CHAPTER 6
Differential Equations

C H A P T E R 6
Differential Equations

Section 6.1 Slope Fields and Euler's Method

1. Differential equation: $y' = 4y$

Solution: $y = Ce^{4x}$

Check: $y' = 4Ce^{4x} = 4y$

2. Differential Equation: $3y' + 4y = e^{-x}$

$y = e^{-x}$

$y' = -e^{-x}$

Check: $3(-e^{-x}) + 4(e^{-x}) = -3e^{-x} + 4e^{-x} = e^{-x}$

3. Differential equation: $y' = \dfrac{2xy}{x^2 - y^2}$

Solution: $x^2 + y^2 = Cy$

Check: $2x + 2yy' = Cy'$

$$y' = \frac{-2x}{(2y - C)}$$

$$y' = \frac{-2xy}{2y^2 - Cy}$$

$$= \frac{-2xy}{2y^2 - (x^2 + y^2)}$$

$$= \frac{-2xy}{y^2 - x^2}$$

$$= \frac{2xy}{x^2 - y^2}$$

4. Differential Equation: $\dfrac{dy}{dx} = \dfrac{xy}{y^2 - 1}$

Solution: $y^2 - 2 \ln y = x^2$

Check: $2yy' - \dfrac{2}{y}y' = 2x$

$$\left(y - \frac{1}{y}\right)y' = x$$

$$y' = \frac{x}{y - \dfrac{1}{y}}$$

$$y' = \frac{xy}{y^2 - 1}$$

5. Differential equation: $y'' + y = 0$

Solution: $y = C_1 \cos x + C_2 \sin x$

Check: $y' = -C_1 \sin x + C_2 \cos x$

$$y'' = -C_1 \cos x - C_2 \sin x$$

$$y'' + y = -C_1 \cos x - C_2 \sin x + C_1 \cos x + C_2 \sin x = 0$$

6. Differential equation: $y'' + 2y' + 2y = 0$

Solution: $y = C_1 e^{-x} \cos x + C_2 e^{-x} \sin x$

Check: $y' = -(C_1 + C_2)e^{-x} \sin x + (-C_1 + C_2)e^{-x} \cos x$

$$y'' = 2\,C_1 e^{-x} \sin x - 2C_2 e^{-x} \cos x$$

$$\begin{aligned} y'' + 2y' + 2y &= 2C_1 e^{-x} \sin x - 2C_2 e^{-x} \cos x + \\ &\quad 2(-(C_1 + C_2)e^{-x} \sin x + (-C_1 + C_2)e^{-x} \cos x) + 2(C_1 e^{-x} \cos x + C_2 e^{-x} \sin x) \\ &= (2C_1 - 2C_1 - 2C_2 + 2C_2)e^{-x} \sin x + (-2C_2 - 2C_1 + 2C_2 + 2C_1)e^{-x} \cos x = 0 \end{aligned}$$

7. Differential Equation: $y'' + y = \tan x$

$$y = -\cos x \ln|\sec x + \tan x|$$

$$y' = (-\cos x)\frac{1}{\sec x + \tan x}(\sec x \cdot \tan x + \sec^2 x) + \sin x \ln|\sec x + \tan x|$$

$$= \frac{(-\cos x)}{\sec x + \tan x}(\sec x)(\tan x + \sec x) + \sin x \ln|\sec x + \tan x|$$

$$= -1 + \sin x \ln|\sec x + \tan x|$$

$$y'' = (\sin x)\frac{1}{\sec x + \tan x}(\sec x \cdot \tan x + \sec^2 x) + \cos x \ln|\sec x + \tan x|$$

$$= (\sin x)(\sec x) + \cos x \ln|\sec x + \tan x|$$

Substituting,

$$y'' + y = (\sin x)(\sec x) + \cos x \ln|\sec x + \tan x| - \cos x \ln|\sec x + \tan x|$$

$$= \tan x.$$

8. $y = \frac{2}{3}(e^{-2x} + e^x)$

$y' = \frac{2}{3}(-2e^{-2x} + e^x)$

$y'' = \frac{2}{3}(4e^{-2x} + e^x)$

Substituting, $y'' + 2y' = \frac{2}{3}(4e^{-2x} + e^x) + 2\left(\frac{2}{3}\right)(-2e^{-2x} + e^x) = 2e^x.$

9. $y = \sin x \cos x - \cos^2 x$

$y' = -\sin^2 x + \cos^2 x + 2 \cos x \sin x$

$= -1 + 2\cos^2 x + \sin 2x$

Differential Equation:

$$2y + y' = 2(\sin x \cos x - \cos^2 x) + (-1 + 2\cos^2 x + \sin 2x)$$

$$= 2 \sin x \cos x - 1 + \sin 2x$$

$$= 2 \sin 2x - 1$$

Initial condition:

$$y\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4}\cos\frac{\pi}{4} - \cos^2\frac{\pi}{4} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} - \left(\frac{\sqrt{2}}{2}\right)^2 = 0$$

10. $y = \frac{1}{2}x^2 - 4\cos x + 2$

$y' = x + 4 \sin x$

Differential Equation:

$$y' = x + 4\sin x$$

Initial condition:

$$y(0) = \frac{1}{2}(0)^2 - 4\cos(0) + 2 = -4 + 2 = -2$$

11. $y = 6e^{-2x^2}$

$y' = 6e^{-2x^2}(-4x) = -24xe^{-2x^2}$

Differential Equation:

$$y' = -24xe^{-2x^2} = -4x(6e^{-2x^2}) = -4xy$$

Initial condition:

$$y(0) = 6e^{-2(0)^2} = 6e^0 = 6(1) = 6$$

12. $y = e^{-\cos x}$

$y' = e^{-\cos x}(\sin x) = \sin x \cdot e^{-\cos x}$

Differential Equation:

$$y' = \sin x \cdot e^{-\cos x} = \sin x(y) = y \sin x$$

Initial condition:

$$y\left(\frac{\pi}{2}\right) = e^{-\cos(\pi/2)} = e^0 = 1$$

In Exercises 13–18, the differential equation is $y^{(4)} - 16y = 0$.

13.
$$y = 3\cos x$$
$$y^{(4)} = 3\cos x$$
$$y^{(4)} - 16y = -45\cos x \neq 0,$$

No

14.
$$y = 3\cos 2x$$
$$y^{(4)} = 48\cos 2x$$
$$y^{(4)} - 16y = 48\cos 2x - 48\cos 2x = 0,$$

Yes

15.
$$y = e^{-2x}$$
$$y^{(4)} = 16e^{-2x}$$
$$y^{(4)} - 16y = 16e^{-2x} - 16e^{-2x} = 0,$$

Yes

16.
$$y = 5\ln x$$
$$y^{(4)} = -\frac{30}{x^4}$$
$$y^{(4)} - 16y = -\frac{30}{x^4} - 80\ln x \neq 0,$$

No

17.
$$y = C_1e^{2x} + C_2e^{-2x} + C_3\sin 2x + C_4\cos 2x$$
$$y^{(4)} = 16C_1e^{2x} + 16C_2e^{-2x} + 16C_3\sin 2x + 16C_4\cos 2x$$
$$y^{(4)} - 16y = 0,$$

Yes

18.
$$y = 3e^{2x} - 4\sin 2x$$
$$y^{(4)} = 48e^{2x} - 64\sin 2x$$
$$y^{(4)} - 16y = (48e^{2x} - 64\sin 2x) - 16(3e^{2x} - 4\sin 2x) = 0,$$

Yes

In 19–24, the differential equation is $xy' - 2y = x^3e^x$.

19. $y = x^2,\ y' = 2x$

$xy' - 2y = x(2x) - 2(x^2) = 0 \neq x^3e^x,$

No

20. $y = x^2e^x,\ y' = x^2e^x + 2xe^x = e^x(x^2 + 2x)$

$xy' - 2y = x(e^x(x^2 + 2x)) - 2(x^2e^x) = x^3e^x,$

Yes

21. $y = x^2(2 + e^x),\ \ y' = x^2(e^x) + 2x(2 + e^x)$

$xy' - 2y = x[x^2e^x + 2xe^x + 4x] - 2[x^2e^x + 2x^2] = x^3e^x,$

Yes

22. $y = \sin x, y' = \cos x$

$xy' - 2y = x(\cos x) - 2(\sin x) \neq x^3e^x,$

No

23. $y = \ln x, y' = \dfrac{1}{x}$

$xy' - 2y = x\left(\dfrac{1}{x}\right) - 2\ln x \neq x^3e^x,$

No

24. $y = x^2e^x - 5x^2, y' = x^2e^x + 2xe^x - 10x$

$xy' - 2y = x[x^2e^x + 2xe^x - 10x] - 2[x^2e^x - 5x^2] = x^3e^x,$

Yes

25. $y = Ce^{-x/2}$ passes through $(0, 3)$.

$3 = Ce^0 = C \implies C = 3$

Particular solution: $y = 3e^{-x/2}$

26. $y(x^2 + y) = C$ passes through $(0, 2)$.

$2(0 + 2) = C \implies C = 4$

Particular solution: $y(x^2 + y) = 4$

27. $y^2 = Cx^3$ passes through $(4, 4)$.

$16 = C(64) \implies C = \frac{1}{4}$

Particular solution: $y^2 = \frac{1}{4}x^3$ or $4y^2 = x^3$

28. $2x^2 - y^2 = C$ passes through $(3, 4)$.

$2(9) - 16 = C \implies C = 2$

Particular solution: $2x^2 - y^2 = 2$

29. Differential equation: $4yy' - x = 0$

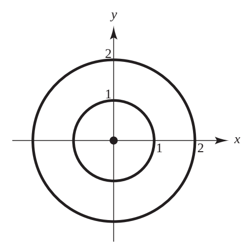
General solution: $4y^2 - x^2 = C$

Particular solutions: $C = 0$, Two intersecting lines
$C = \pm 1$, $C = \pm 4$, Hyperbolas

30. Differential equation: $yy' + x = 0$

General solution: $x^2 + y^2 = C$

Particular solutions: $C = 0$, Point $C = 1$, $C = 4$, Circles



31. Differential equation: $y' + 2y = 0$

General solution: $y = Ce^{-2x}$

$y' + 2y = C(-2)e^{-2x} + 2(Ce^{-2x}) = 0$

Initial condition: $y(0) = 3, 3 = Ce^0 = C$

Particular solution: $y = 3e^{-2x}$

32. Differential equation: $3x + 2yy' = 0$

General solution: $3x^2 + 2y^2 = C$

$$6x + 4yy' = 0$$
$$2(3x + 2yy') = 0$$
$$3x + 2yy' = 0$$

Initial condition: $y(1) = 3$: $3(1)^2 + 2(3)^2$
$= 3 + 18 = 21 = C$

Particular solution: $3x^2 + 2y^2 = 21$

33. Differential equation: $y'' + 9y = 0$

General solution: $y = C_1 \sin 3x + C_2 \cos 3x$

$$y' = 3C_1 \cos 3x - 3C_2 \sin 3x,$$
$$y'' = -9C_1 \sin 3x - 9C_2 \cos 3x$$
$$y'' + 9y = (-9C_1 \sin 3x - 9C_2 \cos 3x) + 9(C_1 \sin 3x + C_2 \cos 3x) = 0$$

Initial conditions: $y\left(\dfrac{\pi}{6}\right) = 2,\ y'\left(\dfrac{\pi}{6}\right) = 1$

$$2 = C_1 \sin\left(\frac{\pi}{2}\right) + C_2 \cos\left(\frac{\pi}{2}\right) \implies C_1 = 2$$

$$y' = 3C_1 \cos 3x - 3C_2 \sin 3x$$

$$1 = 3C_1 \cos\left(\frac{\pi}{2}\right) - 3C_2 \sin\left(\frac{\pi}{2}\right) = -3C_2 \implies C_2 = -\frac{1}{3}$$

Particular solution: $y = 2\sin 3x - \dfrac{1}{3}\cos 3x$

34. Differential equation: $xy'' + y' = 0$

General solution: $y = C_1 + C_2 \ln x$

$y' = C_2\left(\frac{1}{x}\right), y'' = -C_2\left(\frac{1}{x^2}\right)$

$xy'' + y' = x\left(-C_2\frac{1}{x^2}\right) + C_2\frac{1}{x} = 0$

Initial conditions: $y(2) = 0,\ y'(2) = \frac{1}{2}$

$0 = C_1 + C_2 \ln 2$

$y' = \frac{C_2}{x}$

$\frac{1}{2} = \frac{C_2}{2} \implies C_2 = 1,\ C_1 = -\ln 2$

Particular solution: $y = -\ln 2 + \ln x = \ln \frac{x}{2}$

35. Differential equation: $x^2y'' - 3xy' + 3y = 0$

General solution: $y = C_1x + C_2x^3$

$y' = C_1 + 3C_2x^2, y'' = 6C_2x$

$x^2y'' - 3x\,y' + 3y = x^2(6C_2x) - 3x(C_1 + 3C_2x^2) + 3(C_1x + C_2x^3) = 0$

Initial conditions: $y(2) = 0,\ y'(2) = 4$

$0 = 2C_1 + 8C_2$

$y' = C_1 + 3C_2x^2$

$4 = C_1 + 12C_2$

$$\left.\begin{array}{l} C_1 + 4C_2 = 0 \\ C_1 + 12C_2 = 4 \end{array}\right\} \quad C_2 = \tfrac{1}{2},\quad C_1 = -2$$

Particular solution: $y = -2x + \frac{1}{2}x^3$

36. Differential equation: $9y'' - 12y' + 4y = 0$

General solution: $y = e^{2x/3}(C_1 + C_2x)$

$$y' = \frac{2}{3}e^{2x/3}(C_1 + C_2x) + C_2e^{2x/3} = e^{2x/3}\left(\frac{2}{3}C_1 + C_2 + \frac{2}{3}C_2x\right)$$

$$y'' = \frac{2}{3}e^{2x/3}\left(\frac{2}{3}C_1 + C_2 + \frac{2}{3}C_2x\right) + e^{2x/3}\frac{2}{3}C_2 = \frac{2}{3}e^{2x/3}\left(\frac{2}{3}C_1 + 2C_2 + \frac{2}{3}C_2x\right)$$

$$9y'' - 12y' + 4y = 9\left(\frac{2}{3}e^{2x/3}\right)\left(\frac{2}{3}C_1 + 2C_2 + \frac{2}{3}C_2x\right) - 12(e^{2x/3})\left(\frac{2}{3}C_1 + C_2 + \frac{2}{3}C_2x\right) + 4(e^{2x/3})(C_1 + C_2x) = 0$$

Initial conditions: $y(0) = 4,\ y(3) = 0$

$0 = e^2(C_1 + 3C_2)$

$4 = (1)(C_1 + 0) \implies C_1 = 4$

$0 = e^2(4 + 3C_2) \implies C_2 = -\frac{4}{3}$

Particular solution: $y = e^{2x/3}\left(4 - \frac{4}{3}x\right)$

37. $\frac{dy}{dx} = 3x^2$

$y = \int 3x^2\,dx = x^3 + C$

38. $\frac{dy}{dx} = x^3 - 4x$

$y = \int (x^3 - 4x)\,dx = \frac{x^4}{4} - 2x^2 + C$

39. $\dfrac{dy}{dx} = \dfrac{x}{1 + x^2}$

$$y = \int \frac{x}{1 + x^2}\,dx = \frac{1}{2}\ln(1 + x^2) + C$$

$(u = 1 + x^2,\ du = 2x\,dx)$

40. $\dfrac{dy}{dx} = \dfrac{e^x}{1 + e^x}$

$$y = \int \frac{e^x}{1 + e^x}\,dx = \ln(1 + e^x) + C$$

41. $\dfrac{dy}{dx} = \dfrac{x - 2}{x} = 1 - \dfrac{2}{x}$

$$y = \int \left[1 - \frac{2}{x}\right] dx$$

$$= x - 2\ln|x| + C = x - \ln x^2 + C$$

42. $\dfrac{dy}{dx} = x \cos x^2$

$$y = \int x\cos(x^2)\,dx = \frac{1}{2}\sin(x^2) + C$$

$(u = x^2,\ du = 2x\,dx)$

43. $\dfrac{dy}{dx} = \sin 2x$

$$y = \int \sin 2x\,dx = -\frac{1}{2}\cos 2x + C$$

$(u = 2x,\ du = 2\,dx)$

44. $\dfrac{dy}{dx} = \tan^2 x = \sec^2 x - 1$

$$y = \int (\sec^2 x - 1)\,dx = \tan x - x + C$$

45. $\dfrac{dy}{dx} = x\sqrt{x - 3}$

Let $u = \sqrt{x - 3}$, then $x = u^2 + 3$ and $dx = 2u\,du$.

$$y = \int x\sqrt{x - 3}\,dx = \int (u^2 + 3)(u)(2u)\,du$$

$$= 2\int (u^4 + 3u^2)\,du = 2\left(\frac{u^5}{5} + u^3\right) + C = \frac{2}{5}(x - 3)^{5/2} + 2(x - 3)^{3/2} + C$$

46. $\dfrac{dy}{dx} = x\sqrt{5 - x}$. Let $u = \sqrt{5 - x}$, $u^2 = 5 - x$, $dx = -2u\,du$.

$$y = \int x\sqrt{5 - x}\,dx = \int (5 - u^2)u(-2u)\,du$$

$$= \int (-10u^2 + 2u^4)\,du$$

$$= \frac{-10u^3}{3} + \frac{2u^5}{5} + C$$

$$= -\frac{10}{3}(5 - x)^{3/2} + \frac{2}{5}(5 - x)^{5/2} + C$$

47. $\dfrac{dy}{dx} = xe^{x^2}$

$$y = \int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + C$$

$(u = x^2,\ du = 2x\,dx)$

48. $\dfrac{dy}{dx} = 5e^{-x/2}$

$$y = \int 5e^{-x/2}\,dx = 5(-2)\int e^{-x/2}\left(-\frac{1}{2}\right)dx$$

$$= -10e^{-x/2} + C$$

49.

x	-4	-2	0	2	4	8
y	2	0	4	4	6	8
dy/dx	-2	Undef.	0	$\frac{1}{2}$	$\frac{2}{3}$	1

50.

x	-4	-2	0	2	4	8
y	2	0	4	4	6	8
dy/dx	-6	-2	-4	-2	-2	0

51.

x	-4	-2	0	2	4	8
y	2	0	4	4	6	8
dy/dx	$-2\sqrt{2}$	-2	0	0	$-2\sqrt{2}$	-8

52.

x	-4	-2	0	2	4	8
y	2	0	4	4	6	8
dy/dx	$\sqrt{3}$	0	$-\sqrt{3}$	$-\sqrt{3}$	0	$\sqrt{3}$

53. $\dfrac{dy}{dx} = \cos(2x)$

For $x = \pi$, $\dfrac{dy}{dx} = 1$. Matches (b).

54. $\dfrac{dy}{dx} = \dfrac{1}{2}\sin x$

For $x = 0$, $\dfrac{dy}{dx} = 0$. Matches (c).

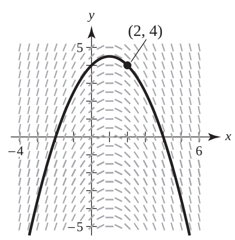
55. $\dfrac{dy}{dx} = e^{-2x}$

As $x \to \infty$, $\dfrac{dy}{dx} \to 0$. Matches (d).

56. $\dfrac{dy}{dx} = \dfrac{1}{x}$

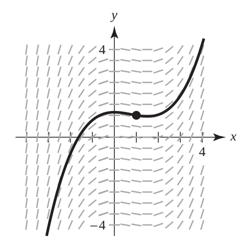
For $x = 0$, $\dfrac{dy}{dx}$ is undefined (vertical tangent). Matches (a).

57. (a), (b)



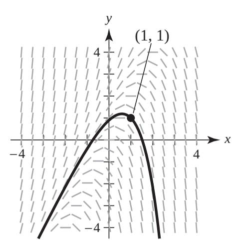
(c) As $x \to \infty$, $y \to -\infty$
As $x \to -\infty$, $y \to -\infty$

58. (a), (b)



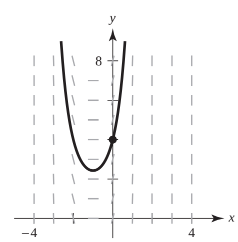
(c) As $x \to \infty$, $y \to \infty$
As $x \to -\infty$, $y \to -\infty$

59. (a), (b)



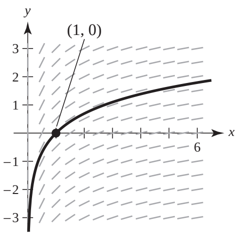
(c) As $x \to \infty$, $y \to -\infty$
As $x \to -\infty$, $y \to -\infty$

60. (a),



(c) As $x \to \infty$, $y \to \infty$
As $x \to -\infty$, $y \to \infty$

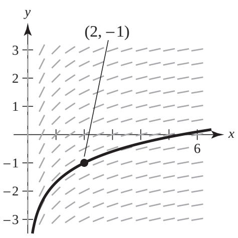
61. (a) $y' = \dfrac{1}{x}, y(1) = 0$



As $x \to \infty, y \to \infty$

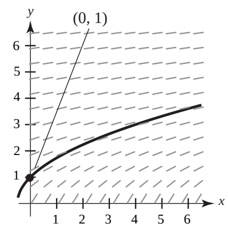
[Note: The solution is $y = \ln x$.]

(b) $y' = \dfrac{1}{x}, y(2) = -1$



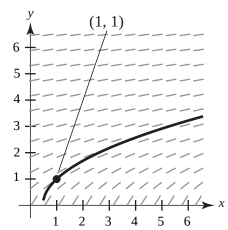
As $x \to \infty, y \to \infty$

62. (a) $y' = \dfrac{1}{y}, y(0) = 1$



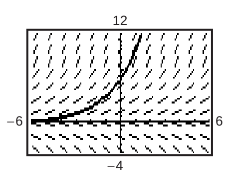
As $x \to \infty, y \to \infty$

(b) $y' = \dfrac{1}{y}, y(1) = 1$

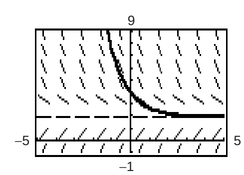


As $x \to \infty, y \to \infty$

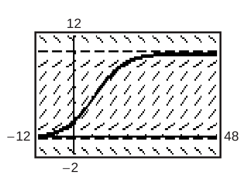
63. $\dfrac{dy}{dx} = 0.5y, y(0) = 6$



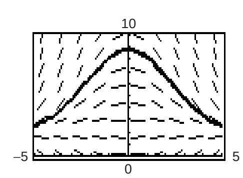
64. $\dfrac{dy}{dx} = 2 - y, y(0) = 4$



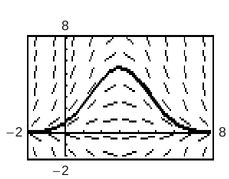
65. $\dfrac{dy}{dx} = 0.02y(10 - y), y(0) = 2$



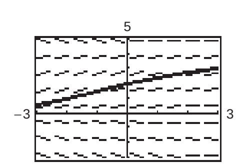
66. $\dfrac{dy}{dx} = 0.2x(2 - y), y(0) = 9$



67. Slope field for $y' = 0.4y(3 - x)$ with solution passing through $(0, 1)$.



68. Slope field for $y' = \dfrac{1}{2}e^{-x/8} \sin \dfrac{\pi y}{4}$ with solution passing through $(0, 2)$.



69. $y' = x + y, \quad y(0) = 2, \quad n = 10, \quad h = 0.1$

$y_1 = y_0 + hF(x_0, y_0) = 2 + (0.1)(0 + 2) = 2.2$

$y_2 = y_1 + hF(x_1, y_1) = 2.2 + (0.1)(0.1 + 2.2) = 2.43$, etc.

n	0	1	2	3	4	5	6	7	8	9	10
x_n	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y_n	2	2.2	2.43	2.693	2.992	3.332	3.715	4.146	4.631	5.174	5.781

70. $y' = x + y, \quad y(0) = 2, \quad n = 20, \quad h = 0.05$

$y_1 = y_0 + hF(x_0, y_0) = 2 + (0.05)(0 + 2) = 2.1$

$y_2 = y_1 + hF(x_1, y_1) = 2.1 + (0.05)(0.05 + 2.1) = 2.2075$, etc.

The table shows the values for $n = 0, 2, 4, \ldots, 20$.

n	0	2	4	6	8	10	12	14	16	18	20
x_n	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y_n	2	2.208	2.447	2.720	3.032	3.387	3.788	4.240	4.749	5.320	5.960

71. $y' = 3x - 2y, \quad y(0) = 3, \quad n = 10, \quad h = 0.05$

$y_1 = y_0 + hF(x_0, y_0) = 3 + (0.05)(3(0) - 2(3)) = 2.7$

$y_2 = y_1 + hF(x_1, y_1) = 2.7 + (0.05)(3(0.05) - 2(2.7)) = 2.4375$, etc.

n	0	1	2	3	4	5	6	7	8	9	10
x_n	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
y_n	3	2.7	2.438	2.209	2.010	1.839	1.693	1.569	1.464	1.378	1.308

72. $y' = 0.5x(3 - y), \quad y(0) = 1, \quad n = 5, \quad h = 0.4$

$y_1 = y_0 + hF(x_0, y_0) = 1 + (0.4)(0.5(0)(3 - 1)) = 1$

$y_2 = y_1 + hF(x_1, y_1) = 1 + (0.4)(0.5(0.4)(3 - 1)) = 1.16$, etc.

n	0	1	2	3	4	5
x_n	0	0.4	0.8	1.2	1.6	2.0
y_n	1	1	1.16	1.454	1.825	2.201

73. $y' = e^{xy}, \quad y(0) = 1, \quad n = 10, \quad h = 0.1$

$y_1 = y_0 + hF(x_0, y_0) = 1 + (0.1)e^{0(1)} = 1.1$

$y_2 = y_1 + hF(x_1, y_1) = 1.1 + (0.1)e^{(0.1)(1.1)} \approx 1.2116$, etc.

n	0	1	2	3	4	5	6	7	8	9	10
x_n	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y_n	1	1.1	1.212	1.339	1.488	1.670	1.900	2.213	2.684	3.540	5.958

74. $y' = \cos x + \sin y, \quad y(0) = 5, \quad n = 10, \quad h = 0.1$

$y_1 = y_0 + hF(x_0, y_0) = 5 + (0.1)(\cos 0 + \sin 5) \approx 5.0041$

$y_2 = y_1 + hF(x_1, y_1) = 5.0041 + (0.1)(\cos(0.1) + \sin(5.0041)) \approx 5.0078$, etc.

n	0	1	2	3	4	5	6	7	8	9	10
x_n	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y_n	5	5.004	5.008	5.010	5.010	5.007	4.999	4.985	4.965	4.938	4.903

75. $\dfrac{dy}{dx} = y,\; y = 3e^x,\; y(0) = 3$

x	0	0.2	0.4	0.6	0.8	1.0
$y(x)$ (exact)	3	3.6642	4.4755	5.4664	6.6766	8.1548
$y(x)$ ($h = 0.2$)	3	3.6000	4.3200	5.1840	6.2208	7.4650
$y(x)$ ($h = 0.1$)	3	3.6300	4.3923	5.3147	6.4308	7.7812

76. $\dfrac{dy}{dx} = \dfrac{2x}{y},\; y = \sqrt{2x^2 + 4},\; y(0) = 2$

x	0	0.2	0.4	0.6	0.8	1.0
$y(x)$ (exact)	2	2.0199	2.0785	2.1726	2.2978	2.4495
$y(x)$ ($h = 0.2$)	2	2.000	2.0400	2.1184	2.2317	2.3751
$y(x)$ ($h = 0.1$)	2	2.0100	2.0595	2.1460	2.2655	2.4131

77. $\dfrac{dy}{dx} = y + \cos x,\; y = \dfrac{1}{2}(\sin x - \cos x + e^x),\; y(0) = 0$

x	0	0.2	0.4	0.6	0.8	1.0
$y(x)$ (exact)	0	0.2200	0.4801	0.7807	0.1231	0.5097
$y(x)$ ($h = 0.2$)	0	0.2000	0.4360	0.7074	0.0140	0.3561
$y(x)$ ($h = 0.1$)	0	0.2095	0.4568	0.7418	0.0649	0.4273

78. As h increases (from 0.1 to 0.2), the error increases.

79. $\dfrac{dy}{dt} = -\dfrac{1}{2}(y - 72),\; y(0) = 140,\; h = 0.1$

(a)

t	0	1	2	3
Euler	140	112.7	96.4	86.6

(b) $y = 72 + 68e^{-t/2}$ exact

t	0	1	2	3
Exact	140	113.24	97.016	87.173

80. $\dfrac{dy}{dt} = -\dfrac{1}{2}(y - 72),\; y(0) = 140,\; h = 0.05$

(a)

t	0	1	2	3
Euler	140	112.98	96.7	86.9

(b) $y = 72 + 68e^{-t/2}$ exact

t	0	1	2	3
Exact	140	113.24	97.016	87.173

The approximations are better using $h = 0.05$.

81. A general solution of order n has n arbitrary constants while in a particular solution initial conditions are given in order to solve for all these constants.

82. A slope field for the differential equation $y' = F(x, y)$ consists of small line segments at various points (x, y) in the plane. The line segment equals the slope $y' = F(x, y)$ of the solution y at the point (x, y).

83. Consider $y' = F(x, y)$, $y(x_0) = y_0$. Begin with a point (x_0, y_0) that satisfies the initial condition, $y(x_0) = y_0$. Then using a step size of h, find the point $(x_1, y_1) = (x_0 + h, y_0 + hF(x_0, y_0))$. Continue generating the sequence of points $(x_{n+1}, y_{n+1}) = (x_n + h, y_n + hF(x_n, y_n))$.

84. $y = Ce^{kx}$

$$\frac{dy}{dx} = Cke^{kx}$$

Since $dy/dx = 0.07y$, we have $Cke^{kx} = 0.07Ce^{kx}$. Thus, $k = 0.07$.

C cannot be determined.

85. False. Consider Example 2. $y = x^3$ is a solution to $xy' - 3y = 0$, but $y = x^3 + 1$ is not a solution.

86. True

87. True

88. False. The slope field could represent many different differential equations, such as $y' = 2x + 4y$.

89. $\dfrac{dy}{dx} = -2y,\ y(0) = 4,\ y = 4e^{-2x}$ solution

(a)

x	0	0.2	0.4	0.6	0.8	1.0
y	4	2.6813	1.7973	1.2048	0.8076	0.5413
y_1	4	2.5600	1.6384	1.0486	0.6711	0.4295
y_2	4	2.4000	1.4400	0.8640	0.5184	0.3110
e_1	0	0.1213	0.1589	0.1562	0.1365	0.1118
e_2	0	0.2813	0.3573	0.3408	0.2892	0.2303
r		0.4312	0.4447	0.4583	0.4720	0.4855

(b) If h is halved, then the error is approximately halved ($r \approx 0.5$).

(c) When $h = 0.05$, the errors will again be approximately halved.

90. $\dfrac{dy}{dx} = x - y,\ y(0) = 1, y = x - 1 + 2e^{-x}$ solution

(a)

x	0	0.2	0.4	0.6	0.8	1.0
y	1	0.8375	0.7406	0.6976	0.6987	0.7358
y_1	1	0.8200	0.7122	0.6629	0.6609	0.6974
y_2	1	0.8000	0.6800	0.6240	0.6192	0.6554
e_1	0	0.0175	0.0284	0.0347	0.0378	0.0384
e_2	0	0.0375	0.0606	0.0736	0.0795	0.0804
r		0.47	0.47	0.47	0.48	0.48

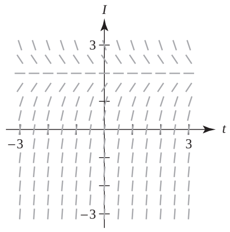
(b) If h is halved, then the error is halved ($r \approx 0.5$).

(c) When $h = 0.05$, the error will again be approximately halved.

91. (a) $L\dfrac{dI}{dt} + RI = E(t)$

$$4\frac{dI}{dt} + 12I = 24$$

$$\frac{dI}{dt} = \frac{1}{4}(24 - 12I) = 6 - 3I$$



(b) As $t \to \infty$, $I \to 2$. That is, $\lim_{t\to\infty} I(t) = 2$. In fact, $I = 2$ is a solution to the differential equation.

92.
$$y = e^{kt}$$
$$y' = ke^{kt}$$
$$y'' = k^2e^{kt}$$
$$y'' - 16y = 0$$
$$k^2e^{kt} - 16e^{kt} = 0$$
$$k^2 - 16 = 0 \qquad \text{(since } e^{kt} \neq 0\text{)}$$
$$k = \pm 4$$

93. $y = A\sin\omega t$

$y' = A\omega\cos\omega t$

$y'' = -A\omega^2\sin\omega t$

$y'' + 16y = 0$

$-A\omega^2\sin\omega t + 16A\sin\omega t = 0$

$A\sin\omega t[16 - \omega^2] = 0$

If $A \neq 0$, then $\omega = \pm 4$ radians/sec.

94.
$$f(x) + f''(x) = -xg(x)f'(x), \qquad g(x) \geq 0$$
$$2f(x)f'(x) + 2f'(x)f''(x) = -2xg(x)[f'(x)]^2$$
$$\frac{d}{dx}[f(x)^2 + f'(x)^2] = -2x\,g(x)[f'(x)]^2$$

For $x < 0$, $-2x\,g(x)[f'(x)]^2 \geq 0$

For $x > 0$, $-2x\,g(x)[f'(x)]^2 \leq 0$

Thus, $f(x)^2 + f'(x)^2$ is increasing for $x < 0$ and decreasing for $x > 0$.

$f(x)^2 + f'(x)^2$ has a maximum at $x = 0$. Thus, it is bounded by its value at $x = 0$, $f(0)^2 + f'(0)^2$. Thus, f (and f') is bounded.

95. Let the vertical line $x = k$ cut the graph of the solution $y = f(x)$ at (k, t).

The tangent line at (k, t) is

$$y - t = f'(k)(x - k)$$

Since $y' + p(x)y = q(x)$, we have

$$y - t = [q(k) - p(k)t](x - k)$$

For any value of t, this line passes through the point $\left(k + \dfrac{1}{p(k)}, \dfrac{q(k)}{p(k)}\right)$.

To see this, note that

$$\frac{q(k)}{p(k)} - t \stackrel{?}{=} [q(k) - p(k)t]\left(k + \frac{1}{p(k)} - k\right)$$
$$\stackrel{?}{=} q(k)k - p(k)tk + \frac{q(k)}{p(k)} - t - kq(k) + p(k)kt$$
$$= \frac{q(k)}{p(k)} - t.$$

Section 6.2 Differential Equations: Growth and Decay

1. $\dfrac{dy}{dx} = x + 2$

$$y = \int (x+2)\,dx = \frac{x^2}{2} + 2x + C$$

2. $\dfrac{dy}{dx} = 4 - x$

$$y = \int (4-x)\,dx = 4x - \frac{x^2}{2} + C$$

3.
$$\begin{aligned}
\frac{dy}{dx} &= y + 2\\
\frac{dy}{y+2} &= dx\\
\int \frac{1}{y+2}\,dy &= \int dx\\
\ln|y+2| &= x + C_1\\
y + 2 &= e^{x+C_1} = Ce^x\\
y &= Ce^x - 2
\end{aligned}$$

4.
$$\begin{aligned}
\frac{dy}{dx} &= 4 - y\\
\frac{dy}{4-y} &= dx\\
\int \frac{-1}{4-y}\,dy &= \int -dx\\
\ln|4-y|\,dy &= -x + C_1\\
4 - y &= e^{-x+C_1} = Ce^{-x}\\
y &= 4 - Ce^{-x}
\end{aligned}$$

5.
$$\begin{aligned}
y' &= \frac{5x}{y}\\
yy' &= 5x\\
\int yy'\,dx &= \int 5x\,dx\\
\int y\,dy &= \int 5x\,dx\\
\frac{1}{2}y^2 &= \frac{5}{2}x^2 + C_1\\
y^2 - 5x^2 &= C
\end{aligned}$$

6.
$$\begin{aligned}
y' &= \frac{\sqrt{x}}{3y}\\
3yy' &= \sqrt{x}\\
\int 3yy'\,dx &= \int \sqrt{x}\,dx\\
\frac{3y^2}{2} &= \frac{2}{3}x^{3/2} + C_1\\
9y^2 - 4x^{3/2} &= C
\end{aligned}$$

7.
$$\begin{aligned}
y' &= \sqrt{x}\,y\\
\frac{y'}{y} &= \sqrt{x}\\
\int \frac{y'}{y}\,dx &= \int \sqrt{x}\,dx\\
\int \frac{dy}{y} &= \int \sqrt{x}\,dx\\
\ln|y| &= \frac{2}{3}x^{3/2} + C_1\\
y &= e^{(2/3)x^{3/2} + C_1}\\
&= e^{C_1}e^{(2/3)x^{3/2}}\\
&= Ce^{(2/3)x^{3/2}}
\end{aligned}$$

8.
$$\begin{aligned}
y' &= x(1+y)\\
\frac{y'}{1+y} &= x\\
\int \frac{y'}{1+y}\,dx &= \int x\,dx\\
\int \frac{dy}{1+y} &= \int x\,dx\\
\ln(1+y) &= \frac{x^2}{2} + C_1\\
1 + y &= e^{(x^2/2)+C_1}\\
y &= e^{C_1}e^{x^2/2} - 1\\
&= Ce^{x^2/2} - 1
\end{aligned}$$

9. $(1 + x^2)y' - 2xy = 0$

$$\begin{aligned}
y' &= \frac{2xy}{1+x^2}\\
\frac{y'}{y} &= \frac{2x}{1+x^2}\\
\int \frac{y'}{y}\,dx &= \int \frac{2x}{1+x^2}\,dx\\
\int \frac{dy}{y} &= \int \frac{2x}{1+x^2}\,dx\\
\ln|y| &= \ln(1+x^2) + C_1\\
\ln|y| &= \ln(1+x^2) + \ln C\\
\ln|y| &= \ln[C(1+x^2)]\\
y &= C(1+x^2)
\end{aligned}$$

10.
$$\begin{aligned}
xy + y' &= 100x \\
y' &= 100x + xy = x(100 - y) \\
\frac{y'}{100 - y} &= x \\
\int \frac{y'}{100 - y}\, dx &= \int x\, dx \\
\int \frac{1}{100 - y}\, dy &= \int x\, dx \\
-\ln(100 - y) &= \frac{x^2}{2} + C_1 \\
\ln(100 - y) &= -\frac{x^2}{2} - C_1 \\
100 - y &= e^{-(x^2/2) - C_1} \\
-y &= e^{-C_1}e^{-x^2/2} - 100 \\
y &= 100 - Ce^{-x^2/2}
\end{aligned}$$

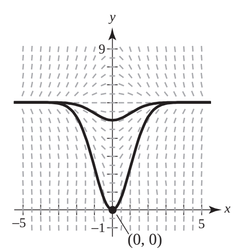
11.
$$\begin{aligned}
\frac{dQ}{dt} &= \frac{k}{t^2} \\
\int \frac{dQ}{dt}\, dt &= \int \frac{k}{t^2}\, dt \\
\int dQ &= -\frac{k}{t} + C \\
Q &= -\frac{k}{t} + C
\end{aligned}$$

12.
$$\begin{aligned}
\frac{dP}{dt} &= k(10 - t) \\
\int \frac{dP}{dt}\, dt &= \int k(10 - t)\, dt \\
\int dP &= -\frac{k}{2}(10 - t)^2 + C \\
P &= -\frac{k}{2}(10 - t)^2 + C
\end{aligned}$$

13.
$$\begin{aligned}
\frac{dN}{ds} &= k(250 - s) \\
\int \frac{dN}{ds}\, ds &= \int k(250 - s)\, ds \\
\int dN &= -\frac{k}{2}(250 - s)^2 + C \\
N &= -\frac{k}{2}(250 - s)^2 + C
\end{aligned}$$

14.
$$\begin{aligned}
\frac{dy}{dx} &= kx(L - y) \\
\frac{1}{L - y}\frac{dy}{dx} &= kx \\
\int \frac{1}{L - y}\frac{dy}{dx}\, dx &= \int kx\, dx \\
\int \frac{1}{L - y}\, dy &= \frac{kx^2}{2} + C_1 \\
-\ln(L - y) &= \frac{kx^2}{2} + C_1 \\
L - y &= e^{-(kx^2/2) - C_1} \\
-y &= -L + e^{-C_1}e^{-kx^2/2} \\
y &= L - Ce^{-kx^2/2}
\end{aligned}$$

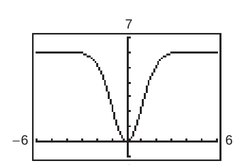
15. (a)



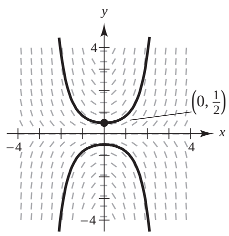
(b) $\dfrac{dy}{dx} = x(6 - y), \quad (0, 0)$

$$\begin{aligned}
\frac{dy}{y - 6} &= -x\, dx \\
\ln|y - 6| &= \frac{-x^2}{2} + C \\
y - 6 &= e^{-x^2/2 + C} = C_1e^{-x^2/2} \\
y &= 6 + C_1e^{-x^2/2}
\end{aligned}$$

$(0, 0)$: $0 = 6 + C_1 \Rightarrow C_1 = -6 \Rightarrow y = 6 - 6e^{-x^2/2}$



16. (a)



(b) $\dfrac{dy}{dx} = xy, \ \left(0, \dfrac{1}{2}\right)$

$$\frac{dy}{y} = x\,dx$$

$$\ln|y| = \frac{x^2}{2} + C$$

$$y = e^{x^2/2+C} = C_1 e^{x^2/2}$$

$$\left(0, \frac{1}{2}\right):\ \frac{1}{2} = C_1 e^0 \implies C_1 = \frac{1}{2} \implies y = \frac{1}{2}e^{x^2/2}$$

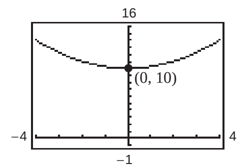
17. $\dfrac{dy}{dt} = \dfrac{1}{2}t, \ (0, 10)$

$$\int dy = \int \frac{1}{2}t\,dt$$

$$y = \frac{1}{4}t^2 + C$$

$$10 = \frac{1}{4}(0)^2 + C \implies C = 10$$

$$y = \frac{1}{4}t^2 + 10$$



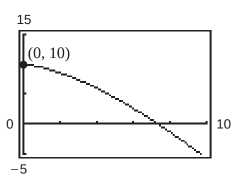
18. $\dfrac{dy}{dt} = -\dfrac{3}{4}\sqrt{t}, \ (0, 10)$

$$\int dy = \int -\frac{3}{4}\sqrt{t}\,dt$$

$$y = -\frac{1}{2}t^{3/2} + C$$

$$10 = -\frac{1}{2}(0)^{3/2} + C \implies C = 10$$

$$y = -\frac{1}{2}t^{3/2} + 10$$



19. $\dfrac{dy}{dt} = -\dfrac{1}{2}y, \ (0, 10)$

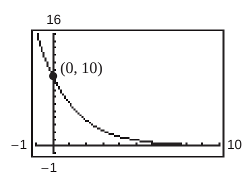
$$\int \frac{dy}{y} = \int -\frac{1}{2}\,dt$$

$$\ln|y| = -\frac{1}{2}t + C_1$$

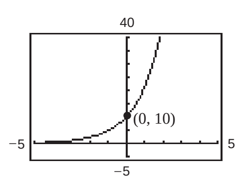
$$y = e^{-(t/2)+C_1} = e^{C_1}e^{-t/2} = Ce^{-t/2}$$

$$10 = Ce^0 \implies C = 10$$

$$y = 10e^{-t/2}$$



20. $\dfrac{dy}{dt} = \dfrac{3}{4}y, \ (0, 10)$



$$\int \frac{dy}{y} = \int \frac{3}{4}\,dt$$

$$\ln y = \frac{3}{4}t + C_1$$

$$y = e^{(3/4)t+C_1}$$

$$= e^{C_1}e^{(3/4)t} = Ce^{3t/4}$$

$$10 = Ce^0 \implies C = 10$$

$$y = 10e^{3t/4}$$

21. $\dfrac{dy}{dx} = ky$

$y = Ce^{kx}$ (Theorem 5.16)

$(0, 4)$: $4 = Ce^0 = C$

$(3, 10)$: $10 = 4e^{3k} \implies k = \dfrac{1}{3}\ln\left(\dfrac{5}{2}\right)$

When $x = 6$, $y = 4e^{1/3\ln(5/2)(6)} = 4e^{\ln(5/2)^2}$

$$= 4\left(\frac{5}{2}\right)^2 = 25.$$

22. $\dfrac{dN}{dt} = kN$

$N = Ce^{kt}$ (Theorem 5.16)

$(0, 250)$: $C = 250$

$(1, 400)$: $400 = 250e^{k} \implies k = \ln \dfrac{400}{250} = \ln \dfrac{8}{5}$

When $t = 4$, $N = 250e^{4\ln(8/5)} = 250e^{\ln(8/5)^4}$

$$= 250\left(\frac{8}{5}\right)^4 = \frac{8192}{5}.$$

23. $\dfrac{dV}{dt} = kV$

$V = Ce^{kt}$ (Theorem 5.16)

$(0, 20{,}000)$: $C = 20{,}000$

$(4, 12{,}500)$: $12{,}500 = 20{,}000e^{4k} \implies k = \dfrac{1}{4}\ln\left(\dfrac{5}{8}\right)$

When $t = 6$, $V = 20{,}000e^{1/4\ln(5/8)(6)} = 20{,}000e^{\ln(5/8)^{3/2}}$

$$= 20{,}000\left(\frac{5}{8}\right)^{3/2} \approx 9882.118.$$

24. $\dfrac{dP}{dt} = kP$

$P = Ce^{kt}$ (Theorem 5.16)

$(0, 5000)$: $C = 5000$

$(1, 4750)$: $4750 = 5000e^{k} \implies k = \ln\left(\dfrac{19}{20}\right)$

When $t = 5$, $P = 5000e^{\ln(19/20)(5)}$

$$= 5000\left(\frac{19}{20}\right)^5 \approx 3868.905.$$

25. $y = Ce^{kt}$, $\left(0, \dfrac{1}{2}\right)$, $(5, 5)$

$$C = \frac{1}{2}$$

$$y = \frac{1}{2}e^{kt}$$

$$5 = \frac{1}{2}e^{5k}$$

$$k = \frac{\ln 10}{5}$$

$$y = \frac{1}{2}e^{(\ln 10/5)t} = \frac{1}{2}(10^{t/5}) \text{ or } y \approx \frac{1}{2}e^{0.4605t}$$

26. $y = Ce^{kt}$, $(0, 4)$, $\left(5, \dfrac{1}{2}\right)$

$$C = 4$$

$$y = 4e^{kt}$$

$$\frac{1}{2} = 4e^{5k}$$

$$k = \frac{\ln(1/8)}{5} \approx -0.4159$$

$$y = 4e^{-0.4159t}$$

27. $y = Ce^{kt}$, $(1, 1)$, $(5, 5)$

$$\begin{aligned}
1 &= Ce^{k} \\
5 &= Ce^{5k} \\
5Ce^{k} &= Ce^{5k} \\
5e^{k} &= e^{5k} \\
5 &= e^{4k} \\
k &= \frac{\ln 5}{4} \approx 0.4024 \\
y &= Ce^{0.4024t} \\
1 &= Ce^{0.4024} \\
C &\approx 0.6687 \quad (C = 5^{-1/4}) \\
y &\approx 0.6687e^{0.4024t}
\end{aligned}$$

28. $y = Ce^{kt}$, $\left(3, \dfrac{1}{2}\right)$, $(4, 5)$

$$\begin{aligned}
\frac{1}{2} &= Ce^{3k} \\
5 &= Ce^{4k} \\
2Ce^{3k} &= \frac{1}{5}Ce^{4k} \\
10e^{3k} &= e^{4k} \\
10 &= e^{k} \\
k &= \ln 10 \approx 2.3026 \\
y &= Ce^{2.3026t} \\
5 &= Ce^{2.3026(4)} \\
C &\approx 0.0005 \\
y &= 0.0005e^{2.3026t}
\end{aligned}$$

29. In the model $y = Ce^{kt}$, C represents the initial value of y (when $t = 0$). k is the proportionality constant.

30. $y' = \dfrac{dy}{dt} = ky$

31. $\dfrac{dy}{dx} = \dfrac{1}{2}xy$

$\dfrac{dy}{dx} > 0$ when $xy > 0$. Quadrants I and III.

32. $\dfrac{dy}{dx} = \dfrac{1}{2}x^2y$

$\dfrac{dy}{dx} > 0$ when $y > 0$. Quadrants I and II.

33. Since the initial quantity is 10 grams,

$$y = 10e^{kt}.$$

Since the half-life is 1599 years,

$$5 = 10e^{k(1599)}$$

$$k = \frac{1}{1599}\ln\left(\frac{1}{2}\right).$$

Thus, $y = 10e^{[\ln(1/2)/1599]t}$.

When $t = 1000$, $y = 10e^{[\ln(1/2)/1599](1000)} \approx 6.48$ g.

When $t = 10{,}000$, $y \approx 0.13$ g.

34. Since the half-life is 1599 years,

$$\frac{1}{2} = 1e^{k(1599)}$$

$$k = \frac{1}{1599}\ln\left(\frac{1}{2}\right).$$

Since there are 1.5 g after 1000 years,

$$1.5 = Ce^{[\ln(1/2)/1599](1000)}$$

$$C \approx 2.314.$$

Hence, the initial quantity is approximately 2.314 g.

When $t = 10{,}000$, $y = 2.314e^{[\ln(1/2)/1599](10{,}000)}$

≈ 0.03 g.

35. Since the half-life is 1599 years,

$$\frac{1}{2} = 1e^{k(1599)}$$

$$k = \frac{1}{1599}\ln\left(\frac{1}{2}\right).$$

Since there are 0.5 gram after 10,000 years,

$$0.5 = Ce^{[\ln(1/2)/1599](10{,}000)}$$

$$C \approx 38.158.$$

Hence, the initial quantity is approximately 38.158 g.

When $t = 1000$, $y = 38.158e^{[\ln(1/2)/1599](1000)}$

≈ 24.74 g.

36. Since the half-life is 5715 years,

$$\frac{1}{2} = 1e^{k(5715)}$$

$$k = \frac{1}{5715}\ln\left(\frac{1}{2}\right).$$

Since there are 2 grams after 10,000 years,

$$2 = Ce^{[\ln(1/2)/5715](10{,}000)}$$

$$C \approx 6.726.$$

Hence, the initial quantity is approximately 6.726 g.

When $t = 1000$, $y = 6.726e^{[\ln(1/2)/5715](1000)}$

≈ 5.96 g.

37. Since the initial quantity is 5 grams, $C = 5$.

Since the half-life is 5715 years,

$$2.5 = 5e^{k(5715)}$$

$$k = \frac{1}{5715}\ln\left(\frac{1}{2}\right).$$

When $t = 1000$ years, $y = 5e^{[\ln(1/2)/5715](1000)}$

≈ 4.43 g.

When $t = 10{,}000$ years, $y = 5e^{[\ln(1/2)/5715](10{,}000)}$

≈ 1.49 g.

38. Since the half-life is 5715 years,

$$\frac{1}{2} = 1e^{k(5715)}$$

$$k = \frac{1}{5715}\ln\left(\frac{1}{2}\right).$$

Since there are 3.2 grams when $t = 1000$ years,

$$3.2 = Ce^{[\ln(1/2)/5715](1000)}$$

$$C \approx 3.613.$$

Thus, the initial quantity is approximately 3.613 g.

When $t = 10{,}000$, $y = 3.613e^{[\ln(1/2)/5715](10{,}000)}$

≈ 1.07 g.

39. Since the half-life is 24,100 years,

$$\frac{1}{2} = 1e^{k(24,100)}$$

$$k = \frac{1}{24,100}\ln\left(\frac{1}{2}\right).$$

Since there are 2.1 grams after 1000 years,

$$2.1 = Ce^{[\ln(1/2)/24,100](1000)}$$

$$C \approx 2.161.$$

Thus, the initial quantity is approximately 2.161 g.

When $t = 10,000$, $y = 2.161e^{[\ln(1/2)/24,100](10,000)}$

$$\approx 1.62 \text{ g.}$$

40. Since the half-life is 24,100 years,

$$\frac{1}{2} = 1e^{k(24,100)}$$

$$k = \frac{1}{24,100}\ln\left(\frac{1}{2}\right).$$

Since there are 0.4 grams after 10,000 years,

$$0.4 = Ce^{[\ln(1/2)/24,100](10,000)}$$

$$C \approx 0.533.$$

Thus, the initial quantity is approximately 0.533 g.

When $t = 1000$, $y = 0.533e^{[\ln(1/2)/24,100](1000)}$

$$\approx 0.52 \text{ g.}$$

41. $y = Ce^{kt}$

$$\frac{1}{2}C = Ce^{k(1599)}$$

$$k = \frac{1}{1599}\ln\left(\frac{1}{2}\right)$$

When $t = 100$, $y = Ce^{[\ln(1/2)/1599](100)}$

$$\approx 0.9576\,C$$

Therefore, 95.76% remains after 100 years.

42.

$$y = Ce^{kt}$$

$$\frac{1}{2}C = Ce^{k(5715)}$$

$$k = \frac{1}{5715}\ln\left(\frac{1}{2}\right)$$

$$0.15C = Ce^{[\ln(1/2)/5715]t}$$

$$\ln(0.15) = \frac{\ln\left(\frac{1}{2}\right)t}{5715}$$

$$t \approx 15,641.8 \text{ years}$$

43. Since $A = 1000e^{0.06t}$, the time to double is given by $2000 = 1000e^{0.06t}$ and we have

$$2 = e^{0.06t}$$

$$\ln 2 = 0.06t$$

$$t = \frac{\ln 2}{0.06} \approx 11.55 \text{ years.}$$

Amount after 10 years: $A = 1000e^{(0.06)(10)} \approx \$1822.12$

44. Since $A = 20,000e^{0.055t}$, the time to double is given by $40,000 = 20,000e^{0.055t}$ and we have

$$2 = e^{0.055t}$$

$$\ln 2 = 0.055t$$

$$t = \frac{\ln 2}{0.055} \approx 12.6 \text{ years.}$$

Amount after 10 years: $A = 20,000e^{(0.055)(10)} \approx \$34,665.06$

45. Since $A = 750e^{rt}$ and $A = 1500$ when $t = 7.75$, we have the following.

$$1500 = 750e^{7.75r}$$

$$r = \frac{\ln 2}{7.75} \approx 0.0894 = 8.94\%$$

Amount after 10 years: $A = 750e^{0.0894(10)} \approx \$1833.67$

46. Since $A = 10,000e^{rt}$ and $A = 20,000$ when $t = 5$, we have the following.

$$20,000 = 10,000e^{5r}$$

$$r = \frac{\ln 2}{5} \approx 0.1386 = 13.86\%$$

Amount after 10 years: $A = 10,000e^{[(\ln 2)/5](10)} = \$40,000$

47. Since $A = 500e^{rt}$ and $A = 1292.85$ when $t = 10$, we have the following.

$$1292.85 = 500e^{10r}$$
$$r = \frac{\ln(1292.85/500)}{10} \approx 0.0950 = 9.50\%$$

The time to double is given by

$$1000 = 500e^{0.0950t}$$
$$t = \frac{\ln 2}{0.095} \approx 7.30 \text{ years.}$$

48. Since $A = 2000e^{rt}$ and $A = 5436.56$ when $t = 10$, we have the following.

$$5436.56 = 2000e^{10r}$$
$$r = \frac{\ln(5436.56/2000)}{10} \approx 0.10 = 10\%$$

The time to double is given by

$$4000 = 2000e^{0.10t}$$
$$t = \frac{\ln 2}{0.10} \approx 6.93 \text{ years.}$$

49. $500{,}000 = P\left(1 + \frac{0.075}{12}\right)^{(12)(20)}$

$$P = 500{,}000\left(1 + \frac{0.075}{12}\right)^{-240}$$
$$\approx \$112{,}087.09$$

50. $500{,}000 = P\left(1 + \frac{0.06}{12}\right)^{(12)(40)}$

$$P = 500{,}000(1.005)^{-480} \approx \$45{,}631.04$$

51. $500{,}000 = P\left(1 + \frac{0.08}{12}\right)^{(12)(35)}$

$$P = 500{,}000\left(1 + \frac{0.08}{12}\right)^{-420}$$
$$= \$30{,}688.87$$

52. $500{,}000 = P\left(1 + \frac{0.09}{12}\right)^{(12)(25)}$

$$P = 500{,}000\left(1 + \frac{0.09}{12}\right)^{-300}$$
$$\approx \$53{,}143.92$$

53. (a) $2000 = 1000(1 + 0.07)^t$

$$2 = 1.07^t$$
$$\ln 2 = t \ln 1.07$$
$$t = \frac{\ln 2}{\ln 1.07} \approx 10.24 \text{ years}$$

(b) $2000 = 1000\left(1 + \frac{0.07}{12}\right)^{12t}$

$$2 = \left(1 + \frac{0.007}{12}\right)^{12t}$$
$$\ln 2 = 12t \ln\left(1 + \frac{0.07}{12}\right)$$
$$t = \frac{\ln 2}{12 \ln(1 + (0.07/12))} \approx 9.93 \text{ years}$$

(c) $2000 = 1000\left(1 + \frac{0.07}{365}\right)^{365t}$

$$2 = \left(1 + \frac{0.07}{365}\right)^{365t}$$
$$\ln 2 = 365t \ln\left(1 + \frac{0.07}{365}\right)$$
$$t = \frac{\ln 2}{365 \ln(1 + (0.07/365))} \approx 9.90 \text{ years}$$

(d) $2000 = 1000e^{(0.07)t}$

$$2 = e^{0.07t}$$
$$\ln 2 = 0.07t$$
$$t = \frac{\ln 2}{0.07} \approx 9.90 \text{ years}$$

54. (a)
$$\begin{aligned} 2000 &= 1000(1 + 0.6)^t \\ 2 &= 1.06^t \\ \ln 2 &= t \ln 1.06 \\ t &= \frac{\ln 2}{\ln 1.06} \approx 11.90 \text{ years} \end{aligned}$$

(b)
$$\begin{aligned} 2000 &= 1000\left(1 + \frac{0.06}{12}\right)^{12t} \\ 2 &= \left(1 + \frac{0.06}{12}\right)^{12t} \\ \ln 2 &= 12t \ln\left(1 + \frac{0.06}{12}\right) \\ t &= \frac{1}{12}\frac{\ln 2}{\ln\left(1 + \frac{0.06}{12}\right)} \approx 11.58 \text{ years} \end{aligned}$$

(c)
$$\begin{aligned} 2000 &= 1000\left(1 + \frac{0.06}{365}\right)^{365t} \\ 2 &= \left(1 + \frac{0.06}{365}\right)^{365t} \\ \ln 2 &= 365t \ln\left(1 + \frac{0.06}{365}\right) \\ t &= \frac{1}{365}\frac{\ln 2}{\ln\left(1 + \frac{0.06}{365}\right)} \approx 11.55 \text{ years} \end{aligned}$$

(d)
$$\begin{aligned} 2000 &= 1000e^{0.06t} \\ 2 &= e^{0.06t} \\ \ln 2 &= 0.06t \\ t &= \frac{\ln 2}{0.06} \approx 11.55 \text{ years} \end{aligned}$$

55. (a)
$$\begin{aligned} 2000 &= 1000(1 + 0.085)^t \\ 2 &= 1.085^t \\ \ln 2 &= t \ln 1.085 \\ t &= \frac{\ln 2}{\ln 1.085} \approx 8.50 \text{ years} \end{aligned}$$

(b)
$$\begin{aligned} 2000 &= 1000\left(1 + \frac{0.085}{12}\right)^{12t} \\ 2 &= \left(1 + \frac{0.085}{12}\right)^{12t} \\ \ln 2 &= 12t \ln\left(1 + \frac{0.085}{12}\right) \\ t &= \frac{1}{12}\frac{\ln 2}{\ln\left(1 + \frac{0.085}{12}\right)} \approx 8.18 \text{ years} \end{aligned}$$

(c)
$$\begin{aligned} 2000 &= 1000\left(1 + \frac{0.085}{365}\right)^{365t} \\ 2 &= \left(1 + \frac{0.085}{365}\right)^{365t} \\ \ln 2 &= 365t \ln\left(1 + \frac{0.085}{365}\right) \\ t &= \frac{1}{365}\frac{\ln 2}{\ln\left(1 + \frac{0.085}{365}\right)} \approx 8.16 \text{ years} \end{aligned}$$

(d)
$$\begin{aligned} 2000 &= 1000e^{0.085t} \\ 2 &= e^{0.085t} \\ \ln 2 &= 0.085t \\ t &= \frac{\ln 2}{0.085} \approx 8.15 \text{ years} \end{aligned}$$

56. (a)
$$\begin{aligned} 2000 &= 1000(1 + 0.055)^t \\ 2 &= 1.055^t \\ \ln 2 &= t \ln 1.055 \\ t &= \frac{\ln 2}{\ln 1.055} \approx 12.95 \text{ years} \end{aligned}$$

(b)
$$\begin{aligned} 2000 &= 1000\left(1 + \frac{0.055}{12}\right)^{12t} \\ 2 &= \left(1 + \frac{0.055}{12}\right)^{12t} \\ \ln 2 &= 12t \ln\left(1 + \frac{0.055}{12}\right) \\ t &= \frac{1}{12}\frac{\ln 2}{\ln\left(1 + \frac{0.055}{12}\right)} \approx 12.63 \text{ years} \end{aligned}$$

(c)
$$\begin{aligned} 2000 &= 1000\left(1 + \frac{0.055}{365}\right)^{365t} \\ 2 &= \left(1 + \frac{0.055}{365}\right)^{365t} \\ \ln 2 &= 365t \ln\left(1 + \frac{0.055}{365}\right) \\ t &= \frac{1}{365}\frac{\ln 2}{\ln\left(1 + \frac{0.055}{365}\right)} \approx 12.60 \text{ years} \end{aligned}$$

(d)
$$\begin{aligned} 2000 &= 1000e^{0.055t} \\ 2 &= e^{0.055t} \\ \ln 2 &= 0.055t \\ t &= \frac{\ln 2}{0.055} \approx 12.60 \text{ years} \end{aligned}$$

57. (a) $P = Ce^{kt} = Ce^{-0.009t}$

$P(1) = 7.7 = Ce^{-0.009(1)}$

$C \approx 7.7696$

$P = 7.7696e^{-0.009t}$

(b) For $t = 15$, $P = 7.7696e^{-0.009(15)} \approx 6.79$ million.

(c) If $k < 0$, the population is decreasing.

58. (a) $P = Ce^{kt} = Ce^{0.018t}$

$P(1) = 12.7 = Ce^{0.018(1)}$

$C \approx 12.4734$

$P = 12.4734e^{0.018t}$

(b) For $t = 15$, $P = 12.4734e^{0.018(15)} \approx 16.34$ million.

(c) For $k > 0$, the population is increasing.

59. (a) $P = Ce^{kt} = Ce^{0.026t}$

$5.2 = P(1) = Ce^{0.026(1)}$

$C \approx 5.0665$

$P = 5.0665e^{0.026t}$

(b) For $t = 15$, $P = 5.0665e^{0.026(15)} \approx 7.48$ million.

(c) For $k > 0$, the population is increasing.

60. (a) $P = Ce^{kt} = Ce^{-0.002t}$

$P(1) = 3.6 = Ce^{-0.002(1)}$

$C \approx 3.6072$

$P = 3.6072e^{-0.002t}$

(b) For $t = 15$, $P = 3.6072e^{-0.002(15)} \approx 3.5$ million.

(c) For $k < 0$, the population is decreasing.

61. (a) $N = 100.1596(1.2455)^t$

(b) $N = 400$ when $t = 6.3$ hours (graphing utility)

Analytically,

$$\begin{aligned} 400 &= 100.1596(1.2455)^t \\ 1.2455^t &= \frac{400}{100.1596} = 3.9936 \\ t \ln 1.2455 &= \ln 3.9936 \\ t &= \frac{\ln 3.9936}{\ln 1.2455} \approx 6.3 \text{ hours.} \end{aligned}$$

62. (a) Let $y = Ce^{kt}$.

At time 2: $125 = Ce^{k(2)} \implies C = 125e^{-2k}$

At time 4: $350 = Ce^{k(4)} \implies 350 = (125e^{-2k})(e^{4k}) \implies \frac{14}{5} = e^{2k}$

$$\begin{aligned} 2k &= \ln \tfrac{14}{5} \\ k &= \tfrac{1}{2} \ln \tfrac{14}{5} \approx 0.5148 \end{aligned}$$

$C = 125e^{-2k} = 125e^{-2(1/2)\ln(14/5)} = 125\left(\frac{5}{14}\right) = \frac{625}{14} \approx 44.64$

Approximately 45 bacteria at time 0.

(b) $y = \frac{625}{14}e^{(1/2)\ln(14/5)t} \approx 44.64e^{0.5148t}$

(c) When $t = 8$, $y = \frac{625}{14}e^{(1/2)\ln(14/5)8} = \frac{625}{14}\left(\frac{14}{5}\right)^4 = 2744.$

(d) $25{,}000 = \frac{625}{14}e^{(1/2)\ln(14/5)t} \implies t \approx 12.29$ hours

63. (a) $19 = 30(1 - e^{20k})$

$30e^{20k} = 11$

$k = \dfrac{\ln(11/30)}{20} \approx -0.0502$

$N \approx 30(1 - e^{-0.0502t})$

(b) $25 = 30(1 - e^{-0.0502t})$

$e^{-0.0502t} = \dfrac{1}{6}$

$t = \dfrac{-\ln 6}{-0.0502} \approx 36$ days

64. (a) $20 = 30(1 - e^{30k})$

$30e^{30k} = 10$

$k = \dfrac{\ln(1/3)}{30} = \dfrac{-\ln 3}{30} \approx -0.0366$

$N \approx 30(1 - e^{-0.0366t})$

(b) $25 = 30(1 - e^{-0.0366t})$

$e^{-0.0366t} = \dfrac{1}{6}$

$t = \dfrac{-\ln 6}{-0.0366} \approx 49$ days

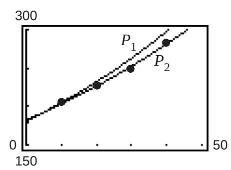
65. (a) $P_1 = Ce^{kt} = 181e^{kt}$

$205 = 181e^{10k} \implies k = \dfrac{1}{10}\ln\left(\dfrac{205}{181}\right) \approx 0.01245$

$P_1 \approx 181e^{0.01245t} \approx 181(1.01253)^t$

(b) Using a graphing utility, $P_2 \approx 182.3248(1.01091)^t$

(c)



The model P_2 fits the data better.

(d) Using the model P_2,

$320 = 182.3248(1.01091)^t$

$\dfrac{320}{182.3248} = (1.01091)^t$

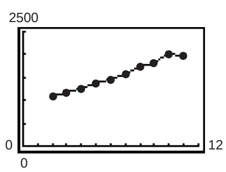
$t = \dfrac{\ln(320/182.3248)}{\ln(1.01091)} \approx 51.8$ years, or 2011.

66. (a) $R = 941.6088(1.0756)^t$

$= 941.6088e^{0.0729t}$

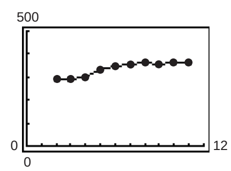
$I = 0.14164t^4 - 3.9288t^3 + 36.599t^2 - 120.82t + 417.0$

(b)

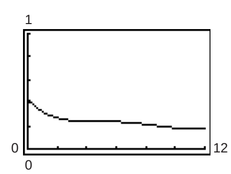


According to the model, $R'(t) \approx 68.6e^{0.0729t}$.

(c)



(d) $P(t) = \dfrac{I}{R}$



67. $\beta(I) = 10 \log_{10} \dfrac{I}{I_0}, I_0 = 10^{-16}$

(a) $\beta(10^{-14}) = 10 \log_{10} \dfrac{10^{-14}}{10^{-16}} = 20$ decibels

(b) $\beta(10^{-9}) = 10 \log_{10} \dfrac{10^{-9}}{10^{-16}} = 70$ decibels

(c) $\beta(10^{-6.5}) = 10 \log_{10} \dfrac{10^{-6.5}}{10^{-16}} = 95$ decibels

(d) $\beta(10^{-4}) = 10 \log_{10} \dfrac{10^{-4}}{10^{-16}} = 120$ decibels

68. $93 = 10 \log_{10} \dfrac{I}{10^{-16}} = 10(\log_{10} I + 16)$

$-6.7 = \log_{10} I \implies I = 10^{-6.7}$

$80 = 10 \log_{10} \dfrac{I}{10^{-16}} = 10(\log_{10} I + 16)$

$-8 = \log_{10} I \implies I = 10^{-8}$

Percentage decrease: $\left(\dfrac{10^{-6.7} - 10^{-8}}{10^{-6.7}}\right)(100) \approx 95\%$

69. $A(t) = V(t)e^{-0.10t} = 100{,}000e^{0.8\sqrt{t}}e^{-0.10t} = 100{,}000e^{0.8\sqrt{t}-0.10t}$

$$\frac{dA}{dt} = 100{,}000\left(\frac{0.4}{\sqrt{t}} - 0.10\right)e^{0.8\sqrt{t}-0.10t} = 0 \text{ when } 16.$$

The timber should be harvested in the year 2014, $(1998 + 16)$. **Note:** You could also use a graphing utility to graph $A(t)$ and find the maximum of $A(t)$. Use the viewing rectangle $0 \le x \le 30$ and $0 \le y \le 600{,}000$.

70. $R = \dfrac{\ln I - \ln I_0}{\ln 10} = \dfrac{\ln I - 0}{\ln 10}, I = e^{R \ln 10} = 10^R$

(a) $8.3 = \dfrac{\ln I - \ln I_0}{\ln 10}$

$I = 10^{8.3} \approx 199{,}526{,}231.5$

(b) $2R = \dfrac{\ln I - \ln I_0}{\ln 10}$

$$\begin{aligned} I &= e^{2R \ln 10} \\ &= e^{2R \ln 10} \\ &= (e^{R \ln 10})^2 \\ &= (10^R)^2 \end{aligned}$$

Increases by a factor of $e^{2R \ln 10}$ or 10^R.

(c) $R = \dfrac{\ln I - \ln I_0}{\ln 10}$

$$\frac{dR}{dI} = \frac{1}{I \ln(10)}$$

71. Since $\dfrac{dy}{dt} = k(y - 80)$

$$\int \frac{1}{y - 80}\,dy = \int k\,dt$$

$\ln(y - 80) = kt + C.$

When $t = 0$, $y = 1500$. Thus, $C = \ln 1420$.

When $t = 1$, $y = 1120$. Thus,

$$k(1) + \ln 1420 = \ln(1120 - 80)$$

$$k = \ln 1040 - \ln 1420 = \ln\frac{104}{142}.$$

Thus, $y = 1420e^{[\ln(104/142)]t} + 80$.

When $t = 5$, $y \approx 379.2°$.

72. $\dfrac{dy}{dt} = k(y - 20)$

$y = 20 + Ce^{kt}$ (See Example 6)

$160 = 20 + Ce^{k(0)} \implies C = 140$

$60 = 20 + 140e^{k(5)}$

$$\frac{2}{7} = e^{5k}$$

$$k = \frac{1}{5}\ln\left(\frac{2}{7}\right) \approx -0.25055$$

$30 = 20 + 140e^{(1/5)\ln(2/7)t}$

$$\frac{1}{14} = e^{\ln(2/7)^{t/s}} = \left(\frac{2}{7}\right)^{t/s}$$

$$\ln\frac{1}{14} = \frac{t}{5}\ln\frac{2}{7}$$

$$t = \frac{5\ln\frac{1}{14}}{\ln\frac{2}{7}} = \frac{5\ln 14}{\ln\frac{7}{2}} \approx 10.53 \text{ minutes}$$

It will take $10.53 - 5 = 5.53$ minutes longer.

73. False. If $y = Ce^{kt}$, $y' = Cke^{kt} \neq$ constant.

74. True

75. True

76. True

Section 6.3 Separation of Variables and the Logistic Equation

1. $\dfrac{dy}{dx} = \dfrac{x}{y}$

$$\int y\,dy = \int x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C_1$$

$$y^2 - x^2 = C$$

2. $\dfrac{dy}{dx} = \dfrac{x^2 + 2}{3y^2}$

$$\int 3y^2\,dy = \int (x^2 + 2)\,dx$$

$$y^3 = \frac{x^3}{3} + 2x + C$$

3. $\dfrac{dr}{ds} = 0.05r$

$$\int \frac{dr}{r} = \int 0.05\,ds$$

$$\ln|r| = 0.05s + C_1$$

$$r = e^{0.05s + C_1} = Ce^{0.05s}$$

4. $\dfrac{dr}{ds} = 0.05s$

$$\int dr = \int 0.05s\,ds$$

$$r = 0.025s^2 + C$$

5. $(2 + x)y' = 3y$

$$\int \frac{dy}{y} = \int \frac{3}{2 + x}\,dx$$

$$\ln|y| = 3\ln|2 + x| + \ln C = \ln|C(2 + x)^3|$$

$$y = C(x + 2)^3$$

6. $xy' = y$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln y = \ln x + \ln C = \ln Cx$$

$$y = Cx$$

7. $yy' = \sin x$

$$\int y\,dy = \int \sin x\,dx$$

$$\frac{y^2}{2} = -\cos x + C_1$$

$$y^2 = -2\cos x + C$$

8. $y\dfrac{dy}{dx} = 6\cos \pi x$

$$\int y\,dy = \int 6\cos \pi x\,dx$$

$$\frac{y^2}{2} = \frac{6}{\pi}\sin \pi x + C_1$$

$$y^2 = \frac{12}{\pi}\sin \pi x + C$$

9. $\sqrt{1 - 4x^2}\,y' = x$

$$dy = \frac{x}{\sqrt{1 - 4x^2}}\,dx$$

$$\int dy = \int \frac{x}{\sqrt{1 - 4x^2}}\,dx$$

$$= -\frac{1}{8}\int (1 - 4x^2)^{-1/2}(-8x\,dx)$$

$$y = -\frac{1}{4}(1 - 4x^2)^{1/2} + C$$

10. $\sqrt{x^2 - 9}\,\dfrac{dy}{dx} = 5x$

$$\int dy = \int \frac{5x}{\sqrt{x^2 - 9}}\,dx$$

$$y = 5(x^2 - 9)^{1/2} + C$$

11. $y \ln x - xy' = 0$

$$\int \frac{dy}{y} = \int \frac{\ln x}{x}\,dx \quad \left(u = \ln x,\ du = \frac{dx}{x}\right)$$

$$\ln|y| = \frac{1}{2}(\ln x)^2 + C_1$$

$$y = e^{(1/2)(\ln x)^2 + C_1} = Ce^{(\ln x)^2/2}$$

12. $4y\,\dfrac{dy}{dx} = 3e^x$

$$\int 4y\,dy = \int 3e^x\,dx$$

$$2y^2 = 3e^x + C$$

13. $yy' - e^x = 0$

$$\int y\,dy = \int e^x\,dx$$

$$\frac{y^2}{2} = e^x + C_1$$

$$y^2 = 2e^x + C$$

Initial condition: $y(0) = 4,\ 16 = 2 + C,\ C = 14$

Particular solution: $y^2 = 2e^x + 14$

14. $\sqrt{x} + \sqrt{y}\,y' = 0$

$$\int y^{1/2}\,dy = -\int x^{1/2}\,dx$$

$$\frac{2}{3}y^{3/2} = -\frac{2}{3}x^{3/2} + C_1$$

$$y^{3/2} + x^{3/2} = C$$

Initial condition: $y(1) = 4,$

$(4)^{3/2} + (1)^{3/2} = 8 + 1 = 9 = C$

Particular solution: $y^{3/2} + x^{3/2} = 9$

15. $y(x + 1) + y' = 0$

$$\int \frac{dy}{y} = -\int (x + 1)\,dx$$

$$\ln|y| = -\frac{(x + 1)^2}{2} + C_1$$

$$y = Ce^{-(x+1)^2/2}$$

Initial condition: $y(-2) = 1,\ 1 = Ce^{-1/2},\ C = e^{1/2}$

Particular solution: $y = e^{[1-(x+1)^2]/2} = e^{-(x^2+2x)/2}$

16. $2xy' - \ln x^2 = 0$

$$2x\,\frac{dy}{dx} = 2\ln x$$

$$\int dy = \int \frac{\ln x}{x}\,dx$$

$$y = \frac{(\ln x)^2}{2} + C$$

$y(1) = 2:\ 2 = C$

$$y = \frac{1}{2}(\ln x)^2 + 2$$

17. $y(1 + x^2)y' = x(1 + y^2)$

$$\frac{y}{1 + y^2}\,dy = \frac{x}{1 + x^2}\,dx$$

$$\frac{1}{2}\ln(1 + y^2) = \frac{1}{2}\ln(1 + x^2) + C_1$$

$$\ln(1 + y^2) = \ln(1 + x^2) + \ln C = \ln[C(1 + x^2)]$$

$$1 + y^2 = C(1 + x^2)$$

$y(0) = \sqrt{3}:\ 1 + 3 = C \implies C = 4$

$$1 + y^2 = 4(1 + x^2)$$

$$y^2 = 3 + 4x^2$$

18. $y\sqrt{1 - x^2}\,\dfrac{dy}{dx} = x\sqrt{1 - y^2}$

$$\int (1 - y^2)^{-1/2}y\,dy = \int (1 - x^2)^{-1/2}x\,dx$$

$$-(1 - y^2)^{1/2} = -(1 - x^2)^{1/2} + C$$

$y(0) = 1:\quad 0 = -1 + C \implies C = 1$

$$\sqrt{1 - y^2} = \sqrt{1 - x^2} - 1$$

19. $\dfrac{du}{dv} = uv \sin v^2$

$$\int \frac{du}{u} = \int v \sin v^2\,dv$$

$$\ln|u| = -\frac{1}{2}\cos v^2 + C_1$$

$$u = Ce^{-(\cos v^2)/2}$$

Initial condition: $u(0) = 1,\ C = \dfrac{1}{e^{-1/2}} = e^{1/2}$

Particular solution: $u = e^{(1-\cos v^2)/2}$

20. $\dfrac{dr}{ds} = e^{r-2s}$

$$\int e^{-r}dr = \int e^{-2s}\,ds$$

$$-e^{-r} = -\frac{1}{2}e^{-2s} + C$$

$$r(0) = 0\colon\ -1 = -\frac{1}{2} + C \Rightarrow C = -\frac{1}{2}$$

$$-e^{-r} = -\frac{1}{2}e^{-2s} - \frac{1}{2}$$

$$e^{-r} = \frac{1}{2}e^{-2s} + \frac{1}{2}$$

$$-r = \ln\left(\frac{1}{2}e^{-2s} + \frac{1}{2}\right) = \ln\left(\frac{1 + e^{-2s}}{2}\right)$$

$$r = \ln\left(\frac{2}{1 + e^{-2s}}\right)$$

21. $dP - kP\,dt = 0$

$$\int \frac{dP}{P} = k\int dt$$

$$\ln|P| = kt + C_1$$

$$P = Ce^{kt}$$

Initial condition: $P(0) = P_0,\ P_0 = Ce^0 = C$

Particular solution: $P = P_0e^{kt}$

22. $dT + k(T - 70)\,dt = 0$

$$\int \frac{dT}{T - 70} = -k\int dt$$

$$\ln(T - 70) = -kt + C_1$$

$$T - 70 = Ce^{-kt}$$

Initial condition: $T(0) = 140;$
$140 - 70 = 70 = Ce^0 = C$

Particular solution: $T - 70 = 70e^{-kt},\ T = 70(1 + e^{-kt})$

23. $\dfrac{dy}{dx} = \dfrac{-9x}{16y}$

$$\int 16y\,dy = -\int 9x\,dx$$

$$8y^2 = \frac{-9}{2}x^2 + C$$

Initial condition: $y(1) = 1,\ 8 = -\dfrac{9}{2} + C,\ C = \dfrac{25}{2}$

Particular solution: $8y^2 = \dfrac{-9}{2}x^2 + \dfrac{25}{2}$

$$16y^2 + 9x^2 = 25$$

24. $\dfrac{dy}{dx} = \dfrac{2y}{3x}$

$$\int \frac{3}{y}\,dy = \int \frac{2}{x}\,dx$$

$$\ln y^3 = \ln x^2 + \ln C$$

$$y^3 = Cx^2$$

Initial condition: $y(8) = 2,\ 2^3 = C(8^2),\ C = \dfrac{1}{8}$

Particular solution: $8y^3 = x^2,\ y = \dfrac{1}{2}x^{2/3}$

25. $m = \dfrac{dy}{dx} = \dfrac{0 - y}{(x + 2) - x} = -\dfrac{y}{2}$

$$\int \frac{dy}{y} = \int -\frac{1}{2}\,dx$$

$$\ln|y| = -\frac{1}{2}x + C_1$$

$$y = Ce^{-x/2}$$

26. $m = \dfrac{dy}{dx} = \dfrac{y - 0}{x - 0} = \dfrac{y}{x}$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln y = \ln x + C_1 = \ln x + \ln C = \ln Cx$$

$$y = Cx$$

27. $f(x, y) = x^3 - 4xy^2 + y^3$

$$f(tx, ty) = t^3x^3 - 4txt^2y^2 + t^3y^3$$
$$= t^3(x^3 - 4xy^2 + y^3)$$

Homogeneous of degree 3

28. $f(x, y) = x^3 + 3x^2y^2 - 2y^2$

$$f(tx, ty) = t^3x^3 + 3t^4x^2y^2 - 2t^2y^2$$

Not homogeneous

29. $f(x, y) = \dfrac{x^2y^2}{\sqrt{x^2 + y^2}}$

$$f(tx, ty) = \frac{t^4x^2y^2}{\sqrt{t^2x^2 + t^2y^2}} = t^3\frac{x^2y^2}{\sqrt{x^2 + y^2}}$$

Homogeneous of degree 3

30. $f(x, y) = \dfrac{xy}{\sqrt{x^2 + y^2}}$

$$f(tx, ty) = \frac{tx\,ty}{\sqrt{t^2x^2 + t^2y^2}}$$
$$= \frac{t^2xy}{t\sqrt{x^2 + y^2}} = t\frac{xy}{\sqrt{x^2 + y^2}}$$

Homogeneous of degree 1

31. $f(x, y) = 2\ln xy$

$$f(tx, ty) = 2\ln[txty]$$
$$= 2\ln[t^2xy] = 2(\ln t^2 + \ln xy)$$

Not homogeneous

32. $f(x, y) = \tan(x + y)$

$$f(tx, ty) = \tan(tx + ty) = \tan[t(x + y)]$$

Not homogeneous

33. $f(x, y) = 2\ln\dfrac{x}{y}$

$$f(tx, ty) = 2\ln\frac{tx}{ty} = 2\ln\frac{x}{y}$$

Homogeneous degree 0

34. $f(x, y) = \tan\dfrac{y}{x}$

$$f(tx, ty) = \tan\frac{ty}{tx} = \tan\frac{y}{x}$$

Homogeneous of degree 0

35. $y' = \dfrac{x + y}{2x},\ y = vx$

$$v + x\frac{dv}{dx} = \frac{x + vx}{2x}$$
$$x\frac{dv}{dx} = \frac{1 + v}{2} - v = \frac{1 - v}{2}$$
$$2\int\frac{dv}{1 - v} = \int\frac{dx}{x}$$
$$-\ln(1 - v)^2 = \ln|x| + \ln C = \ln|Cx|$$
$$\frac{1}{(1 - v)^2} = |Cx|$$
$$\frac{1}{[1 - (y/x)]^2} = |Cx|$$
$$\frac{x^2}{(x - y)^2} = |Cx|$$
$$|x| = C(x - y)^2$$

36. $y' = \dfrac{(x^3 + y^3)}{xy^2}$

$$xy^2\,dy = (x^3 + y^3)\,dx$$
$$y = vx, \quad dy = x\,dv + v\,dx$$
$$x(vx)^2(x\,dv + v\,dx) = (x^3 + (vx)^3)\,dx$$
$$x^4v^2\,dv + x^3v^3\,dx = x^3\,dx + v^3x^3\,dx$$
$$xv^2\,dv = dx$$
$$\int v^2\,dv = \int\frac{1}{x}\,dx$$
$$\frac{v^3}{3} = \ln|x| + C$$
$$\left(\frac{y}{x}\right)^3 = 3\ln|x| + C$$
$$y^3 = 3x^3\ln|x| + Cx^3$$

37.
$$y' = \frac{x - y}{x + y}, y = vx$$
$$v + x\frac{dv}{dx} = \frac{x - xv}{x + xv}$$
$$v\,dx + x\,dv = \frac{1 - v}{1 + v}dx$$
$$x\,dv = \left(\frac{1 - v}{1 + v} - v\right)dx = \frac{1 - 2v - v^2}{1 + v}\,dx$$
$$\int \frac{v + 1}{v^2 + 2v - 1}\,dv = -\int \frac{dx}{x}$$
$$\frac{1}{2}\ln\left|v^2 + 2v - 1\right| = -\ln|x| + \ln C_1 = \ln\left|\frac{C_1}{x}\right|$$
$$\left|v^2 + 2v - 1\right| = \frac{C}{x^2}$$
$$\left|\frac{y^2}{x^2} + 2\frac{y}{x} - 1\right| = \frac{C}{x^2}$$
$$\left|y^2 + 2xy - x^2\right| = C$$

38.
$$y' = \frac{x^2 + y^2}{2xy}, y = vx$$
$$v + x\frac{dv}{dx} = \frac{x^2 + v^2x^2}{2x^2v}$$
$$2v\,dx + 2x\,dv = \frac{1 + v^2}{v}dx$$
$$\int \frac{2v}{v^2 - 1}\,dv = -\int \frac{dx}{x}$$
$$\ln(v^2 - 1) = -\ln x + \ln C = \ln\frac{C}{x}$$
$$v^2 - 1 = \frac{C}{x}$$
$$\frac{y^2}{x^2} - 1 = \frac{C}{x}$$
$$y^2 - x^2 = Cx$$

39.
$$y' = \frac{xy}{x^2 - y^2}, y = vx$$
$$v + x\frac{dv}{dx} = \frac{x^2v}{x^2 - x^2v^2}$$
$$v\,dx + x\,dv = \frac{v}{1 - v^2}dx$$
$$x\,dv = \left(\frac{v}{1 - v^2} - v\right)dx = \left(\frac{v^3}{1 - v^2}\right)dx$$
$$\int \frac{1 - v^2}{v^3}\,dv = \int \frac{dx}{x}$$
$$-\frac{1}{2v^2} - \ln|v| = \ln|x| + \ln C_1 = \ln|C_1x|$$
$$\frac{-1}{2v^2} = \ln|C_1xv|$$
$$\frac{-x^2}{2y^2} = \ln|C_1y|$$
$$y = Ce^{-x^2/2y^2}$$

40.
$$y' = \frac{2x + 3y}{x}, y = vx$$
$$v + x\frac{dv}{dx} = \frac{2x + 3vx}{x} = 2 + 3v$$
$$x\frac{dv}{dx} = 2 + 2v \implies \int \frac{dv}{1 + v} = 2\int \frac{dx}{x}$$
$$\ln|1 + v| = \ln x^2 + \ln C = \ln x^2C$$
$$1 + v = x^2C$$
$$1 + \frac{y}{x} = x^2C$$
$$\frac{y}{x} = x^2C - 1$$
$$y = Cx^3 - x$$

41.
$$x\,dy - (2xe^{-y/x} + y)\,dx = 0, y = vx$$
$$x(v\,dx + x\,dv) - (2xe^{-v} + vx)\,dx = 0$$
$$\int e^v\,dv = \int \frac{2}{x}\,dx$$
$$e^v = \ln C_1x^2$$
$$e^{y/x} = \ln C_1 + \ln x^2$$
$$e^{y/x} = C + \ln x^2$$

Initial condition: $y(1) = 0, 1 = C$

Particular solution: $e^{y/x} = 1 + \ln x^2$

42. $-y^2\,dx + x(x + y)\,dy = 0,\ y = vx$

$$-x^2v^2\,dx + (x^2 + x^2v)(v\,dx + x\,dv) = 0$$

$$\int \frac{1 + v}{v}\,dv = -\int \frac{dx}{x}$$

$$v + \ln v = -\ln x + \ln C_1 = \ln \frac{C_1}{x}$$

$$v = \ln \frac{C_1}{xv}$$

$$\frac{C_1}{vx} = e^v$$

$$\frac{C_1}{y} = e^{y/x}$$

$$y = Ce^{-y/x}$$

Initial condition: $y(1) = 1,\ 1 = Ce^{-1} \implies C = e$

Particular solution: $y = e^{1 - y/x}$

43. $\left(x \sec \frac{y}{x} + y\right) dx - x\,dy = 0,\ y = vx$

$$(x \sec v + xv)dx - x(v\,dx + x\,dv) = 0$$

$$(\sec v + v)\,dx = v\,dx + x\,dv$$

$$\int \cos v\,dv = \int \frac{dx}{x}$$

$$\sin v = \ln x + \ln C_1$$

$$x = Ce^{\sin v}$$

$$= Ce^{\sin(y/x)}$$

Initial condition: $y(1) = 0,\ 1 = Ce^0 = C$

Particular solution: $x = e^{\sin(y/x)}$

44. $(2x^2 + y^2)\,dx + xy\,dy = 0$

Let $y = vx,\ dy = x\,dv + v\,dx.$

$$(2x^2 + v^2x^2)\,dx + x(vx)(x\,dv + v\,dx) = 0$$

$$(2x^2 + 2x^2v^2)\,dx + x^3v\,dv = 0$$

$$(2 + 2v^2)\,dx = -xv\,dv$$

$$\frac{-2}{x}\,dx = \frac{v}{1 + v^2}\,dv$$

$$-2 \ln x = \frac{1}{2}\ln(1 + v^2) + C_1$$

$$\ln x^{-2} = \ln(1 + v^2)^{1/2} + \ln C$$

$$x^{-2} = C(1 + v^2)^{1/2}$$

$$\frac{1}{x^2} = C\left(1 + \frac{y^2}{x^2}\right)^{1/2} = \frac{C}{x}(x^2 + y^2)^{1/2}$$

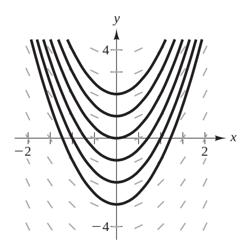
$$\frac{1}{x} = C(x^2 + y^2)^{1/2}$$

$y(1) = 0$: $1 = C(1 + 0) \implies C = 1$

$$\frac{1}{x} = \sqrt{x^2 + y^2}$$

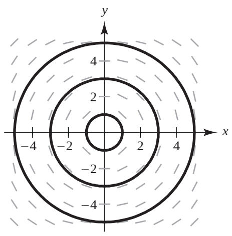
$$1 = x\sqrt{x^2 + y^2}$$

45. $\dfrac{dy}{dx} = x$



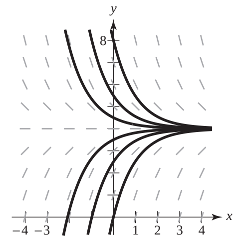
$$y = \int x\,dx = \frac{1}{2}x^2 + C$$

46. $\dfrac{dy}{dx} = -\dfrac{x}{y}$



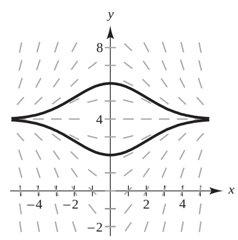
$$y\,dy = -x\,dx$$
$$\frac{y^2}{2} = \frac{-x^2}{2} + C_1$$
$$y^2 + x^2 = C$$

47. $\dfrac{dy}{dx} = 4 - y$



$$\int \frac{dy}{4-y} = \int dx$$
$$\ln|4-y| = -x + C_1$$
$$4 - y = e^{-x+C_1}$$
$$y = 4 + Ce^{-x}$$

48. $\dfrac{dy}{dx} = 0.25x(4-y)$



$$\frac{dy}{4-y} = 0.25x\,dx$$
$$\int \frac{dy}{y-4} = \int -0.25x\,dx = -\frac{1}{4}\int x\,dx$$
$$\ln|y-4| = -\frac{1}{8}x^2 + C_1$$
$$y - 4 = e^{C_1-(1/8)x^2} = Ce^{-(1/8)x^2}$$
$$y = 4 + Ce^{-(1/8)x^2}$$

49. (a) Euler's Method gives $y(1) \approx 0.1602$.

(b) $$\frac{dy}{dx} = -6xy$$
$$\int \frac{dy}{y} = \int -6x$$
$$\ln|y| = -3x^2 + C_1$$
$$y = Ce^{-3x^2}$$
$$y(0) = 5 \implies C = 5$$
$$y = 5e^{-3x^2}$$

(c) At $x = 1$, $y = 5e^{-3(1)} \approx 0.2489$.

Error: $0.2489 - 0.1602 \approx 0.0887$

50. (a) Euler's Method gives $y(1) \approx 0.2622$.

(b) $$\frac{dy}{dx} = -6xy^2$$
$$\int \frac{dy}{y^2} = \int -6x\,dx$$
$$\frac{-1}{y} = -3x^2 + C_1$$
$$y = \frac{1}{3x^2 + C}$$
$$3 = \frac{1}{C} \implies C = \frac{1}{3}$$
$$y = \frac{1}{3x^2 + \frac{1}{3}} = \frac{3}{9x^2 + 1}$$

(c) At $x = 1$, $y = \dfrac{3}{9(1)+1} = \dfrac{3}{10} = 0.3$.

Error: $0.3 - 0.2622 = 0.0378$

51. (a) Euler's Method gives $y(2) \approx 3.0318$.

(b) $\dfrac{dy}{dx} = \dfrac{2x + 12}{3y^2 - 4}$

$$\int (3y^2 - 4)\,dy = \int (2x + 12)\,dx$$

$$y^3 - 4y = x^2 + 12x + C$$

$$y(1) = 2:\; 2^3 - 4(2) = 1 + 12 + C \implies C = -13$$

$$y^3 - 4y = x^2 + 12x - 13$$

(c) For $x = 2$,

$$y^3 - 4y = 2^2 + 12(2) - 13 = 15$$

$$y^3 - 4y - 15 = 0$$

$$(y - 3)(y^2 + 3y + 5) = 0 \implies y = 3.$$

Error: $3.0318 - 3 = 0.0318$

52. (a) Euler's Method gives $y(1.5) \approx 1.7270$.

(b) $\dfrac{dy}{dx} = 2x(1 + y^2)$

$$\int \frac{dy}{1 + y^2} = \int 2x\,dx$$

$$\arctan y = x^2 + C$$

$$\arctan(0) = 1^2 + C \implies C = -1$$

$$\arctan(y) = x^2 - 1$$

$$y = \tan(x^2 - 1)$$

(c) At $x = 1.5$, $y = \tan(1.5^2 - 1) \approx 3.0096$.

53. $\dfrac{dy}{dt} = ky, \quad y = Ce^{kt}$

$y(0) = y_0 = C$ initial amount

$$\frac{y_0}{2} = y_0 e^{k(1599)}$$

$$k = \frac{1}{1599}\ln\left(\frac{1}{2}\right)$$

$$y = Ce^{[\ln(1/2)/1599]t}$$

When $t = 25$, $y = 0.989C$ or 98.9%.

54. $\dfrac{dy}{dt} = ky, y = Ce^{kt}$

Initial conditions: $y(0) = 20, y(1) = 16$

$$20 = Ce^0 = C$$

$$16 = 20e^k$$

$$k = \ln\frac{4}{5}$$

Particular solution: $y = 20e^{t\ln(4/5)}$

When 75% has been changed:

$$5 = 20e^{t\ln(4/5)}$$

$$\frac{1}{4} = e^{t\ln(4/5)}$$

$$t = \frac{\ln(1/4)}{\ln(4/5)} \approx 6.2 \text{ hr}$$

55. $\dfrac{dy}{dx} = k(y - 4)$

The direction field satisfies $(dy/dx) = 0$ along $y = 4$; but not along $y = 0$. Matches (a).

56. $\dfrac{dy}{dx} = k(x - 4)$

The direction field satisfies $(dy/dx) = 0$ along $x = 4$: Matches (b).

57. $\dfrac{dy}{dx} = ky(y - 4)$

The direction field satisfies $(dy/dx) = 0$ along $y = 0$ and $y = 4$. Matches (c).

58. $\dfrac{dy}{dx} = ky^2$

The direction field satisfies $(dy/dx) = 0$ along $y = 0$, and grows more positive as y increases. Matches (d).

59. $$\frac{dw}{dt} = k(1200 - w)$$

$$\int \frac{dw}{1200 - w} = \int k\,dt$$

$$\ln|1200 - w| = -kt + C_1$$

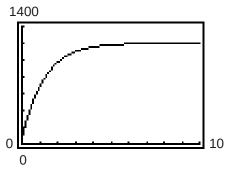
$$1200 - w = e^{-kt+C_1} = Ce^{-kt}$$

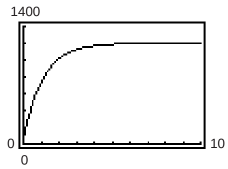
$$w = 1200 - Ce^{-kt}$$

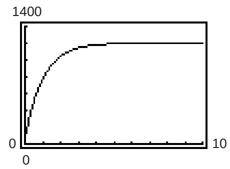
$w(0) = 60 = 1200 - C \implies C = 1200 - 60 = 1140$

$w = 1200 - 1140e^{-kt}$

(a)







(b) $k = 0.8$: $t = 1.31$ years

$k = 0.9$: $t = 1.16$ years

$k = 1.0$: $t = 1.05$ years

(c) Maximum weight: 1200 pounds

$$\lim_{t\to\infty} w = 1200$$

60. From Exercise 101:

$w = 1200 - Ce^{-kt}, k = 1$

$w = 1200 - Ce^{-t}$

$w(0) = w_0 = 1200 - C \implies C = 1200 - w_0$

$w = 1200 - (1200 - w_0)e^{-t}$

61. Given family (circles): $x^2 + y^2 = C$

$$2x + 2yy' = 0$$

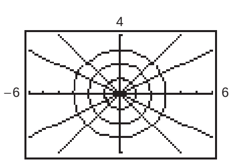
$$y' = -\frac{x}{y}$$

Orthogonal trajectory (lines): $y' = \dfrac{y}{x}$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln|y| = \ln|x| + \ln K$$

$$y = Kx$$



62. Given family (hyperbolas): $x^2 - 2y^2 = C$

$$2x - 4yy' = 0$$

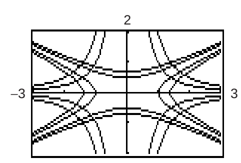
$$y' = \frac{x}{2y}$$

Orthogonal trajectory: $y' = \dfrac{-2y}{x}$

$$\int \frac{dy}{y} = -\int \frac{2}{x}\,dx$$

$$\ln y = -2\ln x + \ln k$$

$$y = kx^{-2} = \frac{k}{x^2}$$



63. Given family (parabolas): $x^2 = Cy$

$$2x = Cy'$$

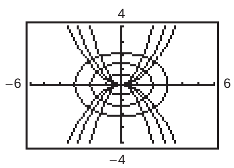
$$y' = \frac{2x}{C} = \frac{2x}{x^2/y} = \frac{2y}{x}$$

Orthogonal trajectory (ellipses): $y' = -\frac{x}{2y}$

$$2\int y\,dy = -\int x\,dx$$

$$y^2 = -\frac{x^2}{2} + K_1$$

$$x^2 + 2y^2 = K$$



64. Given family (parabolas): $y^2 = 2Cx$

$$2yy' = 2C$$

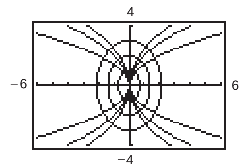
$$y' = \frac{C}{y} = \frac{y^2}{2x}\left(\frac{1}{y}\right) = \frac{y}{2x}$$

Orthogonal trajectory (ellipse): $y' = -\frac{2x}{y}$

$$\int y\,dy = -\int 2x\,dx$$

$$\frac{y^2}{2} = -x^2 + K_1$$

$$2x^2 + y^2 = K$$



65. Given family: $y^2 = Cx^3$

$$2yy' = 3Cx^2$$

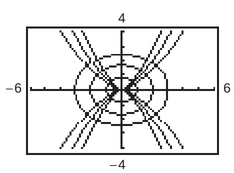
$$y' = \frac{3Cx^2}{2y} = \frac{3x^2}{2y}\left(\frac{y^2}{x^3}\right) = \frac{3y}{2x}$$

Orthogonal trajectory (ellipses): $y' = -\frac{2x}{3y}$

$$3\int y\,dy = -2\int x\,dx$$

$$\frac{3y^2}{2} = -x^2 + K_1$$

$$3y^2 + 2x^2 = K$$



66. Given family (exponential functions): $y = Ce^x$

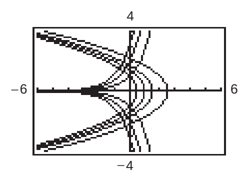
$$y' = Ce^x = y$$

Orthogonal trajectory (parabolas): $y' = -\frac{1}{y}$

$$\int y\,dy = -\int dx$$

$$\frac{y^2}{2} = -x + K_1$$

$$y^2 = -2x + K$$



67. $y = \dfrac{12}{1 + e^{-x}}$

Since $y(0) = 6$, it matches (c) or (d).

Since (d) approaches its horizontal asymptote slower than (c), it matches (d).

68. $y = \dfrac{12}{1 + 3e^{-x}}$

Since $y(0) = \dfrac{12}{4} = 3$, it matches (a).

69. $y = \dfrac{12}{1 + \frac{1}{2}e^{-x}}$

Since $y(0) = \dfrac{12}{\left(\frac{3}{2}\right)} = 8$, it matches (b).

70. $y = \dfrac{12}{1 + e^{-2x}}$

Since $y(0) = 6$, it matches (c) or (d).

Since y approaches $L = 12$ faster for (c), it matches (c).

71. $P(t) = \dfrac{1500}{1 + 24e^{-0.75t}}$

(a) $k = 0.75$

(b) $L = 1500$

(c) $P(0) = \dfrac{1500}{1 + 24} = 60$

(d) $$750 = \frac{1500}{1 + 24e^{-0.75t}}$$
$$1 + 24e^{-0.75t} = 2$$
$$e^{-0.75t} = \frac{1}{24}$$
$$-0.75t = \ln\left(\frac{1}{24}\right) = -\ln 24$$
$$t = \frac{\ln 24}{0.75} \approx 4.2374$$

(e) $\dfrac{dP}{dt} = 0.75P\left(1 - \dfrac{P}{1500}\right), \qquad P(0) = 60$

72. $P(t) = \dfrac{5000}{1 + 39e^{-0.2t}}$

(a) $k = 0.2$

(b) $L = 5000$

(c) $P(0) = \dfrac{5000}{1 + 39} = 125$

(d) $$2500 = \frac{5000}{1 + 39e^{-0.2t}}$$
$$1 + 39e^{-0.2t} = 2$$
$$e^{-0.2t} = \frac{1}{39}$$
$$-0.2t = \ln\left(\frac{1}{39}\right) = -\ln 39$$
$$t = \frac{\ln 39}{0.2} \approx 18.3178$$

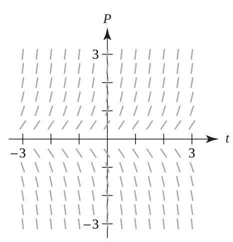
(e) $\dfrac{dP}{dt} = 0.2P\left(1 - \dfrac{P}{5000}\right), \qquad P(0) = 125$

73. $\dfrac{dP}{dt} = 3P\left(1 - \dfrac{P}{100}\right)$

(a) $k = 3$

(b) $L = 100$

(c)



(d) $$\frac{d^2P}{dt^2} = 3P'\left(1 - \frac{P}{100}\right) + 3P\left(\frac{-P'}{100}\right)$$
$$= 3\left[3P\left(1 - \frac{P}{100}\right)\right]\left(1 - \frac{P}{100}\right) - \frac{3P}{100}\left[3P\left(1 - \frac{P}{100}\right)\right]$$
$$= 9P\left(1 - \frac{P}{100}\right)\left(1 - \frac{P}{100} - \frac{P}{100}\right)$$
$$= 9P\left(1 - \frac{P}{100}\right)\left(1 - \frac{2P}{100}\right)$$

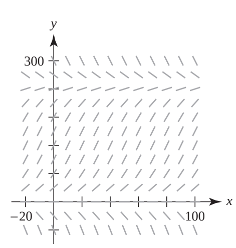
$\dfrac{d^2P}{dt^2} = 0$ for $P = 50$, and by the first Derivative Test, this is a maximum. $\left(\text{Note: } P = 50 = \dfrac{L}{2} = \dfrac{100}{2}\right)$

74. $$\frac{dP}{dt} = 0.1P - 0.0004P^2 = 0.1P(1 - 0.004P)$$
$$= 0.1P\left(1 - \frac{P}{250}\right)$$

(a) $k = 0.1 = \dfrac{1}{10}$

(b) $L = 250$

(c)



(d) $P = \dfrac{250}{2} = 125$. (Same argument as in Exercise 73)

75. $\dfrac{dy}{dt} = y\left(1 - \dfrac{y}{40}\right), \qquad y(0) = 8$

$k = 1,\ L = 40$

$y = \dfrac{L}{1 + be^{-kt}} = \dfrac{40}{1 + be^{-t}}$

$y(0) = 8\colon\ 8 = \dfrac{40}{1 + b} \implies b = 4$

Solution: $y = \dfrac{40}{1 + 4e^{-t}}$

76. $\dfrac{dy}{dt} = 1.2y\left(1 - \dfrac{y}{8}\right), \quad y(0) = 5$

$k = 1.2, \ L = 8$

$y = \dfrac{L}{1 + be^{-kt}} = \dfrac{8}{1 + be^{-1.2t}}$

$y(0) = 5\colon\ 5 = \dfrac{8}{1 + b} \implies 1 + b = \dfrac{8}{5} \implies b = \dfrac{3}{5}$

Solution: $y = \dfrac{8}{1 + \frac{3}{5}e^{-1.2t}}$

77. $\dfrac{dy}{dt} = \dfrac{4y}{5} - \dfrac{y^2}{150} = \dfrac{4}{5}y\left(1 - \dfrac{y}{120}\right), \quad y(0) = 8$

$k = \dfrac{4}{5} = 0.8, \ L = 120$

$y = \dfrac{L}{1 + be^{-kt}} = \dfrac{120}{1 + be^{-0.8t}}$

$y(0) = 8\colon\ 8 = \dfrac{120}{1 + b} \implies b = 14$

Solution: $y = \dfrac{120}{1 + 14e^{-0.8t}}$

78. $\dfrac{dy}{dt} = \dfrac{3y}{20} - \dfrac{y^2}{1600} = \dfrac{3}{20}y\left(1 - \dfrac{y}{240}\right); \quad y(0) = 15$

$k = \dfrac{3}{20}, \ L = 240$

$y = \dfrac{L}{1 + be^{-kt}} = \dfrac{240}{1 + be^{(-3/20)t}}$

$y(0) = 15\colon\ 15 = \dfrac{240}{1 + b} \implies b = 15$

Solution: $y = \dfrac{240}{1 + 15e^{(-3/20)t}}$

79. (a) $y = \dfrac{L}{1 + be^{-kt}}, \ L = 200, \ y(0) = 25$

$$25 = \frac{200}{1 + b} \implies b = 7$$

$$39 = \frac{200}{1 + 7e^{-k(2)}}$$

$$1 + 7e^{-2k} = \frac{200}{39}$$

$$e^{-2k} = \frac{23}{39}$$

$$k = -\frac{1}{2}\ln\left(\frac{23}{39}\right) = \frac{1}{2}\ln\left(\frac{39}{23}\right) \approx 0.2640$$

$$y = \frac{200}{1 + 7e^{-0.2640t}}$$

(b) For $t = 5$, $y \approx 70$ panthers.

(c)
$$100 = \frac{200}{1 + 7e^{-0.264t}}$$

$$1 + 7e^{-0.264t} = 2$$

$$-0.264t = \ln\left(\frac{1}{7}\right)$$

$$t \approx 7.37 \text{ years}$$

(d) $\dfrac{dy}{dt} = ky\left(1 - \dfrac{y}{L}\right) = 0.264y\left(1 - \dfrac{y}{200}\right), \quad y(0) = 25$

Using Euler's Method, $y \approx 220.5$ when $t = 6$.

(e) y is increasing most rapidly where $y = 200/2 = 100$, corresponds to $t \approx 7.37$ years.

80. (a) $y = \dfrac{L}{1 + be^{-kt}},\ L = 10,\ y(0) = 1$

$$1 = \frac{10}{1 + b} \Rightarrow b = 9$$

$$2 = \frac{10}{1 + 9e^{-2k}}$$

$$1 + 9e^{-2k} = 5$$

$$e^{-2k} = \frac{4}{9}$$

$$k = -\frac{1}{2}\ln\left(\frac{4}{9}\right) = \frac{1}{2}\ln\left(\frac{9}{4}\right) = \ln\left(\frac{3}{2}\right) \approx 0.405476$$

$$y = \frac{10}{1 + 9e^{-0.40547t}}$$

Note: $y = \dfrac{10}{1 + 9e^{-t\ln(3/2)}} = \dfrac{10}{1 + 9\left(\frac{3}{2}\right)^{-t}} = \dfrac{10}{1 + 9\left(\frac{2}{3}\right)^{t}}$

(b) For $t = 5$, $y \approx 4.58$ grams.

(c) $8 = \dfrac{10}{1 + 9e^{-t\ln(3/2)}} \Rightarrow 72e^{-t\ln(3/2)} = 2$

$$\Rightarrow \left(\frac{3}{2}\right)^t = 36$$

$$\Rightarrow t \approx 8.84 \text{ hours}$$

(d) $\dfrac{dy}{dt} = \ln\left(\dfrac{3}{2}\right)y\left(1 - \dfrac{y}{10}\right)$

$$\approx 0.40547y\left(1 - \frac{y}{10}\right)$$

t	0	1	2	3	4	5
Exact	1.0	1.4286	2.0	2.7273	3.6	4.5763
Euler	1.0	1.3649	1.8428	2.4523	3.2028	4.0855

For $t = 5$, Euler's Method gives $y \approx 4.09$ grams.

(e) The weight is increasing most rapidly when $y = 10/2 = 5$, corresponding to $t \approx 5.42$ hours.

81. A differential equation can be solved by separation of variables if it can be written in the form

$$M(x) + N(y)\frac{dy}{dx} = 0.$$

To solve a separable equation, rewrite as,

$$M(x)\,dx = -N(y)\,dy$$

and integrate both sides.

82. $M(x, y)\,dx + N(x, y)\,dy = 0$, where M and N are homogeneous functions of the same degree. See Example 7a.

83. Two families of curves are mutually orthogonal if each curve in the first family intersects each curve in the second family at right angles.

84. (a) $$\frac{dv}{dt} = k(W - v)$$

$$\int \frac{dv}{W - v} = \int k\,dt$$

$$-\ln|W - v| = kt + C_1$$

$$v = W - Ce^{-kt}$$

Initial conditions:

$W = 20$, $v = 0$ when $t = 0$, and

$v = 5$ when $t = 1$.

$C = 20$, $k = -\ln(3/4)$

Particular solution:

$$v = 20(1 - e^{\ln(3/4)t}) = 20\left(1 - \left(\frac{3}{4}\right)^t\right)$$

or

$$v \approx 20(1 - e^{-0.2877t})$$

(b) $$s = \int 20(1 - e^{-0.2877t})\,dt$$

$$\approx 20\left[t + 3.4761e^{-0.2877t}\right] + C$$

Since $s(0) = 0$, $C \approx -69.5$ and we have $s \approx 20t + 69.5(e^{-0.2877t} - 1)$.

85. False. $\dfrac{dy}{dx} = \dfrac{x}{y}$ is separable, but $y = 0$ is not a solution.

86. True

$$\frac{dy}{dx} = (x - 2)(y + 1)$$

87. False

$$f(tx, ty) = t^2x^2 + t^2xy + 2$$

$$\neq t^2 f(x, y)$$

88. True

$$x^2 + y^2 = 2Cy \qquad x^2 + y^2 = 2Kx$$

$$\frac{dy}{dx} = \frac{x}{C - y} \qquad \frac{dy}{dx} = \frac{K - x}{y}$$

$$\frac{x}{C - y} \cdot \frac{K - x}{y} = \frac{Kx - x^2}{Cy - y^2}$$

$$= \frac{2Kx - 2x^2}{2Cy - 2y^2}$$

$$= \frac{x^2 + y^2 - 2x^2}{x^2 + y^2 - 2y^2}$$

$$= \frac{y^2 - x^2}{x^2 - y^2}$$

$$= -1$$

89. $$y = \frac{1}{1 + be^{-kt}}$$

$$y' = \frac{-1}{(1 + be^{-kt})^2}(-bke^{-kt})$$

$$= \frac{k}{(1 + be^{-kt})} \cdot \frac{be^{-kt}}{(1 + be^{-kt})}$$

$$= \frac{k}{(1 + be^{-kt})} \cdot \frac{1 + be^{-kt} - 1}{(1 + be^{-kt})}$$

$$= \frac{k}{(1 + be^{-kt})} \cdot \left(1 - \frac{1}{1 + be^{-kt}}\right)$$

$$= ky(1 - y)$$

90. $fg' + gf' = f'g'$ Product Rule

$(f - f')g' + gf' = 0$

$$g' + \frac{f'}{f - f'}g = 0$$

Need $f - f' = e^{x^2} - 2xe^{x^2} = (1 - 2x)e^{x^2} \neq 0$, so avoid $x = \frac{1}{2}$.

$$\frac{g'}{g} = \frac{f'}{f' - f} = \frac{2xe^{x^2}}{(2x - 1)e^{x^2}} = 1 + \frac{1}{2x - 1}$$

$$\ln|g(x)| = x + \frac{1}{2}\ln|2x - 1| + C_1$$

$$g(x) = Ce^x|2x - 1|^{1/2}$$

Hence there exists g and interval (a, b), as long as $\frac{1}{2} \notin (a, b)$.

Section 6.4 First-Order Linear Differential Equations

1. $x^3y' + xy = e^x + 1$

$$y' + \frac{1}{x^2}y = \frac{1}{x^3}(e^x + 1)$$

Linear

2. $2xy - y'\ln x = y$

$(\ln x)y' + (1 - 2x)y = 0$

$$y' + \frac{(1 - 2x)}{\ln x}y = 0$$

Linear

3. $y' + y\cos x = xy^2$

Not linear, because of the xy^2-term.

4. $\dfrac{1 - y'}{y} = 3x$

$1 - y' = 3xy$

$y' + 3xy = 1$

Linear

5. $\dfrac{dy}{dx} + \left(\dfrac{1}{x}\right)y = 3x + 4$

Integrating factor: $e^{\int (1/x)\,dx} = e^{\ln x} = x$

$$xy = \int x(3x + 4)\,dx = x^3 + 2x^2 + C$$

$$y = x^2 + 2x + \frac{C}{x}$$

6. $\dfrac{dy}{dx} + \dfrac{2}{x}y = 3x + 2$

Integrating factor: $e^{\int 2/x\,dx} = e^{\ln x^2} = x^2$

$$x^2y = \int x^2(3x + 2)\,dx = \frac{3}{4}x^4 + \frac{2x^3}{3} + C$$

$$y = \frac{3}{4}x^2 + \frac{2}{3}x + \frac{C}{x^2}$$

7. $y - y = 10$

Integrating factor: $e^{\int -1\,dx} = e^{-x}$

$e^{-x}y' - e^{-x}y = 10e^{-x}$

$$ye^{-x} = \int 10e^{-x}\,dx = -10e^{-x} + C$$

$$y = -10 + Ce^x$$

8. $y' + 2xy = 4x$

Integrating factor: $e^{\int 2x\,dx} = e^{x^2}$

$$ye^{x^2} = \int 4xe^{x^2}\,dx = 2e^{x^2} + C$$

$$y = 2 + Ce^{-x^2}$$

9. $(y + 1)\cos x\,dx = dy$

$y' = (y + 1)\cos x = y\cos x + \cos x$

$y' - (\cos x)y = \cos x$

Integrating factor: $e^{\int -\cos x\,dx} = e^{-\sin x}$

$y'e^{-\sin x} - (\cos x)e^{-\sin x}y = (\cos x)e^{-\sin x}$

$$ye^{-\sin x} = \int (\cos x)e^{-\sin x}\,dx$$

$$= -e^{-\sin x} + C$$

$$y = -1 + Ce^{\sin x}$$

10. $[(y - 1)\sin x]\,dx - dy = 0$

$y' - (\sin x)y = -\sin x$

Integrating factor: $e^{\int -\sin x\,dx} = e^{\cos x}$

$$ye^{\cos x} = \int -\sin x e^{\cos x}\,dx = e^{\cos x} + C$$

$$y = 1 + Ce^{-\cos x}$$

11. $(x - 1)y' + y = x^2 - 1$

$$y' + \left(\frac{1}{x - 1}\right)y = x + 1$$

Integrating factor: $e^{\int [1/(x-1)]\,dx} = e^{\ln|x-1|} = x - 1$

$$y(x - 1) = \int (x^2 - 1)\,dx = \frac{1}{3}x^3 - x + C_1$$

$$y = \frac{x^3 - 3x + C}{3(x - 1)}$$

12. $y' + 3y = e^{3x}$

Integrating factor: $e^{\int 3\,dx} = e^{3x}$

$$ye^{3x} = \int e^{3x}e^{3x}dx = \int e^{6x}dx = \frac{1}{6}e^{6x} + C$$

$$y = \frac{1}{6}e^{3x} + Ce^{-3x}$$

13. $y' - 3x^2y = e^{x^3}$

Integrating factor: $e^{-\int 3x^2\,dx} = e^{-x^3}$

$$ye^{-x^3} = \int e^{x^3}e^{-x^3}\,dx = \int dx = x + C$$

$$y = (x + C)e^{x^3}$$

14. $y' - y = \cos x$

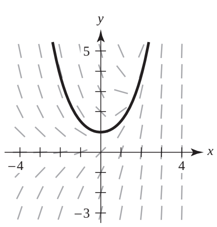
Integrating factor: $e^{\int -1\,dx} = e^{-x}$

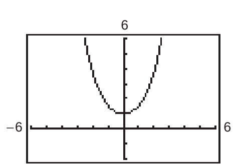
$$ye^{-x} = \int e^{-x}\cos x\,dx$$

$$= \frac{1}{2}e^{-x}(-\cos x + \sin x) + C$$

$$y = \frac{1}{2}(\sin x - \cos x) + Ce^{x}$$

15. (a),(c)





(b) $\dfrac{dy}{dx} = e^x - y$

$\dfrac{dy}{dx} + y = e^x$ Integrating factor: $e^{\int dx} = e^x$

$e^xy' + e^xy = e^{2x}$

$$(ye^x) = \int e^{2x}\,dx$$

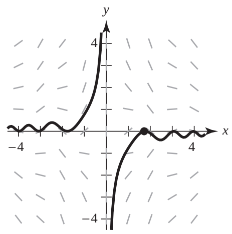
$$ye^x = \frac{1}{2}e^{2x} + C$$

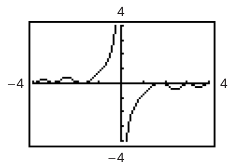
$$y(0) = 1 \implies 1 = \frac{1}{2} + C \implies C = \frac{1}{2}$$

$$ye^x = \frac{1}{2}e^{2x} + \frac{1}{2}$$

$$y = \frac{1}{2}e^x + \frac{1}{2}e^{-x} = \frac{1}{2}(e^x + e^{-x})$$

16. (a),(c)





(b) $y' + \frac{1}{x}y = \sin x^2,\ P(x) = \frac{1}{x},\ Q(x) = \sin x^2$

$u(x) = e^{\int (1/x)\, dx} = e^{\ln x} = x$

$y'x + y = x \sin x^2$

$$yx = \int x \sin x^2\, dx = -\frac{1}{2}\cos x^2 + C$$

$$y = \frac{1}{x}\left[-\frac{1}{2}\cos x^2 + C\right]$$

$$0 = \frac{1}{\sqrt{\pi}}\left[-\frac{1}{2}\cos \pi + C\right] \Rightarrow C = -\frac{1}{2}$$

$$y = \frac{1}{x}\left[-\frac{1}{2}\cos x^2 - \frac{1}{2}\right]$$

17. $y' \cos^2 x + y - 1 = 0$

$y' + (\sec^2 x)y = \sec^2 x$

Integrating factor: $e^{\int \sec^2 x\, dx} = e^{\tan x}$

$$ye^{\tan x} = \int \sec^2 x e^{\tan x}\, dx = e^{\tan x} + C$$

$$y = 1 + Ce^{-\tan x}$$

Initial condition: $y(0) = 5, C = 4$

Particular solution: $y = 1 + 4e^{-\tan x}$

18. $x^3 y' + 2y = e^{1/x^2}$

$y' + \left(\frac{2}{x^3}\right)y = \frac{1}{x^3}e^{1/x^2}$

Integrating factor: $e^{\int (2/x^3)\, dx} = e^{-(1/x^2)}$

$$ye^{-1/x^2} = \int \frac{1}{x^3}\, dx = -\frac{1}{2x^2} + C_1$$

$$y = e^{1/x^2}\left(\frac{Cx^2 - 1}{2x^2}\right)$$

Initial condition: $y(1) = e, C = 3$

Particular solution: $y = e^{1/x^2}\left(\frac{3x^2 - 1}{2x^2}\right)$

19. $y' + y \tan x = \sec x + \cos x$

Integrating factor: $e^{\int \tan x\, dx} = e^{\ln|\sec x|} = \sec x$

$$y \sec x = \int \sec x(\sec x + \cos x)\, dx = \tan x + x + C$$

$$y = \sin x + x \cos x + C \cos x$$

Initial condition: $y(0) = 1, 1 = C$

Particular solution: $y = \sin x + (x + 1)\cos x$

20. $y' + y \sec x = \sec x$

Integrating factor: $e^{\int \sec x\, dx} = e^{\ln|\sec x + \tan x|} = \sec x + \tan x$

$$y(\sec x + \tan x) = \int (\sec x + \tan x) \sec x\, dx$$

$$= \sec x + \tan x + C$$

$$y = 1 + \frac{C}{\sec x + \tan x}$$

Initial condition: $y(0) = 4, 4 = 1 + \frac{C}{1 + 0}, C = 3$

Particular solution:

$$y = 1 + \frac{3}{\sec x + \tan x} = 1 + \frac{3 \cos x}{1 + \sin x}$$

21. $y' + \left(\frac{1}{x}\right)y = 0$

Integrating factor: $e^{\int(1/x)\,dx} = e^{\ln|x|} = x$

Separation of variables:

$$\frac{dy}{dx} = -\frac{y}{x}$$

$$\int \frac{1}{y}\,dy = \int -\frac{1}{x}\,dx$$

$$\ln y = -\ln x + \ln C$$

$$\ln xy = \ln C$$

$$xy = C$$

Initial condition: $y(2) = 2, C = 4$

Particular solution: $xy = 4$

22. $y' + (2x - 1)y = 0$

Integrating factor: $e^{\int(2x-1)\,dx} = e^{x^2-x}$

$$ye^{x^2-x} = C$$

$$y = Ce^{x-x^2}$$

Separation of variables:

$$\int \frac{1}{y}\,dy = \int (1 - 2x)\,dx$$

$$\ln y + \ln C_1 = x - x^2$$

$$yC_1 = e^{x-x^2}$$

$$y = Ce^{x-x^2}$$

Initial condition: $y(1) = 2, 2 = C$

Particular solution: $y = 2e^{x-x^2}$

23. $x\,dy = (x + y + 2)\,dx$

$$\frac{dy}{dx} = \frac{x+y+2}{x} = \frac{y}{x} + 1 + \frac{2}{x}$$

$$\frac{dy}{dx} - \frac{1}{x}y = 1 + \frac{2}{x} \qquad \text{Linear}$$

$$u(x) = e^{\int -(1/x)\,dx} = \frac{1}{x}$$

$$y = x\int\left(1 + \frac{2}{x}\right)\frac{1}{x}\,dx = x\int\left(\frac{1}{x} + \frac{2}{x^2}\right)dx$$

$$= x\left[\ln|x| + \frac{-2}{x} + C\right]$$

$$= -2 + x\ln|x| + Cx$$

$$y(1) = 10 = -2 + C \implies C = 12$$

$$y = -2 + x\ln|x| + 12x$$

24. $2xy' - y = x^3 - x$

$$\frac{dy}{dx} - \frac{1}{2x}y = \frac{x^2}{2} - \frac{1}{2} \qquad \text{Linear}$$

$$u(x) = e^{\int -(1/2x)\,dx} = \frac{1}{x^{1/2}}$$

$$y = x^{1/2}\int\left(\frac{x^2}{2} - \frac{1}{2}\right)\frac{1}{x^{1/2}}\,dx = x^{1/2}\int\left(\frac{x^{3/2}}{2} - \frac{x^{-1/2}}{2}\right)dx$$

$$= x^{1/2}\left[\frac{x^{5/2}}{5} - x^{1/2} + C\right]$$

$$= \frac{x^3}{5} - x + C\sqrt{x}$$

$$y(4) = 2 = \frac{64}{5} - 4 + 2C \implies C = -\frac{17}{5}$$

$$y = \frac{x^3}{5} - x - \frac{17}{5}\sqrt{x}$$

25. $y' + 3x^2y = x^2y^3$

$n = 3, Q = x^2, P = 3x^2$

$$y^{-2}e^{\int(-2)3x^2\,dx} = \int(-2)x^2e^{\int(-2)3x^2\,dx}\,dx$$

$$y^{-2}e^{-2x^3} = -\int 2x^2e^{-2x^3}\,dx$$

$$y^{-2}e^{-2x^3} = \frac{1}{3}e^{-2x^3} + C$$

$$y^{-2} = \frac{1}{3} + Ce^{2x^3}$$

$$\frac{1}{y^2} = Ce^{2x^3} + \frac{1}{3}$$

26. $y' + xy = xy^{-1}$

$n = -1, Q = x, P = x, e^{\int 2x\,dx} = e^{x^2}$

$$y^2e^{x^2} = \int 2xe^{x^2}\,dx = e^{x^2} + C$$

$$y^2 = 1 + Ce^{-x^2}$$

27. $y' + \left(\frac{1}{x}\right)y = xy^2$

$$n = 2, Q = x, P = x^{-1}$$

$$e^{\int -(1/x)\,dx} = e^{-\ln|x|} = x^{-1}$$

$$y^{-1}x^{-1} = \int -x(x^{-1})\,dx = -x + C$$

$$\frac{1}{y} = -x^2 + Cx$$

$$y = \frac{1}{Cx - x^2}$$

28. $y' + \left(\frac{1}{x}\right)y = x\sqrt{y}$

$$n = \frac{1}{2}, Q = x, P = x^{-1}$$

$$e^{(1/2)(1/x)\,dx} = e^{(1/2)\ln x} = \sqrt{x}$$

$$y^{1/2}x^{1/2} = \int \frac{1}{2}x^{1/2}(x)\,dx$$

$$= \frac{1}{5}x^{5/2} + C_1 = \frac{x^{5/2} + C}{5}$$

$$y = \frac{(x^{5/2} + C)^2}{25x}$$

29. $y' - y = e^x\sqrt[3]{y},\ n = \frac{1}{3},\ Q = e^x,\ P = -1$

$$e^{\int -(2/3)\,dx} = e^{-(2/3)x}$$

$$y^{2/3}e^{-(2/3)x} = \int \frac{2}{3}e^x e^{-(2/3)x}\,dx = \int \frac{2}{3}e^{(1/3)x}\,dx$$

$$y^{2/3}e^{-(2/3)x} = 2e^{(1/3)x} + C$$

$$y^{2/3} = 2e^x + Ce^{2x/3}$$

30. $yy' - 2y^2 = e^x$

$$y' - 2y = e^x y^{-1}$$

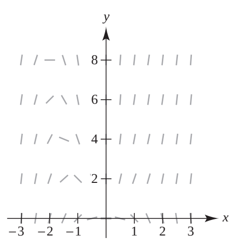
$$n = -1, Q = e^x, P = -2$$

$$e^{\int 2(-2)\,dx} = e^{-4x}$$

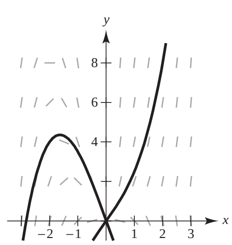
$$y^2e^{-4x} = \int 2e^{-4x}e^x\,dx = -\frac{2}{3}e^{-3x} + C$$

$$y^2 = -\frac{2}{3}e^x + Ce^{4x}$$

31. (a)



(c)



(b) $\frac{dy}{dx} - \frac{1}{x}y = x^2$

Integrating factor: $e^{-1/x\,dx} = e^{-\ln x} = \frac{1}{x}$

$$\frac{1}{x}y' - \frac{1}{x^2}y = x$$

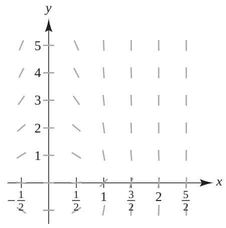
$$\left(\frac{1}{x}y\right) = \int x\,dx = \frac{x^2}{2} + C$$

$$y = \frac{x^3}{2} + Cx$$

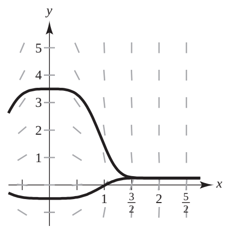
$(-2, 4)$: $4 = \frac{-8}{2} - 2C \implies C = -4 \implies y = \frac{x^3}{2} - 4x = \frac{1}{2}x(x^2 - 8)$

$(2, 8)$: $8 = \frac{8}{2} + 2C \implies C = 2 \implies y = \frac{x^3}{2} + 2x = \frac{1}{2}x(x^2 + 4)$

32. (a)



(c)



(b) $y' + 4x^3y = x^3$

Integrating factor: $e^{\int 4x^3\,dx} = e^{x^4}$

$y'e^{x^4} + 4x^3ye^{x^4} = x^3e^{x^4}$

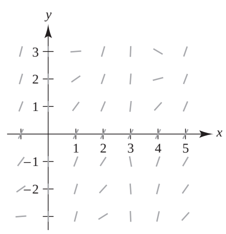
$$ye^{x^4} = \int x^3e^{x^4}\,dx = \frac{1}{4}e^{x^4} + C$$

$$y = \frac{1}{4} + Ce^{-x^4}$$

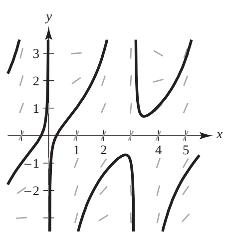
$\left(0, \frac{7}{2}\right)$: $\frac{7}{2} = \frac{1}{4} + C \Rightarrow C = \frac{13}{4} \Rightarrow y = \frac{1}{4} + \frac{13}{4}e^{-x^4}$

$\left(0, -\frac{1}{2}\right)$: $-\frac{1}{2} = \frac{1}{4} + C \Rightarrow C = -\frac{3}{4} \Rightarrow y = \frac{1}{4} - \frac{3}{4}e^{-x^4}$

33. (a)



(c)



(b) $y' + (\cot x)y = 2$

Integrating factor: $e^{\int \cot x\,dx} = e^{\ln|\sin x|} = \sin x$

$y' \sin x + (\cos x)y = 2 \sin x$

$$y \sin x = \int 2 \sin x\,dx = -2 \cos x + C$$

$$y = -2 \cot x + C \csc x$$

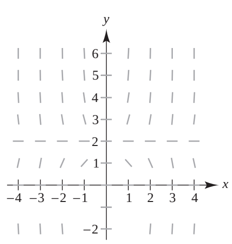
$(1, 1)$: $1 = -2 \cot 1 + C \csc 1 \Rightarrow C = \dfrac{1 + 2 \cot 1}{\csc 1} = \sin 1 + 2 \cos 1$

$$y = -2 \cot x + (\sin 1 + 2 \cos 1) \csc x$$

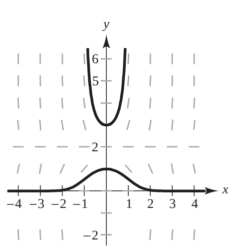
$(3, -1)$: $-1 = -2 \cot 3 + C \csc 3 \Rightarrow C = \dfrac{2 \cot 3 - 1}{\csc 3} = 2 \cos 3 - \sin 3$

$$y = -2 \cot x + (2 \cos 3 - \sin 3) \csc x$$

34. (a)



(c)



(b) $y' + 2xy = xy^2$

Bernoulli equation, $n = 2$ letting $z = y^{1-2} = y^{-1}$, you obtain $e^{-2x\,dx} = e^{-x^2}$ and $\int(-1)xe^{-x^2}dx = \frac{1}{2}e^{-x^2}$. The solution is:

$$y^{-1}e^{-x^2} = \frac{1}{2}e^{-x^2} + C$$

$$\frac{1}{y} = \frac{1}{2} + Ce^{x^2} = \frac{1 + 2Ce^{x^2}}{2}$$

$$y = \frac{2}{1 + 2Ce^{x^2}}$$

$(0, 3)$: $3 = \dfrac{2}{1 + 2C} \Rightarrow 1 + 2C = \dfrac{2}{3} \Rightarrow C = -\dfrac{1}{6}$

$$y = \frac{2}{1 - (e^{x^2}/3)} = \frac{6}{3 - e^{x^2}}$$

$(0, 1)$: $1 = \dfrac{2}{1 + 2C} \Rightarrow 1 + 2C = 2 \Rightarrow C = \dfrac{1}{2}$

$$y = \frac{2}{1 + e^{x^2}}$$

35. $$\frac{dP}{dt} = kP + N, N \text{ constant}$$

$$\frac{dP}{kP+N} = dt$$

$$\int \frac{1}{kP+N}\,dP = \int dt$$

$$\frac{1}{k}\ln(kP+N) = t + C_1$$

$$\ln(kP+N) = kt + C_2$$

$$kP + N = e^{kt+C_2}$$

$$P = \frac{C_3e^{kt} - N}{k}$$

$$P = Ce^{kt} - \frac{N}{k}$$

When $t = 0$: $P = P_0$

$$P_0 = C - \frac{N}{k} \implies C = P_0 + \frac{N}{k}$$

$$P = \left(P_0 + \frac{N}{k}\right)e^{kt} - \frac{N}{k}$$

36. $$\frac{dA}{dt} = rA + P$$

$$\frac{dA}{rA+P} = dt$$

$$\int \frac{dA}{rA+P} = \int dt$$

$$\frac{1}{r}\ln(rA+P) = t + C_1$$

$$\ln(rA+P) = rt + C_2$$

$$rA + P = e^{rt+C_2}$$

$$A = \frac{C_3e^{rt} - P}{r}$$

$$A = Ce^{rt} - \frac{P}{r}$$

When $t = 0$: $A = 0$

$$0 = C - \frac{P}{r} \implies C = \frac{P}{r}$$

$$A = \frac{P}{r}(e^{rt} - 1)$$

37. (a) $A = \dfrac{P}{r}(e^{rt} - 1)$

$$A = \frac{100{,}000}{0.06}(e^{0.06(5)} - 1) \approx 583{,}098.01$$

(b) $A = \dfrac{250{,}000}{0.05}(e^{0.05(10)} - 1) \approx 3{,}243{,}606.35$

38. $$800{,}000 = \frac{75{,}000}{0.08}(e^{0.08t} - 1)$$

$$1.85333333 = e^{0.08t}$$

$$t = \frac{\ln(1.85333333)}{0.08} \approx 7.71 \text{ years}$$

39. (a) $\dfrac{dQ}{dt} = q - kQ$, q constant

(b) $Q' + kQ = q$

Let $P(t) = k$, $Q(t) = q$, then the integrating factor is $u(t) = e^{kt}$.

$$Q = e^{-kt}\int qe^{kt}\,dt = e^{-kt}\left(\frac{q}{k}e^{kt} + C\right) = \frac{q}{k} + Ce^{-kt}$$

When $t = 0$: $Q = Q_0$

$$Q_0 = \frac{q}{k} + C \implies C = Q_0 - \frac{q}{k}$$

$$Q = \frac{q}{k} + \left(Q_0 - \frac{q}{k}\right)e^{-kt}$$

(c) $\displaystyle\lim_{t\to\infty} Q = \frac{q}{k}$

40. (a) $\dfrac{dN}{dt} = k(40 - N)$

(b) $N' + kN = 40k$

Integrating factor: e^{kt}

$$Ne^{kt} = \int 40ke^{kt}\,dt = 40e^{kt} + C$$

$$N = 40 + Ce^{-kt}$$

(c) For $t = 1$, $N = 10$:

$$10 = 40 + Ce^{-k} \implies -30 = Ce^{-k}$$

For $t = 20$, $N = 19$:

$$19 = 40 + Ce^{-20k} \implies -21 = Ce^{-20k}$$

$$\frac{30}{21} = \frac{e^{-k}}{e^{-20k}} = e^{19k}$$

$$\ln\left(\frac{10}{7}\right) = 19k \implies k = \frac{1}{19}\ln\left(\frac{10}{7}\right) \approx 0.0188$$

$$-30 = Ce^{-k} \implies C = -30e^{k} \approx -30.5685$$

$$N = 40 - 30.5685e^{-0.0188t}$$

41. Let Q be the number of pounds of concentrate in the solution at any time t. Since the number of gallons of solution in the tank at any time t is $v_0 + (r_1 - r_2)t$ and since the tank loses r_2 gallons of solution per minute, it must lose concentrate at the rate

$$\left[\frac{Q}{v_0 + (r_1 - r_2)t}\right]r_2.$$

The solution gains concentrate at the rate $r_1 q_1$. Therefore, the net rate of change is

$$\frac{dQ}{dt} = q_1 r_1 - \left[\frac{Q}{v_0 + (r_1 - r_2)t}\right]r_2 \quad \text{or} \quad \frac{dQ}{dt} + \frac{r_2 Q}{v_0 + (r_1 - r_2)t} = q_1 r_1.$$

42. From Exercise 41, and using $r_1 = r_2 = r$,

$$\frac{dQ}{dt} + \frac{rQ}{v_0} = q_1 r.$$

43. (a) $Q' + \dfrac{r^2 Q}{v_0 + (r_1 - r_2)t} = q_1 r_1$

$Q(0) = q_0$, $q_0 = 25$, $q_1 = 0$, $v_0 = 200$,

$r_1 = 10$, $r_2 = 10$, $Q' + \dfrac{1}{20}Q = 0$

$$\int \frac{1}{Q}\,dQ = \int -\frac{1}{20}\,dt$$

$$\ln Q = -\frac{1}{20}t + \ln C_1$$

$$Q = Ce^{-(1/20)t}$$

Initial condition: $Q(0) = 25$, $C = 25$

Particular solution: $Q = 25e^{-(1/20)t}$

(b) $15 = 25e^{-(1/20)t}$

$$\ln\left(\frac{3}{5}\right) = -\frac{1}{20}t$$

$$t = -20\ln\left(\frac{3}{5}\right) \approx 10.2 \text{ min}$$

(c) $\displaystyle\lim_{t\to\infty} 25e^{-(1/20)t} = 0$

44. (a) $Q' + \dfrac{r^2 Q}{v_0 + (r_1 - r_2)t} = q_1 r_1$

$Q(0) = q_0 = 25$, $q_1 = 0.04$, $v_0 = 200$,

$r_1 = 10$, $r_2 = 10$, $Q' + \dfrac{1}{20}Q = 0.4$

Integrating factor: $e^{1/20t}$

$$Qe^{-1/20t} = \int 0.4e^{(1/20)t}\,dt = 8e^{1/20t} + C$$

$$Q = 8 + Ce^{(-1/2)t}$$

$$Q = (0) = 25 = 8 + C \implies C = 17$$

$$Q = 8 + 17e^{(-1/20t)}$$

(b) $15 = 8 + 17e^{(-1/20)t}$

$7 = 17e^{-1/20t}$

$$\ln\left(\frac{7}{17}\right) = -\frac{1}{20}t \implies t = -20\ln\left(\frac{7}{17}\right) \approx 17.75 \text{ min}$$

(c) $\displaystyle\lim_{t\to\infty} Q(t) = 8$ lbs

45. (a) The volume of the solution in the tank is given by $v_0 + (r_1 - r_2)t$. Therefore, $100 + (5 - 3)t = 200$ or $t = 50$ minutes.

(b) $Q' + \dfrac{r_2 Q}{v_0 + (r_1 - r_2)t} = q_1 r_1$

$Q(0) = q_0,\ q_0 = 0,\ q_1 = 0.5,\ v_0 = 100,\ r_1 = 5,\ r_2 = 3,\ Q' + \dfrac{3}{100 + 2t}Q = 2.5$

Integrating factor: $e^{\int [3/(100+2t)]\,dt} = (50 + t)^{3/2}$

$$Q(50 + t)^{3/2} = \int 2.5(50 + t)^{3/2}\,dt = (50 + t)^{5/2} + C$$

$$Q = (50 + t) + C(50 + t)^{-3/2}$$

Initial condition: $Q(0) = 0,\ 0 = 50 + C(50^{-3/2}),\ C = -50^{5/2}$

Particular solution: $Q = (50 + t) - 50^{-5/2}(50 + t)^{-3/2}$

$$Q(50) = 100 - 50^{5/2}(100)^{-3/2} = 100 - \frac{25}{\sqrt{2}} \approx 82.32 \text{ lbs}$$

46. (a) The volume of the solution is given by $v_0 + (r_1 - r_2)t = 100 + (5 - 3)t = 200 \implies t = 50$ minutes.

(b) $Q' + \dfrac{r_2 Q}{v_0 + (r_1 - r_2)t} = q_1 r_1$

$Q(0) = q_0 = 0,\ q_1 = 1,\ v_0 = 100,\ r_1 = 5,\ r_2 = 3$

$Q' + \dfrac{3Q}{100 + 2t} = 5$

Integrating factor is $(50 + t)^{3/2}$, as in #43.

$$Q(50 + t)^{3/2} = \int 5(50 + t)^{3/2}\,dt = 2(50 + t)^{5/2} + C$$

$$Q = 2(50 + t) + C(50 + t)^{-3/2}$$

$Q(0) = 0{:}\ 0 = 100 + C(50)^{-3/2} \implies C = -100(50)^{3/2} = -2(50)^{5/2}$

$Q = 2(50 + t) - 2(50)^{5/2}(50 + t)^{-3/2}$

When $t = 50$, $Q = 200 - 2(50)^{5/2}(100)^{-3/2} \approx 164.64$ lbs (double the answer to #43)

47. From Example 6,

$$\frac{dv}{dt} + \frac{kv}{m} = g$$

$$v = \frac{mg}{k}(1 - e^{-kt/m}), \qquad \text{Solution}$$

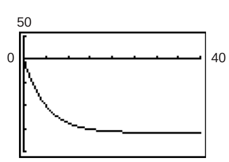
$g = -32,\ mg = -8,\ v(5) = -101,\ m = \dfrac{-8}{g} = \dfrac{1}{4}$ implies that

$$-101 = \frac{-8}{k}(1 - e^{-5k/(1/4)}).$$

Using a graphing utility, $k \approx 0.050165$, and

$$v = -159.47(1 - e^{-0.2007t}).$$

As $t \to \infty$, $v \to -159.47$ ft/sec. The graph of v is shown below.



48. $s(t) = \displaystyle\int v(t)\,dt$

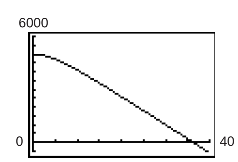
$$= \int -159.47(1 - e^{-0.2007t})\,dt$$

$$= -159.47t - 794.57e^{-0.2007t} + C$$

$s(0) = 5000 = -794.57 + C \implies C = 5794.57$

$s(t) = -159.47t - 794.57e^{-0.2007t} + 5794.57$

The graph of $s(t)$ is shown below.



$s(t) = 0$ when $t \approx 36.33$ sec.

49. $L\dfrac{dI}{dt} + RI = E_0,\ I' + \dfrac{R}{L}I = \dfrac{E_0}{L}$

Integrating factor: $e^{\int (R/L)\,dt} = e^{Rt/L}$

$$I\,e^{Rt/L} = \int \frac{E_0}{L}e^{Rt/L}\,dt = \frac{E_0}{R}e^{Rt/L} + C$$

$$I = \frac{E_0}{R} + Ce^{-Rt/L}$$

50. $I(0) = 0,\ E_0 = 120$ volts, $R = 600$ ohms, $L = 4$ henrys

$$I = \frac{E_0}{R} + Ce^{-Rt/L}$$

$$(0) = \frac{120}{600} + C \implies C = -\frac{1}{5}$$

$$I = \frac{1}{5} - \frac{1}{5}e^{-150t}$$

$$\lim_{t\to\infty} I = \frac{1}{5}\text{ amp}$$

$$(0.90)\frac{1}{5} = 0.18 = \frac{1}{5}(1 - e^{-150t})$$

$$0.9 = 1 - e^{-150t}$$

$$e^{-150t} = 0.1$$

$$-150t = \ln(0.1)$$

$$t = \frac{\ln(0.1)}{-150} \approx 0.0154\text{ sec}$$

51. $\dfrac{dy}{dx} + P(x)y = Q(x)$ Standard form

$u(x) = e^{\int P(x)\,dx}$ Integrating factor

52. $y' + P(x)y = Q(x)y^n$ Standard form

Let $z = y^{1-n}$ $(n \neq 0, 1)$. Multiplying by $(1-n)y^{-n}$ produces

$$(1-n)y^{-n}y' + (1-n)P(x)y^{1-n} = (1-n)Q(x)$$

$$z' + (1-n)P(x)z = (1-n)Q(x). \qquad \text{Linear}$$

53. $y' - 2x = 0$

$$\int dy = \int 2x\,dx$$

$$y = x^2 + C$$

Matches c.

54. $y' - 2y = 0$

$$\int \frac{dy}{y} = \int 2\,dx$$

$$\ln y = 2x + C_1$$

$$y = Ce^{2x}$$

Matches d.

55. $y' - 2xy = 0$

$$\int \frac{dy}{y} = \int 2x\,dx$$

$$\ln y = x^2 + C_1$$

$$y = Ce^{x^2}$$

Matches a.

56. $y' - 2xy = x$

$$\int \frac{dy}{2y+1} = \int x\,dx$$

$$\frac{1}{2}\ln(2y+1) = \frac{1}{2}x^2 + C_1$$

$$2y + 1 = C_2e^{x^2}$$

$$y = -\frac{1}{2} + Ce^{x^2}$$

Matches b.

57. $e^{2x+y}\,dx - e^{x-y}\,dy = 0$

Separation of variables:

$$e^{2x}e^{y}\,dx = e^{x}e^{-y}\,dy$$

$$\int e^{x}\,dx = \int e^{-2y}\,dy$$

$$e^{x} = -\frac{1}{2}e^{-2y} + C_1$$

$$2e^{x} + e^{-2y} = C$$

58. $(x + 1)\,dx - (y^2 + 2y)\,dy = 0$

Separation of variables:

$$\int (x + 1)\,dx = \int (y^2 + 2y)\,dy$$

$$\frac{1}{2}x^2 + x = \frac{1}{3}y^3 + y^2 + C_1$$

$$3x^2 + 6x - 2y^3 - 6y^2 = C$$

59. $(y \cos x - \cos x)\,dx + dy = 0$

Separation of variables:

$$\int \cos x\,dx = \int \frac{-1}{y - 1}\,dy$$

$$\sin x = -\ln(y - 1) + \ln C$$

$$\ln(y - 1) = -\sin x + \ln C$$

$$y = Ce^{-\sin x} + 1$$

60. $y' = 2x\sqrt{1 - y^2}$

Separation of variables:

$$\int \frac{1}{\sqrt{1 - y^2}}\,dy = \int 2x\,dx$$

$$\arcsin y = x^2 + C$$

$$y = \sin(x^2 + C)$$

61. $(3y^2 + 4xy)\,dx + (2xy + x^2)\,dy = 0$

Homogeneous: $y = vx$, $dy = v\,dx + x\,dv$

$$(3v^2x^2 + 4vx^2)\,dx + (2vx^2 + x^2)(v\,dx + x\,dv) = 0$$

$$\int \frac{5}{x}\,dx + \int \left(\frac{2v + 1}{v^2 + v}\right)dv = 0$$

$$\ln x^5 + \ln|v^2 + v| = \ln C$$

$$x^5(v^2 + v) = C$$

$$x^3y^2 + x^4y = C$$

62. $(x + y)\,dx - x\,dy = 0$

Linear: $y' - \frac{1}{x}y = 1$

Integrating factor: $e^{\int -(1/x)\,dx} = e^{\ln|x^{-1}|} = \frac{1}{x}$

$$y\frac{1}{x} = \int \frac{1}{x}\,dx = \ln|x| + C$$

$$y = x(\ln|x| + C)$$

63. $(2y - e^{x})\,dx + x\,dy = 0$

Linear: $y' + \left(\frac{2}{x}\right)y = \frac{1}{x}e^{x}$

Integrating factor: $e^{\int (2/x)\,dx} = e^{\ln x^2} = x^2$

$$yx^2 = \int x^2\frac{1}{x}e^{x}\,dx = e^{x}(x - 1) + C$$

$$y = \frac{e^{x}}{x^2}(x - 1) + \frac{C}{x^2}$$

64. $(y^2 + xy)\,dx - x^2\,dy = 0$

Homogeneous: $y = vx$, $dy = v\,dx + x\,dv$

$$(v^2x^2 + vx^2)\,dx - x^2(v\,dx + x\,dv) = 0$$

$$v^2\,dx - x\,dv = 0$$

$$\int \frac{1}{x}\,dx = \int \frac{1}{v^2}\,dv$$

$$\ln x = -\frac{1}{v} + C$$

$$y = \frac{x}{C - \ln|x|}$$

65. $(x^2y^4 - 1)\,dx + x^3y^3\,dy = 0$

$$y' + \left(\frac{1}{x}\right)y = x^{-3}y^{-3}$$

Bernoulli: $n = -3,\ Q = x^{-3},\ P = x^{-1},$

$$e^{\int (4/x)\,dx} = e^{\ln x^4} = x^4$$

$$y^4x^4 = \int 4(x^{-3})(x^4)\,dx = 2x^2 + C$$

$$x^4y^4 - 2x^2 = C$$

66. $y\,dx + (3x + 4y)\,dy = 0$

Homogeneous: $x = vy,\ dx = v\,dy + y\,dv$

$$y(v\,dy + y\,dv) + (3vy + 4y)\,dy = 0$$

$$\int \frac{1}{v+1}\,dv = \int -\frac{4}{y}\,dy$$

$$\ln|v + 1| = -\ln y^4 + \ln C$$

$$y^4(v + 1) = C$$

$$y^3(x + y) = C$$

67. $3(y - 4x^2)\,dx = -x\,dy$

$$x\frac{dy}{dx} = -3y + 12x^2$$

$$y' + \frac{3}{x}y = 12x$$

Integrating factor: $e^{\int (3/x)\,dx} = e^{3\ln x} = x^3$

$$y'x^3 + \frac{3}{x}x^3y = 12x(x^3) = 12x^4$$

$$yx^3 = \int 12x^4\,dx = \frac{12}{5}x^5 + C$$

$$y = \frac{12}{5}x^2 + \frac{C}{x^3}$$

68. $x\,dx + (y + e^y)(x^2 + 1)\,dy = 0$

Separation of variables:

$$\int \frac{x}{x^2+1}\,dx = \int -(y + e^y)\,dy$$

$$\frac{1}{2}\ln(x^2 + 1) = -\frac{1}{2}y^2 - e^y + C_1$$

$$\ln(x^2 + 1) + y^2 + 2e^y = C$$

69. False. The equation contains $\sqrt{y}$.

70. True. $y' + (x - e^x)y = 0$ is linear.

Review Exercises for Chapter 6

1. $y = x^3,\ y' = 3x^2$

$$x^2y' + 3y = x^2[3x^2] + 3[x^3] = 3(x^4 + x^3) \neq 6x^3$$

Not a solution

2.
$$y = 2\sin 2x$$
$$y' = 4\cos 2x$$
$$y'' = -8\sin 2x$$
$$y''' = -16\cos 2x$$
$$y''' - 8y = -16\cos 2x - 8(2\sin 2x) \neq 0$$

Not a solution

3. $\dfrac{dy}{dx} = 2x^2 + 5$

$$y = \int (2x^2 + 5)\,dx = \frac{2x^3}{3} + 5x + C$$

4. $\dfrac{dy}{dx} = x^3 - 2x$

$$y = \int (x^3 - 2x)\,dx = \frac{x^4}{4} - x^2 + C$$

5. $\dfrac{dy}{dx} = \cos 2x$

$$y = \int \cos 2x\,dx = \frac{1}{2}\sin 2x + C$$

6. $\dfrac{dy}{dx} = 2\sin x$

$$y = \int 2\sin x\,dx = -2\cos x + C$$

7. $\dfrac{dy}{dx} = 2x\sqrt{x-7}$

$y = \displaystyle\int 2x\sqrt{x-7}\,dx$

Let $u = x - 7$, $du = dx$, $x = u + 7$:

$$\begin{aligned} y &= \int 2(u+7)u^{1/2}\,du \\ &= \frac{4}{5}u^{5/2} + \frac{28}{3}u^{3/2} + C \\ &= \frac{4}{5}(x-7)^{5/2} + \frac{28}{3}(x-7)^{3/2} + C \\ &= \frac{4}{15}(x-7)^{3/2}(3x+14) + C \end{aligned}$$

8. $\dfrac{dy}{dx} = 3e^{-x/3}$

$y = \displaystyle\int 3e^{-x/3}\,dx = -9e^{-x/3} + C$

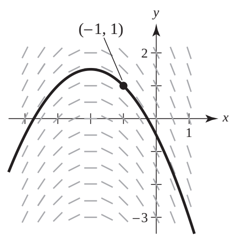
9. $\dfrac{dy}{dx} = \dfrac{2x}{y}$

x	-4	-2	0	2	4	8
y	2	0	4	4	6	8
dy/dx	-4	Undef.	0	1	$4/3$	2

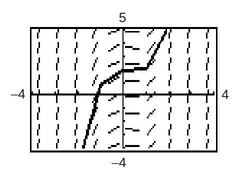
10. $\dfrac{dy}{dx} = x\sin\left(\dfrac{\pi y}{4}\right)$

x	-4	-2	0	2	4	8
y	2	0	4	4	6	8
dy/dx	-4	0	0	0	-4	0

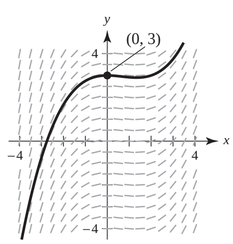
11. $y' = -x - 2$, $(-1, 1)$



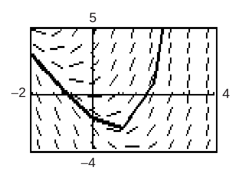
12. $y' = 2x^2 - x$, $(0, 2)$



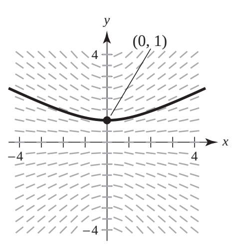
13. $y' = \dfrac{1}{4}x^2 - \dfrac{1}{3}x$, $(0, 3)$



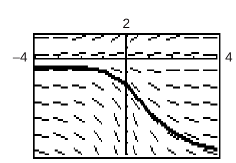
14. $y' = y + 3x$, $(2, 1)$



15. $y' = \dfrac{xy}{x^2+4}$, $(0, 1)$



16. $y' = \dfrac{y}{x^2+1}$, $(0, -2)$



17. $\dfrac{dy}{dx} = 6 - x$

$$y = \int (6 - x)\,dx = 6x - \frac{x^2}{2} + C$$

18. $\dfrac{dy}{dx} = y + 6$

$$\int \frac{dy}{y+6} = \int dx$$

$$\ln|y + 6| = x + C_1$$

$$|y + 6| = e^{x + C_1} = Ce^x$$

$$y = -6 + Ce^x$$

19. $\dfrac{dy}{dx} = (3 + y)^2$

$$\int (3 + y)^{-2}\,dy = \int dx$$

$$-(3 + y)^{-1} = x + C$$

$$3 + y = \frac{-1}{x + C}$$

$$y = -3 - \frac{1}{x + C}$$

20. $\dfrac{dy}{dx} = 4\sqrt{y}$

$$\int y^{-1/2}\,dy = \int 4\,dx$$

$$2y^{1/2} = 4x + C_1$$

$$y^{1/2} = 2x + C \qquad \left(C = \frac{C_1}{2}\right)$$

$$y = (2x + C)^2$$

21. $(2 + x)y' - xy = 0$

$$(2 + x)\frac{dy}{dx} = xy$$

$$\frac{1}{y}\,dy = \frac{x}{2 + x}dx$$

$$\frac{1}{y}\,dy = \left(1 - \frac{2}{2 + x}\right)dx$$

$$\ln|y| = x - 2\ln|2 + x| + C_1$$

$$y = Ce^x(2 + x)^{-2} = \frac{Ce^x}{(2 + x)^2}$$

22. $xy' - (x + 1)y = 0$

$$x\frac{dy}{dx} = (x + 1)y$$

$$\int \frac{dy}{y} = \int \frac{x + 1}{x}\,dx$$

$$\ln|y| = x + \ln|x| + C_1$$

$$y = Cxe^x$$

23. $y = Ce^{kt}$

$\left(0, \frac{3}{4}\right)$: $\dfrac{3}{4} = C$

$(5, 5)$: $\quad 5 = \dfrac{3}{4}e^{k(5)}$

$$\frac{20}{3} = e^{5k}$$

$$k = \frac{1}{5}\ln\left(\frac{20}{3}\right)$$

$$y = \frac{3}{4}e^{[\ln(20/3)/5]t} \approx \frac{3}{4}e^{0.379t}$$

24. $y = Ce^{kt}$

$\left(2, \frac{3}{2}\right)$: $\dfrac{3}{2} = Ce^{2k} \implies C = \dfrac{3}{2}e^{-2k}$

$(4, 5)$: $5 = Ce^{4k} = \left(\dfrac{3}{2}e^{-2k}\right)e^{4k} = \dfrac{3}{2}e^{2k}$

$$\frac{10}{3} = e^{2k} \implies k = \frac{1}{2}\ln\left(\frac{10}{3}\right)$$

Hence, $C = \dfrac{3}{2}e^{-2(1/2)\ln(10/3)} = \dfrac{3}{2}\left(\dfrac{3}{10}\right) = \dfrac{9}{20}$.

Finally, $y = \dfrac{9}{20}e^{1/2\,\ln(10/3)t}$.

25. $y = Ce^{kt}$

$(0, 5)$: $C = 5$

$\left(5, \frac{1}{6}\right)$: $\frac{1}{6} = 5e^{5k}$

$$k = \frac{1}{5}\ln\left(\frac{1}{30}\right) = \frac{-\ln 30}{5}$$

$y = 5e^{[-t\ln 30]/5} \approx 5e^{-0.680t}$

26. $y = Ce^{kt}$

$(1, 9)$: $9 = Ce^{k} \implies C = 9e^{-k}$

$(6, 2)$: $2 = Ce^{6k} \implies 2 = (9e^{-k})e^{6k} = 9e^{5k}$

$$k = \frac{1}{5}\ln\left(\frac{2}{9}\right) \approx -0.3008$$

Hence, $C = 9e^{-1/5\ln(2/9)} = 9\left(\frac{2}{9}\right)^{-1/5} \approx 12.15864.$

Finally, $y \approx 12.1586e^{-0.3008t}$.

27. $\frac{dP}{dh} = kp$, $P(0) = 30$

$$P(h) = 30e^{kh}$$

$$P(18{,}000) = 30e^{18{,}000k} = 15$$

$$k = \frac{\ln(1/2)}{18{,}000} = \frac{-\ln 2}{18{,}000}$$

$$P(h) = 30e^{-(h\ln 2)/18{,}000}$$

$$P(35{,}000) = 30e^{-(35{,}000\ln 2)/18{,}000} \approx 7.79 \text{ inches}$$

28. $y = Ce^{kt} = 5e^{kt}$

$$2.5 = 5e^{k(1599)}$$

$$k = \frac{1}{1599}\ln\left(\frac{1}{2}\right) \approx -0.000433$$

When $t = 600$, $y = 5e^{-0.000433(600)} \approx 3.86$ g.

29. $S = Ce^{k/t}$

(a) $S = 5$ when $t = 1$

$$5 = Ce^{k}$$

$$\lim_{t\to\infty} Ce^{k/t} = C = 30$$

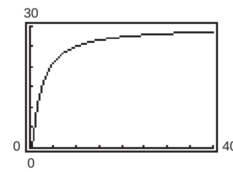
$$5 = 30e^{k}$$

$$k = \ln\tfrac{1}{6} \approx -1.7918$$

$$S = 30\left(\tfrac{1}{6}\right) \approx 30e^{-1.7918/t}$$

(b) When $t = 5$, $S \approx 20.9646$ which is 20,965 units.

(c)



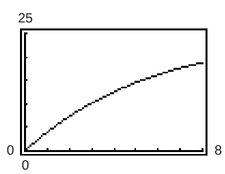
30. $S = 25(1 - e^{kt})$

(a) $4 = 25(1 - e^{k(1)}) \implies 1 - e^{k} = \frac{4}{25} \implies e^{k} = \frac{21}{25} \implies k = \ln\left(\frac{21}{25}\right) \approx -0.1744$

(b) 25,000 units $\left(\lim_{t\to\infty} S = 25\right)$

(c) When $t = 5$, $S \approx 14.545$ which is 14,545 units.

(d)



31.

$$P = Ce^{0.015t}$$

$$2C = Ce^{0.015t}$$

$$2 = e^{0.015t}$$

$$\ln 2 = 0.015t$$

$$t = \frac{\ln 2}{0.015} \approx 46.21 \text{ years}$$

32. (a) $\dfrac{dy}{ds} = -0.012y,\ s > 50$

$$\frac{-1}{0.012}\int \frac{dy}{y} = \int ds$$

$$\frac{-1}{0.012}\ln y = s + C_1$$

$$y = Ce^{-0.012s}$$

When $s = 50$, $y = 28 = Ce^{-0.012(50)} \implies C = 28e^{0.6}$

$$y = 28e^{0.6-0.012s},\ s > 50.$$

(b)

Speed(s)	50	55	60	65	70
Miles per Gallon (y)	28	26.4	24.8	23.4	22.0

33. $\dfrac{dy}{dx} = \dfrac{x^2+3}{x}$

$$\int dy = \int\left(x + \frac{3}{x}\right)dx$$

$$y = \frac{x^2}{2} + 3\ln|x| + C$$

34. $\dfrac{dy}{dx} = \dfrac{e^{-2x}}{1+e^{-2x}}$

$$\int dy = \int \frac{e^{-2x}}{1+e^{-2x}}\,dx = -\frac{1}{2}\int \frac{-2e^{-2x}}{1+e^{-2x}}\,dx$$

$$y = -\frac{1}{2}\ln(1+e^{-2x}) + C$$

35. $y' - 2xy = 0$

$$\frac{dy}{dx} = 2xy$$

$$\int \frac{1}{y}\,dy = \int 2x\,dx$$

$$\ln|y| = x^2 + C_1$$

$$e^{x^2+C_1} = y$$

$$y = Ce^{x^2}$$

36. $y' - e^y \sin x = 0$

$$\frac{dy}{dx} = e^y \sin x$$

$$\int e^{-y}\,dy = \int \sin x\,dx$$

$$-e^{-y} = -\cos x + C_1$$

$$e^y = \frac{1}{\cos x + C} \qquad (C = -C_1)$$

$$y = \ln\left|\frac{1}{\cos x + C}\right| = -\ln|\cos x + C|$$

37. $\dfrac{dy}{dx} = \dfrac{x^2+y^2}{2xy}$ (homogeneous differential equation)

$$(x^2+y^2)\,dx - 2xy\,dy = 0$$

Let $y = vx$, $dy = x\,dv + v\,dx$.

$$(x^2 + v^2x^2)\,dx - 2x(vx)(x\,dv + v\,dx) = 0$$

$$(x^2 + v^2x^2 - 2x^2v^2)\,dx - 2x^3v\,dv = 0$$

$$(x^2 - x^2v^2)\,dx = 2x^3v\,dv$$

$$(1 - v^2)\,dx = 2xv\,dv$$

$$\int \frac{dx}{x} = \int \frac{2v}{1-v^2}\,dv$$

$$\ln|x| = -\ln|1-v^2| + C_1 = -\ln|1-v^2| + \ln C$$

$$x = \frac{C}{1-v^2} = \frac{C}{1-(y/x)^2} = \frac{Cx^2}{x^2-y^2}$$

$$1 = \frac{Cx}{x^2-y^2} \quad \text{or} \quad C_1 = \frac{x}{x^2-y^2}$$

38. $\dfrac{dy}{dx} = \dfrac{3(x + y)}{x}$ (homogeneous differential equation)

$$3(x + y)\,dx - x\,dy = 0$$

Let $y = vx$, $dy = x\,dv + v\,dx$.

$$\begin{aligned}
3(x + vx)\,dx - x(x\,dv + v\,dx) &= 0\\
(3x + 2vx)\,dx - x^2\,dv &= 0\\
(3 + 2v)\,dx &= x\,dv\\
\int \frac{1}{x}\,dx &= \int \frac{1}{3 + 2v}\,dv\\
\ln|x| = \frac{1}{2}\ln|3 + 2v| + C_1 &= \ln(3 + 2v)^{1/2} + \ln C_2\\
x &= C_2(3 + 2v)^{1/2}\\
x^2 = C(3 + 2v) &= C\left(3 + 2\left(\frac{y}{x}\right)\right)\\
x^3 = C(3x + 2y) &= 3Cx + 2Cy\\
y &= \frac{x^3 - 3Cx}{2C}
\end{aligned}$$

39. $y = C_1x + C_2x^3$

$y' = C_1 + 3C_2x^2$

$y'' = 6C_2x$

$$\begin{aligned}
x^2y'' - 3xy' + 3y &= x^2(6C_2x) - 3x(C_1 + 3C_2x^2) + 3(C_1x + C_2x^3)\\
&= 6C_2x^3 - 3C_1x - 9C_2x^3 + 3C_1x + 3C_2x^3 = 0
\end{aligned}$$

$x = 2, y = 0$: $0 = 2C_1 + 8C_2 \implies C_1 = -4C_2$

$x = 2, y' = 4$: $4 = C_1 + 12C_2$

$$4 = (-4C_2) + 12C_2 = 8C_2 \implies C_2 = \tfrac{1}{2},\ C_1 = -2$$

$y = -2x + \frac{1}{2}x^3$

40. $\dfrac{dv}{dt} = kv - 9.8$

(a)
$$\begin{aligned}
\int \frac{dv}{kv - 9.8} &= \int dt\\
\frac{1}{k}\ln|kv - 9.8| &= t + C_1\\
\ln|kv - 9.8| &= kt + C_2\\
kv - 9.8 &= e^{kt + C_2} = C_3e^{kt}\\
v &= \frac{1}{k}\left[9.8 + C_3e^{kt}\right]
\end{aligned}$$

At $t = 0$, $v_0 = \dfrac{1}{k}(9.8 + C_3) \implies C_3 = kv_0 - 9.8$

$$v = \frac{1}{k}[9.8 + (kv_0 - 9.8)e^{kt}].$$

Note that $k < 0$ since the object is moving downward.

(b) $\displaystyle\lim_{t\to\infty} v(t) = \frac{9.8}{k}$

(c)
$$\begin{aligned}
s(t) &= \int \frac{1}{k}[9.8 + (kv_0 - 9.8)e^{kt}]\,dt\\
&= \frac{1}{k}\left[9.8t + \frac{1}{k}(kv_0 - 9.8)e^{kt}\right] + C\\
&= \frac{9.8t}{k} + \frac{1}{k^2}(kv_0 - 9.8)e^{kt} + C
\end{aligned}$$

$$s(0) = \frac{1}{k^2}(kv_0 - 9.8) + C \implies C = s_0 - \frac{1}{k^2}(kv_0 - 9.8)$$

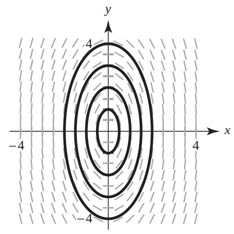
$$\begin{aligned}
s(t) &= \frac{9.8t}{k} + \frac{1}{k^2}(kv_0 - 9.8)e^{kt} + s_0 - \frac{1}{k^2}(kv_0 - 9.8)\\
&= \frac{9.8t}{k} + \frac{1}{k^2}(kv_0 - 9.8)(e^{kt} - 1) + s_0
\end{aligned}$$

41. $\dfrac{dy}{dx} = \dfrac{-4x}{y}$

$$\int y\,dy = \int -4x\,dx$$

$$\frac{y^2}{2} = -2x^2 + C_1$$

$4x^2 + y^2 = C$ ellipses



42. $\dfrac{dy}{dx} = 3 - 2y$

$$\int \frac{dy}{2y-3} = \int -dx$$

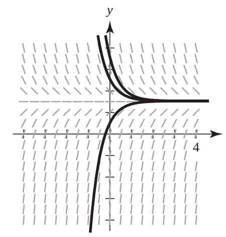
$$\frac{1}{2}\ln|2y-3| = -x + C_1$$

$$\ln|2y-3| = -2x + 2C_1$$

$$|2y-3| = C_2 e^{-2x}$$

$$2y = 3 + C_2 e^{-2x}$$

$$y = \frac{3}{2} + Ce^{-2x}$$



43. $P(t) = \dfrac{7200}{1 + 44e^{-0.55t}}$

(a) $k = 0.55$

(b) $L = 7200$

(c) $P(0) = \dfrac{7200}{1+44} = 160$

(d) $3600 = \dfrac{7200}{1 + 44e^{-0.55t}}$

$$1 + 44e^{-0.55t} = 2$$

$$e^{-0.55t} = \frac{1}{44}$$

$$t = \frac{-1}{0.55}\ln\left(\frac{1}{44}\right) \approx 6.88 \text{ yrs.}$$

(e) $\dfrac{dP}{dt} = 0.55P\left(1 - \dfrac{P}{7200}\right)$

44. $P(t) = \dfrac{4800}{1 + 14e^{-0.15t}}$

(a) $k = 0.15$

(b) $L = 4800$

(c) $P(0) = \dfrac{4800}{1+14} = 320$

(d) $2400 = \dfrac{4800}{1 + 14e^{-0.15t}}$

$$14e^{-0.15t} = 1$$

$$t = -\frac{1}{0.15}\ln\left(\frac{1}{14}\right) \approx 17.59 \text{ yrs.}$$

(e) $\dfrac{dP}{dt} = 0.15P\left(1 - \dfrac{P}{4800}\right)$

45. (a) $L = 20{,}400$, $y(0) = 1200$, $y(1) = 2000$

$$y = \frac{20{,}400}{1 + be^{-kt}}$$

$$y(0) = 1200 = \frac{20{,}400}{1+b} \implies b = 16$$

$$y(1) = 2000 = \frac{20{,}400}{1 + 16e^{-k}}$$

$$16e^{-k} = \frac{46}{5}$$

$$k = -\ln\frac{23}{40} = \ln\frac{40}{23} \approx 0.553$$

$$y = \frac{20{,}400}{1 + 16e^{-0.553t}}$$

(b) $y(8) \approx 17{,}118$ trout

(c) $10{,}000 = \dfrac{20{,}400}{1 + 16e^{-0.553t}} \implies t \approx 4.94$ yrs.

46. $\dfrac{dy}{dt} = 0.553y\left(1 - \dfrac{y}{20{,}400}\right)$, $y(0) = 1200$

Use Euler's method with $h = 1$.

t	0	2	4	6	8
Exact	1200	3241	7414	12,915	17,117
Euler	1200	2743	5853	10,869	16,170

Euler's method gives $y(8) \approx 16{,}170$ trout.

47. $y' - y = 8$

$P(x) = -1,\ Q(x) = 8$

$u(x) = e^{\int -dx} = e^{-x}$

$$y = \frac{1}{e^{-x}}\int 8e^{-x}\,dx$$

$$= e^{x}[-8e^{-x} + C]$$

$$= -8 + Ce^{x}$$

48. $e^{x}y' + 4e^{x}y = 1$

$y' + 4y = e^{-x}$

$P(x) = 4,\ Q(x) = e^{-x}$

$u(x) = e^{\int 4\,dx} = e^{4x}$

$$y = \frac{1}{e^{4x}}\int e^{-x}e^{4x}\,dx$$

$$= e^{-4x}\left[\frac{1}{3}e^{3x} + C\right]$$

$$= \frac{1}{3}e^{-x} + Ce^{-4x}$$

49. $4y' = e^{x/y} + y$

$y' - \frac{1}{4}y = \frac{1}{4}e^{x/4}$

$P(x) = -\frac{1}{4},\ Q(x) = \frac{1}{4}e^{x/4}$

$u(x) = e^{\int -(1/4)\,dx} = e^{-(1/4)x}$

$$y = \frac{1}{e^{-(1/4)x}}\int \frac{1}{4}e^{x/4}e^{-(1/4)x}\,dx$$

$$= e^{(1/4)x}\left[\frac{1}{4}x + C\right]$$

$$= \frac{1}{4}xe^{x/4} + Ce^{x/4}$$

50. $\frac{dy}{dx} - \frac{5y}{x^2} = \frac{1}{x^2}$

$P(x) = -\frac{5}{x^2},\ Q(x) = \frac{1}{x^2}$

$u(x) = e^{\int -(5/x^2)dx} = e^{5/x}$

$$y = \frac{1}{e^{5/x}}\int \frac{1}{x^2}e^{5/x}\,dx$$

$$= \frac{1}{e^{5/x}}\left[-\frac{1}{5}e^{5/x} + C\right]$$

$$= -\frac{1}{5} + Ce^{-5/x}$$

51. $(x - 2)y' + y = 1$

$\frac{dy}{dx} + \frac{1}{x-2}y = \frac{1}{x-2}$

$P(x) = \frac{1}{x-2},\ Q(x) = \frac{1}{x-2}$

$u(x) = e^{\int (1/x-2)\,dx} = e^{\ln|x-2|} = x - 2$

$$y = \frac{1}{x-2}\int\left(\frac{1}{x-2}\right)(x-2)\,dx$$

$$= \frac{1}{x-2}[x + c]$$

52. $(x + 3)y' + 2y = 2(x + 3)^2$

$\frac{dy}{dx} + \frac{2}{x+3}y = 2(x + 3)$

$P(x) = \frac{2}{x+3},\ Q(x) = 2(x + 3)$

$u(x) = e^{\int (2/x+3)\,dx} = e^{2\ln(x+3)} = (x + 3)^2$

$$y = \frac{1}{(x+3)^2}\int 2(x + 3)(x + 3)^2\,dx$$

$$= \frac{1}{(x+3)^2}\left[\frac{(x+3)^4}{2} + C\right]$$

$$= \frac{(x+3)^2}{2} + \frac{C}{(x+3)^2}$$

53. $(3y + \sin 2x)\,dx - dy = 0$

$y' - 3y = \sin 2x$

Integrating factor: $e^{\int -3\,dx} = e^{-3x}$

$$ye^{-3x} = \int e^{-3x}\sin 2x\,dx$$

$$= \frac{1}{13}e^{-3x}(-3\sin 2x - 2\cos 2x) + C$$

$$y = -\frac{1}{13}(3\sin 2x + 2\cos 2x) + Ce^{3x}$$

54. $dy = (y\tan x + 2e^{x})\,dx$

$\frac{dy}{dx} - (\tan x)y = 2e^{x}$

Integrating factor: $e^{-\int \tan x\,dx} = e^{\ln|\cos x|} = \cos x$

$$y\cos x = \int 2e^{x}\cos x\,dx = e^{x}(\cos x + \sin x) + C$$

$$y = e^{x}(1 + \tan x) + C\sec x$$

55. $y' + 5y = e^{5x}$

Integrating factor: $e^{\int 5\,dx} = e^{5x}$

$$ye^{5x} = \int e^{10x}\,dx = \frac{1}{10}e^{10x} + C$$

$$y = \frac{1}{10}e^{5x} + Ce^{-5x}$$

56. $y' - \left(\frac{a}{x}\right)y = bx^3$

Integrating factor: $e^{-\int (a/x)\,dx} = e^{-a\ln x} = x^{-a}$

$$yx^{-a} = \int bx^3(x^{-a})\,dx = \frac{b}{4-a}x^{4-a} + C$$

$$y = \frac{bx^4}{4-a} + Cx^a$$

57. $y' + y = xy^2$ Bernoulli equation

$n = 2$, let $z = y^{1-2} = y^{-1}$, $z' = -y^{-2}y'$.

$$(-y^{-2})y' + (-y^{-2})y = -x$$

$$z' - z = -x \qquad \text{Linear equation}$$

$$u(x) = e^{\int -dx} = e^{-x}$$

$$z = \frac{1}{e^{-x}}\int -xe^{-x}\,dx = e^x[xe^{-x} + e^{-x} + C]$$

$$y^{-1} = x + 1 + Ce^x$$

$$y = \frac{1}{x + 1 + Ce^x}$$

58. $y' + 2xy = xy^2$ Bernoulli equation

$n = 2$, let $z = y^{1-2} = y^{-1}$, $z' = -y^{-2}y'$.

$$(-y^{-2})y' + 2xy(-y^{-2}) = -x$$

$$z' - 2xz = -x \qquad \text{Linear equation}$$

$$u(x) = e^{\int -2x\,dx} = e^{-x^2}$$

$$z = \frac{1}{e^{-x^2}}\int(-x)e^{-x^2}\,dx = e^{x^2}\left[\frac{1}{2}e^{-x^2} + C\right]$$

$$\frac{1}{y} = \frac{1}{2} + Ce^{x^2}$$

$$y = \frac{1}{\frac{1}{2} + Ce^{x^2}} = \frac{2}{1 + C_1e^{x^2}}$$

59. $y' + \frac{1}{x}y = \frac{y^3}{x^2}$ Bernoulli equation

$n = 3$, let $z = y^{1-3} = y^{-2}$, $z' = -2y^{-3}y'$.

$$(-2y^{-3})y' + \frac{1}{x}y(-2y^{-3}) = \frac{-2}{x^2}$$

$$z' - \frac{2}{x}z = \frac{-2}{x^2} \qquad \text{Linear equation}$$

$$u(x) = e^{\int -(2/x)\,dx} = e^{-2\ln x} = x^{-2}$$

$$z = \frac{1}{x^{-2}}\int\frac{-2}{x^2}(x^{-2})\,dx = x^2\left[\frac{2x^{-3}}{3} + C\right]$$

$$\frac{1}{y^2} = \frac{2}{3x} + Cx^2$$

60. $xy' + y = xy^2$

$y' + \frac{1}{x}y = y^2$ Bernoulli Equation

$n = 2$, let $z = y^{1-2} = y^{-1}$, $z' = -y^{-2}y'$.

$$-y^{-2}y' + \frac{1}{x}y(-y^{-2}) = y^2(-y^{-2})$$

$$z' - \frac{1}{x}z = -1 \qquad \text{Linear equation}$$

$$u(x) = e^{\int -(1/x)\,dx} = \frac{1}{x}$$

$$z = x\int -\frac{1}{x}\,dx = -x[\ln|x| + C]$$

$$\frac{1}{y} = -x\ln x + Cx$$

$$y = \frac{1}{Cx - x\ln x}$$

61. Answers will vary. Sample Answer: $(x^2 + 3y^2)\,dx - 2xy\,dy = 0$

Solution: Let $y = vx$, $dy = x\,dv + v\,dx$.

$$(x^2 + 3v^2x^2)\,dx - 2x(vx)(x\,dv + v\,dx) = 0$$

$$(x^2 + v^2x^2)\,dx - 2x^3v\,dv = 0$$

$$(1 + v^2)\,dx = 2\,xv\,dv$$

$$\int\frac{dx}{x} = \int\frac{2v}{1+v^2}\,dv$$

$$\ln|x| = \ln|1 + v^2| + C_1$$

$$x = C(1 + v^2) = C\left(1 + \frac{y^2}{x^2}\right)$$

$$x^3 = C(x^2 + y^2)$$

62. Answers will vary.

63. Answers will vary.

Sample Answer: $x^3y' + 2x^2y = 1$

$$y' + \frac{2}{x}y = \frac{1}{x^3}$$

$$u(x) = e^{\int (2/x)\,dx} = x^2$$

$$y = \frac{1}{x^2}\int \frac{1}{x^3}(x^2)\,dx = \frac{1}{x^2}[\ln|x| + C]$$

64. Answers will vary.

Problem Solving for Chapter 6

1. (a)
$$\frac{dy}{dt} = y^{1.01}$$

$$\int y^{-1.01}\,dy = \int dt$$

$$\frac{y^{-0.01}}{-0.01} = t + C_1$$

$$\frac{1}{y^{0.01}} = -0.01t + C$$

$$y^{0.01} = \frac{1}{C - 0.01t}$$

$$y = \frac{1}{(C - 0.01t)^{100}}$$

$y(0) = 1$: $1 = \dfrac{1}{C^{100}} \Rightarrow C = 1$

Hence, $y = \dfrac{1}{(1 - 0.01t)^{100}}$.

For $T = 100$, $\lim\limits_{t\to T^-} y = \infty$.

(b)
$$\int y^{-(1+\varepsilon)}\,dy = \int k\,dt$$

$$\frac{y^{-\varepsilon}}{-\varepsilon} = kt + C_1$$

$$y^{-\varepsilon} = -\varepsilon kt + C$$

$$y = \frac{1}{(C - \varepsilon kt)^{1/\varepsilon}}$$

$$y(0) = y_0 = \frac{1}{C^{1/\varepsilon}} \Rightarrow C^{1/\varepsilon} = \frac{1}{y_0} \Rightarrow C = \left(\frac{1}{y_0}\right)^{\varepsilon}$$

Hence, $y = \dfrac{1}{\left(\dfrac{1}{y_0^{\varepsilon}} - \varepsilon kt\right)^{1/\varepsilon}}$.

For $t \to \dfrac{1}{y_0^{\varepsilon}\varepsilon k}$, $y \to \infty$.

2. Since $\dfrac{dy}{dt} = k(y - 20)$,

$$\int \frac{1}{y - 20}\,dy = \int k\,dt$$

$$\ln|y - 20| = kt + C$$

$$y = Ce^{kt} + 20.$$

When $t = 0$, $y = 72$. Therefore, $C = 52$.

When $t = 1$, $y = 48$. Therefore, $48 = 52e^k + 20$, $e^k = \frac{28}{52} = \frac{7}{13}$, and $k = \ln\frac{7}{13}$. Thus, $y = 52e^{[\ln(7/13)]t} + 20$.

When $t = 5$, $y = 52e^{5\ln(7/13)} + 20 \approx 22.35°$.

3. (a) $\dfrac{dS}{dt} = k_1 S(L - S)$

$S = \dfrac{L}{1 + Ce^{-kt}}$ is a solution because

$$\frac{dS}{dt} = -L(1 + Ce^{-kt})^{-2}(-Cke^{-kt})$$

$$= \frac{LC\,ke^{-kt}}{(1 + Ce^{-kt})^2}$$

$$= \left(\frac{k}{L}\right)\frac{L}{1 + Ce^{-kt}} \cdot \frac{C\,Le^{-kt}}{1 + Ce^{-kt}}$$

$$= \left(\frac{k}{L}\right)\frac{L}{1 + Ce^{-kt}} \cdot \left(L - \frac{L}{1 + Ce^{-kt}}\right)$$

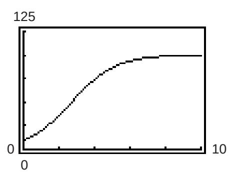
$$= k_1 S(L - S), \text{ where } k_1 = \frac{k}{L}.$$

$L = 100$. Also, $S = 10$ when $t = 0 \implies C = 9$. And, $S = 20$ when $t = 1 \implies k = -\ln\frac{4}{9}$.

Particular Solution: $s = \dfrac{100}{1 + 9e^{\ln(4/9)t}}$

$$= \frac{100}{1 + 9e^{-0.8109t}}$$

(c)



(b) $\dfrac{dS}{dt} = k_1 S(100 - S)$

$$\frac{d^2S}{dt^2} = k_1\left[S\left(-\frac{dS}{dt}\right) + (100 - S)\frac{dS}{dt}\right]$$

$$= k_1(100 - 2S)\frac{dS}{dt}$$

$$= 0 \text{ when } S = 50 \text{ or } \frac{dS}{dt} = 0.$$

Choosing $S = 50$, we have:

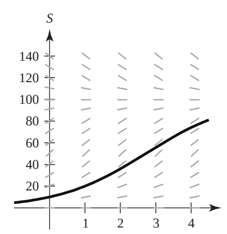
$$50 = \frac{100}{1 + 9e^{\ln(4/9)t}}$$

$$2 = 1 + 9e^{\ln(4/9)t}$$

$$\frac{\ln(1/9)}{\ln(4/9)} = t$$

$t \approx 2.7$ months (This is the inflection point.)

(d)



(e) Sales will decrease toward the line $S = L$.

4. (a) $\dfrac{dy}{dt} = k\ln\left(\dfrac{L}{y}\right)y$

$$\frac{dy}{y[\ln L - \ln y]} = k\,dt$$

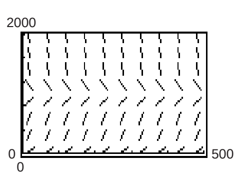
$$\ln[\ln L - \ln y] = -kt + C_1$$

$$\ln\frac{L}{y} = Ce^{-kt}$$

$$\frac{L}{y} = e^{Ce^{-kt}}$$

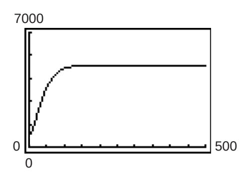
$$y = Le^{-Ce^{-kt}}$$

(b)



(c) As $t \to \infty$, $y \to L$, the carrying capacity.

(d) $y_0 = 500 = 5000e^{-C} \implies e^C = 10 \implies C = \ln 10$



$$\frac{dy}{dt} = k\ln\left(\frac{L}{y}\right)y$$

$$\frac{d^2y}{dt^2} = k\ln\left(\frac{L}{y}\right)\frac{dy}{dt} + ky\frac{1}{(L/y)}\left(\frac{-L}{y^2}\right)\frac{dy}{dt}$$

$$= k\frac{dy}{dt}\left[\ln\left(\frac{L}{y}\right) - 1\right]$$

$$= k^2\ln\left(\frac{L}{y}\right)y\left[\ln\left(\frac{L}{y}\right) - 1\right]$$

Hence, $\dfrac{d^2y}{dt^2} = 0$ when $\ln\left(\dfrac{L}{y}\right) = 1 \implies \dfrac{L}{y} = e \implies y = \dfrac{L}{e}$.

$y = \dfrac{L}{e} = \dfrac{5000}{e} \approx 1839.4$ and $t \approx 41.7$.

The graph is concave upward on $(0, 41.7)$ and downward on $(41.7, \infty)$.

5. Let $u = \frac{1}{2}k\left(t - \frac{\ln b}{k}\right)$.

$$1 + \tanh u = 1 + \frac{e^4 - e^{-u}}{e^u + e^{-u}} = \frac{2}{1 + e^{-2u}}$$

$$e^{-2u} = e^{-k(t-(\ln b/k))} = e^{\ln b}e^{-kt} = be^{-kt}$$

Finally,

$$\begin{aligned}\frac{1}{2}L\left[1 + \tanh\left(\frac{1}{2}k\left(t - \frac{\ln b}{k}\right)\right)\right] &= \frac{L}{2}[1 + \tanh u]\\ &= \frac{L}{2}\frac{2}{1 + be^{-kt}}\\ &= \frac{L}{1 + be^{-kt}}.\end{aligned}$$

The graph of the logistics function is just a shift of the graph of the hyperbolic tangent, as shown in Section 5.10.

6. $k = \left(\frac{1}{12}\right)^2\pi$

$g = 32$

$x^2 + (y - 6)^2 = 36$ Equation of tank

$$x^2 = 36 - (y - 6)^2 = 12y - y^2$$

$A(h) = (12h - h^2)\pi$ Area of cross section

$$A(h)\frac{dh}{dt} = -k\sqrt{2gh}$$

$$(12h - h^2)\pi\frac{dh}{dt} = -\frac{1}{144}\pi\sqrt{64h}$$

$$(12h - h^2)\frac{dh}{dt} = -\frac{1}{18}h^{1/2}$$

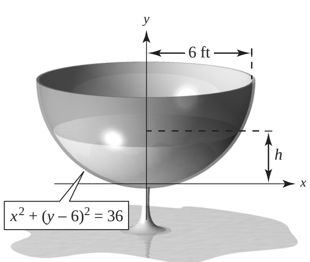
$$\int(18h^{3/2} - 216h^{1/2})\,dh = \int dt$$

$$\frac{36}{5}h^{5/2} - 144h^{3/2} = t + C$$

$$\frac{h^{3/2}}{5}(36h - 720) = t + C$$

When $h = 6$, $t = 0$ and $C = \frac{6^{3/2}}{5}(-504) \approx -1481.45$.

The tank is completely drained when $h = 0 \implies t = 1481.45$ seconds ≈ 24 minutes and 41 seconds.



7. (a) $A(h)\frac{dh}{dt} = -k\sqrt{2gh}$

$$\pi r^2\frac{dh}{dt} = -k\sqrt{64h}$$

$$h^{-1/2}\,dh = \frac{-8k}{\pi r^2}\,dt = -C\,dt, \quad C = \frac{8k}{\pi r^2}$$

$$2\sqrt{h} = -Ct + C_1$$

$2\sqrt{18} = C_1$ (at $t = 0$, $h = 18$)

Hence, $2\sqrt{h} = -Ct + 6\sqrt{2}$.

At $t = 30(60) = 1800$, $h = 12$:

$$2\sqrt{12} = -1800\,C + 6\sqrt{2}$$

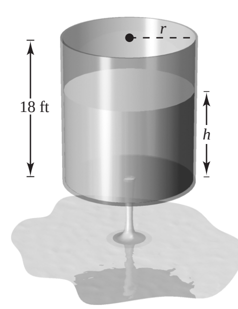
$$\frac{6\sqrt{2} - 4\sqrt{3}}{1800} = C \approx 0.000865$$

Hence, $2\sqrt{h} = -0.000865t + 6\sqrt{2}$.

$h = 0 \implies t = \frac{6\sqrt{2}}{0.000865} \approx 9809.1$ seconds (2 hr, 43 min, 29 sec)

(b) $t = 3600$ sec $\implies 2\sqrt{h} = -0.000865(3600) + 6\sqrt{2}$

$$\implies h \approx 7.21 \text{ feet}$$



8. $$A(h)\frac{dh}{dt} = -k\sqrt{2gh}$$

$$\pi 64\frac{dh}{dt} = \frac{-\pi}{36}8\sqrt{h}$$

$$\int h^{-1/2}\,dh = \int \frac{-1}{288}\,dt$$

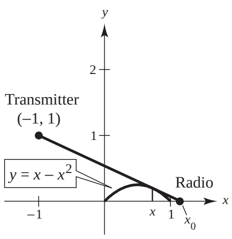
$$2\sqrt{h} = \frac{-t}{288} + C$$

$$h = 20\colon 2\sqrt{20} = C = 4\sqrt{5}$$

$$2\sqrt{h} = \frac{-t}{288} + 4\sqrt{5}$$

$$h = 0 \implies t = 4\sqrt{5}(288) \approx 2575.95 \text{ sec}$$

9. Let the radio receiver be located at $(x_0, 0)$. The tangent line to $y = x - x^2$ joins $(-1, 1)$ and $(x_0, 0)$.



(a) If (x, y) is the point of tangency on $y = x - x^2$, then

$$1 - 2x = \frac{y - 1}{x + 1} = \frac{x - x^2 - 1}{x + 1}$$

$$x - 2x^2 + 1 - 2x = x - x^2 - 1$$

$$x^2 + 2x - 2 = 0$$

$$x = \left(\frac{-2 \pm \sqrt{4 + 8}}{2}\right) = -1 + \sqrt{3}$$

$$y = x - x^2 = 3\sqrt{3} - 5.$$

$$\text{Then } \frac{1 - 0}{-1 - x_0} = \frac{1 - 3\sqrt{3} + 5}{-1 + 1 - \sqrt{3}} = \frac{6 - 3\sqrt{3}}{-\sqrt{3}}$$

$$\sqrt{3} = (1 + x_0)\left(6 - 3\sqrt{3}\right)$$

$$= 6 - 3\sqrt{3} + x_0\left(6 - 3\sqrt{3}\right)$$

$$x_0 = \frac{4\sqrt{3} - 6}{6 - 3\sqrt{3}} \approx 1.155.$$

(b) Now let the transmitter be located at $(-1, h)$.

$$1 - 2x = \frac{y - h}{x + 1} = \frac{x - x^2 - h}{x + 1}$$

$$x - 2x^2 + 1 - 2x = x - x^2 - h$$

$$x^2 + 2x - h - 1 = 0$$

$$x = \frac{\left(-2 \pm \sqrt{4 + 4(h + 1)}\right)}{2}$$

$$= -1 + \sqrt{2 + h}$$

$$y = x - x^2$$

$$= 3\sqrt{2 + h} - h - 4$$

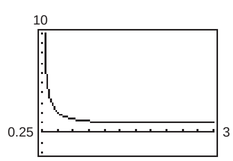
$$\text{Then, } \frac{h - 0}{-1 - x_0} = \frac{h - \left(3\sqrt{2 + h} - h - 4\right)}{-1 - \left(-1 + \sqrt{2 + h}\right)}$$

$$= \frac{2h + 4 - 3\sqrt{2 + h}}{-\sqrt{2 + h}}$$

$$\frac{x_0 + 1}{h} = \frac{\sqrt{2 + h}}{2h + 4 - 3\sqrt{2 + h}}$$

$$x_0 = \frac{h\sqrt{2 + h}}{2h + 4 - 3\sqrt{2 + h}} - 1.$$

(c)



There is a vertical asymptote at $h = \frac{1}{4}$, which is the height of the mountain.

10. $\dfrac{ds}{dt} = 3.5 - 0.019s$

(a) $$\int \frac{-ds}{3.5 - 0.019s} = -\int dt$$

$$\frac{1}{0.019} \ln|3.5 - 0.019s| = -t + C_1$$

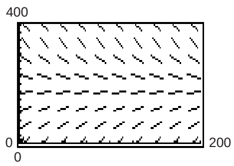
$$\ln|3.5 - 0.019s| = -0.019t + C_2$$

$$3.5 - 0.019s = C_3e^{-0.019t}$$

$$-0.019s = 3.5 - C_3e^{-0.019t}$$

$$s = 184.21 - Ce^{-0.019t}$$

(b)



(c) As $t \to \infty$, $Ce^{-0.019t} \to 0$, and $s \to 184.21$.

11. (a) $$\int \frac{dC}{C} = \int -\frac{R}{V}\,dt$$

$$\ln|C| = -\frac{R}{V}t + K_1$$

$$C = Ke^{-Rt/V}$$

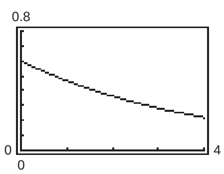
Since $C = C_0$ when $t = 0$, it follows that $K = C_0$ and the function is $C = C_0e^{-Rt/V}$.

(b) Finally, as $t \to \infty$, we have

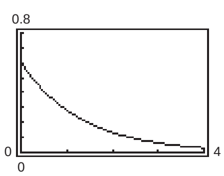
$$\lim_{t\to\infty} C = \lim_{t\to\infty} C_0e^{-Rt/V} = 0.$$

12. From Exercise 39, we have $C = C_0e^{-Rt/V}$.

(a) For $V = 2$, $R = 0.5$, and $C_0 = 0.6$, we have $C = 0.6e^{-0.25t}$.



(b) For $V = 2$, $R = 1.5$, and $C_0 = 0.6$, we have $C = 0.6e^{-0.75t}$.



13. (a) $$\int \frac{1}{Q - RC}\,dC = \int \frac{1}{V}\,dt$$

$$-\frac{1}{R}\ln|Q - RC| = \frac{t}{V} + K_1$$

$$Q - RC = e^{-R[(t/V) + K_1]}$$

$$C = \frac{1}{R}\left(Q - e^{-R[(t/V) + K_1]}\right)$$

$$= \frac{1}{R}\left(Q - Ke^{-Rt/V}\right)$$

Since $C = 0$ when $t = 0$, it follows that $K = Q$ and we have $C = \dfrac{Q}{R}(1 - e^{-Rt/V})$.

(b) As $t \to \infty$, the limit of C is Q/R.