Homework 9

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1

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\begin{aligned}
&\text{不妨设} Z_1 = \alpha X + \beta Y, Z_2 = \alpha X - \beta Y \\
&E(Z_1) = (\alpha + \beta)\mu, E(Z_2) = (\alpha - \beta)\mu \\
&Var(Z_1) = Var(Z_2) = (\alpha^2 + \beta^2)\sigma^2 \\
&E(Z_1 Z_2) = E(\alpha^2 X^2 - \beta^2 Y^2) = (\alpha^2 - \beta^2)(\sigma^2 + \mu^2) \\
&Cov(\alpha X + \beta Y, \alpha X - \beta Y) = E(Z_1 Z_2) - E(Z_1)E(Z_2) \\
&= (\alpha^2 - \beta^2)(\sigma^2 + \mu^2) - \mu^2(\alpha^2 - \beta^2) \\
&= (\alpha^2 - \beta^2)\sigma^2 = 0 \\
&\implies \alpha = \pm\beta
\end{aligned}
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2

2.1

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\begin{aligned}
&\because S = 2 \times 2 \times \frac{1}{2} = 2 \\
&\therefore f(X, Y) = \frac{1}{2} \\
&\quad E(X, Y) = \iint\limits_{|x|+|y|\leqslant 1} xy\frac{1}{2}dxdy = \frac{1}{2}\int_{-1}^{1}\int_{-1+|y|}^{1-|y|} xydxdy \\
&= \frac{1}{2}\int_{-1}^{1} 0dy = 0
\end{aligned}
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\begin{aligned}
&f_x(x) = \int_{|y|<1-|x|} \frac{1}{2}dy = 1 - |x| \\
&f_Y(y) = 1 - |y| \\
&E(x) = \int_{-1}^{1} x f_X(x)dx = 0 \\
&E(y) = \int_{-1}^{1} y f_y(y)dy = 0 \\
&Cov(X, Y) = E(XY) - E(X)E(Y) = 0
\end{aligned}
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2.2

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\begin{aligned}
&\because f(x,y) = \frac{1}{2} \neq f_X(x)f_Y(y) = (1-|x|)(1-|y|) \\
&\therefore \text{X 与 Y 不相互独立}
\end{aligned}
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3

选 C.

(A) 因为 X、Y 相互独立且服从标准正态分布, (X,Y) 服从二元正态分布

(B) 独立正态分布的商为柯西分布

(C)X,Y 不能准确对应联合分布函数, 因而不能推出相互独立

(D)$X+Y \sim N(0,2),\ X-Y \sim N(0,2);\ Cov(X+Y,X-Y)=0 \implies$ 相互独立

4

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\begin{aligned}
H(x) &= -\int_{-\infty}^{+\infty} \lambda e^{\lambda x} I_{[0,\infty)}(x) ln(\lambda e^{-\lambda x}) dx \\
&= -\int_{0}^{+\infty} \lambda ln\lambda e^{-\lambda x} dx + \int_{0}^{+\infty} \lambda^2 x e^{-\lambda x} dx \\
&= ln\lambda e^{-\lambda x}|_{x\to 0}^{x\to +\infty} + (-e^{-\lambda x} - \lambda x e^{-\lambda x})|_{x\to 0}^{x\to +\infty} \\
&= (ln\lambda - \lambda x - 1)e^{-\lambda x}|_{x\to 0}^{x\to +\infty} \\
&= 1 - ln\lambda
\end{aligned}
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5

5.1

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\begin{aligned}
E(F_n(x)) &= \frac{1}{n} E(\sum_{k=1}^{n} I(U_k \leqslant x)) = \frac{1}{n}\sum_{k=1}^{n} E(I(U_k \leqslant x)) \\
&= \frac{1}{n}\sum_{k=1}^{n} x = x \\
D(F_n(x)) &= E(F_n(x) - x)^2 \\
&= E(F_n(x))^2 - x^2 \\
&= E(\sum_{k=1}^{n} I(U_k \leqslant x))^2 - x^2 \\
&= \frac{1}{n^2} E[\sum_{k=1}^{n}\sum_{j=1}^{n} I(U_k \leqslant x) I(U_j \leqslant x)] \\
&= \frac{1}{n^2}[nx + (n^2-n)x^2] - x^2 \\
&= \frac{1}{n}(x - x^2)
\end{aligned}
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5.2

$$\begin{aligned}
Cov(F_n(x), F_n(y)) =& E(XY) - E(X)E(Y) \\
=& \frac{1}{n^2} E[\sum_{k=1}^{n} \sum_{j=1}^{n} I(U_k \leqslant x) I(U_j \leqslant x)] - xy \\
=& \frac{1}{n^2} [n min(x, y) + (n^2 - n) - xy] - xy \\
=& \frac{1}{n} [min(x, y) - xy]
\end{aligned}$$