$$

The sieve of Eratosthenes removes all such M_p for each p in increasing order, starting from $p = 2$.

Formally, define the remaining set after sieving up to $p = P$ as

$$S_P = \mathbb{N} \setminus \bigcup_{p \leq P} M_p.$$

Then, the primes are obtained in the limit:

$$\mathbb{P} = \lim_{P \to \infty} S_P.$$

This process removes all integers with finite factorizations involving smaller primes, leaving only those which cannot be decomposed—the primes themselves.

3 An Analogous Sieve on the Real Numbers

Now, consider the real line $\mathbb{R}$. For each positive integer q, define the set of rationals with denominator q:

$$R_q = \left\{ \frac{p}{q} \in \mathbb{Q} \mid p \in \mathbb{Z} \right\}.$$

These sets form a countable union:

$$\mathbb{Q} = \bigcup_{q=1}^{\infty} R_q.$$

We define a filtration of $\mathbb{R}$ analogous to the prime sieve:

$$T_Q = \mathbb{R} \setminus \bigcup_{q \le Q} R_q.$$

Then, as $Q \to \infty$, the limit set is

$$\lim_{Q \to \infty} T_Q = \mathbb{R} \setminus \mathbb{Q},$$

which is precisely the set of irrational numbers.

This "irrational sieve" removes all points that can be expressed as ratios of finite integers, just as the Eratosthenes sieve removes all numbers that can be expressed as products of smaller integers.

4 Structural Analogy

Both sieves share the following structural properties:

1. **Base domain:** The natural numbers $\mathbb{N}$ vs. the real numbers $\mathbb{R}$.
2. **Finite representability:** Integers with finite factorizations (composites) correspond to real numbers with finite rational representations.
3. **Filtering rule:** Removal of elements generated by finite combinations (products or rational divisions).
4. **Limit structure:** The remaining sets (primes and irrationals) are those that cannot be expressed by any finite process within the generating rule.

Thus, primes and irrationals both emerge as residual entities beyond finite generative closure:

$$\text{Primes: } \mathbb{P} = \mathbb{N} \setminus \bigcup_p M_p, \quad \text{Irrationals: } \mathbb{R} \setminus \bigcup_q R_q.$$

5 Conclusion and Outlook

Both sieves can be seen as instances of a broader concept: *a filtration process that removes all finitely generable elements to reveal the infinitary residue.* In this sense, primes and irrationals occupy analogous positions in their respective domains.

This analogy opens potential investigations into:

- Information-theoretic measures of "finite generability".
- Topological or measure-theoretic parallels between $\mathbb{P}$ and $\mathbb{R} \setminus \mathbb{Q}$.
- Computational analogs of "sieve complexity" across discrete and continuous systems.

Acknowledgement

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