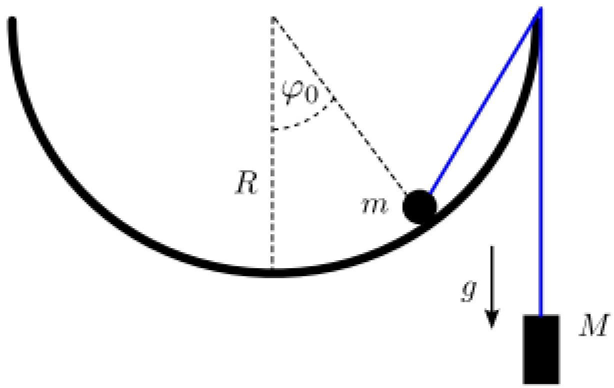
Codes used to type and visualise the answers can be found here

Question

In a cylindrical fixed trough with radius R, a point mass m is held on a rope as shown in the figure, with the rope passing over the trough and the other end attached to a mass M hanging vertically. The acceleration due to gravity is g. Initially, the mass m is at rest at an angle ϕ_0 ($-\pi/2 \leq \phi_0 \leq \pi/2$). Study the motion of the two bodies as a function of the given parameters.

In [1]:
```
from IPython.display import Image
Image(filename='figures/image_6.png')
```

Out[1]:



Answer

There are various ways to obtain the equation of motion for this system. In our case, we will be using force analysis in the relevant directions.

The dynamics of m and M is related by the rope. Let us define y as the distance of M from the nearest end of the cylinder, and L as the length of the rope. L, y, ϕ and R must be related by

$$L = y + 2R\sin\left(\frac{\pi}{4} - \phi/2\right)$$

Taking the time derivative twice, we obtain

$$0 = \dot{y} - R\dot{\phi}\cos\left(\frac{\pi}{4} - \phi/2\right)$$
$$0 = \ddot{y} - R\ddot{\phi}\cos\left(\frac{\pi}{4} - \phi/2\right) - \frac{1}{2}R\dot{\phi}^2\sin\left(\frac{\pi}{4} - \phi/2\right)$$

M dynamic

The dynamic of M only involves the gravitational force and the rope tension T (assuming the rope is stretched ($T > 0$)), giving the following relation

$$M\ddot{y} = Mg - T$$

Notice that based on our assumption, this equation is only valid for $g > \ddot{y}$.

m dynamic

The dynamic of m involves gravitational force, rope tension, normal force, and (if applicable) the friction between its surface and the cylinder. A natural coordinate would be one that tangential and perpendicular to the cylinder surface. In this frame, assuming m touches the cylinder ($N > 0$), the equation of motion is

$$\begin{aligned} mR\ddot{\phi} &= T\cos\left(\frac{\pi}{4} - \phi/2\right) - mg\sin(\phi) - f \\ 0 &= N + T\sin\left(\frac{\pi}{4} - \phi/2\right) - mg\cos(\phi) \end{aligned}$$

where we treat f as a force in the $-\phi$ direction. In truth, f (if exists at all) can be in either two directions, depending on the mass relation between m and M.

Tangential motion

First thing we can do is to obtain the dynamics of ϕ exclusively. This can easily be done by solving the M dynamic for T and substituting it into the ϕ dynamics, resulting in

$$\begin{aligned} mR\ddot{\phi} &= M(g - \ddot{y})\cos\left(\frac{\pi}{4} - \phi/2\right) - mg\sin(\phi) - f \\ mR\ddot{\phi} &= Mg\cos\left(\frac{\pi}{4} - \phi/2\right) - MR\ddot{\phi}\cos^2\left(\frac{\pi}{4} - \phi/2\right) - \frac{1}{4}MR\dot{\phi}^2\cos(\phi) - mg\sin(\phi) - f \end{aligned}$$

Thus

$$\begin{aligned} \left(m + M\cos^2\left(\frac{\pi}{4} - \phi/2\right)\right)R\ddot{\phi} &= g\left(M\cos\left(\frac{\pi}{4} - \phi/2\right) - m\sin(\phi)\right) - \frac{1}{4}MR\dot{\phi}^2\cos(\phi) - f \\ \ddot{\phi} &= \frac{g}{R}\frac{M\cos\left(\frac{\pi}{4} - \phi/2\right) - m\sin(\phi)}{M\cos^2\left(\frac{\pi}{4} - \phi/2\right) + m} - \frac{1}{R}\frac{\frac{1}{4}MR\dot{\phi}^2\cos(\phi) + f}{M\cos^2\left(\frac{\pi}{4} - \phi/2\right) + m} \end{aligned}$$

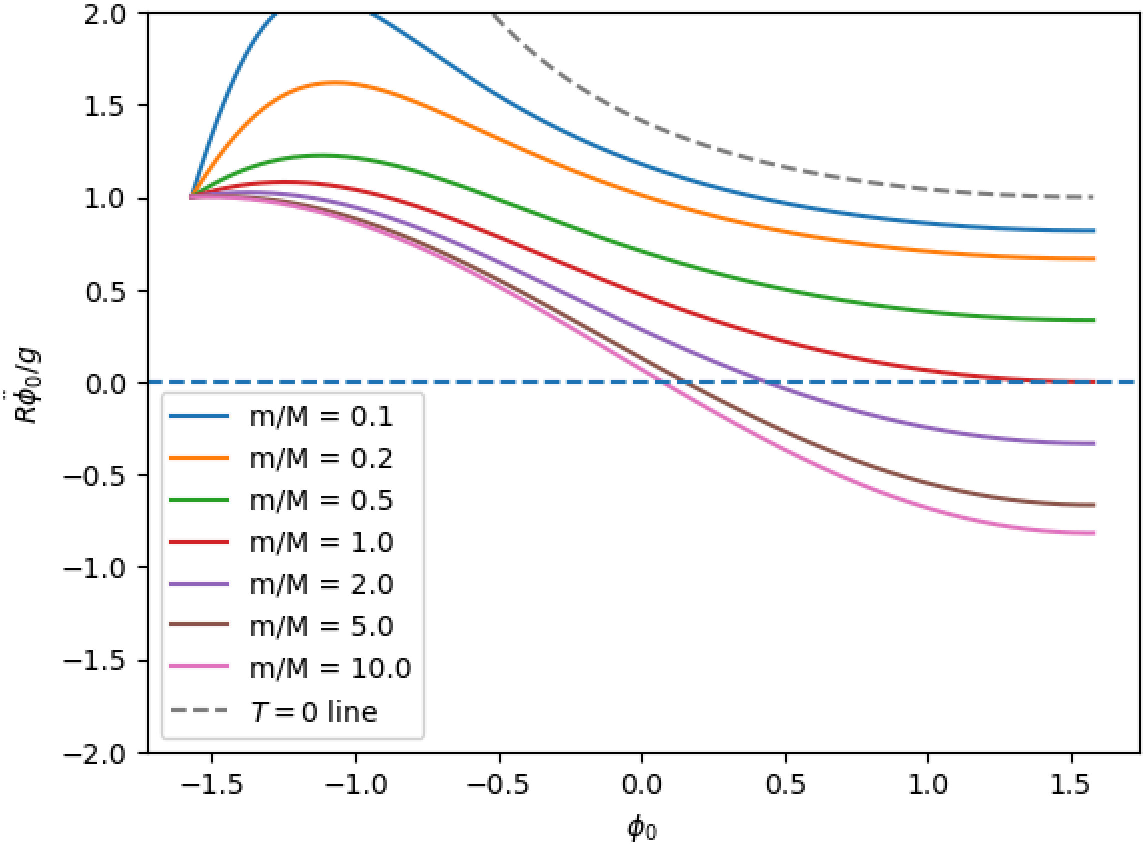
We can see here that at ϕ_0, where $\dot{\phi}_0 = 0$, the direction of motion of the system depends entirely on ϕ_0 and the mass relation between M and m. To let us see more clearly, here we show a plot of $R\ddot{\phi}/g$ for different m/M as a function of ϕ, assuming no friction.

In [2]:

```
import numpy as np
import matplotlib.pyplot as plt

phi = np.arange(-np.pi/2, np.pi/2 + 0.01, 0.01)
mpM = np.array([0.1, 0.2, 0.5, 1.0, 2.0, 5.0, 10.0])

for mM in mpM:
    num = (np.cos(np.pi/4 - phi/2) - mM * np.sin(phi))
    den = (np.cos(np.pi/4 - phi/2)**2 + mM)
    acc = num / den
    plt.plot(phi, acc, label = f'm/M = {mM}')
plt.plot(phi, 1 / np.cos(np.pi/4 - phi/2), label = r'$T = 0$ line', linestyle = 'dashed')
plt.legend()
plt.axhline(0, linestyle = 'dashed')
plt.xlabel(r'$\phi_0$')
plt.ylabel(r'$R\ddot{\phi}_0/g$')
plt.ylim(-2, 2)
plt.show()
```



We can see that m can only move in the negative ϕ direction if $m > M$, and this (reasonably) only happen for $\phi_0 > 0$. For $\phi_0 < 0$, the system will always move in the positive ϕ direction. Notice also that in every cases, we always have $T > 0$, ensuring the validity of our assumptions.

Can m be launched

In some parameter region, we could end up in the scenario where m will be ejected from the cylinder surface, and our assumption about tangential motion brokes. This happens when

$$N = mg\cos(\phi) - T\sin\left(\frac{\pi}{4} - \phi/2\right) < 0$$

We can substitute T from the tangential equation of motion, resulting in

$$\begin{aligned} N &= mg\cos(\phi) - T\sin\left(\frac{\pi}{4} - \phi/2\right) \\ N &= mg\cos(\phi) - mg\tan\left(\frac{\pi}{4} - \phi/2\right)\left[R\ddot{\phi}/g + \sin(\phi)\right] \\ \frac{N}{mg} &= \cos\phi - \tan\left(\frac{\pi}{4} - \phi/2\right)\left[R\ddot{\phi}/g + \sin(\phi)\right] \end{aligned}$$

We could make further observation that

$$
\begin{aligned}
\tan\left(\frac{\pi}{4} - \phi/2\right) &= \frac{\tan\left(\frac{\pi}{4}\right) - \tan(\phi/2)}{1 + \tan\left(\frac{\pi}{4}\right)\tan(\phi/2)} \\
&= \frac{1 - \tan(\phi/2)}{1 + \tan(\phi/2)} \\
&= \frac{\cos(\phi/2) - \sin(\phi/2)}{\cos(\phi/2) + \sin(\phi/2)} \\
&= \frac{\cos(\phi/2) - \sin(\phi/2)}{\cos(\phi/2) + \sin(\phi/2)} \frac{\cos(\phi/2) + \sin(\phi/2)}{\cos(\phi/2) + \sin(\phi/2)} \\
&= \frac{\cos^2(\phi/2) - \sin^2(\phi/2)}{\cos^2(\phi/2) + \sin^2(\phi/2) + 2\cos(\phi/2)\sin(\phi/2)} \\
&= \frac{\cos(\phi)}{1 + \sin(\phi)}
\end{aligned}
$$

leading to

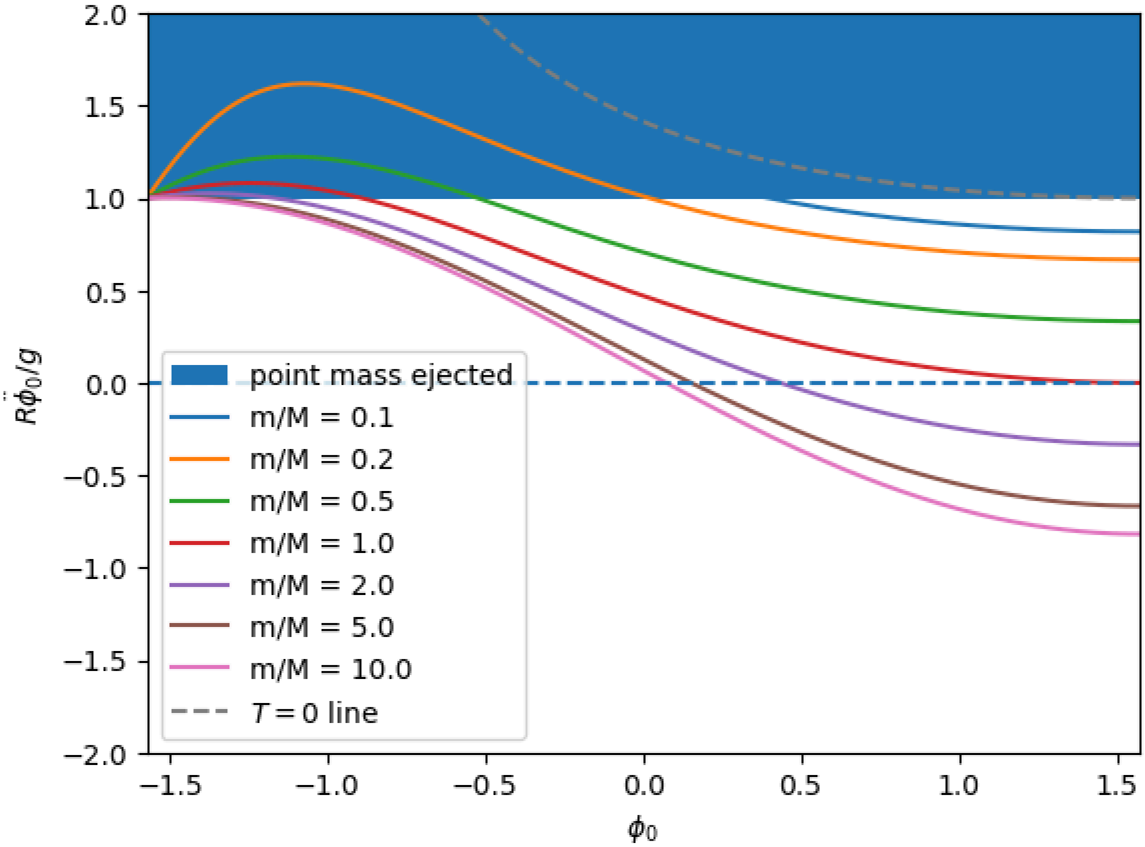
$$
\frac{N}{mg\cos(\phi)} = 1 - \frac{R\ddot{\phi}/g + \sin(\phi)}{1 + \sin(\phi)} \implies N = \frac{mg\cos(\phi)}{1 + \sin(\phi)}[1 - R\ddot{\phi}/g]
$$

The coefficient is always positive in the open range $(-\pi/2, \pi/2)$, which means that N will only be negative when $R\ddot{\phi}/g > 1$. Notice that this results is generic for any cases (as long as $T > 0$), meaning that even if m does not get ejected in the beginnning of the motion, it will still be ejected as long as $R\ddot{\phi}/g$ exceed 1.

We can combine this observation with the previous plot for $R\ddot{\phi}_0/g$, highlighting the region where m will spontaneously get ejected in the beginning of the motion.

In [3]:
```
phi = np.arange(-np.pi/2, np.pi/2 + 0.01, 0.01)
mpM = np.array([0.1, 0.2, 0.5, 1.0, 2.0, 5.0, 10.0])

plt.fill_between(phi, phi/phi, 2*phi/phi, label = f'point mass ejected')
for mM in mpM:
    num = (np.cos(np.pi/4 - phi/2) - mM * np.sin(phi))
    den = (np.cos(np.pi/4 - phi/2)**2 + mM)
    acc = num / den
    plt.plot(phi, acc, label = f'm/M = {mM}')
plt.plot(phi, 1 / np.cos(np.pi/4 - phi/2),
         label = r'$T = 0$ line', linestyle = 'dashed')
plt.legend()
plt.axhline(0, linestyle = 'dashed')
plt.xlabel(r'$\phi_0$')
plt.ylabel(r'$R\ddot{\phi}_0/g$')
plt.ylim(-2, 2)
plt.xlim(-np.pi/2, np.pi/2)
plt.show()
```



We can also see that the bigger M is compared to m, the smaller the range of ϕ_0 at which m will not be spontaneously ejected.