Todo list

Chapter 1

Introduction

1.1 What is analysis?

Real analysis is the analysis of real numbers, sequences and series of real numbers and real-valued functions (functions which have range as real numbers).

1.2 Examples

The examples in this section will show why understanding of real numbers is important. We'll see situations where if we don't really understand what these real numbers are, we won't be to make correct decisions.

1.2.1 division by zero

$ac = bc \implies a = b$. But, this does not work when $c = 0$. What is this example telling us? Whenever we are cancelling like how we did above, we are ruling out that $c = 0$ for all practical purposes.Because cancellation here actually means division by c, we must make sure $c \neq 0$.

1.2.2 divergent series

Take $S = 1 + \frac{1}{2} + \frac{1}{4} + +\frac{1}{8} + \ldots$, what is S?. We can use this trick: Multiply both sides by 2. We get $2S = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \ldots$.

$$2S = 2 + S \implies S = 2.$$

If we have another sum $S = 1 + 2 + 4 + 8 + 16 + \ldots$, using the same trick leads to

$$2S = 2 + 4 + 8 + 16 + \cdots \implies 2S + 1 = S \implies S = -1.$$

Clearly, that should not be the case. My initial guess is that we cannot apply this trick for a divergent series but can for a convergent series. Why, I need to find out.

EOB need answer

1.2.3 Divergent sequence

Let x be a real number and let L define the limit as follows:

$$L = \lim_{n \to \infty} x^n.$$

Let $n = m + 1$.

$$\therefore L = \lim_{m+1 \to \infty} x^{m+1}.$$

$$L = x \lim_{m+1 \to \infty} x^m.$$

Is it allowed to do take x out of the limit?

$$\because m + 1 \implies \infty, m \implies \infty.$$

$$L = x \lim_{m \to \infty} x^m.$$

$$L = xL.$$

Either $x = 1$ or $L = 0$. This means $L = \lim_{n\to\infty} x^n = 0, x \neq 1$. But, this does not make sense because clearly when instantiated for $x = 2$, $L = \lim_{n\to\infty} 2^n \neq 0$ My guess is that the part where we move x out of the limit so easily should not be allowed.

1.2.4

1.2.5

1.2.6 Interchanging of integrals

We sometimes use this trick when integrating $\int\int f(x,y)dxdy = \int\int f(x,y)dydx$. But this too can lead to issues sometimes. Look at the example in the book for specific example.

EOB need answer on interchanging integrals

$$\int_0^\infty \int_0^1 \left(e^{-xy} - xye^{-xy} dydx\right) = \int_0^1 \int_0^\infty \left(e^{-xy} - xye^{-xy} dxdy\right).$$

We'll see in the book that the results of the two are different (LHS and RHS is not the same).

1.2.7 Interchanging limits

1.2.8

1.2.9

Chapter 2

Natural numbers

Just to reiterate the different kinds of numbers we'll deal with in this book, here are they in increasing order of sophistication. Natural numbers, $\mathbb{N}$, integers, $\mathbb{Z}$, rational numbers, $\mathbb{Q}$, real numbers, $\mathbb{R}$ and complex numbers, $\mathbb{C}$.

2.1 The Peano axioms

The Peano axioms is one of the ways of defining natural numbers. There are other approaches such as using sets. We define natural numbers as that set of elements for which the following axioms hold:

2.1.1 0 is a natural number.

2.1.2 If n is a natural number, then $n{+}{+}$ is also a natural number

Note, we assume that there exists an operation with symbol ++ that means increment. This axiom allows us to move forward in our count.

2.1.3 0 is not the successor for any natural number, i.e., $n++ \neq 0 \forall n$

This ensures we don't circle back to 0. Imagine if we defined $(0++)++$ as 0. Axioms 2.1.2 and 2.1.1 would still hold but we know we don't want that in our natural number system. This axiom ensures that.

2.1.4 $n \neq m \implies n++ \neq m++$

This axiom ensures that we don't reach an upper bound. If we define $2++$ and $3++$ both as 3, none of the above axioms are violated and yet we know we don't want an upper bound to our natural number system. Take the next number,

$(3++)++ = 3++ = 3$, we have reached an upper bound for us. Another way to state this axiom is in its contrapositive form. $n++ = m++ \implies n = m$.

2.1.5 Principle of Mathematical Induction

It might be surprising right now why we are including this as an axiom because it is part of the larger logic scheme. Why do we need to explicitly say this. My guess is that it is needed because there are other systems where this principle does not hold. The principle is if some property P holds for 0, that is, if $(P(0))$ is true and $P(n) \implies P(n+1)$ is true, then property P is true for all elements in the number system.

find why math induction principle is needed as a Peano axiom to define natural numbers

This axiom allows us to define properties of objects for all elements.

Note our definition of natural numbers is axiomatic and not *constructive.* What I mean by that is we have not defined *what* a natural number is but rather properties that a natural number holds. Any number system that follows the above mentioned axioms is a natural number system.

2.1.6 Recursive definition of sequence: Proposition

Suppose for each natural number n, we have some function $f_n : \mathbb{N} \mapsto \mathbb{N}$. Let c be a natural number. Then, we can assign a unique natural number a_n to each natural number n such that $a_0 = c$ and $a_{\{n++\}=f_n(a_n)}$ for each natural number n.

commentary The goal is to show that each value of the sequence a_i is defined only once and because we are looking at a recursive defintion we do not revisit a_i again when going forward.

proof We'll prove this by induction. Take the base case, $a_0 = c$. Because of axiom 2.1.3, going forward, any a_{m++} will not redefine a_0. For example, if axiom 2.1.3 was actually violated and say $3{+}{+} = 0$. Then, $a_{3++} = f_3(a_3) = a_0$. Note, we redefined a_0 without this axiom.

Let's assume that a_n by the given recursive formula uniquely defined a_n. We need to this also holds for $a_{(n+1)}$. Any natural number after $n++$, say we call it $m++$ will redefine $n++$ because of axiom 2.1.4. Note, this axiom was introduced to ensure the successors are always bigger and that's what this axiom ensures. Because we cannot circle back, the previously defined a_i remains as is.

2.2 Addition

2.2.1 Definition: Addition of natural numbers

Let m be a natural number, we define addition of zero to m as $0+m := m$. We give a recursive defintion for adding. Let's say we know how to perform $n+m$. Therefore, adding $n++$ to m is defined as $(n++)+m := (n+m)++$.

For example, say, we need to add $2+5$. $(1++)+5 = (1+5)++ = ((0++)+5)++ = ((0+5)++)++ = (5++)++ = 6++ = 7$

Appendix

2.3 Statements

Exercises

2.3.1

$$(x \vee y) \wedge \neg(x \wedge y)$$

The negation of the above statement is

$$\neg((x \vee y) \wedge \neg(x \wedge y)).$$

$$\neg(x \vee y) \vee (x \wedge y).$$

$$(\neg x \wedge \neg y) \vee (x \wedge y).$$

2.3.2

2.3.3

G iven: $x \implies y \wedge \neg x \implies \neg y$

To show: $x \iff y$

Proof: We need to show $y \implies x$.

2.3.4

Given: $x \implies y \wedge \neg y \implies \neg x$

To show: $x \iff y$

Proof: If y is true, we cannot conclude anything about x hence no we have not shown $x \iff y$.

complete this logic exercise as current proof is wrong