Todo list

Chapter 1

Introduction

1.1 What is analysis?

Real analysis is the analysis of real numbers, sequences and series of real numbers and real-valued functions (functions which have range as real numbers).

1.2 Examples

The examples in this section will show why understanding of real numbers is important. We'll see situations where if we don't really understand what these real numbers are, we won't be to make correct decisions.

1.2.1 division by zero

$ac = bc \implies a = b$. But, this does not work when $c = 0$. What is this example telling us? Whenever we are cancelling like how we did above, we are ruling out that $c = 0$ for all practical purposes.Because cancellation here actually means division by c, we must make sure $c \neq 0$.

1.2.2 divergent series

Take $S = 1 + \frac{1}{2} + \frac{1}{4} + +\frac{1}{8} + \dots$, what is S?. We can use this trick: Multiply both sides by 2. We get $2S = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \dots$.

$$2S = 2 + S \implies S = 2.$$

If we have another sum $S = 1 + 2 + 4 + 8 + 16 + \dots$, using the same trick leads to

$$2S = 2 + 4 + 8 + 16 + \cdots \implies 2S + 1 = S \implies S = -1.$$

Clearly, that should not be the case. My initial guess is that we cannot apply this trick for a divergent series but can for a convergent series. Why, I need to find out.

EOB need answer

1.2.3 Divergent sequence

Let x be a real number and let L define the limit as follows:

$$L = \lim_{n \to \infty} x^n.$$

Let $n = m + 1$.

$$\therefore L = \lim_{m+1 \to \infty} x^{m+1}.$$

$$L = x \lim_{m+1 \to \infty} x^m.$$

Is it allowed to do take x out of the limit?

$$\because m + 1 \implies \infty, m \implies \infty.$$

$$L = x \lim_{m \to \infty} x^m.$$

$$L = xL.$$

Either $x = 1$ or $L = 0$. This means $L = \lim_{n \to \infty} x^n = 0, x \neq 1$. But, this does not make sense because clearly when instantiated for $x = 2$, $L = \lim_{n \to \infty} 2^n \neq 0$ My guess is that the part where we move x out of the limit so easily should not be allowed.

1.2.4

1.2.5

1.2.6 Interchanging of integrals

We sometimes use this trick when integrating $\int \int f(x, y) dx dy = \int \int f(x, y) dy dx$. But this too can lead to issues sometimes. Look at the example in the book for specific example.

EOB need answer on interchanging integrals

$$\int_0^\infty \int_0^1 \left(e^{-xy} - xye^{-xy} dy dx \right) = \int_0^1 \int_0^\infty \left(e^{-xy} - xye^{-xy} dx dy \right).$$

We'll see in the book that the results of the two are different (LHS and RHS is not the same).

1.2.7 Interchanging limits

1.2.8

1.2.9

Chapter 2

Natural numbers

Just to reiterate the different kinds of numbers we'll deal with in this book, here are they in increasing order of sophistication. Natural numbers, $\mathbb{N}$, integers, $\mathbb{Z}$, rational numbers, $\mathbb{Q}$, real numbers, $\mathbb{R}$ and complex numbers, $\mathbb{C}$.

Axiom and definition counts in the chapter

Commentary: Since axioms and definitions are the most important part of a book, we'll keep a track of the number of axioms and definitions in each chapter. 5 axioms for Peano's axioms to define natural numbers, 1 definition of addition, , 1 definition of positive numbers, 1 defintion of ordering (greater than).

2.1 The Peano axioms

The Peano axioms is one of the ways of defining natural numbers. There are other approaches such as using sets. We define natural numbers as that set of elements for which the following axioms hold:

2.1.1 0 is a natural number.

2.1.2 If n is a natural number, then $n{+}{+}$ is also a natural number

Note, we assume that there exists an operation with symbol ++ that means increment. This axiom allows us to move forward in our count.

2.1.3 0 is not the successor for any natural number, i.e., $n++ \neq 0 \forall n$

This ensures we don't circle back to 0. Imagine if we defined $(0++)++$ as 0. Axioms 2.1.2 and 2.1.1 would still hold but we know we don't want that in our natural number system. This axiom ensures that.

2.1.4 $n \neq m \implies n++ \neq m++$

This axiom ensures that we don't reach an upper bound. If we define $2++$ and $3++$ both as 3, none of the above axioms are violated and yet we know we don't want an upper bound to our natural number system. Take the next number, $(3++)++ = 3++ = 3$, we have reached an upper bound for us. Another way to state this axiom is in its contrapositive form. $n++ = m++ \implies n = m$.

2.1.5 Principle of Mathematical Induction

It might be surprising right now why we are including this as an axiom because it is part of the larger logic scheme. Why do we need to explicitly say this. My guess is that it is needed because there are other systems where this principle does not hold. The principle is if some property P holds for 0, that is, if $(P(0))$ is true and $P(n) \implies P(n+1)$ is true, then property P is true for all elements in the number system.

find why math induction principle is needed as a Peano axiom to define natural numbers

This axiom allows us to define properties of objects for all elements.

Commentary:

Why do we fix other variables and induct on only one?

Note our definition of natural numbers is axiomatic and not *constructive.* What I mean by that is we have not defined *what* a natural number is but rather properties that a natural number holds. Any number system that follows the above mentioned axioms is a natural number system.

2.1.6 Recursive definition of sequence: Proposition

Suppose for each natural number n, we have some function $f_n : \mathbb{N} \mapsto \mathbb{N}$. Let c be a natural number. Then, we can assign a unique natural number a_n to each natural number n such that $a_0 = c$ and $a_{\{n++\}=f_n(a_n)}$ for each natural number n.

commentary The goal is to show that each value of the sequence a_i is defined only once and because we are looking at a recursive defintion we do not revisit a_i again when going forward.

proof We'll prove this by induction. Take the base case, $a_0 = c$. Because of axiom 2.1.3, going forward, any a_{m++} will not redefine a_0. For example, if axiom 2.1.3 was actually violated and say $3{+}{+} = 0$. Then, $a_{3++} = f_3(a_3) = a_0$. Note, we redefined a_0 without this axiom.

Let's assume that a_n by the given recursive formula uniquely defined a_n. We need to this also holds for $a_{(n+1)}$. Any natural number after $n++$, say we call it $m++$ will redefine $n++$ because of axiom 2.1.4. Note, this axiom was introduced to ensure the successors are always bigger and that's what this axiom ensures. Because we cannot circle back, the previously defined a_i remains as is.

2.2 Addition

2.2.1 Definition: Addition of natural numbers

Let m be a natural number, we define addition of zero to m as $0 + m := m$. We give a recursive defintion for adding. Let's say we know how to perform $n + m$. Therefore, adding $n++$ to m is defined as $(n++) + m := (n + m)++$.

For example, say, we need to add $2 + 5$. $(1++) + 5 = (1 + 5)++ = ((0++) + 5)++ = ((0 + 5)++)++ = (5++)++ = 6++ = 7$

Commentary: This definition is enough to deduce everything about addition of natural numbers!

2.2.2 Lemma

For any natural number n, $n + 0 = n$.

Commentary: Note, in our recursive defintion, we have defined only $0{+}n = n$ and we don't know yet that for any $a, b \in \mathbb{N}$, $a + b = b + a$.

Proof: Take base case, $n = 0$. $0{+}0 = 0$. This is true by our recursive defition's base case. Assume, $n + 0 = n$ for n. We need to show $(n++) + 0 = n++$ and we are done with our proof by induction.

$(n++) + 0 = (n + 0)++$ by our defintion of addition. $(n + 0)++ = n++$ because of our induction assumption for n. Hence, proved.

2.2.3 Lemma

For any natural numbers n, m $n + (m++) = (n + m)++$.

Commentary: Again, we cannot deduce this from our recursive defintion $(n++) + m = (n + m)++$ because we do not know $a + b = b + a$.

Proof: We'll try this by induction on n (and keep m fixed). Let's take the base case at $n = 0$. $0 + (m++) = m++ = (0 + m)++$. The stamenent holds for the base case.

Assume, $n + (m++) = (n + m)++$. We now need to show that $(n++) + (m++) = ((n++) + m)++$

$(n++) + (m++) = (n + (m++))++$ by the recursive defintion of addition. $(n + (m++))++ = ((n + m)++)++$ by the the induction step taken for n. $((n++) + m)++$ by the defintion of recursive defntion of addition. Hence, proved.

Corollary

$n++ = n + 1$

Commentary: Why would this hold because of Lemma 2.2.2 and Lemma 2.2.3.

Proof: $n+1 = n+(0++) = (n+0)++$ by Lemma 2.2.3. $(n+0)++ = n++$ because of Lemma 2.2.2. Hence, proved.

2.2.4 Proposition

Addition is commutative. For any natural numbers n, m, $n + m = m + n$.

Proof: We can prove this by induction. (note, anytime there's a proof of applying it for all numbers, induction really helps). We'll fix m. Take $n = 0$. $0 + m = m$ by defition of base case for addition. $m + 0 = m$ by Lemma 2.2.2. Base case is done.

Assume $n + m = m + n$. This is our induction step. We now need to show it holds for $n{+}{+}$, i.e., $(n++)+m = m+(n++)$. LHS, $(n++)+m = (n+m)++$ by the recursive definition of addition. $(n + m)++ = (m + n)++$ by our induction step. RHS, $m + (n++) = (m + n)++$ by Lemma 2.2.3.

LHS = RHS. Hence proved. $\square$

2.2.5 Proposition

Addition is associative. For any natural numbers a, b, c, we have $(a + b) + c = a + (b + c)$.

Write proof for Addition is associative

2.2.6 Proposition

Cancellation law. Let a, b, c be natural numbers such that $a + b = a + c$. Then we have $b = c$.

Commentary: Note, we do not have the concept of negative numbers or subtraction yet.

Proof: We'll prove this by induction. Fix b, c. Take base case, $a = 0$. $0 + b = 0 + c \implies b = c$ because of base case of definition of addition.

Assume $a + b = a + c \implies b = c$. We need to show $(a++) + b = (a++) + c \implies b = c$.

$$(a++) + b = (a++) + c.$$

$$(a + b) + + = (a + c) + +. \qquad \text{(by defintion of addition)}$$

$$b + + = c + +. \qquad \text{(by assumption of the induction step)}$$

$$b = c.$$

Hence, proved by induction. □

2.2.7 Definition

Positive natural numbers. A natural number n is said to be positive iff it is not equal to 0.

2.2.8 Proposition

If a is positive and b is a natural number, then $a + b$ is positive.

Proof: We'll use induction on b. Let $b = 0$. $a + b = a + 0 = a$. Base case holds. Assume $a + b$ is positive. Then we need to show $a + (b + +)$ is positive. $a + (b + +) = (a + b) + +$ by Lemma 2.2.3. $(a + b) + +$ cannot be zero because of Axiom 2.3 ($n + + \neq 0$). Hence, $(a + b) + +$ is positive. This closes the induction. □

2.2.9 Corollary

If a and b are natural numbers such that $a + b = 0$ then $a = b = 0$.

Proof: We'll prove this by contradiction. Assume $a + b = 0$ but $a \neq 0$ or $b \neq 0$. In either case, $a + b = 0$ where one of them is not zero. But, by Proposition 2.2.8 that cannot be the case as $a + b$ must be positive (not zero). Hence, contradiction and $a = b = 0$. □

2.2.10 Lemma

Let a be a positive number. Then there exists exactly one natural number b such that $b + + = a$.

Commentary: Note, this means there is exactly one element behind a. There seems to be an order between the natural numbers.

2.2.11 Definition

Ordering of natural numbers. Let n and m be natural numbers. We say n is greater than or equal to m iff we have $n = m + a$ for some natural number a. We write greater than or equal to as $n \geq m$ or $m \leq n$. We say that n is strictly greater than m iff $n \geq m$ and $n \neq m$.

Commentary: Note, how we have used the concept of some a existing such that adding it to m gives us n. This also helps us show why there is no largest natural number because for every n, $n++ > n$. Hence, for the number $n++$, there's another larger number $(n++)++$.

2.2.12 Proposition

Basic properties of order for natural numbers. Let $a, b, c \in \mathbb{N}$. Then

1. (Order is reflexive) $a \geq a$
2. (Order is transitive) If $a \geq b$ and $b \geq c$, then $a \geq c$
3. (Order is anti-symmetric) If $a \geq b$ and $b \geq$, then $a = b$.
4. (Addition preserves order) $a \geq b$ iff $a + c \geq b + c$
5. $a < b$ iff $a++ \leq b$.
6. $a < b$ iff $b = a + d$ for some positive number d.

Proof:

1. 1 $a \geq a$ iff there exists n such that $a = a + n$. Let $n = 0$. Therefore, $a = a + 0 = a$. Hence, proved.
2. It is given that $a \geq b$ and $b \geq c$. We need to show this implies $a \geq c$. There must bexist m such that $a = b+m$ and n such that $b = c+n$. Substituting value of b, $a = c+n+m \implies a = k+c$ where k is some natural number. Hence, by definition of ordering of natural numbers, $a \geq c$. $\square$
3. $a = b + m$ for some m and $b = a + k$ for some k. Substitute value of b, $a = a + k + m \implies a + 0 = a + (k + m) \implies 0 = k + m$ by cancellation law (Proposition 2.2.6). The only way this holds is iff $k = m = 0$. This is because there are four situations $k, m \neq 0 \vee k = m = 0, \vee k = 0 \wedge m \neq 0 \vee k \neq 0 \wedge m = 0$. In all situations except $k = m = 0$, we'll have a situation where Axiom 2.1.3 is violated because we'll have an equation of form $n++ = 0$. Hence, $a = b + m = 0 = b \implies a = b$. $\square$
4. Assume $a \geq b$. This means $a = b + m$ for some m. Adding c on both sides, (for which we'll need to prove this logic TODO), $a + c = b + m + c \implies (a + c) = m + (b + c)$. (We have only rearranged the terms

using Proposition 2.2.4). Therefore, by definition of greater than, we have $a + c \geq b + c$. Now assume $a + c \geq b + c$. We need to show $a \geq b$. $a+c = b+c+d$ for some d. By cancellation law, $a = b+d$. Hence, $a \geq b$. We are done with our proof now.

5. Assume $a < b$. We need to show $a{+}{+} \leq b$. Or in other words $a+c = b$ for some c and $a \neq b$. We'll prove this by contradiction. Assume $\neg\,(a + c \neq b) \wedge (a \neq b) = a + c \neq b \vee a = b$. We'll show that in both cases of or, we have a contradiction. $a + + \leq b \implies (a++) + m = b = (a + m) + + = b$. Since $a = b$ (by our assumption, this is the second case of or), $(b + m) + + = b$. In both cases, when $m = 0 \vee m \neq 0$, we have a contradiction as $b + + \neq b$ as this would imply $0 + + = 0$. (This is a contradiction of Axiom 2.1.3). Now, we'll show the other case of or also leads to contradiction, i.e., $a + c \neq b$ will lead to contradiction. Since $a + + \leq b \implies (a++) + c = b \implies (a + c) + + = b \implies (c + +) + a = b$. But there does not exist a number b such that $a + c = b$ (Note, here there is no difference between c and $c + +$ for us because all we care about is some number in N. Hence, contradiction. Hence, our initial assumption $\neg\,(a + c \neq b) \wedge (a \neq b)$ must be wrong. Therefore, $a + c = b$
a $\neq b$. Hence, we are done with left-to-right side for our proof. We now need to show right to left follows. A

2.2.13 Proposition:

Trichotomy of order of natural numbers. Let a and b be natural numbers. Then exactly one of the following statements is true: $a < b, a = b, a > b$.

Commentary: This is a great example to show how we can pin down what we want to say in exact terms using mathematics. We need to show that exactly one of three statements is true. We first show that no two statements can be true together. We also show that the three together also does not hold true. Then we show at least one of them is true. In simple terms, we show one statement will definitely be true and two statements cannot be true together. Hence, only one statement will be true.

Proof: We show first no more than one statement is true. If $a < b$, by definition of greater than $a \neq b$. If $a > b$, by definition of smaller than $a \neq b$. If $a < b$ and $a > b$, then by Proposition 2.2.12(3), $a = b$ which is a contradiction. Hence, no more than one statement can be true.

Now, we show at least one of the statements is true. We'll use induction. Fix b and induct on a. Base case, $a = 0$. $0 \leq b$ for all b. This is because $b = a \vee b \neq a$. If $b = a$, $0 \leq b$ holds. If $b \neq a$, and we know $a = 0$, therefore, b must be some positive number and hence $b > 0$. Therefore, we show that in both cases, $0 \leq b$. Now suppose we have proven the proposition for a and now we want to prove the proposition for $a + +$. There are three cases: $a < b, a = b, a > b$. If $a > b$, by proposition 2.2.12(e), we get $a + + \leq b$. This case holds. (Note, we show that $a + + = b \vee a + + < b$. I know we want to show only one statement holds overall but this part of the proof is about at least one of the statemnets is true, if both are true we don't care). If $a = b$. $a = b \implies a++ = b+1$. By defintion of greater than, $a++ \geq b$. Therefore, $a++ > b$ or $a++ = b$. If $a++ = b \implies a++ = a$ which cannot be the case. Therefore, $a++ > b$. We are done with this case as well. If $a < b$, then by Proposition 2.2.12(5), $a++ \leq b$ and in both cases we are done with our induction. Hence proved.

2.2.14 Proposition: Strong principle of induction

Suppose there exists a natural number m_0. Suppose we also have a property $P(m)$ for some natural number m such that the following as true: If $P(m^{'})$ is true for all $m_0 \leq m^{'} < m$, then $P(m)$ is true. Then we conclude P is true for all $m \geq m_0$.

Commentary : Is strong principle of induction stronger than our Principle of Induction Axiom (Proposition 2.1.5)itself? If strong principle of induction is proved using Principle of Induction Axiom then it's not really a stronger claim than Principle of Induction Axiom. Then, why call it strong?

The proposition hinges on the if then implication. It says that if proposition holds till m, it holds for $m++$. Note, this part is given, i.e., the implication itself is given as true. Then, we can conclude that P is true for all m.

There are no numbers before m_0. And we are given that if $P(m')$ is true for all numbers before m, then $P(m)$ is true. So, $P(m_0)$ is true.

Let's talk about $P(m_1)$ then. It is given to us that if $P(m')$ is true for all numbers before m_1, it implies $P(m_1$ is true. We know $P(m_0)$ is true. Hence, $P(m_1)$ is true.

This process of induction does feel like a trick but it hinges on a given if-then statement where it states that P is true for all numbers before m, P is true for m. Again, this is a given to us.

Exercises

2.2.1

Q: Prove Proposition 2.2.5, i.e., $(a + b) + c = a + (b + c)$.

Proof: We'll prove this by induction. Fix b, c. Take $a = 0$ for base case. $(0 + b) + c = b + c$. This was LHS. RHS side, $0 + (b + c) = b + c$. Base case holds. We'll take the induction step. Assume the given property holds for a. We need to show $((a++) + b) + c = (a++) + (b + c)$. Take LHS, $((a++) + b) + c = ((a + b)++) + c = ((a + b) + c)++$. We only used the definition of addition for this manipulation. Take RHS, we'll get $(a + (b + c))$++ after similar manipulation. By the inductive step we know $a+(b+c) = (a+b)+c = k$ for some k. LHS and RHS are equal. $k++ = k++$. Hence, induction closes here.

2.2.2

Q: Prove that if a is a positive number then there exists exactly one number b such that $b++ = a$.

Commentary: I have a feeling for contradiction here. Clearly, I have to show it in two parts, first that there exists such an number b such that $b = a++$ and that b is unique. For the first part, we don't need a contradiction. Hint says *use induction*, so we'll try induction.

Proof: Base case, take $a = 1$. Clearly, there exists $b = 0$ since $0++ = 1$. There isn't any other number so we are good with the base case. Induction assumption step. Say, there exists exactly one number b for a. We need to show there exists a number c for $a++$. $a++ = (b++)++$. Here $b++ = c$. Since, we know there exists only one b, $b++$ is also unique.

not satisfied with proof, try again

2.2.4

why(1): b is a natural number. It either is 0 or a positive number.

why(2): We need to show $a > b \implies a++ > b$. $a > b$, by defintion means there exists n such that $a = b + n$. Adding 1 on both sides, $a++ = b + (n++)$. Clearly, by definition of greater than we can say $a++ > b$.

why(3): We need to show $a = b \implies a++ > b$. $a++ = b++ \implies a++ = b + 1$. Clearly, by defintion of greater than $a++ > b$. $\square$

2.2.5

Q: Prove Proposition 2.2.14 Strong principle of induction

Proof: $(P(m_0)$ is true. $(P(m)$ is true for all $m \geq m_i \geq m_0$ implies $P(m++)$ is true$)) \implies P(m)$ is true $\forall m \in \mathbb{N}$.

Commetary: We'll use induction for this proof. The base case will be that $P(0)$ is true (we need to show that) and also as part of the base case, we need to show that $P(0)$ is true $\implies$, $P(0++)$ is true and that this implies $P(m)$ is true for all $m \in \mathbb{N}$. Then, as part of the induction step, we need to show if we assume $A, B \implies C$ to be true for m, then $A, B \implies C$ is also true. Note, here A is $P(m_0)$ is true. B is $P(m_i) \implies P(m++) \forall m \geq m_i \geq m_0$ and C is $P(m) \forall m \in \mathbb{N}$ is true. Note, as part of the induction step as for the base case, we will assume that the first part of B is true and show that the second part, i.e., the implication follows from it.

Take the base case $m_0 = 0$. It is given to use that A, B is true and we need to show C follows from them. B is true, therefore the implication of it is true, i.e., $P(0++)$ is true. From this, we can say that $P(1)$ is true. So, again we have $P(0)$ is true and by B we will get $P(1++)$ is true. This way, we can show $P(m)$ is true $\forall m \in \mathbb{N}$. We are done with the base case.

Now, assume $A, B \implies C$ is true for m. We need to show $A, B \implies C$ is also true. What do we get when we assume $A, B \implies C$ is true? Well, the exact statement that we need to prove. No point in repeating it here. What we need to show is that given $P(m_0)$ is true, and $P(m++) \implies P((m++)++)$ is true, tells us that $P(m++)$ is true ∀m++∈ ℕ. Any $m \in \mathbb{N}$ is written as $(((0++)++)++)$... where ++ operation is repeated m number of times. We are given $P(1) \implies P(2)$ and so on and so forth. So, $P(0) \implies P(1)$ and by using the same logic we'll show that $P(1) \implies P(2)$. In this way, since every positive number is some $m++$, we have shown that C is true $\forall m++ \in \mathbb{N}$. Hence, induction closed.

very bad proof, not satisfied, try again.

2.2.6

Given: $P(m++) \implies P(m)$. $P(n)$ is true.

Show: $P(m)$ is true $\forall m \leq n$.

Proof: Assume base case is about $n = 0$. We are given $P(0++) \implies P(0)$ is true. $P(m)$ is true trivially as there is $m\neg \leq 0$ Base case holds. Assume, $A \implies B, C$ is true and together they imply C is true. We need to show the same for $m+++$. Here $n = m++$. And as part of given we have $P((n++)++) \implies P(n++)$, we need to show that $P(m)$ is true $m \leq n++$. Now, because of the given, $P(m)$ is true $\forall m \leq n$. We just need to show it also holds for $n = m++$, But that is also given as part of the assumption that $P(n++)$ is true. Hence, we are done with the induction. $\Box$.

bad proof, not convinced

Chapter 3

Set Theory

Definition and Axiom count

There are 11 axioms used to construct set theory. These 11 axioms are called *Zernelo-Frankel* axioms. There is a 12th axiom called the *Axiom of Choice* that we don't need now but will need at some point of time. There are 13 definitions in this chapter. Note, these also include the defintion of functions, types of functions as well because the axioms make use of them in their statements. While it is true that we can have infinite definitions and hence makes it redundant to keep a track of them but without defintions,

Why is Axiom of Choice needed/not needed now?

Why are definitions secondary to axioms?

Commentary: What does obeying the axiom of substitution mean? Take example, for two natural numbers, a, b we have $a = b$. Then, any other equation where a is referred, I can substitute b for it. Similarly, even for sets, we have the concept of equality. For any two sets, A, B we have defined equality, $A = B$. Therefore, we can write $x \in A$ as $x \in B$.

Find if axiom of substitution is defined for operation or elements

3.1 Axiom

If A is a set, then A is also an object. In other words, given two sets A, B it is meaningful to ask if $A \in B$ or $B \in A$.

3.2 Axiom

Empty set

3.3 Axiom

Singelton and pair sets

Commentary: This axiom is about for a given object a, there exists a set S such that $S = a$, i.e., a is the only element of S. Another way of stating the same thing is by saying that if $y \in S \implies y = a$. Similary, we have a pair set property for a set. Given two elements a, b, there exists a set S whose only elements are a, b.

There is only one singleton set for each object a by defintion 3.1.4 (equality definition). Say, there were two singleton sets for a. Every element y of S_1 would belong to S_2 because $y = a$ and by definition of S_2 being a singleton set, each $y \in S_2 \iff y = a$. Similarly, every element of S_2 belongs to S_1. Hence, $S_1 = S_2$.

3.4 Axiom: Union of sets

Given two sets A, B, there exists a set called /*union* and written as $A \cup B$ which elements belong to A or B or both.

In set theory, why is union an axiom but intersection a definition?

3.1.12 Remark

If A, B, A_1 are three sets and $A = A_1$, then $A \cup B = A_1 \cup B$. By definition of equality, any $x \in A \implies x \in A_1$ and vice versa. Any element $x \in A \cup B$ implies $x \in A \vee x \in B$ by defintion of pairwise union. Since, $x \in A \implies x \in A_1$, we will replace the belongs to part. We can do this because by the defintion of equality, $\in$ allows axiom of substitution. Therefore, $x \in A \cup B \implies x \in A_1 \cup B$ and vice versa. Hence, $A \cup B = A_1 \cup B$.

Commentary: This axiom allows us to create larger sets, i.e., sets with more than two elements.

3.1.13 Lemma

$(A \cup B) \cup C = A \cup (B \cup C)$ (associativity)

Commentary: To prove this we have a statement which needs us to show all elements on the side of the set X also belong to the set Y on the right hand side and vice versa.

Proof: Let $x \in (A \cup B) \cup C$. We need to show $x \in A \cup (B \cup C)$. First, say $x \in A \cup B$. $x \in A \cup (B \cup C) = x \in A \vee x \in B \cup C$. If $x \in A \implies x \in A \vee x \in B \cup C$. If $x \in B$, we need to show $x \in A \vee x \in B \cup C$ holds. If $x \in B \implies x \in B \cup C$. (why?) because $x \in B \cup C = x \in B \vee x \in C$ and $x \in B$ holds.

Now, we need to show if $x \in C \implies x \in A \cup (B \cup C)$. Again $x \in C \implies x \in B \cup C$. We are done with this second part as well.

Similarly, we can show from LHS to RHS as well. Hence proved. □

3.1.18 Proposition

Sets are partially ordered by set inclusion. If $A \subseteq B$, and $B \subseteq C$, then $A \subseteq C$.

3.5 Axiom: Axiom of specification

Commentary : This axiom allows us to create subsets from larger sets.

Given a set S and a property P, there exists a subset of S, such that elements of $x \in S$, for which $P(x)$ is true, belong to this subset.

$y \in \{x \in A : P(x)\} \iff y \in A \wedge P(y)$. This is how we mathematically we capture the above statement.

3.1.23 Definition (Intersection)

The *intersection* of two sets written as $S_1 \cap S_2$ is defined to be the set $S_1 \cap S_2 := \{x \in S_1 : x \in S_2\}$. Note, how beautifully we used the axiom of specification which is a more general form of axiom where we talk about a generic property of the element and separate the set according to that property. Belongingness is also a property. This axiom, thus, gives us the ability to define the concept of intersection.

3.1.28 Sets form a boolean algebra

Commentary: Is it the case that there are other objects that form a boolean algebra. What properties are quired for an object to satisfy boolean algebra? What lies at the heart of boolean algebra?

Sets form a boolean algebra means what?

Why does the complementation relation $A \implies X/A$ create this duality in Morgan's Laws, i.e., unions convert into intersectin and vice versa?

3.6 Axiom of replacement

Commentary: This axiom helps us in converting one element into another of a set. Say, we have a set $S = 1, 2, 3$ and we want to convert it to $S = 4, 5, 6$, none of the existing axioms will allow us to do that. Why are existing axioms not enough?

Suppose that for any object $x \in S$ and y, there exists a property $P(x, y)$ such that for every x, there exists at most one y such that $P(x, y)$ is true, then, there exists a set $\{y : P(x, y) for some x \in S\}$ such that for any object z in this set we have $z \in \{y : P(x, y) for some x \in S\} \iff P(x, z) is true for some x \in S$.

How will you combine axiom of separation with axiom of replacement? Let's define a set using axiom of separation $\{x : P(x)\}$. Now, how to apply axiom of replacement on this. Assume $\{f(x) : x \in A; P(x)\}$.

3.7 Axiom: Infinity

Commentary: This axiom says that there exists a set N whose elements satisfy the property of Peano's axioms. Hence, this axiom helps us show that N is a set.

Exercises

3.1.1

Reflexive, symmetric and transitive is followed by the defintion of equality. Reflexive here means $A = A$, symmetric means $A_1 = A_2 \implies A_2 = A_1$ and transitive means $A_1 = A_2 \wedge A_2 = A_3 \implies A_1 = A_3$. Reflexive is simple because each $x \in A \implies x \in A$. Symmetric because the if part gives us $A_1 = A_2$. And by the defintion of equality, $A_2 = A_1$ because any element $x \in A_2 \implies xinA_1$ by defintion of equality of the given statement. Transitivity. Let $A_1 = A_2$ and $A_2 = A_3$. We need to show $A_1 = A_3$. Let $x \in A_1$. Since $A_1 = A_2$, $x \in A_2$ and similarly $x \in A_3$. Going in reverse direction also gives us the same situation, i.e. $x \in A_3 \implies x \in A_1$. Hence, by defintion of equality $A_1 = A_3$. $\square$

3.1.2

eq, 1,2,3 prove sets $\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}$ are distinct, i.e., none of them are equal. This is easy so leaving it. Just to show that I am capable, we'll prove $\phi, \{\phi\}$ are not equal. By definition of ϕ, there does not exists an $x \in \phi$ but $\phi \in \{\phi\}$. Hence, not equal.

3.1.3

3.1.11

Show that axiom of replacement implies axiom of specification. Given a set S and a property $P(x, y)$ pertaining to x, y such that for every x, there exists at least one y such that $P(x, y)$ is true, then there exists a set $\{y : P(x, y) for some x \in S\}$. Axiom of specification is about separating the set into two depending on a property P and which $x \in S$ holds true for P and which ones don't. Let us define a set with those y such that $P(x, y)$ as that $P(x)$ is true. Therefore $y \in S$ only. Hence, we show that axiom of specification is implied from axiom of replacement. $\square$

Definition 3.6.1 Equal cardinality

Two sets X and Y are said to have *equal cardinality* iff there exists a bijection between X and Y.

Note, we are moving to the next chapter because there is not much to gain from this chapter apart from exercising how to rigorously write mathematics. I know this is the motive of this book but we need to be pragmatic and reach the heart of this book that starts from real numbers.

Chapter 4

Integers and rationals

Definition and axiom count

4.1 Integers

We need to define subtraction and we need to define integers. subtraction is an operation and integers is an entity. From the natural numbers construction, we saw, we can define entities by the properties or axioms they hold. Therefore, we will define integers as simply a notation $a - -b$. Later on, we'll see that we can just replace $--$ with $-$.

4.1.1 Definition: Integers

An integer is an expression of the form $a - -b$, where a, b are natural numbers. Two integers $a - -b$ and $c - -d$ are only equal iff $a + d = b + c$. Note, we are expressing the notion of subtraction using the notion of addition only because the sign $--$ is just a symbol at this point of time without any meaning of an operator as such. We took four natural numbers and defined a relation between them.

4.1.2 Addition and multiplication of integers

The sum of two integers $a - -b$ and $c - -d$ is defined as $(a - -b) + (c - -d) := (a + b) - -(c + d)$

The product of two integers $a - -b$ and $c - -d$ is defined as $(a - -b) * (c - -d) := ac - -bd$. Note, how this is similar to as if the integer is composed of two parts and when two integers interact via an operation, only the parts that are of the same type interact.

4.1.3 Addition and Multiplication are well defined for integers

We need to show if a, b, a_1, b_1, c, d are natural numbers and $a - -b = a_1 - -b_1$, then, $(a - -b) + (c - -d) = (a_1 - -b_1) + (c - -d)$ and $(a - -b) * (c - -d) = (a_1 - b_1) * (c - -d)$ and we also need to show the symmetric-ness of these two equations.

integer as a pair of natural numbers integer has a negation subtraction is addition with the negation of integer a - b = a + (-b) = (a–0) + (0–b) = a – b 7 - 3 = a + (-3) = (a–0) + (0–3) = 7 – 3 but here still we haven't reached the stage of taking away or basically we haven't been able to represent 7 – 3 rather than what it is already written as.

Remark on how subtraction is conceived from addition

Note, there is no subtraction in this world defined by Tao. Here's how we get the same quality though. First, we have the concept of negation of an integer. Note, we have defined an integer as a pair of natural numbers written as $a - -b$. We define the negation of an integer as $-(a - -b) = (b - -a)$. We define subtraction as addition of the negation of the integer, i.e., $a - b = a + (-b) = (a - -0) + (0 - -b)$. But, this doesn't tell us how to actually do the reduction of value that we think of. Say, $7 - 3 = (7 - -0) + (0 - -3)$. Sure, we have expanded this by how do actually get 4? Note, two integers $a - -b$ and $c - -d$ are equal iff $a + d = b + c$. Therefore, $7 - -3 = 4 - -0$ because $7 + 0 = 3 + 4$.

4.2 Rationals

Rationals are not well ordered

Well ordered means that you can pinpoint the smallest element in a set. You cannot do that with a set of rational numbers. Consider $\{n : 0 < n < 1, n \in \mathbb{Q}\}$. For any element you choose, you can find an even smaller element. There is an infinite descent when it comes to rational numbers.

Rationals don't contain all numbers

We complete this chapter by showing that rationals cannot represent all numbers. There does not exist a number x such that $x^2 = 2$. If we are able to do this proof and that why such numbers mentioned above don't exist, we have gotten the most out of this chapter. Prove principle of infinite descent

4.4.4 Proposition

There does not exist any rational number x for which $x^2 = 2$

Proof: Clearly, $x \neq 0$. If it were so, we'll have $0^2 = 2 \implies 0 * 0 = 0 \neq 2$. Assume x is positive. We can assume so because if x is actually negative, we can replace x with $-x$ as $x^2 = (-x)^2$. Therefore, there exists $x = \frac{p}{q}$ such that for some p, x, $\left(\frac{p}{q}\right)^2 = 2$ which we can rearrage as $p^2 = 2q^2$.

We define a natural number p to be *even* if there exists a natural number k such that $p = 2k$ and odd if there exists k such that $p = 2k + 1$. Every natural number is either even or odd but not both. The structure is $A \vee B \wedge \neg(A \wedge B)$. We can show it in two steps. Let A be the statement that p is even and B be p is odd. We'll first show A and B cannot be true together. And then we'll show one of them is atleast true. Let p be both even and odd. Then, there exists k such that p is even and there exists l such that p is odd. Therefore, $2k = 2l + 1$ by defintion of even and odd. Either $k = l \vee k > l \vee k < l$. If $k = l$, $0 = 1$. Contradiction. If $k > l$, there exists n such that $k = l + n$. $2l + 2n = 2l + 1 \implies n + n = 1$. Now, $n \neq 0$, so it is a positive number. There cannot be a positive number n such that $n + n = 1$ as for any n, $n + n > 1$. Therefore, contradiction and $k > l$ cannot be the case. Similarly, we can show it cannot be the case that $k < l$. Therefore, we've shown p cannot be both even and odd. Now, we need to show that atleast one of the statements is true, i.e., for any n natural number, there exists k such that either $n = 2k$ or $n = 2k + 1$.

We'll show that atleast one of the statements holds by induction. Base case $n = 0$. At $k = 0$. $2 * 0 = 0$. Base case holds. Assume there exists k such that n is either odd or even. Now, we need to show that $n++$ is also either odd or even. If n is odd, $n = 2k + 1$. Then $n + 1 = 2k + 2 \implies n++ = 2(k + 1)$. Therefore, $n++$ is even. If n is even, we show $n++$ is odd. Therefore, $n++$ is either odd or even. We close induction here.

Therefore, we showed that every natural number p is either odd or even.

Now, we show that if p is odd then p^2 is also odd and if p is even, p^2 is also even. Assume p is odd. Therefore, there exists k such that $p = 2k + 1$. Now, $p^2 = p*p = (2k+1)(2k+1) = (2k+1)2k + (2k+1)1 = 4k + 2k + 2k + 1 = 8k + 1$. Therefore, p^2 is also odd as at $k_1 = 8k$, it satisfies the defintion of odd number. Similarly, we can show that if p is even, p^2 is even.

Now, we go back to our proof we started with at the beginning. We have two important lemmas with us. Every p, nautural number, is either odd or even but not both and that if p is odd then p^2 is also odd. If p is odd p^2 is also odd. But, we are given that $p^2 = 2q^2$. Therefore, p cannot be mk$p$ is even. Therefore, there exists k such that $p = 2k$. Inserting $p = 2k$ into $p^2 = 2q^2$, we get $q^2 = 2k^2$. Since the solution pair p, q and q, k share the same equation and it's just the variable name change, they both will satisfy our equation $x^2 = 2$. Thus, if we rewrite p, q as $p_1 = q, q_1 = k$, we moved from one solution p, q to another p_1, q_1 of our equation. The catch here is that as we move from one solution to another, the value of p keeps on decreasing and p is positive integer. But, we cannot have p decreasing forever. This contradicts the principle of infinite descent. Hence, our initial assumption, that there exists such an x such that $x^2 = 2$ is false. $\square$. Note, this is a momentous moment because we have shown rationals cannot represent all numbers.

Exercises

4.1.1

Show that the definition of integers is both reflexive and symmetric. Reflexivity is a property of a relation R, i.e., if xRx, then the property is reflexive. Here, the relation R is $a--b = c--b \iff a+d = b+c$. Then, does this relation hold when both $a-b$ and $c-d$ is the same. Let this new integer be $l--m$ that represents both of them. Then, by the definition of R, $l--m = l--m \iff l+m = m+l$. Clearly, the definition holds as $l+m = m+l$ by the symmetric property of addition.

Symmetric means that $jRk \iff kRj$. Here, R holds iff $a+d = c+b$. Now, jRk here is $a+d = c+b$ and clearly this can be written as $d+a = b+c$ which is the defintion of $b--a$. Hence, $jRk \implies kRj$ and similarly we can show $kRj \implies jRk$. $\square$.

Chapter 5

Real numbers

5.1 Cauchy sequence

5.1.8 Definition: Cauchy sequence

A sequence of rational numbers is said to a cauchy sequence iff for every $\epsilon > 0$, the sequence is eventually ϵ-steady. In other words, for every rational number $\epsilon > 0$, there exists a $N \geq 0$ such that $d(a_i, a_j) \leq \epsilon$ for every $i, j \geq N$.

5.1.11 Proposition

The sequence $a_n := 1/n$ is a cauchy sequence.

Commentary : We need to show that for any given $\epsilon > 0$, there exists a natural number N such that $|\frac{1}{a_i} - \frac{1}{a_j}| \leq \epsilon \forall i, j \geq$N.

Proof: Assume we have an arbitrary $\epsilon > 0$. We need to find an appropriate N such that $\forall i, j \geq N$, $|\frac{1}{a_i} - \frac{1}{a_j}| \leq \epsilon$. Note, $i \geq N \therefore \frac{1}{i} > 0$. (why (1)?). Similarly, $j \geq N \therefore \frac{1}{j} \leq \frac{1}{N}$. (why 2?). $\therefore |\frac{1}{i} - \frac{1}{j}| \leq \frac{1}{N}$. Now, it should be enough to show that $\frac{1}{N} \leq \epsilon$ or $N > \frac{1}{\epsilon}$. We know that for every ration, there exists a natural number bigger than it by proposition 4.4.1. $\square$

Answer to why (1): $i \geq N \implies \frac{1}{i} \geq 0$ Assume $\frac{1}{i} < 0$. Therefore, by defintion of $<$, there must exists a positive rational m such that $\frac{1}{i} + \frac{l}{m} = 0$. By the defintion of addition in fractions, $\frac{m+il}{im} = \frac{0}{1}$. By the defintion of equality of rationals, $m + il = 0 * im \implies m + il = 0$. $l, m \neq 0$ as it's a positive rational number. Therefore, we have a case where two positive rationals sum up to 0 and that is not possible. Hence, a contradiction. Therefore, $\frac{1}{i} \geq 0$. Why 2 will also have a similar reasoning.

Note, how difficult it is to show that a sequence is a cauchy sequence. We need to explicitly find a N and then show that $\forall i, j \geq N$, their absolute difference is $\leq \epsilon \forall \epsilon > 0$.

Lemma: 5.1.15 Cauchy sequences are bounded

Every cauchy sequence is bounded.

Commentary: Bounded means that the absolute value of every element in the sequnece is bounded by some rational M. Cauchy sequence means that the difference between the two consecutive elements in the sequence gets smaller and smaller.

Proof: Since, we have a cauchy sequence, we know we will have a number n after which the sequence is 1-steady. So, we have two sequences now. One before 1-steady and one after 1-steady. First one is bounded because it is finite. But, why would the second 1-steady sequence be bounded? 1-steady means that the difference between any two elements in the sequence is less than or equal to 1. An infinite sequence is bounded if there exists M such that $a_i \leq M \forall n \geq 0$. Clearly, because any consecutive $d(a_i, a_j) \leq 1$, $|a_i - a_j| \leq 1$. If $a_i - a_j$ is positive, we have $a_j \leq 1 - a_i$. If negative, we have $-(a_i - a_j) \leq 1 \implies a_j - a_i \leq 1 \implies a_j \leq 1 + a_i$. In both cases, we found an upper bound for the element. Hence, the second part is also bounded. ☐

5.2 Equivalent Cauchy Sequence

Definition 5.2.6 Equivalent sequence

Two sequences $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are said to equivalent iff for every $\epsilon > 0$, there exists $N \geq 0$ such that $|a_i - b_i| \leq \epsilon \forall i \geq N$. In other words, for every rational $\epsilon > 0$, the two sequences are ϵ-close.

Exercises

5.2.1

Given two sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$, $(a_n)_{n=1}^{\infty}$ is a cauchy sequence iff $(b_n)_{n=1}^{\infty}$ is a cauchy sequence.

Commentary: What is the difference between a sequence and a cauchy sequence? In a cauchy sequence, the difference between any two numbers in the sequence keeps on decreasing, whereas, a sequence is just a sequence. Two sequences are equivalent when the difference between the two keeps on decreasing as well. Clearly, if we want two sequences to be equivalent, i.e., the difference to keep on decreasing between the two, both of them individually also need to keep on decreasing because imagine if one of them is increasing, the other one keeps on decreasing, so the difference between the last element of the two sequence will keep on increasing and then they cannot be equivalent which we know they are. So, one increasing and other decreasing is not possible. What about both increasing? No, we don't need to consider this situation because our iff condition is saying that if one of them is decreasing, the other one has to be too which we just discussed.

Proof: We'll show A implies B and B implies A. This is the structure of our proof. Assume $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence. Also, assume $(b_n)_{n=1}^{\infty}$ is not a Cauchy sequence. Since two two sequences are equivalent, for every $\epsilon \exists N$ such that $|a_n - b_n| \leq \epsilon \forall n \geq N$. Assume for ϵ_1, $(b_n)_{n=1}^{\infty}$ does not have a corresponding N for which this sequence can be a Cauchy sequence. Since, a_n sequence is Cauchy, let it become cauchy at n_1. Then, $|a_m - a_n| \leq \epsilon \forall m, n \geq n_1$. For the same ϵ, $|b_m - b_n| \geq \epsilon$.

We know that for every ϵ, there exists N such that $|a_n - b_n| \leq \epsilon \forall n \geq N$. Take two values $m, n \geq N$, for both of them, we have $|a_n - b_n| \leq \epsilon$ and $|a_m - b_m| \leq \epsilon$. Adding these two, we get $|a_n - b_n| + |a_m - b_m| \leq 2\epsilon$. We'll take these case by case, simple case first. Assuming that both values turn out to be positive, we have $a_n - b_n + a_m - b_m \leq 2\epsilon$. Since the sequence a_n is a cauchy sequence, we can find such m, n that $a_m - a_n \leq epsilon$. Basically, I will show here that this will lead to b_n 's sequence satisfying the cauchy sequence definition and because of that we have a contradiction as we assumed b_n 's sequence to not be a cauchy sequence.

Complete proof

5.2.2

Let $\epsilon > 0$ and the two sequences $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are ϵ-close. We need to show that a_n is bounded iff b_n is bounded.

Commentary: Here, the difference between the two sequences is eventually between ϵ. Now, if one of the sequences is bounded, the other one has to be bounded too. This is because say one is bounded and the other is not. For any point of time in the sequence for a_n (which we assume to be bounded), b_n can be any value because it is not bounded. This will mean it will get difficult to keep their absolute difference in the given ϵ range. Hence, b_n's sequence will also be bounded.

Proof: Since, the two sequences are ϵ-close, there exists N such that $|a_n - b_n| \leq \epsilon \forall n \geq N$. We'll assume that to be N only. Also assume a_n is bounded. Let that value of bound be M. $\forall n \geq 0, a_n \leq M$ and let b_n be not bounded. Take $|a_N - b_N| \leq \epsilon \implies |M - b_N| \leq \epsilon$. But at value $b_N = M + \epsilon + 1$, this equation will not be satisfied. Therefore every b_N must be smaller than this value. Hence, b_n 's sequence is also bounded.

What I am not satisfied with is that after N every number b_n in the sequence will be smaller than $M + \epsilon + 1$, yes, that's true, but this does not stop $b_n > M$ for some $n < N$. Oh wait, ϵ-close means that for all $|a_i - b_i| \leq \epsilon$. Then, our proof will work because $|M - b_i| \leq \epsilon$ needs to hold for all i. And this will given us our bound for b_n's sequence.

Appendix

5.3 Statements

Exercises

5.3.1

$$(x \vee y) \wedge \neg(x \wedge y)$$

The negation of the above statement is

$$\neg((x \vee y) \wedge \neg(x \wedge y)).$$

$$\neg(x \vee y) \vee (x \wedge y).$$

$$(\neg x \wedge \neg y) \vee (x \wedge y).$$

5.3.2

5.3.3

G iven: $x \implies y \wedge \neg x \implies \neg y$

To show: $x \iff y$

Proof: We need to show $y \implies x$.

5.3.4

Given: $x \implies y \wedge \neg y \implies \neg x$

To show: $x \iff y$

Proof: If y is true, we cannot conclude anything about x hence no we have not shown $x \iff y$.

complete this logic exercise as current proof is wrong