Crisp Set is based on assumption that every proposition is either true or false / Two Valued)

We Can extend this proposition to multivalued eg for 3 Valued true (1), false (0) & indeterminacy (1/2)

By applying Diff operations on let's Say 3 Valued proposition

a	b	AND ($\wedge$)	OR ($\vee$)	NOT ($\neg a$)	IMPLICATION ($\Rightarrow$)	EQUAL (=)
0	0	0	0	1	1	1
0	1/2	0	1/2	1	1	1/2
0	1	0	1	1	1	0
1/2	0	0	1/2	1/2	1/2	1/2
1/2	1/2	1/2	1/2	1/2	1/2	1
1/2	1	1/2	1	1/2	1	1/2
1	0	0	1	0	0	0
1	1/2	1/2	1	0	1/2	1/2
1	1	1	1	0	1	1

$$\boxed{a \Rightarrow b \equiv \bar{a} \vee b}$$

Three - Valued Logic

Symbol	Connective	Usuage	Defination
$\neg$	NOT	$\neg P$	$1 - T(P)$
$\vee$	OR	$P \vee Q$	$\max\{T(P), T(Q)\}$
$\wedge$	AND	$P \wedge Q$	$\min\{T(P), T(Q)\}$
$\Rightarrow$	IMPLICATION	$(P \Rightarrow Q)$ or $(\neg P \vee Q)$	$\max\{(1-T(P)), T(Q)\}$
$=$	EQUALITY	$(P = Q)$ or $[(P \Rightarrow Q) \wedge (Q \Rightarrow P)]$	$1 - \lvert T(P) - T(Q)\rvert$

eg P: Mary is efficient $T(P) = 0.8$
Q: Ram is Efficient $T(Q) = 0.6$

- Mary is not Efficient $\neg P = 1 - T(P) = 0.2$
- Mary is Efficient & So is Ram $(P \wedge Q) = \underline{0.6}$
- Either Mary or Ram $P \vee Q = 0.8$
- If Mary is efficient then So is RAM $(P \Rightarrow Q) = (\neg P \vee Q) = 0.6$

The main Diff b/w fuzzy & Crisp proposition is its range in truth Values. The degree of truth of each proposition is lie in interval $[0, 1]$ both inclusive.

Canonical representation of fuzzy proposition

Let Say X is Universe of discourse of five people. Intelligent of $x \in X$

Intelligent $\{(x_1, 0.3), (x_2, 0.4), (x_3, 0.1), (x_4, 0.6), (x_5, 0.9)\}$

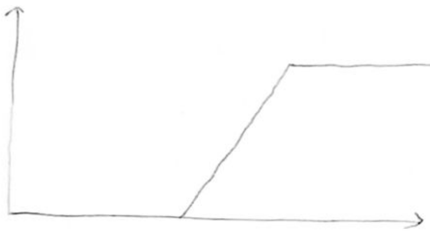
Fuzzy proposition is P: x is intelligent

So Canonical form of this proposition P is expressed as $P : \mathcal{V}$ is F

$\mathcal{V} \Rightarrow$ An element that takes values v from some Universal Set V

$F \Rightarrow$ It is a fuzzy Set on V OR

Particular element v, this element belongs to F with membership grade $\mu_F(v)$



Given Value v in proposition P, $T(P)$ denotes the truth of proposition P.

Fuzzy rule :- A fuzzy implication (if-then rule) assume the form

Antecedent / premise $\leftarrow$ If x is A then Y is B $\Longrightarrow$ Consequence / Conclusion

Where A & B are two linguistic Variables define over fuzzy Set X & Y

Fuzzy implication is denoted by $R : A \rightarrow B$. It is a binary fuzzy relation on product of $A \times B$.

eg. P & T are two Universe of discourse

$P = \{1, 2, 3, 4\}$ $T = \{10, 15, 20, 25, 30, 35, 40, 45, 50\}$

Linguistic Variables are given as $T_{HIGH} = \{(20, 0.2), (25, 0.4), (30, 0.6), (35, 0.6), (40, 0.7), (45, 0.8), (50, 0.8)\}$

$P_{LOW} = \{(1, 0.8), (2, 0.8), (3, 0.6), (4, 0.4)\}$

So for $T_{HIGH} \xrightarrow{R} P_{LOW}$

$$R = \begin{array}{c} \\ 20 \\ 25 \\ 30 \\ 35 \\ 40 \\ 45 \\ 50 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.2 \\ 0.4 & 0.4 & 0.4 & 0.4 \\ 0.6 & 0.6 & 0.6 & 0.4 \\ 0.6 & 0.6 & 0.6 & 0.4 \\ \boxed{0.7 \;\; 0.7 \;\; 0.6 \;\; 0.4} \\ 0.8 & 0.8 & 0.6 & 0.4 \\ 0.8 & 0.8 & 0.6 & 0.4 \end{bmatrix} \end{array}$$

So for temp 40

$P_{LOW} = \{(1, 0.7), (2, 0.7), (3, 0.6), (4, 0.4)\}$

Ways to Compute fuzzy rule $A \to B$

a) A Coupled with B $\quad R: A \to B = A \times B = \int_{X \times Y} \mu_A(x) * \mu_B(y) \Big| (x, y)$

where $*$ is a T-norm operator

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Frequently used T-norm operators are

- Minimum $\quad T_{min}(a,b) = \min(a,b) = a \wedge b$
- Algebra Product $\quad T_{AP}(a,b) = ab$
- Bounded product $\quad T_{bp}(a,b) = 0 \vee (a+b-1)$
- Drastic product

$$T_{dp} = \begin{cases} a & \text{if } b = 1 \\ b & \text{if } a = 1 \\ 0 & \text{if } a, b < 1 \end{cases}$$

So for $R: A \to B$

Min operator $\quad R_m = A \times B = \int_{X \times Y} \mu_A(x) \wedge \mu_B(y) \Big| (x,y)$ or $f_{min}(a,b) = a \wedge b$

[Mamdani rule]

Algebra Product operator

$$R_{AP} = A \times B = \int_{X \times Y} \mu_A(x) \cdot \mu_B(y) \Big| (x,y) \quad \text{or} \quad f_{ap}(a,b) = ab$$

[Larsen rule]

Bounded product operator

$$R_{bp} = A \times B = \int_{X \times Y} \mu_A(x) \odot \mu_B(y) \Big| (x,y) = \int_{X \times Y} 0 \vee (\mu_A(x) + \mu_B(y) - 1) \Big| (x,y)$$

$$f_{bp}(a,b) = 0 \vee (a+b-1)$$

Drastic Product operator

$$R_{dp} = A \times B = \int_{X \times Y} \mu_A(x) \hat{\cdot} \mu_B(y) \Big| (x,y)$$

$$f_{dp}(a,b) = \begin{cases} a & \text{if } b = 1 \\ b & \text{if } a = 1 \\ 0 & \text{if otherwise} \end{cases}$$

---

b) A entrails B

i) $R: A \to B = \bar{A} \cup B$ Material Implication

ii) $R: A \to B = \bar{A} \cup (A \cup B)$ Propositional Calculus

iii) $R: A \to B = (\bar{A} \cap \bar{B}) \cup B$ Extended propositional Calculus

Zadeh arithmetic rule

$$R_{za} = \bar{A} \cup B = \int_{X \times Y} 1 \wedge (1 - \mu_A(x) + \mu_B(y)) \,|\, (x, y)$$

$$f_{za}(a, b) = 1 \wedge (1 - a + b)$$

Zadeh min max rule

$$R_{mn} = \bar{A} \cup (A \cap B) = \int_{X \times Y} (1 - \mu_A(x)) \vee (\mu_A(x) \wedge \mu_B(y)) \,|\, (x, y)$$

$$f_{mn}(a, b) = (1 - a) \vee (a \wedge b)$$

Boolean fuzzy rule

$$R_{bf} = \bar{A} \cup B = \int_{X \times Y} (1 - \mu_A(x)) \vee \mu_B(x) \,|\, (x, y)$$

$$f_{bf}(a, b) = (1 - a) \vee b$$

Goguen fuzzy rule

$$R_{gf} = \int_{X \times Y} \mu_A(x) * \mu_B(y) \,|\, (x, y) \qquad a * b \begin{cases} 1 & \text{if } a \le b \\ \frac{b}{a} & \text{if } a > b \end{cases}$$

Zadeh max-min rule If x is A then Y is B

$$R_{mm} = (A \times B) \cup (\bar{A} \times Y)$$

where Y is Universe of ~~disclouse~~ discourse with memb. values $y \in Y$ is 1

$\mu_Y(y) = 1 \;\forall\; y \in Y$

$\boxed{Y = \{(1,1), (2,1), (3,1), (4,1)\}}$

$X = \{a, b, c, d\}$ $Y = \{1, 2, 3, 4\}$ can also be said like this just for reference

$A = \{(a, 0.0), (b, 0.8), (c, 0.6), (d, 1.0)\}$ $B = \{(1, 0.2), (2, 1.0), (3, 0.8), (4, 0.0)\}$

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$$A\times B = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline a & 0.0 & 0.0 & 0.0 & 0.0 \\ b & 0.2 & 0.8 & 0.8 & 0.0 \\ c & 0.2 & 0.6 & 0.6 & 0.0 \\ d & 0.2 & 1.0 & 0.8 & 0.0 \end{array}$$

$$\bar{A}\times Y = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline a & 1 & 1 & 1 & 1 \\ b & 0.2 & 0.2 & 0.2 & 0.2 \\ c & 0.4 & 0.4 & 0.4 & 0.4 \\ d & 0 & 0 & 0 & 0 \end{array}$$

x

$\int$

I

$$R_{mn} = (A\times B)\cup(\bar{A}\times Y) = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline a & 1 & 1 & 1 & 1 \\ b & 0.2 & 0.8 & 0.8 & 0.2 \\ c & 0.4 & 0.6 & 0.6 & 0.4 \\ d & 0.2 & 1.0 & 0.8 & 0 \end{array}$$

eg. If x is A then Y is B Else Y is C

$R = (A\times B)\cup(\bar{A}\times C)$ then Simply Calculate it

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a Fuzzy Inferences

b

1. Modus Ponens $\underbrace{P,\ P\Rightarrow Q,}_{\text{true}} \Longleftrightarrow Q$

2. Modus Tollens $\underbrace{P\Rightarrow Q,\ \neg Q,}_{\text{true}} \Longleftrightarrow \neg P$

3. Chain rule $\underbrace{P\Rightarrow Q,\ Q\Rightarrow R}_{\text{true}} \Longleftrightarrow P\Rightarrow R$

$P\wedge(P\Rightarrow Q)$

$P\wedge(\bar{P}\vee Q)$

$(P\wedge\bar{P})\vee(P\wedge Q)$

$0\vee(P\wedge Q)$

$(P\wedge Q)$ $P\rightarrow$ true $\Rightarrow$ 1

$1\wedge Q$

Q

Inferring procedures in fuzzy logic

- Generalised Modus Ponens (GMP)

  If x is A then Y is B

  x is A'

  Y is B'

- Generalised Modus Tollens (GMT)

  If x is A then Y is B

  X is B'

  x is A'

Here A, B, A' & B' are fuzzy Sets. To compute memb func A' & B' the max-min composition of B' & A' with $R(x,y)$ is used

$B' = A' \circ R(x,y)$ $\quad \mu_{B'}(y) = \max[\min(\mu_{A'}(x), \mu_R(x,y))]$

$A' = B' \circ R(x,y)$ $\quad \mu_{A'}(x) = \max[\min(\mu_{B'}(y), \mu_R(x,y))]$

eg. If x is A then Y is B

x is A'

Y is B'

Pollens

$X = \{x_1, x_2, x_3\}$ $Y = \{y_1, y_2\}$

$A = \{(x_1, 0.5), (x_2, 1), (x_3, 0.6)\}$

$B = \{(y_1, 1), (y_2, 0.4)\}$

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$B' = A' \circ R(x, y)$

$A' = \{(x_1, 0.6), (x_2, 0.9), (x_3, 0.7)\}$

where $R(x,y) = (A \times B) \cup (\bar{A} \times Y)$ Calculate this the composition with A'

Refer $\boxed{\text{Slide 46}}$ $\boxed{\text{Slide 50}}$ Week2

eg. If x is A then Y is B

Y is B'

---

x is A'

$x = \{x_1, x_2, x_3\}$ $Y = \{y_1, y_2\}$

$A = \{(x_1, 0.5), (x_2, 1), (x_3, 0.6)\}$

$B = \{(y_1, 1), (y_2, 0.4)\}$

$B' = \{(y_1, 0.9), (y_2, 0.7)\}$

$A' = B' \circ R(x, y)$

$R(x,y) = (A \times B) \cup (\bar{A} \times Y)$ Calculate this then composition with B'

Defuzzification ⇒ fuzzy to crisp

↳ 1) Lambda-Cut Method ⇒ In this fuzzy set A converted to crisp set A_λ where $(0 \le \lambda \le 1)$. The value of Lambda-cut set A_λ is x, the memb. value corresponding to x is greater than or equal to specified λ.

eg. $A = \{(x_1, 0.9), (x_2, 0.5), (x_3, 0.2), (x_4, 0.3)\}$ $\lambda = 0.6$

$A_{0.6} = \{(x_1, 1), (x_2, 0), (x_3, 0), (x_4, 0)\}$
$= \{x_1\}$

Values greater than or equal to λ is 1

eg. Refer Slide-66

3. Centroid Methods

Lambda-Cut on a fuzzy relation

eg. $R = \begin{bmatrix} 1 & 0.2 & 0.3 \\ 0.5 & 0.9 & 0.6 \\ 0.4 & 0.8 & 0.7 \end{bmatrix}$ $R_0 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $R_{0.2} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $R_{0.9} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

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Some properties of fuzzy sets

a) $(A \cup B)_\lambda = A_\lambda \cup B_\lambda$ b) $(A \cap B)_\lambda = A_\lambda \cap B_\lambda$ c) $(\overline{A})_\lambda \neq \overline{A}_\lambda$

Except for Value $\lambda = 0.5$

d) for any $\lambda \leq \alpha$ where $\alpha \in [0, 1]$

$A_\alpha \subseteq A_\lambda$

Same properties Can be Used on fuzzy relations.

Output of fuzzy System

Let Say We have n-rules

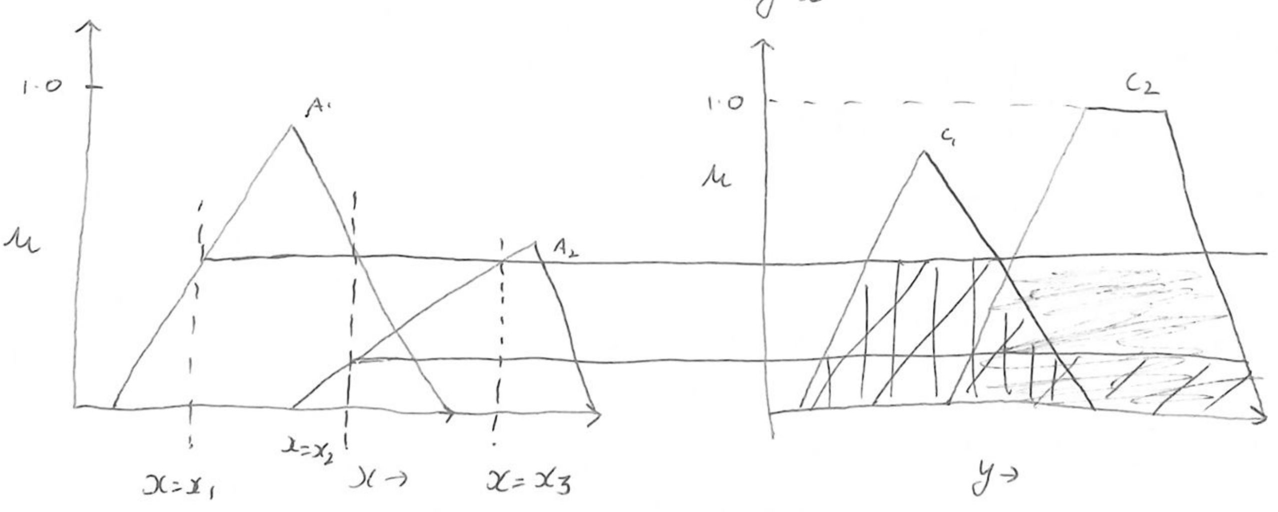
R_1: If x is A_1 then Y is B_1
R_2: If x is A_2 then Y is B_2
- - - - - - - - -
R_n: If x is A_n then Y is B_n

Then output Y for given Input $x = x_1$ is possibally $B = B_1 \cup B_2 \cdots \cup B_n$

Illustration

R_1: If x is A_1 then Y is C_1
R_2: If x is A_2 then Y is C_2

Output fuzzy Set $C = C_1 \cup C_2$ for any x



||||| ⇒ for x_1

////// for x_2

for x_3

Refer Slide 73, 74, 75

Maxima Methods

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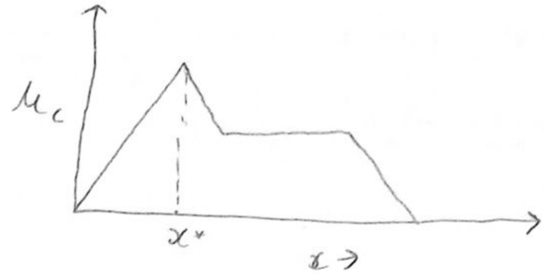
a) Height Method ⇒ Based on Max-membership principle

$\mu_c(x^*) \geq \mu_c(x)$

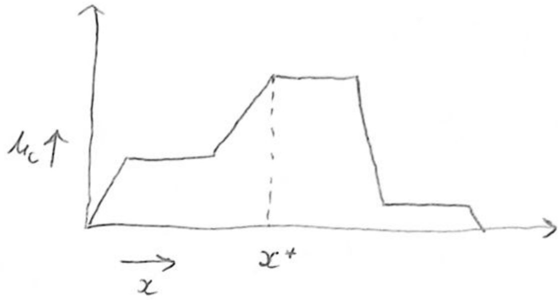
where $x \in X$



x^* ⇒ Height of the output fuzzy set c
Applicable on when height is Unique.

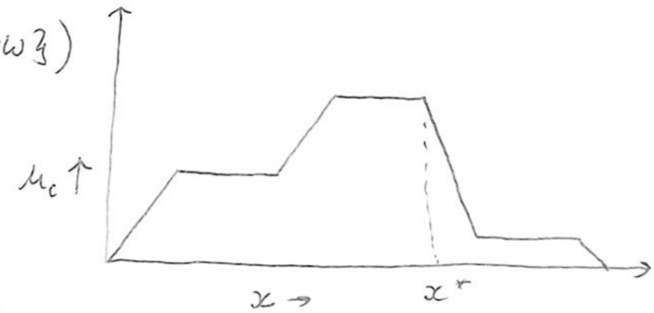
b) First of Maxima

$x^* = \min(x \mid C(x) = \max_w C\{w\})$



c) Last of Maxima

$x^* = \max(x \mid C(x) = \max_w C\{w\})$



d) Mean of Maxima

$$x^* = \frac{\sum_{x_i \in M}(x_i)}{|M|}$$

where

$M = \{x_i \mid \mu(x_i) = h(c)\}$ where $h(c)$ is height of fuzzy set

eg. $Y = \{(15, 0.5), (20, 0.8), (25, 0.8), (30, 0.5), (35, 0.3)\}$

$$x^* = \frac{20+25}{2} = 22.5$$

eg.

$$x^* = \frac{a+b}{2}$$

Note:- MoM is also Synonymous to middle of maxima

Refer 86 Slide

MoM is also a general method of height

3. Centroid Methods

a) Center of Gravity Method (CoG) ⇒ find a point x where a Vertical line slice aggregate into two equal masses.

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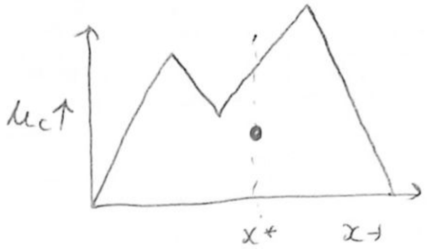
CoG $x^* = \dfrac{\int x \cdot \mu_c(x)\, d(x)}{\int \mu_c(x)\, d(x)}$

⇓

x-Coordinate of CoG



$\int \mu_c(x)\, d(x)$ ⇒ Area bounded by Curve μ_c

If μ_c is defined as discrete memb. func the CoG $x^* = \dfrac{\sum x_i\, \mu_c(x_i)}{\sum \mu_c(x_i)}$ $i = 1$ to n

x_i ⇒ Sample element

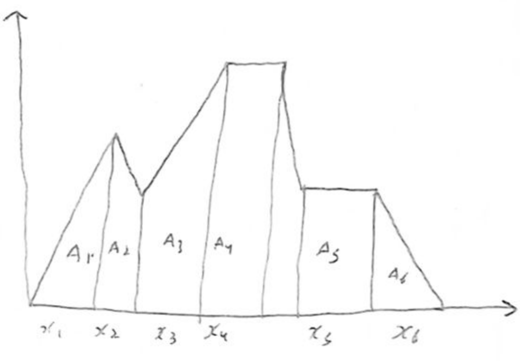
n ⇒ number of Samples

Geometrical Method of Calculation

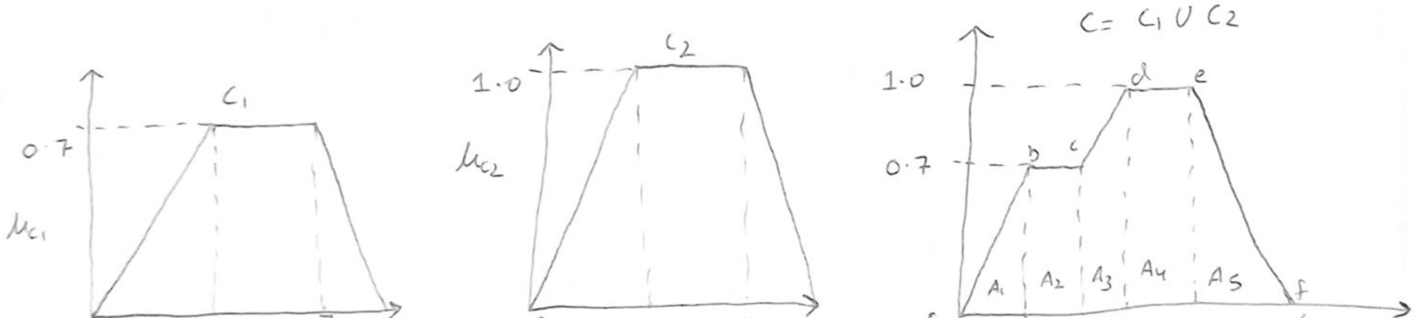
a) Divide entire region into Smaller region of triangles, trapezoid etc

b) Let A_i & x_i denotes area & c.g of i^{th} position

$$x^* = \frac{\sum_{i=1}^{n} x_i (A_i)}{\sum_{i=1}^{n} (A_i)}$$



eg.



$$\mu_c(x) = \begin{cases} 0.35x & 0 \le x < 2 \\ 0.7 & 2 \le x < 2.7 \\ x - 2 & 2.7 \le x < 3 \\ 1 & 3 \le x < 4 \\ (-.5x + 3) & 4 \le x \le 6 \end{cases}$$

Using line equation formula We Calculate this lines

Slide 92-93

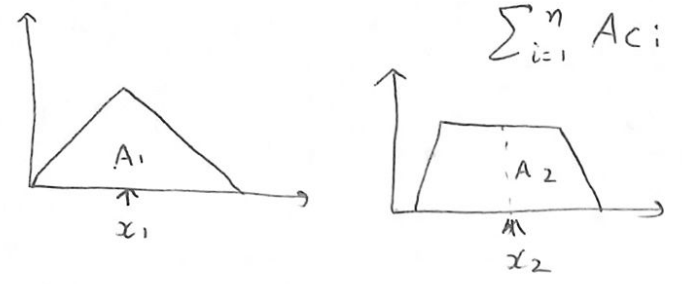
Using this & above formula We Calculate x^*

b) Center of Sum (COS) If output fuzzy set $C = C_1 \cup C_2 \cdots \cup C_n$

$$x^* = \frac{\sum_{i=1}^{n} x_i A_{ci}}{\sum_{i=1}^{n} A_{ci}}$$

$A_{ci} \Rightarrow$ Area bounded by ~~region~~ fuzzy set C_i

$x_i \Rightarrow$ Geometric Center of Area A_{C_i}



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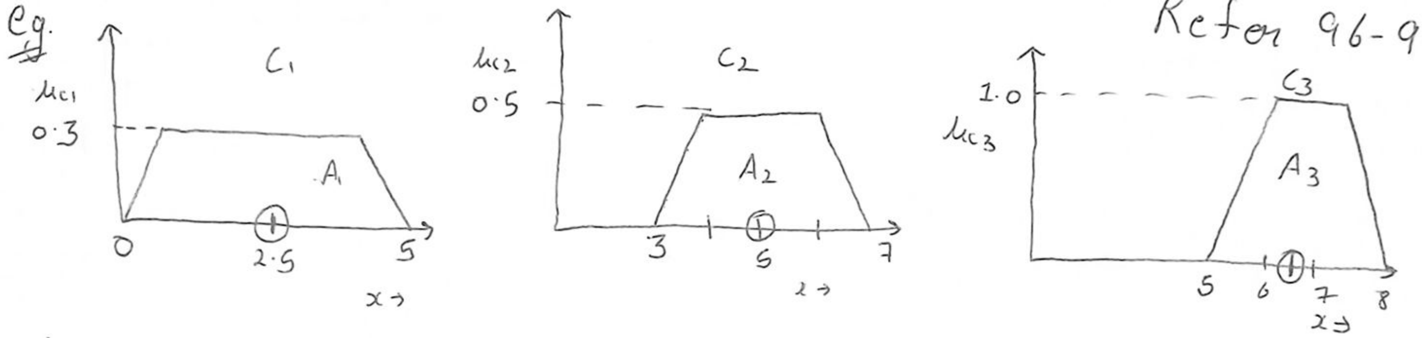
In CoG Overlap area only Counted Once whereas in CoS it is Counted twice

Refer 96-97

eg.



$A_{c_1} = \frac{1}{2} \times (0.3) \times (3+5)$, $x_1 = 2.5$

$A_{c_2} = \frac{1}{2} \times (0.5) \times (4+2)$, $x_2 = 5$

$A_{c_3} = \frac{1}{2} \times (1) \times (3+1)$, $x_3 = 6.5$

$$x^* = \frac{4 \times 0.3 \times 2.5 + 3 \times 0.5 \times 5 + 2 \times 6.5}{4 \times 0.3 + 3 \times 0.5 + 2 \times 1.0}$$

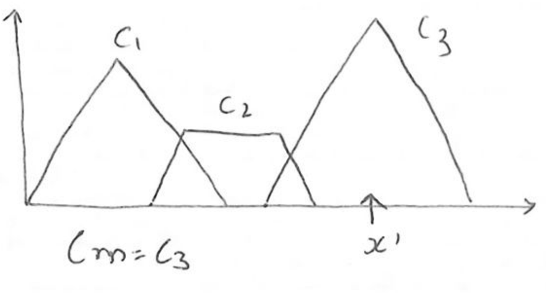
---

c) Center of Largest area :- Center of gravity of the Sub region with the largest area used to Calculate defuzzified value

$$x^* = \frac{\int \mu_{cm}(x) \cdot x' \, d(x)}{\int \mu_{cm}(x) \, dx}$$

$C_m \Rightarrow$ Region of largest area

$x' \Rightarrow$ Center of gravity of C_3

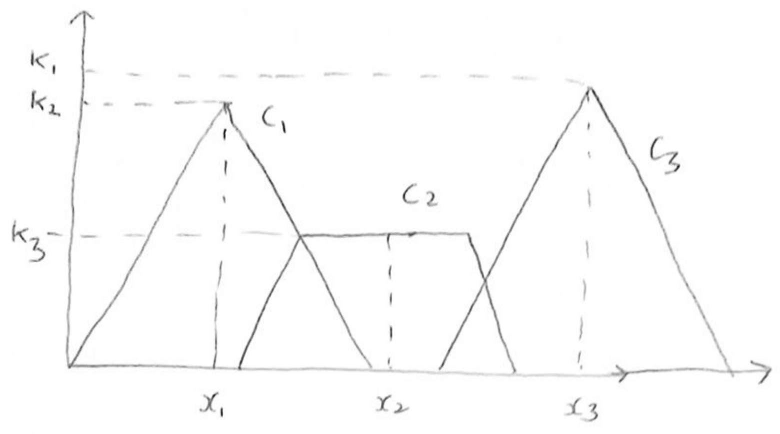


3. Weighted average Methord ⇒ Sugeno ~~deff~~ defuzzification

Used only for Symmetrical output membership functions

$$x^* = \frac{\sum_{i=1}^{n} \mu_{c_i}(x_i) \cdot x_i}{\sum_{i=1}^{n} \mu_{c_i}(x_i)}$$

where $c_1, c_2 \ldots c_2$ are Output fuzzy Set & x_i is value where middle of fuzzy Set c_i is observed



Exercise 102 → 107