Question

Two bodies of mass m can move along a straight line, and the interaction is given by a pair potential $V(x_1 - x_2)$. Let us assume the symmetry $V(x) = V(-x)$. The potential is unknown, and we intend to obtain information about it, using a thought experiment which is motivated by particle physics. We initialize the two particles far apart, with velocities v and $-v$. We don't have an apparatus to follow the entire motion, but we can measure the trajectories of the two particles after a long time. Let us assume that the two masses don't stick together, so the two particles either pass each other or there is a full reflection. If there was no interaction, then the trajectory of one of the particles could be described in a suitable coordinate system with the equation $x = vt$. In the case of the interaction let us assume that a long time before scattering, the trajectory is $x = vt$ to a good approximation. After scattering we find $x = vt + d$ to a good approximation. What can the sign of the shift d tell us about whether the potential is attractive or repulsive? Given the velocity dependence of the displacement d, how can we deduce the shape of the potential? Let us examine some simple cases!

Answer

One equation that we can use to constraint is the conservation of energy

$$E = \frac{1}{2}v_1^2 + \frac{1}{2}v_2^2 + \Phi(x_1 - x_2)$$

Assuming the centre of mass of this system is at $x = 0$, we can conclude that $v_1 = v_2 = v$ and $x_1 - x_2 = 2x_1 = 2x$, leading to a simpler equation

$$E = v^2 + V(x)$$

where we have defined $\Phi(2x) = V(x)$.

Logical constraint

Knowing that far before scattering, $v = v_0$ (Let us use v for the velocity and v_0 for the initial velocity), we can conclude that $\lim_{x \to \infty} V(x) = 0$. There are several potential form that can satisfy this requirement, such as $V(x) = 1/|x|, 1/x^2, \exp(-x^2)$, and so on. Knowing that $v \to v_0$ as $V(x) \to 0$, we can write the energy equation as

$$v_0^2 = v^2 + V(x) \implies v^2 = v_0^2 - V(x)$$

This gives a very useful insight. An **attractive** potential ($V(x) = -|V(x)|$) will make v^2 always bigger than v_0, which should correspond to **positive** d in $x = v_0 t + d$. On the other hand, the repulsive potential is a bit tricky since we can have $v^2 = 0$, in which the particle may just go in the opposite direction afterwards. However, we can confidently make a remark that for **repulsive** potential with $V(0) < v_0^2$ (possible for the like of $\exp(-x^2)$), we will always have $v < v_0$ and thus correspond to **negative** d.

Analytic

Analytically, considering only the 'safe' potential, we will have

$$\frac{dx}{dt} = v = \sqrt{v_0^2 - V(x)} \implies \int_{x_0}^{x} \frac{dx'}{\sqrt{1 - V(x')/v_0^2}} = \int_0^t v_0 \, dt'$$

We could numerically integrate both sides to obtain a definite expression of $x = v_0 t + d$.

Approximation

Let us consider the case where $V(0) \ll v_0^2$. The integral can be approximated as

$$\int_{x_0}^{x} dx' \left[1 + \frac{V(x')}{2v_0^2} \right] = v_0 t \implies x - x_0 = v_0 t - \frac{1}{2v_0^2} \int_{x_0}^{x} dx' \, V(x')$$

This is incredible, as we get to know that

1. d is inversely proportional to v_0^2
2. We can obtain the shape of the potential by taking the derivative of d with respect to x, as $d'(x) = -\frac{V(x)}{2v_0^2}$.

Numerical test

Let us simulate the motion for several example of potential. In this case, $dv/dt = -V'(x)/2 = F(x)$

In [28]:
```python
import numpy as np

def Euler(x0, v0, times, acc):
    xs = np.zeros_like(times)
    vs = np.zeros_like(times)
    xs[0] = x0 + 0
    vs[0] = v0 + 0
    i = 0
    for t0, t1 in zip(times[:-1], times[1:]):
        i += 1
        dt = t1-t0
        xs[i] = xs[i-1] + vs[i-1] * dt
        vs[i] = vs[i-1] + acc(xs[i]) * dt
    return xs, vs
```

In [29]:
```python
func = [lambda x: 2*x*np.exp(-x**2),
        lambda x: -2*x*np.exp(-x**2),
        lambda x: 0]
v0s = np.array([5, 10, 20, 40])
x0 = -100
times = np.arange(0, 40, 0.01)

X = np.zeros((3, 4, len(times)))
V = np.zeros((3, 4, len(times)))

for i, f in enumerate(func):
    for j, v0 in enumerate(v0s):
        xs, vs = Euler(x0, v0, times, f)
        X[i][j] = xs + 0
        V[i][j] = vs + 0
```

In [30]:
```python
import matplotlib.pyplot as plt

fig, ax = plt.subplots(nrows = 1, ncols = 3, figsize = (18, 4))
```
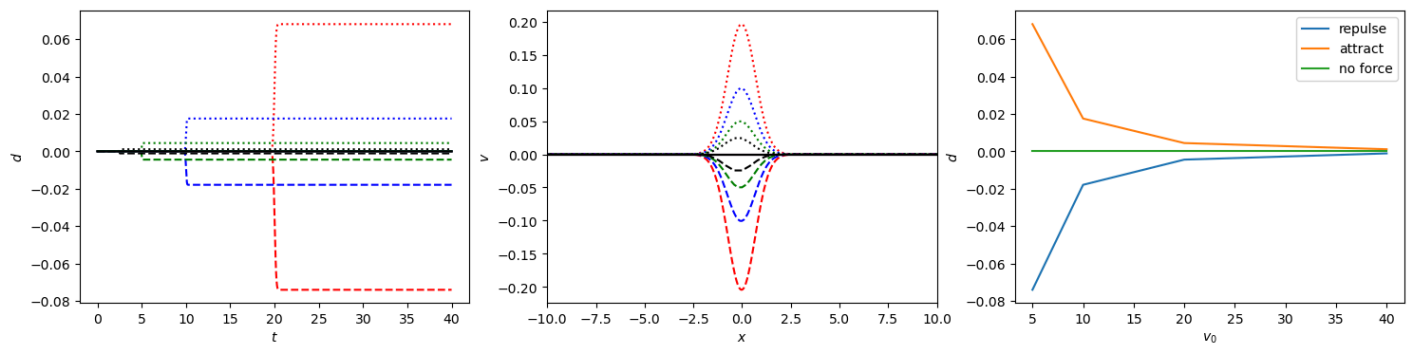
```
ax = ax.flatten()

line = ['dashed', 'dotted', 'solid']
name = ['repulse', 'attract', 'no force']
clrs = ['r', 'b', 'g', 'k']

for i, f in enumerate(func):
    for j, v0 in enumerate(v0s):
        ax[0].plot(times, X[i][j]-x0-v0*times, linestyle = line[i],
                    color = clrs[j], label = f'{name[i]}, v0 = {v0}')
        ax[1].plot(X[i][j], V[i][j]-v0, linestyle = line[i],
                    color = clrs[j], label = f'{name[i]}, v0 = {v0}')
    ax[2].plot(v0s, X[i,:,-1]-x0-v0s*times[-1], label = f'{name[i]}')
ax[1].set_xlim(-10, 10)
ax[0].set_xlabel(r'$t$')
ax[0].set_ylabel(r'$d$')
ax[1].set_xlabel(r'$x$')
ax[1].set_ylabel(r'$v$')
ax[2].set_xlabel(r'$v_0$')
ax[2].set_ylabel(r'$d$')
ax[2].legend()
plt.show()
```


We have shown in the plot above that

1. repulsive potential (in this case $V(x) = 2\exp(-x^2) = \exp(-x^2) \implies F(x) = 2x\exp(-x^2)$)
   produce negative d, while the attractive one ($V(x) = -2\exp(-x^2) \implies F(x) = -2x\exp(-x^2)$)
   produce positive d.
2. For our choice of initial velocity (5 to 40), we get symmetric $v(x)$ between the attractive and repulsive
   force, representing the equation $v = v_0 - \dfrac{V(x)}{2v_0}$. This might not be the case for smaller value of v_0.
3. In the third plot, albeit a bit ambiguously, we can see that $d \sim 1/v_0^2$.

Can we obtain V(x) from d?

Now let us see if we can truly obtain $V(x)$ from d data throughout the run. Note that the part of data that is
most important is around $x = 0$, as that is where $V(x)$ influence the velocity the most.
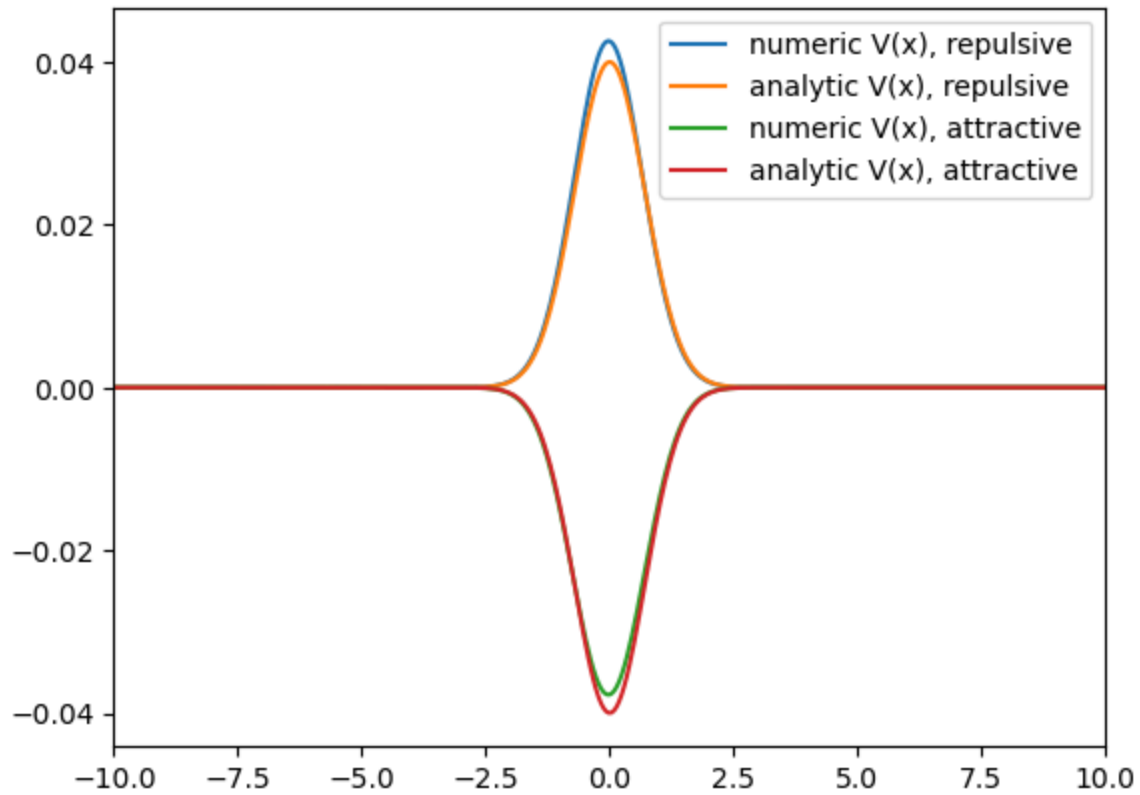
```
In [31]:  der = -((X[0,0,1:]-v0s[0]*times[1:])-(X[0,0,:-1]-v0s[0]*times[:-1]))
          der = der / (X[0,0,1:] - X[0,0,:-1])
          plt.plot(X[0,0,:-1], der, label = 'numeric V(x), repulsive')
          plt.plot(X[0,0], 2 * np.exp(-X[0,0]**2) / (2 * (v0s[0]**2)),
                  label = 'analytic V(x), repulsive')

          der = -((X[1,0,1:]-v0s[0]*times[1:])-(X[1,0,:-1]-v0s[0]*times[:-1]))
          der = der / (X[1,0,1:] - X[1,0,:-1])
          plt.plot(X[1,0,:-1], der, label = 'numeric V(x), attractive')
          plt.plot(X[1,0], -2 * np.exp(-X[1,0]**2) / (2 * (v0s[0]**2)),
```

```
                label = 'analytic V(x), attractive')

plt.legend()
plt.xlim(-10, 10)
plt.show()
```


We obtain an astonishingly accurate potential!