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A GLOBAL OPTIMIZATION METHOD FOR PACKING PROBLEMS

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ABSTRACT

The packing optimization problem is to seek the best way of placing a given set of rectangular boxes within a rectangular container with minimal volume. Current packing optimization methods (Chen *et al*. [2], Li and Chang [6]) are either difficult to find an optimal solution or required to use too many extra 0-1 variables to solve the problems. This paper proposes a new method for finding the global optimum of the packing problem within tolerable error based on piecewise linearization techniques, which is more computationally efficient than current methods.

Keywords: packing, optimization, piecewise linearization

1. INTRODUCTION

The packing optimization problem is to seek a container placing a given set of small rectangular boxes. This problem is referring to as a container loading problem. Packing boxes into a container is a fundamental material handling activity in the manufacturing and distribution industries. This problem has been extensively applied in many related studies, such as knapsack [4], cutting stock [5], assortment problems [1,6], pallet loading [8,10] and container loading problems [2,9] etc. In addition, researchers have dealt with various related problems. For instance, Dowsland [3] proposed a heuristic method for solving 3D packing problems, Chen *et al*. [2] formulated a mixed integer program for container loading problems, and Li and Chang [6] developed a method for finding the approximate global optimum of the assortment problem. However, Li and Chang's method [6] requires to use numerous 0-1 variables to linearize the polynomial objective function in their model, which would cause heavy computational burden in solving packing problems. Moreover, Chen *et al*.'s [2] approach can only find a local optimum of packing problems with the nonlinear objective function xyz.

This paper proposes a new method for solving packing optimization problems. Two advantages of this method are listed below:

1. It can find the solution which can be as close as possible to the global optimum, instead of obtaining a local optimum.
2. It adopts less 0-1 variables to reformulate the packing problem than used in Li and Chang's model [6].

2. PROBLEM FORMULATION

Given n rectangular boxes with fixed lengths, widths, and heights, a packing optimization problem is to allocate these n boxes within a rectangular container which has minimal volume. Denote x, y, and z as the width, length, and height of the container $(x>0, y>0, z>0)$, the packing optimization problem is stated as follows:

Minimize xyz

subject to

1. All of n boxes are non-overlapping.
2. All of n boxes are within the range of x, y, and z.
3. $\underline{x} \le x \le \overline{x}$, $\underline{y} \le y \le \overline{y}$, and $\underline{z} \le z \le \overline{z}$ ($\underline{x}$, $\underline{y}$, $\underline{z}$, $\overline{x}$, $\overline{y}$, and $\overline{z}$ are constants).

The related terminologies used in the packing model, referring to Chen *et al*. [2], are described below:

(p_i, q_i, r_i): Dimension of box *i*, p_i is the length, q_i is the width and r_i is the height, p_i, q_i, and r_i are constants. $i \in J$, $J=\{1, 2, 3, .., n\}$ is the set of given boxes.

(x, y, z): Continuous variables indicating the length, width, and height of the container.

(x_i, y_i, z_i): Continuous variables indicating the

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coordinates of the front-left bottom corner of box i.

(l_{xi}, l_{yi}, l_{zi}): Binary variables indicating whether the length of box i is parallel to the X-axis, Y-axis, or Z-axis. For example, the value of l_{xi} is equal to 1 if the length of box i is parallel to the X-axis; otherwise, it is equal to 0. It is clear that $l_{xi} + l_{yi} + l_{zi} = 1$.

(w_{xi}, w_{yi}, w_{zi}): Binary variables indicating whether the width of box i is parallel to the X-axis, Y-axis, or Z-axis. For example, the value of w_{xi} is equal to 1 if the width of box i is parallel to the X-axis; otherwise, it is equal to 0. $w_{xi} + w_{yi} + w_{zi} = 1$.

(h_{xi}, h_{yi}, h_{zi}): Binary variables indicating whether the height of box i is parallel to the X-, Y-, or Z-axis. For example, the value of h_{xi} is equal to 1 if the height of box i is parallel to the X-axis; otherwise, it is equal to 0. $h_{xi} + h_{yi} + h_{zi} = 1$.

For a pair of boxes (i,k) where $i<k$, there is a set of 0-1 vectors $(a_{ik}, b_{ik}, c_{ik}, d_{ik}, e_{ik}, f_{ik})$ defined as:

a_{ik} =1 if box i is on the left of box k, therwise $a_{ik} = 0$.

b_{ik} =1 if box i is on the right of box k,otherwise $b_{ik} = 0$.

c_{ik} =1 if box i is behind box k, otherwise $c_{ik} = 0$.

d_{ik} =1 if box i is in front of box k,otherwise $d_{ik} = 0$.

e_{ik} =1 if box i is below box k, otherwise $e_{ik} = 0$.

f_{ik} =1 if box i is above box k, otherwise $f_{ik} = 0$.

The front-left bottom corner of the container is fixed at the origin. The interpretation of these variables is illustrated in Figure 1. Figure 1 contains two boxes i and k, where box i is located with its length along the X-axis and the width parallel to the Z-axis, and box k is located with its length along the Z-axis and the width parallel to the X-axis. We then have l_{xi}, w_{zi}, h_{yi}, l_{zk}, w_{xk}, and h_{yk} all equal to 1.

The packing problem can then be formulated as follows, referring to Chen *et al*. [2] and Li and Chang [6] :

Model 1:

$$\text{Minimize } \; xyz \tag{1}$$

subject to

$$x_i + p_i l_{xi} + q_i w_{xi} + r_i h_{xi} \le x_k + (1 - a_{ik})M \quad \forall i,k \in J, i < k \tag{2}$$

$$x_k + p_k l_{xk} + q_k w_{xk} + r_k h_{xk} \le x_i + (1 - b_{ik})M, \quad \forall i,k \in J, i < k \tag{3}$$

$$y_i + p_i l_{yi} + q_i w_{yi} + r_i h_{yi} \le y_k + (1 - c_{ik})M, \quad \forall i,k \in J, i < k \tag{4}$$

$$y_k + p_k l_{yk} + q_k w_{yk} + r_k h_{yk} \le y_i + (1 - d_{ik})M, \quad \forall i,k \in J, i < k \tag{4}$$

$$z_i + p_i l_{zi} + q_i w_{zi} + r_i h_{zi} \le z_k + (1 - e_{ik})M, \quad \forall i,k \in J, i < k \tag{6}$$

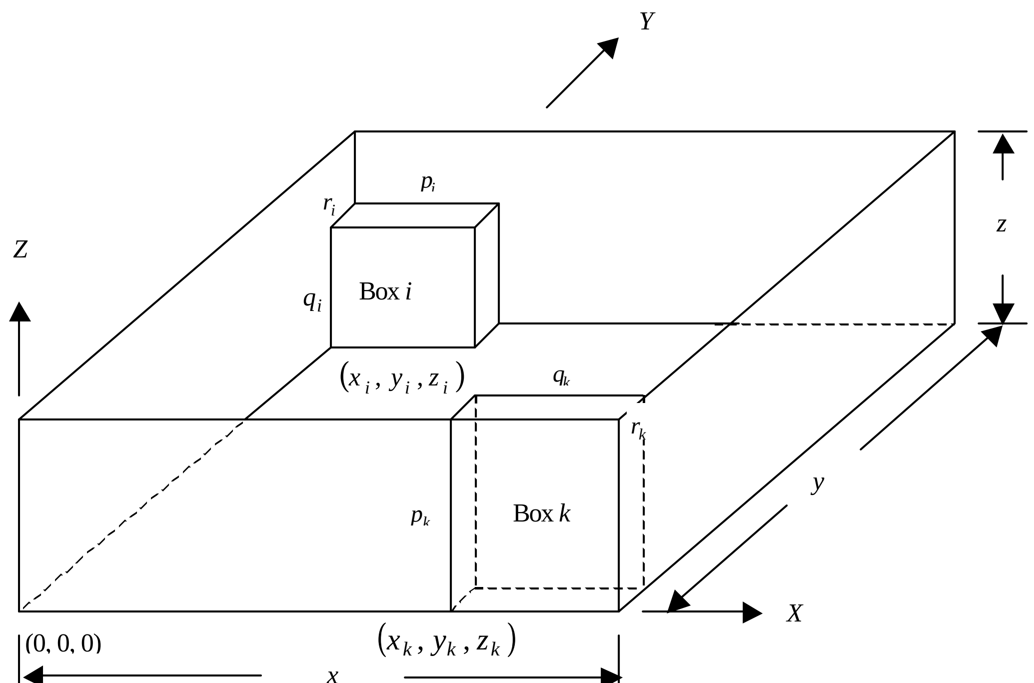

Figure 1. Graphical illustration

$$z_k + p_k l_{zk} + q_k w_{zk} + r_k h_{zk} \le z_i + (1 - f_{ik})M, \quad \forall i,k \in J, i < k \tag{7}$$

$$a_{ik} + b_{ik} + c_{ik} + d_{ik} + e_{ik} + f_{ik} \ge 1, \quad \forall i,k \in J, i < k \tag{8}$$

$$x_i + p_i l_{xi} + q_i w_{xi} + r_i h_{xi} \le x, \quad \forall i \in J \tag{9}$$

$$y_i + p_i l_{yi} + q_i w_{yi} + r_i h_{yi} \le y, \quad \forall i \in J \tag{10}$$

$$z_i + p_i l_{zi} + q_i w_{zi} + r_i h_{zi} \le z, \quad \forall i \in J \tag{11}$$

$$l_{xi} + l_{yi} + l_{zi} = 1, \quad \forall i \in J \tag{12}$$

$$w_{xi} + w_{yi} + w_{zi} = 1, \quad \forall i \in J \tag{13}$$

$$h_{xi} + h_{yi} + h_{zi} = 1, \quad \forall i \in J \tag{14}$$

$$l_{xi} + w_{xi} + h_{xi} = 1, \quad \forall i \in J \tag{15}$$

$$l_{yi} + w_{yi} + h_{yi} = 1, \quad \forall i \in J \tag{16}$$

$$l_{zi} + w_{zi} + h_{zi} = 1, \quad \forall i \in J \tag{17}$$

where $l_{xi}, l_{yi}, l_{zi}, w_{xi}, w_{yi}, w_{zi}, h_{xi}, h_{yi}, h_{zi}, a_{ik}, b_{ik}, c_{ik}, d_{ik}, e_{ik}$ and f_{ik} are 0-1 variables, $M =$, max $\{\bar{x}, \bar{y}, \bar{z}\}$ $x_i, y_i, z_i > 0$, $\underline{x} \le x \le \bar{x}, \underline{y} \le y \le \bar{y}, \underline{z} \le z \le \bar{z}$, and $\underline{x}, \underline{y}, \underline{z}, \bar{x}, \bar{y}$, and $\bar{z}$ are Constants.

The objective of this model is to minimize the volume of the container. The constraints (2)-(8) are non-overlapping conditions used to ensure that none of these n boxes overlaps with each other. Constraints (9)-(11) ensure that all boxes are within the enveloping container. Constraints (12)-(17) describe the allocation restrictions among logic variables. For instance, (12) implies that the length of carton i is parallel to one of the axes. (15) implies that only one of the length, the width and the height of carton i is parallel to X-axis. Using the constraints (12)-(17), we can eliminate from the model the following five 0-1 variables $l_{yi}, w_{xi}, w_{zi}, h_{xi}$, and h_{yi}. Model 1 is then fully converted into Model 2 below [2]:

Model 2

Minimize xyz

subject to

$$q_i(l_{zi} - w_{yi} + h_{zi}) + r_i(1 - l_{xi} - l_{zi} + w_{yi} - h_{zi}) + x_i + p_i l_{xi} \le x_k + (1 - a_{ik})M$$

$$q_k(l_{zk} - w_{yk} + h_{zk}) + r_k(1 - l_{xk} - l_{zk} + w_{yk} - h_{zk}) + x_k + p_k l_{xk} \le x_i + (1 - b_{ik})M$$

$$y_i + p_i(1 - l_{xi} - l_{zi}) + q_i w_{yi} + r_i(l_{xi} + l_{zi} - w_{yi}) \le y_k + (1 - c_{ik})M$$

$$y_k + p_k(1 - l_{xk} - l_{zk}) + q_k w_{yk} + r_k(l_{xk} + l_{zk} - w_{yk}) \le y_i + (1 - d_{ik})M$$

$$z_i + p_i l_{zi} + q_i(1 - l_{zi} - h_{zi}) + r_i h_{zi} \le z_k + (1 - e_{ik})M$$

$$z_k + p_k l_{zk} + q_k(1 - l_{zk} - h_{zk}) + r_k h_{zk} \le z_i + (1 - f_{ik})M$$

$$a_{ik} + b_{ik} + c_{ik} + d_{ik} + e_{ik} + f_{ik} \ge 1$$

$$x_i + p_i l_{xi} + q_i(l_{zi} - w_{yi} + h_{zi}) + r_i(1 - l_{xi} - l_{zi} + w_{yi} - h_{zi}) \le x$$

$$y_i + p_i(1 - l_{xi} - l_{zi}) + q_i w_{yi} + r_i(l_{xi} + l_{zi} - w_{yi}) \le y$$

$$z_i + p_i l_{zi} + q_i(1 - l_{zi} - h_{zi}) + r_i h_{zi} \le z$$

where all variables are the same as in Model 1.

Model 2 is a nonlinear mixed 0-1 program which is difficult to be solved by current optimization methods. Chen *et al*. [2] can only solve Model 2 where the objective function is linear. For solving Model 2 by Li and Chang's method [6], x, y, and z are requested to be rewritten as follows:

$$x = \bar{e}_x \sum_{g=1}^{G} 2^{g-1} q_g + e_x$$

$$y = \bar{e}_y \sum_{h=1}^{H} 2^{h-1} d_h + e_y$$

$$z = \bar{e}_z \sum_{l=1}^{L} 2^{l-1} w_l + e_z$$

where e_x, e_y, and e_z are small positive variables. $\bar{e}_x$, $\bar{e}_y$, and $\bar{e}_z$ are the pre-specified constants which are the upper bounds of e_x, e_y, and e_z respectively. q_g, d_h, and w_l are 0-1 variables, and G, H, and L are integers which denote the number of required 0-1 variables for representing x, y, and z.

The major difficulty of Li and Chang's method is that it involves $G+H+L$ additional 0-1 variables. The smaller the tolerable errors (i.e., e_x, e_y, and e_z) we specify, the larger the size of G, H, and L it becomes, and the longer the computation time it requires.

3. PROPOSED METHOD

Denote F as a feasible set of Model 2 in which $x \ge y \ge z$, F={(2')~(11'), $x \ge y \ge z$ }.First, consider following fact:

An optimization program P1: {Minimize Obj1 = xyz, subject to $\underline{x} \le x \le \overline{x}$, $\underline{y} \le y \le \overline{y}$, $\underline{z} \le z \le \overline{z}$, $x, y, z \in F$ } is equivalent to the program below.

P2:{Minimize Obj2 = $\ln x + \ln y + \ln z$, subject to $\underline{x} \le x \le \overline{x}$, $\underline{y} \le y \le \overline{y}$, $\underline{z} \le z \le \overline{z}$, $x, y, z \in F$ }.

Following propositions discuss the proposed approach of linearizing the logarithmic terms $\ln x$, $\ln y$, and $\ln z$.

Proposition 1

A logarithm function $\ln x$, $0 < a_1 \le x \le a_m$, as shown in Figure2, can be approximately expressed as

$$\ln x \doteq \ln \hat{x} = \ln a_1 + s_1(x - a_1) + \sum_{j=2}^{m-1} \frac{s_j - s_{j-1}}{2}\left(\left|x - a_j\right| + x - a_j\right) \tag{18}$$

where $a_j, j = 1,2,\ldots,m$, are the break points of $\ln x$, $a_j < a_{j+1}$; $|\bullet|$ is the absolute value of x; and $s_j, j = 1,2,\ldots,m-1$, are the slopes of line segments between a_j and a_{j+1}, $s_j = \dfrac{(\ln a_{j+1} - \ln a_j)}{(a_{j+1} - a_j)}$, for j=1,2,...,m-1.

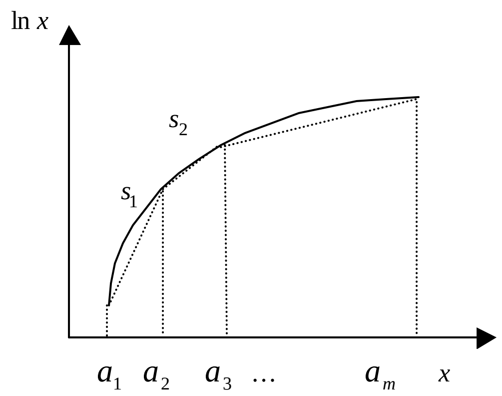

Figure 2. Graphical illustration

This proposition can be examined as follows:

If $x = a_1$ then $\ln x = \ln a_1$.

If $x \le a_2$ then $\ln x = \ln a_1 + s_1(x - a_1)$.

If $x \le a_3$ then

$$\ln x = \ln a_1 + s_1(x - a_1) + \frac{s_2 - s_1}{2}\left(\left|x - a_2\right| + x - a_2\right).$$

Similarly, logarithm functions $\ln y$ and $\ln z$, can be approximately linearized as

$$\ln y \doteq \ln \hat{y} = \ln b_1 + t_1(y - b_1) + \sum_{j=2}^{m-1} \frac{t_j - t_{j-1}}{2}\left(\left|y - b_j\right| + y - b_j\right) \tag{19}$$

where $t_j = \dfrac{(\ln b_{j+1} - \ln b_j)}{(b_{j+1} - b_j)}$, $0 < b_1 \le y \le b_m$, and $b_1, b_2, \ldots, b_m$ are its break points, $b_j < b_{j+1}$, for j=1,2,...,m-1;

$$\ln z \doteq \ln \hat{z} = \ln c_1 + r_1(z - c_1) + \sum_{j=2}^{m-1} \frac{r_j - r_{j-1}}{2}\left(\left|z - c_j\right| + z - c_j\right) \tag{20}$$

where $r_j = \dfrac{(\ln c_{j+1} - \ln c_j)}{(c_{j+1} - c_j)}$, $0 < c_1 \le z \le c_m$, and $c_1, c_2, \ldots, c_m$ are its break points, $c_j < c_{j+1}$, for j=1,2,...,m-1.

Proposition 2

Since $\ln x$, $\ln y$, and $\ln z$ are concave functions, it is clear that $\ln x \ge \ln \hat{x}$, $\ln y \ge \ln \hat{y}$, and $\ln z \ge \ln \hat{z}$. $\ln \hat{x}$, $\ln \hat{y}$, and $\ln \hat{z}$ are specified in (18), (19), and (20) respectively.

We then have the following propositions.

Proposition 3 (Lower bound)

Consider the following program

P3 : { Minimize Obj3 = $\ln \hat{x}$ + $\ln \hat{y}$ + $\ln \hat{z}$, subject to $\underline{x} \le x \le \overline{x}$, $\underline{y} \le y \le \overline{y}$, $\underline{z} \le z \le \overline{z}$, $x, y, z \in F$ }.

Program P3 provides a lower bound on Program P2 by referring to Proposition 2. Now we discuss the way to linearize $\ln \hat{x}$. Consider the following proposition,

Proposition 4

$\ln \hat{x}$ in (18) can be fully linearized as follows.

$$\ln \hat{x} = \ln a_1 + s_1(x - a_1) + \sum_{j=2}^{m-1}(s_j - s_{j-1})(a_j u_j + x - a_j - w_j) \tag{21}$$

where
(i) $-a_m u_j \le x - a_j \le a_m(1-u_j)$, for j =2, 3, . ., m;
(ii) $-a_m u_j \le w_j \le a_m u_j$, for j =2, 3, . ., m;
(iii) $a_m(u_j - 1) + x \le w_j \le a_m(1-u_j) + x$,
for j =2, 3, . ., m;
(iv) $u_j \ge u_{j-1}$, $u_j \in (0,1)$, for j =2, 3, . ., m.

Proof:
If $x - a_j \ge 0$ then u_j =0 and w_j =0 based on (i) and (ii); which results in $a_j u_j + x - a_j - w_j = \frac{1}{2}(\left| x - a_j \right| + x - a_j)$ If $x - a_j < 0$ then u_j =1 and $w_j = x$ based on (i) and (iii); which results in $a_j u_j + x - a_j - w_j = \left| x - a_j \right| + x - a_j$.
Therefore, $\ln \hat{x}$ in (18) is equivalent to (21). Now we consider condition (iv).Since $a_{j-1} < a_j$, if $x < a_j$ (i.e. $u_j = 1$) then $x < a_{j+1}$ and we have $u_{j+1} = 1$.
If $x > a_{j+1}$ (i.e. $u_{j+1} = 0$) then $x > a_j$, and we have $u_j = 0$.
Therefore, it is true that $u_j \ge u_{j-1}$.Condition (iv) is used to accelerate the computational speed of solving the problem.Similarly, $\ln \hat{y}$ and $\ln \hat{z}$ can be piecewisely linearized in the same way.

In the proposed method, $3.5n^2 - 0.5n + 12m$ constraints and $3n^2 + n + 3m$ binary variables are used to solve the packing optimization problems where n is the number of boxes and m is the number of break points. Therefore, the computational time required to solve the problem depends on both the problem size and specified tolerable error. Additionally, the higher level computer can also improve the computation time.

4. SOLUTION ALGORITHM

From the basis of above discussion, the solution algorithm of solving the packing problem is proposed below:
Let S_i, T_i, and U_i be respectively a set of break points of $\ln x$, $\ln y$, and $\ln z$ at i' th iteration. Denote e as a tolerable error.

Step 1 Initialization
Let i=1, $S_1 = \{\underline{x}, \bar{x}\}$, $T_1 = \{\underline{y}, \bar{y}\}$, $U_1 = \{\underline{z}, \bar{z}\}$, $\underline{x} > 0$, $\underline{y} > 0$, $\underline{z} > 0$.

Solving the linear program $\underset{(x,y,z)}{Min}$ $Obj(x(2)) + Obj(y(2)) + Obj(z(2))$ $= \ln a_1 + \frac{\ln \bar{x} - \ln \underline{x}}{\bar{x} - \underline{x}}(x - \underline{x}) + \ln b_1 + \frac{\ln \bar{y} - \ln \underline{y}}{\bar{y} - \underline{y}}(y - \underline{y}) + \ln c_1 + \frac{\ln \bar{z} - \ln \underline{z}}{\bar{z} - \underline{z}}(z - \underline{z})$ .subject to $(x, y, z) \in F$, $\underline{x} \le x \le \bar{x}$, $\underline{y} \le y \le \bar{y}$, $\underline{z} \le z \le \bar{z}$.
Let the solution be $(x(2), y(2), z(2))$.
If $\left| Obj(x(2)) - \ln x(2) \right| < e$, $\left| Obj(y(2)) - \ln y(2) \right| < e$, and $\left| Obj(z(2)) - \ln z(2) \right| < e$ then terminate the process. The optimal solution is $(x(2), y(2), z(2))$.
Otherwise, go to Step 2.

Step 2
Let $S_i = S_{i-1} \cup \{x(i)\}$, $T_i = T_{i-1} \cup \{y(i)\}$, $U_i = U_{i-1} \cup \{z(i)\}$. "U" means union.
Denote the number of elements in S_i, T_i, and U_i as m_i. Solving the following linear mixed 0-1 program,

$$\underset{(x,y,z)}{Min} \; Obj(x(i+1)) + Obj(y(i+1)) + Obj(z(i+1))$$
$$= \ln a_1 + s_1(x - a_1) + \sum_{j=2}^{m_i}(s_j - s_{j-1})(a_j u_j + x - a_j - w_j)$$
$$+ \ln b_1 + t_1(y - b_1) + \sum_{j=2}^{m_i}(t_j - t_{j-1})(b_j v_j + y - b_j - q_j)$$
$$+ \ln c_1 + r_1(z - c_1) + \sum_{j=2}^{m_i}(r_j - r_{j-1})(c_j o_j + z - c_j - p_j)$$

s.t. $(x, y, z) \in F$, for all j following constraints should be satisfied:
$-\bar{x}u_j \le x - a_j \le \bar{x}(1-u_j)$, $-\bar{x}u_j \le w_j \le \bar{x}u_j$,
$\bar{x}(u_j - 1) + x \le w_j \le \bar{x}(1-u_j) + x$, $u_j \ge u_{j-1}$,
$-\bar{y}v_j \le y - b_j \le \bar{y}(1-v_j)$, $-\bar{y}v_j \le q_j \le \bar{y}v_j$,
$\bar{y}(v_j - 1) + y \le q_j \le \bar{y}(1-v_j) + y$, $v_j \ge v_{j-1}$,
$-\bar{z}o_j \le z - c_j \le \bar{z}(1-o_j)$, $-\bar{z}o_j \le p_j \le \bar{z}o_j$,
$\bar{z}(o_j - 1) + z \le p_j \le \bar{z}(1-o_j) + z$, $o_j \ge o_{j-1}$,
where u_j, v_j, and o_j are 0-1 variables, $w_j, q_j, p_j \ge 0$, $a_1, a_2, \ldots, a_{m_i} \in S_i$, $a_1 = \underline{x} < a_2 < \ldots < a_{m_i} = \bar{x}$, $b_1, b_2, \ldots, b_{m_i} \in T_i$, $b_1 = \underline{y} < b_2 < \ldots < b_{m_i} = \bar{y}$, $c_1, c_2, \ldots, c_{m_i} \in U_i$, $c_1 = \underline{z} < c_2 < \ldots < c_{m_i} = \bar{z}$, for j =2, 3, . ., m_i.
Let the solution be $(x(i+1), y(i+1), z(i+1))$.
If $\left| Obj(x(i+1)) - \ln x(i+1) \right| < e$, $\left| Obj(y(i+1)) - \ln y(i+1) \right| < e$, and $\left| Obj(z(i+1)) - \ln z(i+1) \right| < e$ then terminate the process, and ($x(i+1)$, $y(i+1)$ $z(i+1)$) is the optimal solution.
Otherwise, let $i = i + 1$ and reiterate Step 2.

Proposition 5 (Convergence)
The above algorithm (run with $e \doteq 0$) terminates with the incumbent solution $(\hat{x}^*, \hat{y}^*, \hat{z}^*)$ being optimum to the packing problem (1)-(17) when $i \to \infty$.

Proof:

Let $\{[l_x^i, u_x^i]\}$ express the sequence $[a_0^i, a_1^i]$, $[a_1^i, a_2^i]$, …, $[a_{m_r-1}^i, a_{m_r}^i]$, $\{[l_y^i, u_y^i]\}$ express the sequence $[b_0^i, b_1^i]$, $[b_1^i, b_2^i]$, …, $[b_{n_r-1}^i, b_{n_r}^i]$ and $\{[l_z^i, u_z^i]\}$ express the sequence $[c_0^i, c_1^i]$, $[c_1^i, c_2^i]$, …, $[c_{n_r-1}^i, c_{n_r}^i]$ where $a_0^i < a_1^i < ... < a_{m_i}^i$, $b_0^i < b_1^i < ... < b_{m_i}^i$ and $c_0^i < c_1^i < ... < c_{m_i}^i$.

Since each sequence $\{a_k^i\}\{a_{k+1}^i\}$, $\{b_k^i\}\{b_{k+1}^i\}$, and $\{c_k^i\}\{c_{k+1}^i\}$ are monotone and bounded, it is obvious that $\{[l_x^i, u_x^i]\}$, $\{[l_y^i, u_y^i]\}$, and $\{[l_z^i, u_z^i]\}$ converge to some intervals $[l_x^i, u_x^i]$, $[l_y^i, u_y^i]$, and $[l_z^i, u_z^i]$. When $i \to \infty$, by the concavity of $\ln \hat{x}$, $\ln \hat{y}$ and $\ln \hat{z}$ in (18) and the Mean Value Theorem, we have to validate the problem size solvable by the proposed method, two other problems are examined as shown in Table 2. To simplify the computational processes, all boxes in these two problems are cubes which are $\hat{x}^* = l_x^* = u_x^*$, $\hat{y}^* = l_y^* = u_y^*$, and $\hat{z}^* = l_z^* = u_z^*$ as $i \to \infty$. Which means $\ln \hat{x}^* = \ln x^*$, $\ln \hat{y}^* = \ln y^*$, and $\ln \hat{z}^* = \ln z^*$. By referring to Proposition 3, $\ln \hat{x}^*$, $\ln \hat{y}^*$, and $\ln \hat{z}^*$ are the lower bound of Program P2, $(\hat{x}^*, \hat{y}^*, \hat{z}^*)$ is then the optimal solution to the problem (1)-(17).

5. NUMERICAL EXAMPLES

Consider five packing optimization problems: Some given boxes are required to be packed within a rectangular container which has minimal volume. The dimensions of the boxes are given in Table 1. Here we solve the example problems using the proposed method by LINGO[7] (LINDO SYSTEMS INC., 1999, a common-used optimization package).

Proposed method solves the example problems by specifying $e = 0.1$ and obtains the global solution within the tolerable error. Although the proposed method spends much less time than Li and Chang's method [6] for finding the optimal solution, the proposed method is still difficult for finding the global optimum for the packing problem with large scale in rational time. This might remain for further study. Figure 3 is the graphical representation of problem 2's result obtained by proposed method.
chosen arbitrarily. The obtained solutions are global optima within tolerable error $e = 0.1$.

6. CONCLUSIONS

This paper proposes a new method to solve packing problems. By piecewisely linearizing the nonlinear objective function in the packing problems, the proposed method reformulates the original problem as a linear mixed 0-1 program. Solving the linear mixed 0-1 program iteratively, the proposed method can finally find a global optimum. Numerical examples demonstrate that the proposed method can effectively obtain the global optimum within a tolerable error.

Table 1. Computational results of proposed method

Problem number	Box number	p_i	q_i	r_i	x_i	y_i	z_i	Objective value	CPU time (hh:mm:ss)
1	1	25	8	6	0	10	0	3200	00:00:03
	2	20	10	5	5	0	0		
	3	16	7	3	9	0	5		
2	1	25	8	6	0	0	0	4368	00:00:11
	2	20	10	5	8	0	0		
	3	16	7	3	8	10	2		
	4	15	12	6	16	11	0		
3	1	25	8	6	0	0	0	5040	00:26:45
	2	20	10	5	20	8	0		
	3	16	7	3	8	0	0		
	4	15	12	6	8	8	0		
	5	22	8	3	8	0	3		

Table 2. Computational results of proposed method

Problem number	Box number	p_i	q_i	r_i	x_i	y_i	z_i	Objective value	CPU time (hh:mm:ss)
4	1	1	1	1	4	0	0	140	00:00:05
	2	1	1	1	0	4	1		
	3	1	1	1	6	4	2		
	4	2	2	2	4	3	2		
	5	2	2	2	4	3	0		
	6	3	3	3	4	0	1		
	7	4	4	4	0	0	0		
5	1	1	1	1	0	5	4	240	00:01:27
	2	1	1	1	0	1	0		
	3	1	1	1	0	0	0		
	4	2	2	2	0	3	3		
	5	2	2	2	1	1	0		
	6	3	3	3	0	0	2		
	7	3	3	3	0	3	0		
	8	5	5	5	3	0	0		

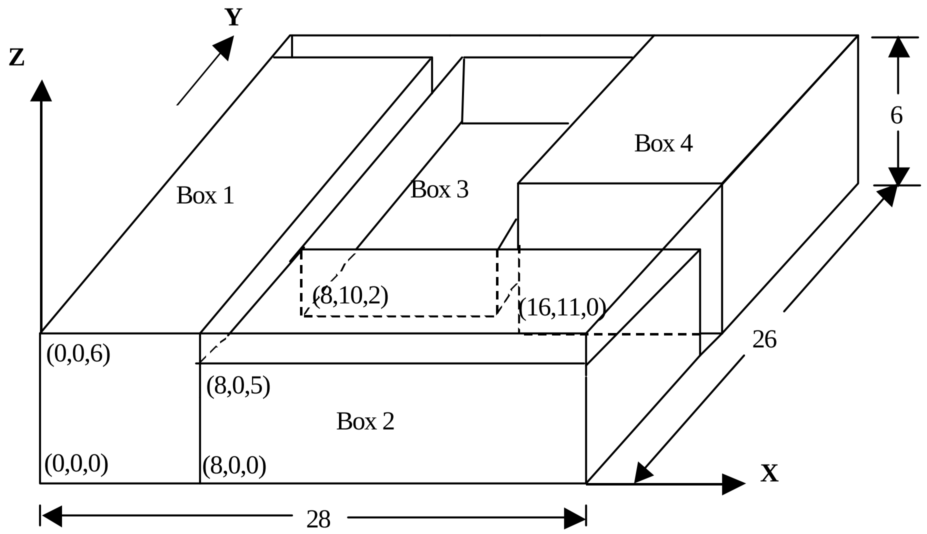

Figure 3. The graphical representation of 4 boxes

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封裝問題之最佳化方法

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摘要

封裝最佳化之研究係設計一最小體積的容器以擺置一組小四方體。Chen 等[2]的模式只適合求解線性封裝問題；Li 及 Chang[6]提出可求得近似全域最佳解的方法，但所需使用的0-1變數太多。本文提出一新方法求解封裝最佳化問題，以逐斷線性規劃法將一非線性目標式轉換為線性式，於誤差值範圍內求得全域最佳解。此方法之優點為：(1) 可找出全域最佳解。(2) 計算時間較短。本研究並以實例驗證此方法的有效性與正確性。

關鍵詞：封裝、最佳化、逐斷線性規劃法

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