# Todo list

# Chapter 1

# Introduction

## 1.1 What is analysis?

Real analysis is the analysis of real numbers, sequences and series of real numbers and real-valued functions (functions which have range as real numbers).

## 1.2 Examples

The examples in this section will show why understanding of real numbers is important. We'll see situations where if we don't really understand what these real numbers are, we won't be to make correct decisions.

### 1.2.1 division by zero

$ac = bc \implies a = b$. But, this does not work when $c = 0$. What is this example telling us? Whenever we are cancelling like how we did above, we are ruling out that $c = 0$ for all practical purposes.Because cancellation here actually means division by $c$, we must make sure $c \neq 0$.

### 1.2.2 divergent series

Take $S = 1 + \frac{1}{2} + \frac{1}{4} + +\frac{1}{8} + \dots$, what is $S$?. We can use this trick: Multiply both sides by 2. We get $2S = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \dots$.

$$2S = 2 + S \implies S = 2.$$

If we have another sum $S = 1 + 2 + 4 + 8 + 16 + \dots$, using the same trick leads to

$$2S = 2 + 4 + 8 + 16 + \cdots \implies 2S + 1 = S \implies S = -1.$$

Clearly, that should not be the case. My initial guess is that we cannot apply this trick for a divergent series but can for a convergent series. Why, I need to find out.

EOB need answer

### 1.2.3 Divergent sequence

Let $x$ be a real number and let $L$ define the limit as follows:

$$L = \lim_{n \to \infty} x^n.$$

Let $n = m + 1$.

$$\therefore L = \lim_{m+1 \to \infty} x^{m+1}.$$

$$L = x \lim_{m+1 \to \infty} x^m.$$

Is it allowed to do take $x$ out of the limit?

$$\because m + 1 \implies \infty, m \implies \infty.$$

$$L = x \lim_{m \to \infty} x^m.$$

$$L = xL.$$

Either $x = 1$ or $L = 0$. This means $L = \lim_{n \to \infty} x^n = 0, x \neq 1$. But, this does not make sense because clearly when instantiated for $x = 2$, $L = \lim_{n \to \infty} 2^n \neq 0$ My guess is that the part where we move $x$ out of the limit so easily should not be allowed.

### 1.2.4

### 1.2.5

### 1.2.6 Interchanging of integrals

We sometimes use this trick when integrating $\int \int f(x, y) dxdy = \int \int f(x, y) dydx$. But this too can lead to issues sometimes. Look at the example in the book for specific example.

EOB need answer on interchanging integrals

$$\int_0^\infty \int_0^1 \left(e^{-xy} - xye^{-xy} dydx\right) = \int_0^1 \int_0^\infty \left(e^{-xy} - xye^{-xy} dxdy\right).$$

We'll see in the book that the results of the two are different (LHS and RHS is not the same).

### 1.2.7 Interchanging limits

### 1.2.8

### 1.2.9

# Chapter 2

# Natural numbers

Just to reiterate the different kinds of numbers we'll deal with in this book, here are they in increasing order of sophistication. Natural numbers, $\mathbb{N}$, integers, $\mathbb{Z}$, rational numbers, $\mathbb{Q}$, real numbers, $\mathbb{R}$ and complex numbers, $\mathbb{C}$.

## Axiom and definition counts in the chapter

**Commentary:** Since axioms and definitions are the most important part of a book, we'll keep a track of the number of axioms and definitions in each chapter. 5 axioms for Peano's axioms to define natural numbers, 1 definition of addition, , 1 definition of positive numbers, 1 defintion of ordering (greater than).

## 2.1 The Peano axioms

The Peano axioms is one of the ways of defining natural numbers. There are other approaches such as using sets. We define natural numbers as that set of elements for which the following axioms hold:

### 2.1.1 0 is a natural number.

### 2.1.2 If $n$ is a natural number, then $n{+}{+}$ is also a natural number

Note, we assume that there exists an operation with symbol ++ that means increment. This axiom allows us to move forward in our count.

### 2.1.3 0 is not the successor for any natural number, i.e., $n++ \neq 0 \forall n$

This ensures we don't circle back to 0. Imagine if we defined $(0++)++$ as 0. Axioms 2.1.2 and 2.1.1 would still hold but we know we don't want that in our natural number system. This axiom ensures that.

### 2.1.4 $n \neq m \implies n++ \neq m++$

This axiom ensures that we don't reach an upper bound. If we define $2++$ and $3++$ both as 3, none of the above axioms are violated and yet we know we don't want an upper bound to our natural number system. Take the next number, $(3++)++ = 3++ = 3$, we have reached an upper bound for us. Another way to state this axiom is in its contrapositive form. $n++ = m++ \implies n = m$.

### 2.1.5 Principle of Mathematical Induction

It might be surprising right now why we are including this as an axiom because it is part of the larger logic scheme. Why do we need to explicitly say this. My guess is that it is needed because there are other systems where this principle does not hold. The principle is if some property $P$ holds for 0, that is, if $(P(0))$ is true and $P(n) \implies P(n+1)$ is true, then property $P$ is true for all elements in the number system.

find why math induction principle is needed as a Peano axiom to define natural numbers

This axiom allows us to define properties of objects for all elements.

**Commentary:**

Why do we fix other variables and induct on only one?

**Note our definition of natural numbers is axiomatic and not *constructive.*** What I mean by that is we have not defined *what* a natural number is but rather properties that a natural number holds. Any number system that follows the above mentioned axioms is a natural number system.

### 2.1.6 Recursive definition of sequence: Proposition

Suppose for each natural number $n$, we have some function $f_n : \mathbb{N} \mapsto \mathbb{N}$. Let $c$ be a natural number. Then, we can assign a unique natural number $a_n$ to each natural number $n$ such that $a_0 = c$ and $a_{\{n++\}=f_n(a_n)}$ for each natural number $n$.

**commentary** The goal is to show that each value of the sequence $a_i$ is defined only once and because we are looking at a recursive defintion we do not revisit $a_i$ again when going forward.

**proof** We'll prove this by induction. Take the base case, $a_0 = c$. Because of axiom 2.1.3, going forward, any $a_{m++}$ will not redefine $a_0$. For example, if axiom 2.1.3 was actually violated and say $3{+}{+} = 0$. Then, $a_{3++} = f_3(a_3) = a_0$. Note, we redefined $a_0$ without this axiom.

Let's assume that $a_n$ by the given recursive formula uniquely defined $a_n$. We need to this also holds for $a_{(n+1)}$. Any natural number after $n++$, say we call it $m++$ will redefine $n++$ because of axiom 2.1.4. Note, this axiom was introduced to ensure the successors are always bigger and that's what this axiom ensures. Because we cannot circle back, the previously defined $a_i$ remains as is.

## 2.2 Addition

### 2.2.1 Definition: Addition of natural numbers

Let $m$ be a natural number, we define addition of zero to $m$ as $0 + m := m$. We give a recursive defintion for adding. Let's say we know how to perform $n + m$. Therefore, adding $n++$ to $m$ is defined as $(n++) + m := (n + m)++$.

For example, say, we need to add $2 + 5$. $(1++) + 5 = (1 + 5)++ = ((0++) + 5)++ = ((0 + 5)++)++ = (5++)++ = 6++ = 7$

**Commentary:** This definition is enough to deduce everything about addition of natural numbers!

### 2.2.2 Lemma

For any natural number $n$, $n + 0 = n$.

**Commentary:** Note, in our recursive defintion, we have defined only $0{+}n = n$ and we don't know yet that for any $a, b \in \mathbb{N}$, $a + b = b + a$.

**Proof:** Take base case, $n = 0$. $0{+}0 = 0$. This is true by our recursive defition's base case. Assume, $n + 0 = n$ for $n$. We need to show $(n++) + 0 = n++$ and we are done with our proof by induction.

$(n++) + 0 = (n + 0)++$ by our defintion of addition. $(n + 0)++ = n++$ because of our induction assumption for $n$. Hence, proved.

### 2.2.3 Lemma

For any natural numbers $n, m$ $n + (m++) = (n + m)++$.

**Commentary:** Again, we cannot deduce this from our recursive defintion $(n++) + m = (n + m)++$ because we do not know $a + b = b + a$.

**Proof:** We'll try this by induction on $n$ (and keep $m$ fixed). Let's take the base case at $n = 0$. $0 + (m + +) = m + + = (0 + m) + +$. The stamenent holds for the base case.

Assume, $n + (m + +) = (n + m) + +$. We now need to show that $(n + +) + (m + +) = ((n + +) + m) + +$

$(n + +) + (m + +) = (n + (m + +)) + +$ by the recursive defintion of addition. $(n + (m + +)) + + = ((n + m) + +) + +$ by the the induction step taken for $n$. $((n + +) + m) + +$ by the defintion of recursive defntion of addition. Hence, proved.

### Corollary

$n + + = n + 1$

**Commentary:** Why would this hold because of Lemma 2.2.2 and Lemma 2.2.3.

**Proof:** $n+1 = n+(0 + +) = (n + 0)++$ by Lemma 2.2.3. $(n + 0)++ = n++$ because of Lemma 2.2.2. Hence, proved.

### 2.2.4 Proposition

Addition is commutative. For any natural numbers $n, m$, $n + m = m + n$.

**Proof:** We can prove this by induction. (note, anytime there's a proof of applying it for all numbers, induction really helps). We'll fix $m$. Take $n = 0$. $0 + m = m$ by defition of base case for addition. $m + 0 = m$ by Lemma 2.2.2. Base case is done.

Assume $n + m = m + n$. This is our induction step. We now need to show it holds for $n$++, i.e., $(n + +)+m = m+(n + +)$. LHS, $(n + +)+m = (n + m)++$ by the recursive definition of addition. $(n + m) + + = (m + n) + +$ by our induction step. RHS, $m + (n + +) = (m + n) + +$ by Lemma 2.2.3.

LHS = RHS. Hence proved. □

### 2.2.5 Proposition

Addition is associative. For any natural numbers $a, b, c$, we have $(a + b) + c = a + (b + c)$.

Write proof for Addition is associative

### 2.2.6 Proposition

Cancellation law. Let $a, b, c$ be natural numbers such that $a + b = a + c$. Then we have $b = c$.

**Commentary:** Note, we do not have the concept of negative numbers or subtraction yet.

**Proof:** We'll prove this by induction. Fix $b, c$. Take base case, $a = 0$. $0 + b = 0 + c \implies b = c$ because of base case of definition of addition.

Assume $a + b = a + c \implies b = c$. We need to show $(a++) + b = (a++) + c \implies b = c$.

$$(a++) + b = (a++) + c.$$
$$(a + b) ++ = (a + c) ++. \qquad \text{(by defintion of addition)}$$
$$b ++ = c ++. \qquad \text{(by assumption of the induction step)}$$
$$b = c.$$

Hence, proved by induction. □

### 2.2.7 Definition

Positive natural numbers. A natural number $n$ is said to be positive iff it is not equal to 0.

### 2.2.8 Proposition

If $a$ is positive and $b$ is a natural number, then $a + b$ is positive.

**Proof:** We'll use induction on $b$. Let $b = 0$. $a + b = a + 0 = a$. Base case holds. Assume $a + b$ is positive. Then we need to show $a + (b++)$ is positive. $a + (b++) = (a + b)++$ by Lemma 2.2.3. $(a + b)++$ cannot be zero because of Axiom 2.3 ($n++ \neq 0$ ). Hence, $(a + b)++$ is positive. This closes the induction. □

### 2.2.9 Corollary

If $a$ and $b$ are natural numbers such that $a + b = 0$ then $a = b = 0$.

**Proof:** We'll prove this by contradiction. Assume $a + b = 0$ but $a \neq 0$ or $b \neq 0$. In either case, $a + b = 0$ where one of them is not zero. But, by Proposition 2.2.8 that cannot be the case as $a + b$ must be positive (not zero). Hence, contradiction and $a = b = 0$. □

### 2.2.10 Lemma

Let $a$ be a positive number. Then there exists exactly one natural number $b$ such that $b++ = a$.

**Commentary:** Note, this means there is exactly one element behind $a$. There seems to be an order between the natural numbers.

### 2.2.11 Definition

Ordering of natural numbers. Let $n$ and $m$ be natural numbers. We say $n$ is greater than or equal to $m$ iff we have $n = m + a$ for some natural number $a$. We write greater than or equal to as $n \geq m$ or $m \leq n$. We say that $n$ is strictly greater than $m$ iff $n \geq m$ and $n \neq m$.

**Commentary:** Note, how we have used the concept of some $a$ existing such that adding it to $m$ gives us $n$. This also helps us show why there is no largest natural number because for every $n$, $n++ > n$. Hence, for the number $n++$, there's another larger number $(n++)++$.

### 2.2.12 Proposition

Basic properties of order for natural numbers. Let $a, b, c \in \mathbb{N}$. Then

1. (Order is reflexive) $a \geq a$
2. (Order is transitive) If $a \geq b$ and $b \geq c$, then $a \geq c$
3. (Order is anti-symmetric) If $a \geq b$ and $b \geq$, then $a = b$.
4. (Addition preserves order) $a \geq b$ iff $a + c \geq b + c$
5. $a < b$ iff $a++ \leq b$.
6. $a < b$ iff $b = a + d$ for some positive number $d$.

**Proof:**

1. 1 $a \geq a$ iff there exists $n$ such that $a = a + n$. Let $n = 0$. Therefore, $a = a + 0 = a$. Hence, proved.
2. It is given that $a \geq b$ and $b \geq c$. We need to show this implies $a \geq c$. There must bexist $m$ such that $a = b+m$ and $n$ such that $b = c+n$. Substituting value of $b$, $a = c+n+m \implies a = k+c$ where $k$ is some natural number. Hence, by definition of ordering of natural numbers, $a \geq c$. $\square$
3. $a = b + m$ for some $m$ and $b = a + k$ for some $k$. Substitute value of $b$, $a = a + k + m \implies a + 0 = a + (k + m) \implies 0 = k + m$ by cancellation law (Proposition 2.2.6). The only way this holds is iff $k = m = 0$. This is because there are four situations $k, m \neq 0 \vee k = m = 0, \vee k = 0 \wedge m \neq 0 \vee k \neq 0 \wedge m = 0$. In all situations except $k = m = 0$, we'll have a situation where Axiom 2.1.3 is violated because we'll have an equation of form $n++ = 0$. Hence, $a = b + m = 0 = b \implies a = b$. $\square$
4. Assume $a \geq b$. This means $a = b + m$ for some $m$. Adding $c$ on both sides, (for which we'll need to prove this logic TODO), $a + c = b + m + c \implies (a + c) = m + (b + c)$. (We have only rearranged the terms

using Proposition 2.2.4). Therefore, by definition of greater than, we have $a + c \geq b + c$. Now assume $a + c \geq b + c$. We need to show $a \geq b$. $a+c = b+c+d$ for some $d$. By cancellation law, $a = b+d$. Hence, $a \geq b$. We are done with our proof now.

5. Assume $a < b$. We need to show $a{+}{+} \leq b$. Or in other words $a+c = b$ for some $c$ and $a \neq b$. We'll prove this by contradiction. Assume $\neg\,(a + c \neq b) \wedge (a \neq b) = a + c \neq b \vee a = b$. We'll show that in both cases of or, we have a contradiction. $a + + \leq b \implies (a + +) + m = b = (a + m) + + = b$. Since $a = b$ (by our assumption, this is the second case of or), $(b + m) + + = b$. In both cases, when $m = 0 \vee m \neq 0$, we have a contradiction as $b + + \neq b$ as this would imply $0 + + = 0$. (This is a contradiction of Axiom 2.1.3). Now, we'll show the other case of or also leads to contradiction, i.e., $a + c \neq b$ will lead to contradiction. Since $a + + \leq b \implies (a + +) + c = b \implies (a + c) + + = b \implies (c + +) + a = b$. But there does not exist a number $b$ such that $a + c = b$ (Note, here there is no difference between $c$ and $c + +$ for us because all we care about is some number in N. Hence, contradiction. Hence, our initial assumption $\neg\,(a + c \neq b) \wedge (a \neq b)$ must be wrong. Therefore, $a + c = b$
a $\neq b$. Hence, we are done with left-to-right side for our proof. We now need to show right to left follows. A

### 2.2.13 Proposition:

Trichotomy of order of natural numbers. Let $a$ and $b$ be natural numbers. Then exactly one of the following statements is true: $a < b, a = b, a > b$.

**Commentary:** This is a great example to show how we can pin down what we want to say in exact terms using mathematics. We need to show that exactly one of three statements is true. We first show that no two statements can be true together. We also show that the three together also does not hold true. Then we show at least one of them is true. In simple terms, we show one statement will definitely be true and two statements cannot be true together. Hence, only one statement will be true.

**Proof:** We show first no more than one statement is true. If $a < b$, by definition of greater than $a \neq b$. If $a > b$, by definition of smaller than $a \neq b$. If $a < b$ and $a > b$, then by Proposition 2.2.12(3), $a = b$ which is a contradiction. Hence, no more than one statement can be true.

Now, we show at least one of the statements is true. We'll use induction. Fix $b$ and induct on $a$. Base case, $a = 0$. $0 \leq b$ for all $b$. This is because $b = a \vee b \neq a$. If $b = a$, $0 \leq b$ holds. If $b \neq a$, and we know $a = 0$, therefore, $b$ must be some positive number and hence $b > 0$. Therefore, we show that in both cases, $0 \leq b$. Now suppose we have proven the proposition for $a$ and now we want to prove the proposition for $a + +$. There are three cases: $a < b, a = b, a > b$. If $a > b$, by proposition 2.2.12($e$), we get $a + + \leq b$. This case holds. (Note, we show that $a + + = b \vee a + + < b$. I know we want to show only one statement holds overall but this part of the proof is about at least one of the statemnets is true, if both are true we don't care). If $a = b$. $a = b \implies a++ = b+1$. By defintion of greater than, $a++ \geq b$. Therefore, $a++ > b$ or $a++ = b$. If $a++ = b \implies a++ = a$ which cannot be the case. Therefore, $a++ > b$. We are done with this case as well. If $a < b$, then by Proposition 2.2.12(5), $a++ \leq b$ and in both cases we are done with our induction. Hence proved.

### 2.2.14 Proposition: Strong principle of induction

Suppose there exists a natural number $m_0$. Suppose we also have a property $P(m)$ for some natural number $m$ such that the following as true: If $P(m^{‘})$ is true for all $m_0 \leq m^{‘} < m$, then $P(m)$ is true. Then we conclude $P$ is true for all $m \geq m_0$.

**Commentary** : Is strong principle of induction stronger than our Principle of Induction Axiom (Proposition 2.1.5)itself? If strong principle of induction is proved using Principle of Induction Axiom then it's not really a stronger claim than Principle of Induction Axiom. Then, why call it strong?

The proposition hinges on the if then implication. It says that if proposition holds till $m$, it holds for $m++$. Note, this part is given, i.e., the implication itself is given as true. Then, we can conclude that $P$ is true for all $m$.

There are no numbers before $m_0$. And we are given that if $P(m‘)$ is true for all numbers before $m$, then $P(m)$ is true. So, $P(m_0)$ is true.

Let's talk about $P(m_1)$ then. It is given to us that if $P(m‘)$ is true for all numbers before $m_1$, it implies $P(m_1$ is true. We know $P(m_0)$ is true. Hence, $P(m_1)$ is true.

**This process of induction does feel like a trick but it hinges on a given if-then statement where it states that $P$ is true for all numbers before $m$, $P$ is true for $m$. Again, this is a given to us.**

## Exercises

### 2.2.1

**Q:** Prove Proposition 2.2.5, i.e., $(a + b) + c = a + (b + c)$.

**Proof:** We'll prove this by induction. Fix $b, c$. Take $a = 0$ for base case. $(0 + b) + c = b + c$. This was LHS. RHS side, $0 + (b + c) = b + c$. Base case holds. We'll take the induction step. Assume the given property holds for $a$. We need to show $((a++) + b) + c = (a++) + (b + c)$. Take LHS, $((a++) + b) + c = ((a + b)++) + c = ((a + b) + c)++$. We only used the definition of addition for this manipulation. Take RHS, we'll get $(a + (b + c))$++ after similar manipulation. By the inductive step we know $a+(b+c) = (a+b)+c = k$ for some $k$. LHS and RHS are equal. $k++ = k++$. Hence, induction closes here.

### 2.2.2

**Q:** Prove that if $a$ is a positive number then there exists exactly one number $b$ such that $b{+}{+} = a$.

**Commentary:** I have a feeling for contradiction here. Clearly, I have to show it in two parts, first that there exists such an number $b$ such that $b = a{+}{+}$ and that $b$ is unique. For the first part, we don't need a contradiction. Hint says *use induction*, so we'll try induction.

**Proof:** Base case, take $a = 1$. Clearly, there exists $b = 0$ since $0{+}{+} = 1$. There isn't any other number so we are good with the base case. Induction assumption step. Say, there exists exactly one number $b$ for $a$. We need to show there exists a number $c$ for $a{+}{+}$. $a{+}{+} = (b{+}{+}){+}{+}$. Here $b{+}{+} = c$. Since, we know there exists only one $b$, $b{+}{+}$ is also unique.

not satisfied with proof, try again

### 2.2.4

why(1): $b$ is a natural number. It either is 0 or a positive number.

why(2): We need to show $a > b \implies a{+}{+} > b$. $a > b$, by defintion means there exists $n$ such that $a = b + n$. Adding 1 on both sides, $a{+}{+} = b + (n{+}{+})$. Clearly, by definition of greater than we can say $a{+}{+} > b$.

why(3): We need to show $a = b \implies a{+}{+} > b$. $a{+}{+} = b{+}{+} \implies a{+}{+} = b + 1$. Clearly, by defintion of greater than $a{+}{+} > b$. $\square$

### 2.2.5

**Q: Prove Proposition 2.2.14 Strong principle of induction**

**Proof:** $(P(m_0)$ is true. $(P(m)$ is true for all $m \geq m_i \geq m_0$ implies $P(m{+}{+})$ is true$)) \implies P(m)$ is true $\forall m \in \mathbb{N}$.

**Commetary:** We'll use induction for this proof. The base case will be that $P(0)$ is true (we need to show that) and also as part of the base case, we need to show that $P(0)$ is true $\implies$, $P(0{+}{+})$ is true and that this implies $P(m)$ is true for all $m \in \mathbb{N}$. Then, as part of the induction step, we need to show if we assume $A, B \implies C$ to be true for $m$, then $A, B \implies C$ is also true. Note, here $A$ is $P(m_0)$ is true. $B$ is $P(m_i) \implies P(m{+}{+})\forall m \geq m_i \geq m_0$ and $C$ is $P(m)\forall m \in \mathbb{N}$ is true. Note, as part of the induction step as for the base case, we will assume that the first part of $B$ is true and show that the second part, i.e., the implication follows from it.

Take the base case $m_0 = 0$. It is given to use that $A, B$ is true and we need to show $C$ follows from them. $B$ is true, therefore the implication of it is true, i.e., $P(0{+}{+})$ is true. From this, we can say that $P(1)$ is true. So, again we have $P(0)$ is true and by $B$ we will get $P(1{+}{+})$ is true. This way, we can show $P(m)$ is true $\forall m \in \mathbb{N}$. We are done with the base case.

Now, assume $A, B \implies C$ is true for $m$. We need to show $A, B \implies C$ is also true. What do we get when we assume $A, B \implies C$ is true? Well, the exact statement that we need to prove. No point in repeating it here. What we need to show is that given $P(m_0)$ is true, and $P(m++) \implies P((m++)++)$ is true, tells us that $P(m++)$ is true ∀m++∈ $\mathbb{N}$. Any $m \in \mathbb{N}$ is written as $(((0++)++)++)$... where ++ operation is repeated $m$ number of times. We are given $P(1) \implies P(2)$ and so on and so forth. So, $P(0) \implies P(1)$ and by using the same logic we'll show that $P(1) \implies P(2)$. In this way, since every positive number is some $m++$, we have shown that $C$ is true $\forall m++ \in \mathbb{N}$. Hence, induction closed.

very bad proof, not satisfied, try again.

### 2.2.6

**Given:** $P(m++) \implies P(m)$. $P(n)$ is true.

**Show:** $P(m)$ is true $\forall m \leq n$.

**Proof:** Assume base case is about $n = 0$. We are given $P(0++) \implies P(0)$ is true. $P(m)$ is true trivially as there is $m \neg \leq 0$ Base case holds. Assume, $A \implies B, C$ is true and together they imply $C$ is true. We need to show the same for $m+++$. Here $n = m++$. And as part of given we have $P((n++)++) \implies P(n++)$, we need to show that $P(m)$ is true $m \leq n++$. Now, because of the given, $P(m)$ is true $\forall m \leq n$. We just need to show it also holds for $n = m++$, But that is also given as part of the assumption that $P(n++)$ is true. Hence, we are done with the induction. $\square$.

bad proof, not convinced

# Chapter 3

# Set Theory

## Definition and Axiom count

**Commentary:** What does obeying the axiom of substitution mean? Take example, for two natural numbers, $a, b$ we have $a = b$. Then, any other equation where $a$ is referred, I can substitute $b$ for it. Similarly, even for sets, we have the concept of equality. For any two sets, $A, B$ we have defined equality, $A = B$. Therefore, we can write $x \in A$ as $x \in B$.

## 3.1 Axiom

If $A$ is a set, then $A$ is also an object. In other words, given two sets $A, B$ it is meaningful to ask if $A \in B$ or $B \in A$.

# Appendix

## 3.2 Statements

**Exercises**

### 3.2.1

$$(x \lor y) \land \neg(x \land y)$$

The negation of the above statement is

$$\neg((x \lor y) \land \neg(x \land y)).$$

$$\neg(x \lor y) \lor (x \land y).$$

$$(\neg x \land \neg y) \lor (x \land y).$$

### 3.2.2

### 3.2.3

**G** iven: $x \implies y \land \neg x \implies \neg y$

**To show:** $x \iff y$

**Proof:** We need to show $y \implies x$.

### 3.2.4

**Given:** $x \implies y \land \neg y \implies \neg x$

**To show:** $x \iff y$

**Proof:** If $y$ is true, we cannot conclude anything about $x$ hence no we have not shown $x \iff y$.

complete this logic exercise as current proof is wrong