Chapter 7: Analytic Trigonometry

7.1 Exercises

[1] $\csc\theta - \sin\theta = \dfrac{1}{\sin\theta} - \sin\theta = \dfrac{1-\sin^2\theta}{\sin\theta} = \dfrac{\cos^2\theta}{\sin\theta} = \dfrac{\cos\theta}{\sin\theta}\cdot\cos\theta = \cot\theta\,\cos\theta$

[2] $\sin x + \cos x\,\cot x = \sin x + \cos x\cdot\dfrac{\cos x}{\sin x} = \dfrac{\sin^2 x + \cos^2 x}{\sin x} = \dfrac{1}{\sin x} = \csc x$

[3] $\dfrac{\sec^2 2u - 1}{\sec^2 2u} = 1 - \dfrac{1}{\sec^2 2u} = 1 - \cos^2 2u = \sin^2 2u$

[4] $\tan t + 2\cos t\,\csc t = \dfrac{\sin t}{\cos t} + \dfrac{2\cos t}{\sin t} = \dfrac{\sin^2 t + 2\cos^2 t}{\cos t\,\sin t} =$

$$\frac{1-\cos^2 t + 2\cos^2 t}{\cos t\,\sin t} = \frac{1+\cos^2 t}{\cos t\,\sin t} = \frac{1}{\cos t\,\sin t} + \frac{\cos t}{\sin t} = \sec t\,\csc t + \cot t$$

[5] $\dfrac{\csc^2\theta}{1+\tan^2\theta} = \dfrac{\csc^2\theta}{\sec^2\theta} = \dfrac{1/\sin^2\theta}{1/\cos^2\theta} = \dfrac{\cos^2\theta}{\sin^2\theta} = \left(\dfrac{\cos\theta}{\sin\theta}\right)^2 = \cot^2\theta$

[6] $(\tan u + \cot u)(\cos u + \sin u) =$

$\left(\dfrac{\sin u}{\cos u} + \dfrac{\cos u}{\sin u}\right)(\cos u + \sin u) = \left(\dfrac{\sin^2 u + \cos^2 u}{\cos u\,\sin u}\right)(\cos u + \sin u) =$

$$\left(\frac{1}{\cos u\,\sin u}\right)(\cos u + \sin u) = \frac{\cos u}{\cos u\,\sin u} + \frac{\sin u}{\cos u\,\sin u} = \frac{1}{\sin u} + \frac{1}{\cos u} = \csc u + \sec u$$

[7] $\dfrac{1+\cos 3t}{\sin 3t} + \dfrac{\sin 3t}{1+\cos 3t} = \dfrac{(1+\cos 3t)^2 + \sin^2 3t}{\sin 3t\,(1+\cos 3t)} =$

$$\frac{1+2\cos 3t + \cos^2 3t + \sin^2 3t}{\sin 3t\,(1+\cos 3t)} = \frac{2+2\cos 3t}{\sin 3t\,(1+\cos 3t)} = \frac{2(1+\cos 3t)}{\sin 3t\,(1+\cos 3t)} = 2\csc 3t$$

[8] $\tan^2\alpha - \sin^2\alpha = \dfrac{\sin^2\alpha}{\cos^2\alpha} - \sin^2\alpha = \sin^2\alpha\left(\dfrac{1}{\cos^2\alpha} - 1\right) = (\sec^2\alpha - 1)\sin^2\alpha = \tan^2\alpha\,\sin^2\alpha$

[9] $\dfrac{1}{1-\cos\gamma} + \dfrac{1}{1+\cos\gamma} = \dfrac{1+\cos\gamma + 1 - \cos\gamma}{1-\cos^2\gamma} = \dfrac{2}{\sin^2\gamma} = 2\csc^2\gamma$

[10] $\dfrac{1+\csc 3\beta}{\sec 3\beta} - \cot 3\beta = \dfrac{1}{\sec 3\beta} + \dfrac{\csc 3\beta}{\sec 3\beta} - \cot 3\beta = \cos 3\beta + \dfrac{\cos 3\beta}{\sin 3\beta} - \cot 3\beta = \cos 3\beta$

[11] $(\sec u - \tan u)(\csc u + 1) = \left(\dfrac{1}{\cos u} - \dfrac{\sin u}{\cos u}\right)\left(\dfrac{1}{\sin u} + 1\right) =$

$$\left(\frac{1-\sin u}{\cos u}\right)\left(\frac{1+\sin u}{\sin u}\right) = \frac{1-\sin^2 u}{\cos u\,\sin u} = \frac{\cos^2 u}{\cos u\,\sin u} = \frac{\cos u}{\sin u} = \cot u$$

[12] $\dfrac{\cot\theta - \tan\theta}{\sin\theta + \cos\theta} = \dfrac{\dfrac{\cos\theta}{\sin\theta} - \dfrac{\sin\theta}{\cos\theta}}{\sin\theta + \cos\theta} = \dfrac{\dfrac{\cos^2\theta - \sin^2\theta}{\sin\theta\,\cos\theta}}{\sin\theta + \cos\theta} = \dfrac{(\cos\theta + \sin\theta)(\cos\theta - \sin\theta)}{\sin\theta\,\cos\theta\,(\sin\theta + \cos\theta)} =$

$$\frac{\cos\theta - \sin\theta}{\sin\theta\,\cos\theta} = \frac{\cos\theta}{\sin\theta\,\cos\theta} - \frac{\sin\theta}{\sin\theta\,\cos\theta} = \frac{1}{\sin\theta} - \frac{1}{\cos\theta} = \csc\theta - \sec\theta$$

[13] $\csc^4 t - \cot^4 t = (\csc^2 t + \cot^2 t)(\csc^2 t - \cot^2 t) = (\csc^2 t + \cot^2 t)(1) = \csc^2 t + \cot^2 t$

[14] $\cos^4 2\theta + \sin^2 2\theta = (\cos^2 2\theta)^2 + \sin^2 2\theta = (1 - \sin^2 2\theta)^2 + \sin^2 2\theta =$

$$1 - 2\sin^2 2\theta + \sin^4 2\theta + \sin^2 2\theta = 1 - \sin^2 2\theta + \sin^4 2\theta = \cos^2 2\theta + \sin^4 2\theta$$

[15] $\dfrac{\cos\beta}{1-\sin\beta}=\dfrac{\cos\beta}{1-\sin\beta}\cdot\dfrac{1+\sin\beta}{1+\sin\beta}=\dfrac{\cos\beta\,(1+\sin\beta)}{1-\sin^2\beta}=\dfrac{\cos\beta\,(1+\sin\beta)}{\cos^2\beta}=\dfrac{1+\sin\beta}{\cos\beta}=$

$\dfrac{1}{\cos\beta}+\dfrac{\sin\beta}{\cos\beta}=\sec\beta+\tan\beta$

[16] $\dfrac{1}{\csc y-\cot y}=\dfrac{1}{\csc y-\cot y}\cdot\dfrac{\csc y+\cot y}{\csc y+\cot y}=\dfrac{\csc y+\cot y}{\csc^2 y-\cot^2 y}=\dfrac{\csc y+\cot y}{1}=\csc y+\cot y$

[17] $\dfrac{\tan^2 x}{\sec x+1}=\dfrac{\sec^2 x-1}{\sec x+1}=\dfrac{(\sec x+1)(\sec x-1)}{\sec x+1}=\sec x-1=\dfrac{1}{\cos x}-1=\dfrac{1-\cos x}{\cos x}$

[18] $\dfrac{\cot x}{\csc x+1}=\dfrac{\cot x}{\csc x+1}\cdot\dfrac{\csc x-1}{\csc x-1}=\dfrac{\cot x\,(\csc x-1)}{\csc^2 x-1}=\dfrac{\cot x\,(\csc x-1)}{\cot^2 x}=\dfrac{\csc x-1}{\cot x}$

[19] $\dfrac{\cot 4u-1}{\cot 4u+1}=\dfrac{\dfrac{1}{\tan 4u}-1}{\dfrac{1}{\tan 4u}+1}=\dfrac{\dfrac{1-\tan 4u}{\tan 4u}}{\dfrac{1+\tan 4u}{\tan 4u}}=\dfrac{1-\tan 4u}{1+\tan 4u}$

[20] $\dfrac{1+\sec 4x}{\sin 4x+\tan 4x}=\dfrac{1+\dfrac{1}{\cos 4x}}{\sin 4x+\dfrac{\sin 4x}{\cos 4x}}=\dfrac{\dfrac{\cos 4x+1}{\cos 4x}}{\dfrac{\sin 4x\,\cos 4x+\sin 4x}{\cos 4x}}=\dfrac{\cos 4x+1}{\sin 4x\,(\cos 4x+1)}=$

$\dfrac{1}{\sin 4x}=\csc 4x$

[21] $\sin^4 r-\cos^4 r=(\sin^2 r-\cos^2 r)(\sin^2 r+\cos^2 r)=(\sin^2 r-\cos^2 r)(1)=\sin^2 r-\cos^2 r$

[22] $\sin^4\theta+2\sin^2\theta\,\cos^2\theta+\cos^4\theta=(\sin^2\theta+\cos^2\theta)^2=(1)^2=1$

[23] $\tan^4 k-\sec^4 k=(\tan^2 k-\sec^2 k)(\tan^2 k+\sec^2 k)=(-1)(\sec^2 k-1+\sec^2 k)=$

$(-1)(2\sec^2 k-1)=1-2\sec^2 k$

[24] $\sec^4 u-\sec^2 u=\sec^2 u\,(\sec^2 u-1)=(1+\tan^2 u)(\tan^2 u)=\tan^2 u+\tan^4 u$

[25] $(\sec t+\tan t)^2=\left(\dfrac{1}{\cos t}+\dfrac{\sin t}{\cos t}\right)^2=\left(\dfrac{1+\sin t}{\cos t}\right)^2=\dfrac{(1+\sin t)^2}{\cos^2 t}=$

$\dfrac{(1+\sin t)^2}{1-\sin^2 t}=\dfrac{(1+\sin t)^2}{(1+\sin t)(1-\sin t)}=\dfrac{1+\sin t}{1-\sin t}$

[26] $\text{RS}=(1-\sin^4\gamma)\sec^4\gamma=\sec^4\gamma-\dfrac{\sin^4\gamma}{\cos^4\gamma}=\sec^4\gamma-\tan^4\gamma=$

$(\sec^2\gamma-\tan^2\gamma)(\sec^2\gamma+\tan^2\gamma)=(1)(\sec^2\gamma+\tan^2\gamma)=\sec^2\gamma+\tan^2\gamma=\text{LS}$

[27] $(\sin^2\theta+\cos^2\theta)^3=(1)^3=1$

[28] $\dfrac{\sin t}{1-\cos t}=\dfrac{\sin t}{1-\cos t}\cdot\dfrac{1+\cos t}{1+\cos t}=\dfrac{\sin t\,(1+\cos t)}{1-\cos^2 t}=\dfrac{\sin t\,(1+\cos t)}{\sin^2 t}=\dfrac{1+\cos t}{\sin t}=$

$\dfrac{1}{\sin t}+\dfrac{\cos t}{\sin t}=\csc t+\cot t$

[29] $\dfrac{1+\csc\beta}{\cot\beta+\cos\beta}=\dfrac{1+\dfrac{1}{\sin\beta}}{\dfrac{\cos\beta}{\sin\beta}+\cos\beta}=\dfrac{\dfrac{\sin\beta+1}{\sin\beta}}{\dfrac{\cos\beta+\cos\beta\,\sin\beta}{\sin\beta}}=\dfrac{\sin\beta+1}{\cos\beta\,(1+\sin\beta)}=\dfrac{1}{\cos\beta}=\sec\beta$

[30] $\dfrac{\cos^3 x-\sin^3 x}{\cos x-\sin x}=\dfrac{(\cos x-\sin x)(\cos^2 x+\cos x\,\sin x+\sin^2 x)}{\cos x-\sin x}=1+\sin x\,\cos x$

[31] $(\csc t-\cot t)^4(\csc t+\cot t)^4=$

$\big[(\csc t-\cot t)(\csc t+\cot t)\big]^4=(\csc^2 t-\cot^2 t)^4=(1)^4=1$

[32] $(a\ \cos t - b\ \sin t)^2 + (a\ \sin t + b\ \cos t)^2 =$

$(a^2\ \cos^2 t - 2ab\ \cos t\ \sin t + b^2\ \sin^2 t) + (a^2\ \sin^2 t + 2ab\ \sin t\ \cos t + b^2\ \cos^2 t) =$

$a^2(\cos^2 t + \sin^2 t) + b^2(\sin^2 t + \cos^2 t) = a^2 + b^2$

[33] $\text{RS} = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\ \tan\beta} = \dfrac{\dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\sin\beta}{\cos\beta}}{1 - \dfrac{\sin\alpha}{\cos\alpha}\cdot\dfrac{\sin\beta}{\cos\beta}} = \dfrac{\dfrac{\sin\alpha\ \cos\beta + \cos\alpha\ \sin\beta}{\cos\alpha\ \cos\beta}}{\dfrac{\cos\alpha\ \cos\beta - \sin\alpha\ \sin\beta}{\cos\alpha\ \cos\beta}} =$

$\dfrac{\sin\alpha\ \cos\beta + \cos\alpha\ \sin\beta}{\cos\alpha\ \cos\beta - \sin\alpha\ \sin\beta} = \text{LS}$

Note: We could obtain the RS by dividing

the numerator and denominator of the LS by $(\cos\alpha\ \cos\beta)$.

[34] $\dfrac{\tan u - \tan v}{1 + \tan u\ \tan v} = \dfrac{\dfrac{1}{\cot u} - \dfrac{1}{\cot v}}{1 + \dfrac{1}{\cot u}\cdot\dfrac{1}{\cot v}} = \dfrac{\dfrac{\cot v - \cot u}{\cot u\ \cot v}}{\dfrac{\cot u\ \cot v + 1}{\cot u\ \cot v}} = \dfrac{\cot v - \cot u}{\cot u\ \cot v + 1}$

[35] $\dfrac{\tan\alpha}{1 + \sec\alpha} + \dfrac{1 + \sec\alpha}{\tan\alpha} = \dfrac{\tan^2\alpha + (1 + \sec\alpha)^2}{(1 + \sec\alpha)\tan\alpha} = \dfrac{\sec^2\alpha - 1 + 1 + 2\sec\alpha + \sec^2\alpha}{(1 + \sec\alpha)\tan\alpha} =$

$\dfrac{2\sec^2\alpha + 2\sec\alpha}{(1 + \sec\alpha)\tan\alpha} = \dfrac{2\sec\alpha\,(\sec\alpha + 1)\cot\alpha}{1 + \sec\alpha} = \dfrac{2}{\cos\alpha}\cdot\dfrac{\cos\alpha}{\sin\alpha} = \dfrac{2}{\sin\alpha} = 2\csc\alpha$

[36] $\dfrac{\csc x}{1 + \csc x} - \dfrac{\csc x}{1 - \csc x} = \dfrac{\csc x(1 - \csc x) - \csc x(1 + \csc x)}{1 - \csc^2 x} =$

$\dfrac{\csc x - \csc^2 x - \csc x - \csc^2 x}{1 - \csc^2 x} = \dfrac{-2\csc^2 x}{-\cot^2 x} = \dfrac{2/\sin^2 x}{\cos^2 x/\sin^2 x} = \dfrac{2}{\cos^2 x} = 2\sec^2 x$

[37] $\dfrac{1}{\tan\beta + \cot\beta} = \dfrac{1}{\dfrac{\sin\beta}{\cos\beta} + \dfrac{\cos\beta}{\sin\beta}} = \dfrac{1}{\dfrac{\sin^2\beta + \cos^2\beta}{\cos\beta\ \sin\beta}} = \sin\beta\ \cos\beta$

[38] $\dfrac{\cot y - \tan y}{\sin y\ \cos y} = \dfrac{\dfrac{\cos y}{\sin y} - \dfrac{\sin y}{\cos y}}{\sin y\ \cos y} = \dfrac{\cos^2 y - \sin^2 y}{\sin^2 y\ \cos^2 y} =$

$\dfrac{\cos^2 y}{\sin^2 y\ \cos^2 y} - \dfrac{\sin^2 y}{\sin^2 y\ \cos^2 y} = \dfrac{1}{\sin^2 y} - \dfrac{1}{\cos^2 y} = \csc^2 y - \sec^2 y$

[39] $\sec\theta + \csc\theta - \cos\theta - \sin\theta = \dfrac{1}{\cos\theta} - \cos\theta + \dfrac{1}{\sin\theta} - \sin\theta = \dfrac{1 - \cos^2\theta}{\cos\theta} + \dfrac{1 - \sin^2\theta}{\sin\theta} =$

$\dfrac{\sin^2\theta}{\cos\theta} + \dfrac{\cos^2\theta}{\sin\theta} = \sin\theta\cdot\dfrac{\sin\theta}{\cos\theta} + \cos\theta\cdot\dfrac{\cos\theta}{\sin\theta} = \sin\theta\ \tan\theta + \cos\theta\ \cot\theta$

[40] $\sin^3 t + \cos^3 t = (\sin t + \cos t)(\sin^2 t - \sin t\ \cos t + \cos^2 t) = (1 - \sin t\ \cos t)(\sin t + \cos t)$

[41] $\text{RS} = \sec^4\phi - 4\tan^2\phi = (\sec^2\phi)^2 - 4\tan^2\phi = (1 + \tan^2\phi)^2 - 4\tan^2\phi =$

$1 + 2\tan^2\phi + \tan^4\phi - 4\tan^2\phi = 1 - 2\tan^2\phi + \tan^4\phi = (1 - \tan^2\phi)^2 = \text{LS}$

[42] $\cos^4 w + 1 - \sin^4 w = \cos^4 w + 1 - (1 - \cos^2 w)^2 =$

$\cos^4 w + 1 - (1 - 2\cos^2 w + \cos^4 w) = 2\cos^2 w$

[43] $\dfrac{\cot(-t)+\tan(-t)}{\cot t}=\dfrac{-\cot t-\tan t}{\cot t}=-\dfrac{\cot t}{\cot t}-\dfrac{\tan t}{\cot t}=-(1+\tan^2 t)=-\sec^2 t$

[44] $\dfrac{\csc(-t)-\sin(-t)}{\sin(-t)}=\dfrac{-\csc t+\sin t}{-\sin t}=\dfrac{\csc t}{\sin t}-\dfrac{\sin t}{\sin t}=\csc^2 t-1=\cot^2 t$

[45] $\log 10^{\tan t}=\log_{10} 10^{\tan t}=\tan t$, since $\log_a a^x=x$

[46] $10^{\log|\sin t|}=10^{\log_{10}|\sin t|}=|\sin t|$, since $a^{\log_a x}=x$

[47] $\ln\cot x=\ln(\cot x)=\ln(\tan x)^{-1}=-\ln(\tan x)=-\ln\tan x$

[48] $\ln\sec\theta=\ln(\sec\theta)=\ln(\cos\theta)^{-1}=-\ln(\cos\theta)=-\ln\cos\theta$

[49] $\ln|\sec\theta+\tan\theta|=\ln\left|\dfrac{(\sec\theta+\tan\theta)(\sec\theta-\tan\theta)}{\sec\theta-\tan\theta}\right|=\ln\left|\dfrac{\sec^2\theta-\tan^2\theta}{\sec\theta-\tan\theta}\right|=$

$\ln\left|\dfrac{1}{\sec\theta-\tan\theta}\right|=\ln|1|-\ln|\sec\theta-\tan\theta|=-\ln|\sec\theta-\tan\theta|\ \{\ln 1=0\}$

[50] $\ln|\csc x-\cot x|=\ln\left|\dfrac{(\csc x-\cot x)(\csc x+\cot x)}{\csc x+\cot x}\right|=\ln\left|\dfrac{\csc^2 x-\cot^2 x}{\csc x+\cot x}\right|=$

$\ln\left|\dfrac{1}{\csc x+\cot x}\right|=\ln|1|-\ln|\csc x+\cot x|=-\ln|\csc x+\cot x|$

[51] $\cos^2 t=1-\sin^2 t \Rightarrow \cos t=\pm\sqrt{1-\sin^2 t}$. Hence, choose any t such that $\cos t<0$.
Using $t=\pi$, LS $=\cos\pi=-1$. RS $=\sqrt{1-\sin^2\pi}=1$. Since $-1\neq 1$, LS $\neq$ RS.

[52] $(\sin t+\cos t)^2=\sin^2 t+2\cos t\,\sin t+\cos^2 t$. Hence, choose any t except 0 or $\frac{\pi}{2}$ and their coterminal angles. Using $t=\pi$, LS $=\sqrt{\sin^2\pi+\cos^2\pi}=1$.
RS $=\sin\pi+\cos\pi=0+(-1)=-1$. Since $1\neq -1$, LS $\neq$ RS.

[53] $\sqrt{\sin^2 t}=|\sin t|=\pm\sin t$. Hence, choose any t such that $\sin t<0$.
Using $t=\frac{3\pi}{2}$, LS $=\sqrt{(-1)^2}=1$. RS $=\sin\frac{3\pi}{2}=-1$. Since $1\neq -1$, LS $\neq$ RS.

[54] $\sec^2 t=\tan^2 t+1 \Rightarrow \sec t=\pm\sqrt{\tan^2 t+1}$. Hence, choose any t such that $\sec t<0$.
Using $t=\frac{3\pi}{4}$, LS $=\sec\frac{3\pi}{4}=-\sqrt{2}$. RS $=\sqrt{(-1)^2+1}=\sqrt{2}$.
Since $-\sqrt{2}\neq\sqrt{2}$, LS $\neq$ RS.

[55] $(\sin\theta+\cos\theta)^2=\sin^2\theta+2\sin\theta\,\cos\theta+\cos^2\theta$. Hence, choose any θ such that $\sin\theta\,\cos\theta\neq 0$. Using $\theta=\frac{\pi}{4}$, LS $=(\frac{1}{2}\sqrt{2}+\frac{1}{2}\sqrt{2})^2=(\sqrt{2})^2=2$.
RS $=(\frac{1}{2}\sqrt{2})^2+(\frac{1}{2}\sqrt{2})^2=\frac{1}{2}+\frac{1}{2}=1$. Since $2\neq 1$, LS $\neq$ RS.

[56] $\log(1/\sin t)=-\log\sin t\neq(\log\sin t)^{-1}$. $\log\sin t$ is defined if $\sin t>0$. If $\sin t>0$, $\log\sin t\leq 0$ since $\sin t\leq 1$ and $-\log\sin t\geq 0$. $(\log\sin t)^{-1}<0$ when defined so LS is never equal to RS. Using $t=\frac{\pi}{6}$, LS $=\log 2$. RS $=1/\log\frac{1}{2}=-1/\log 2$.
Since $\log 2\neq -1/\log 2$, LS $\neq$ RS.

[57] $\cos(-t)=-\cos t$. Choose any t such that $\cos t\neq-\cos t$, i.e., any t such that $\cos t\neq 0$. Using $t=\pi$, LS $=\cos(-\pi)=-1$. RS $=-\cos\pi=-(-1)=1$.
Since $-1\neq 1$, LS $\neq$ RS.

[58] From the unit circle, $\sin(t+\pi) = -\sin t$. Choose any t such that $-\sin t \neq \sin t$, i.e., any t such that $\sin t \neq 0$. Using $t = \frac{\pi}{2}$, LS $= \sin\frac{3\pi}{2} = -1$.

RS $= \sin\frac{\pi}{2} = 1$. Since $-1 \neq 1$, LS $\neq$ RS.

[59] Don't confuse $\cos(\sec t) = 1$ with $\cos t \cdot \sec t = 1$. Choose any t such that $\sec t \neq 2\pi n$.

Using $t = \frac{\pi}{4}$, LS $= \cos(\sec\frac{\pi}{4}) = \cos\sqrt{2} \neq 1 =$ RS.

[60] Don't confuse $\cot(\tan\theta) = 1$ with $\cot\theta \cdot \tan\theta = 1$. Choose any θ such that

$\tan\theta \neq \frac{\pi}{4} + \pi n$. Using $\theta = \frac{\pi}{4}$, LS $= \cot(\tan\frac{\pi}{4}) = \cot 1 \neq 1 =$ RS.

[61] LS $= (\sec x + \tan x)^2 = \sec^2 x + 2\sec x\,\tan x + \tan^2 x$

$= (\tan^2 x + 1) + 2\sec x\,\tan x + \tan^2 x = 2\tan^2 x + 2\sec x\,\tan x + 1$

RS $= 2\tan x(\tan x + \sec x) = 2\tan^2 x + 2\tan x\,\sec x$

Thus, LS $=$ RS $+ 1$, and the equation *is not* an identity.

[62] LS $= \dfrac{\tan^2 x}{\sec x - 1} = \dfrac{\tan^2 x}{\sec x - 1}\cdot\dfrac{\sec x + 1}{\sec x + 1} = \dfrac{(\tan^2 x)(\sec x + 1)}{(\sec x - 1)(\sec x + 1)}$

$= \dfrac{(\tan^2 x)(\sec x + 1)}{\sec^2 x - 1} = \dfrac{(\tan^2 x)(\sec x + 1)}{\tan^2 x} = \sec x + 1 \neq \sec x =$ RS

Thus, LS $=$ RS $+ 1$, and the equation *is not* an identity.

[63] LS $= \cos x(\tan x + \cot x) = \cos x\left(\dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x}\right) = \sin x + \dfrac{\cos^2 x}{\sin x}$

$= \dfrac{\sin^2 x + \cos^2 x}{\sin x} = \dfrac{1}{\sin x} = \csc x =$ RS, so the equation *is* an identity.

[64] LS $= \csc^2 x + \sec^2 x = \dfrac{1}{\sin^2 x} + \dfrac{1}{\cos^2 x} = \dfrac{\cos^2 x + \sin^2 x}{\sin^2 x\,\cos^2 x}$

$= \dfrac{1}{\sin^2 x\,\cos^2 x} = \dfrac{1}{\sin^2 x}\cdot\dfrac{1}{\cos^2 x} = \csc^2 x\,\sec^2 x =$ RS, so the equation *is* an identity.

Note: Exer. 65–68: Use $\sqrt{a^2 - x^2} = a\cos\theta$ because

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2(1 - \sin^2\theta)} = \sqrt{a^2\cos^2\theta} = |\,a\,|\,|\cos\theta\,| = a\cos\theta$$

since $\cos\theta > 0$ if $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $a > 0$.

[65] $(a^2 - x^2)^{3/2} = (\sqrt{a^2 - x^2})^3 = (a\cos\theta)^3 = a^3\cos^3\theta$

[66] $\dfrac{\sqrt{a^2 - x^2}}{x} = \dfrac{a\cos\theta}{a\sin\theta} = \cot\theta$

[67] $\dfrac{x^2}{\sqrt{a^2 - x^2}} = \dfrac{a^2\sin^2\theta}{a\cos\theta} = a\cdot\dfrac{\sin\theta}{\cos\theta}\cdot\sin\theta = a\tan\theta\,\sin\theta$

[68] $\dfrac{1}{x\sqrt{a^2 - x^2}} = \dfrac{1}{(a\sin\theta)(a\cos\theta)} = \dfrac{1}{a^2}\csc\theta\,\sec\theta$

Note: Exer. 69–72: Use $\sqrt{a^2 + x^2} = a\sec\theta$ because

$$\sqrt{a^2 + x^2} = \sqrt{a^2 + a^2\tan^2\theta} = \sqrt{a^2(1 + \tan^2\theta)} = \sqrt{a^2\sec^2\theta} = |\,a\,|\,|\sec\theta\,| =$$

$a\sec\theta$ since $\sec\theta > 0$ if $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $a > 0$.

$\boxed{69}$ See the *Note* on the preceding page. $\sqrt{a^2+x^2} = a\sec\theta$

$\boxed{70}$ $\dfrac{1}{\sqrt{a^2+x^2}} = \dfrac{1}{a\sec\theta} = \frac{1}{a}\cos\theta$

$\boxed{71}$ $\dfrac{1}{x^2+a^2} = \dfrac{1}{(\sqrt{a^2+x^2})^2} = \dfrac{1}{(a\sec\theta)^2} = \dfrac{1}{a^2\sec^2\theta} = \dfrac{1}{a^2}\cos^2\theta$

$\boxed{72}$ $\dfrac{(x^2+a^2)^{3/2}}{x} = \dfrac{(\sqrt{a^2+x^2})^3}{x} = \dfrac{(a\sec\theta)^3}{a\tan\theta} = a^2\sec^2\theta\cdot\dfrac{1/\cos\theta}{\sin\theta/\cos\theta} = a^2\sec^2\theta\csc\theta$

Note: Exer. 73–76: Use $\sqrt{x^2-a^2} = a\tan\theta$ because

$$\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2(\sec^2\theta-1)} = \sqrt{a^2\tan^2\theta} = |a|\,|\tan\theta| =$$

$$a\tan\theta \text{ since } \tan\theta > 0 \text{ if } 0<\theta<\tfrac{\pi}{2} \text{ and } a>0.$$

$\boxed{73}$ $\sqrt{x^2-a^2} = a\tan\theta$

$\boxed{74}$ $\dfrac{1}{x^2\sqrt{x^2-a^2}} = \dfrac{1}{(a^2\sec^2\theta)(a\tan\theta)} = \dfrac{1}{a^3}\cos^2\theta\cot\theta$

$\boxed{75}$ $x^3\sqrt{x^2-a^2} = (a^3\sec^3\theta)(a\tan\theta) = a^4\sec^3\theta\tan\theta$

$\boxed{76}$ $\dfrac{\sqrt{x^2-a^2}}{x^2} = \dfrac{a\tan\theta}{a^2\sec^2\theta} = \dfrac{1}{a}\cdot\dfrac{\sin\theta/\cos\theta}{1/\cos\theta}\cdot\dfrac{1}{\sec\theta} = \frac{1}{a}\sin\theta\cos\theta$

$\boxed{77}$ The graph of f appears to be that of $y = g(x) = -1$.

$$\frac{\sin^2 x - \sin^4 x}{(1-\sec^2 x)\cos^4 x} = \frac{\sin^2 x(1-\sin^2 x)}{-\tan^2 x\cos^4 x} = \frac{\sin^2 x\cos^2 x}{-(\sin^2 x/\cos^2 x)\cos^4 x} = \frac{\sin^2 x\cos^2 x}{-\sin^2 x\cos^2 x} = -1$$

$\boxed{78}$ The graph of f appears to be that of $y = g(x) = \sin x$.

$$\frac{\sin x - \sin^3 x}{\cos^4 x + \cos^2 x\sin^2 x} = \frac{\sin x(1-\sin^2 x)}{\cos^2 x(\cos^2 x+\sin^2 x)} = \frac{\sin x\cos^2 x}{\cos^2 x(1)} = \sin x$$

$\boxed{79}$ The graph of f appears to be that of $y = g(x) = \cos x$.

$$\sec x(\sin x\cos x + \cos^2 x) - \sin x = \sec x\cos x(\sin x + \cos x) - \sin x =$$

$$(\sin x + \cos x) - \sin x = \cos x$$

$\boxed{80}$ The graph of f appears to be that of $y = g(x) = 1$.

$$\frac{\sin^3 x + \sin x\cos^2 x}{\csc x} + \frac{\cos^3 x + \cos x\sin^2 x}{\sec x} =$$

$$\frac{\sin x(\sin^2 x+\cos^2 x)}{\csc x} + \frac{\cos x(\cos^2 x+\sin^2 x)}{\sec x} = \frac{\sin x}{\csc x} + \frac{\cos x}{\sec x} = \sin^2 x + \cos^2 x = 1$$

7.2 Exercises

$\boxed{1}$ In $[0, 2\pi)$, $\sin x = -\dfrac{\sqrt{2}}{2}$ only if $x = \frac{5\pi}{4}, \frac{7\pi}{4}$. All solutions would include these angles plus all angles coterminal with them. Hence, $x = \frac{5\pi}{4} + 2\pi n, \frac{7\pi}{4} + 2\pi n$.

$\boxed{2}$ $\cos t = -1 \Rightarrow t = \pi + 2\pi n$, or, equivalently, $(2n+1)\pi$.

$\boxed{3}$ $\tan\theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3} + \pi n$.

$\boxed{4}$ $\cot\alpha = -\frac{1}{\sqrt{3}} \Rightarrow \tan\alpha = -\sqrt{3} \Rightarrow \alpha = \frac{2\pi}{3} + \pi n.$

$\boxed{5}$ $\sec\beta = 2 \Rightarrow \cos\beta = \frac{1}{2} \Rightarrow \beta = \frac{\pi}{3} + 2\pi n, \frac{5\pi}{3} + 2\pi n.$

$\boxed{6}$ $\csc\gamma = \sqrt{2} \Rightarrow \sin\gamma = \frac{1}{\sqrt{2}} \Rightarrow \gamma = \frac{\pi}{4} + 2\pi n, \frac{3\pi}{4} + 2\pi n.$

$\boxed{7}$ $\sin x = \frac{\pi}{2}$ has no solution since $\frac{\pi}{2} > 1$, which is not in the range $[-1, 1]$.

$\boxed{8}$ $\cos x = -\frac{\pi}{3}$ has no solution since $-\frac{\pi}{3} < -1$, which is not in the range $[-1, 1]$.

$\boxed{9}$ $\cos\theta = \frac{1}{\sec\theta}$ is true for all values for which the equation is defined.

★ All θ *except* $\theta = \frac{\pi}{2} + \pi n$

$\boxed{10}$ $\csc\theta\sin\theta = 1$ is true for all values for which the equation is defined.

★ All θ *except* $\theta = \pi n$

$\boxed{11}$ $2\cos 2\theta - \sqrt{3} = 0 \Rightarrow \cos 2\theta = \frac{\sqrt{3}}{2} \Rightarrow 2\theta = \frac{\pi}{6} + 2\pi n, \frac{11\pi}{6} + 2\pi n \Rightarrow$

$\theta = \frac{\pi}{12} + \pi n, \frac{11\pi}{12} + \pi n$

$\boxed{12}$ $2\sin 3\theta + \sqrt{2} = 0 \Rightarrow \sin 3\theta = -\frac{\sqrt{2}}{2} \Rightarrow 3\theta = \frac{5\pi}{4} + 2\pi n, \frac{7\pi}{4} + 2\pi n \Rightarrow$

$\theta = \frac{5\pi}{12} + \frac{2\pi}{3}n, \frac{7\pi}{12} + \frac{2\pi}{3}n$

$\boxed{13}$ $\sqrt{3}\tan\frac{1}{3}t = 1 \Rightarrow \tan\frac{1}{3}t = \frac{1}{\sqrt{3}} \Rightarrow \frac{1}{3}t = \frac{\pi}{6} + \pi n \Rightarrow t = \frac{\pi}{2} + 3\pi n$

$\boxed{14}$ $\cos\frac{1}{4}x = -\frac{\sqrt{2}}{2} \Rightarrow \frac{1}{4}x = \frac{3\pi}{4} + 2\pi n, \frac{5\pi}{4} + 2\pi n \Rightarrow x = 3\pi + 8\pi n, 5\pi + 8\pi n$

$\boxed{15}$ $\sin(\theta + \frac{\pi}{4}) = \frac{1}{2} \Rightarrow \theta + \frac{\pi}{4} = \frac{\pi}{6} + 2\pi n, \frac{5\pi}{6} + 2\pi n \Rightarrow \theta = -\frac{\pi}{12} + 2\pi n, \frac{7\pi}{12} + 2\pi n$

$\boxed{16}$ $\cos(x - \frac{\pi}{3}) = -1 \Rightarrow x - \frac{\pi}{3} = \pi + 2\pi n \Rightarrow x = \frac{4\pi}{3} + 2\pi n$

$\boxed{17}$ $\sin(2x - \frac{\pi}{3}) = \frac{1}{2} \Rightarrow 2x - \frac{\pi}{3} = \frac{\pi}{6} + 2\pi n, \frac{5\pi}{6} + 2\pi n \Rightarrow 2x = \frac{\pi}{2} + 2\pi n, \frac{7\pi}{6} + 2\pi n \Rightarrow$

$x = \frac{\pi}{4} + \pi n, \frac{7\pi}{12} + \pi n$

$\boxed{18}$ $\cos(4x - \frac{\pi}{4}) = \frac{\sqrt{2}}{2} \Rightarrow 4x - \frac{\pi}{4} = \frac{\pi}{4} + 2\pi n, \frac{7\pi}{4} + 2\pi n \Rightarrow$

$4x = \frac{\pi}{2} + 2\pi n, 2\pi + 2\pi n$ {or just $2\pi n$} $\Rightarrow x = \frac{\pi}{8} + \frac{\pi}{2}n, \frac{\pi}{2}n$

$\boxed{19}$ $2\cos t + 1 = 0 \Rightarrow \cos t = -\frac{1}{2} \Rightarrow t = \frac{2\pi}{3} + 2\pi n, \frac{4\pi}{3} + 2\pi n$

$\boxed{20}$ $\cot\theta + 1 = 0 \Rightarrow \cot\theta = -1 \Rightarrow \theta = \frac{3\pi}{4} + \pi n$

$\boxed{21}$ $\tan^2 x = 1 \Rightarrow \tan x = \pm 1 \Rightarrow x = \frac{\pi}{4} + \pi n, \frac{3\pi}{4} + \pi n$, or simply $\frac{\pi}{4} + \frac{\pi}{2}n$

$\boxed{22}$ $4\cos\theta - 2 = 0 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3} + 2\pi n, \frac{5\pi}{3} + 2\pi n$

$\boxed{23}$ $(\cos\theta - 1)(\sin\theta + 1) = 0 \Rightarrow \cos\theta = 1$ or $\sin\theta = -1 \Rightarrow \theta = 2\pi n$ or $\theta = \frac{3\pi}{2} + 2\pi n$

$\boxed{24}$ $2\cos x = \sqrt{3} \Rightarrow \cos x = \frac{\sqrt{3}}{2} \Rightarrow x = \frac{\pi}{6} + 2\pi n, \frac{11\pi}{6} + 2\pi n$

$\boxed{25}$ $\sec^2\alpha - 4 = 0 \Rightarrow \sec^2\alpha = 4 \Rightarrow \sec\alpha = \pm 2 \Rightarrow$

$\alpha = \frac{\pi}{3} + 2\pi n, \frac{5\pi}{3} + 2\pi n, \frac{2\pi}{3} + 2\pi n, \frac{4\pi}{3} + 2\pi n$, or simply $\frac{\pi}{3} + \pi n, \frac{2\pi}{3} + \pi n$

$\boxed{26}$ $3 - \tan^2\beta = 0 \Rightarrow \tan^2\beta = 3 \Rightarrow \tan\beta = \pm\sqrt{3} \Rightarrow \beta = \frac{\pi}{3} + \pi n, \frac{2\pi}{3} + \pi n$

$\boxed{27}$ $\sqrt{3} + 2\sin\beta = 0 \Rightarrow \sin\beta = -\frac{\sqrt{3}}{2} \Rightarrow \beta = \frac{4\pi}{3} + 2\pi n, \frac{5\pi}{3} + 2\pi n$

[28] $4\sin^2 x - 3 = 0 \Rightarrow \sin^2 x = \frac{3}{4} \Rightarrow \sin x = \pm\frac{\sqrt{3}}{2} \Rightarrow x = \frac{\pi}{3} + \pi n, \frac{2\pi}{3} + \pi n$

[29] $\cot^2 x - 3 = 0 \Rightarrow \cot^2 x = 3 \Rightarrow \cot x = \pm\sqrt{3} \Rightarrow x = \frac{\pi}{6} + \pi n, \frac{5\pi}{6} + \pi n$

[30] $(\sin t - 1)\cos t = 0 \Rightarrow \sin t = 1$ or $\cos t = 0 \Rightarrow$
$t = \frac{\pi}{2} + 2\pi n$ or $t = \frac{\pi}{2} + \pi n$, or simply $\frac{\pi}{2} + \pi n$

[31] $(2\sin\theta + 1)(2\cos\theta + 3) = 0 \Rightarrow \sin\theta = -\frac{1}{2}$ or $\cos\theta = -\frac{3}{2} \Rightarrow$
$\theta = \frac{7\pi}{6} + 2\pi n, \frac{11\pi}{6} + 2\pi n$ $\{\cos\theta = -\frac{3}{2}$ has no solutions$\}$

[32] $(2\sin u - 1)(\cos u - \sqrt{2}) = 0 \Rightarrow \sin u = \frac{1}{2}$ or $\cos u = \sqrt{2} \Rightarrow$
$u = \frac{\pi}{6} + 2\pi n, \frac{5\pi}{6} + 2\pi n$ $\{\cos u = \sqrt{2}$ has no solutions$\}$

[33] $\sin 2x\,(\csc 2x - 2) = 0 \Rightarrow 1 - 2\sin 2x = 0 \Rightarrow \sin 2x = \frac{1}{2} \Rightarrow$
$2x = \frac{\pi}{6} + 2\pi n, \frac{5\pi}{6} + 2\pi n \Rightarrow x = \frac{\pi}{12} + \pi n, \frac{5\pi}{12} + \pi n$

[34] $\tan\alpha + \tan^2\alpha = 0 \Rightarrow \tan\alpha(1 + \tan\alpha) = 0 \Rightarrow \tan\alpha = 0, -1 \Rightarrow \alpha = \pi n, \frac{3\pi}{4} + \pi n$

[35] $\cos(\ln x) = 0 \Rightarrow \ln x = \frac{\pi}{2} + \pi n \Rightarrow x = e^{(\pi/2) + \pi n}$

[36] $\ln(\sin x) = 0 \Rightarrow \sin x = 1 \Rightarrow x = \frac{\pi}{2} + 2\pi n$

[37] $\cos(2x - \frac{\pi}{4}) = 0 \Rightarrow 2x - \frac{\pi}{4} = \frac{\pi}{2} + \pi n \Rightarrow 2x = \frac{3\pi}{4} + \pi n \Rightarrow x = \frac{3\pi}{8} + \frac{\pi}{2}n.$
x will be in the interval $[0, 2\pi)$ if $n = 0, 1, 2,$ or 3. Thus, $x = \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$.

[38] $\sin(3x - \frac{\pi}{4}) = 1 \Rightarrow 3x - \frac{\pi}{4} = \frac{\pi}{2} + 2\pi n \Rightarrow 3x = \frac{3\pi}{4} + 2\pi n \Rightarrow x = \frac{\pi}{4} + \frac{2\pi}{3}n.$
x will be in the interval $[0, 2\pi)$ if $n = 0, 1,$ or 2. Thus, $x = \frac{\pi}{4}, \frac{11\pi}{12}, \frac{19\pi}{12}$.

[39] $2 - 8\cos^2 t = 0 \Rightarrow \cos^2 t = \frac{1}{4} \Rightarrow \cos t = \pm\frac{1}{2} \Rightarrow t = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

[40] $\cot^2\theta - \cot\theta = 0 \Rightarrow \cot\theta(\cot\theta - 1) = 0 \Rightarrow \cot\theta = 0, 1 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{4}, \frac{5\pi}{4}$

[41] $2\sin^2 u = 1 - \sin u \Rightarrow 2\sin^2 u + \sin u - 1 = 0 \Rightarrow (2\sin u - 1)(\sin u + 1) = 0 \Rightarrow$
$\sin u = \frac{1}{2}, -1 \Rightarrow u = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

[42] $2\cos^2 t + 3\cos t + 1 = 0 \Rightarrow (2\cos t + 1)(\cos t + 1) = 0 \Rightarrow$
$\cos t = -\frac{1}{2}, -1 \Rightarrow t = \frac{2\pi}{3}, \frac{4\pi}{3}, \pi$

[43] $\tan^2 x\,\sin x = \sin x \Rightarrow \tan^2 x\,\sin x - \sin x = 0 \Rightarrow \sin x(\tan^2 x - 1) = 0 \Rightarrow$
$\sin x = 0$ or $\tan x = \pm 1 \Rightarrow x = 0, \pi, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

[44] $\sec\beta\,\csc\beta = 2\csc\beta \Rightarrow \sec\beta\,\csc\beta - 2\csc\beta = 0 \Rightarrow \csc\beta(\sec\beta - 2) = 0 \Rightarrow$
$\csc\beta = 0$ or $\sec\beta = 2 \Rightarrow \beta = \frac{\pi}{3}, \frac{5\pi}{3}$ $\{\csc\beta = 0$ has no solutions$\}$

[45] $2\cos^2\gamma + \cos\gamma = 0 \Rightarrow \cos\gamma(2\cos\gamma + 1) = 0 \Rightarrow \cos\gamma = 0, -\frac{1}{2} \Rightarrow$
$\gamma = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}$

[46] $\sin x - \cos x = 0 \Rightarrow \sin x = \cos x \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}$

[47] $\sin^2\theta + \sin\theta - 6 = 0 \Rightarrow (\sin\theta + 3)(\sin\theta - 2) = 0 \Rightarrow \sin\theta = -3, 2.$
There are *no solutions* for either equation.

[48] $2\sin^2 u + \sin u - 6 = 0 \Rightarrow (2\sin u - 3)(\sin u + 2) = 0 \Rightarrow \sin u = \frac{3}{2}, -2.$
There are *no solutions* for either equation.

[49] $1-\sin t=\sqrt{3}\cos t$ • Square both sides to obtain an equation in either sin or cos.

$(1-\sin t)^2=(\sqrt{3}\cos t)^2 \Rightarrow 1-2\sin t+\sin^2 t=3\cos^2 t \Rightarrow$

$\sin^2 t-2\sin t+1=3(1-\sin^2 t) \Rightarrow 4\sin^2 t-2\sin t-2=0 \Rightarrow$

$2\sin^2 t-\sin t-1=0 \Rightarrow (2\sin t+1)(\sin t-1)=0 \Rightarrow \sin t=-\frac{1}{2}, 1 \Rightarrow$

$t=\frac{7\pi}{6}, \frac{11\pi}{6}, \frac{\pi}{2}$. Since each side of the equation was squared,

the solutions must be checked in the original equation. $\frac{7\pi}{6}$ is an extraneous solution.

[50] $\cos\theta-\sin\theta=1 \Rightarrow \cos\theta=1+\sin\theta \Rightarrow \cos^2\theta=1+2\sin\theta+\sin^2\theta \Rightarrow$

$1-\sin^2\theta=1+2\sin\theta+\sin^2\theta \Rightarrow 2\sin^2\theta+2\sin\theta=0 \Rightarrow 2\sin\theta(\sin\theta+1)=0 \Rightarrow$

$\sin\theta=0, -1 \Rightarrow \theta=0, \pi, \frac{3\pi}{2}$. π is an extraneous solution.

[51] $\cos\alpha+\sin\alpha=1 \Rightarrow \cos\alpha=1-\sin\alpha \Rightarrow \cos^2\alpha=1-2\sin\alpha+\sin^2\alpha \Rightarrow$

$1-\sin^2\alpha=1-2\sin\alpha+\sin^2\alpha \Rightarrow 2\sin^2\alpha-2\sin\alpha=0 \Rightarrow$

$2\sin\alpha(\sin\alpha-1)=0 \Rightarrow \sin\alpha=0, 1 \Rightarrow \alpha=0, \pi, \frac{\pi}{2}$. π is an extraneous solution.

[52] $\sqrt{3}\sin t+\cos t=1 \Rightarrow \sqrt{3}\sin t=1-\cos t \Rightarrow 3\sin^2 t=1-2\cos t+\cos^2 t \Rightarrow$

$3(1-\cos^2 t)=1-2\cos t+\cos^2 t \Rightarrow 4\cos^2 t-2\cos t-2=0 \Rightarrow$

$2\cos^2 t-\cos t-1=0 \Rightarrow (2\cos t+1)(\cos t-1)=0 \Rightarrow \cos t=-\frac{1}{2}, 1 \Rightarrow$

$t=\frac{2\pi}{3}, \frac{4\pi}{3}, 0$. $\frac{4\pi}{3}$ is an extraneous solution.

[53] $2\tan t-\sec^2 t=0 \Rightarrow 2\tan t-(1+\tan^2 t)=0 \Rightarrow \tan^2 t-2\tan t+1=0 \Rightarrow$

$(\tan t-1)^2=0 \Rightarrow \tan t=1 \Rightarrow t=\frac{\pi}{4}, \frac{5\pi}{4}$

[54] $\tan\theta+\sec\theta=1 \Rightarrow \sec^2\theta=(1-\tan\theta)^2 \Rightarrow 1+\tan^2\theta=1-2\tan\theta+\tan^2\theta \Rightarrow$

$2\tan\theta=0 \Rightarrow \theta=0, \pi$. π is an extraneous solution.

[55] $\cot\alpha+\tan\alpha=\csc\alpha\sec\alpha \Rightarrow \frac{\cos\alpha}{\sin\alpha}+\frac{\sin\alpha}{\cos\alpha}=\frac{1}{\sin\alpha\cos\alpha} \Rightarrow$

$\frac{\cos^2\alpha+\sin^2\alpha}{\sin\alpha\cos\alpha}=\frac{1}{\sin\alpha\cos\alpha}$. This is an identity and is true for *all numbers in* $[0, 2\pi)$

except $0, \frac{\pi}{2}, \pi$, and $\frac{3\pi}{2}$ since these values make the original equation undefined.

[56] $\sin x+\cos x\cot x=\csc x \Rightarrow \sin x+\cos x\cdot\frac{\cos x}{\sin x}=\frac{1}{\sin x} \Rightarrow \frac{\sin^2 x+\cos^2 x}{\sin x}=\frac{1}{\sin x}$.

This is an identity and is true for *all numbers in* $[0, 2\pi)$ *except* 0 and π

since these values make the original equation undefined.

[57] $2\sin^3 x+\sin^2 x-2\sin x-1=0 \Rightarrow \sin^2 x(2\sin x+1)-1(2\sin x+1)=0 \Rightarrow$

$(\sin^2 x-1)(2\sin x+1)=0 \Rightarrow \sin x=\pm 1, -\frac{1}{2} \Rightarrow x=\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$

[58] $\sec^5\theta=4\sec\theta \Rightarrow \sec\theta(\sec^4\theta-4)=0 \Rightarrow \sec\theta=0$ or $\sec^2\theta=\pm 2 \Rightarrow$

$\sec\theta=\pm\sqrt{2}$ {since $\sec\theta\neq 0$ and $\sec^2\theta\neq -2$} $\Rightarrow \theta=\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

[59] $2\tan t\csc t+2\csc t+\tan t+1=0 \Rightarrow 2\csc t(\tan t+1)+1(\tan t+1) \Rightarrow$

$(2\csc t+1)(\tan t+1)=0 \Rightarrow \csc t=-\frac{1}{2}$ or $\tan t=-1 \Rightarrow$

$t=\frac{3\pi}{4}, \frac{7\pi}{4}$ {since $\csc t\neq -\frac{1}{2}$}

[60] $2\sin v\csc v - \csc v = 4\sin v - 2 \Rightarrow 2\sin v\csc v - \csc v - 4\sin v + 2 = 0 \Rightarrow$

$\csc v(2\sin v - 1) - 2(2\sin v - 1) = 0 \Rightarrow (\csc v - 2)(2\sin v - 1) = 0 \Rightarrow$

$\csc v = 2$ or $\sin v = \frac{1}{2} \Rightarrow v = \frac{\pi}{6}, \frac{5\pi}{6}$. The equations are equivalent.

[61] $\sin^2 t - 4\sin t + 1 = 0 \Rightarrow \sin t = \dfrac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}$.

$(2+\sqrt{3}) > 1$ is not in the range of the sine, so $\sin t = 2 - \sqrt{3} \Rightarrow$

$t = 15°30'$ or $164°30'$ {to the nearest ten minutes}

[62] $\cos^2 t - 4\cos t + 2 = 0 \Rightarrow \cos t = \dfrac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}$.

$(2+\sqrt{2}) > 1$ is not in the range of the cosine, so $\cos t = 2 - \sqrt{2} \Rightarrow$

$t = 54°10'$ or $305°50'$ {to the nearest ten minutes}

[63] $\tan^2\theta + 3\tan\theta + 2 = 0 \Rightarrow (\tan\theta + 1)(\tan\theta + 2) = 0 \Rightarrow$

$\tan\theta = -1, -2 \Rightarrow \theta = 135°, 315°, 116°30', 296°30'$

[64] $2\tan^2 x - 3\tan x - 1 = 0 \Rightarrow$

$\tan x = \dfrac{3 \pm \sqrt{17}}{4} \Rightarrow x = 60°40', 240°40', 164°20', 344°20'$

[65] $12\sin^2 u - 5\sin u - 2 = 0 \Rightarrow (3\sin u - 2)(4\sin u + 1) = 0 \Rightarrow \sin u = \frac{2}{3}, -\frac{1}{4} \Rightarrow$

$u = 41°50', 138°10', 194°30', 345°30'$

[66] $5\cos^2\alpha + 3\cos\alpha - 2 = 0 \Rightarrow (5\cos\alpha - 2)(\cos\alpha + 1) = 0 \Rightarrow \cos\alpha = \frac{2}{5}, -1 \Rightarrow$

$\alpha = 66°30', 293°30', 180°$

[67] $y > 12.5 \Rightarrow 25\cos\frac{\pi}{15}t > 12.5 \Rightarrow \cos\frac{\pi}{15}t > \frac{1}{2} \Rightarrow -\frac{\pi}{3} < \frac{\pi}{15}t < \frac{\pi}{3} \Rightarrow$

$-5 < t < 5 \Rightarrow y > 12.5$ for about $5 - (-5) = 10$ minutes of each 30-minute period.

[68] The low temperature will be below -4°F when $T < -4$.

$T < -4 \Rightarrow 36\sin\left[\frac{2\pi}{365}(t-101)\right] + 14 < -4 \Rightarrow \sin\left[\frac{2\pi}{365}(t-101)\right] < -\frac{1}{2} \Rightarrow$

$\frac{7\pi}{6} < \frac{2\pi}{365}(t-101) < \frac{11\pi}{6} \Rightarrow \frac{2555}{12} < t - 101 < \frac{4015}{12} \Rightarrow \frac{3767}{12} < t < \frac{5227}{12} \Rightarrow$

$313\frac{11}{12} < t < 435\frac{7}{12} \Rightarrow T < -4$ for $435\frac{7}{12} - 313\frac{11}{12} = 121\frac{2}{3}$ days.

[69] (b) July: $T(7) = 83$°F; October: $T(10) = 56.5$°F.

(c) Graph $Y_1 = 26.5\sin(\frac{\pi}{6}x - \frac{2\pi}{3}) + 56.5$ and $Y_2 = 69$. Their graphs intersect at $t \approx 4.94$, 9.06 on $[1, 13]$. The average high temperature is above 69°F approximately May through September.

(d) A sine function is periodic and varies between a maximum and minimum value. Average monthly high temperatures are also seasonal with a 12-month period. Therefore, a sine function is a reasonable function to model these temperatures. See *Figure 69.*

[1, 25, 5] by [0, 100, 10]

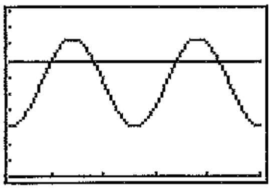

Figure 69

[1, 25, 5] by [0, 100, 10]

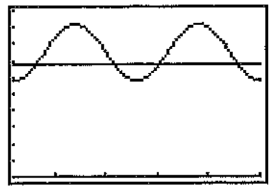

Figure 70

[70] (b) April: $T(4) = 75$°F; December: $T(12) \approx 60.3$°F.

(c) Graph $Y_1 = 17\cos\left(\frac{\pi}{6}x - \frac{7\pi}{6}\right) + 75$ and $Y_2 = 67$. Their graphs intersect at $t \approx 3.06$, 10.94 on the interval [1, 13]. The average high temperature is below 67°F approximately November through March.

[71] $I = \frac{1}{2}I_M$ and $D = 12 \Rightarrow \frac{1}{2}I_M = I_M \sin^3 \frac{\pi}{12}t \Rightarrow \sin^3 \frac{\pi}{12}t = \frac{1}{2} \Rightarrow$

$\sin\frac{\pi}{12}t = \sqrt[3]{\frac{1}{2}} \Rightarrow \frac{\pi}{12}t \approx 0.9169$ and 2.2247 { $\pi - 0.9169 \approx 2.2247$ is the reference angle for 0.9169 in QII. } $\Rightarrow t \approx 3.50$ and $t \approx 8.50$

[72] $I = \frac{1}{2}I_M$ and $D = 12 \Rightarrow \frac{1}{2}I_M = I_M \sin^2 \frac{\pi}{12}t \Rightarrow \sin^2 \frac{\pi}{12}t = \frac{1}{2} \Rightarrow \sin\frac{\pi}{12}t = \pm\sqrt{\frac{1}{2}}$.

The sine is positive since if $0 \le t \le D$, then $0 \le \frac{\pi t}{D} \le \pi$.

Thus, $\frac{\pi}{12}t = \frac{\pi}{4}$ and $\frac{\pi}{12}t = \frac{3\pi}{4} \Rightarrow t = 3$ and $t = 9$.

[73] (a) $I > 0.75\,I_M \Rightarrow I_M \sin^3 \frac{\pi}{12}t > 0.75\,I_M \Rightarrow \sin^3 \frac{\pi}{12}t > \frac{3}{4} \Rightarrow \sin\frac{\pi}{12}t > \sqrt[3]{\frac{3}{4}} \Rightarrow$

$1.1398 < \frac{\pi}{12}t < 2.0018 \Rightarrow 4.3538 < t < 7.6462$, or approximately 3.29 hours.

(b) $I > 0.75\,I_M \Rightarrow I_M \sin^2 \frac{\pi}{12}t > 0.75\,I_M \Rightarrow \sin^2 \frac{\pi}{12}t > \frac{3}{4} \Rightarrow \sin\frac{\pi}{12}t > \frac{1}{2}\sqrt{3} \Rightarrow$

$\frac{\pi}{3} < \frac{\pi}{12}t < \frac{2\pi}{3} \Rightarrow 4 < t < 8$, or 4 hours.

[74] (a) On the surface, $x = 0$. Thus, $T = T_0 e^{-\lambda(0)} \sin(\omega t - \lambda(0)) = T_0 \sin\omega t$.

Since the period is 24 hours, $24 = \frac{2\pi}{\omega}$, or $\omega = \frac{\pi}{12}$.

The formula for the temperature at the surface is then $T = T_0 \sin\frac{\pi}{12}t$.

(b) T will be a minimum when $\sin\frac{\pi}{12}t$ equals -1.

$\sin\frac{\pi}{12}t = -1 \Rightarrow \frac{\pi}{12}t = \frac{3\pi}{2} + 2\pi n \Rightarrow t = 18 + 24n$ for $n = 0, 1, 2, \ldots$.

(c) If $\lambda = 2.5$ and $x = 1$, then $T = T_0 e^{-2.5} \sin\left(\frac{\pi}{12}t - 2.5\right)$. As in part (b),

$\sin\left(\frac{\pi}{12}t - 2.5\right) = -1 \Rightarrow \frac{\pi}{12}t - \frac{5}{2} = -\frac{\pi}{2} + 2\pi n \Rightarrow \frac{\pi}{12}t = \frac{5-\pi}{2} + 2\pi n \Rightarrow$

$t = \frac{6(5-\pi)}{\pi} + 24n$ for $n = 0, 1, 2, \ldots \approx 3.55 + 24n$ for $n = 0, 1, 2, \ldots$.

{ If $\frac{3\pi}{2}$ is used instead of $-\frac{\pi}{2}$, then $n = -1, 0, 1, 2, \ldots$ }

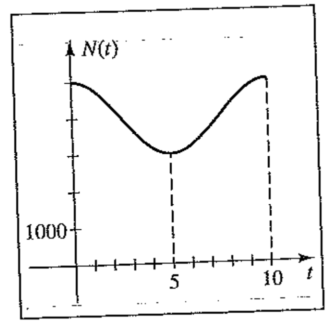

Figure 75

$\boxed{75}$ (a) $N(t) = 1000\cos\frac{\pi}{5}t + 4000$, amplitude $= 1000$,

period $= \frac{2\pi}{\pi/5} = 10$ years

(b) $N > 4500 \Rightarrow 1000\cos\frac{\pi}{5}t + 4000 > 4500 \Rightarrow$

$\cos\frac{\pi}{5}t > \frac{1}{2} \Rightarrow 0 \le \frac{\pi}{5}t < \frac{\pi}{3}$ and $\frac{5\pi}{3} < \frac{\pi}{5}t \le 2\pi \Rightarrow$

$0 \le t < \frac{5}{3}$ and $\frac{25}{3} < t \le 10$

$\boxed{76}$ $F > 55{,}000 \Rightarrow 26{,}000\sin\left[\frac{\pi}{6}(t-5.5)\right] + 34{,}000 > 55{,}000 \Rightarrow$

$\sin\left[\frac{\pi}{6}(t-5.5)\right] > \frac{21}{26} \Rightarrow$ {approximate values} $0.94 < \frac{\pi}{6}(t-5.5) < 2.20 \Rightarrow$

$1.8 < t - 5.5 < 4.2 \Rightarrow 7.3 < t < 9.7 \Rightarrow F > 55{,}000$ for about 2.4 months.

$\boxed{77}$ $\frac{1}{2} + \cos x = 0 \Rightarrow \cos x = -\frac{1}{2} \Rightarrow x = -\frac{4\pi}{3}, -\frac{2\pi}{3}, \frac{2\pi}{3}$, and $\frac{4\pi}{3}$ {for x in $[-2\pi, 2\pi]$}

for A, B, C, and D, respectively. The corresponding y values are found by using

$y = \frac{1}{2}x + \sin x$ with each of the above values. The points are:

$A(-\frac{4\pi}{3}, -\frac{2\pi}{3} + \frac{1}{2}\sqrt{3})$, $B(-\frac{2\pi}{3}, -\frac{\pi}{3} - \frac{1}{2}\sqrt{3})$, $C(\frac{2\pi}{3}, \frac{\pi}{3} + \frac{1}{2}\sqrt{3})$, and $D(\frac{4\pi}{3}, \frac{2\pi}{3} - \frac{1}{2}\sqrt{3})$

$\boxed{78}$ $4\cos 2x - \sin 2x = 0 \Rightarrow 4\cos 2x = \sin 2x \Rightarrow 4 = \tan 2x \Rightarrow$

$2x \approx 1.3258 + \pi n \Rightarrow x \approx 0.6629 + \frac{\pi}{2}n$ for $n = 0, 1, 2, \ldots,$

or $0.66, 2.23, 3.80, 5.38, \ldots$.

$\boxed{79}$ $-10 = 20\sin(60\pi t - 6\pi) \Rightarrow \sin(60\pi t - 6\pi) = -\frac{1}{2} \Rightarrow 60\pi t_1 - 6\pi = \frac{7\pi}{6} + 2\pi n$ or

$60\pi t_2 - 6\pi = \frac{11\pi}{6} + 2\pi n \Rightarrow 60t_1 = \frac{43}{6} + 2n$ or $60t_2 = \frac{47}{6} + 2n \Rightarrow t_1 = \frac{43}{360} + \frac{1}{30}n$ or

$t_2 = \frac{47}{360} + \frac{1}{30}n$. $t_1 > 0 \Rightarrow \frac{1}{30}n > -\frac{43}{360} \Rightarrow n > -\frac{43}{12}$. If $n = -3$, then $t_1 = \frac{7}{360}$.

$t_2 > 0 \Rightarrow \frac{1}{30}n > -\frac{47}{360} \Rightarrow n > -\frac{47}{12}$. If $n = -3$, then $t_1 = \frac{11}{360}$. Thus, $t = \frac{7}{360}$ sec.

$\boxed{80}$ $20 = 40\sin(100\pi t - 4\pi) \Rightarrow \sin(100\pi t - 4\pi) = \frac{1}{2} \Rightarrow 100\pi t_1 - 4\pi = \frac{\pi}{6} + 2\pi n$ or

$100\pi t_2 - 4\pi = \frac{5\pi}{6} + 2\pi n \Rightarrow 100t_1 = \frac{25}{6} + 2n$ or $100t_2 = \frac{29}{6} + 2n \Rightarrow t_1 = \frac{25}{600} + \frac{1}{50}n$

or $t_2 = \frac{29}{600} + \frac{1}{50}n$. $t_1 > 0 \Rightarrow \frac{1}{50}n > -\frac{25}{600} \Rightarrow n > -\frac{25}{12}$. If $n = -2$, then $t_1 = \frac{1}{600}$.

$t_2 > 0 \Rightarrow \frac{1}{50}n > -\frac{29}{600} \Rightarrow n > -\frac{29}{12}$. If $n = -2$, then $t_2 = \frac{5}{600}$. Thus, $t = \frac{1}{600}$ sec.

$\boxed{81}$ Graph $y = \cos x$ and $y = 0.3$ on the same coordinate plane.

The points of intersection are located at $x \approx 1.27, 5.02$, and $\cos x$ is less than 0.3

between these values. Therefore, $\cos x \ge 0.3$ on $[0, 1.27] \cup [5.02, 2\pi]$.

$[0, 2\pi, \pi/4]$ by $[-2.09, 2.09]$

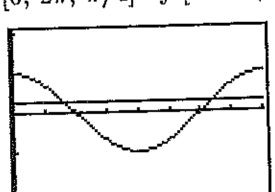

Figure 81

$[0, 2\pi, \pi/4]$ by $[-2.09, 2.09]$

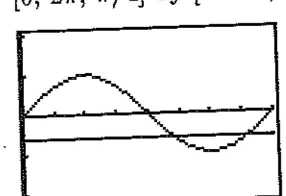

Figure 82

[82] Graph $y = \sin x$ and $y = -0.6$ on the same coordinate plane.

The points of intersection are located at $x \approx 3.79,\ 5.64$.

From *Figure 82*, we see that $\sin x < -0.6$ on $(3.79,\ 5.64)$.

[83] Graph $y = \cos 3x$ and $y = \sin x$ on the same coordinate plane.

The points of intersection are located at $x \approx 0.39,\ 1.96,\ 2.36,\ 3.53,\ 5.11,\ 5.50$.

From the graph, we see that $\cos 3x$ is less than $\sin x$ on

$(0.39,\ 1.96) \cup (2.36,\ 3.53) \cup (5.11,\ 5.50)$.

$[0,\ 2\pi,\ \pi/4]$ by $[-2.09,\ 2.09]$

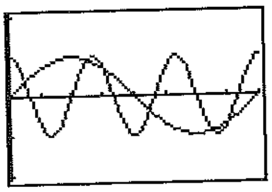

Figure 83

$[0,\ 2\pi,\ \pi/4]$ by $[-2.09,\ 2.09]$

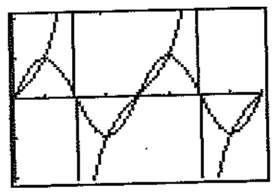

Figure 84

[84] Graph $y = \tan x$ and $y = \sin 2x$ on the same coordinate plane.

The points of intersection are located at $x \approx 0,\ 0.79,\ 2.36,\ \pi,\ 3.93,\ 5.50,\ 2\pi$.

$\tan x$ is undefined at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. From the graph, we see that $\tan x \leq \sin 2x$ on

$[0,\ 0.79] \cup (\frac{\pi}{2},\ 2.36] \cup [\pi,\ 3.93] \cup (\frac{3\pi}{2},\ 5.50]$.

[85] (a) The largest zero occurs when $x \approx 0.6366$.

(b) As x becomes large, the graph of $f(x) = \cos(1/x)$ approaches the horizontal asymptote $y = 1$.

(c) There appears to be an infinite number of zeros on $[0,\ c]$ for any $c > 0$.

$[0,\ 3]$ by $[-1.5,\ 1.5]$

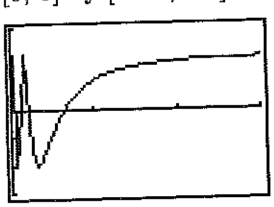

Figure 85

$[0,\ 3]$ by $[-1.5,\ 1.5]$

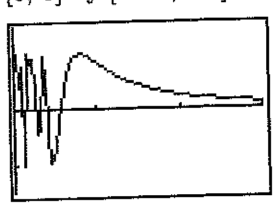

Figure 86

[86] (a) The largest zero occurs when $x \approx 0.5642$.

(b) As x becomes large, the graph of $f(x) = \sin(1/x^2)$ approaches the horizontal asymptote $y = 0$.

(c) There appears to be an infinite number of zeros on $[0,\ c]$ for any $c > 0$.

Note: Exer. 87–90: Graph $Y_1 = M$ and $Y_2 = \theta + e\sin\theta$ and approximate the value of θ such that $Y_1 = Y_2$.

$\boxed{87}$ Mercury: $Y_1 = 5.241$ and $Y_2 = \theta + 0.206\sin\theta$ intersect when $\theta \approx 5.400$ (radians).

[0, 12] by [0, 8]

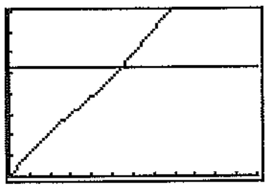

Figure 87

[0, 12] by [0, 8]

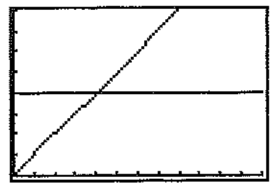

Figure 88

$\boxed{88}$ Mars: $Y_1 = 4.028$ and $Y_2 = \theta + 0.093\sin\theta$ intersect when $\theta \approx 4.104$.

$\boxed{89}$ Earth: $Y_1 = 3.611$ and $Y_2 = \theta + 0.0167\sin\theta$ intersect when $\theta \approx 3.619$.

[0, 12] by [0, 8]

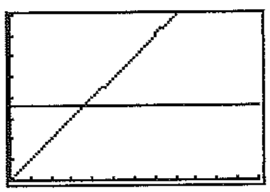

Figure 89

[0, 0.3, 0.1] by [0, 0.2, 0.1]

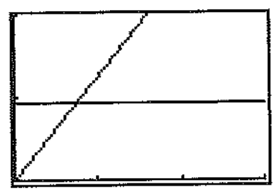

Figure 90

$\boxed{90}$ Pluto: $Y_1 = 0.09424$ and $Y_2 = \theta + 0.255\sin\theta$ intersect when $\theta \approx 0.075$.

$\boxed{91}$ Graph $y = \sin 2x$ and $y = 2 - x^2$. From the graph, we see that there are two points of intersection. The x-coordinates of these points are $x \approx -1.48, 1.08$.

$[-\pi, \pi, \pi/4]$ by $[-2.09, 2.09]$

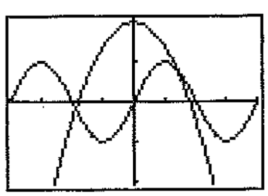

Figure 91

$[-\pi, \pi, \pi/4]$ by $[-2.09, 2.09]$

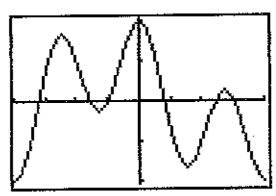

Figure 92

$\boxed{92}$ Graph $y = \cos^3 x + \cos 3x - \sin^3 x$.

The graph has x-intercepts at $x \approx -2.51, -1.22, -0.79, 0.63, 1.92, 2.36$.

93 Graph $y = \ln(1 + \sin^2 x)$ and $y = \cos x$. From the graph, we see that there are two points of intersection. The x-coordinates of these points are $x \approx \pm 1.00$.

$[-\pi, \pi, \pi/4]$ by $[-2.09, 2.09]$

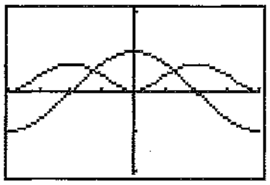

Figure 93

$[-\pi, \pi, \pi/4]$ by $[-2.09, 2.09]$

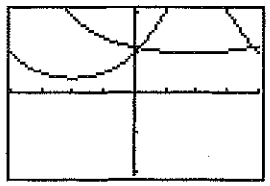

Figure 94

94 Graph $y = e^{\sin x}$ and $y = \sec(\frac{1}{3}x - \frac{1}{2})$. From the graph, we see that there are two points of intersection. The x-coordinates of these points are $x \approx 0.11, 3.01$.

95 Graph $y = 3\cos^4 x - 2\cos^3 x + \cos x - 1$.

The graph has x-intercepts at $x \approx \pm 0.64, \pm 2.42$.

$[-\pi, \pi, \pi/4]$ by $[-2.09, 2.09]$

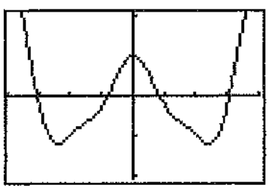

Figure 95

$[-\pi, \pi, \pi/4]$ by $[-2.09, 2.09]$

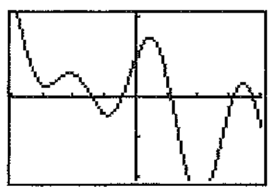

Figure 96

96 Graph $y = \cos 2x + \sin 3x - \tan\frac{1}{3}x$.

The graph has x-intercepts at $x \approx -1.13, -0.36, 0.88, 2.45, 2.94$.

97 (a) $9.8 = 9.8066(1 - 0.00264\cos 2\theta) \Rightarrow 0.00264\cos 2\theta = 1 - \frac{9.8}{9.8066} \Rightarrow$

$\cos 2\theta = \frac{0.0066}{(9.8066)(0.00264)} \Rightarrow 2\theta \approx 75.2^\circ \Rightarrow \theta \approx 37.6^\circ$

(b) At the equator, $g_0 = 9.8066(1 - 0.00264\cos 0^\circ) = 9.8066(0.99736)$.

At $\theta = 0^\circ$, $W = kg \Rightarrow 150 = kg_0 \Rightarrow k = \dfrac{150}{g_0} \Rightarrow W = \dfrac{150}{g_0}g$.

$W = 150.5 \Rightarrow 150.5 = \frac{150}{0.99736}(1 - 0.00264\cos 2\theta) \Rightarrow$

$0.00264\cos 2\theta = 1 - \frac{150.5(0.99736)}{150} \Rightarrow \cos 2\theta \approx -0.2593 \Rightarrow$

$2\theta \approx 105.0^\circ \Rightarrow \theta \approx 52.5^\circ$

7.3 Exercises

1 (a) $\sin 46^\circ 37' = \cos(90^\circ - 46^\circ 37') = \cos 43^\circ 23'$

(b) $\cos 73^\circ 12' = \sin(90^\circ - 73^\circ 12') = \sin 16^\circ 48'$

(c) $\tan\frac{\pi}{6} = \cot(\frac{\pi}{2} - \frac{\pi}{6}) = \cot(\frac{3\pi}{6} - \frac{\pi}{6}) = \cot\frac{2\pi}{6} = \cot\frac{\pi}{3}$

(d) $\sec 17.28^\circ = \csc(90^\circ - 17.28^\circ) = \csc 72.72^\circ$

$\boxed{2}$ (a) $\tan 24°12' = \cot(90° - 24°12') = \cot 65°48'$

(b) $\sin 89°41' = \cos(90° - 89°41') = \cos 0°19'$

(c) $\cos\frac{\pi}{3} = \sin(\frac{\pi}{2} - \frac{\pi}{3}) = \sin(\frac{3\pi}{6} - \frac{2\pi}{6}) = \sin\frac{\pi}{6}$

(d) $\cot 61.87° = \tan(90° - 61.87°) = \tan 28.13°$

$\boxed{3}$ (a) $\cos\frac{7\pi}{20} = \sin(\frac{\pi}{2} - \frac{7\pi}{20}) = \sin\frac{3\pi}{20}$ (b) $\sin\frac{1}{4} = \cos(\frac{\pi}{2} - \frac{1}{4}) = \cos(\frac{2\pi - 1}{4})$

(c) $\tan 1 = \cot(\frac{\pi}{2} - 1) = \cot(\frac{\pi - 2}{2})$ (d) $\csc 0.53 = \sec(\frac{\pi}{2} - 0.53)$

$\boxed{4}$ (a) $\sin\frac{\pi}{12} = \cos(\frac{\pi}{2} - \frac{\pi}{12}) = \cos\frac{5\pi}{12}$ (b) $\cos 0.64 = \sin(\frac{\pi}{2} - 0.64)$

(c) $\tan\sqrt{2} = \cot(\frac{\pi}{2} - \sqrt{2})$ (d) $\sec 1.2 = \csc(\frac{\pi}{2} - 1.2)$

$\boxed{5}$ (a) $\cos\frac{\pi}{4} + \cos\frac{\pi}{6} = \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} = \frac{\sqrt{2} + \sqrt{3}}{2}$

(b) $\cos\frac{5\pi}{12} = \cos(\frac{\pi}{4} + \frac{\pi}{6}) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} - \sin\frac{\pi}{4}\sin\frac{\pi}{6} = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6} - \sqrt{2}}{4}$

$\boxed{6}$ (a) $\sin\frac{2\pi}{3} + \sin\frac{\pi}{4} = \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} = \frac{\sqrt{3} + \sqrt{2}}{2}$

(b) $\sin\frac{11\pi}{12} = \sin(\frac{2\pi}{3} + \frac{\pi}{4}) = \sin\frac{2\pi}{3}\cos\frac{\pi}{4} + \cos\frac{2\pi}{3}\sin\frac{\pi}{4} =$

$$\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} + (-\tfrac{1}{2})\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{6} - \sqrt{2}}{4}$$

$\boxed{7}$ (a) $\tan 60° + \tan 225° = \sqrt{3} + 1$

(b) $\tan 285° = \tan(60° + 225°) =$

$$\frac{\tan 60° + \tan 225°}{1 - \tan 60° \tan 225°} = \frac{\sqrt{3} + 1}{1 - (\sqrt{3})(1)}\cdot\frac{1 + \sqrt{3}}{1 + \sqrt{3}} = \frac{4 + 2\sqrt{3}}{-2} = -2 - \sqrt{3}$$

$\boxed{8}$ (a) $\cos 135° - \cos 60° = -\frac{\sqrt{2}}{2} - \frac{1}{2} = \frac{-\sqrt{2} - 1}{2}$

(b) $\cos 75° = \cos(135° - 60°) = \cos 135°\cos 60° + \sin 135°\sin 60° =$

$$-\frac{\sqrt{2}}{2}\cdot\frac{1}{2} + \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{6} - \sqrt{2}}{4}$$

$\boxed{9}$ (a) $\sin\frac{3\pi}{4} - \sin\frac{\pi}{6} = \frac{\sqrt{2}}{2} - \frac{1}{2} = \frac{\sqrt{2} - 1}{2}$

(b) $\sin\frac{7\pi}{12} = \sin(\frac{3\pi}{4} - \frac{\pi}{6}) = \sin\frac{3\pi}{4}\cos\frac{\pi}{6} - \cos\frac{3\pi}{4}\sin\frac{\pi}{6} = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - (-\frac{\sqrt{2}}{2})\cdot\frac{1}{2} = \frac{\sqrt{6} + \sqrt{2}}{4}$

$\boxed{10}$ (a) $\tan\frac{3\pi}{4} - \tan\frac{\pi}{6} = -1 - \frac{\sqrt{3}}{3} = \frac{-3 - \sqrt{3}}{3}$

(b) $\tan\frac{7\pi}{12} = \tan(\frac{3\pi}{4} - \frac{\pi}{6}) = \dfrac{\tan\frac{3\pi}{4} - \tan\frac{\pi}{6}}{1 + \tan\frac{3\pi}{4}\tan\frac{\pi}{6}} = \dfrac{-1 - \sqrt{3}/3}{1 + (-1)\cdot(\sqrt{3}/3)}\cdot\dfrac{3}{3} =$

$$\frac{-3 - \sqrt{3}}{3 - \sqrt{3}}\cdot\frac{3 + \sqrt{3}}{3 + \sqrt{3}} = \frac{-12 - 6\sqrt{3}}{6} = -2 - \sqrt{3}$$

$\boxed{11}$ $\cos 48°\cos 23° + \sin 48°\sin 23° = \cos(48° - 23°) = \cos 25°$

$\boxed{12}$ $\cos 13°\cos 50° - \sin 13°\sin 50° = \cos(13° + 50°) = \cos 63°$

$\boxed{13}$ $\cos 10°\sin 5° - \sin 10°\cos 5° = \sin(5° - 10°) = \sin(-5°)$

$\boxed{14}$ $\sin 57°\cos 4° + \cos 57°\sin 4° = \sin(57° + 4°) = \sin 61°$

$\boxed{15}$ $\cos 3\sin(-2) - \cos 2\sin 3 = \sin(-2)\cos 3 - \cos(-2)\sin 3 = \sin(-2 - 3) = \sin(-5)$

$\boxed{16}$ $\sin(-5)\cos 2 + \cos 5 \sin(-2) = \sin(-5)\cos(-2) + \cos(-5)\sin(-2) =$

$$\sin[-2 + (-5)] = \sin(-7)$$

$\boxed{17}$ (a) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = \frac{3}{5}\cdot\frac{15}{17} + \frac{4}{5}\cdot\frac{8}{17} = \frac{77}{85}$

(b) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \frac{4}{5}\cdot\frac{15}{17} - \frac{3}{5}\cdot\frac{8}{17} = \frac{36}{85}$

(c) Since the sine and cosine of $(\alpha+\beta)$ are positive, $(\alpha+\beta)$ is in QI.

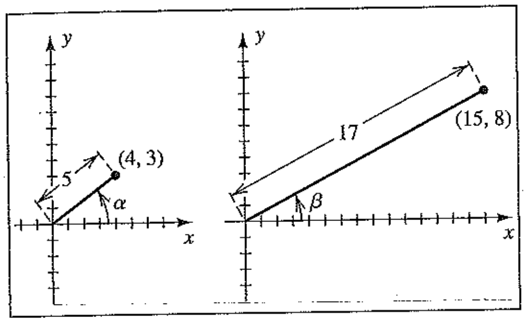

Figure 17

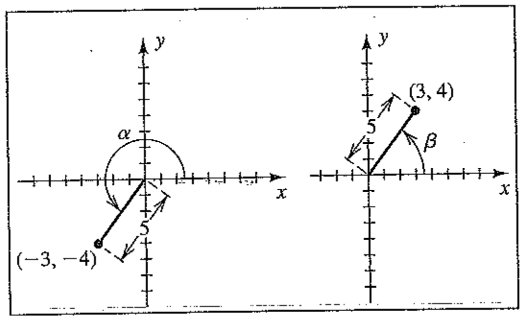

Figure 19

$\boxed{18}$ (a) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = \frac{12}{13}\cdot\frac{4}{5} + \frac{5}{13}\cdot\frac{3}{5} = \frac{63}{65}$

(b) $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \dfrac{\frac{12}{5} + \frac{3}{4}}{1 - \frac{12}{5}\cdot\frac{3}{4}} \cdot \dfrac{20}{20} = \dfrac{48+15}{20-36} = -\dfrac{63}{16}$

(c) Since the sine is positive and the tangent is negative, $(\alpha+\beta)$ is in QII.

$\boxed{19}$ (a) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = (-\frac{4}{5})\cdot\frac{3}{5} + (-\frac{3}{5})\cdot\frac{4}{5} = -\frac{24}{25}$

(b) $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \dfrac{\frac{4}{3} + \frac{4}{3}}{1 - \frac{4}{3}\cdot\frac{4}{3}} \cdot \dfrac{9}{9} = \dfrac{12+12}{9-16} = -\dfrac{24}{7}$

(c) Since the sine and tangent of $(\alpha+\beta)$ are negative, $(\alpha+\beta)$ is in QIV.

$\boxed{20}$ (a) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = \frac{7}{25}\cdot(-\frac{3}{5}) + (-\frac{24}{25})\cdot(-\frac{4}{5}) = \frac{75}{125} = \frac{3}{5}$

(b) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = (-\frac{24}{25})\cdot(-\frac{3}{5}) - \frac{7}{25}\cdot(-\frac{4}{5}) = \frac{100}{125} = \frac{4}{5}$

(c) $\tan(\alpha+\beta) = \dfrac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \dfrac{3/5}{4/5} = \dfrac{3}{4}$

(d) $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta = \frac{7}{25}\cdot(-\frac{3}{5}) - (-\frac{24}{25})\cdot(-\frac{4}{5}) = -\frac{117}{125}$

(e) $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta = (-\frac{24}{25})\cdot(-\frac{3}{5}) + \frac{7}{25}\cdot(-\frac{4}{5}) = \frac{44}{125}$

(f) $\tan(\alpha-\beta) = \dfrac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)} = \dfrac{-117/125}{44/125} = -\dfrac{117}{44}$

$\boxed{21}$ (a) $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta =$

$$\left(-\frac{\sqrt{21}}{5}\right)\cdot(-\tfrac{3}{5}) - (-\tfrac{2}{5})\cdot(-\tfrac{4}{5}) = \frac{3\sqrt{21}-8}{25} \approx 0.23$$

(b) $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta =$

$$(-\tfrac{2}{5})\cdot(-\tfrac{3}{5}) + \left(-\frac{\sqrt{21}}{5}\right)\cdot(-\tfrac{4}{5}) = \frac{4\sqrt{21}+6}{25} \approx 0.97$$

(c) Since the sine and cosine of $(\alpha-\beta)$ are positive, $(\alpha-\beta)$ is in QI.

$\boxed{22}$ (a) $\sin(\alpha+\beta)=\sin\alpha\,\cos\beta+\cos\alpha\,\sin\beta=$

$$\tfrac{2}{3}\cdot(-\tfrac{1}{3})+\left(-\frac{\sqrt{5}}{3}\right)\cdot\left(\frac{2\sqrt{2}}{3}\right)=\frac{-2-2\sqrt{10}}{9}\approx-0.92$$

(b) $$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\,\tan\beta}=\frac{-\frac{2}{\sqrt{5}}+(-2\sqrt{2})}{1-\left(-\frac{2}{\sqrt{5}}\right)\cdot(-2\sqrt{2})}\cdot\frac{\sqrt{5}}{\sqrt{5}}=\frac{-2-2\sqrt{10}}{\sqrt{5}-4\sqrt{2}}\approx 2.43$$

(c) Since the sine is negative and the tangent is positive, $(\alpha+\beta)$ is in QIII.

$\boxed{23}$ $\sin(\theta+\pi)=\sin\theta\,\cos\pi+\cos\theta\,\sin\pi=\sin\theta(-1)+\cos\theta(0)=-\sin\theta$

$\boxed{24}$ $\sin(x+\frac{\pi}{2})=\sin x\,\cos\frac{\pi}{2}+\cos x\,\sin\frac{\pi}{2}=\sin x(0)+\cos x(1)=\cos x$. Alternatively, we could use the cofunction identity: $\sin(x+\frac{\pi}{2})=\cos[\frac{\pi}{2}-(x+\frac{\pi}{2})]=\cos(-x)=\cos x$

$\boxed{25}$ $\sin(x-\frac{5\pi}{2})=\sin x\,\cos\frac{5\pi}{2}-\cos x\,\sin\frac{5\pi}{2}=\sin x(0)-\cos x(1)=-\cos x$

$\boxed{26}$ $\sin(\theta-\frac{3\pi}{2})=\sin\theta\,\cos\frac{3\pi}{2}-\cos\theta\,\sin\frac{3\pi}{2}=\sin\theta(0)-\cos\theta(-1)=\cos\theta$

$\boxed{27}$ $\cos(\theta-\pi)=\cos\theta\,\cos\pi+\sin\theta\,\sin\pi=\cos\theta(-1)+\sin\theta(0)=-\cos\theta$

$\boxed{28}$ $\cos(x+\frac{\pi}{2})=\sin[\frac{\pi}{2}-(x+\frac{\pi}{2})]=\sin(-x)=-\sin x$

$\boxed{29}$ $\cos(x+\frac{3\pi}{2})=\cos x\,\cos\frac{3\pi}{2}-\sin x\,\sin\frac{3\pi}{2}=\cos x(0)-\sin x(-1)=\sin x$

$\boxed{30}$ $\cos(\theta-\frac{5\pi}{2})=\cos\theta\,\cos\frac{5\pi}{2}+\sin\theta\,\sin\frac{5\pi}{2}=\cos\theta(0)+\sin\theta(1)=\sin\theta$

$\boxed{31}$ $$\tan(x-\tfrac{\pi}{2})=\frac{\sin(x-\frac{\pi}{2})}{\cos(x-\frac{\pi}{2})}=\frac{\sin x\,\cos\frac{\pi}{2}-\cos x\,\sin\frac{\pi}{2}}{\cos x\,\cos\frac{\pi}{2}+\sin x\,\sin\frac{\pi}{2}}=\frac{-\cos x}{\sin x}=-\cot x$$

$\boxed{32}$ $$\tan(\pi-\theta)=\frac{\tan\pi-\tan\theta}{1+\tan\pi\,\tan\theta}=\frac{0-\tan\theta}{1+(0)\tan\theta}=-\tan\theta$$

$\boxed{33}$ $\tan(\theta+\frac{\pi}{2})=\cot[\frac{\pi}{2}-(\theta+\frac{\pi}{2})]=\cot(-\theta)=-\cot\theta$

$\boxed{34}$ $$\tan(x+\pi)=\frac{\tan x+\tan\pi}{1-\tan x\,\tan\pi}=\frac{\tan x+0}{1-(\tan x)(0)}=\tan x$$

$\boxed{35}$ $$\sin(\theta+\tfrac{\pi}{4})=\sin\theta\,\cos\tfrac{\pi}{4}+\cos\theta\,\sin\tfrac{\pi}{4}=\frac{\sqrt{2}}{2}\sin\theta+\frac{\sqrt{2}}{2}\cos\theta=\frac{\sqrt{2}}{2}(\sin\theta+\cos\theta)$$

$\boxed{36}$ $$\cos(\theta+\tfrac{\pi}{4})=\cos\theta\,\cos\tfrac{\pi}{4}-\sin\theta\,\sin\tfrac{\pi}{4}=\frac{\sqrt{2}}{2}\cos\theta-\frac{\sqrt{2}}{2}\sin\theta=\frac{\sqrt{2}}{2}(\cos\theta-\sin\theta)$$

$\boxed{37}$ $$\tan(u+\tfrac{\pi}{4})=\frac{\tan u+\tan\frac{\pi}{4}}{1-\tan u\,\tan\frac{\pi}{4}}=\frac{\tan u+1}{1-\tan u(1)}=\frac{1+\tan u}{1-\tan u}$$

$\boxed{38}$ $$\tan(x-\tfrac{\pi}{4})=\frac{\tan x-\tan\frac{\pi}{4}}{1+\tan x\,\tan\frac{\pi}{4}}=\frac{\tan x-1}{1+\tan x(1)}=\frac{\tan x-1}{\tan x+1}$$

$\boxed{39}$ $\cos(u+v)+\cos(u-v)=(\cos u\,\cos v-\sin u\,\sin v)+(\cos u\,\cos v+\sin u\,\sin v)=$

$$2\cos u\,\cos v$$

$\boxed{40}$ $\sin(u+v)+\sin(u-v)=(\sin u\,\cos v+\cos u\,\sin v)+(\sin u\,\cos v-\cos u\,\sin v)=$

$$2\sin u\,\cos v$$

[41] $\sin(u+v)\cdot\sin(u-v) = (\sin u\,\cos v + \cos u\,\sin v)\cdot(\sin u\,\cos v - \cos u\,\sin v) =$

$\sin^2 u\,\cos^2 v - \cos^2 u\,\sin^2 v = \sin^2 u\,(1-\sin^2 v) - (1-\sin^2 u)\sin^2 v =$

$\sin^2 u - \sin^2 u\,\sin^2 v - \sin^2 v + \sin^2 u\,\sin^2 v = \sin^2 u - \sin^2 v$

[42] $\cos(u+v)\cdot\cos(u-v) = (\cos u\,\cos v - \sin u\,\sin v)\cdot(\cos u\,\cos v + \sin u\,\sin v) =$

$\cos^2 u\,\cos^2 v - \sin^2 u\,\sin^2 v = \cos^2 u\,(1-\sin^2 v) - (1-\cos^2 u)\sin^2 v =$

$\cos^2 u - \cos^2 u\,\sin^2 v - \sin^2 v + \cos^2 u\,\sin^2 v = \cos^2 u - \sin^2 v$

[43] $$\frac{1}{\cot\alpha - \cot\beta} = \frac{1}{\dfrac{\cos\alpha}{\sin\alpha} - \dfrac{\cos\beta}{\sin\beta}} = \frac{1}{\dfrac{\cos\alpha\,\sin\beta - \cos\beta\,\sin\alpha}{\sin\alpha\,\sin\beta}} = \frac{\sin\alpha\,\sin\beta}{\sin(\beta-\alpha)}$$

[44] $$\frac{1}{\tan\alpha + \tan\beta} = \frac{1}{\dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\sin\beta}{\cos\beta}} = \frac{1}{\dfrac{\sin\alpha\,\cos\beta + \sin\beta\,\cos\alpha}{\cos\alpha\,\cos\beta}} = \frac{\cos\alpha\,\cos\beta}{\sin(\alpha+\beta)}$$

[45] $\sin(u+v+w) = \sin[(u+v)+w]$

$= \sin(u+v)\cos w + \cos(u+v)\sin w$

$= (\sin u\,\cos v + \cos u\,\sin v)\cos w + (\cos u\,\cos v - \sin u\,\sin v)\sin w$

$= \sin u\,\cos v\,\cos w + \cos u\,\sin v\,\cos w + \cos u\,\cos v\,\sin w - \sin u\,\sin v\,\sin w$

[46] $$\tan(u+v+w) = \tan\big[(u+v)+w\big] = \frac{\tan(u+v) + \tan w}{1 - \tan(u+v)\tan w} =$$

$$\frac{\dfrac{\tan u + \tan v}{1-\tan u\,\tan v} + \tan w}{1 - \left(\dfrac{\tan u + \tan v}{1 - \tan u\,\tan v}\right)\tan w} = \frac{\dfrac{\tan u + \tan v + \tan w - \tan u\,\tan v\,\tan w}{1-\tan u\,\tan v}}{\dfrac{(1-\tan u\,\tan v) - \tan u\,\tan w - \tan v\,\tan w}{1 - \tan u\,\tan v}} =$$

$$\frac{\tan u + \tan v + \tan w - \tan u\,\tan v\,\tan w}{1 - (\tan u\,\tan v + \tan u\,\tan w + \tan v\,\tan w)}$$

[47] $$\cot(u+v) = \frac{\cos(u+v)}{\sin(u+v)} = \frac{(\cos u\,\cos v - \sin u\,\sin v)(1/\sin u\,\sin v)}{(\sin u\,\cos v + \cos u\,\sin v)(1/\sin u\,\sin v)} = \frac{\cot u\,\cot v - 1}{\cot v + \cot u}$$

[48] $\alpha + \beta = 90^\circ \;\Rightarrow\; \alpha = 90^\circ - \beta.\ \sin^2\alpha + \cos^2\alpha = 1 \;\Rightarrow$

$\sin^2\alpha + \cos^2(90^\circ - \beta) = 1 \;\Rightarrow\; \sin^2\alpha + \sin^2\beta = 1$ since $\cos(90^\circ - \beta) = \sin\beta$

[49] $\sin(u-v) = \sin\big[u + (-v)\big] = \sin u\,\cos(-v) + \cos u\,\sin(-v) = \sin u\,\cos v - \cos u\,\sin v$

[50] $$\tan(u-v) = \tan\big[u+(-v)\big] = \frac{\tan u + \tan(-v)}{1 - \tan u\,\tan(-v)} = \frac{\tan u - \tan v}{1 + \tan u\,\tan v}$$

[51] $$\frac{f(x+h)-f(x)}{h} = \frac{\cos(x+h) - \cos x}{h} = \frac{\cos x\,\cos h - \sin x\,\sin h - \cos x}{h} =$$

$$\frac{\cos x\,\cos h - \cos x}{h} - \frac{\sin x\,\sin h}{h} = \cos x\left(\frac{\cos h - 1}{h}\right) - \sin x\left(\frac{\sin h}{h}\right)$$

[52] $$\frac{f(x+h)-f(x)}{h} = \frac{\tan(x+h) - \tan x}{h} = \frac{\dfrac{\tan x + \tan h}{1 - \tan x\,\tan h} - \tan x}{h} =$$

$$\frac{\tan x + \tan h - \tan x + \tan^2 x\,\tan h}{h(1 - \tan x\,\tan h)} = \frac{\tan h\,(\tan^2 x + 1)}{h(1 - \tan x\,\tan h)} =$$

$$\sec^2 x\left(\frac{\sin h}{h}\right)\left(\frac{1}{\cos h}\right)\left(\frac{1}{1 - \tan x\,\tan h}\right) = \sec^2 x\left(\frac{\sin h}{h}\right)\frac{1}{\cos h - \sin h\,\tan x}$$

$\boxed{53}$ (a) Both sides, $\sin 63^\circ - \sin 57^\circ$ and $\sin 3^\circ$, are approximately equal to 0.0523.

(b) $\text{LS} = \sin(\alpha+\beta) - \sin(\alpha-\beta)$

$= (\sin\alpha\cos\beta + \cos\alpha\sin\beta) - (\sin\alpha\cos\beta - \cos\alpha\sin\beta)$

$= 2\cos\alpha\sin\beta \qquad \text{RS} = \sin\beta$

For $2\cos\alpha\sin\beta$ to equal $\sin\beta$, $2\cos\alpha$ must equal 1, so $\cos\alpha = \frac{1}{2}$ and $\alpha = 60^\circ$.

(c) $\alpha = 60^\circ$ and $\beta = 3^\circ$.

$\boxed{54}$ (a) Both sides, $\sin 35^\circ + \sin 25^\circ$ and $\cos 5^\circ$, are approximately equal to 0.9962.

(b) $\text{LS} = \sin(\alpha+\beta) + \sin(\alpha-\beta)$

$= (\sin\alpha\cos\beta + \cos\alpha\sin\beta) + (\sin\alpha\cos\beta - \cos\alpha\sin\beta)$

$= 2\sin\alpha\cos\beta \qquad \text{RS} = \cos\beta$

For $2\sin\alpha\cos\beta$ to equal $\cos\beta$, $2\sin\alpha$ must equal 1, so $\sin\alpha = \frac{1}{2}$ and $\alpha = 30^\circ$.

(c) $\alpha = 30^\circ$ and $\beta = 5^\circ$.

$\boxed{55}$ $\sin 4t\cos t = \sin t\cos 4t \Rightarrow \sin 4t\cos t - \sin t\cos 4t = 0 \Rightarrow \sin(4t-t) = 0 \Rightarrow$
$\sin 3t = 0 \Rightarrow 3t = \pi n \Rightarrow t = \frac{\pi}{3}n$. In $[0, \pi)$, $t = 0, \frac{\pi}{3}, \frac{2\pi}{3}$.

$\boxed{56}$ $\cos 5t\cos 3t = \frac{1}{2} + \sin(-5t)\sin 3t \Rightarrow \cos 5t\cos 3t + \sin 5t\sin 3t = \frac{1}{2} \Rightarrow$
$\cos(5t-3t) = \frac{1}{2} \Rightarrow \cos 2t = \frac{1}{2} \Rightarrow 2t = \frac{\pi}{3} + 2\pi n, \frac{5\pi}{3} + 2\pi n \Rightarrow$
$t = \frac{\pi}{6} + \pi n, \frac{5\pi}{6} + \pi n$. In $[0, \pi)$, $t = \frac{\pi}{6}, \frac{5\pi}{6}$.

$\boxed{57}$ $\cos 5t\cos 2t = -\sin 5t\sin 2t \Rightarrow \cos 5t\cos 2t + \sin 5t\sin 2t = 0 \Rightarrow$
$\cos(5t-2t) = 0 \Rightarrow \cos 3t = 0 \Rightarrow 3t = \frac{\pi}{2} + \pi n \Rightarrow t = \frac{\pi}{6} + \frac{\pi}{3}n$.
In $[0, \pi)$, $t = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$.

$\boxed{58}$ $\sin 3t\cos t + \cos 3t\sin t = -\frac{1}{2} \Rightarrow \sin(3t+t) = -\frac{1}{2} \Rightarrow \sin 4t = -\frac{1}{2} \Rightarrow$
$4t = \frac{7\pi}{6} + 2\pi n, \frac{11\pi}{6} + 2\pi n \Rightarrow t = \frac{7\pi}{24} + \frac{\pi}{2}n, \frac{11\pi}{24} + \frac{\pi}{2}n$. In $[0, \pi)$, $t = \frac{7\pi}{24}, \frac{19\pi}{24}, \frac{11\pi}{24}, \frac{23\pi}{24}$.

$\boxed{59}$ $\tan 2t + \tan t = 1 - \tan 2t\tan t \Rightarrow \dfrac{\tan 2t + \tan t}{1 - \tan 2t\tan t} = 1 \Rightarrow \tan(2t+t) = 1 \Rightarrow$
$\tan 3t = 1 \Rightarrow 3t = \frac{\pi}{4} + \pi n \Rightarrow t = \frac{\pi}{12} + \frac{\pi}{3}n$. In $[0, \pi)$, $t = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}$.
However, $\tan 2t$ is undefined if $t = \frac{3\pi}{4}$, so exclude this value of t.

$\boxed{60}$ $\tan t - \tan 4t = 1 + \tan 4t\tan t \Rightarrow \dfrac{\tan t - \tan 4t}{1 + \tan t\tan 4t} = 1 \Rightarrow \tan(t-4t) = 1 \Rightarrow$
$\tan(-3t) = 1 \Rightarrow -\tan 3t = 1 \Rightarrow \tan 3t = -1 \Rightarrow 3t = \frac{3\pi}{4} + \pi n \Rightarrow t = \frac{\pi}{4} + \frac{\pi}{3}n$.
In $[0, \pi)$, $t = \frac{\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12}$.

$\boxed{61}$ (a) $f(x) = \sqrt{3}\cos 2x + \sin 2x$ • $A = \sqrt{(\sqrt{3})^2 + 1^2} = 2$. $\tan C = \dfrac{1}{\sqrt{3}} \Rightarrow C = \frac{\pi}{6}$.
$f(x) = 2\cos(2x - \frac{\pi}{6}) = 2\cos[2(x - \frac{\pi}{12})]$

(b) amplitude $= 2$, period $= \frac{2\pi}{2} = \pi$, phase shift $= \frac{\pi}{12}$ (See *Figure 61.*)

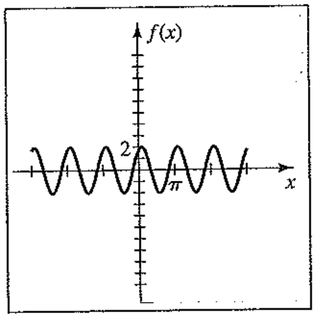

Figure 61

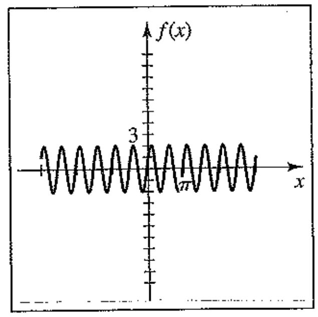

Figure 62

[62] (a) $f(x) = \cos 4x + \sqrt{3}\sin 4x$ • $A = \sqrt{1^2 + (\sqrt{3})^2} = 2$. $\tan C = \frac{\sqrt{3}}{1} \Rightarrow C = \frac{\pi}{3}$.

$f(x) = 2\cos(4x - \frac{\pi}{3}) = 2\cos\left[4(x - \frac{\pi}{12})\right]$

(b) amplitude $= 2$, period $= \frac{2\pi}{4} = \frac{\pi}{2}$, phase shift $= \frac{\pi}{12}$

[63] (a) $f(x) = 2\cos 3x - 2\sin 3x$ • $A = \sqrt{2^2 + 2^2} = 2\sqrt{2}$. $\tan C = \frac{-2}{2} = -1 \Rightarrow$

$C = -\frac{\pi}{4}$. $f(x) = 2\sqrt{2}\cos(3x + \frac{\pi}{4}) = 2\sqrt{2}\cos\left[3(x + \frac{\pi}{12})\right]$

(b) amplitude $= 2\sqrt{2}$, period $= \frac{2\pi}{3}$, phase shift $= -\frac{\pi}{12}$

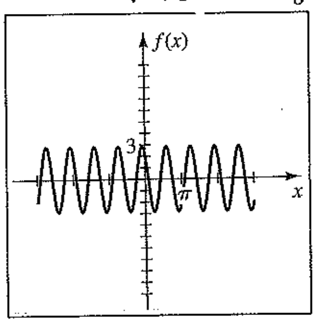

Figure 63

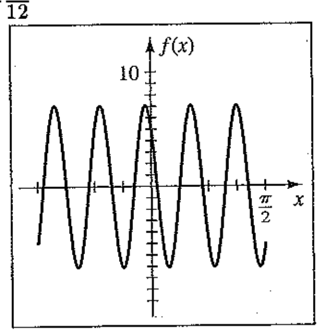

Figure 64

[64] (a) $f(x) = 5\cos 10x - 5\sin 10x$ • $A = \sqrt{5^2 + (-5)^2} = 5\sqrt{2}$.

$\tan C = \frac{-5}{5} = -1 \Rightarrow C = -\frac{\pi}{4}$. $f(x) = 5\sqrt{2}\cos(10x + \frac{\pi}{4}) = 5\sqrt{2}\cos\left[10(x + \frac{\pi}{40})\right]$

(b) amplitude $= 5\sqrt{2}$, period $= \frac{2\pi}{10} = \frac{\pi}{5}$, phase shift $= -\frac{\pi}{40}$

[65] $y = 50\sin 60\pi t + 40\cos 60\pi t$ • $A = \sqrt{50^2 + 40^2} = 10\sqrt{41}$. $\tan C = \frac{50}{40} \Rightarrow$

$C = \tan^{-1}\frac{5}{4} \approx 0.8961$. $y = 10\sqrt{41}\cos(60\pi t - \tan^{-1}\frac{5}{4}) \approx 10\sqrt{41}\cos(60\pi t - 0.8961)$.

[66] $y = 10\sin(120\pi t - \frac{\pi}{2}) + 5\sin 120\pi t$ •

$\sin(120\pi t - \frac{\pi}{2}) = \sin 120\pi t\,\cos\frac{\pi}{2} - \cos 120\pi t\,\sin\frac{\pi}{2} = -\cos 120\pi t$.

Now, $y = -10\cos 120\pi t + 5\sin 120\pi t = -(10\cos 120\pi t - 5\sin 120\pi t)$ { a, the coefficient of the cosine term, must be positive to apply the formula in Example 6. }.

$A = \sqrt{10^2 + (-5)^2} = 5\sqrt{5}$. $\tan C = \frac{-5}{10} \Rightarrow C = \tan^{-1}(-\frac{1}{2}) \approx -0.4636$.

$y = -5\sqrt{5}\cos\left[120\pi t - \tan^{-1}(-\frac{1}{2})\right] \approx -5\sqrt{5}\cos(120\pi t + 0.4636)$.

[67] (a) $y = 2\cos t + 3\sin t$; $A = \sqrt{2^2 + 3^2} = \sqrt{13}$; $\tan C = \frac{3}{2} \Rightarrow C \approx 0.98$;

$y = \sqrt{13}\cos(t - C)$; amplitude $= \sqrt{13}$, period $= \frac{2\pi}{1} = 2\pi$

(b) $y = 0 \ \Rightarrow \ \cos(t - C) = 0 \ \Rightarrow$

$t = C + \frac{\pi}{2} + \pi n \approx 2.5536 + \pi n$ for every nonnegative integer n.

68 $4 = \text{amplitude} \ \Rightarrow \ 4 = \sqrt{(y_0)^2 + \left(\frac{v_0}{\omega}\right)^2} \ \Rightarrow \ 4 = \sqrt{1^2 + \left(\frac{v_0}{2}\right)^2}$ $\{y_0 = 1$ and

$\omega = 2\} \ \Rightarrow \ 16 = 1 + \frac{v_0^2}{4} \ \Rightarrow \ v_0^2 = 60 \ \Rightarrow \ v_0 = \pm 2\sqrt{15}$ ft/sec

69 (a) $p(t) = A \sin \omega t + B \sin(\omega t + \tau)$

$= A \sin \omega t + B(\sin \omega t \cos \tau + \cos \omega t \sin \tau)$

$= (B \sin \tau) \cos \omega t + (A + B \cos \tau) \sin \omega t$

$= a \cos \omega t + b \sin \omega t$ with $a = B \sin \tau$ and $b = A + B \cos \tau$

(b) $C^2 = (B \sin \tau)^2 + (A + B \cos \tau)^2$

$= B^2 \sin^2 \tau + A^2 + 2AB \cos \tau + B^2 \cos^2 \tau$

$= A^2 + B^2(\sin^2 \tau + \cos^2 \tau) + 2AB \cos \tau$

$= A^2 + B^2 + 2AB \cos \tau$

70 (a) Using $C^2 = A^2 + B^2 + 2AB \cos \tau$ from Exercise 69(b) and letting $A = B$ yields $C^2 = 2A^2 + 2A^2 \cos \tau$, or $C^2 = 2A^2(1 + \cos \tau)$. If the amplitude C of p is zero, then $1 + \cos \tau = 0$, i.e., $\cos \tau = -1$. Hence, $\tau = \pi$.

(b) Destructive interference occurs if $C < A$. If A, $C > 0$, then $C < A \ \Rightarrow$ $C^2 < A^2 \ \Rightarrow \ 2A^2(1 + \cos \tau) < A^2 \ \Rightarrow \ 1 + \cos \tau < \frac{1}{2} \ \Rightarrow \ \cos \tau < -\frac{1}{2} \ \Rightarrow$ $\frac{2\pi}{3} < \tau < \frac{4\pi}{3}$.

71 (a) $C^2 = A^2 + B^2 + 2AB \cos \tau \le A^2 + B^2 + 2AB$, since $\cos \tau \le 1$ and $A > 0$, $B > 0$. Thus, $C^2 \le (A + B)^2$, and hence $C \le A + B$.

(b) $C = A + B$ if $\cos \tau = 1$, or $\tau = 0, 2\pi$.

(c) Constructive interference will occur if $C > A$. $C > A \ \Rightarrow \ C^2 > A^2 \ \Rightarrow$ $A^2 + B^2 + 2AB \cos \tau > A^2 \ \Rightarrow \ B^2 + 2AB \cos \tau > 0 \ \Rightarrow \ B(B + 2A \cos \tau) > 0$. Since $B > 0$, the product will be positive if $B + 2A \cos \tau > 0$, i.e., $\cos \tau > -\frac{B}{2A}$.

72 (a) $A = B = 2$, $\omega_1 = 1$, $\omega_2 = 20$, and $\tau = 3 \ \Rightarrow \ p(t) = 2 \sin t + 2 \sin(20t + 3)$.

(b) The tone of the tuning fork with the shortest period will fade in and out at intervals equal to the period of the tuning fork with the longest period.

$[-2\pi, 2\pi, \pi/2]$ by $[-4.19, 4.19]$

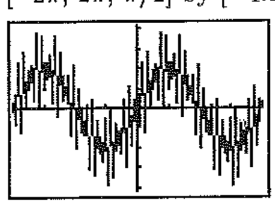

Figure 72

73 Graph $y = 3\sin 2t + 2\sin(4t+1)$. Constructive interference will occur when $y > 3$ or $y < -3$. From the graph, we see that this occurs on the intervals

$$(-2.97, -2.69), (-1.00, -0.37), (0.17, 0.46), \text{ and } (2.14, 2.77).$$

$[-\pi, \pi, \pi/4]$ by $[-5, 5]$

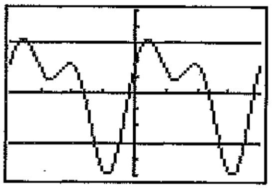

Figure 73

$[-\pi, \pi, \pi/4]$ by $[-5, 5]$

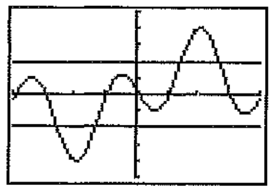

Figure 74

74 Graph $y = 2\sin t + 2\sin(3t+3)$. Constructive interference will occur when $y > 2$ or $y < -2$. From the graph, we see that this occurs on the intervals

$$(-2.01, -1.05) \text{ and } (1.13, 2.10).$$

7.4 Exercises

1 From *Figure 1*, $\sin\theta = \frac{4}{5}$ and $\cos\theta = \frac{3}{5}$. Thus, $\sin 2\theta = 2\sin\theta\cos\theta = 2(\frac{4}{5})(\frac{3}{5}) = \frac{24}{25}$.

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = (\tfrac{3}{5})^2 - (\tfrac{4}{5})^2 = -\tfrac{7}{25}. \quad \tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{24/25}{-7/25} = -\tfrac{24}{7}.$$

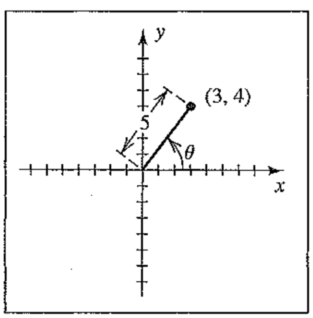

Figure 1

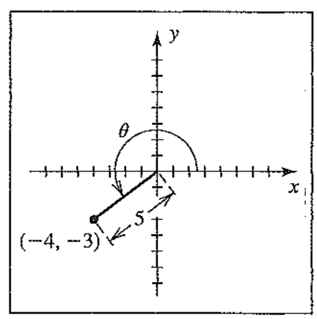

Figure 2

2 From *Figure 2*, $\sin\theta = -\frac{3}{5}$ and $\cos\theta = -\frac{4}{5}$.
Thus, $\sin 2\theta = 2\sin\theta\cos\theta = 2(-\frac{3}{5})(-\frac{4}{5}) = \frac{24}{25}$.

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = (-\tfrac{4}{5})^2 - (-\tfrac{3}{5})^2 = \tfrac{7}{25}. \quad \tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{24/25}{7/25} = \tfrac{24}{7}.$$

3 From *Figure 3*, $\sin\theta = \sqrt{8}/3 = \frac{2}{3}\sqrt{2}$ and $\cos\theta = -\frac{1}{3}$.
Thus, $\sin 2\theta = 2\sin\theta\cos\theta = 2(\frac{2}{3}\sqrt{2})(-\frac{1}{3}) = -\frac{4}{9}\sqrt{2}$.
$\cos 2\theta = \cos^2\theta - \sin^2\theta = (-\frac{1}{3})^2 - (\frac{2}{3}\sqrt{2})^2 = \frac{1}{9} - \frac{8}{9} = -\frac{7}{9}$.

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{-4\sqrt{2}/9}{-7/9} = \frac{4\sqrt{2}}{7}.$$

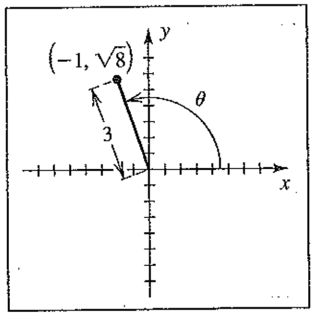

Figure 3

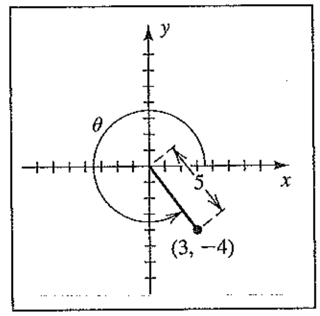

Figure 4

[4] From *Figure 4*, $\sin\theta = -\frac{4}{5}$ and $\cos\theta = \frac{3}{5}$.

Thus, $\sin 2\theta = 2\sin\theta\,\cos\theta = 2(-\frac{4}{5})(\frac{3}{5}) = -\frac{24}{25}$.

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = (\tfrac{3}{5})^2 - (-\tfrac{4}{5})^2 = -\tfrac{7}{25}. \quad \tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{-24/25}{-7/25} = \frac{24}{7}.$$

[5] $\sec\theta = \frac{5}{4} \Rightarrow \cos\theta = \frac{4}{5}$. θ acute implies that $\frac{\theta}{2}$ is acute, so all functions of $\frac{\theta}{2}$ are positive. $\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-\frac{4}{5}}{2}} = \sqrt{\frac{\frac{1}{5}}{2}} = \sqrt{\frac{1}{10}\cdot\frac{10}{10}} = \frac{\sqrt{10}}{10}$.

$$\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}} = \sqrt{\frac{1+\frac{4}{5}}{2}} = \sqrt{\frac{\frac{9}{5}}{2}} = \sqrt{\frac{9}{10}\cdot\frac{10}{10}} = \frac{3\sqrt{10}}{10}.$$

$$\tan\frac{\theta}{2} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \frac{\sqrt{10}/10}{3\sqrt{10}/10} = \frac{1}{3}.$$

[6] From *Figure 6*, $\cos\theta = \frac{4}{5}$ and $\frac{\theta}{2}$ is in QIV. Hence, $\sin\frac{\theta}{2} < 0$ and $\cos\frac{\theta}{2} > 0$.

$$\sin\frac{\theta}{2} = -\sqrt{\frac{1-\cos\theta}{2}} = -\sqrt{\frac{1-\frac{4}{5}}{2}} = -\sqrt{\frac{\frac{1}{5}}{2}} = -\sqrt{\frac{1}{10}\cdot\frac{10}{10}} = -\frac{\sqrt{10}}{10}.$$

$$\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}} = \sqrt{\frac{1+\frac{4}{5}}{2}} = \sqrt{\frac{\frac{9}{5}}{2}} = \sqrt{\frac{9}{10}\cdot\frac{10}{10}} = \frac{3\sqrt{10}}{10}.$$

$$\tan\frac{\theta}{2} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \frac{-\sqrt{10}/10}{3\sqrt{10}/10} = -\frac{1}{3}.$$

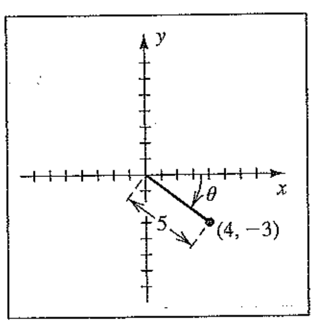

Figure 6

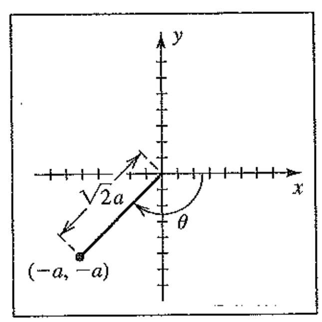

Figure 7

[7] From *Figure 7* ($a > 0$), $\cos\theta = -\frac{a}{\sqrt{2}\,a} = -\frac{\sqrt{2}}{2}$ and $\frac{\theta}{2}$ is in QIV.

$$\sin\frac{\theta}{2} = -\sqrt{\frac{1-\cos\theta}{2}} = -\sqrt{\frac{1+\sqrt{2}/2}{2}} = -\sqrt{\frac{2+\sqrt{2}}{4}} = -\tfrac{1}{2}\sqrt{2+\sqrt{2}}.$$

(cont.)

$$\cos\frac{\theta}{2}=\sqrt{\frac{1+\cos\theta}{2}}=\sqrt{\frac{1-\sqrt{2}/2}{2}}=\sqrt{\frac{2-\sqrt{2}}{4}}=\tfrac{1}{2}\sqrt{2-\sqrt{2}}.$$

$$\tan\frac{\theta}{2}=\frac{1-\cos\theta}{\sin\theta}=\frac{1+\sqrt{2}/2}{-\sqrt{2}/2}\cdot\frac{2}{2}=\frac{2+\sqrt{2}}{-\sqrt{2}}=-\sqrt{2}-1.$$

$\boxed{8}$ $\sec\theta=-4 \;\Rightarrow\; \cos\theta=-\frac{1}{4}$. $180^\circ<\theta<270^\circ \;\Rightarrow\; 90^\circ<\frac{\theta}{2}<135^\circ \;\Rightarrow\; \frac{\theta}{2}$ is in QII.

$$\sin\frac{\theta}{2}=\sqrt{\frac{1-\cos\theta}{2}}=\sqrt{\frac{1+\frac{1}{4}}{2}}=\sqrt{\frac{\frac{5}{4}}{2}}=\sqrt{\tfrac{5}{8}\cdot\tfrac{2}{2}}=\frac{\sqrt{10}}{4}.$$

$$\cos\frac{\theta}{2}=-\sqrt{\frac{1+\cos\theta}{2}}=-\sqrt{\frac{1-\frac{1}{4}}{2}}=-\sqrt{\frac{\frac{3}{4}}{2}}=-\sqrt{\tfrac{3}{8}\cdot\tfrac{2}{2}}=-\frac{\sqrt{6}}{4}.$$

$$\tan\frac{\theta}{2}=\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}=\frac{\sqrt{10}/4}{-\sqrt{6}/4}=-\sqrt{\tfrac{5}{3}\cdot\tfrac{3}{3}}=-\frac{\sqrt{15}}{3}.$$

$\boxed{9}$ (a) $\cos 67^\circ 30'=\sqrt{\frac{1+\cos 135^\circ}{2}}=\sqrt{\frac{1-\sqrt{2}/2}{2}}=\sqrt{\frac{2-\sqrt{2}}{4}}=\frac{1}{2}\sqrt{2-\sqrt{2}}.$

(b) $\sin 15^\circ=\sqrt{\frac{1-\cos 30^\circ}{2}}=\sqrt{\frac{1-\sqrt{3}/2}{2}}=\sqrt{\frac{2-\sqrt{3}}{4}}=\frac{1}{2}\sqrt{2-\sqrt{3}}.$

(c) $\tan\frac{3\pi}{8}=\dfrac{1-\cos\frac{3\pi}{4}}{\sin\frac{3\pi}{4}}=\dfrac{1+\sqrt{2}/2}{\sqrt{2}/2}\cdot\dfrac{2}{2}=\dfrac{2+\sqrt{2}}{\sqrt{2}}=\sqrt{2}+1.$

$\boxed{10}$ (a) $\cos 165^\circ=-\sqrt{\frac{1+\cos 330^\circ}{2}}=-\sqrt{\frac{1+\sqrt{3}/2}{2}}=-\sqrt{\frac{2+\sqrt{3}}{4}}=-\frac{1}{2}\sqrt{2+\sqrt{3}}.$

(b) $\sin 157^\circ 30'=\sqrt{\frac{1-\cos 315^\circ}{2}}=\sqrt{\frac{1-\sqrt{2}/2}{2}}=\sqrt{\frac{2-\sqrt{2}}{4}}=\frac{1}{2}\sqrt{2-\sqrt{2}}.$

(c) $\tan\frac{\pi}{8}=\dfrac{1-\cos\frac{\pi}{4}}{\sin\frac{\pi}{4}}=\dfrac{1-\sqrt{2}/2}{\sqrt{2}/2}\cdot\dfrac{2}{2}=\dfrac{2-\sqrt{2}}{\sqrt{2}}=\sqrt{2}-1.$

$\boxed{11}$ $\sin 10\theta=\sin(2\cdot 5\theta)=2\sin 5\theta\,\cos 5\theta$

$\boxed{12}$ $\cos^2 3x-\sin^2 3x=\cos(2\cdot 3x)=\cos 6x$

$\boxed{13}$ $4\sin\frac{x}{2}\cos\frac{x}{2}=2\cdot 2\sin\frac{x}{2}\cos\frac{x}{2}=2\sin\left(2\cdot\frac{x}{2}\right)=2\sin x$

$\boxed{14}$ $\dfrac{\sin^2 2\alpha}{\sin^2\alpha}=\dfrac{(2\sin\alpha\,\cos\alpha)^2}{\sin^2\alpha}=\dfrac{4\sin^2\alpha\,\cos^2\alpha}{\sin^2\alpha}=4\cos^2\alpha=4(1-\sin^2\alpha)=4-4\sin^2\alpha$

$\boxed{15}$ $(\sin t+\cos t)^2=\sin^2 t+2\sin t\,\cos t+\cos^2 t=1+\sin 2t$

$\boxed{16}$ $\csc 2u=\dfrac{1}{\sin 2u}=\dfrac{1}{2\sin u\,\cos u}=\dfrac{1}{2}\csc u\,\sec u$

$\boxed{17}$ $\sin 3u=\sin(2u+u)=\sin 2u\,\cos u+\cos 2u\,\sin u$

$$=(2\sin u\,\cos u)\cos u+(1-2\sin^2 u)\sin u=2\sin u\,\cos^2 u+\sin u-2\sin^3 u$$
$$=2\sin u(1-\sin^2 u)+\sin u-2\sin^3 u=2\sin u-2\sin^3 u+\sin u-2\sin^3 u$$
$$=3\sin u-4\sin^3 u=\sin u(3-4\sin^2 u)$$

$\boxed{18}$ $\sin 4t=\sin(2\cdot 2t)=2\sin 2t\,\cos 2t=2(2\sin t\,\cos t)(1-2\sin^2 t)=$

$$4\sin t\,\cos t\,(1-2\sin^2 t)$$

[19] $\cos 4\theta = \cos(2\cdot 2\theta) = 2\cos^2 2\theta - 1 = 2(2\cos^2\theta - 1)^2 - 1 =$
$2(4\cos^4\theta - 4\cos^2\theta + 1) - 1 = 8\cos^4\theta - 8\cos^2\theta + 1$

[20] $\cos 6t = \cos(2\cdot 3t) = 2\cos^2 3t - 1$
$= 2(4\cos^3 t - 3\cos t)^2 - 1$ { using Example 2 }
$= 2(16\cos^6 t - 24\cos^4 t + 9\cos^2 t) - 1$
$= 32\cos^6 t - 48\cos^4 t + 18\cos^2 t - 1$

[21] $\sin^4 t = (\sin^2 t)^2 = \left(\frac{1-\cos 2t}{2}\right)^2 = \frac{1}{4}(1 - 2\cos 2t + \cos^2 2t) =$
$\frac{1}{4} - \frac{1}{2}\cos 2t + \frac{1}{4}\left(\frac{1+\cos 4t}{2}\right) = \frac{1}{4} - \frac{1}{2}\cos 2t + \frac{1}{8} + \frac{1}{8}\cos 4t = \frac{3}{8} - \frac{1}{2}\cos 2t + \frac{1}{8}\cos 4t$

[22] $\cos^4 x - \sin^4 x = (\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x) = \cos^2 x - \sin^2 x = \cos 2x$

[23] $\sec 2\theta = \frac{1}{\cos 2\theta} = \frac{1}{2\cos^2\theta - 1} = \frac{1}{2\left(\frac{1}{\sec^2\theta}\right) - 1} = \frac{1}{\frac{2-\sec^2\theta}{\sec^2\theta}} = \frac{\sec^2\theta}{2-\sec^2\theta}$

[24] $\cot 2u = \frac{1}{\tan 2u} = \frac{1-\tan^2 u}{2\tan u} = \frac{1 - \frac{1}{\cot^2 u}}{\frac{2}{\cot u}} \cdot \frac{\cot^2 u}{\cot^2 u} = \frac{\cot^2 u - 1}{2\cot u}$

[25] $2\sin^2 2t + \cos 4t = 2\sin^2 2t + \cos(2\cdot 2t) = 2\sin^2 2t + (1 - 2\sin^2 2t) = 1$

[26] $\tan\theta + \cot\theta = \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\,\sin\theta} = \frac{2}{2\sin\theta\,\cos\theta} = \frac{2}{\sin 2\theta} = 2\csc 2\theta$

[27] $\tan 3u = \tan(2u + u) = \frac{\tan 2u + \tan u}{1 - \tan 2u\,\tan u} = \frac{\frac{2\tan u}{1-\tan^2 u} + \tan u}{1 - \frac{2\tan u}{1-\tan^2 u}\cdot\tan u} =$

$$\frac{\frac{2\tan u + \tan u - \tan^3 u}{1-\tan^2 u}}{\frac{1-\tan^2 u - 2\tan^2 u}{1-\tan^2 u}} = \frac{3\tan u - \tan^3 u}{1 - 3\tan^2 u} = \frac{\tan u\,(3-\tan^2 u)}{1-3\tan^2 u}$$

[28] $\frac{1+\sin 2v + \cos 2v}{1+\sin 2v - \cos 2v} = \frac{1 + 2\sin v\,\cos v + 2\cos^2 v - 1}{1 + 2\sin v\,\cos v - 1 + 2\sin^2 v} =$
$\frac{2\cos^2 v + 2\sin v\,\cos v}{2\sin^2 v + 2\sin v\,\cos v} = \frac{2\cos v\,(\cos v + \sin v)}{2\sin v\,(\sin v + \cos v)} = \cot v$

[29] $\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta} = \frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta} = \csc\theta - \cot\theta$

[30] $\tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta}\cdot\frac{1-\cos\theta}{1-\cos\theta} = \frac{(1-\cos\theta)^2}{1-\cos^2\theta} = \frac{1 - 2\cos\theta + \cos^2\theta}{\sin^2\theta}$
$= \frac{1}{\sin^2\theta} - \frac{2\cos\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \csc^2\theta - 2\cot\theta\,\csc\theta + \cot^2\theta$
$= (1 + \cot^2\theta) - 2\cot\theta\,\csc\theta + \cot^2\theta = 1 - 2\cot\theta\,\csc\theta + 2\cot^2\theta$

$\boxed{31}$ $\cos^4\frac{\theta}{2} = \left(\cos^2\frac{\theta}{2}\right)^2 = \left(\frac{1+\cos\theta}{2}\right)^2 = \frac{1+2\cos\theta+\cos^2\theta}{4} = \frac{1}{4}+\frac{1}{2}\cos\theta+\frac{1}{4}\left(\frac{1+\cos 2\theta}{2}\right) =$

$\frac{1}{4}+\frac{1}{2}\cos\theta+\frac{1}{8}+\frac{1}{8}\cos 2\theta = \frac{3}{8}+\frac{1}{2}\cos\theta+\frac{1}{8}\cos 2\theta$

$\boxed{32}$ $\cos^4 2x = (\cos^2 2x)^2 = \left(\frac{1+\cos 4x}{2}\right)^2 = \frac{1+2\cos 4x+\cos^2 4x}{4} =$

$\frac{1}{4}+\frac{1}{2}\cos 4x+\frac{1}{4}\left(\frac{1+\cos 8x}{2}\right) = \frac{1}{4}+\frac{1}{2}\cos 4x+\frac{1}{8}+\frac{1}{8}\cos 8x = \frac{3}{8}+\frac{1}{2}\cos 4x+\frac{1}{8}\cos 8x$

$\boxed{33}$ $\sin^4 2x = (\sin^2 2x)^2 = \left(\frac{1-\cos 4x}{2}\right)^2 = \frac{1-2\cos 4x+\cos^2 4x}{4} =$

$\frac{1}{4}-\frac{1}{2}\cos 4x+\frac{1}{4}\left(\frac{1+\cos 8x}{2}\right) = \frac{1}{4}-\frac{1}{2}\cos 4x+\frac{1}{8}+\frac{1}{8}\cos 8x = \frac{3}{8}-\frac{1}{2}\cos 4x+\frac{1}{8}\cos 8x$

$\boxed{34}$ $\sin^4\frac{\theta}{2} = \left(\sin^2\frac{\theta}{2}\right)^2 = \left(\frac{1-\cos\theta}{2}\right)^2 = \frac{1-2\cos\theta+\cos^2\theta}{4} = \frac{1}{4}-\frac{1}{2}\cos\theta+\frac{1}{4}\left(\frac{1+\cos 2\theta}{2}\right) =$

$\frac{1}{4}-\frac{1}{2}\cos\theta+\frac{1}{8}+\frac{1}{8}\cos 2\theta = \frac{3}{8}-\frac{1}{2}\cos\theta+\frac{1}{8}\cos 2\theta$

$\boxed{35}$ $\sin 2t+\sin t = 0 \;\Rightarrow\; 2\sin t\,\cos t+\sin t = 0 \;\Rightarrow\; \sin t\,(2\cos t+1) = 0 \;\Rightarrow$

$\sin t = 0$ or $\cos t = -\frac{1}{2} \;\Rightarrow\; t = 0,\ \pi$ or $\frac{2\pi}{3},\ \frac{4\pi}{3}$

$\boxed{36}$ $\cos t-\sin 2t = 0 \;\Rightarrow\; \cos t-2\sin t\,\cos t = 0 \;\Rightarrow\; \cos t\,(1-2\sin t) = 0 \;\Rightarrow$

$\cos t = 0$ or $\sin t = \frac{1}{2} \;\Rightarrow\; t = \frac{\pi}{2},\ \frac{3\pi}{2}$ or $\frac{\pi}{6},\ \frac{5\pi}{6}$

$\boxed{37}$ $\cos u+\cos 2u = 0 \;\Rightarrow\; \cos u+2\cos^2 u-1 = 0 \;\Rightarrow\; (2\cos u-1)(\cos u+1) = 0 \;\Rightarrow$

$\cos u = \frac{1}{2},\ -1 \;\Rightarrow\; u = \frac{\pi}{3},\ \frac{5\pi}{3},\ \pi$

$\boxed{38}$ $\cos 2\theta-\tan\theta = 1 \;\Rightarrow\; 1-2\sin^2\theta-\frac{\sin\theta}{\cos\theta} = 1 \;\Rightarrow\; 2\sin^2\theta\,\cos\theta+\sin\theta = 0 \;\Rightarrow$

$\sin\theta\,(2\sin\theta\,\cos\theta+1) = 0 \;\Rightarrow\; \sin\theta = 0$ or $\sin 2\theta = -1 \;\Rightarrow$

$\theta = 0,\ \pi$ or $2\theta = \frac{3\pi}{2},\ \frac{7\pi}{2}$ {If $\theta \in [0,\ 2\pi)$, then $2\theta \in [0,\ 4\pi)$.} $\Rightarrow\; \theta = 0,\ \pi,\ \frac{3\pi}{4},\ \frac{7\pi}{4}$

$\boxed{39}$ $\tan 2x = \tan x \;\Rightarrow\; 2x = x+\pi n \;\Rightarrow\; x = \pi n \;\Rightarrow\; x = 0,\ \pi.$

Another approach is: $\tan 2x = \tan x \;\Rightarrow\; \frac{\sin 2x}{\cos 2x} = \frac{\sin x}{\cos x} \;\Rightarrow$

$\sin 2x\,\cos x = \sin x\,\cos 2x \;\Rightarrow\; \sin 2x\,\cos x-\sin x\,\cos 2x = 0 \;\Rightarrow$

$\sin(2x-x) = 0 \;\Rightarrow\; \sin x = 0 \;\Rightarrow\; x = 0,\ \pi.$

$\boxed{40}$ $\tan 2t-2\cos t = 0 \;\Rightarrow\; \frac{\sin 2t}{\cos 2t}-2\cos t = 0 \;\Rightarrow$

$\frac{2\sin t\,\cos t}{\cos 2t}-\frac{(1-2\sin^2 t)(2\cos t)}{\cos 2t} = 0 \;\Rightarrow\; 2\sin t\,\cos t-2\cos t+4\cos t\,\sin^2 t = 0 \;\Rightarrow$

$2\cos t\,(2\sin^2 t+\sin t-1) = 0 \;\Rightarrow\; (2\cos t)(2\sin t-1)(\sin t+1) \;\Rightarrow$

$\cos t = 0$ or $\sin t = \frac{1}{2},\ -1 \;\Rightarrow\; t = \frac{\pi}{2},\ \frac{3\pi}{2}$ or $t = \frac{\pi}{6},\ \frac{5\pi}{6},\ \frac{3\pi}{2}$

$\boxed{41}$ $\sin\frac{1}{2}u+\cos u = 1 \;\Rightarrow\; \sin\frac{1}{2}u+(1-2\sin^2\frac{1}{2}u) = 1 \;\Rightarrow\; \sin\frac{1}{2}u-2\sin^2\frac{1}{2}u = 0 \;\Rightarrow$

$\sin\frac{1}{2}u\,(1-2\sin\frac{1}{2}u) = 0 \;\Rightarrow\; \sin\frac{1}{2}u = 0,\ \frac{1}{2} \;\Rightarrow\; \frac{1}{2}u = 0,\ \frac{\pi}{6},\ \frac{5\pi}{6} \;\Rightarrow\; u = 0,\ \frac{\pi}{3},\ \frac{5\pi}{3}$

$\boxed{42}$ $2-\cos^2 x = 4\sin^2\frac{1}{2}x \;\Rightarrow\; 2-\cos^2 x = 4\left(\frac{1-\cos x}{2}\right) \;\Rightarrow\; 2-\cos^2 x = 2-2\cos x \;\Rightarrow$

$\cos^2 x-2\cos x = 0 \;\Rightarrow\; \cos x\,(\cos x-2) = 0 \;\Rightarrow\; \cos x = 0\ \{\cos x \neq 2\} \;\Rightarrow$

$x = \frac{\pi}{2},\ \frac{3\pi}{2}$

$\boxed{43}$ $\sqrt{a^2+b^2}\sin(u+v) = \sqrt{a^2+b^2}\sin u\cos v + \sqrt{a^2+b^2}\cos u\sin v$

$= a\sin u + b\cos u$ {equate coefficients of $\sin u$ and $\cos u$} $\Rightarrow$

$a = \sqrt{a^2+b^2}\cos v$ and $b = \sqrt{a^2+b^2}\sin v \Rightarrow$

$\cos v = \dfrac{a}{\sqrt{a^2+b^2}}$ and $\sin v = \dfrac{b}{\sqrt{a^2+b^2}}$. Since $0 < u < \frac{\pi}{2}$, $\sin u > 0$ and $\cos v > 0$.

Now $a > 0$ and $b > 0$ combine with the above to imply that $\cos v > 0$ and $\sin v > 0$.

Thus, $0 < v < \frac{\pi}{2}$.

$\boxed{44}$ $\sqrt{8^2+15^2} = 17$. $\sin v = \frac{15}{17}$ and $\cos v = \frac{8}{17} \Rightarrow v = \approx 1.08$ radians, or $v \approx 62°$.

$8\sin u + 15\cos u \approx 17\sin(u+1.08)$.

$\boxed{45}$ (a) $\cos 2x + 2\cos x = 0 \Rightarrow 2\cos^2 x + 2\cos x - 1 = 0 \Rightarrow$

$\cos x = \dfrac{-2 \pm \sqrt{12}}{4} = \dfrac{-1 \pm \sqrt{3}}{2} \approx 0.366 \left\{ \cos x \neq \dfrac{-1-\sqrt{3}}{2} \right\} \Rightarrow x \approx 1.20$ and 5.09.

(b) $\sin 2x + \sin x = 0 \Rightarrow 2\sin x\cos x + \sin x = 0 \Rightarrow \sin x(2\cos x + 1) = 0 \Rightarrow$

$\sin x = 0$ or $\cos x = -\frac{1}{2} \Rightarrow x = 0, \pi, 2\pi$ or $\frac{2\pi}{3}, \frac{4\pi}{3}$.

$P(\frac{2\pi}{3}, -1.5)$, $Q(\pi, -1)$, $R(\frac{4\pi}{3}, -1.5)$

$\boxed{46}$ (a) $\cos x - \sin 2x = 0 \Rightarrow \cos x - 2\sin x\cos x = 0 \Rightarrow \cos x(1 - 2\sin x) = 0 \Rightarrow$

$\cos x = 0$ or $\sin x = \frac{1}{2} \Rightarrow x = -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$ or $-\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$.

(b) $\sin x + 2\cos 2x = 0 \Rightarrow \sin x + 2(1 - 2\sin^2 x) = 0 \Rightarrow$

$4\sin^2 x - \sin x - 2 = 0 \Rightarrow \sin x = \dfrac{1 \pm \sqrt{33}}{8} \approx 0.843, -0.593 \Rightarrow$

$x \approx 1.00, -5.28, 2.14, -4.14, -0.63, 5.65\ 3.78, -2.51$

$\boxed{47}$ (a) $\cos 3x - 3\cos x = 0 \Rightarrow 4\cos^3 x - 3\cos x - 3\cos x = 0 \Rightarrow$

$4\cos^3 x - 6\cos x = 0 \Rightarrow 2\cos x(2\cos^2 x - 3) = 0 \Rightarrow \cos x = 0, \pm\sqrt{3/2} \Rightarrow$

$x = -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$ $\{\cos x \neq \pm\sqrt{3/2}\}$

(b) $\sin 3x - \sin x = 0 \Rightarrow 3\sin x - 4\sin^3 x - \sin x = 0 \Rightarrow 4\sin^3 x - 2\sin x = 0 \Rightarrow$

$2\sin x(2\sin^2 x - 1) = 0 \Rightarrow \sin x = 0, \pm 1/\sqrt{2} \Rightarrow$

$x = 0, \pm\pi, \pm 2\pi, \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}, \pm\frac{5\pi}{4}, \pm\frac{7\pi}{4}$

$\boxed{48}$ $\sin 4x - 4\sin x = 0 \Rightarrow 4\sin x\cos x(1 - 2\sin^2 x) - 4\sin x = 0 \Rightarrow$

$4\sin x[\cos x(1 - 2\sin^2 x) - 1] = 0 \Rightarrow 4\sin x[\cos x(2\cos^2 x - 1) - 1] = 0 \Rightarrow$

$4\sin x(2\cos^3 x - \cos x - 1) = 0 \Rightarrow 4\sin x(\cos x - 1)(2\cos^2 x + 2\cos x + 1) = 0 \Rightarrow$

$\sin x = 0$ or $\cos x = 1$ $\{2\cos^2 x + 2\cos x + 1 \neq 0\} \Rightarrow x = 0, \pm\pi, \pm 2\pi$

$\boxed{49}$ (a) Let $y = \overline{BC}$. Form a right triangle with hypotenuse y, side opposite θ, 20, and side adjacent θ, x. $\sin\theta = \dfrac{20}{y} \Rightarrow y = \dfrac{20}{\sin\theta}$. $\cos\theta = \frac{x}{y} \Rightarrow$

$x = y\cos\theta = \dfrac{20\cos\theta}{\sin\theta}$. Now $d = (40 - x) + y = 40 - \dfrac{20\cos\theta}{\sin\theta} + \dfrac{20}{\sin\theta} =$

$20\left(\dfrac{1-\cos\theta}{\sin\theta}\right) + 40 = 20\tan\dfrac{\theta}{2} + 40.$

(b) $50 = 20\tan\frac{\theta}{2} + 40 \Rightarrow \tan\frac{\theta}{2} = \frac{1}{2} \Rightarrow \frac{1-\cos\theta}{\sin\theta} = \frac{1}{2} \Rightarrow 2 - 2\cos\theta = \sin\theta \Rightarrow$

$4 - 8\cos\theta + 4\cos^2\theta = \sin^2\theta = 1 - \cos^2\theta \Rightarrow 5\cos^2\theta - 8\cos\theta + 3 = 0 \Rightarrow$

$(5\cos\theta - 3)(\cos\theta - 1) = 0 \Rightarrow \cos\theta = \frac{3}{5}, 1.$ $\{\cos\theta = 1 \Rightarrow \theta = 0$ and 0 is extraneous$\}$. $\cos\theta = \frac{3}{5} \Rightarrow \sin\theta = \frac{4}{5}$ and $y = \frac{20}{4/5} = 25$.

$\cos\theta = \frac{x}{y}$ and $\cos\theta = \frac{3}{5} \Rightarrow \frac{x}{25} = \frac{3}{5} \Rightarrow x = 15,$

which means that B would be 25 miles from A.

$\boxed{50}$ $R = 150$ and $v = 80 \Rightarrow 150 = \frac{80^2}{16}\sin\theta\cos\theta \Rightarrow \frac{3}{8} = \frac{1}{2}(2\sin\theta\cos\theta) \Rightarrow$

$\sin 2\theta = \frac{3}{4} \Rightarrow 2\theta \approx 48.59^\circ$ or $131.41^\circ \Rightarrow \theta \approx 24.30^\circ$ or 65.70°.

$\boxed{51}$ (a) From Example 8, the area A of a cross section is

$A = \frac{1}{2}(\text{side})^2(\text{sine of included angle}) = \frac{1}{2}(\frac{1}{2})^2\sin\theta = \frac{1}{8}\sin\theta.$

The volume $V = (\text{length of gutter})(\text{area of cross section}) = 20(\frac{1}{8}\sin\theta) = \frac{5}{2}\sin\theta.$

(b) $V = 2 \Rightarrow \frac{5}{2}\sin\theta = 2 \Rightarrow \sin\theta = \frac{4}{5} \Rightarrow \theta \approx 53.13^\circ$.

$\boxed{52}$ (a) Let D denote the center of the circle. $\angle ACB = 180 - \phi$.

$\triangle DAC$ is a right triangle since the circle is tangent to the highway.

$$\angle DCA = \frac{1}{2}\angle ACB = 90^\circ - \frac{\phi}{2}, \text{ so } \angle CDA = \frac{\phi}{2}.\ \tan\frac{\phi}{2} = \frac{d}{R} \Rightarrow d = R\tan\frac{\phi}{2}.$$

(b) $20 = R\tan\frac{45^\circ}{2} \Rightarrow 20 = R\left(\frac{1-\cos 45^\circ}{\sin 45^\circ}\right) \Rightarrow$

$$R = \frac{20\sin 45^\circ}{1-\cos 45^\circ} = \frac{20\cdot\sqrt{2}/2}{1-\sqrt{2}/2}\cdot\frac{2}{2} = \frac{20\sqrt{2}}{2-\sqrt{2}}\cdot\frac{2+\sqrt{2}}{2+\sqrt{2}} = \frac{20(2\sqrt{2}+2)}{2} =$$

$20\sqrt{2} + 20 \approx 48.28$ ft. The length of the curbing can be treated as an arc length. The radius is R and the central angle is 45°. $s = r\theta = (20 + 20\sqrt{2})(\frac{\pi}{4}) \approx 37.92$ ft.

$\boxed{53}$ (a) Let $y = \overline{DB}$ and x denote the distance from D to the midpoint of $\overline{BC}$.

$\sin\frac{\theta}{2} = \frac{b/2}{y} \Rightarrow y = \frac{b}{2}\cdot\frac{1}{\sin(\theta/2)}$ and $\tan\frac{\theta}{2} = \frac{b/2}{x} \Rightarrow x = \frac{b}{2}\cdot\frac{\cos(\theta/2)}{\sin(\theta/2)}$.

$$l = (a-x) + y = a - \frac{b}{2}\cdot\frac{\cos(\theta/2)}{\sin(\theta/2)} + \frac{b}{2}\cdot\frac{1}{\sin(\theta/2)} = a + \frac{b}{2}\cdot\frac{1-\cos(\theta/2)}{\sin(\theta/2)} =$$

$$a + \frac{b}{2}\tan\left(\frac{\theta/2}{2}\right) = a + \frac{b}{2}\tan\frac{\theta}{4}.$$

(b) $a = 10$ mm, $b = 6$ mm, and $\theta = 156^\circ \Rightarrow l = 10 + 3\tan 39^\circ \approx 12.43$ mm.

$\boxed{54}$ (a) $f(t) = \sin^2\omega t = \frac{1-\cos 2\omega t}{2} = \frac{1}{2} - \frac{1}{2}\cos 2\omega t$.

Now $\cos 2\omega t$ has period $\frac{2\pi}{2\omega} = \frac{\pi}{\omega}$. Since $0 \le t \le 2(\frac{\pi}{\omega})$, we see that f will complete 2 cycles on the given interval. Thus, the average value is $c = \frac{1}{2}$.

(b) $r = RI^2 = RI_0^2\sin^2\omega t = RI_0^2(\frac{1}{2} - \frac{1}{2}\cos 2\omega t) = \frac{1}{2}RI_0^2 - \frac{1}{2}RI_0^2\cos 2\omega t$.

Thus, the average rate at which heat is produced is $\frac{1}{2}RI_0^2$.

[55] The graph of f appears to be that of $y = g(x) = \tan x$.

$$\frac{\sin 2x + \sin x}{\cos 2x + \cos x + 1} = \frac{2\sin x\,\cos x + \sin x}{(2\cos^2 x - 1) + \cos x + 1} = \frac{\sin x(2\cos x + 1)}{\cos x(2\cos x + 1)} = \frac{\sin x}{\cos x} = \tan x$$

[56] The graph of f appears to be that of $y = g(x) = \cos x$.

$$\frac{\sin x(1 + \cos 2x)}{\sin 2x} = \frac{\sin x\left[1 + (2\cos^2 x - 1)\right]}{2\sin x\,\cos x} = \frac{2\cos^2 x}{2\cos x} = \cos x$$

[57] Graph $Y_1 = \tan(0.5x + 1)$ and $Y_2 = \sin 0.5x$ on $[-2\pi, 2\pi]$ in Dot mode. There are two points of intersection. They occur at $x \approx -3.55, 5.22$.

$[-2\pi, 2\pi, \pi/2]$ by $[-4, 4]$

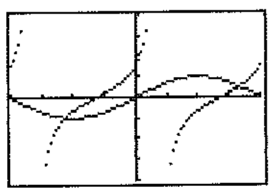

Figure 57

$[-\pi/2, \pi/2, \pi/4]$ by $[-4, 4]$

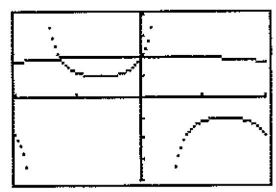

Figure 58

[58] Graph $Y_1 = 1/\cos(2x + 1)$ and $Y_2 = \cos(0.5x) + 1$ on $[-\pi/2, \pi/2]$ in Dot mode. There are two points of intersection. They occur at $x \approx -1.00, 0.02$.

[59] Graph $Y_1 = 1/\sin(0.25x + 1)$ and $Y_2 = 1.5 - \cos 2x$ on $[-\pi, \pi]$. There are four points of intersection. They occur at $x \approx -2.03, -0.72, 0.58, 2.62$.

$[-\pi, \pi, \pi/4]$ by $[-4, 4]$

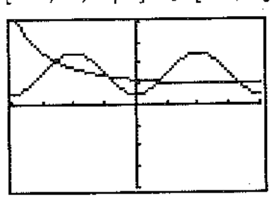

Figure 59

$[-\pi, \pi, \pi/4]$ by $[-4, 4]$

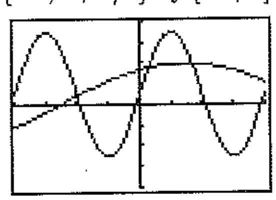

Figure 60

[60] Graph $Y_1 = 3\sin(2x) + 0.5$ and $Y_2 = 2\sin(\frac{1}{2}x + 1)$ on $[-\pi, \pi]$. There are three points of intersection. They occur at $x \approx -1.56, 0.22, 1.31$.

[61] Graph $Y_1 = 2/\tan(.25x)$ and $Y_2 = 1 - 1/\cos(.5x)$ on $[-2\pi, 2\pi]$ in Dot mode. There is one point of intersection. It occurs at $x \approx -2.59$.

$[-2\pi, 2\pi, \pi/2]$ by $[-4, 4]$

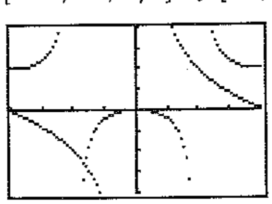

Figure 61

$[-\pi, \pi, \pi/4]$ by $[-4, 4]$

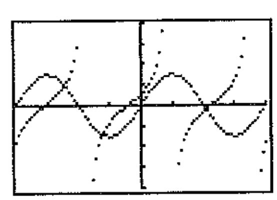

Figure 62

$\boxed{62}$ Graph $Y_1 = \tan(1.5x + .5)$ and $Y_2 = 1.5\sin(2x)$ on $[-\pi, \pi]$ in Dot mode. There are three points of intersection. They occur at $x \approx -1.92, -0.97, 1.63$. See *Figure 62.*

7.5 Exercises

Note: We will reference the product-to-sum formulas as [P1] – [P4] and the sum-to-product formulas as [S1] – [S4] in the order they appear in the text. The formulas $\cos(-kx) = \cos kx$ and $\sin(-kx) = -\sin kx$ will be used without mention.

$\boxed{1}$ $\sin 7t \sin 3t =$ [P4] $\frac{1}{2}[\cos(7t - 3t) - \cos(7t + 3t)] = \frac{1}{2}\cos 4t - \frac{1}{2}\cos 10t$

$\boxed{2}$ $\sin(-4x)\cos 8x =$ [P1] $\frac{1}{2}[\sin(-4x + 8x) + \sin(-4x - 8x)] =$

$$\tfrac{1}{2}\sin 4x + \tfrac{1}{2}\sin(-12x) = \tfrac{1}{2}\sin 4x - \tfrac{1}{2}\sin 12x$$

$\boxed{3}$ $\cos 6u \cos(-4u) =$ [P3] $\frac{1}{2}\left\{\cos[6u + (-4u)] + \cos[6u - (-4u)]\right\} = \frac{1}{2}\cos 2u + \frac{1}{2}\cos 10u$

$\boxed{4}$ $\cos 4t \sin 6t =$ [P2] $\frac{1}{2}[\sin(4t + 6t) - \sin(4t - 6t)] = \frac{1}{2}\sin 10t - \frac{1}{2}\sin(-2t) =$

$$\tfrac{1}{2}\sin 10t + \tfrac{1}{2}\sin 2t$$

$\boxed{5}$ $2\sin 9\theta \cos 3\theta =$ [P1] $2 \cdot \frac{1}{2}[\sin(9\theta + 3\theta) + \sin(9\theta - 3\theta)] = \sin 12\theta + \sin 6\theta$

$\boxed{6}$ $2\sin 7\theta \sin 5\theta =$ [P4] $2 \cdot \frac{1}{2}[\cos(7\theta - 5\theta) - \cos(7\theta + 5\theta)] = \cos 2\theta - \cos 12\theta$

$\boxed{7}$ $3\cos x \sin 2x =$ [P2] $3 \cdot \frac{1}{2}[\sin(x + 2x) - \sin(x - 2x)] = \frac{3}{2}\sin 3x - \frac{3}{2}\sin(-x) =$

$$\tfrac{3}{2}\sin 3x + \tfrac{3}{2}\sin x$$

$\boxed{8}$ $5\cos u \cos 5u =$ [P3] $5 \cdot \frac{1}{2}[\cos(u + 5u) + \cos(u - 5u)] = \frac{5}{2}\cos 6u + \frac{5}{2}\cos(-4u) =$

$$\tfrac{5}{2}\cos 6u + \tfrac{5}{2}\cos 4u$$

$\boxed{9}$ $\sin 6\theta + \sin 2\theta =$ [S1] $2\sin\dfrac{6\theta + 2\theta}{2}\cos\dfrac{6\theta - 2\theta}{2} = 2\sin 4\theta \cos 2\theta$

$\boxed{10}$ $\sin 4\theta - \sin 8\theta =$ [S2] $2\cos\dfrac{4\theta + 8\theta}{2}\sin\dfrac{4\theta - 8\theta}{2} = 2\cos 6\theta \sin(-2\theta) = -2\cos 6\theta \sin 2\theta$

$\boxed{11}$ $\cos 5x - \cos 3x =$ [S4] $-2\sin\dfrac{5x + 3x}{2}\sin\dfrac{5x - 3x}{2} = -2\sin 4x \sin x$

$\boxed{12}$ $\cos 5t + \cos 6t =$ [S3] $2\cos\dfrac{5t + 6t}{2}\cos\dfrac{5t - 6t}{2} = 2\cos\frac{11}{2}t \cos(-\frac{1}{2}t) = 2\cos\frac{11}{2}t \cos\frac{1}{2}t$

$\boxed{13}$ $\sin 3t - \sin 7t =$ [S2] $2\cos\dfrac{3t + 7t}{2}\sin\dfrac{3t - 7t}{2} = 2\cos 5t \sin(-2t) = -2\cos 5t \sin 2t$

$\boxed{14}$ $\cos\theta - \cos 5\theta =$ [S4] $-2\sin\dfrac{\theta + 5\theta}{2}\sin\dfrac{\theta - 5\theta}{2} = -2\sin 3\theta \sin(-2\theta) = 2\sin 3\theta \sin 2\theta$

$\boxed{15}$ $\cos x + \cos 2x =$ [S3] $2\cos\dfrac{x + 2x}{2}\cos\dfrac{x - 2x}{2} = 2\cos\frac{3}{2}x \cos(-\frac{1}{2}x) = 2\cos\frac{3}{2}x \cos\frac{1}{2}x$

$\boxed{16}$ $\sin 8t + \sin 2t =$ [S1] $2\sin\dfrac{8t + 2t}{2}\cos\dfrac{8t - 2t}{2} = 2\sin 5t \cos 3t$

$\boxed{17}$ $\dfrac{\sin 4t + \sin 6t}{\cos 4t - \cos 6t} = \dfrac{\text{[S1] } 2\sin 5t \cos(-t)}{\text{[S4] } -2\sin 5t \sin(-t)} = \dfrac{\cos t}{\sin t} = \cot t$

$\boxed{18}$ $\dfrac{\sin\theta + \sin 3\theta}{\cos\theta + \cos 3\theta} = \dfrac{\text{[S1] } 2\sin 2\theta \cos(-\theta)}{\text{[S3] } 2\cos 2\theta \cos(-\theta)} = \dfrac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta$

$\boxed{19}$ $\dfrac{\sin u + \sin v}{\cos u + \cos v} = \dfrac{[\text{S1}]\ 2\sin\frac{1}{2}(u+v)\ \cos\frac{1}{2}(u-v)}{[\text{S3}]\ 2\cos\frac{1}{2}(u+v)\ \cos\frac{1}{2}(u-v)} = \tan\frac{1}{2}(u+v)$

$\boxed{20}$ $\dfrac{\sin u - \sin v}{\cos u - \cos v} = \dfrac{[\text{S2}]\ 2\cos\frac{1}{2}(u+v)\ \sin\frac{1}{2}(u-v)}{[\text{S4}]\ -2\sin\frac{1}{2}(u+v)\ \sin\frac{1}{2}(u-v)} = -\cot\frac{1}{2}(u+v)$

$\boxed{21}$ $\dfrac{\sin u - \sin v}{\sin u + \sin v} = \dfrac{[\text{S2}]\ 2\cos\frac{1}{2}(u+v)\ \sin\frac{1}{2}(u-v)}{[\text{S1}]\ 2\sin\frac{1}{2}(u+v)\ \cos\frac{1}{2}(u-v)} = \cot\frac{1}{2}(u+v)\tan\frac{1}{2}(u-v) = \dfrac{\tan\frac{1}{2}(u-v)}{\tan\frac{1}{2}(u+v)}$

$\boxed{22}$ $\dfrac{\cos u - \cos v}{\cos u + \cos v} = \dfrac{[\text{S4}]\ -2\sin\frac{1}{2}(u+v)\ \sin\frac{1}{2}(u-v)}{[\text{S3}]\ 2\cos\frac{1}{2}(u+v)\ \cos\frac{1}{2}(u-v)} = -\tan\frac{1}{2}(u+v)\tan\frac{1}{2}(u-v)$

$\boxed{23}$ $4\cos x\ \cos 2x\ \sin 3x = 2\cos 2x\,(2\sin 3x\ \cos x) = 2\cos 2x\,([\text{P1}]\ \sin 4x + \sin 2x) =$
$(2\cos 2x\ \sin 4x) + (2\cos 2x\ \sin 2x) = \big[[\text{P2}]\ \sin 6x - \sin(-2x)\big] + ([\text{P2}]\ \sin 4x - \sin 0) =$
$\sin 2x + \sin 4x + \sin 6x$

$\boxed{24}$ $\dfrac{\cos t + \cos 4t + \cos 7t}{\sin t + \sin 4t + \sin 7t} = \dfrac{\cos 4t + [\text{S3}]\ 2\cos 4t\ \cos(-3t)}{\sin 4t + [\text{S1}]\ 2\sin 4t\ \cos(-3t)} = \dfrac{\cos 4t\,(1 + 2\cos 3t)}{\sin 4t\,(1 + 2\cos 3t)} = \cot 4t$

$\boxed{25}$ $(\sin ax)(\cos bx) = [\text{P1}]\ \frac{1}{2}\big[\sin(ax + bx) + \sin(ax - bx)\big] = \frac{1}{2}\sin\big[(a+b)x\big] + \frac{1}{2}\sin\big[(a-b)x\big]$

$\boxed{26}$ $(\cos au)(\cos bu) = [\text{P3}]\ \frac{1}{2}\big[\cos(au + bu) + \cos(au - bu)\big] =$
$\frac{1}{2}\cos\big[(a+b)u\big] + \frac{1}{2}\cos\big[(a-b)u\big]$

$\boxed{27}$ $\sin 5t + \sin 3t = 0 \Rightarrow [\text{S1}]\ 2\sin 4t\ \cos t = 0 \Rightarrow \sin 4t = 0$ or $\cos t = 0 \Rightarrow$
$4t = \pi n$ or $t = \frac{\pi}{2} + \pi n \Rightarrow t = \frac{\pi}{4}n$ { which includes $t = \frac{\pi}{2} + \pi n$ }

$\boxed{28}$ $\sin t + \sin 3t = \sin 2t \Rightarrow [\text{S1}]\ 2\sin 2t\ \cos(-t) = \sin 2t \Rightarrow$
$\sin 2t\,(2\cos t - 1) = 0 \Rightarrow \sin 2t = 0$ or $\cos t = \frac{1}{2} \Rightarrow$
$2t = \pi n$ (i.e., $t = \frac{\pi}{2}n$) or $t = \frac{\pi}{3} + 2\pi n,\ \frac{5\pi}{3} + 2\pi n$

$\boxed{29}$ $\cos x = \cos 3x \Rightarrow \cos x - \cos 3x = 0 \Rightarrow [\text{S4}]\ -2\sin 2x\ \sin(-x) = 0 \Rightarrow$
$\sin 2x = 0$ or $\sin x = 0 \Rightarrow 2x = \pi n$ or $x = \pi n \Rightarrow x = \frac{\pi}{2}n$ { which includes $x = \pi n$ }

$\boxed{30}$ $\cos 4x - \cos 3x = 0 \Rightarrow [\text{S4}]\ -2\sin\frac{7}{2}x\ \sin\frac{1}{2}x = 0 \Rightarrow \sin\frac{7}{2}x = 0$ or $\sin\frac{1}{2}x = 0 \Rightarrow$
$\frac{7}{2}x = \pi n$ or $\frac{1}{2}x = \pi n \Rightarrow x = \frac{2\pi}{7}n$ or $x = 2\pi n \Rightarrow$
$x = \frac{2\pi}{7}n$ { which includes $x = 2\pi n$ }

$\boxed{31}$ $\cos 3x + \cos 5x = \cos x \Rightarrow [\text{S3}]\ 2\cos 4x\ \cos(-x) - \cos x = 0 \Rightarrow$
$\cos x\,(2\cos 4x - 1) = 0 \Rightarrow \cos x = 0$ or $\cos 4x = \frac{1}{2} \Rightarrow$
$x = \frac{\pi}{2} + \pi n$ or $4x = \frac{\pi}{3} + 2\pi n,\ \frac{5\pi}{3} + 2\pi n \Rightarrow x = \frac{\pi}{2} + \pi n,\ \frac{\pi}{12} + \frac{\pi}{2}n,\ \frac{5\pi}{12} + \frac{\pi}{2}n$

$\boxed{32}$ $\cos 3x = -\cos 6x \Rightarrow \cos 3x + \cos 6x = 0 \Rightarrow [\text{S3}]\ 2\cos\frac{9}{2}x\ \cos(-\frac{3}{2}x) = 0 \Rightarrow$
$\frac{9}{2}x = \frac{\pi}{2} + \pi n$ or $\frac{3}{2}x = \frac{\pi}{2} + \pi n \Rightarrow x = \frac{\pi}{9} + \frac{2\pi}{9}n$ or $x = \frac{\pi}{3} + \frac{2\pi}{3}n \Rightarrow$
$x = \frac{\pi}{9} + \frac{2\pi}{9}n$ { which includes $x = \frac{\pi}{3} + \frac{2\pi}{3}n$ }

$\boxed{33}$ $\sin 2x - \sin 5x = 0 \Rightarrow [\text{S2}]\ 2\cos\frac{7}{2}x\ \sin(-\frac{3}{2}x) = 0 \Rightarrow \frac{7}{2}x = \frac{\pi}{2} + \pi n$ or $\frac{3}{2}x = \pi n \Rightarrow$
$x = \frac{\pi}{7} + \frac{2\pi}{7}n$ or $x = \frac{2\pi}{3}n$

$\boxed{34}$ $\sin 5x - \sin x = 2\cos 3x \Rightarrow$ [S2] $2\cos 3x\,\sin 2x - 2\cos 3x = 0 \Rightarrow$

$2\cos 3x\,(\sin 2x - 1) = 0 \Rightarrow \cos 3x = 0$ or $\sin 2x = 1 \Rightarrow$

$3x = \frac{\pi}{2} + \pi n$ or $2x = \frac{\pi}{2} + 2\pi n \Rightarrow x = \frac{\pi}{6} + \frac{\pi}{3}n$ or $x = \frac{\pi}{4} + \pi n$

$\boxed{35}$ $\cos x + \cos 3x = 0 \Rightarrow$ [S3] $2\cos 2x\,\cos(-x) = 0 \Rightarrow \cos 2x = 0$ or $\cos x = 0 \Rightarrow$

$2x = \frac{\pi}{2} + \pi n$ or $x = \frac{\pi}{2} + \pi n \Rightarrow x = \frac{\pi}{4} + \frac{\pi}{2}n$ or $x = \frac{\pi}{2} + \pi n \Rightarrow$

$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{2}$ for $0 \le x \le 2\pi$

$\boxed{36}$ $\sin 4x - \sin x = 0 \Rightarrow$ [S2] $2\cos\frac{5}{2}x\,\sin\frac{3}{2}x \Rightarrow \cos\frac{5}{2}x = 0$ or $\sin\frac{3}{2}x = 0 \Rightarrow$

$\frac{5}{2}x = \frac{\pi}{2} + \pi n$ or $\frac{3}{2}x = \pi n \Rightarrow x = \frac{\pi}{5} + \frac{2\pi}{5}n$ or $x = \frac{2\pi}{3}n \Rightarrow$

$x = \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5}, 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$ for $0 \le x \le 2\pi$. *Note:* $\frac{3\pi}{5} \approx 1.88$, $\frac{2\pi}{3} \approx 2.09$,

$\frac{4\pi}{3} \approx 4.19$, and $\frac{7\pi}{5} \approx 4.40$ are the intercepts that are close together.

$\boxed{37}$ $\sin 3x - \sin x = 0 \Rightarrow$ [S2] $2\cos 2x\,\sin x = 0 \Rightarrow \cos 2x = 0$ or $\sin x = 0 \Rightarrow$

$2x = \frac{\pi}{2} + \pi n$ or $x = \pi n \Rightarrow x = \frac{\pi}{4} + \frac{\pi}{2}n$ or $x = \pi n \Rightarrow$

$x = 0, \pm\pi, \pm 2\pi, \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}, \pm\frac{5\pi}{4}, \pm\frac{7\pi}{4}$ for $-2\pi \le x \le 2\pi$.

$\boxed{38}$ $\cos 4x - \cos x = 0 \Rightarrow$ [S4] $-2\sin\frac{5}{2}x\,\sin\frac{3}{2}x = 0 \Rightarrow \sin\frac{5}{2}x = 0$ or $\sin\frac{3}{2}x = 0 \Rightarrow$

$\frac{5}{2}x = \pi n$ or $\frac{3}{2}x = \pi n \Rightarrow x = \frac{2\pi}{5}n$ or $x = \frac{2\pi}{3}n \Rightarrow$

$x = 0, \pm\frac{2\pi}{5}, \pm\frac{4\pi}{5}, \pm\frac{6\pi}{5}, \pm\frac{8\pi}{5}, \pm 2\pi, \pm\frac{2\pi}{3}, \pm\frac{4\pi}{3}$ for $-2\pi \le x \le 2\pi$.

$\boxed{39}$ $f(x) = \sin\left(\frac{\pi n}{l}x\right)\cos\left(\frac{k\pi n}{l}t\right) =$ [P1] $\frac{1}{2}\left[\sin\frac{\pi n}{l}(x + kt) + \sin\frac{\pi n}{l}(x - kt)\right] =$

$\frac{1}{2}\sin\frac{\pi n}{l}(x + kt) + \frac{1}{2}\sin\frac{\pi n}{l}(x - kt)$

$\boxed{40}$ (a) $p(t) = a\cos\omega_1 t + a\cos\omega_2 t = a(\cos\omega_1 t + \cos\omega_2 t) =$

[S3] $a \cdot 2\cos\left[\frac{1}{2}(\omega_1 + \omega_2)t\right]\cos\left[\frac{1}{2}(\omega_1 - \omega_2)t\right] = 2a\cos\frac{1}{2}(\omega_1 + \omega_2)t\,\cos\frac{1}{2}(\omega_1 - \omega_2)t$

(b) From part (a),

$p(t) = \left[2a\cos\frac{1}{2}(\omega_1 - \omega_2)t\right]\cos\frac{1}{2}(\omega_1 + \omega_2)t = f(t)\cos\frac{1}{2}(\omega_1 + \omega_2)t.$

Since $\omega_1 \approx \omega_2$, $\frac{1}{2}(\omega_1 + \omega_2) \approx \omega_1$, and the period is approximately $2\pi/\omega_1$.

The maximum amplitude occurs when $\cos\frac{1}{2}(\omega_1 - \omega_2)t = 1$, i.e., $f(t) = 2a$.

(c) $p(t) = 0 \Rightarrow \cos 4.5t + \cos 3.5t = 0 \Rightarrow$ [S3] $2\cos 4t\,\cos\frac{1}{2}t = 0 \Rightarrow$

$\left[2\cos\frac{1}{2}t\right]\cos 4t = 0$. From part (b), we want to know when the amplitude is zero.

$2\cos\frac{1}{2}t = 0 \Rightarrow \frac{1}{2}t = \frac{\pi}{2} + \pi n \Rightarrow t = \pi + 2\pi n.$

$A = (-\pi, 0)$ and $B = (\pi, 0)$. Near-silence occurs every 2π units of time.

(d) From the graph, we see that one-half period occurs on the interval from A to B.

$\frac{1}{2}(\textit{period}) = \pi - (-\pi) \Rightarrow \frac{1}{2}(\textit{period}) = 2\pi \Rightarrow \textit{period} = 4\pi.$

[41] (a) Estimating the x-intercepts, we have $x \approx 0,\ \pm 1.05,\ \pm 1.57,\ \pm 2.09,\ \pm 3.14$.

(b) $\sin 4x + \sin 2x = 2\sin 3x\ \cos x = 0 \ \Rightarrow\ \sin 3x = 0$ or $\cos x = 0$.

$\sin 3x = 0 \ \Rightarrow\ 3x = \pi n \ \Rightarrow\ x = \frac{\pi}{3}n \ \Rightarrow\ x = 0,\ \pm\frac{\pi}{3},\ \pm\frac{2\pi}{3},\ \pm\pi$.

$\cos x = 0 \ \Rightarrow\ x = \pm\frac{\pi}{2}$. The x-intercepts are $0,\ \pm\frac{\pi}{3},\ \pm\frac{\pi}{2}, \pm\frac{2\pi}{3},\ \pm\pi$.

$[-\pi,\ \pi,\ \pi/4]$ by $[-2.09,\ 2.09]$ $[-\pi,\ \pi,\ \pi/4]$ by $[-2.09,\ 2.09]$

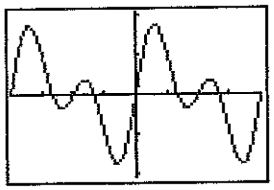

Figure 41

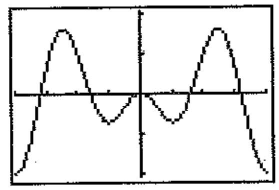

Figure 42

[42] (a) Estimating the x-intercepts, we have $x \approx 0,\ \pm 1.26,\ \pm 2.51$.

(b) $\cos 3x - \cos 2x = -2\sin\frac{5}{2}x\ \sin\frac{1}{2}x = 0 \ \Rightarrow\ \sin\frac{5}{2}x = 0$ or $\sin\frac{1}{2}x = 0$.

$\sin\frac{5}{2}x = 0 \ \Rightarrow\ \frac{5}{2}x = \pi n \ \Rightarrow\ x = \frac{2\pi}{5}n \ \Rightarrow\ x = 0,\ \pm\frac{2\pi}{5},\ \pm\frac{4\pi}{5}$.

$\sin\frac{1}{2}x = 0 \ \Rightarrow\ x = 0$. The x-intercepts are $0,\ \pm\frac{2\pi}{5},\ \pm\frac{4\pi}{5}$.

[43] The graph of f appears to be that of $y = g(x) = \tan 2x$.

$$\frac{\sin x + \sin 2x + \sin 3x}{\cos x + \cos 2x + \cos 3x} = \frac{\sin 2x + (\sin 3x + \sin x)}{\cos 2x + (\cos 3x + \cos x)} = \frac{\sin 2x + 2\sin 2x\ \cos x}{\cos 2x + 2\cos 2x\ \cos x} =$$
$$\frac{\sin 2x(1 + 2\cos x)}{\cos 2x(1 + 2\cos x)} = \frac{\sin 2x}{\cos 2x} = \tan 2x$$

[44] The graph of f appears to be that of $y = g(x) = \cot 2x$.

$$\frac{\cos x - \cos 2x + \cos 3x}{\sin x - \sin 2x + \sin 3x} = \frac{-\cos 2x + (\cos 3x + \cos x)}{-\sin 2x + (\sin 3x + \sin x)} = \frac{-\cos 2x + 2\cos 2x\ \cos x}{-\sin 2x + 2\sin 2x\ \cos x} =$$
$$\frac{\cos 2x(-1 + 2\cos x)}{\sin 2x(-1 + 2\cos x)} = \frac{\cos 2x}{\sin 2x} = \cot 2x$$

7.6 Exercises

[1] (a) $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$ (b) $\cos^{-1}(-\frac{1}{2}) = \frac{2\pi}{3}$ (c) $\tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$

[2] (a) $\sin^{-1}(-\frac{1}{2}) = -\frac{\pi}{6}$ (b) $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$ (c) $\tan^{-1}(-1) = -\frac{\pi}{4}$

[3] (a) $\arcsin\frac{\sqrt{3}}{2} = \frac{\pi}{3}$ (b) $\arccos\frac{\sqrt{2}}{2} = \frac{\pi}{4}$ (c) $\arctan\frac{1}{\sqrt{3}} = \frac{\pi}{6}$

[4] (a) $\arcsin 0 = 0$ (b) $\arccos(-1) = \pi$ (c) $\arctan 0 = 0$

[5] (a) $\sin^{-1}\frac{\pi}{3}$ is <u>not defined</u> since $\frac{\pi}{3} > 1$, i.e., $\frac{\pi}{3} \notin [-1,\ 1]$

(b) $\cos^{-1}\frac{\pi}{2}$ is <u>not defined</u> since $\frac{\pi}{2} > 1$, i.e., $\frac{\pi}{2} \notin [-1,\ 1]$ (c) $\tan^{-1} 1 = \frac{\pi}{4}$

[6] (a) $\arcsin\frac{\pi}{2}$ is <u>not defined</u> since $\frac{\pi}{2} > 1$, i.e., $\frac{\pi}{2} \notin [-1,\ 1]$

(b) $\arccos\frac{\pi}{3}$ is <u>not defined</u> since $\frac{\pi}{3} > 1$, i.e., $\frac{\pi}{3} \notin [-1,\ 1]$ (c) $\arctan\left(-\frac{\sqrt{3}}{3}\right) = -\frac{\pi}{6}$

Note: Exercises 7–10 refer to the boxed properties of $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$.

$\boxed{7}$ (a) $\sin[\arcsin(-\frac{3}{10})] = -\frac{3}{10}$ since $-1 \le -\frac{3}{10} \le 1$

(b) $\cos(\arccos\frac{1}{2}) = \frac{1}{2}$ since $-1 \le \frac{1}{2} \le 1$

(c) $\tan(\arctan 14) = 14$ since $\tan(\arctan x) = x$ for every x

$\boxed{8}$ (a) $\sin(\sin^{-1}\frac{2}{3}) = \frac{2}{3}$ since $-1 \le \frac{2}{3} \le 1$

(b) $\cos[\cos^{-1}(-\frac{1}{5})] = -\frac{1}{5}$ since $-1 \le -\frac{1}{5} \le 1$

(c) $\tan[\tan^{-1}(-9)] = -9$ since $\tan(\arctan x) = x$ for every x

$\boxed{9}$ (a) $\sin^{-1}(\sin\frac{\pi}{3}) = \frac{\pi}{3}$ since $-\frac{\pi}{2} \le \frac{\pi}{3} \le \frac{\pi}{2}$ (b) $\cos^{-1}\left[\cos\left(\frac{5\pi}{6}\right)\right] = \frac{5\pi}{6}$ since $0 \le \frac{5\pi}{6} \le \pi$

(c) $\tan^{-1}\left[\tan\left(-\frac{\pi}{6}\right)\right] = -\frac{\pi}{6}$ since $-\frac{\pi}{2} < -\frac{\pi}{6} < \frac{\pi}{2}$

$\boxed{10}$ (a) $\arcsin\left[\sin\left(-\frac{\pi}{2}\right)\right] = -\frac{\pi}{2}$ since $-\frac{\pi}{2} \le -\frac{\pi}{2} \le \frac{\pi}{2}$

(b) $\arccos(\cos 0) = 0$ since $0 \le 0 \le \pi$ (c) $\arctan(\tan\frac{\pi}{4}) = \frac{\pi}{4}$ since $-\frac{\pi}{2} < \frac{\pi}{4} < \frac{\pi}{2}$

$\boxed{11}$ (a) $\arcsin(\sin\frac{5\pi}{4}) = \arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$

(b) $\arccos(\cos\frac{5\pi}{4}) = \arccos\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$ (c) $\arctan(\tan\frac{7\pi}{4}) = \arctan(-1) = -\frac{\pi}{4}$

$\boxed{12}$ (a) $\sin^{-1}(\sin\frac{2\pi}{3}) = \sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}$ (b) $\cos^{-1}(\cos\frac{4\pi}{3}) = \cos^{-1}(-\frac{1}{2}) = \frac{2\pi}{3}$

(c) $\tan^{-1}(\tan\frac{7\pi}{6}) = \tan^{-1}\frac{\sqrt{3}}{3} = \frac{\pi}{6}$

$\boxed{13}$ (a) $\sin[\cos^{-1}(-\frac{1}{2})] = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$ (b) $\cos(\tan^{-1} 1) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

(c) $\tan[\sin^{-1}(-1)] = \tan(-\frac{\pi}{2})$, which is <u>not defined</u>.

$\boxed{14}$ (a) $\sin(\tan^{-1}\sqrt{3}) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ (b) $\cos(\sin^{-1} 1) = \cos\frac{\pi}{2} = 0$

(c) $\tan(\cos^{-1} 0) = \tan\frac{\pi}{2}$, which is <u>not defined</u>.

$\boxed{15}$ (a) Let $\theta = \sin^{-1}\frac{2}{3}$. From *Figure 15(a)*, $\cot(\sin^{-1}\frac{2}{3}) = \cot\theta = \frac{x}{y} = \frac{\sqrt{5}}{2}$.

(b) Let $\theta = \tan^{-1}(-\frac{3}{5})$. From *Figure 15(b)*, $\sec[\tan^{-1}(-\frac{3}{5})] = \sec\theta = \frac{r}{x} = \frac{\sqrt{34}}{5}$.

(c) Let $\theta = \cos^{-1}(-\frac{1}{4})$. From *Figure 15(c)*, $\csc[\cos^{-1}(-\frac{1}{4})] = \csc\theta = \frac{r}{y} = \frac{4}{\sqrt{15}}$.

Note: Triangles could be used for the figures, and may be easier to work with in class.

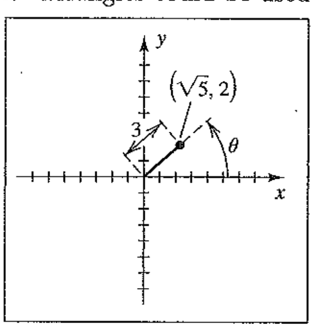

Figure 15(a)

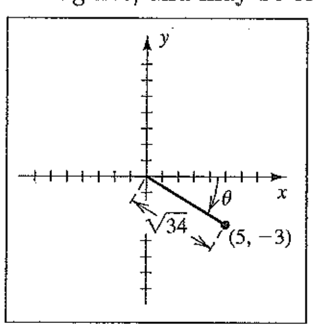

Figure 15(b)

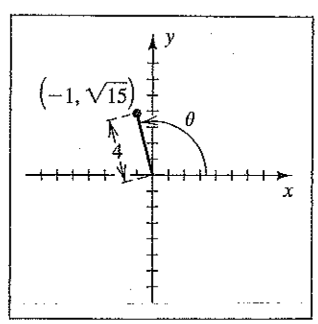

Figure 15(c)

$\boxed{16}$ (a) Let $\theta = \sin^{-1}(-\frac{2}{5})$.

From *Figure 16(a)*, $\cot[\sin^{-1}(-\frac{2}{5})] = \cot\theta = \frac{x}{y} = \frac{\sqrt{21}}{-2} = -\frac{\sqrt{21}}{2}$.

(b) Let $\theta = \tan^{-1}\frac{7}{4}$. From *Figure 16(b)*, $\sec(\tan^{-1}\frac{7}{4}) = \sec\theta = \frac{r}{x} = \frac{\sqrt{65}}{4}$.

(c) Let $\theta = \cos^{-1}\frac{1}{5}$. From *Figure 16(c)*, $\csc(\cos^{-1}\frac{1}{5}) = \csc\theta = \frac{r}{y} = \frac{5}{\sqrt{24}}$.

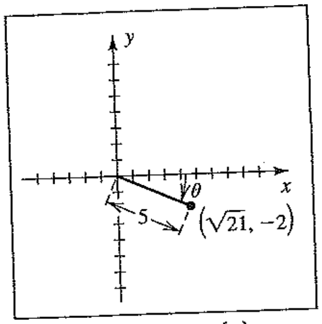

Figure 16(a)

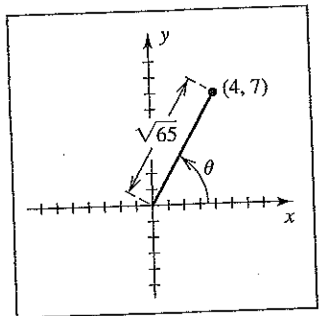

Figure 16(b)

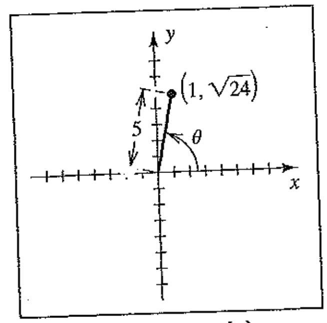

Figure 16(c)

$\boxed{17}$ (a) $\sin(\arcsin\frac{1}{2} + \arccos 0) = \sin(\frac{\pi}{6} + \frac{\pi}{2}) = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$.

(b) Let $\alpha = \arctan(-\frac{3}{4})$ and $\beta = \arcsin\frac{4}{5}$. Using the difference identity for the cosine and figures as in Exercises 15 and 16, we have $\cos[\arctan(-\frac{3}{4}) - \arcsin\frac{4}{5}]$

$$= \cos(\alpha - \beta) = \cos\alpha\,\cos\beta + \sin\alpha\,\sin\beta = \tfrac{4}{5}\cdot\tfrac{3}{5} + (-\tfrac{3}{5})\cdot\tfrac{4}{5} = 0.$$

(c) Let $\alpha = \arctan\frac{4}{3}$ and $\beta = \arccos\frac{8}{17}$. $\tan(\arctan\frac{4}{3} + \arccos\frac{8}{17}) =$

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\,\tan\beta} = \frac{\frac{4}{3} + \frac{15}{8}}{1 - \frac{4}{3}\cdot\frac{15}{8}}\cdot\frac{24}{24} = \frac{32 + 45}{24 - 60} = -\frac{77}{36}.$$

$\boxed{18}$ (a) Let $\alpha = \sin^{-1}\frac{5}{13}$ and $\beta = \cos^{-1}(-\frac{3}{5})$. $\sin[\sin^{-1}\frac{5}{13} - \cos^{-1}(-\frac{3}{5})] =$

$$\sin(\alpha - \beta) = \sin\alpha\,\cos\beta - \cos\alpha\,\sin\beta = \tfrac{5}{13}\cdot(-\tfrac{3}{5}) - \tfrac{12}{13}\cdot\tfrac{4}{5} = -\tfrac{63}{65}.$$

(b) Let $\alpha = \sin^{-1}\frac{4}{5}$ and $\beta = \tan^{-1}\frac{3}{4}$. $\cos(\sin^{-1}\frac{4}{5} + \tan^{-1}\frac{3}{4}) =$

$$\cos(\alpha + \beta) = \cos\alpha\,\cos\beta - \sin\alpha\,\sin\beta = \tfrac{3}{5}\cdot\tfrac{4}{5} - \tfrac{4}{5}\cdot\tfrac{3}{5} = 0.$$

(c) $\tan[\cos^{-1}\frac{1}{2} - \sin^{-1}(-\frac{1}{2})] = \tan[\frac{\pi}{3} - (-\frac{\pi}{6})] = \tan\frac{\pi}{2}$, which is <u>not defined</u>.

$\boxed{19}$ (a) Let $\alpha = \arccos(-\frac{3}{5})$. $\sin[2\arccos(-\frac{3}{5})] = \sin 2\alpha = 2\sin\alpha\,\cos\alpha = 2(\frac{4}{5})(-\frac{3}{5}) = -\frac{24}{25}$.

(b) Let $\alpha = \sin^{-1}\frac{15}{17}$. $\cos(2\sin^{-1}\frac{15}{17}) = \cos 2\alpha = \cos^2\alpha - \sin^2\alpha = (\frac{8}{17})^2 - (\frac{15}{17})^2 = -\frac{161}{289}$.

(c) Let $\alpha = \tan^{-1}\frac{3}{4}$.

$$\tan(2\tan^{-1}\tfrac{3}{4}) = \tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha} = \frac{2\cdot\frac{3}{4}}{1 - (\frac{3}{4})^2}\cdot\frac{16}{16} = \frac{24}{16 - 9} = \frac{24}{7}.$$

[20] (a) Let $\alpha = \tan^{-1}\frac{5}{12}$. $\sin(2\tan^{-1}\frac{5}{12}) = \sin 2\alpha = 2\sin\alpha\cos\alpha = 2\left(\frac{5}{13}\right)\left(\frac{12}{13}\right) = \frac{120}{169}$.

(b) Let $\alpha = \arccos\frac{9}{41}$.

$$\cos(2\arccos\tfrac{9}{41}) = \cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \left(\tfrac{9}{41}\right)^2 - \left(\tfrac{40}{41}\right)^2 = -\tfrac{1519}{1681}.$$

(c) Let $\alpha = \arcsin(-\frac{8}{17})$. $\tan[2\arcsin(-\frac{8}{17})] =$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2\cdot(-\frac{8}{15})}{1-(-\frac{8}{15})^2}\cdot\frac{225}{225} = -\frac{240}{225-64} = -\frac{240}{161}.$$

[21] (a) Let $\alpha = \sin^{-1}(-\frac{7}{25})$. $-\frac{\pi}{2} < \alpha < 0 \;\Rightarrow\; -\frac{\pi}{4} < \frac{1}{2}\alpha < 0$ and $\sin\frac{1}{2}\alpha < 0$.

$$\sin[\tfrac{1}{2}\sin^{-1}(-\tfrac{7}{25})] = \sin\tfrac{1}{2}\alpha = -\sqrt{\frac{1-\cos\alpha}{2}} = -\sqrt{\frac{1-\frac{24}{25}}{2}} = -\sqrt{\tfrac{1}{50}\cdot\tfrac{2}{2}} = -\tfrac{1}{10}\sqrt{2}.$$

(b) Let $\alpha = \tan^{-1}\frac{8}{15}$. $0 < \alpha < \frac{\pi}{2} \;\Rightarrow\; 0 < \frac{1}{2}\alpha < \frac{\pi}{4}$ and $\cos\frac{1}{2}\alpha > 0$.

$$\cos(\tfrac{1}{2}\tan^{-1}\tfrac{8}{15}) = \cos\tfrac{1}{2}\alpha = \sqrt{\frac{1+\cos\alpha}{2}} = \sqrt{\frac{1+\frac{15}{17}}{2}} = \sqrt{\tfrac{16}{17}\cdot\tfrac{17}{17}} = \tfrac{4}{17}\sqrt{17}.$$

(c) Let $\alpha = \cos^{-1}\frac{3}{5}$. $\tan(\frac{1}{2}\cos^{-1}\frac{3}{5}) = \tan\frac{1}{2}\alpha = \dfrac{1-\cos\alpha}{\sin\alpha} = \dfrac{1-\frac{3}{5}}{\frac{4}{5}} = \dfrac{1}{2}$.

[22] (a) Let $\alpha = \cos^{-1}(-\frac{3}{5})$. $\frac{\pi}{2} < \alpha < \pi \;\Rightarrow\; \frac{\pi}{4} < \frac{1}{2}\alpha < \frac{\pi}{2}$ and $\sin\frac{1}{2}\alpha > 0$.

$$\sin[\tfrac{1}{2}\cos^{-1}(-\tfrac{3}{5})] = \sin\tfrac{1}{2}\alpha = \sqrt{\frac{1-\cos\alpha}{2}} = \sqrt{\frac{1-(-\frac{3}{5})}{2}} = \sqrt{\tfrac{4}{5}\cdot\tfrac{5}{5}} = \tfrac{2}{5}\sqrt{5}.$$

(b) Let $\alpha = \sin^{-1}\frac{12}{13}$. $0 < \alpha < \frac{\pi}{2} \;\Rightarrow\; 0 < \frac{1}{2}\alpha < \frac{\pi}{4}$ and $\cos\frac{1}{2}\alpha > 0$.

$$\cos(\tfrac{1}{2}\sin^{-1}\tfrac{12}{13}) = \cos\tfrac{1}{2}\alpha = \sqrt{\frac{1+\cos\alpha}{2}} = \sqrt{\frac{1+\frac{5}{13}}{2}} = \sqrt{\tfrac{9}{13}\cdot\tfrac{13}{13}} = \tfrac{3}{13}\sqrt{13}.$$

(c) Let $\alpha = \tan^{-1}\frac{40}{9}$. $\tan(\frac{1}{2}\tan^{-1}\frac{40}{9}) = \tan\frac{1}{2}\alpha = \dfrac{1-\cos\alpha}{\sin\alpha} = \dfrac{1-\frac{9}{41}}{\frac{40}{41}} = \dfrac{4}{5}$.

[23] Let $\alpha = \tan^{-1}x$. From *Figure 23*, $\sin(\tan^{-1}x) = \sin\alpha = \dfrac{x}{\sqrt{x^2+1}}$.

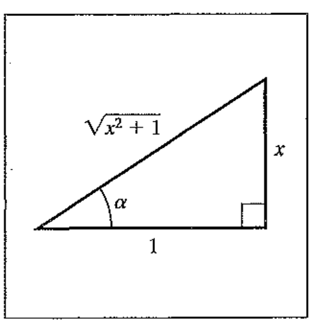

Figure 23

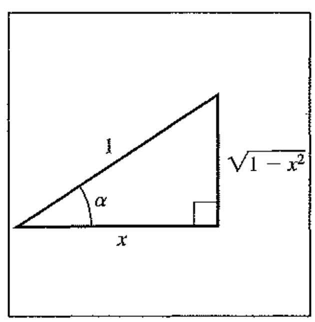

Figure 24

[24] Let $\alpha = \arccos x$. From *Figure 24*, $\tan(\arccos x) = \tan\alpha = \dfrac{\sqrt{1-x^2}}{x}$.

[25] Let $\alpha = \sin^{-1}\dfrac{x}{\sqrt{x^2+4}}$. From *Figure 25*, $\sec\left(\sin^{-1}\dfrac{x}{\sqrt{x^2+4}}\right) = \sec\alpha = \dfrac{\sqrt{x^2+4}}{2}$.

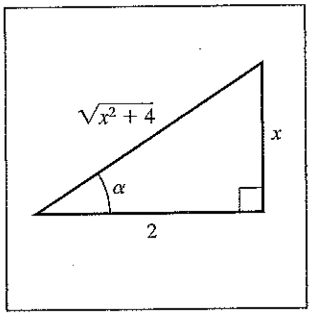

Figure 25

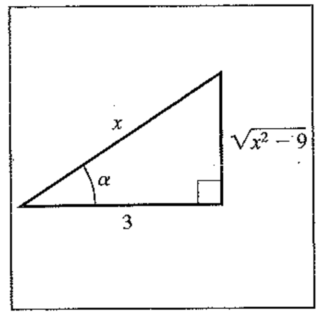

Figure 26

[26] Let $\alpha = \sin^{-1}\dfrac{\sqrt{x^2-9}}{x}$. From *Figure 26*, $\cot\left(\sin^{-1}\dfrac{\sqrt{x^2-9}}{x}\right) = \cot\alpha = \dfrac{3}{\sqrt{x^2-9}}$.

[27] Let $\alpha = \sin^{-1}x$. From *Figure 27*,

$$\sin(2\sin^{-1}x) = \sin 2\alpha = 2\sin\alpha\,\cos\alpha = 2\cdot\frac{x}{1}\cdot\frac{\sqrt{1-x^2}}{1} = 2x\sqrt{1-x^2}.$$

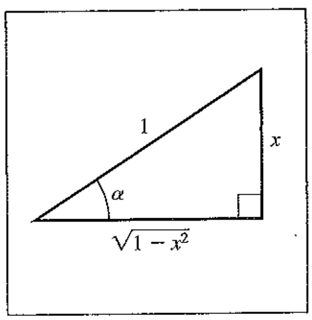

Figure 27

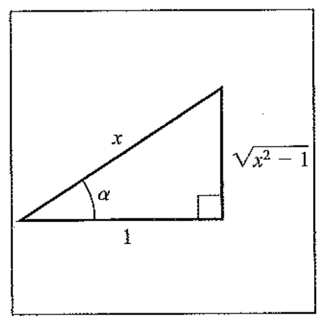

Figure 30

[28] Let $\alpha = \tan^{-1}x$. See *Figure 23*. $\cos(2\tan^{-1}x) = \cos 2\alpha = \cos^2\alpha - \sin^2\alpha =$

$$\left(\frac{1}{\sqrt{x^2+1}}\right)^2 - \left(\frac{x}{\sqrt{x^2+1}}\right)^2 = \frac{1}{x^2+1} - \frac{x^2}{x^2+1} = \frac{1-x^2}{x^2+1}.$$

[29] Let $\alpha = \arccos x$. See *Figure 24*. $0 \le \alpha \le \pi \;\Rightarrow\; 0 \le \frac{1}{2}\alpha \le \frac{\pi}{2}$ and $\cos\frac{1}{2}\alpha > 0$.

$$\cos\left(\tfrac{1}{2}\arccos x\right) = \cos\tfrac{1}{2}\alpha = \sqrt{\frac{1+\cos\alpha}{2}} = \sqrt{\frac{1+x}{2}}.$$

[30] Let $\alpha = \cos^{-1}\frac{1}{x}$. From *Figure 30*,

$$\tan\left(\tfrac{1}{2}\cos^{-1}\tfrac{1}{x}\right) = \tan\tfrac{1}{2}\alpha = \frac{1-\cos\alpha}{\sin\alpha} = \frac{1-\frac{1}{x}}{\frac{\sqrt{x^2-1}}{x}}\cdot\frac{x}{x} = \frac{x-1}{\sqrt{x^2-1}}.$$

[31] (a) See text Figure 2. As $x \to -1^+$, $\sin^{-1}x \to \underline{\;-\frac{\pi}{2}\;}$.

(b) See text Figure 5. As $x \to 1^-$, $\cos^{-1}x \to \underline{\;0\;}$.

(c) See text Figure 8. As $x \to \infty$, $\tan^{-1}x \to \underline{\;\frac{\pi}{2}\;}$.

[32] (a) As $x \to 1^-$, $\sin^{-1}x \to \underline{\;\frac{\pi}{2}\;}$. (b) As $x \to -1^+$, $\cos^{-1}x \to \underline{\;\pi\;}$.

(c) As $x \to -\infty$, $\tan^{-1}x \to \underline{\;-\frac{\pi}{2}\;}$.

$\boxed{33}$ $y = \sin^{-1} 2x$ • horizontally compress $y = \sin^{-1} x$ by a factor of 2

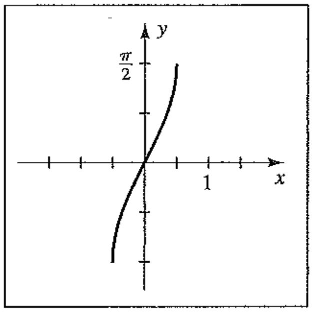

Figure 33

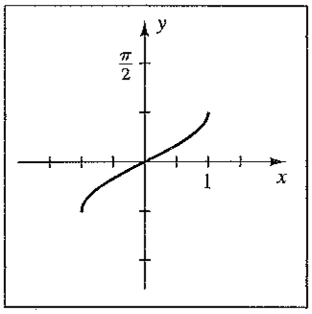

Figure 34

$\boxed{34}$ $y = \frac{1}{2} \sin^{-1} x$ • vertically compress $y = \sin^{-1} x$ by a factor of 2

$\boxed{35}$ $y = \sin^{-1}(x+1)$ • shift $y = \sin^{-1} x$ left 1 unit

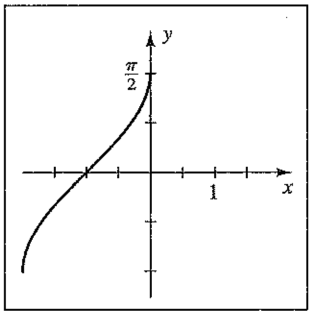

Figure 35

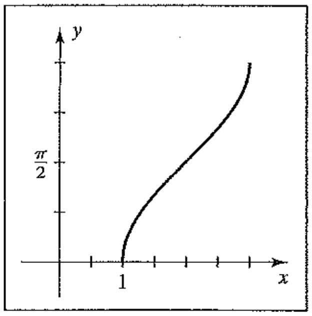

Figure 36

$\boxed{36}$ $y = \sin^{-1}(x-2) + \frac{\pi}{2}$ • shift $y = \sin^{-1} x$ right 2 units and up $\frac{\pi}{2}$ units

$\boxed{37}$ $y = \cos^{-1} \frac{1}{2}x$ • horizontally stretch $y = \cos^{-1} x$ by a factor of 2

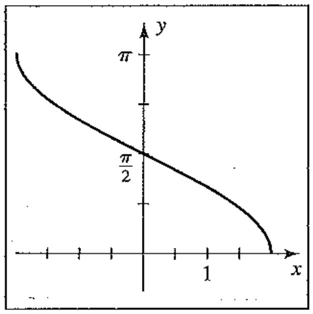

Figure 37

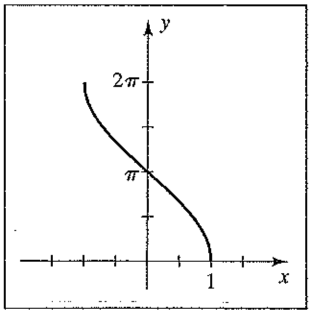

Figure 38

$\boxed{38}$ $y = 2\cos^{-1} x$ • vertically stretch $y = \cos^{-1} x$ by a factor of 2

[39] $y = 2 + \tan^{-1} x$ • shift $y = \tan^{-1} x$ up 2 units

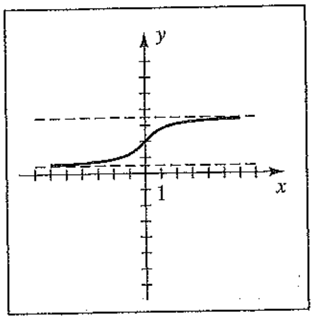

Figure 39

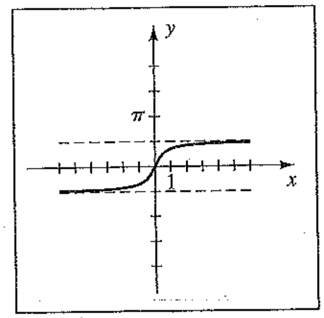

Figure 40

[40] $y = \tan^{-1} 2x$ • horizontally compress $y = \tan^{-1} x$ by a factor of 2

[41] If $\alpha = \arccos x$, then $\cos\alpha = x$, where $0 \le \alpha \le \pi$.

Hence, $y = \sin(\arccos x) = \sin\alpha = \sqrt{1 - \cos^2\alpha} = \sqrt{1 - x^2}$.

Thus, we have the graph of the semicircle $y = \sqrt{1 - x^2}$ on the interval $[-1, 1]$.

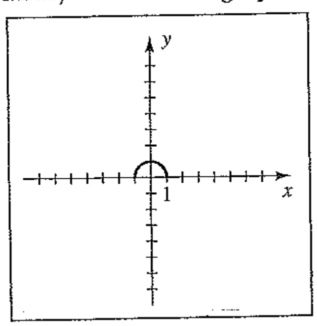

Figure 41

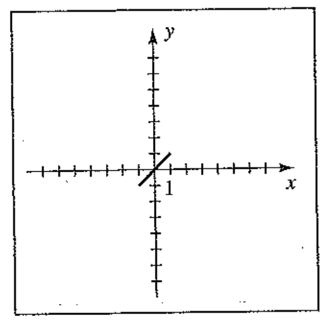

Figure 42

[42] By a property of $\sin^{-1}$, $\sin(\sin^{-1} x) = x$ for $-1 \le x \le 1$.

Thus, we have the graph of the line $y = x$ on the interval $[-1, 1]$.

[43] (a) $-1 \le x - 3 \le 1 \Rightarrow 2 \le x \le 4$

(b) $-\frac{\pi}{2} \le \sin^{-1}(x-3) \le \frac{\pi}{2} \Rightarrow -\frac{\pi}{4} \le \frac{1}{2}\sin^{-1}(x-3) \le \frac{\pi}{4} \Rightarrow -\frac{\pi}{4} \le y \le \frac{\pi}{4}$

(c) $y = \frac{1}{2}\sin^{-1}(x-3) \Rightarrow 2y = \sin^{-1}(x-3) \Rightarrow \sin 2y = x - 3 \Rightarrow x = \sin 2y + 3$

[44] (a) Since the domain of $\tan^{-1}$ is $\mathbb{R}$, x can be any real number.

(b) $-\frac{\pi}{2} < \tan^{-1}(2x+1) < \frac{\pi}{2} \Rightarrow -\frac{3\pi}{2} < 3\tan^{-1}(2x+1) < \frac{3\pi}{2} \Rightarrow -\frac{3\pi}{2} < y < \frac{3\pi}{2}$

(c) $y = 3\tan^{-1}(2x+1) \Rightarrow \frac{1}{3}y = \tan^{-1}(2x+1) \Rightarrow \tan\frac{1}{3}y = 2x + 1 \Rightarrow$

$2x = \tan\frac{1}{3}y - 1 \Rightarrow x = \frac{1}{2}(\tan\frac{1}{3}y - 1)$

[45] (a) $-1 \le \frac{2}{3}x \le 1 \Rightarrow -\frac{3}{2} \le x \le \frac{3}{2}$

(b) $0 \le \cos^{-1}\frac{2}{3}x \le \pi \Rightarrow 0 \le 4\cos^{-1}\frac{2}{3}x \le 4\pi \Rightarrow 0 \le y \le 4\pi$

(c) $y = 4\cos^{-1}\frac{2}{3}x \Rightarrow \frac{1}{4}y = \cos^{-1}\frac{2}{3}x \Rightarrow \cos\frac{1}{4}y = \frac{2}{3}x \Rightarrow x = \frac{3}{2}\cos\frac{1}{4}y$

[46] (a) $-1 \le 3x - 4 \le 1 \Rightarrow 3 \le 3x \le 5 \Rightarrow 1 \le x \le \frac{5}{3}$

(b) $-\frac{\pi}{2} \le \sin^{-1}(3x-4) \le \frac{\pi}{2} \Rightarrow -\pi \le 2\sin^{-1}(3x-4) \le \pi \Rightarrow -\pi \le y \le \pi$

(c) $y = 2\sin^{-1}(3x-4) \Rightarrow \frac{1}{2}y = \sin^{-1}(3x-4) \Rightarrow \sin\frac{1}{2}y = 3x-4 \Rightarrow$

$3x = \sin\frac{1}{2}y + 4 \Rightarrow x = \frac{1}{3}\sin\frac{1}{2}y + \frac{4}{3}$

[47] $y = -3 - \sin x \Rightarrow y+3 = -\sin x \Rightarrow -(y+3) = \sin x \Rightarrow x = \sin^{-1}(-y-3)$

[48] $y = 2 + 3\sin x \Rightarrow y-2 = 3\sin x \Rightarrow \frac{1}{3}(y-2) = \sin x \Rightarrow x = \sin^{-1}\left[\frac{1}{3}(y-2)\right]$

[49] $y = 15 - 2\cos x \Rightarrow 2\cos x = 15 - y \Rightarrow \cos x = \frac{1}{2}(15-y) \Rightarrow x = \cos^{-1}\left[\frac{1}{2}(15-y)\right]$

[50] $y = 6 - 3\cos x \Rightarrow 3\cos x = 6 - y \Rightarrow \cos x = \frac{1}{3}(6-y) \Rightarrow x = \cos^{-1}\left[\frac{1}{3}(6-y)\right]$

[51] $\dfrac{\sin x}{3} = \dfrac{\sin y}{4} \Rightarrow \sin x = \frac{3}{4}\sin y$. The reference angle for x is $x_R = \sin^{-1}(\frac{3}{4}\sin y)$,

where $0 < \frac{3}{4}\sin y \le \frac{3}{4} < 1$. If $0 < x < \frac{\pi}{2}$, then $x = x_R$. If $\frac{\pi}{2} < x < \pi$, then $x = \pi - x_R$.

[52] $\dfrac{4}{\sin x} = \dfrac{7}{\sin y} \Rightarrow \dfrac{\sin x}{4} = \dfrac{\sin y}{7} \Rightarrow \sin x = \frac{4}{7}\sin y$.

The reference angle for x is $x_R = \sin^{-1}(\frac{4}{7}\sin y)$, where $0 < \frac{4}{7}\sin y \le \frac{4}{7} < 1$.

If $0 < x < \frac{\pi}{2}$, then $x = x_R$. If $\frac{\pi}{2} < x < \pi$, then $x = \pi - x_R$.

[53] $\cos^2 x + 2\cos x - 1 = 0 \Rightarrow \cos x = -1 \pm \sqrt{2} \approx 0.4142, -2.4142$.

Since $-2.4142 < -1$, $x = \cos^{-1}(-1+\sqrt{2}) \approx 1.1437$ is one answer.

$x = 2\pi - \cos^{-1}(-1+\sqrt{2}) \approx 2\pi - 1.1437 \approx 5.1395$ is the other.

[54] $\sin^2 x - \sin x - 1 = 0 \Rightarrow \sin x = \dfrac{1 \pm \sqrt{5}}{2} \approx 1.6180, -0.6180$. Since $1.6180 > 1$,

$x_0 = \sin^{-1}\left(\dfrac{1-\sqrt{5}}{2}\right) \approx -0.6662$, but this is not in $[0, 2\pi)$. The reference angle is

$x_R = -x_0 \approx 0.6662$. Since the sine is negative in quadrants III and IV,

the values are $\pi + x_R \approx 3.8078$ and $2\pi - x_R \approx 5.6170$.

[55] $2\tan^2 t + 9\tan t + 3 = 0 \Rightarrow \tan t = \dfrac{-9 \pm \sqrt{81-24}}{4} \Rightarrow t = \tan^{-1}\frac{1}{4}(-9 \pm \sqrt{57})$

$\tan^{-1}\frac{1}{4}(-9+\sqrt{57}) \approx -0.3478$, $\tan^{-1}\frac{1}{4}(-9-\sqrt{57}) \approx -1.3337$

[56] $3\sin^2 t + 7\sin t + 3 = 0 \Rightarrow \sin t = \dfrac{-7 \pm \sqrt{49-36}}{6} \Rightarrow$

$t = \sin^{-1}\frac{1}{6}(-7+\sqrt{13})$ $\{\sin t \ne \frac{1}{6}(-7-\sqrt{13}) < -1\}$; $\sin^{-1}\frac{1}{6}(-7+\sqrt{13}) \approx -0.6013$

[57] $15\cos^4 x - 14\cos^2 x + 3 = 0 \Rightarrow (5\cos^2 x - 3)(3\cos^2 x - 1) = 0 \Rightarrow \cos^2 x = \frac{3}{5}, \frac{1}{3} \Rightarrow$

$\cos x = \pm\frac{1}{5}\sqrt{15}, \pm\frac{1}{3}\sqrt{3} \Rightarrow x = \cos^{-1}(\pm\frac{1}{5}\sqrt{15}), \cos^{-1}(\pm\frac{1}{3}\sqrt{3})$.

$\cos^{-1}\frac{1}{5}\sqrt{15} \approx 0.6847$, $\cos^{-1}(-\frac{1}{5}\sqrt{15}) \approx 2.4569$,

$\cos^{-1}\frac{1}{3}\sqrt{3} \approx 0.9553$, $\cos^{-1}(-\frac{1}{3}\sqrt{3}) \approx 2.1863$

[58] $3\tan^4\theta - 19\tan^2\theta + 2 = 0 \Rightarrow \tan^2\theta = \dfrac{19 \pm \sqrt{361-24}}{6} \Rightarrow$

$\theta = \tan^{-1}\left(\pm\sqrt{\frac{1}{6}(19 \pm \sqrt{337})}\right)$.

$\tan^{-1}\left(\pm\sqrt{\frac{1}{6}(19+\sqrt{337})}\right) \approx \pm 1.1896$, $\tan^{-1}\left(\pm\sqrt{\frac{1}{6}(19-\sqrt{337})}\right) \approx \pm 0.3162$

$\boxed{59}$ $6\sin^3\theta + 18\sin^2\theta - 5\sin\theta - 15 = 0 \Rightarrow 6\sin^2\theta(\sin\theta + 3) - 5(\sin\theta + 3) = 0 \Rightarrow$
$(6\sin^2\theta - 5)(\sin\theta + 3) = 0 \Rightarrow \sin\theta = \pm\frac{1}{6}\sqrt{30} \Rightarrow \theta = \sin^{-1}(\pm\frac{1}{6}\sqrt{30}) \approx \pm 1.1503$

$\boxed{60}$ $6\sin 2x - 8\cos x + 9\sin x - 6 = 0 \Rightarrow 12\sin x\cos x - 8\cos x + 9\sin x - 6 = 0 \Rightarrow$
$4\cos x(3\sin x - 2) + 3(3\sin x - 2) = 0 \Rightarrow (4\cos x + 3)(3\sin x - 2) = 0 \Rightarrow$
$x = \cos^{-1}(-\frac{3}{4}), \sin^{-1}\frac{2}{3} \approx 0.7297.$
However, $\cos^{-1}(-\frac{3}{4})$ is in $(\frac{\pi}{2}, \pi)$, and not in $(-\frac{\pi}{2}, \frac{\pi}{2})$.

$\boxed{61}$ $(\cos x)(15\cos x + 4) = 3 \Rightarrow 15\cos^2 x + 4\cos x - 3 = 0 \Rightarrow$
$(5\cos x + 3)(3\cos x - 1) = 0 \Rightarrow \cos x = -\frac{3}{5}, \frac{1}{3} \Rightarrow$
$x = \cos^{-1}(-\frac{3}{5}) \approx 2.2143, \cos^{-1}\frac{1}{3} \approx 1.2310.$
In $[0, 2\pi)$, we also have $2\pi - \cos^{-1}(-\frac{3}{5}) \approx 4.0689$ and $2\pi - \cos^{-1}\frac{1}{3} \approx 5.0522$.

$\boxed{62}$ $6\sin^2 x = \sin x + 2 \Rightarrow 6\sin^2 x - \sin x - 2 = 0 \Rightarrow$
$(3\sin x - 2)(2\sin x + 1) = 0 \Rightarrow \sin x = \frac{2}{3}, -\frac{1}{2} \Rightarrow$
$x = \sin^{-1}\frac{2}{3} \approx 0.7297, \pi - \sin^{-1}\frac{2}{3} \approx 2.4119, \frac{7\pi}{6} \approx 3.6652, \frac{11\pi}{6} \approx 5.7596.$

$\boxed{63}$ $3\cos 2x - 7\cos x + 5 = 0 \Rightarrow 3(2\cos^2 x - 1) - 7\cos x + 5 = 0 \Rightarrow$
$6\cos^2 x - 7\cos x + 2 = 0 \Rightarrow (3\cos x - 2)(2\cos x - 1) = 0 \Rightarrow \cos x = \frac{2}{3}, \frac{1}{2} \Rightarrow$
$x = \cos^{-1}\frac{2}{3} \approx 0.8411, 2\pi - \cos^{-1}\frac{2}{3} \approx 5.4421, \frac{\pi}{3} \approx 1.0472, \frac{5\pi}{3} \approx 5.2360.$

$\boxed{64}$ $\sin 2x = -1.5\cos x \Rightarrow \sin 2x + 1.5\cos x = 0 \Rightarrow 2\sin x\cos x + 1.5\cos x = 0 \Rightarrow$
$(\cos x)(2\sin x + 1.5) = 0 \Rightarrow \cos x = 0$ or $\sin x = -\frac{3}{4} \Rightarrow$
$x = \frac{\pi}{2} \approx 1.5708, \frac{3\pi}{2} \approx 4.7124, \sin^{-1}(-\frac{3}{4})$ {not in $[0, 2\pi)$},
$2\pi + \sin^{-1}(-\frac{3}{4}) \approx 5.4351, \pi - \sin^{-1}(-\frac{3}{4}) \approx 3.9897.$

$\boxed{65}$ (a) $S = 4, D = 3.5, d = 1 \Rightarrow M = \frac{S}{2}\left(1 - \frac{2}{\pi}\tan^{-1}\frac{d}{D}\right) = \frac{4}{2}\left(1 - \frac{2}{\pi}\tan^{-1}\frac{1}{3.5}\right) \approx 1.65$ m

(b) $d = 4 \Rightarrow M = \frac{S}{2}\left(1 - \frac{2}{\pi}\tan^{-1}\frac{d}{D}\right) = \frac{4}{2}\left(1 - \frac{2}{\pi}\tan^{-1}\frac{4}{3.5}\right) \approx 0.92$ m

(c) $d = 10 \Rightarrow M = \frac{S}{2}\left(1 - \frac{2}{\pi}\tan^{-1}\frac{d}{D}\right) = \frac{4}{2}\left(1 - \frac{2}{\pi}\tan^{-1}\frac{10}{3.5}\right) \approx 0.43$ m

$\boxed{66}$ $M = \frac{S}{2}\left(1 - \frac{2}{\pi}\tan^{-1}\frac{d}{D}\right) \Rightarrow \frac{2}{\pi}\tan^{-1}\frac{d}{D} = 1 - \frac{2M}{S} \Rightarrow \tan^{-1}\frac{d}{D} = \frac{\pi}{2}\left(1 - \frac{2M}{S}\right) \Rightarrow$

$\frac{d}{D} = \tan\left[\frac{\pi}{2}\left(1 - \frac{2M}{S}\right)\right] \Rightarrow D = d\cot\left[\frac{\pi}{2}\left(1 - \frac{2M}{S}\right)\right] = 5\cot\left[\frac{\pi}{2}\left(1 - \frac{2(0.6)}{3}\right)\right] \approx 3.63$ km

$\boxed{67}$ opp $= \frac{1}{2}(30)$ and hyp $= 280 \Rightarrow \sin\theta = \frac{15}{280} \Rightarrow \theta = \sin^{-1}\frac{15}{280} \approx 3.07^\circ$

$\boxed{68}$ $\tan\alpha = \frac{4'}{11'10''} \Rightarrow \alpha = \tan^{-1}\frac{48}{142} \approx 18.7^\circ.$ $\alpha + \beta = 90^\circ \Rightarrow \beta \approx 71.3^\circ.$

$\boxed{69}$ (a) Let β denote the angle by the sailboat with opposite side d and hypotenuse k.
Now $\sin\beta = \frac{d}{k} \Rightarrow \beta = \sin^{-1}\frac{d}{k}$. Using alternate interior angles,
we see that $\alpha + \beta = \theta$. Thus, $\alpha = \theta - \beta = \theta - \sin^{-1}\frac{d}{k}$.

(b) $d = 50$, $k = 210$, and $\theta = 53.4^\circ \Rightarrow \alpha = 53.4^\circ - \sin^{-1}\frac{50}{210} \approx 39.63^\circ$, or 40°.

$\boxed{70}$ (a) Draw a line from the art critic's eyes to the painting. This forms two right triangles with opposite sides 8 {upper △} and 2 {lower △} and adjacent side x. Let α be the angle of elevation to the top of the painting and β be the angle of depression to the bottom of the painting.

Since $\tan\alpha = \frac{8}{x}$ and $\tan\beta = \frac{2}{x}$, $\theta = \alpha + \beta = \tan^{-1}\frac{8}{x} + \tan^{-1}\frac{2}{x}$.

(b) $\tan\theta = \tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\,\tan\beta} = \dfrac{8/x + 2/x}{1-(8/x)(2/x)} \cdot \dfrac{x^2}{x^2} = \dfrac{8x+2x}{x^2-16}$

$= \dfrac{10x}{x^2-16} \Rightarrow \theta = \tan^{-1}\left(\dfrac{10x}{x^2-16}\right)$. Note that if $0 < x < 4$, $\dfrac{10x}{x^2-16} < 0$

and $90^\circ < \theta < 180^\circ$, *not* $-90^\circ < \theta < 0^\circ$ since $0^\circ < \theta < 180^\circ$ in any triangle.

If $x = 4$, $\dfrac{10x}{x^2-16}$ is undefined and $\theta = 90^\circ$. If $x > 4$, $\dfrac{10x}{x^2-16} > 0$ and $0 < \theta < 90^\circ$.

(c) $45^\circ = \tan^{-1}\left(\dfrac{10x}{x^2-16}\right) \Rightarrow \tan 45^\circ = \dfrac{10x}{x^2-16} \Rightarrow (1)(x^2-16) = 10x \Rightarrow$

$x^2 - 10x - 16 = 0 \Rightarrow x = \dfrac{10 \pm \sqrt{164}}{2} = \{x > 0\}\ x = 5 + \sqrt{41} \approx 11.4$ ft.

Note: The following is a general outline that can be used for verifying trigonometric identities involving inverse trigonometric functions.

(1) Define angles and their ranges—make sure the range of values for one side of the equation is equal to the range of values for the other side.

(2) Choose a trigonometric function T that is one-to-one on the range of values listed in part (1).

(3) Show that $T(\text{LS}) = T(\text{RS})$. Note that $T(\text{LS}) = T(\text{RS}) \not\Rightarrow \text{LS} = \text{RS}$.

(4) Conclude that since T is one-to-one on the range of values, LS = RS.

$\boxed{71}$ Let $\alpha = \sin^{-1}x$ and $\beta = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$ with $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$ and $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$.

Thus, $\sin\alpha = x$ and $\sin\beta = x$.

Since the sine function is one-to-one on $(-\frac{\pi}{2}, \frac{\pi}{2})$, we have $\alpha = \beta$.

$\boxed{72}$ Let $\alpha = \arccos x$ and $\beta = \arccos\sqrt{1-x^2}$.

Since $0 \le x \le 1$, we have $0 \le \alpha \le \frac{\pi}{2}$ and $0 \le \beta \le \frac{\pi}{2}$, and hence $0 \le \alpha + \beta \le \pi$.

Thus, $\cos(\alpha+\beta) = \cos\alpha\,\cos\beta - \sin\alpha\,\sin\beta = x\cdot\sqrt{1-x^2} - \sqrt{1-x^2}\cdot x = 0$.

Since the cosine function is one-to-one on $[0, \pi]$, we have $\alpha + \beta = \frac{\pi}{2}$.

$\boxed{73}$ Let $\alpha = \arcsin(-x)$ and $\beta = \arcsin x$ with $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$ and $-\frac{\pi}{2} \le \beta \le \frac{\pi}{2}$.

Thus, $\sin\alpha = -x$ and $\sin\beta = x$. Consequently, $\sin\alpha = -\sin\beta = \sin(-\beta)$.

Since the sine function is one-to-one on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, we have $\alpha = -\beta$.

$\boxed{74}$ Let $\alpha = \arccos(-x)$ and $\beta = \pi - \arccos x$ with $0 \le \alpha \le \pi$ and $0 \le \beta \le \pi$ since

$0 \le \arccos x \le \pi \Rightarrow 0 \ge -\arccos x \ge -\pi \Rightarrow \pi \ge \pi - \arccos x \ge 0$.

Thus, $\cos\alpha = -x$ and $\cos\beta = \cos(\pi - \arccos x) = \cos\pi \cdot x + \sin\pi \cdot \sin(\arccos x) = -x$.

Since the cosine function is one-to-one on $[0, \pi]$, we have $\alpha = \beta$.

$\boxed{75}$ Let $\alpha = \arctan x$ and $\beta = \arctan(1/x)$.

Since $x > 0$, we have $0 < \alpha < \frac{\pi}{2}$ and $0 < \beta < \frac{\pi}{2}$, and hence $0 < \alpha + \beta < \pi$.

Thus, $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\,\tan\beta} = \dfrac{x + (1/x)}{1 - x \cdot (1/x)} = \dfrac{x + (1/x)}{0}$.

Since the denominator is 0, $\tan(\alpha + \beta)$ is undefined and hence $\alpha + \beta = \frac{\pi}{2}$.

$\boxed{76}$ Let $\alpha = \cos^{-1} x$ and $\beta = \cos^{-1}(2x^2 - 1)$. Since $0 \le x \le 1$, $0 \le \alpha \le \frac{\pi}{2}$ and $0 \le 2\alpha \le \pi$.

Also, $0 \le x \le 1 \Rightarrow 0 \le 2x^2 \le 2 \Rightarrow -1 \le 2x^2 - 1 \le 1$ and $0 \le \beta \le \pi$.

Thus, $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = (x)^2 - (\sqrt{1 - x^2})^2 = x^2 - (1 - x^2) = 2x^2 - 1$ and

$\cos\beta = 2x^2 - 1$. Since the cosine function is one-to-one on $[0, \pi]$, we have $2\alpha = \beta$.

$\boxed{77}$ The domain of $\sin^{-1}(x - 1)$ is $[0, 2]$ and the domain of $\cos^{-1}\frac{1}{2}x$ is $[-2, 2]$.

The domain of f is the intersection of $[0, 2]$ and $[-2, 2]$, i.e., $[0, 2]$.

From the graph, we see that the function is increasing and its range is $[-\frac{\pi}{2}, \pi]$.

$[-3, 6]$ by $[-2, 4]$

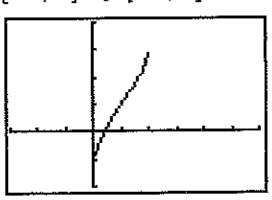

Figure 77

$[-3, 12]$ by $[-4, 6]$

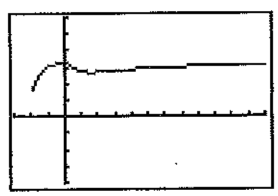

Figure 78

$\boxed{78}$ The domain of $\frac{1}{2}\tan^{-1}(1 - 2x)$ is $(-\infty, \infty)$ and the domain of $3\tan^{-1}\sqrt{x + 2}$ is $[-2, \infty)$. The domain of f is $[-2, \infty)$. The minimum value of approximately $\frac{1}{2}\tan^{-1} 5 \approx 0.69$ occurs at $x = -2$. The maximum value of the function does not occur at $x \approx -0.13$. Rather for large x, $\frac{1}{2}\tan^{-1}(1 - 2x)$ approaches $-\frac{\pi}{4}$ and $3\tan^{-1}\sqrt{x + 2}$ approaches $\frac{3\pi}{2}$. Thus, the function increases asymptotically to $\frac{5\pi}{4} \approx 3.93$. The range of the function is $[\frac{1}{2}\tan^{-1} 5, \frac{5\pi}{4})$.

$\boxed{79}$ Graph $y = \sin^{-1} 2x$ and $y = \tan^{-1}(1 - x)$.

From the graph, we see that there is one solution at $x \approx 0.29$.

[−3, 3] by [−2, 2]

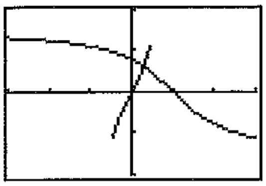

Figure 79

[−6, 6] by [−4.5, 4.5]

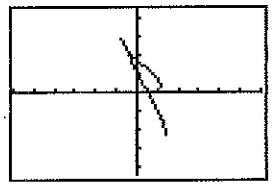

Figure 80

$\boxed{80}$ Graph $y = \cos^{-1}(x - \frac{1}{5})$ and $y = 2\sin^{-1}(\frac{1}{2} - x)$.

From the graph, we see that there is one solution at $x \approx -0.39$.

$\boxed{81}$ From the graph, we see that when $f(\theta) = 0.2$, $\theta \approx 1.25$, or approximately 72°.

[0, $\pi/2$, 0.2] by [0, 1.05, 0.2]

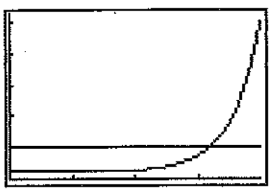

Figure 81

[$-\pi/2$, $\pi/2$, 0.5] by [−1.05, 1.05, 0.5]

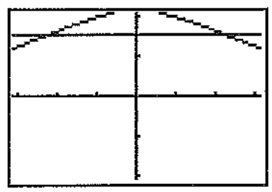

Figure 82

$\boxed{82}$ (a) $\phi = \sin^{-1}(\sin 23.5^\circ \sin 51.7^\circ + \cos 23.5^\circ \cos 51.7^\circ \cos H)$

(b) From *Figure 82*, we find that $\phi = 45^\circ = \frac{\pi}{4} \approx 0.785398$ at $H \approx \pm 0.8044$. Since 6 hours corresponds to $\frac{\pi}{2}$, 1 hour corresponds to $\frac{\pi}{12}$. $\pm 0.8044 \div \frac{\pi}{12} \approx \pm 3.07$ hr $\approx$ ± 3 hr and 4 min. The times are approximately 8:56 A.M. and 3:04 P.M.

$\boxed{83}$ Actual distance between x-ticks is equal to $x_A = \frac{3 \text{ units}}{3 \text{ ticks}} = 1$ unit between ticks.

Actual distance between y-ticks is equal to $y_A = \frac{2 \text{ units}}{2 \text{ ticks}} = 1$ unit between ticks.

The ratio is $m_A = \frac{y_A}{x_A} = \frac{1}{1} = 1$. The graph will make an angle of $\theta = \tan^{-1} 1 = 45^\circ$.

[0, 3] by [0, 2]

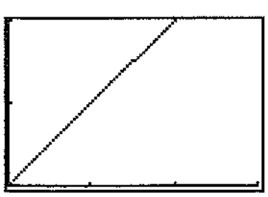

Figure 83

[0, 6] by [0, 2]

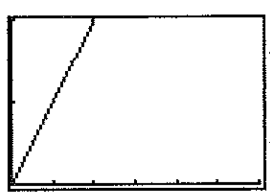

Figure 84

$\boxed{84}$ $x_A = \frac{3 \text{ units}}{6 \text{ ticks}} = \frac{1}{2}$, $y_A = \frac{2 \text{ units}}{2 \text{ ticks}} = 1 \Rightarrow m_A = \frac{1}{1/2} = 2 \Rightarrow \theta = \tan^{-1} 2 \approx 63.4^\circ$.

[85] $x_A = \frac{3 \text{ units}}{3 \text{ ticks}} = 1,\ y_A = \frac{2 \text{ units}}{4 \text{ ticks}} = \frac{1}{2} \Rightarrow m_A = \frac{1/2}{1} = \frac{1}{2} \Rightarrow \theta = \tan^{-1}\frac{1}{2} \approx 26.6°.$

[0, 3] by [0, 4]

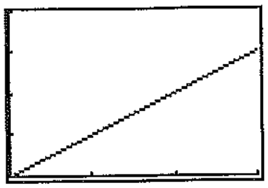

Figure 85

[0, 2] by [0, 2]

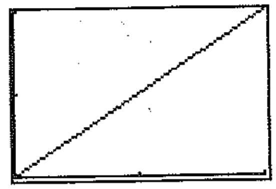

Figure 86

[86] $x_A = \frac{3 \text{ units}}{2 \text{ ticks}} = \frac{3}{2},\ y_A = \frac{2 \text{ units}}{2 \text{ ticks}} = 1 \Rightarrow m_A = \frac{1}{3/2} = \frac{2}{3} \Rightarrow \theta = \tan^{-1}\frac{2}{3} \approx 33.7°.$

Chapter 7 Review Exercises

[1] $(\cot^2 x + 1)(1 - \cos^2 x) = (\csc^2 x)(\sin^2 x) = 1$

[2] $\cos\theta + \sin\theta\,\tan\theta = \cos\theta + \sin\theta \cdot \frac{\sin\theta}{\cos\theta} = \frac{\cos^2\theta + \sin^2\theta}{\cos\theta} = \frac{1}{\cos\theta} = \sec\theta$

[3] $\dfrac{(\sec^2\theta - 1)\cot\theta}{\tan\theta\,\sin\theta + \cos\theta} = \dfrac{(\tan^2\theta)\cot\theta}{\frac{\sin\theta}{\cos\theta}\cdot\sin\theta + \cos\theta} = \dfrac{\tan\theta}{\frac{\sin^2\theta + \cos^2\theta}{\cos\theta}} = \dfrac{\sin\theta\,/\cos\theta}{1/\cos\theta} = \sin\theta$

[4] $(\tan x + \cot x)^2 = \left(\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right)^2 = \left(\frac{\sin^2 x + \cos^2 x}{\cos x\,\sin x}\right)^2 = \frac{1}{\cos^2 x\,\sin^2 x} = \sec^2 x\,\csc^2 x$

[5] $\frac{1}{1+\sin t}\cdot\frac{1-\sin t}{1-\sin t} = \frac{1-\sin t}{1-\sin^2 t} = \frac{1-\sin t}{\cos^2 t} = \frac{1-\sin t}{\cos t}\cdot\frac{1}{\cos t} =$

$\left(\frac{1}{\cos t} - \frac{\sin t}{\cos t}\right)\cdot\sec t = (\sec t - \tan t)\sec t$

[6] $\dfrac{\sin(\alpha-\beta)}{\cos(\alpha+\beta)} = \dfrac{(\sin\alpha\,\cos\beta - \cos\alpha\,\sin\beta)\ /\ \cos\alpha\,\cos\beta}{(\cos\alpha\,\cos\beta - \sin\alpha\,\sin\beta)\ /\ \cos\alpha\,\cos\beta} = \dfrac{\tan\alpha - \tan\beta}{1 - \tan\alpha\,\tan\beta}$

[7] $\tan 2u = \dfrac{2\tan u}{1-\tan^2 u} = \dfrac{2\cdot\frac{1}{\cot u}}{1 - \frac{1}{\cot^2 u}} = \dfrac{\frac{2}{\cot u}}{\frac{\cot^2 u - 1}{\cot^2 u}} = \dfrac{2\cot u}{\cot^2 u - 1} = \dfrac{2\cot u}{(\csc^2 u - 1) - 1} =$

$\dfrac{2\cot u}{\csc^2 u - 2}$

[8] $\cos^2\frac{v}{2} = \dfrac{1+\cos v}{2} = \dfrac{1 + \frac{1}{\sec v}}{2} = \dfrac{\frac{\sec v + 1}{\sec v}}{2} = \dfrac{1+\sec v}{2\sec v}$

[9] $\dfrac{\tan^3\phi - \cot^3\phi}{\tan^2\phi + \csc^2\phi} = \dfrac{(\tan\phi - \cot\phi)[(\tan^2\phi + \tan\phi\,\cot\phi + \cot^2\phi)]}{[\tan^2\phi + (1 + \cot^2\phi)]} = \tan\phi - \cot\phi$

$\boxed{10}$ $\text{LS} = \dfrac{\sin u + \sin v}{\csc u + \csc v} = \dfrac{\sin u + \sin v}{\dfrac{1}{\sin u} + \dfrac{1}{\sin v}} = \dfrac{\sin u + \sin v}{\dfrac{\sin v + \sin u}{\sin u \sin v}} = \sin u \sin v$

$\text{RS} = \dfrac{1 - \sin u \sin v}{-1 + \csc u \csc v} = \dfrac{1 - \sin u \sin v}{-1 + \dfrac{1}{\sin u \sin v}} = \dfrac{1 - \sin u \sin v}{\dfrac{1 - \sin u \sin v}{\sin u \sin v}} = \sin u \sin v$

Since the LS and RS equal the same expression and the steps are reversible,

the identity is verified.

$\boxed{11}$ $\left(\dfrac{\sin^2 x}{\tan^4 x}\right)^3 \left(\dfrac{\csc^3 x}{\cot^6 x}\right)^2 = \left(\dfrac{\sin^6 x}{\tan^{12} x}\right)\left(\dfrac{\csc^6 x}{\cot^{12} x}\right) = \dfrac{(\sin x \csc x)^6}{(\tan x \cot x)^{12}} = \dfrac{(1)^6}{(1)^{12}} = 1$

$\boxed{12}$ $\dfrac{\cos\gamma}{1 - \tan\gamma} + \dfrac{\sin\gamma}{1 - \cot\gamma} = \dfrac{\cos\gamma}{\dfrac{\cos\gamma - \sin\gamma}{\cos\gamma}} + \dfrac{\sin\gamma}{\dfrac{\sin\gamma - \cos\gamma}{\sin\gamma}} = \dfrac{\cos^2\gamma}{\cos\gamma - \sin\gamma} + \dfrac{\sin^2\gamma}{\sin\gamma - \cos\gamma} =$

$\dfrac{\cos^2\gamma - \sin^2\gamma}{\cos\gamma - \sin\gamma} = \dfrac{(\cos\gamma + \sin\gamma)(\cos\gamma - \sin\gamma)}{\cos\gamma - \sin\gamma} = \cos\gamma + \sin\gamma$

$\boxed{13}$ $\dfrac{\cos(-t)}{\sec(-t) + \tan(-t)} = \dfrac{\cos t}{\sec t - \tan t} = \dfrac{\cos t}{\dfrac{1}{\cos t} - \dfrac{\sin t}{\cos t}} = \dfrac{\cos t}{\dfrac{1 - \sin t}{\cos t}} = \dfrac{\cos^2 t}{1 - \sin t} =$

$\dfrac{1 - \sin^2 t}{1 - \sin t} = \dfrac{(1 - \sin t)(1 + \sin t)}{1 - \sin t} = 1 + \sin t$

$\boxed{14}$ $\dfrac{\cot(-t) + \csc(-t)}{\sin(-t)} = \dfrac{-\cot t - \csc t}{-\sin t} = \dfrac{\dfrac{\cos t}{\sin t} + \dfrac{1}{\sin t}}{\sin t} = \dfrac{\cos t + 1}{\sin^2 t} =$

$\dfrac{\cos t + 1}{1 - \cos^2 t} = \dfrac{\cos t + 1}{(1 - \cos t)(1 + \cos t)} = \dfrac{1}{1 - \cos t}$

$\boxed{15}$ $\sqrt{\dfrac{1 - \cos t}{1 + \cos t}} = \sqrt{\dfrac{(1 - \cos t)}{(1 + \cos t)} \cdot \dfrac{(1 - \cos t)}{(1 - \cos t)}} = \sqrt{\dfrac{(1 - \cos t)^2}{1 - \cos^2 t}} = \sqrt{\dfrac{(1 - \cos t)^2}{\sin^2 t}} =$

$\dfrac{|1 - \cos t|}{|\sin t|} = \dfrac{1 - \cos t}{|\sin t|}$, since $(1 - \cos t) \geq 0$.

$\boxed{16}$ $\sqrt{\dfrac{1 - \sin\theta}{1 + \sin\theta}} = \sqrt{\dfrac{(1 - \sin\theta)}{(1 + \sin\theta)} \cdot \dfrac{(1 + \sin\theta)}{(1 + \sin\theta)}} = \sqrt{\dfrac{1 - \sin^2\theta}{(1 + \sin\theta)^2}} = \sqrt{\dfrac{\cos^2\theta}{(1 + \sin\theta)^2}} =$

$\dfrac{|\cos\theta|}{|1 + \sin\theta|} = \dfrac{|\cos\theta|}{1 + \sin\theta}$, since $(1 + \sin\theta) \geq 0$.

$\boxed{17}$ $\cos(x - \frac{5\pi}{2}) = \cos x \cos\frac{5\pi}{2} + \sin x \sin\frac{5\pi}{2} = \cos x\,(0) + \sin x\,(1) = \sin x$

$\boxed{18}$ $\tan(x + \frac{3\pi}{4}) = \dfrac{\tan x + \tan\frac{3\pi}{4}}{1 - \tan x \tan\frac{3\pi}{4}} = \dfrac{\tan x - 1}{1 + \tan x}$

$\boxed{19}$ $\frac{1}{4}\sin 4\beta = \frac{1}{4}\sin(2 \cdot 2\beta) = \frac{1}{4}(2\sin 2\beta \cos 2\beta) = \frac{1}{2}(2\sin\beta\cos\beta)(\cos^2\beta - \sin^2\beta) =$

$\sin\beta\cos^3\beta - \cos\beta\sin^3\beta$

$\boxed{20}$ $\tan\frac{1}{2}\theta = \dfrac{1 - \cos\theta}{\sin\theta} = \dfrac{1}{\sin\theta} - \dfrac{\cos\theta}{\sin\theta} = \csc\theta - \cot\theta$

$\boxed{21}$ $\sin 8\theta = 2\sin 4\theta\, \cos 4\theta = 2\,(2\sin 2\theta\, \cos 2\theta)(1 - 2\sin^2 2\theta)$
$= 8\sin\theta\, \cos\theta\,(1 - 2\sin^2\theta)[1 - 2\,(2\sin\theta\, \cos\theta)^2]$
$= 8\sin\theta\, \cos\theta\,(1 - 2\sin^2\theta)(1 - 8\sin^2\theta\, \cos^2\theta)$

$\boxed{22}$ Let $\alpha = \arctan x$ and $\beta = \arctan\dfrac{2x}{1-x^2}$. Because $-1 < x < 1$, $-\frac{\pi}{4} < \alpha < \frac{\pi}{4}$.
Thus, $\tan\alpha = x$ and $\tan\beta = \dfrac{2x}{1-x^2} = \dfrac{2\tan\alpha}{1-\tan^2\alpha} = \tan 2\alpha$. Since the tangent function is one-to-one on $(-\frac{\pi}{2}, \frac{\pi}{2})$, we have $\beta = 2\alpha$ or, equivalently, $\alpha = \frac{1}{2}\beta$.

$\boxed{23}$ $2\cos^3\theta - \cos\theta = 0 \;\Rightarrow\; \cos\theta\,(2\cos^2\theta - 1) = 0 \;\Rightarrow\; \cos\theta = 0,\ \pm\dfrac{\sqrt{2}}{2} \;\Rightarrow$
$\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{4}, \frac{7\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$

$\boxed{24}$ $2\cos\alpha + \tan\alpha = \sec\alpha \;\Rightarrow\; \{\text{multiply by } \cos\alpha\}\ 2\cos^2\alpha + \sin\alpha = 1 \;\Rightarrow$
$2(1 - \sin^2\alpha) + \sin\alpha = 1 \;\Rightarrow\; 2\sin^2\alpha - \sin\alpha - 1 = 0 \;\Rightarrow$
$(2\sin\alpha + 1)(\sin\alpha - 1) = 0 \;\Rightarrow\; \sin\alpha = -\frac{1}{2}, 1 \;\Rightarrow\; \alpha = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{\pi}{2}$.
However, $\tan\frac{\pi}{2}$ is undefined so exclude $\frac{\pi}{2}$ and $\alpha = \frac{7\pi}{6}, \frac{11\pi}{6}$.

$\boxed{25}$ $\sin\theta = \tan\theta \;\Rightarrow\; \sin\theta - \dfrac{\sin\theta}{\cos\theta} = 0 \;\Rightarrow\; \sin\theta\left(1 - \dfrac{1}{\cos\theta}\right) = 0 \;\Rightarrow$
$\sin\theta = 0$ or $\cos\theta = 1 \;\Rightarrow\; \theta = 0, \pi$ or $\theta = 0 \;\Rightarrow\; \theta = 0, \pi$

$\boxed{26}$ $\csc^5\theta - 4\csc\theta = 0 \;\Rightarrow\; \csc\theta\,(\csc^4\theta - 4) = 0 \;\Rightarrow$
$\csc^2\theta = 2\ \{\csc\theta \neq 0,\ \csc^2\theta \neq -2\} \;\Rightarrow\; \csc\theta = \pm\sqrt{2} \;\Rightarrow\; \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

$\boxed{27}$ $2\cos^3 t + \cos^2 t - 2\cos t - 1 = 0 \;\Rightarrow\; \cos^2 t\,(2\cos t + 1) - 1(2\cos t + 1) = 0 \;\Rightarrow$
$(\cos^2 t - 1)(2\cos t + 1) = 0 \;\Rightarrow\; \cos t = \pm 1, -\frac{1}{2} \;\Rightarrow\; t = 0, \pi, \frac{2\pi}{3}, \frac{4\pi}{3}$

$\boxed{28}$ $\cos x\, \cot^2 x = \cos x \;\Rightarrow\; \cos x\, \cot^2 x - \cos x = 0 \;\Rightarrow\; \cos x\,(\cot^2 x - 1) = 0 \;\Rightarrow$
$\cos x = 0$ or $\cot x = \pm 1 \;\Rightarrow\; x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$

$\boxed{29}$ $\sin\beta + 2\cos^2\beta = 1 \;\Rightarrow\; \sin\beta + 2(1 - \sin^2\beta) = 1 \;\Rightarrow\; 2\sin^2\beta - \sin\beta - 1 = 0 \;\Rightarrow$
$(2\sin\beta + 1)(\sin\beta - 1) = 0 \;\Rightarrow\; \sin\beta = -\frac{1}{2}, 1 \;\Rightarrow\; \beta = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{\pi}{2}$

$\boxed{30}$ $\cos 2x + 3\cos x + 2 = 0 \;\Rightarrow\; 2\cos^2 x + 3\cos x + 1 = 0 \;\Rightarrow$
$(2\cos x + 1)(\cos x + 1) = 0 \;\Rightarrow\; \cos x = -\frac{1}{2}, -1 \;\Rightarrow\; x = \frac{2\pi}{3}, \frac{4\pi}{3}, \pi$

$\boxed{31}$ $2\sec u\, \sin u + 2 = 4\sin u + \sec u \;\Rightarrow\; 2\sec u\, \sin u - 4\sin u - \sec u + 2 = 0 \;\Rightarrow$
$2\sin u\,(\sec u - 2) - 1(\sec u - 2) = 0 \;\Rightarrow\; (2\sin u - 1)(\sec u - 2) = 0 \;\Rightarrow$
$\sin u = \frac{1}{2}$ or $\sec u = 2 \;\Rightarrow\; u = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{3}$

$\boxed{32}$ $\tan 2x\, \cos 2x = \sin 2x \;\Rightarrow\; \sin 2x = \sin 2x$. This is an identity and is true for all values of x in $[0, 2\pi)$ except those that make $\tan 2x$ undefined, or, equivalently, those that make $\cos 2x$ equal to 0. $\cos 2x = 0 \;\Rightarrow\; 2x = \frac{\pi}{2} + \pi n \;\Rightarrow\; x = \frac{\pi}{4} + \frac{\pi}{2}n$.
Hence, the solutions are all x in $[0, 2\pi)$ except $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

$\boxed{33}$ $2\cos 3x\,\cos 2x = 1 - 2\sin 3x\,\sin 2x \;\Rightarrow\; 2\cos 3x\,\cos 2x + 2\sin 3x\,\sin 2x = 1 \;\Rightarrow$

$2(\cos 3x\,\cos 2x + \sin 3x\,\sin 2x) = 1 \;\Rightarrow\; \cos(3x - 2x) = \frac{1}{2} \;\Rightarrow\; \cos x = \frac{1}{2} \;\Rightarrow$

$x = \frac{\pi}{3}, \frac{5\pi}{3}$

$\boxed{34}$ $\sin x\,\cos 2x + \cos x\,\sin 2x = 0 \;\Rightarrow\; \sin(x + 2x) = 0 \;\Rightarrow\; \sin 3x = 0 \;\Rightarrow$

$3x = \pi n \;\Rightarrow\; x = \frac{\pi}{3}n \;\Rightarrow\; x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$

$\boxed{35}$ $\cos \pi x + \sin \pi x = 0 \;\Rightarrow\; \sin \pi x = -\cos \pi x \;\Rightarrow\; \tan \pi x = -1 \;\Rightarrow\; \pi x = \frac{3\pi}{4} + \pi n \;\Rightarrow$

$x = \frac{3}{4} + n \;\Rightarrow\; x = \frac{3}{4}, \frac{7}{4}, \frac{11}{4}, \frac{15}{4}, \frac{19}{4}, \frac{23}{4}$

$\boxed{36}$ $\sin 2u = \sin u \;\Rightarrow\; 2\sin u\,\cos u = \sin u \;\Rightarrow\; 2\sin u\,\cos u - \sin u = 0 \;\Rightarrow$

$\sin u(2\cos u - 1) = 0 \;\Rightarrow\; \sin u = 0$ or $\cos u = \frac{1}{2} \;\Rightarrow\; u = 0, \pi, \frac{\pi}{3}, \frac{5\pi}{3}$

$\boxed{37}$ $2\cos^2 \frac{1}{2}\theta - 3\cos\theta = 0 \;\Rightarrow\; 2\left(\dfrac{1+\cos\theta}{2}\right) - 3\cos\theta = 0 \;\Rightarrow$

$(1 + \cos\theta) - 3\cos\theta = 0 \;\Rightarrow\; 1 - 2\cos\theta = 0 \;\Rightarrow\; \cos\theta = \frac{1}{2} \;\Rightarrow\; \theta = \frac{\pi}{3}, \frac{5\pi}{3}$

$\boxed{38}$ $\sec 2x\,\csc 2x = 2\csc 2x \;\Rightarrow\; \sec 2x\,\csc 2x - 2\csc 2x = 0 \;\Rightarrow$

$\csc 2x(\sec 2x - 2) = 0 \;\Rightarrow\; \cos 2x = \frac{1}{2}\;\{\csc 2x \neq 0\} \;\Rightarrow$

$2x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3} \;\Rightarrow\; x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

$\boxed{39}$ $\sin 5x = \sin 3x \;\Rightarrow\; \sin 5x - \sin 3x = 0 \;\Rightarrow\; 2\cos\dfrac{5x+3x}{2}\sin\dfrac{5x-3x}{2} = 0 \;\Rightarrow$

$\cos 4x\,\sin x = 0 \;\Rightarrow\; 4x = \frac{\pi}{2} + \pi n$ or $x = \pi n \;\Rightarrow\; x = \frac{\pi}{8} + \frac{\pi}{4}n$ or $x = 0, \pi \;\Rightarrow$

$x = 0, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \pi, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$

$\boxed{40}$ $\cos 3x = -\cos 2x \;\Rightarrow\; \cos 3x + \cos 2x = 0 \;\Rightarrow\; 2\cos\dfrac{3x+2x}{2}\cos\dfrac{3x-2x}{2} = 0 \;\Rightarrow$

$\cos\frac{5}{2}x\,\cos\frac{1}{2}x = 0 \;\Rightarrow\; \frac{5}{2}x = \frac{\pi}{2} + \pi n$ or $\frac{1}{2}x = \frac{\pi}{2} + \pi n \;\Rightarrow$

$x = \frac{\pi}{5} + \frac{2\pi}{5}n$ or $x = \pi + 2\pi n \;\Rightarrow\; x = \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5}$

$\boxed{41}$ $\cos 75^\circ = \cos(45^\circ + 30^\circ) = \cos 45^\circ\cos 30^\circ - \sin 45^\circ \sin 30^\circ = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$

$\boxed{42}$ $\tan 285^\circ = \tan(225^\circ + 60^\circ) =$

$$\frac{\tan 225^\circ + \tan 60^\circ}{1 - \tan 225^\circ \tan 60^\circ} = \frac{1+\sqrt{3}}{1-\sqrt{3}}\cdot\frac{1+\sqrt{3}}{1+\sqrt{3}} = \frac{4+2\sqrt{3}}{-2} = -2-\sqrt{3}$$

$\boxed{43}$ $\sin 195^\circ = \sin(135^\circ + 60^\circ)$

$= \sin 135^\circ \cos 60^\circ + \cos 135^\circ \sin 60^\circ = \frac{\sqrt{2}}{2}\cdot\frac{1}{2} + (-\frac{\sqrt{2}}{2})\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{2}-\sqrt{6}}{4}$

$\boxed{44}$ $\csc\dfrac{\pi}{8} = \dfrac{1}{\sin(\frac{1}{2}\cdot\frac{\pi}{4})} = \dfrac{1}{\sqrt{\dfrac{1-\cos\frac{\pi}{4}}{2}}} = \dfrac{1}{\sqrt{\dfrac{1-\sqrt{2}/2}{2}}} = \dfrac{1}{\sqrt{\dfrac{2-\sqrt{2}}{4}}} = \dfrac{2}{\sqrt{2-\sqrt{2}}}$

$\boxed{45}$ $\sin(\theta + \phi) = \sin\theta\,\cos\phi + \cos\theta\,\sin\phi = \frac{3}{5}\cdot\frac{8}{17} + \frac{4}{5}\cdot\frac{15}{17} = \frac{84}{85}$

$\boxed{46}$ $\cos(\theta + \phi) = \cos\theta\,\cos\phi - \sin\theta\,\sin\phi = \frac{4}{5}\cdot\frac{8}{17} - \frac{3}{5}\cdot\frac{15}{17} = -\frac{13}{85}$

$\boxed{47}$ $\tan(\phi + \theta) = \tan(\theta + \phi) = \dfrac{\sin(\theta+\phi)}{\cos(\theta+\phi)} = \dfrac{84/85}{-13/85} = -\dfrac{84}{13}$

[48] $\tan(\theta-\phi)=\dfrac{\tan\theta-\tan\phi}{1+\tan\theta\,\tan\phi}=\dfrac{\frac{3}{4}-\frac{15}{8}}{1+\frac{3}{4}\cdot\frac{15}{8}}\cdot\dfrac{32}{32}=\dfrac{24-60}{32+45}=-\dfrac{36}{77}$

[49] $\sin(\phi-\theta)=\sin\phi\,\cos\theta-\cos\phi\,\sin\theta=\frac{15}{17}\cdot\frac{4}{5}-\frac{8}{17}\cdot\frac{3}{5}=\frac{36}{85}$

[50] First recognize the relationship to Exercise 49.

$$\sin(\theta-\phi)=\sin\left[-(\phi-\theta)\right]=-\sin(\phi-\theta)=-\tfrac{36}{85}$$

[51] $\sin 2\phi=2\sin\phi\,\cos\phi=2\cdot\frac{15}{17}\cdot\frac{8}{17}=\frac{240}{289}$

[52] $\cos 2\phi=\cos^2\phi-\sin^2\phi=(\frac{8}{17})^2-(\frac{15}{17})^2=-\frac{161}{289}$

[53] $\tan 2\theta=\dfrac{2\tan\theta}{1-\tan^2\theta}=\dfrac{2\cdot\frac{3}{4}}{1-(\frac{3}{4})^2}\cdot\dfrac{16}{16}=\dfrac{24}{16-9}=\dfrac{24}{7}$

[54] $\sin\frac{1}{2}\theta=\sqrt{\dfrac{1-\cos\theta}{2}}=\sqrt{\dfrac{1-\frac{4}{5}}{2}}=\sqrt{\dfrac{\frac{1}{5}}{2}}=\sqrt{\frac{1}{10}\cdot\frac{10}{10}}=\frac{1}{10}\sqrt{10}$

[55] $\tan\frac{1}{2}\theta=\dfrac{1-\cos\theta}{\sin\theta}=\dfrac{1-\frac{4}{5}}{\frac{3}{5}}=\dfrac{\frac{1}{5}}{\frac{3}{5}}=\dfrac{1}{3}$

[56] $\cos\frac{1}{2}\phi=\sqrt{\dfrac{1+\cos\phi}{2}}=\sqrt{\dfrac{1+\frac{8}{17}}{2}}=\sqrt{\dfrac{\frac{25}{17}}{2}}=\sqrt{\frac{25}{34}\cdot\frac{34}{34}}=\frac{5}{34}\sqrt{34}$

[57] (a) $\sin 7t\,\sin 4t=$ [P4] $\frac{1}{2}[\cos(7t-4t)-\cos(7t+4t)]=\frac{1}{2}\cos 3t-\frac{1}{2}\cos 11t$

(b) $\cos\frac{1}{4}u\,\cos(-\frac{1}{6}u)=$ [P3] $\frac{1}{2}\left\{\cos\left[\frac{1}{4}u+(-\frac{1}{6}u)\right]+\cos\left[\frac{1}{4}u-(-\frac{1}{6}u)\right]\right\}=$

$$\tfrac{1}{2}(\cos\tfrac{2}{24}u+\cos\tfrac{10}{24}u)=\tfrac{1}{2}\cos\tfrac{1}{12}u+\tfrac{1}{2}\cos\tfrac{5}{12}u$$

(c) $6\cos 5x\,\sin 3x=$ [P2] $6\cdot\frac{1}{2}[\sin(5x+3x)-\sin(5x-3x)]=3\sin 8x-3\sin 2x$

(d) $4\sin 3\theta\,\cos 7\theta=$ [P1] $4\cdot\frac{1}{2}[\sin(3\theta+7\theta)+\sin(3\theta-7\theta)]=2\sin 10\theta-2\sin 4\theta$

[58] (a) $\sin 8u+\sin 2u=$ [S1] $2\sin\dfrac{8u+2u}{2}\cos\dfrac{8u-2u}{2}=2\sin 5u\,\cos 3u$

(b) $\cos 3\theta-\cos 8\theta=$ [S4] $-2\sin\dfrac{3\theta+8\theta}{2}\sin\dfrac{3\theta-8\theta}{2}=-2\sin\frac{11}{2}\theta\,\sin(-\frac{5}{2}\theta)=$

$$2\sin\tfrac{11}{2}\theta\,\sin\tfrac{5}{2}\theta$$

(c) $\sin\frac{1}{4}t-\sin\frac{1}{5}t=$ [S2] $2\cos\dfrac{\frac{1}{4}t+\frac{1}{5}t}{2}\sin\dfrac{\frac{1}{4}t-\frac{1}{5}t}{2}=2\cos\frac{9}{40}t\,\sin\frac{1}{40}t$

(d) $3\cos 2x+3\cos 6x=$ [S3] $3\cdot 2\cos\dfrac{2x+6x}{2}\cos\dfrac{2x-6x}{2}=6\cos 4x\,\cos(-2x)=$

$$6\cos 4x\,\cos 2x$$

[59] $\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)=\frac{\pi}{6}$

[60] $\arcsin\left(\dfrac{\sqrt{2}}{2}\right)=\frac{\pi}{4}$

[61] $\arctan\sqrt{3}=\frac{\pi}{3}$

[62] $\arccos(\tan\frac{3\pi}{4})=\arccos(-1)=\pi$

[63] $\arcsin\left(\sin\dfrac{5\pi}{4}\right)=\arcsin\left(-\dfrac{\sqrt{2}}{2}\right)=-\frac{\pi}{4}$

[64] $\cos^{-1}\left(\cos\dfrac{5\pi}{4}\right)=\cos^{-1}\left(-\dfrac{\sqrt{2}}{2}\right)=\dfrac{3\pi}{4}$

[65] $\sin\left[\arccos\left(-\dfrac{\sqrt{3}}{2}\right)\right]=\sin\frac{5\pi}{6}=\frac{1}{2}$

[66] $\tan(\tan^{-1}2)=2$ since $\tan(\arctan x)=x$ for every x

$\boxed{67}$ $\sec(\sin^{-1}\frac{3}{2})$ is not defined since $\frac{3}{2} > 1$ $\boxed{68}$ $\cos^{-1}(\sin 0) = \cos^{-1} 0 = \frac{\pi}{2}$.

$\boxed{69}$ Let $\alpha = \sin^{-1}\frac{15}{17}$ and $\beta = \sin^{-1}\frac{8}{17}$.

$\cos(\sin^{-1}\frac{15}{17} - \sin^{-1}\frac{8}{17}) = \cos(\alpha - \beta) = \cos\alpha\,\cos\beta + \sin\alpha\,\sin\beta = \frac{8}{17}\cdot\frac{15}{17} + \frac{15}{17}\cdot\frac{8}{17} = \frac{240}{289}$.

$\boxed{70}$ Let $\alpha = \sin^{-1}\frac{4}{5}$. $\cos(2\sin^{-1}\frac{4}{5}) = \cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = (\frac{3}{5})^2 - (\frac{4}{5})^2 = -\frac{7}{25}$.

$\boxed{71}$ $y = \cos^{-1} 3x$ • horizontally compress $y = \cos^{-1} x$ by a factor of 3

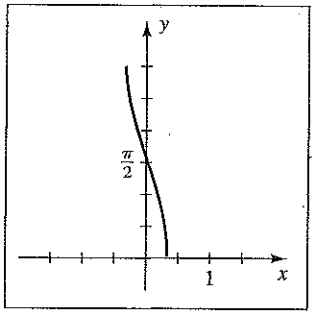

Figure 71

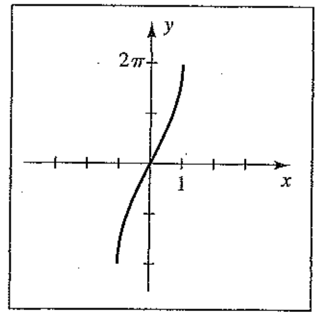

Figure 72

$\boxed{72}$ $y = 4\sin^{-1} x$ • vertically stretch $y = \sin^{-1} x$ by a factor of 4

$\boxed{73}$ $y = 1 - \sin^{-1} x = -\sin^{-1} x + 1$ •

reflect $y = \sin^{-1} x$ through the x-axis and shift it up 1 unit.

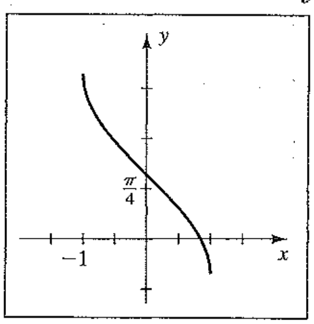

Figure 73

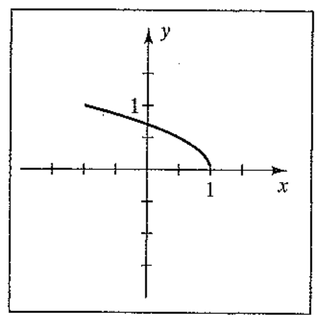

Figure 74

$\boxed{74}$ If $\alpha = \cos^{-1} x$, then $\cos\alpha = x$, where $0 \le \alpha \le \pi$.

Hence, $y = \sin\left(\frac{1}{2}\cos^{-1} x\right) = \sin\frac{1}{2}\alpha = \sqrt{\frac{1-\cos\alpha}{2}} = \sqrt{\frac{1-x}{2}}$.

Thus, we have the graph of the half-parabola $y = \sqrt{\frac{1}{2}(1-x)}$ on the interval $[-1, 1]$.

$\boxed{75}$ $\cos(\alpha + \beta + \gamma) = \cos[(\alpha + \beta) + \gamma] = \cos(\alpha + \beta)\cos\gamma - \sin(\alpha + \beta)\sin\gamma =$

$(\cos\alpha\,\cos\beta - \sin\alpha\,\sin\beta)\cos\gamma - (\sin\alpha\,\cos\beta + \cos\alpha\,\sin\beta)\sin\gamma =$

$\cos\alpha\,\cos\beta\,\cos\gamma - \sin\alpha\,\sin\beta\,\cos\gamma - \sin\alpha\,\cos\beta\,\sin\gamma - \cos\alpha\,\sin\beta\,\sin\gamma$

$\boxed{76}$ (a) $t = -\frac{\pi}{2b} \Rightarrow F = A\left[\cos\left(-\frac{\pi}{2}\right) - a\cos\left(-\frac{3\pi}{2}\right)\right] = A(0 - a\cdot 0) = 0$

$t = \frac{\pi}{2b} \Rightarrow F = A\left(\cos\frac{\pi}{2} - a\cos\frac{3\pi}{2}\right) = A(0 - a\cdot 0) = 0$

(b) $a = \frac{1}{3} \Rightarrow \sin 3bt = \sin bt \Rightarrow \sin 3bt - \sin bt = 0 \Rightarrow$

[S2] $2\cos\dfrac{3bt + bt}{2}\sin\dfrac{3bt - bt}{2} = 0 \Rightarrow \cos 2bt \sin bt = 0 \Rightarrow$

$\cos 2bt = 0$ or $\sin bt = 0 \Rightarrow 2bt = \frac{\pi}{2} + \pi n$ or $bt = \pi n \Rightarrow$

$t = \frac{\pi}{4b} + \frac{\pi}{2b}n$ or $t = \frac{\pi}{b}n$. Since $-\frac{\pi}{2b} < t < \frac{\pi}{2b}$, $t = \pm\frac{\pi}{4b}$, 0.

(c) Using the values from part (b),

$t = 0 \Rightarrow F = A\left(\cos 0 - \frac{1}{3}\cos 0\right) = A\left(1 - \frac{1}{3}\right) = \frac{2}{3}A$. $t = \pm\frac{\pi}{4b} \Rightarrow$

$F = A\left[\cos\left(\pm\frac{\pi}{4}\right) - \frac{1}{3}\cos\left(\pm\frac{3\pi}{4}\right)\right] = A\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{6}\right) = \frac{4\sqrt{2}}{6}A = \frac{2}{3}\sqrt{2}\,A.$

The second value is $\sqrt{2}$ times the first, hence $\frac{2}{3}\sqrt{2}\,A$ is the maximum force.

$\boxed{77}$ $\cos x - \cos 2x + \cos 3x = 0 \Rightarrow (\cos x + \cos 3x) - \cos 2x \Rightarrow$

[S3] $2\cos\dfrac{x + 3x}{2}\cos\dfrac{x - 3x}{2} - \cos 2x = 0 \Rightarrow 2\cos 2x\cos x - \cos 2x = 0 \Rightarrow$

$\cos 2x\,(2\cos x - 1) = 0 \Rightarrow \cos 2x = 0$ or $\cos x = \frac{1}{2} \Rightarrow$

$2x = \frac{\pi}{2} + \pi n$ (or $x = \frac{\pi}{4} + \frac{\pi}{2}n$) or $x = \frac{\pi}{3} + 2\pi n$, $\frac{5\pi}{3} + 2\pi n$.

In the figure on $[-2\pi, 2\pi]$, $x = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}, \pm\frac{5\pi}{4}, \pm\frac{7\pi}{4}, \pm\frac{\pi}{3}, \pm\frac{5\pi}{3}$.

$\boxed{78}$ (a) Bisect θ to form two right triangles. $\tan\frac{1}{2}\theta = \dfrac{\frac{1}{2}x}{d} \Rightarrow x = 2d\tan\frac{1}{2}\theta$.

(b) Using part (a) with $x = 0.5$ ft and $\theta = 0.0005$ radian,

we have $d = \dfrac{x}{2\tan\frac{1}{2}\theta} \approx 1000$ ft, so $d \le 1000$ ft.

$\boxed{79}$ (a) Bisect θ to form two right triangles.

$\cos\frac{1}{2}\theta = \dfrac{r}{d + r} \Rightarrow d + r = \dfrac{r}{\cos\frac{1}{2}\theta} \Rightarrow d = r\sec\frac{1}{2}\theta - r = r\left(\sec\frac{1}{2}\theta - 1\right)$.

(b) $d = 300$ and $r = 4000 \Rightarrow \cos\frac{1}{2}\theta = \dfrac{r}{d + r} = \frac{4000}{4300} \Rightarrow \frac{1}{2}\theta \approx 21.5^\circ \Rightarrow \theta \approx 43^\circ$.

$\boxed{80}$ (a) $\tan\theta = \frac{h}{w} = \frac{400}{80} = 5 \Rightarrow \theta = \tan^{-1}5 \approx 78.7^\circ$

(b) $\tan\theta = \frac{h}{w} = \frac{55}{30} = \frac{11}{6} \Rightarrow \theta = \tan^{-1}\frac{11}{6} \approx 61.4^\circ$

Chapter 7 Discussion Exercises

[1] $\dfrac{\tan x}{1-\cot x}+\dfrac{\cot x}{1-\tan x}=\dfrac{\frac{\sin x}{\cos x}}{1-\frac{\cos x}{\sin x}}+\dfrac{\frac{\cos x}{\sin x}}{1-\frac{\sin x}{\cos x}}=$

$\dfrac{\sin^2 x}{\cos x\,(\sin x-\cos x)}+\dfrac{\cos^2 x}{\sin x\,(\cos x-\sin x)}=\dfrac{\sin^2 x}{\cos x\,(\sin x-\cos x)}-\dfrac{\cos^2 x}{\sin x\,(\sin x-\cos x)}=$

$\dfrac{\sin^3 x-\cos^3 x}{\cos x\,\sin x\,(\sin x-\cos x)}=\dfrac{(\sin x-\cos x)(\sin^2 x+\sin x\,\cos x+\cos^2 x)}{\cos x\,\sin x\,(\sin x-\cos x)}=$

$\dfrac{1+\sin x\,\cos x}{\cos x\,\sin x}=\dfrac{1}{\cos x\,\sin x}+1=1+\sec x\,\csc x$

[2] $\sqrt{a^2-x^2}=\begin{cases} a\,\cos\theta & \text{if } 0\le\theta\le\pi/2 \text{ or } 3\pi/2\le\theta<2\pi \\ -a\,\cos\theta & \text{if } \pi/2<\theta<3\pi/2\end{cases}$

[3] *Note:* Graphing on a TI-82/83 doesn't really help to solve this problem.

$3\cos 45x+4\sin 45x=5 \;\Rightarrow\;$ { by Example 6 in Section 7.3 }

$5\cos(45x-\tan^{-1}\frac{4}{3})=5 \;\Rightarrow\; \cos(45x-\tan^{-1}\frac{4}{3})=1 \;\Rightarrow\; 45x-\tan^{-1}\frac{4}{3}=2\pi n \;\Rightarrow$

$45x=2\pi n+\tan^{-1}\frac{4}{3} \;\Rightarrow\; x=\dfrac{2\pi n+\tan^{-1}\frac{4}{3}}{45}$. $n=0, 1, \ldots, 44$ will yield x values in $[0, 2\pi)$. *Note:* After using Example 6, you might notice that this is a function with period $2\pi/45$, and it will obtain 45 maximums on an interval of length 2π. The largest value of x is approximately 6.164 { when $n=44$ }.

[4] The difference quotient for the sine function appears to be the cosine function.

$$\begin{aligned}\frac{f(x+h)-f(x)}{h}&=\frac{\sin(x+h)-\sin x}{h}\\&=\frac{\sin x\,\cos h+\cos x\,\sin h-\sin x}{h}\\&=\sin x\left(\frac{\cos h-1}{h}\right)+\cos x\left(\frac{\sin h}{h}\right)\end{aligned}$$

Figure 4 shows the graph of

$y=\dfrac{\sin(x+h)-\sin x}{h}$

for $h=0.5$, 0.1, and 0.001 on the viewing rectangle $[0, 2\pi, \pi/2]$ by $[-1, 1]$.

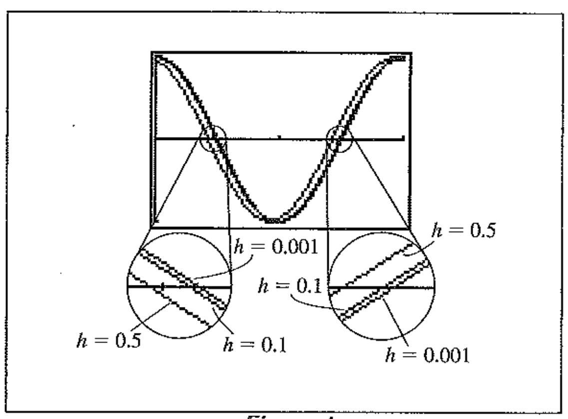

Figure 4

[5] Let $\alpha = \tan^{-1}(\frac{1}{239})$ and $\theta = \tan^{-1}(\frac{1}{5})$. $\frac{\pi}{4} = 4\theta - \alpha \;\Rightarrow\; \frac{\pi}{4} + \alpha = 4\theta$. Both sides are acute angles, and we will show that the tangent of each side is equal to the same value, hence proving the identity.

$$\text{LS} = \tan\left(\tfrac{\pi}{4} + \alpha\right) = \frac{\tan\frac{\pi}{4} + \tan\alpha}{1 - \tan\frac{\pi}{4}\tan\alpha} = \frac{1 + \frac{1}{239}}{1 - 1\cdot\frac{1}{239}} = \frac{\frac{240}{239}}{\frac{238}{239}} = \frac{240}{238} = \frac{120}{119}.$$

$\text{RS} = \tan 4\theta =$

$$\frac{2\tan 2\theta}{1-\tan^2(2\theta)} = \frac{2\cdot\frac{2\tan\theta}{1-\tan^2\theta}}{1-\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)^2} = \frac{2\cdot\frac{\frac{2}{5}}{1-\frac{1}{25}}}{1-\left(\frac{\frac{2}{5}}{1-\frac{1}{25}}\right)^2} = \frac{\frac{\frac{4}{5}}{\frac{24}{25}}}{1-\frac{\frac{4}{25}}{\frac{24\cdot 24}{25\cdot 25}}} = \frac{\frac{5}{6}}{\frac{119}{144}} = \frac{120}{119}.$$

Similarly, for the second relationship, we could write $\frac{\pi}{4} = \alpha + \beta + \gamma$ and show that $\tan\left(\frac{\pi}{4} - \alpha\right) = \tan(\beta + \gamma) = \frac{1}{3}$.

For the third relationship, write $\pi - \tan^{-1} 1 = \tan^{-1} 2 + \tan^{-1} 3 \;\Rightarrow$

$$\tan\left(\pi - \tfrac{\pi}{4}\right) = \tan(\alpha + \beta) \;\Rightarrow\; \tan\tfrac{3\pi}{4} = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\,\tan\beta} \;\Rightarrow\; -1 = \frac{2+3}{1-2\cdot 3} \text{ \{true\}}.$$

[6] (a) The **inverse sawtooth function**, denoted by saw^{-1} or arcsaw, is defined by $y = \text{saw}^{-1} x$ iff $x = \text{saw}\, y$ for $-2 \le x \le 2$ and $-1 \le y \le 1$.

(b) $\text{arcsaw}\,(1.7) = 0.85$ since $\text{saw}\,(0.85) = 1.7$

$\text{arcsaw}\,(-0.8) = -0.4$ since $\text{saw}\,(-0.4) = -0.8$

(c) $\text{saw}\,(\text{saw}^{-1} x) = \text{saw}\,(\text{arcsaw}\, x) = x$ if $-2 \le x \le 2$

$\text{saw}^{-1}\,(\text{saw}\, y) = \text{arcsaw}\,(\text{saw}\, y) = y$ if $-1 \le y \le 1$

(d)

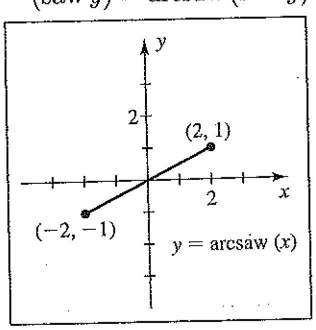

Figure 6