Todo list

Chapter 1

Vector Spaces

Chapter 2

Finite-Dimensional Vector Spaces

2.1 Span and Linear Independence

Test case to create generic vector using *vectorv* command: $v = a_1 v_1 + a_2 v_2 + ... + a_n v_n$

2.2 Bases

A *basis* of V is a list of vectors in V that is linearly independent and spans V.
How do we check if a given list of vectors is a basis or not?
One option is to check if they are linearly independent and span the space.But, we have a more beautiful solution. A criteria that when satisfied ensures that the given list of vectors is a basis.

Criteria for basis 1. *A list $v_1, v_2, ..., v_n$ of vectors span V iff every v in V can be written uniquely in the form*
$v = a_1 v_1 + ... + a_n v_n$

This is similar to the criteria of linear independence which is encoded by the criteria that a list of vectors are linearly independent *iff* their sum equals zero only when all coefficients are zero themselves.
How will we determine if given a list of k vectors for a space with dimension m, and $k >> m$, which m vectors in k form a basis for V, if any? Is there an algorithm for it?
Clearly, there are $C(k, m)$ options available. Given a particular combination, is there an algorithm to determine if they meet the criteria equation of linear independence.

Algorithm to find basis given a list of vectors

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Every subspace of V **is part of a direct sum equal to V 1.** *Suppose V is finite dimensional and U is a subspace of V. Then **there exists** a subspace W of V such that $V = U + W$.*

Because V is finite dimensional, so is U. Let $u_1, u_2, ..., u_m$ be the basis for U. Clearly, this basis can be extended by adding $w_1, w_2, ..., w_k$ such that $u_1, u_2, ..., u_m, w_1, ..., w_k$ span V.

Let $W = span(w_1, ..., w_k)$. We need to show $V = U + W$ and $V = UW = 0$. For any $v \in V$, because $u_1, ..., u_m, w_1, ... w_k$ span V, there exists $a_1, ... a_m, b_1, ..., b_k$ such that $v = a_1 u_1 + ... a_m u_m + b_1 w_1 + ... + b_k w_k = u + w$. $V = U + W$.

Now, we need to show the intersection property.Let $v \in UW$. Then, there exists scalars $a_1, ... a_m$ and $b_1, ..., b_k$ such that

$v = a_1 u_1 + ... + a_m u_m = b_1 w_1 + ... + b_k w_k$ or $a_1 u_1 + ... + a_m u_m + b_1 w1 + b_k w_k = 0$. Since, all vectors are independent, $a_1 = ... = a_m = b_1 + ... b_k$. Thus, $v = 0$. Note, we had taken any arbitrary v part of the intersection...

Hence, proved. $\square$

Exercises 2.B

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Intuitively, there cannot exist such a basis because if there is no polynomial p of degree 2, how can we represent $P_3(R)$. One way to solve this is take any 4 poynomials, assume they are independent, write their general, write their sum (the 4 polynomials) and we won't have a single term with degree 2. Therefore, there cannot exist such a basis. Hence proved.

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We need to show new list is also a basis. The criteria for basis is unique representation. As a helper method, we first show linear independence of the new list. That's easy. Assume non-independence and write independence criteria equation with assumption one of the coefficients is non-zero. Reshuffle all new terms such that they are written using original basis. This leads to contradiction as now a linear combination of original basis with at least one coefficient non-zero on the LHS and 0 on RHS. Since, all coefficients need to be 0, non-independence of the new list is false. Hence, independence shown.

Next, we need to show unique representation using the new new list. Assume two sets of coefficients lead to a common vector. Write the two representations with LHS = RHS. Because, we know the new list is independent, this leads to all corresponding coefficients in the two sets being equal. Hence, unique representation. Hence, proved.

As a side note, any linear combination of the new list can be rewritten as a combination of the original basis and the original basis spans V. Hence, the new list also spans V. Another way to think of it is every v can be written as a linear combination of the original basis and that representation can be written

as a linear combination of the new list. That is the definition of spanning. For every v, there exists a representation using the list.

2.3 Dimension

The dimension of a vector space V is defined as the length of its basis. Every basis for a given vector space has the same length which can be stated below.

commentary How do I imagine the concept of dimension? It's a visual image. Imagine a 3 dimensional cartesian plane. Take some vectors (say you take 2) such that they don't lie on top of each other. This ensures they are linearly independent. However number of directions they can cover is the dimension for that space. So, for example, take two set of vectors. v_1, v_2 and v_3, v_4. v_1, v_2 span a plane. v_3, v_4 also span a plane. If both sets of vectors are linearly independent, then in total they'll cover 4 directions but the space itself is limited to dimension 3. Hence, it must be the case that the at least one of the vectors from the first set is covered in the span of the second set or vice versa.

2.3.1 Basis length does not depend on basis

outline of proof There's a logical statment that will help us. When we say x is less than and equal to y and y is less than or equal to x, i.e., $x \leq y$ and $y \leq x$, we get $x = y$. Example, take two numbers $2 \leq 4$ and $4 \leq 2$, the only way other criteria is fullfilled is is 4 changes to 2.

proof Let B_1, B_2 be two basis for V. B_1 is linearly independent and B_2 spans V. , $len(B_1) \leq len(B_2)$. Note, we are using the property that the length of the linearly independent list is less than or equal to the length of the spanning list. /todoProve this statement without looking at the book. Similarly, we show $len(B_1) \leq len(B_2)$. Hence, $len(B_1) = len(B_2)$.

2.3.2 Changing field changes the dimension of a vector space

Let's take R_2 as the vector space defined over R. The dimension is obviously 2. But, if the same vector space R_2 is defined over field C, the dimension is 1. First of all, $c(x, y) \neq (cx, cy)$. That's not how we can multiply a complex number with a $v \in V$. Actually, the scalar multiplication is defined by the field not the vector space. So, when R_2 is defined over R, $c(x, y) = (cx, cy)$. When defined over C, $c(x, y) = (a + b\iota)(x + y\iota) = (ax - by) + (ay + bx)\iota$ and the we reinterpret it as an element of R_2, i.e., $(ax - by), (ay + bx)$. This reinterpretation is allowed because the scalar multiplication is defined as an operation λv such that the result is still in V and that this holds true $\forall \lambda \in F$, F is the field and $v \in V$.

show $(5,7),(4,3)$ is a basis of F. Basis definition says list must span F and it must be linearly independent. Let's use the independent criteria check. $a(5,7) + b(4,3) = 0$ iff $a = b = 0$. This can be solved by setting up a system of linear equations:

$$5a + 4b = 0 \text{ and } 7a + 3b = 0.$$
$$b = \frac{-5a}{4}.$$
$$7a + 3\frac{-5a}{4} = 0.$$
$$a = 0.$$
$$b = 0.$$

Therefore, they are linearly independent. We know that the length of basis for F_2 is 2. And here this independent list also has that length. So, we don't need to check whether it spans F_2 or not. We have another rule. Basis check needs two criteria, linear independence and spanning. If we know the list in question has the length equal to that of the $\dim V$, we only need to check one of the two criteria.

2.41 Example

Show that $1, (x - 5)^2, (x - 5)^3$ is a basis of subspace U of $P_3(R)$ defined by

$$U = p \in P_3(R) : p'(5) = 0.$$

Subspace contains elements. Here the elements are such p's whose differentiation at $5 = 0$. Clearly, the basis are independent. Their linear combination will be of the form $p(5) = a_1 + a_2(x - 5)^2 + a_3(x - 5)^3$.

outline Remember, basis means they the vectors are independent and span the subspace. We'll show that the $\dim U = 3$. Remember, we came up with a rule that if the basis of the right length, we need to prove only one of the two, independence and spanning. We'll show independence.

proof First, we'll show indepedence.

$$a_1 + a_2(x - 5)^2 + a_3(x - 5)^3 = 0.$$

Right hand side does not have any x^3 term. $c = 0$. If $c = 0$, there is a term x^2 again on left hand side but not on the right hand side. Therefore, $b = 0$. This makes $a = 0$. Hence, all the coefficients are 0.

We're done with the independence criteria. $4 = \dim V \geq \dim U \geq 3$. Clearly, U doesn't contain all the elements of P_3. Take $a_1 + a_2x_2^2 + a_3x_{3+a_4x_4^4}$ such that $p' = 7$. By the very definition of U, it cannot have such polynomials. Hence, $\dim U = 3$ because if we make it 4 then when we extend the basis by even one, it exceeds $\dim V$. So, $\dim U = 3$ and given independent list is of length 3. Hence, proved that U is a basis. $\square$

2.42 Spanning list of the right length is a basis

Suppose V is a finite-dimensional. Then every spanning list of vectors in V with length $\dim V$ is a basis for V.

proof   Let $\dim V = n$ and let $v_1 \dot{v}_n$ span V. The spanning list can be reduced to a basis for V. But, even reduction by 1 will make the $\dim V < 4$. Hence, given list must already be a basis. $\square$

2.43 $\dim(U + V) = \dim U + \dim V - \dim(U \cap V)$

> write proof without consulting book

commentary U and V are like two small barrels inside a big barrel. Dimension represents the minimum elements required from each small barrel to fill it using their span. Either the barrels are non-intersecting or intersecting. If they are non-intersecting, the above statement holds true as then the minimum vectors required to represent $U \cap V$ is trivial as the space is empty. If they are intersecting, there can be basis vectors between the two basis sets of these barrels that are linearly dependent on each other. We uncount them by reducing this sum by the dimension required to represent $U \cap V$.

direct sum

mental model   If you think of the vector space V as a barrel, then there are many small barrels inside it. These are the subspaces of V. Each of these barrels are non-overlapping, i.e., the barrels are indepedent with no common vector. This analogy works for seeing the indepedendence part of the barrels.

  Now, imagine the vector subspace spanned by $(x, 0)$. It's a horizontal line. There's another subspace spanned by $(0, y)$ which is a vertical line. These are two small barrels. The sum of elements from these two barrels can be outside the barrels. Any (x, y) formed by the sum of these two subspaces is not part of the original barrels. But, to represent the new vector from their sum, you need contribution from both the barrels.

2.C Exercises

> comp lete Dimension exercises.

Chapter 3

Linear Maps

3.1 Defintion: Linear Maps

mental model If vector spaces are barrels then linear maps are pipes that connect these barrels. A linear map or it's pipe has one specific property. Say there exists a vector $w = u + v$ in the first barrel (domain). You send w directly from the pipe or send u and v separately and then sum the results up. You'll get the same result. This shows linear maps preserve addition. Linear maps are really good at parallelisation then. More than that if say we consider $u = a_1 v_1 + a_2 v_2$ where v_1, v_2 are basis vectors, the property of a linear pipe say just send each of the basis once and find their mappings $T v_i$. Any $u \in V$, will be a sum of these basis only. Hence, if we know what the basis maps to we know all about where each vector in V will go to in the next barrel W.

3.2 Theorem: Fundametal theorem of linear maps

$$\dim V = \dim nullT + \dim rangeT$$

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mental model Why is this statement obvious? Let's talk about T first. A linear map T gets defined by where the basis vectors of V go. Once you know that, you know the mapping for each vector v in V. This is because of the definition of linear maps. It is clear that the basis vectors that get mapped to the null T, their dimension is com pletely consumed by $nullT$. The question is why would the rest of the left dimensions must map to such a range of T that the range's dimension is the left over dimensions. Another way of stating it is that for each v_i which is a basis vector and $T(v_i) \neq 0$, $T v_i$ must be independent, i.e., all $T v_i$'s must be independent. Why? Let's say that that is not the case and they are not linearly independent. Let $a T v_1 = b T v_2 a T v_1 - a T v_2 =$

$0T(av_1 - bv_2) = 0$. This means $av_1 - bv_2 \in nullT$ but that cannot be the case that we've already included what's there in the null space of T. This leads to contradition.

3.3   Isomorphic spaces

Two vector spaces are the same if there exists an isomorphic linear map (one to one and surjective) between the two vector spaces. What they mean by *same* is that the linear maps just renames the elements of the two vector spaces. Inherently they are the same. Say, for example, take a linear map $L : (i, j) ->$ $i + jx$. The first is a space R_2 and the second space is P_1 and yet they are the same structurally. But, why is even one such a linear map to conclude so much about the two vector spaces? By showing one to one and surjective we show two things, every v in V when goes to W via T it gets mapped to a unique w and no two v 's will have the same output. Hence there is no loss of information. We've only renamed elements. The second surjective shows that all of W is reachablefrom V and because we already have one-to-one mapping, we can go back from W to V easily without changing anything.

find a stronger mental model for fundamental theorem of linear maps