Codes used to type and visualise the answers can be found here

Question

We would like to measure the radius of the Earth. The `mass' of a ceramic mug was measured to be 469.90 grams on the 3rd floor of the North Building of ELTE's Lagymanyos campus in Budapest, and as 469.86 grams on the top of Janos Hill in Budapest using an exceptionally high resolution digital scale. Let us consider the shape of Janos Hill as a right cone with a half-opening angle of 45 degrees. What is the radius of the Earth and its experimental uncertainty?

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Answer

Assumption

Let us start by stating some reasonable assumptions that can be deduced from the question

1. The earth can be modeled as a perfect sphere with uniform (or at least, spherically symmetric) mass distribution. We could then assign a mass M_{earth} and radius R to this model of earth.

2. The digital scale measures the normal force exerted between its surface and the ceramic mug. This normal force should then be equivalent to the gravitational force between the earth system and the mug

$$F^{\text{meas}}(h) = \frac{GM_{\text{earth}}m_{\text{mug}}}{(R+h)^2} = m^{\text{meas}}_{\text{mug}}(h)g_{\text{const}} \implies m^{\text{meas}}_{\text{mug}}(h) = \frac{C}{(R+h)^2}$$

where h is the altitude relative to earth surface. In the practical sense, h can be measured from the sea level.

3. The earth radius is much bigger than either ELTE's building, the peak of the Janos Hill, and their difference.

Analysis

Starting from the above equation, we could perform Taylor expansion to the first order and obtain

$$m(h) \equiv m^{\text{meas}}_{\text{mug}}(h) = \frac{C}{R^2}\left(1+\frac{h}{R}\right)^{-2} = \frac{C}{R^2}\left(1-\frac{2h}{R}+\mathcal{O}\left(\frac{h^2}{R^2}\right)\right)$$

This will then lead to

$$\frac{m(h_1)-m(h_2)}{m(h_1)} = \frac{\frac{C}{R^2}\left(\frac{2(h_2-h_1)}{R}+\mathcal{O}\left(\frac{h^2}{R^2}\right)\right)}{\frac{C}{R^2}\left(1+\mathcal{O}\left(\frac{h}{R}\right)\right)} = 2\frac{h_2-h_1}{R}+\mathcal{O}\left(\frac{h^2}{R^2}\right)$$

where we have grouped both higher order terms containing h_1/R and h_2/R into h/R, owing to the assumption that $h_1 < h_2 \ll R$. Using the last equation, we could estimate the earth radius as

$$R \approx \frac{2[h_2-h_1]m(h_1)}{m(h_1)-m(h_2)} \tag{1}$$

Inserting Data

Now we need some real world data to estimate R. According to the google earth, the North building of ELTE's Lagymanyos campus is built on altitude $h = 110$ m, giving a reasonable estimate of its third floor located at $h_1 = 125$ m, while the top of the Janos hill is located at $h_2 = 530$ m. These data gives

In [1]:
```
# All data in SI
m1 = 469.90e-3
m2 = 469.86e-3
h1 = 125
h2 = 530

R = 2 * (h2 - h1) * m1 / (m1 - m2)
print(f'Earth radius estimate = {R:.0f} m = {R/1000:.1f} km')
```

```
Earth radius estimate = 9515475 m = 9515.5 km
```

Next, we need to account for the uncertainties. There are two sources of uncertainty here:

1. The systematic uncertainty that arise from ignoring higher order terms in the Taylor expansion. However, this uncertainty is in order of h/R and would be shown to be reasonably negligible.
2. The measurement uncertainty, that could arise either from the mass or altitude measurement error. We could separate both uncertainties as coming from $(h_2 - h_1)$ and $m(h_1)/(m(h_1) - m(h_2))$ terms.

Let us assume that the altitude measurement using google earth would give $\Delta h = 1$ m. This means that the altitude measurement will give rise to around

In [2]:
```
dh = 1
rh = (dh + dh) / (h2 - h1)
print(f"Uncertainty from measurement altitude = {100 * rh:.3f} %")
print(f"Equivalent to dR = {(R/1000) * rh:.3f} km")
```

```
Uncertainty from measurement altitude = 0.494 %
Equivalent to dR = 46.990 km
```

which is quite small relative to the estimate. Meanwhile, we can assume that the scale must have come with $\Delta m = 0.005$ g, as its smallest scale seems to be within the second decimal place in grams. This leads to the measurement uncertainty of

In [3]:
```
dm = 0.005e-3
rm = (dm + dm)/abs(m1 - m2) + dm/m1

print(f"Uncertainty from mass measurement = {100 * rm:.3f} %")
print(f"Equivalent to dR = {(R/1000) * rm:.3f} km")
```

```
Uncertainty from mass measurement = 25.001 %
Equivalent to dR = 2378.970 km
```

We can see that the uncertainty mostly come from the $m(h_1) - m(h_2)$ term, which is reasonable because both terms are very close to each other. We can further conclude that all uncertainties are dominated by one single term, and a reasonable estimate of the earth radius from this measurement would then be

In [4]:
```
print(f"Earth Radius Estimate = ({R/1000:.0f} +- {(R/1000) * rm:.0f}) km")
```

```
Earth Radius Estimate = (9515 +- 2379) km
```

The actual earth radius is $6370 - 6378$ km (depending on whether we measure it in the polar or equatorial side), which is quite close to the lower bound of the measurement estimate.