Matrix

The Matrix Equation Ax=b.

homogenious -> x^2+xy=1 -> homo; x+y -> h, x^2+y=nh

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbb{R}^n$, then the **product of A and $\mathbf{x}$**, denoted by $A\mathbf{x}$, is **the linear combination of the columns of A using the corresponding entries in $\mathbf{x}$ as weights**; that is,

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

Note that A**x** is defined only if the number of columns of A equals the number of entries in **x**.

Augmented matrix

$$\left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ & \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{array}\right] = [A \mid b]$$

Coefficient matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ & \vdots & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} = A$$

Ax=b

EXAMPLE For $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in $\mathbb{R}^m$, write the linear combination $3\mathbf{v}_1 - 5\mathbf{v}_2 + 7\mathbf{v}_3$ as a matrix times a vector.

A vector in R^m is simply a list of **m real numbers**, represented as a column or row of numbers.

In $\mathbb{R}^2$, vectors have 2 components (e.g., (x_1, x_2)).

In $\mathbb{R}^3$, vectors have 3 components (e.g., (x_1, x_2, x_3)).

In $\mathbb{R}^m$, vectors have **m components** (e.g., $(x_1, x_2, \ldots, x_m)$).

SOLUTION Place $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ into the columns of a matrix A and place the weights $3, -5$, and 7 into a vector $\mathbf{x}$. That is,

$$3\mathbf{v}_1 - 5\mathbf{v}_2 + 7\mathbf{v}_3 = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix} = A\mathbf{x}$$ ■

linear equations as a vector equation involving a linear combination of vectors. For example, the system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 4 \\ -5x_2 + 3x_3 &= 1 \end{aligned} \tag{1}$$

is equivalent to

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \tag{2}$$

the linear combination on the left side is a matrix times a vector, so that (2) becomes

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \tag{3}$$

Equation (3) has the form $A\mathbf{x} = \mathbf{b}$. Such an equation is called a **matrix equation**, to distinguish it from a vector equation such as is shown in (2).

Let $A = \begin{bmatrix} 1 & 5 & -2 & 0 \\ -3 & 1 & 9 & -5 \\ 4 & -8 & -1 & 7 \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} 3 \\ -2 \\ 0 \\ -4 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -7 \\ 9 \\ 0 \end{bmatrix}$. It can be shown that $\mathbf{p}$ is a solution of $A\mathbf{x} = \mathbf{b}$. Use this fact to exhibit $\mathbf{b}$ as a specific linear combination of the columns of A.

$$\begin{bmatrix} 1 & 5 & -2 & 0 \\ -3 & 1 & 9 & -5 \\ 4 & -8 & -1 & 7 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \\ 0 \\ -4 \end{bmatrix} = \begin{bmatrix} -7 \\ 9 \\ 0 \end{bmatrix}$$

is equivalent to the vector equation

$$3\begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} - 2\begin{bmatrix} 5 \\ 1 \\ -8 \end{bmatrix} + 0\begin{bmatrix} -2 \\ 9 \\ -1 \end{bmatrix} - 4\begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix} = \begin{bmatrix} -7 \\ 9 \\ 0 \end{bmatrix}$$

which expresses $\mathbf{b}$ as a linear combination of the columns of A.

Homogeneous Linear Systems

A system of linear equations is said to be **homogeneous** if it can be written in the form $A\mathbf{x} = \mathbf{0}$, where A is an $m \times n$ matrix and $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$. Such a system $A\mathbf{x} = \mathbf{0}$ *always* has at least one solution, namely, $\mathbf{x} = \mathbf{0}$ (the zero vector in $\mathbb{R}^n$). This

Zero solution is usually called the **trivial solution**.

consistent sol -> 1 or many
non " -> 0 sol

The homogeneous equation A**x** = **0** has a nontrivial solution if and only if the equation has at least one free variable.

$\boxed{0 \ \ |A| \ 0 \ \ \text{-> trivial}}$

$$a_1x + b_1y + c_1z = 0$$

$$a_2x + b_2y + c_2z = 0$$

$$a_3x + b_3y + c_3z = 0$$

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

Non-trivial if $|A| = 0$ -> tricky

Trivial $|A| \neq 0$ $x, y, z = 0$ -> easy

Solve a system of linear equations (consistent system)

$$\begin{array}{rcrcrcrl} x & - & 2y & + & 3z & = & 9 & \quad (1) \\ -x & + & 3y & & & = & -4 & \quad (2) \\ 2x & - & 5y & + & 5z & = & 17 & \quad (3) \end{array}$$

Sol: $(1)+(2)\rightarrow(2)$

$$\begin{array}{rcrcrcrl} x & - & 2y & + & 3z & = & 9 & \\ & & y & + & 3z & = & 5 & \quad (4) \\ 2x & - & 5y & + & 5z & = & 17 & \end{array}$$

$(1)\times(-2)+(3)\rightarrow(3)$

$$\begin{array}{rcrcrcrl} x & - & 2y & + & 3z & = & 9 & \\ & & y & + & 3z & = & 5 & \\ & & -y & - & z & = & -1 & \quad (5) \end{array}$$

$(4)+(5)\rightarrow(5)$

$$\begin{array}{rcrcrcr} x & - & 2y & + & 3z & = & 9 \\ & & y & + & 3z & = & 5 \\ & & & & 2z & = & 4 \end{array} \qquad (6)$$

$(6)\times\frac{1}{2}\rightarrow(6)$

$$\begin{array}{rcrcrcr} x & - & 2y & + & 3z & = & 9 \\ & & y & + & 3z & = & 5 \\ & & & & z & = & 2 \end{array}$$

So the solution is $x=1,\ y=-1,\ z=2$ (only one solution)

Solve a system of linear equations (inconsistent system)

$$
\begin{aligned}
x_1 - 3x_2 + x_3 &= 1 \qquad (1)\\
2x_1 - x_2 - 2x_3 &= 2 \qquad (2)\\
x_1 + 2x_2 - 3x_3 &= -1 \qquad (3)
\end{aligned}
$$

Sol:

$$(1)\times(-2)+(2)\to(2)$$

$$(1)\times(-1)+(3)\to(3)$$

$$
\begin{aligned}
x_1 - 3x_2 + x_3 &= 1\\
5x_2 - 4x_3 &= 0 \qquad (4)\\
5x_2 - 4x_3 &= -2 \qquad (5)
\end{aligned}
$$

$$(4)\times(-1)+(5)\rightarrow(5)$$

$$\begin{array}{rcrcrcr} x_1 & - & 3x_2 & + & x_3 & = & 1 \\ & & 5x_2 & - & 4x_3 & = & 0 \\ & & & & \boxed{0} & \boxed{=} & \boxed{-2} \end{array}$$

(a false statement)

So the system has no solution (an inconsistent system).

Solve a system of linear equations (infinitely many solutions)

-> all 3 line over lapping

$$\begin{array}{rrrrrrl} & & x_2 & - & x_3 & = & 0 \qquad (1) \\ x_1 & & & - & 3x_3 & = & -1 \qquad (2) \\ -x_1 & + & 3x_2 & & & = & 1 \qquad (3) \end{array}$$

Sol: $(1) \leftrightarrow (2)$

$$\begin{array}{rrrrrrl} x_1 & & & - & 3x_3 & = & -1 \qquad (1) \\ & & x_2 & - & x_3 & = & 0 \qquad (2) \\ -x_1 & + & 3x_2 & & & = & 1 \qquad (3) \end{array}$$

$(1) + (3) \rightarrow (3)$

$$\begin{array}{rrrrrrl} x_1 & & & - & 3x_3 & = & -1 \\ & & x_2 & - & x_3 & = & 0 \\ & & 3x_2 & - & 3x_3 & = & 0 \qquad (4) \end{array}$$

$$\begin{array}{cccccc} x_1 & & - & 3x_3 & = & -1 \\ & x_2 & - & x_3 & = & 0 \end{array}$$

$$\Rightarrow x_2 = x_3, \quad x_1 = -1 + 3x_3$$

let $x_3 = t$

then
$$x_1 = 3t - 1,$$
$$x_2 = t, \qquad t \in R$$
$$x_3 = t,$$

So this system has infinitely many solutions.

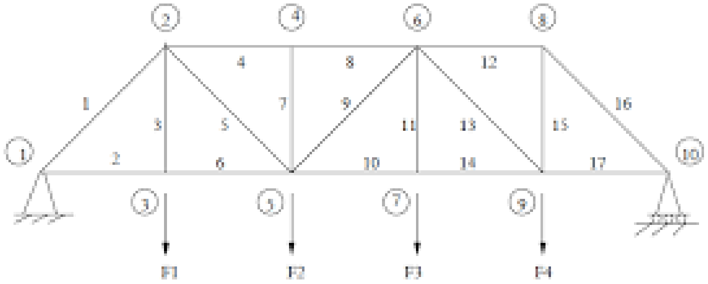
APPLICATIONS OF LINEAR SYSTEMS

Linear systems equations have numerous applications in various fields, such as mathematics, engineering, physics, economics, and computer science. These systems are of the form:

1. Structural Analysis (Engineering)

In civil and mechanical engineering, linear systems of homogeneous equations are used to analyze the equilibrium of forces in structures like bridges, buildings, and trusses. These structures must satisfy force and moment equilibrium, which can often be modeled by homogeneous systems of equations.

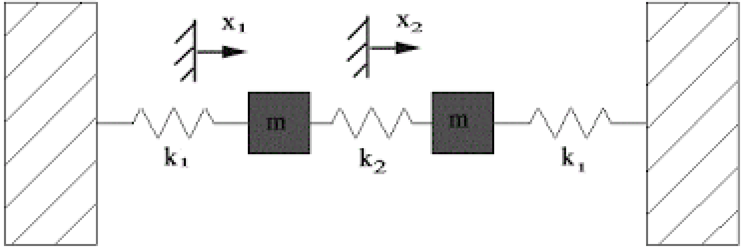
Example: The analysis of statically determinate structures can involve setting up a homogeneous system that ensures the sum of forces and moments equals zero, ensuring that the structure is stable.



2. Vibrations and Eigenvalue Problems (Physics and Engineering)

Homogeneous linear systems play a key role in the study of mechanical and electrical vibrations. In vibrating systems, such as springs and oscillators, the equations governing the motion often lead to systems of homogeneous linear equations.

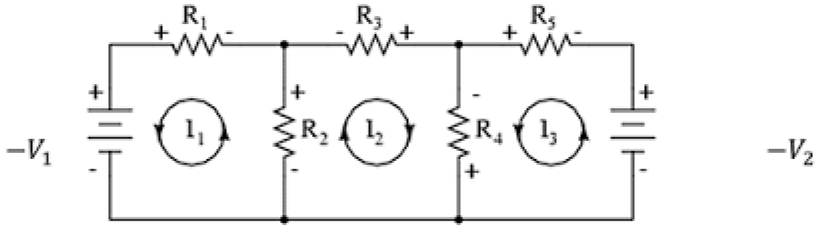
Example: For a mechanical system in free vibration (without external forces), the system's behavior is described by a homogeneous linear system. The non-trivial solutions correspond to the system's natural frequencies and modes of vibration, which are eigenvectors and eigenvalues of the system matrix.



3. Circuit Analysis (Electrical Engineering)

In electrical circuit analysis, especially in mesh and nodal analysis of circuits with resistors, capacitors, and inductors, homogeneous systems of equations can arise when analyzing circuits in steady-state or in the absence of external sources (natural response).

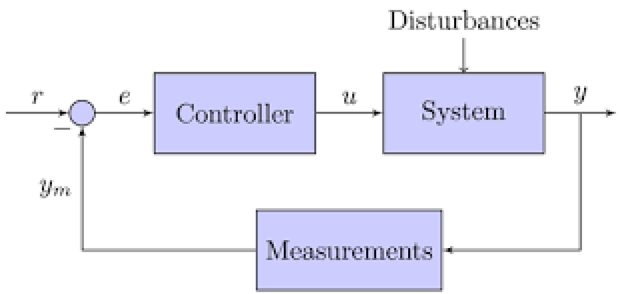
Example: In an RLC circuit (resistor, inductor, capacitor), the system of equations that describes the current flow and voltages can be modeled as a homogeneous system when no external voltage or current source is applied.



4. Control Systems

In control theory, homogeneous systems arise in the stability analysis of **linear dynamic systems**. The **state-space representation** of a control system can often be expressed as a system of homogeneous equations to study the behavior of the system in the absence of input.

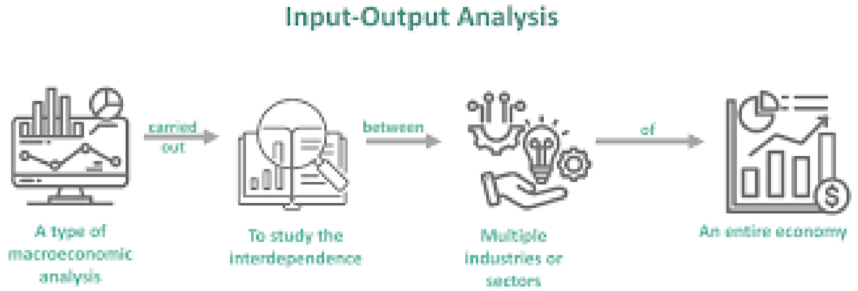
•**Example**: Stability analysis often involves solving homogeneous systems to find the system's eigenvalues, which indicate the system's stability behavior. Systems are stable if all eigenvalues have negative real parts.



5. Economics and Input-Output Models

In economics, homogeneous systems of linear equations are used in **input-output analysis** to model how different sectors of an economy depend on each other. Leontief's **input-output model** uses a system of homogeneous linear equations to study equilibrium in an economic system, where the total output equals the total input for each sector.

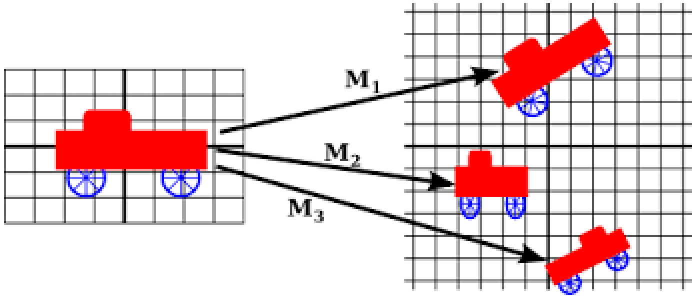
•**Example**: In equilibrium conditions, the amount of goods produced by each sector should exactly match the goods required by other sectors and by consumers. This can be formulated as a homogeneous system to determine consistent production levels.



6. Computer Graphics and 3D Transformations

In computer graphics, **3D transformations** such as rotations, scaling, and translations are often modeled using matrices. When determining **eigenvectors** for these transformations, a homogeneous system of equations arises.

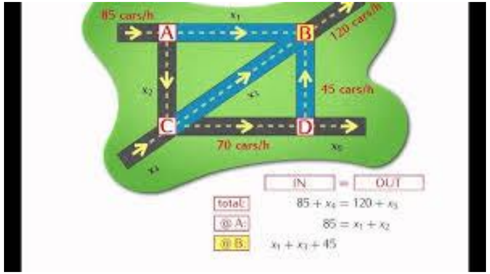
•**Example**: When calculating the principal axes of a 3D object (for example, in character animation or object rotation), eigenvectors and eigenvalues derived from homogeneous systems are used to describe the transformation behavior.



7. Network Flow Problems

In operations research, **network flow problems** involve systems of linear equations that can sometimes be modeled as homogeneous systems, especially when analyzing traffic flow or distribution networks in a steady state.

•**Example**: In an electrical grid or a traffic network, the flow of energy or vehicles must be balanced, which can lead to homogeneous systems to ensure that the inflows and outflows are equal.



8. Quantum Mechanics

In quantum mechanics, the **Schrödinger equation** can often lead to homogeneous systems when solving for **stationary states** (eigenstates) of quantum systems. The resulting eigenvalue problem (e.g., finding allowed energy levels) involves solving homogeneous systems of equations.

•**Example**: When determining the allowed energy states of an electron in an atom, homogeneous systems of equations arise, where the solutions correspond to the possible quantum states of the system.

9. Solving Differential Equations

, -> principal componenet analysis

Homogeneous systems of linear equations are often used to solve systems of **linear differential equations**. In particular, they arise in the **solution of homogeneous linear differential equations** with constant coefficients, often leading to eigenvalue and eigenvector problems.

- **Example**: In systems of linear differential equations, the general solution can often be written in terms of exponentials involving the eigenvalues and eigenvectors of the system matrix, which come from solving a homogeneous system.

10. Data Compression and Principal Component Analysis (PCA)

In statistics and machine learning, homogeneous systems are used in **Principal Component Analysis (PCA)** for dimensionality reduction. PCA involves solving an eigenvalue problem that arises from a homogeneous system to find the principal components of a dataset.

- **Example**: In PCA, the covariance matrix of a dataset is analyzed by solving a homogeneous system, and the principal components (eigenvectors) help reduce the dimensionality while preserving the most important features of the data.