Todo list

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Chapter 1

Vector Spaces

Chapter 2

Finite-Dimensional Vector Spaces

2.1 Span and Linear Independence

Test case to create generic vector using *vectorv* command: $v = a_1 v_1 + a_2 v_2 + ... + a_n v_n$

2.2 Bases

A *basis* of V is a list of vectors in V that is linearly independent and spans V.
How do we check if a given list of vectors is a basis or not?
One option is to check if they are linearly independent and span the space.But, we have a more beautiful solution. A criteria that when satisfied ensures that the given list of vectors is a basis.

Criteria for basis 1. *A list* $v_1, v_2, ..., v_n$ *of vectors span* V *iff every* v *in* V *can be written uniquely in the form*
$v = a_1 v_1 + ... + a_n v_n$

This is similar to the criteria of linear independence which is encoded by the criteria that a list of vectors are linearly independent *iff* their sum equals zero only when all coefficients are zero themselves.

How will we determine if given a list of k vectors for a space with dimension m, and $k >> m$, which m vectors in k form a basis for V, if any? Is there an algorithm for it?
Clearly, there are $C(k, m)$ options available. Given a particular combination, is there an algorithm to determine if they meet the criteria equation of linear independence.

Algorithm to find basis given a list of vectors

Every subspace of$V$ is part of a direct sum equal to V 1. *Suppose V is finite dimensional and U is a subspace of V. Then* ***there exists*** *a subspace W of V such that $V = U + W$.*

Because V is finite dimensional, so is U. Let $u_1, u_2, ..., u_m$ be the basis for U. Clearly, this basis can be extended by adding $w_1, w_2, ..., w_k$ such that $u_1, u_2, ..., u_m, w_1, ..., w_k$ span V.

Let $W = span(w_1, ..., w_k)$. We need to show $V = U + W$ and $V = UW = 0$. For any $v \in V$, because $u_1, ..., u_m, w_1, ...w_k$ span V, there exists $a_1, ...a_m, b_1, ..., b_k$ such that $v = a_1u_1 + ...a_mu_m + b_1w_1 + ... + b_kw_k = u + w$. $V = U + W$.
Now, we need to show the intersection property.Let $v \in UW$. Then, there exists scalars $a_1, ...a_m$ and $b_1, ..., b_k$ such that
$v = a_1u_1 + ... + a_mu_m = b_1w_1 + ... + b_kw_k$ or $a_1u_1 + ... + a_mu_m + b_1w1 + b_kw_k = 0$. Since, all vectors are independent, $a_1 = ... = a_m = b_1 + ...b_k$. Thus, $v = 0$. Note, we had taken any arbitrary v part of the intersection. . .

Hence, proved. □

Exercises 2.B

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Intuitively, there cannot exist such a basis because if there is no polynomial p of degree 2, how can we represent $P_3(R)$. One way to solve this is take any 4 poynomials, assume they are independent, write their general, write their sum (the 4 polynomials) and we won't have a single term with degree 2. Therefore, there cannot exist such a basis. Hence proved.

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We need to show new list is also a basis. The criteria for basis is unique representation. As a helper method, we first show linear independence of the new list. That's easy. Assume non-independence and write independence criteria equation with assumption one of the coefficients is non-zero. Reshuffle all new terms such that they are written using original basis. This leads to contradiction as now a linear combination of original basis with at least one coefficient non-zero on the LHS and 0 on RHS. Since, all coefficients need to be 0, non-independence of the new list is false. Hence, independence shown.
Next, we need to show unique representation using the new new list. Assume two sets of coefficients lead to a common vector. Write the two representations with LHS = RHS. Because, we know the new list is independent, this leads to all corresponding coefficients in the two sets being equal. Hence, unique representation. Hence, proved.

As a side note, any linear combination of the new list can be rewritten as a combination of the original basis and the original basis spans V. Hence, the new list also spans V. Another way to think of it is every v can be written as a linear combination of the original basis and that representation can be written as a linear combination of the new list. That is the definition of spanning. For every v, there exists a representation using the list.

2.3 Dimension

The dimension of a vector space V is defined as the length of its basis. Every basis for a given vector space has the same length which can be stated below.

2.3.1 Basis length does not depend on basis

outline of proof There's a logical statment that will help us. When we say x is less than and equal to y and y is less than or equal to x, i.e., $x \leq y$ and $y \leq x$, we get $x = y$. Example, take two numbers $2 \leq 4$and $4 \leq 2$, the only way other criteria is fullfilled is is 4 changes to 2.

proof Let B_1, B_2 be two basis for V. B_1 is linearly independent and B_2 spans V. , $len(B_1) \leq len(B_2)$. Note, we are using the property that the length of the linearly independent list is less than or equal to the length of the spanning list. /todoProve this statement without looking at the book. Similarly, we show $len(B_1) \leq len(B_2)$. Hence, $len(B_1) = len(B_2)$.

2.3.2 Changing field changes the dimension of a vector space

Let's take R_2 as the vector space defined over R. The dimension is obviously 2. But, if the same vector space R_2 is defined over field C, the dimension is 1. First of all, $c(x, y) \neq (cx, cy)$. That's not how we can multiply a complex number with a $v \in V$. Actually, the scalar multiplication is defined by the field not the vector space. So, when R_2 is defined over R, $c(x, y) = (cx, cy)$. When defined over C, $c(x, y) = (a + b\iota)(x + y\iota) = (ax - by) + (ay + bx)\iota$ and the we reinterpret it as an element of R_2, i.e., $(ax - by), (ay + bx)$. This reinterpretation is allowed because the scalar multiplication is defined as an operation λv such that the result is still in V and that this holds true $\forall \lambda \in F$, F is the field and $v \in V$.

show $(5, 7), (4, 3)$ is a basis of F. Basis definition says list must span F and it must be linearly independent. Let's use the independent criteria check. $a(5, 7) + b(4, 3) = 0$ iff $a = b = 0$. This can be solved by setting up a system of linear equations:

$$5a + 4b = 0 and 7a + 3b = 0.$$

$$b = \frac{-5a}{4}.$$

$$7a + 3\frac{-5a}{4} = 0.$$

$$a = 0.$$

$$b = 0.$$

Therefore, they are linearly independent. We know that the length of basis for F_2 is 2. And here this independent list also has that length. So, we don't need to check whether it spans F_2 or not. We have another rule. Basis check needs two criteria, linear independence and spanning. If we know the list in question has the length equal to that of the $\dim V$, we only need to check one of the two criteria.