Matrix Algebra

Matrix Operations

EXAMPLE If A is a 3×5 matrix and B is a 5×2 matrix, what are the sizes of AB and BA, if they are defined?

SOLUTION Since A has 5 columns and B has 5 rows, the product AB is defined and is a 3×2 matrix:

$$
\begin{array}{ccccc}
A & & B & & AB \\
\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} & &
\begin{bmatrix} * & * \\ * & * \\ * & * \\ * & * \\ * & * \end{bmatrix} & = &
\begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix} \\
3 \times 5 & & 5 \times 2 & & 3 \times 2
\end{array}
$$

Match

Size of AB

The product BA is *not* defined because the 2 columns of B do not match the 3 rows of A.

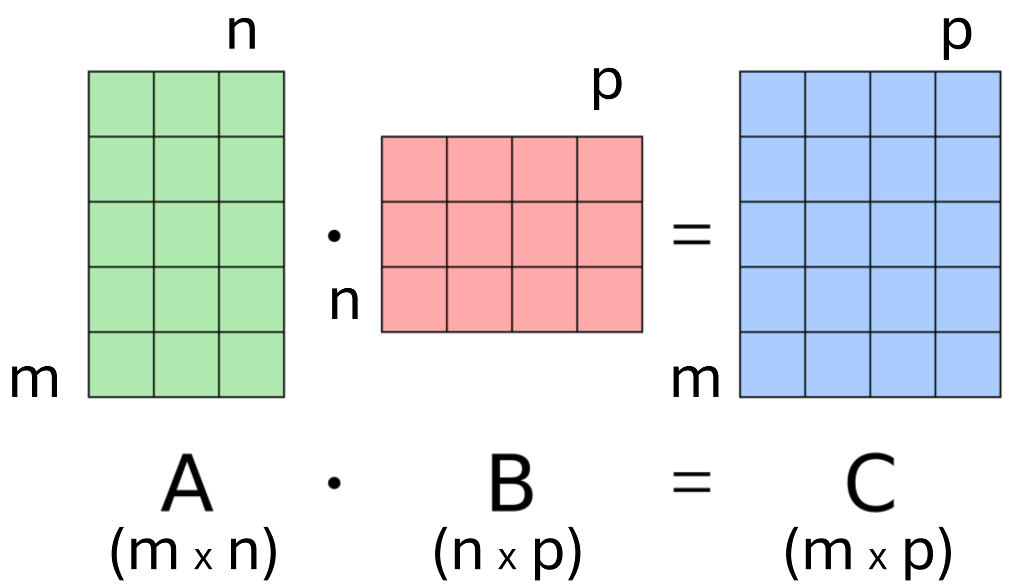
Matrix Operations

Matrix Multiplication

If A is an m × n matrix and B is an n × p matrix, the matrix product C = AB (denoted without multiplication signs or dots) is defined to be the m × p matrix.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{pmatrix}$$



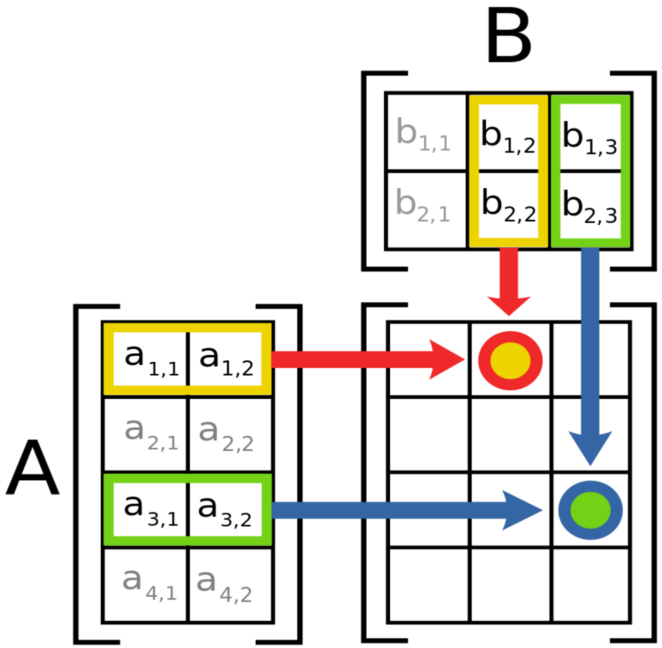
Such that $\boxed{c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj},}$ for i = 1, ..., m and j = 1, ..., p.

Matrix Operations

Matrix Multiplication

Example:

$$
\overset{4\times 2 \text{ matrix}}{\begin{bmatrix} a_{11} & a_{12} \\ \cdot & \cdot \\ a_{31} & a_{32} \\ \cdot & \cdot \end{bmatrix}}
\overset{2\times 3 \text{ matrix}}{\begin{bmatrix} \cdot & b_{12} & b_{13} \\ \cdot & b_{22} & b_{23} \end{bmatrix}}
=
\overset{4\times 3 \text{ matrix}}{\begin{bmatrix} \cdot & c_{12} & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & c_{33} \\ \cdot & \cdot & \cdot \end{bmatrix}}
$$



$$c_{12} = a_{11}b_{12} + a_{12}b_{22}$$
$$c_{33} = a_{31}b_{13} + a_{32}b_{23}$$

Matrix Operations

EXAMPLE Find the entries in the second row of AB, where

$$A = \begin{bmatrix} 2 & -5 & 0 \\ -1 & 3 & -4 \\ 6 & -8 & -7 \\ -3 & 0 & 9 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & -6 \\ 7 & 1 \\ 3 & 2 \end{bmatrix}$$

SOLUTION By the row–column rule, the entries of the second row of AB come from row 2 of A (and the columns of B):

$$\begin{bmatrix} 2 & -5 & 0 \\ -1 & 3 & -4 \\ 6 & -8 & -7 \\ -3 & 0 & 9 \end{bmatrix} \begin{bmatrix} 4 & -6 \\ 7 & 1 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \square & \square \\ -4+21-12 & 6+3-8 \\ \square & \square \\ \square & \square \end{bmatrix} = \begin{bmatrix} \square & \square \\ 5 & 1 \\ \square & \square \\ \square & \square \end{bmatrix}$$

Matrix Operations

Practice Problem

Q. If $A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & -2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 4 & -2 \end{bmatrix}$ BA if possible.

Inverse of a Matrix

Introduction

Recall that the multiplicative inverse of a number such as 5 is 1/5 or 5^{-1}. This inverse satisfies the equations

$$5^{-1} \cdot 5 = 1 \quad \text{and} \quad 5 \cdot 5^{-1} = 1$$

For finding the inverse of a matrix, we need

- Both equations must be true as matrix multiplication is not commutative (i.e., $AB \neq BA$).
- Also, the matrices involved must be square

Inverse of a Matrix

Definition

An $n \times n$ matrix A is said to be **invertible** if there is an $n \times n$ matrix C such that

$$CA = I \quad \text{and} \quad AC = I$$

where $I = I_n$, the $n \times n$ identity matrix. In this case, C is an **inverse** of A. In fact, C is uniquely determined by A, because if B were another inverse of A, then $B = BI = B(AC) = (BA)C = IC = C$. This unique inverse is denoted by A^{-1}, so that

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I$$

A matrix that is *not* invertible is sometimes called a **singular matrix**, and an invertible matrix is called a **nonsingular matrix**.

Matrix Operations

Multiplicative Inverse of a Matrix

- For a square matrix A, the inverse is written A^{-1}.
- When A is multiplied by A^{-1} the result is the identity matrix I.
- Non-square matrices do not have inverses.

Example:

For matrix $A = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$, its inverse is $A^{-1} = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}$ since $AA^{-1} = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Note: Not all square matrices have inverses. A square matrix which has an inverse is called invertible or nonsingular, and a square matrix without an inverse is called noninvertible or singular.

Matrix Operations

Practice Problem

Q. Find the Value of x and y; given,

$$2\begin{bmatrix}1 & 3\\0 & x\end{bmatrix}+\begin{bmatrix}y & 0\\1 & 2\end{bmatrix}=\begin{bmatrix}5 & 6\\1 & 8\end{bmatrix}$$

Steps

Step 1: Find the determinant of the given matrix, say A.

Step 2: Find the cofactor matrix $C_{ij} = (-1)^{i+j} \det(M_{ij})$, where M_{ij} is the (i,j)th minor matrix after removing the ith row and the jth column.

Step 3: Find the transpose of the cofactor matrix to get the adj A.

Step 4: $A^{-1} = \text{adj } A/\det(A)$

Matrix Operations

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Note: Not all square matrices have inverses. A square matrix which has an inverse is called invertible or nonsingular, and a square matrix without an inverse is called noninvertible or singular.

Find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

Solution:

Given,

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

Let us find the determinant of A.

$$|A| = \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 2 \times 2 - 3 \times 1 = 4 - 3 = 1$$

Here, |A| ≠ 0, so the inverse of A exists.

$$adjA = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

Now, A^{-1} = adjA/|A|

Therefore,

$$A^{-1} = \frac{1}{1} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

Calculate the inverse of the matrix

$$A = \begin{bmatrix} 2 & 4 & -6 \\ 7 & 3 & 5 \\ 1 & -2 & 4 \end{bmatrix}$$

Solution:

Given,

$$A = \begin{bmatrix} 2 & 4 & -6 \\ 7 & 3 & 5 \\ 1 & -2 & 4 \end{bmatrix}$$

First, find the determinant of matrix A.

$$A = \begin{vmatrix} 2 & 4 & -6 \\ 7 & 3 & 5 \\ 1 & -2 & 4 \end{vmatrix}$$

$= 2(12 + 10) - 4(28 - 5) - 6(-14 - 3)$

$= 2(22) - 4(23) - 6(-17)$

$= 44 - 92 + 102$

$= 54 \neq 0$

Thus, the inverse matrix exists.

$$M_{11} = \det\begin{pmatrix} 3 & 5 \\ -2 & 4 \end{pmatrix} = 22$$

$$M_{12} = \det\begin{pmatrix} 7 & 5 \\ 1 & 4 \end{pmatrix} = 23$$

$$M_{13} = \det\begin{pmatrix} 7 & 3 \\ 1 & -2 \end{pmatrix} = -17$$

$$M_{21} = \det\begin{pmatrix} 4 & -6 \\ -2 & 4 \end{pmatrix} = 4$$

$$M_{22} = \det\begin{pmatrix} 2 & -6 \\ 1 & 4 \end{pmatrix} = 14$$

$$M_{23} = \det\begin{pmatrix} 2 & 4 \\ 1 & -2 \end{pmatrix} = -8$$

$$M_{31} = \det\begin{pmatrix} 4 & -6 \\ 3 & 5 \end{pmatrix} = 38$$

$$M_{32} = \det\begin{pmatrix} 2 & -6 \\ 7 & 5 \end{pmatrix} = 52$$

$$M_{33} = \det\begin{pmatrix} 2 & 4 \\ 7 & 3 \end{pmatrix} = -22$$

Thus, the minor matrix of A

$$= \begin{bmatrix} 22 & 23 & -17 \\ 4 & 14 & -8 \\ 38 & 52 & -22 \end{bmatrix}$$

Cofactor matrix of A

$$= \begin{bmatrix} 22 & -23 & -17 \\ -4 & 14 & 8 \\ 38 & -52 & -22 \end{bmatrix}$$

Also, adjA

$$= \begin{bmatrix} 22 & -4 & 38 \\ -23 & 14 & -52 \\ -17 & 8 & -22 \end{bmatrix}$$

Therefore, A^{-1} = adjA/|A|

$$A^{-1} = \frac{1}{54} \begin{bmatrix} 22 & -4 & 38 \\ -23 & 14 & -52 \\ -17 & 8 & -22 \end{bmatrix} = \begin{bmatrix} \frac{11}{27} & -\frac{2}{27} & \frac{19}{27} \\ -\frac{23}{54} & \frac{7}{27} & -\frac{26}{27} \\ -\frac{17}{54} & \frac{4}{27} & -\frac{11}{27} \end{bmatrix}$$

If

$$A = \begin{bmatrix} 2 & 1 \\ 7 & 2 \end{bmatrix}$$

, show that $(A^{-1})^{-1} = A$.

Solution:

Given,

$$A = \begin{bmatrix} 2 & 1 \\ 7 & 2 \end{bmatrix}$$

Now,

$$|A| = \begin{vmatrix} 2 & 1 \\ 7 & 2 \end{vmatrix} = 4 - 7 = -3$$

Here, matrix A is non-singular.

$$adjA = \begin{bmatrix} 2 & -1 \\ -7 & 2 \end{bmatrix}$$

Thus,

$$A^{-1} = \frac{adjA}{|A|} = \frac{1}{-3}\begin{bmatrix} 2 & -1 \\ -7 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{7}{3} & -\frac{2}{3} \end{bmatrix}$$

Let $A^{-1} = B$

So,

$$B = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{7}{3} & -\frac{2}{3} \end{bmatrix}$$

And

$$|B| = \begin{vmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{7}{3} & -\frac{2}{3} \end{vmatrix}$$

$= (-2/3)(-2/3) - (1/3)(7/3)$

$= (4/9) - (7/9)$

$= (4 - 7)/9$

$= -3/9$

$= -1/3$

Now,

$$adjB = \begin{bmatrix} -\frac{2}{3} & -\frac{1}{3} \\ -\frac{7}{3} & -\frac{2}{3} \end{bmatrix}$$

Also,

$$B^{-1} = \frac{adjB}{|B|} = \frac{1}{-\frac{1}{3}}\begin{bmatrix} -\frac{2}{3} & -\frac{1}{3} \\ -\frac{7}{3} & -\frac{2}{3} \end{bmatrix} = -3\begin{bmatrix} -\frac{2}{3} & -\frac{1}{3} \\ -\frac{7}{3} & -\frac{2}{3} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 7 & 2 \end{bmatrix}$$

That means $B^{-1} = (A^{-1})^{-1} = A$

Inverse Matrix

Find x, y, z if

$$A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$$

satisfies $A^T = A^{-1}$.

Solution:

Given,

$$A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$$

$A^T = A^{-1}$

$\Rightarrow AA^T = AA^{-1}$

$\Rightarrow AA^T = I$ {since $A^{-1}A = AA^{-1} = I$}

Now,

$$A^T = \begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix}$$

$AA^T = I$

$$\begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix} \begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By performing multiplication on the LHS, we get:

$$\begin{bmatrix} 4y^2 + z^2 & 2y^2 - z^2 & -2y^2 + z^2 \\ 2y^2 - z^2 & x^2 + y^2 + z^2 & x^2 - y^2 - z^2 \\ -2y^2 + z^2 & x^2 - y^2 + z^2 & x^2 + y^2 + z^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By equating the corresponding elements, we have:

$4y^2 + z^2 = 1$ ...(1)

$x^2 + y^2 + z^2 = 1$ ...(2)

$2y^2 - z^2 = 0$ ...(3)

Adding equations (1) and (3), we get:

$4y^2 + z^2 + 2y^2 - z^2 = 1 + 0$

$6y^2 = 1$

$y^2 = 1/6$

$\Rightarrow y = \pm 1/\sqrt{6}$

Substituting the value of y in equation (3), we get:

$z^2 = 2y^2$

$z^2 = 2(1/6)$

$z^2 = 1/3$

$\Rightarrow z = \pm 1/\sqrt{3}$

Substituting the values of y and z in equation (2), we get:

$x^2 = 1 - y^2 - z^2$

$x^2 = 1 - (1/6) - (1/3)$

$x^2 = (6 - 1 - 2)/6$

$x^2 = 3/6$

$x^2 = 1/2$

$\Rightarrow x = \pm 1/\sqrt{2}$

Therefore, $x = \pm 1/\sqrt{2}$, $y = \pm 1/\sqrt{6}$ and $z = \pm 1/\sqrt{3}$.

Matrix Operations

Properties of Matrix Multiplication

The following theorem lists the standard properties of matrix multiplication. Recall that I_m represents the $m \times m$ identity matrix and $I_m\mathbf{x} = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^m$.

Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

a. $A(BC) = (AB)C$ (associative law of multiplication)

b. $A(B + C) = AB + AC$ (left distributive law)

c. $(B + C)A = BA + CA$ (right distributive law)

d. $r(AB) = (rA)B = A(rB)$ for any scalar r

e. $I_mA = A = AI_n$ (identity for matrix multiplication)

Matrix Operations

Powers of a Matrix

If A is an $n \times n$ matrix and if k is a positive integer, then A^k denotes the product of k copies of A:

$$A^k = \underbrace{A \cdots A}_{k}$$

Transpose of a Matrix

Given an $m \times n$ matrix A, the **transpose** of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

Matrix Operations

EXAMPLE Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} -5 & 2 \\ 1 & -3 \\ 0 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -3 & 5 & -2 & 7 \end{bmatrix}$$

Then

$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \quad B^T = \begin{bmatrix} -5 & 1 & 0 \\ 2 & -3 & 4 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 & -3 \\ 1 & 5 \\ 1 & -2 \\ 1 & 7 \end{bmatrix}$$

Matrix Operations

Properties involving Transpose of a Matrix

Let A and B denote matrices whose sizes are appropriate for the following sums and products.

a. $(A^T)^T = A$

b. $(A + B)^T = A^T + B^T$

c. For any scalar r, $(rA)^T = rA^T$

d. $(AB)^T = B^T A^T$

The Matrix Equation

Example

1. Consider a system of linear equations,

$$\begin{aligned} 3x_1 - 2x_2 &= 1 \\ -x_1 + 4x_2 &= 3 \end{aligned}$$

2. We can represent this as an Augmented matrix

$$\begin{bmatrix} 3 & -2 & 1 \\ -1 & 4 & 3 \end{bmatrix}$$

3. We can represent this the Matrix Equation form, A**x** = **b**,

$$\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

The Matrix Equation

Solve system of linear equations,

$$3x_1 - 2x_2 = 1$$
$$-x_1 + 4x_2 = 3$$

https://www.desmos.com/calculator

Vector Plot