Chapter 10: Sequences, Series, and Probability

10.1 Exercises

Note: For Exercises 1–16, the answers are listed in the order a_1, a_2, a_3, a_4; and a_8.

[1] $a_n = 12 - 3n$ ★ $9, 6, 3, 0; -12$

[2] $a_n = \dfrac{3}{5n-2}$ ★ $1, \frac{3}{8}, \frac{3}{13}, \frac{1}{6}; \frac{3}{38}$

[3] $a_n = \dfrac{3n-2}{n^2+1}$ ★ $\frac{1}{2}, \frac{4}{5}, \frac{7}{10}, \frac{10}{17}; \frac{22}{65}$

[4] $a_n = 10 + \frac{1}{n}$ ★ $11, \frac{21}{2}, \frac{31}{3}, \frac{41}{4}; \frac{81}{8}$

[5] $a_n = 9$ ★ $9, 9, 9, 9; 9$

[6] $a_n = \sqrt{2}$ ★ $\sqrt{2}, \sqrt{2}, \sqrt{2}, \sqrt{2}; \sqrt{2}$

[7] $a_n = 2 + (-0.1)^n$ ★ $1.9, 2.01, 1.999, 2.0001; 2.00000001$

[8] $a_n = 4 + (0.1)^n$ ★ $4.1, 4.01, 4.001, 4.0001; 4.00000001$

[9] $a_n = (-1)^{n-1}\dfrac{n+7}{2n}$ ★ $4, -\frac{9}{4}, \frac{5}{3}, -\frac{11}{8}; -\frac{15}{16}$

[10] $a_n = (-1)^n\dfrac{6-2n}{\sqrt{n+1}}$ ★ $-2\sqrt{2}, \frac{2}{3}\sqrt{3}, 0, -\frac{2}{5}\sqrt{5}; -\frac{10}{3}$

[11] $a_n = 1 + (-1)^{n+1}$ ★ $2, 0, 2, 0; 0$

[12] $a_n = (-1)^{n+1} + (0.1)^{n-1}$ ★ $2, -0.9, 1.01, -0.999; -0.9999999$

[13] $a_n = \dfrac{2^n}{n^2+2}$ ★ $\frac{2}{3}, \frac{2}{3}, \frac{8}{11}, \frac{8}{9}; \frac{128}{33}$

[14] $a_n = (n-1)(n-2)(n-3)$ ★ $0, 0, 0, 6; 210$

[15] a_n is the number of decimal places in $(0.1)^n$. ★ $1, 2, 3, 4; 8$

[16] a_n is the number of positive integers less than n^3. ★ $0, 7, 26, 63; 511$

[17] $\left\{\dfrac{1}{\sqrt{n}}\right\} = \dfrac{1}{\sqrt{1}}, \dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{4}}, \dfrac{1}{\sqrt{5}}, \ldots \approx 1, 0.71, 0.58, 0.5, 0.45, \ldots$

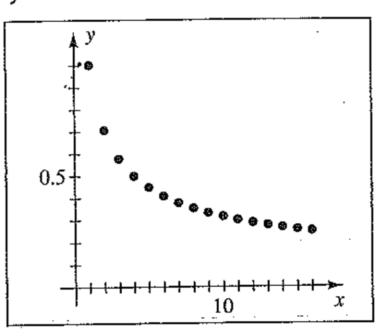


Figure 17

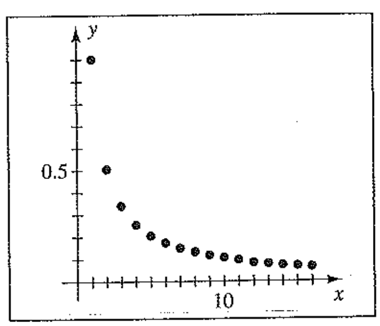


Figure 18

[18] $\left\{\frac{1}{n}\right\} = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots = 1, 0.5, 0.\overline{3}, 0.25, 0.2, \ldots$

[19] $\{(-1)^{n+1}n^2\} = 1 \cdot 1^2, -1 \cdot 2^2, 1 \cdot 3^2, -1 \cdot 4^2, \ldots = 1, -4, 9, -16, \ldots$

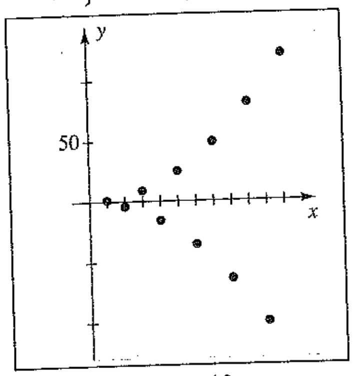


Figure 19

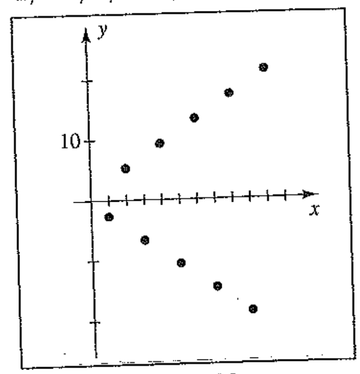


Figure 20

[20] $\{(-1)^n(2n+1)\} = -1 \cdot 3, 1 \cdot 5, -1 \cdot 7, 1 \cdot 9, \ldots = -3, 5, -7, 9, \ldots$

Note: For Exercises 21–28, the answers are listed in the order a_1, a_2, a_3, a_4, a_5.

[21] $a_1 = 2,$ $\quad a_{k+1} = 3a_k - 5$ •

$a_2 = 3a_1 - 5 = 3(2) - 5 = 1,$ $\quad a_3 = 3a_2 - 5 = 3(1) - 5 = -2,$

$a_4 = 3a_3 - 5 = 3(-2) - 5 = -11,$ $\quad a_5 = 3a_4 - 5 = 3(-11) - 5 = -38$

[22] $a_1 = 5,$ $\quad a_{k+1} = 7 - 2a_k$ • ★ 5, −3, 13, −19, 45

[23] $a_1 = -3,$ $\quad a_{k+1} = a_k^2$ • ★ $-3, 3^2, 3^4, 3^8, 3^{16}$

[24] $a_1 = 128,$ $\quad a_{k+1} = \frac{1}{4}a_k$ • ★ $128, 32, 8, 2, \frac{1}{2}$

[25] $a_1 = 5,$ $\quad a_{k+1} = ka_k$ • ★ 5, 5, 10, 30, 120

[26] $a_1 = 3,$ $\quad a_{k+1} = 1/a_k$ •

$a_2 = 1/a_1 = \frac{1}{3}, a_3 = 1/a_2 = 1/(\frac{1}{3}) = 3, a_4 = 1/a_3 = \frac{1}{3}, a_5 = 1/a_4 = 1/(\frac{1}{3}) = 3$

[27] $a_1 = 2,$ $\quad a_{k+1} = (a_k)^k$ • ★ $2, 2, 4, 4^3, 4^{12}$

[28] $a_1 = 2,$ $\quad a_{k+1} = (a_k)^{1/k}$ • ★ $2, 2, 2^{1/2}, 2^{1/6}, 2^{1/24}$

[29] $\{3 + \frac{1}{2}n\}$ • $S_1 = a_1 = 3 + \frac{1}{2} = \frac{7}{2}$. $S_2 = S_1 + a_2 = \frac{7}{2} + 4 = \frac{15}{2}$.

$S_3 = S_2 + a_3 = \frac{15}{2} + \frac{9}{2} = 12$. $S_4 = S_3 + a_4 = 12 + 5 = 17$.

[30] $\{1/n^2\}$ • $S_1 = a_1 = 1$. $S_2 = S_1 + a_2 = 1 + \frac{1}{4} = \frac{5}{4}$.

$S_3 = S_2 + a_3 = \frac{5}{4} + \frac{1}{9} = \frac{49}{36}$. $S_4 = S_3 + a_4 = \frac{49}{36} + \frac{1}{16} = \frac{205}{144}$.

[31] $\{(-1)^n n^{-1/2}\} = \left\{(-1)^n \frac{1}{\sqrt{n}}\right\}$ • $S_1 = a_1 = -1$.

$S_2 = S_1 + a_2 = -1 + \frac{1}{\sqrt{2}}$. $\quad S_3 = S_2 + a_3 = -1 + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}$.

$$S_4 = S_3 + a_4 = -1 + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{2} = -\frac{1}{2} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}.$$

[32] $\{(-1)^n(1/2)^n\}$ • $S_1 = a_1 = -\frac{1}{2}$. $S_2 = S_1 + a_2 = -\frac{1}{2} + \frac{1}{4} = -\frac{1}{4}$.

$S_3 = S_2 + a_3 = -\frac{1}{4} - \frac{1}{8} = -\frac{3}{8}$. $\quad S_4 = S_3 + a_4 = -\frac{3}{8} + \frac{1}{16} = -\frac{5}{16}$.

[33] $\sum_{k=1}^{5} (2k-7) = (-5)+(-3)+(-1)+1+3 = -5$

[34] $\sum_{k=1}^{6} (10-3k) = 7+4+1+(-2)+(-5)+(-8) = -3$

[35] $\sum_{k=1}^{4} (k^2-5) = (-4)+(-1)+4+11 = 10$

[36] $\sum_{k=1}^{10} \left[1+(-1)^k\right] = 0+2+0+2+0+2+0+2+0+2 = 10$

[37] $\sum_{k=0}^{5} k(k-2) = 0+(-1)+0+3+8+15 = 25$

[38] $\sum_{k=0}^{4} (k-1)(k-3) = 3+0+(-1)+0+3 = 5$

[39] $\sum_{k=3}^{6} \frac{k-5}{k-1} = (-1)+\left(-\frac{1}{3}\right)+0+\frac{1}{5} = -\frac{17}{15}$

[40] $\sum_{k=1}^{6} \frac{3}{k+1} = \frac{3}{2}+1+\frac{3}{4}+\frac{3}{5}+\frac{1}{2}+\frac{3}{7} = \frac{669}{140}$

[41] $\sum_{k=1}^{5} (-3)^{k-1} = 1+(-3)+9+(-27)+81 = 61$

[42] $\sum_{k=0}^{4} 3(2^k) = 3+6+12+24+48 = 93$

[43] By (1) of the theorem on the sum of a constant, $\sum_{k=1}^{100} 100 = 100(100) = 10{,}000$.

[44] By (1) of the theorem on the sum of a constant, $\sum_{k=1}^{1000} 5 = 1000(5) = 5000$.

[45] By (2) of the theorem on the sum of a constant,

$$\sum_{k=253}^{571} \tfrac{1}{3} = (571-253+1)(\tfrac{1}{3}) = 319(\tfrac{1}{3}) = \tfrac{319}{3}.$$

[46] By (2) of the theorem on the sum of a constant,

$$\sum_{k=137}^{428} 2.1 = (428-137+1)(2.1) = 292(2.1) = 613.2.$$

[47] $\sum_{j=1}^{7} \frac{1}{2}k^2 = 7(\frac{1}{2}k^2) = \frac{7}{2}k^2$ {note that j, not k, is the summation variable}

[48] $\sum_{k=0}^{5} (3j+2) = 6(3j+2) = 18j+12$ {note that k, not j, is the summation variable}

[49]
$$\begin{aligned}\sum_{k=1}^{n} (a_k-b_k) &= (a_1-b_1)+(a_2-b_2)+\cdots+(a_n-b_n)\\ &= (a_1+a_2+\cdots+a_n)+(-b_1-b_2-\cdots-b_n)\\ &= (a_1+a_2+\cdots+a_n)-(b_1+b_2+\cdots+b_n)\\ &= \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k\end{aligned}$$

[50] $$\sum_{k=1}^{n}(a_k+b_k+c_k) = \sum_{k=1}^{n}\left[(a_k+b_k)+c_k\right]$$
$$= \sum_{k=1}^{n}(a_k+b_k)+\sum_{k=1}^{n}c_k$$
$$= \sum_{k=1}^{n}a_k+\sum_{k=1}^{n}b_k+\sum_{k=1}^{n}c_k$$

[51] As k increases, the terms approach 1.

[52] (a) 3, 3.142546543, 3.141592653, 3.141592654, 3.141592654

(b) When $x_1 = 6$, the terms of the sequence approach 2π.

[53] $a_1 = 0.4$, $a_k = 0.1(3\cdot 2^{k-2}+4) \Rightarrow$

$a_2 = 0.1(3\cdot 2^{2-2}+4) = 0.1(7) = 0.7$ ★ 0.4, 0.7, 1, 1.6, 2.8

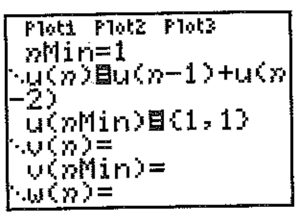
[54] (a) After one day, 1000; two days, 2000; three days, 4000.

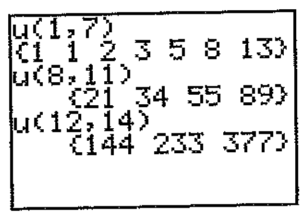
(b) After n days, $a_n = 500(2)^n$.

[55] (a) $a_1 = 1$, $a_2 = 1$, $a_{k+1} = a_k + a_{k-1}$ for $k \geq 2$. So $a_3 = a_2 + a_1 = 1 + 1 = 2$.

The first ten terms are 1, 1, 2, 3, 5, 8, 13, 21, 34, and 55.

The following screens show the assignment needed to generate the Fibonacci sequence on the TI-83 Plus and a listing of the first 14 Fibonacci numbers.





(b) $r_1 = \frac{1}{1} = 1$, $r_2 = \frac{2}{1} = 2$, $r_3 = \frac{3}{2} = 1.5$, $r_4 = \frac{5}{3} = 1.\overline{6}$, $r_5 = \frac{8}{5} = 1.6$,

$r_6 = \frac{13}{8} = 1.625$, $r_7 = \frac{21}{13} \approx 1.6153846$, $r_8 = \frac{34}{21} \approx 1.6190476$,

$r_9 = \frac{55}{34} \approx 1.6176471$, and $r_{10} = \frac{89}{55} \approx 1.6181818$.

[56] $$a_1 = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^1 - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^1 = \frac{1}{2\sqrt{5}}+\frac{1}{2}-\frac{1}{2\sqrt{5}}+\frac{1}{2} = 1.$$

Similarly, $a_2 = 1$, $a_3 = 2$, $a_4 = 3$, $a_5 = 5$, $a_6 = 8$, $a_7 = 13$, and $a_8 = 21$.

[57] (a) Since the amount of chlorine decreases by a factor of 0.20 each day, $a_n = 0.8a_{n-1}$, where a_0 is the initial amount of chlorine in the pool.

(b) Let $a_0 = 7$ and $a_n = 0.8a_{n-1}$.

Day (n)	0	1	2	3	4	5
Chlorine (a_n)	7.00	5.60	4.48	3.58	2.87	2.29

The chlorine level will drop below 3 ppm during the fourth day.

58 (a) $a_0 = 2$, $a_1 = 0.8a_0 + 0.5 = 2.1$, $a_2 = 0.8a_1 + 0.5 = 2.18$, $a_3 = 0.8a_2 + 0.5 = 2.244$, $a_4 = 0.8a_3 + 0.5 = 2.2952$. In general, the next value can be calculated by multiplying the previous value by 0.8 and adding 0.5. Thus, $a_n = 0.8a_{n-1} + 0.5$.

(b) Using the table, $a_{15} \approx 2.482$. By continuing to calculate a_n for larger and larger n, the amount of chlorine appears to level off at 2.5 ppm.

n	a_n	n	a_n	n	a_n
1	2.100	6	2.369	11	2.457
2	2.180	7	2.395	12	2.466
3	2.244	8	2.416	13	2.473
4	2.295	9	2.433	14	2.478
5	2.336	10	2.446	15	2.482

(c) Since we are retaining 80% each day, we are replacing 20%. Because the target amount is 1.5 ppm, we must replace 20% of 1.5 ppm, or equivalently, 0.3 ppm.

Note: You could also solve this part by trial and error—from part (b) we know 0.5 is too large, so decrease that amount until you arrive at 0.3.

59 If $1 \le n \le 4$, then $C(n) = 89.95n$. If $5 \le n \le 9$, then $C(n) = 87.95n$—there are no changes in prices, that is, you don't pay \$89.95 for the first 4 and then \$87.95 for the next one. If $n \ge 10$, then $C(n) = 85.95n$. Summarizing in a piecewise-defined function gives us

$$C(n) = \begin{cases} 89.95n & \text{if } 1 \le n \le 4 \\ 87.95n & \text{if } 5 \le n \le 9 \\ 85.95n & \text{if } n \ge 10 \end{cases}$$

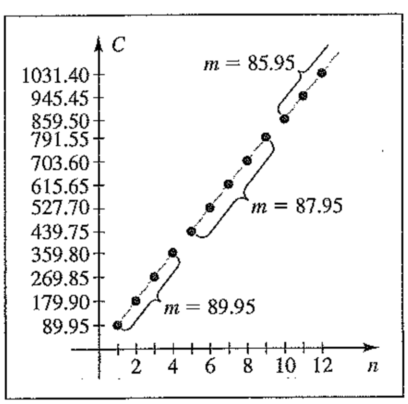


Figure 59

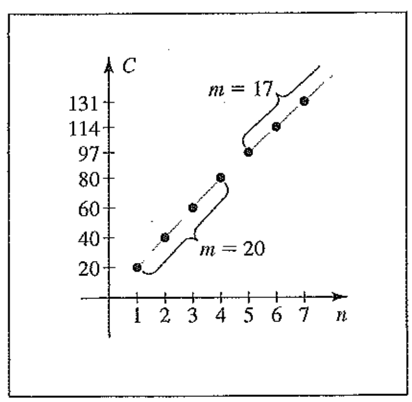


Figure 60

60 Note that in this case the price changes for the DVD players over 4 units—that is, you pay \$20 for the first 4 and then \$17 for all units over 4.

$$C(n) = \begin{cases} 20n & \text{if } 1 \le n \le 4 \\ 80 + 17(n-4) & \text{if } n \ge 5 \end{cases}$$

$\boxed{61}$ $N = 5$ and $x_1 = \frac{5}{2}$ $\Rightarrow$ $x_2 = \frac{1}{2}\left(x_1 + \frac{N}{x_1}\right) = \frac{1}{2}\left(2.5 + \frac{5}{2.5}\right) = 2.25$ $\Rightarrow$

$x_3 = \frac{1}{2}\left(2.25 + \frac{5}{2.25}\right) \approx 2.236111$ $\Rightarrow$ $x_4 \approx 2.236068$ $\Rightarrow$ $x_5 \approx 2.236068$.

Thus, $\sqrt{5} \approx 2.236068$.

$\boxed{62}$ $N = 18$ and $x_1 = \frac{18}{2} = 9$ $\Rightarrow$ $x_2 = \frac{1}{2}\left(x_1 + \frac{N}{x_1}\right) = \frac{1}{2}\left(9 + \frac{18}{9}\right) = 5.5$ $\Rightarrow$

$x_3 \approx 4.386364$ $\Rightarrow$ $x_4 \approx 4.244995$ $\Rightarrow$ $x_5 \approx 4.242641$ $\Rightarrow$ $x_6 \approx 4.242641$.

Thus, $\sqrt{18} \approx 4.242641$.

$\boxed{63}$ $x_1 = 2$ and $x_2 = \frac{1}{3}\sqrt[3]{x_1} + 2$ $\Rightarrow$ $x_2 \approx 2.419974$ $\Rightarrow$ $x_3 \approx 2.447523$ $\Rightarrow$

$x_4 \approx 2.449215$ $\Rightarrow$ $x_5 \approx 2.449319$ $\Rightarrow$ $x_6 \approx 2.449325$.

The root is approximately 2.4493.

$\boxed{64}$ $2x + \dfrac{1}{x^4 + x + 2} = 0$ $\Rightarrow$ $x = -\dfrac{1}{2(x^4 + x + 2)}$ $\Rightarrow$ $x_2 = -\dfrac{1}{2(x_1^4 + x_1 + 2)}$.

$x_1 = 0$ $\Rightarrow$ $x_2 = -0.25$ $\Rightarrow$ $x_3 \approx -0.285078$ $\Rightarrow$ $x_4 \approx -0.290440$ $\Rightarrow$

$x_5 \approx -0.291261$ $\Rightarrow$ $x_6 \approx -0.291386$ $\Rightarrow$ $x_7 \approx -0.291405$.

The root is approximately -0.2914.

$\boxed{65}$ (a) $f(1) = -1 < 0$ and $f(2) \approx 0.30 > 0$.

Thus, f assumes both positive and negative values on [1, 2].

(b) $\log x + x - 2 = 0$ $\Rightarrow$ $x = 2 - \log x$.

$x_1 = 1.5$, $x_2 = 2 - \log x_1 = 2 - \log 1.5 \approx 1.823909$, $x_3 \approx 1.738997$,

$x_4 \approx 1.759701$, $x_5 \approx 1.754561$, $x_6 \approx 1.755832$, $x_7 \approx 1.755517$.

The zero is approximately 1.76. *Figure 65* shows how to obtain the values on a graphing calculator.

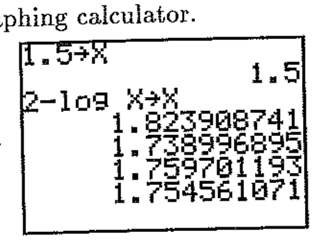


Figure 65

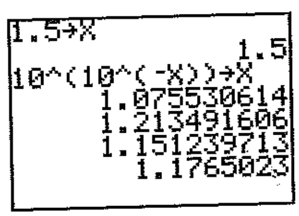


Figure 66

$\boxed{66}$ (a) $f(1) = -0.1 < 0$ and $f(2) \approx 0.29 > 0$.

Thus, f assumes both positive and negative values on [1, 2].

(b) $\log x - 10^{-x} = 0$ $\Rightarrow$ $\log x = 10^{-x}$ $\Rightarrow$ $x = (10)^{10^{-x}}$.

$x_1 = 1.5$, $x_2 = (10)^{10^{-x_1}} = (10)^{10^{-1.5}} \approx 1.075531$, $x_3 \approx 1.213492$,

$x_4 \approx 1.151240$, $x_5 \approx 1.176502$, $x_6 \approx 1.165745$, $x_7 \approx 1.170237$.

The zero is approximately 1.17.

[67] Graph $y = \left(1 + \frac{1}{x} + \frac{1}{2x^2}\right)^x$ on the interval [1, 100].

The graph approaches the horizontal asymptote $y \approx 2.718 \approx e$.

For increasing values of n, the terms of the sequence appear to approximate e.

[1, 100, 10] by [0, 3]

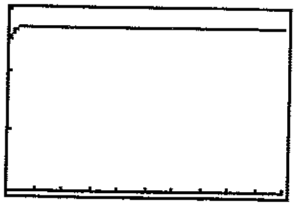


Figure 67

[1, 100, 10] by [0, 3]

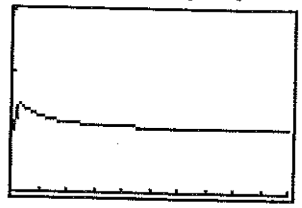


Figure 68

[68] Graph $y = x^{1/x}$ on the interval [1, 100].

The graph approaches the horizontal asymptote $y \approx 1$ from above.

For increasing values of n, the terms of the sequence appear to approximate 1.

[69] Graph $y = \left(\frac{1}{x}\right)^{1/x}$ on the interval [1, 100].

The graph approaches the horizontal asymptote $y \approx 1$ from below.

For increasing values of n, the terms of the sequence appear to approximate 1.

[1, 100, 10] by [0, 3]

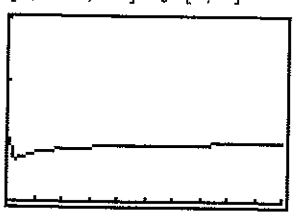


Figure 69

[1, 100, 20] by [0, 5]

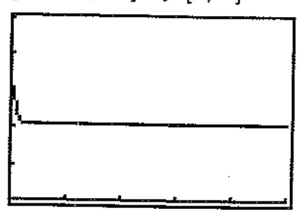


Figure 70

[70] Graph $y = (2.1^x + 1)^{1/x}$ on the interval [1, 100].

The graph approaches the horizontal asymptote $y \approx 2.1$.

For increasing values of n, the terms of the sequence appear to approximate 2.1.

[71] By tracing the graph we see that $a_9 \approx 66.55$ and $a_{10} \approx 113.64$. Thus, $k = 10$.

[0, 20, 5] by [0, 125, 25]

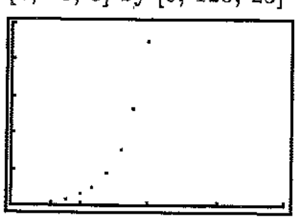


Figure 71

[0, 20, 5] by [0, 125, 25]

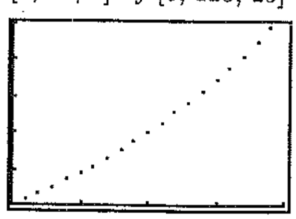


Figure 72

[72] By tracing the graph we see that $a_{16} \approx 98.79$ and $a_{17} \approx 107.73$. Thus, $k = 17$.

[73] By tracing the graph we see that $a_{18} \approx 50.39$ and $a_{19} \approx 255.96$. Thus, $k = 19$.

[0, 20, 5] by [0, 300, 50]

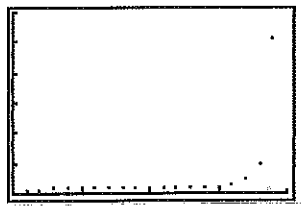


Figure 73

[0, 20, 5] by [0, 375, 50]

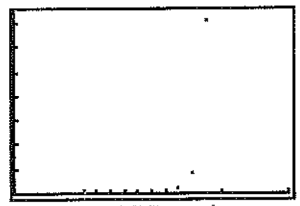


Figure 74

[74] By tracing the graph we see that $a_{14} \approx 42.12$ and $a_{15} \approx 356.37$. Thus, $k = 15$.

[75] (a) Since $c = 0.5$, let the sequence be defined by $a_{k+1} = 0.5a_k(1 - a_k)$. Then, $a_1 = 0.25$, $a_2 = 0.5a_1(1 - a_1) = 0.5(0.25)(1 - 0.25) = 0.09375$. In a similar manner, $a_3 \approx 0.04248$, $a_4 \approx 0.02034$, $\ldots$, $a_{10} \approx 0.00031$, $a_{11} \approx 0.00015$, $a_{12} \approx 0.00008$. The insect population is initially $1000a_1 = 1000(0.25) = 250$. It then becomes approximately 94, 42, 20, and so on, until the population decreases to zero.

(b) The sequence determined is $a_1 = 0.25$, $a_2 \approx 0.28125$, $a_3 \approx 0.30322$, $a_4 \approx 0.31692$, $\ldots$, $a_{18} \approx a_{19} \approx a_{20} \approx 0.33333$. The insect population is initially 250. It then becomes approximately 281, 303, 317, and so on, until the population stabilizes at 333.

(c) The sequence determined is $a_1 = 0.25$, $a_2 \approx 0.51563$, $a_3 \approx 0.68683$, $a_4 \approx 0.59151$, $\ldots$, $a_{40} \approx a_{41} \approx a_{42} \approx 0.63636$. The insect population is initially 250. It then becomes approximately 516, 687, 592, and so on, until the population stabilizes at 636.

[76] (a) One conjecture is that the insect population will decrease to zero as in part (a) of the previous exercise.

(b) Whenever $0 < c < 1$, the insect population decreases to zero.

10.2 Exercises

[1] To show that the given sequence, $-6, -2, 2, \ldots, 4n - 10, \ldots$, is arithmetic, we must show that $a_{k+1} - a_k$ is equal to some constant, which is the common difference.

$a_n = 4n - 10 \Rightarrow$

$$a_{k+1} - a_k = [4(k+1) - 10] - [4(k) - 10] = 4k + 4 - 10 - 4k + 10 = 4$$

[2] $53, 48, 43, \ldots, 58 - 5n, \ldots$ •

$a_n = 58 - 5n \Rightarrow$

$$a_{k+1} - a_k = [58 - 5(k+1)] - [58 - 5k] = 58 - 5k - 5 - 58 + 5k = -5$$

[3] $d = 6 - 2 = 4$; $a_n = 2 + (n-1)(4) = 4n - 2$; $a_5 = 18$; $a_{10} = 38$

[4] $d = 13 - 16 = -3$; $a_n = 16 + (n-1)(-3) = -3n + 19$; $a_5 = 4$; $a_{10} = -11$

[5] $d = 2.7 - 3 = -0.3$; $a_n = 3 + (n-1)(-0.3) = -0.3n + 3.3$; $a_5 = 1.8$; $a_{10} = 0.3$

[6] $d = -4.5 - (-6) = 1.5$; $a_n = -6 + (n-1)(1.5) = 1.5n - 7.5$; $a_5 = 0$; $a_{10} = 7.5$

[7] $d = -3.9 - (-7) = 3.1$; $a_n = -7 + (n-1)(3.1) = 3.1n - 10.1$; $a_5 = 5.4$; $a_{10} = 20.9$

[8] $d = (x-3) - (x-8) = 5$;

$$a_n = x - 8 + (n-1)(5) = x + 5n - 13;\ a_5 = x + 12;\ a_{10} = x + 37$$

[9] An equivalent sequence is $\ln 3,\ 2\ln 3,\ 3\ln 3,\ 4\ln 3, \ldots$; $d = 2\ln 3 - \ln 3 = \ln 3$;

$$a_n = \ln 3 + (n-1)(\ln 3) = n\ln 3 \text{ or } \ln 3^n;\ a_5 = 5\ln 3 \text{ or } \ln 3^5;\ a_{10} = 10\ln 3 \text{ or } \ln 3^{10}$$

[10] An equivalent sequence is $3, 2, 1, 0, \ldots$; $d = 2 - 3 = -1$;

$$a_n = 3 + (n-1)(-1) = -n + 4;\ a_5 = -1;\ a_{10} = -6$$

[11] $a_6 = a_1 + 5d$ and $a_2 = a_1 + d \Rightarrow a_6 - a_2 = 4d$. But $a_6 - a_2 = -11 - 21 = -32$.

Hence, $4d = -32$ and $d = -8$.

[12] $a_{11} = a_1 + 10d$ and $a_4 = a_1 + 3d \Rightarrow a_{11} - a_4 = 7d$. But $a_{11} - a_4 = 35 - 14 = 21$.

Hence, $7d = 21$ and $d = 3$.

[13] $d = a_2 - a_1 = 7.5 - 9.1 = -1.6$. $a_{12} = 9.1 + (11)(-1.6) = -8.5$

[14] $d = 1 - \sqrt{2}$; $a_{11} = (2 + \sqrt{2}) + (10)(1 - \sqrt{2}) = 12 - 9\sqrt{2}$

[15] $d = 2.5$; $a_6 = a_1 + 5d \Rightarrow 2.7 = a_1 + 12.5 \Rightarrow a_1 = -9.8$

[16] $a_9 = 53$ and $a_8 = 47 \Rightarrow d = 6$. $a_8 = a_1 + 7d \Rightarrow 47 = a_1 + 7(6) \Rightarrow a_1 = 5$

[17] $a_3 = 7$ and $a_{20} = 43 \Rightarrow 17d = 36 \Rightarrow d = \frac{36}{17}$. $a_{15} = a_3 + 12d = 7 + 12(\frac{36}{17}) = \frac{551}{17}$.

[18] $a_2 = 1$ and $a_{18} = 49 \Rightarrow 16d = 49 - 1 \Rightarrow d = 3$. $a_{10} = a_2 + 8d = 1 + 8(3) = 25$.

Note: To find the sums in Exercises 19–28, we use the sum formulas

$$\textbf{(1)}\ S_n = \tfrac{n}{2}[2a_1 + (n-1)d] \quad \text{and} \quad \textbf{(2)}\ S_n = \tfrac{n}{2}(a_1 + a_n).$$

[19] Using **(1)** with $a_1 = 40$, $d = -3$, and $n = 30$, we have

$$S_{30} = \tfrac{30}{2}[2(40) + (29)(-3)] = -105.$$

[20] $S_{40} = \frac{40}{2}[2(5) + (39)(0.1)] = 278$

[21] Using **(2)** with $a_1 = -9$, $a_{10} = 15$, and $n = 10$, we have $S_{10} = \frac{10}{2}(-9 + 15) = 30$.

[22] $a_7 = a_1 + 6d \Rightarrow \frac{7}{3} = a_1 + 6(-\frac{2}{3}) \Rightarrow a_1 = \frac{19}{3}$. $S_{15} = \frac{15}{2}\left[2(\frac{19}{3}) + (14)(-\frac{2}{3})\right] = 25$

[23] $\sum_{k=1}^{20}(3k - 5)$ • $a_1 = -2$ and $a_{20} = 55 \Rightarrow S_{20} = \frac{20}{2}(-2 + 55) = 530$

[24] $\sum_{k=1}^{12}(7 - 4k)$ • $a_1 = 3$ and $a_{12} = -41 \Rightarrow S_{12} = \frac{12}{2}[3 + (-41)] = -228$

[25] $\sum_{k=1}^{18}(\frac{1}{2}k + 7)$ • $a_1 = \frac{15}{2}$ and $a_{18} = 16 \Rightarrow S_{18} = \frac{18}{2}(\frac{15}{2} + 16) = \frac{423}{2}$

[26] $\sum_{k=1}^{10} (\frac{1}{4}k + 3)$ • $a_1 = \frac{13}{4}$ and $a_{10} = \frac{11}{2}$ $\Rightarrow$ $S_{10} = \frac{10}{2}(\frac{13}{4} + \frac{11}{2}) = \frac{175}{4}$

[27] $$\sum_{k=126}^{592} (5k + 2j) = \sum_{k=126}^{592} 5k + \sum_{k=126}^{592} 2j$$

$$= 5 \sum_{k=126}^{592} k + 2 \sum_{k=126}^{592} j = 5(\text{SUM1}) + 2(\text{SUM2}).$$

The number of terms is $n = 592 - 126 + 1 = 467$.

To evaluate SUM1, we use **(1)** with $a_1 = 126$, $d = 1$, and $n = 467$ to get:

$$S_{467} = \tfrac{467}{2}[2(126) + (466)(1)] = \tfrac{467}{2}[718] = 167{,}653.$$

SUM2 is just $467 \cdot j$ because j is a constant (k is the index).

Thus, $5(\text{SUM1}) + 2(\text{SUM2}) = 5(167{,}653) + 2(467j) = 838{,}265 + 934j$.

[28] $$\sum_{k=88}^{371} (3j - 2k) = \sum_{k=88}^{371} 3j - \sum_{k=88}^{371} 2k$$

$$= 3 \sum_{k=88}^{371} j - 2 \sum_{k=88}^{371} k = 3(\text{SUM1}) - 2(\text{SUM2}).$$

The number of terms is $n = 371 - 88 + 1 = 284$.

SUM1 is just $284 \cdot j$ because j is a constant (k is the index).

To evaluate SUM2, we use **(1)** with $a_1 = 88$, $d = 1$, and $n = 284$ to get:

$$S_{284} = \tfrac{284}{2}[2(88) + (283)(1)] = \tfrac{284}{2}[459] = 65{,}178.$$

Thus, $3(\text{SUM1}) - 2(\text{SUM2}) = 3(284j) - 2(65{,}178) = 852j - 130{,}356$.

[29] $4 + 11 + 18 + 25 + 32$ • The number of terms is $n = 5$.

The difference in terms is $d = 11 - 4 = 7$.

The general term is $a_n = a_1 + (n-1)d = 4 + (n-1)7 = 4 + 7n - 7 = 7n - 3$.

We can represent the sum in terms of summation notation as $\sum_{n=1}^{5} a_n = \sum_{n=1}^{5} (7n - 3)$.

Sometimes it is easier to use $\sum_{n=0}^{4} a_{n+1}$ than $\sum_{n=1}^{5} a_n$.

In this case, $a_{n+1} = a_1 + nd = 4 + n \cdot 7$,

so an alternate form of the summation notation is $\sum_{n=0}^{4} (4 + 7n)$.

[30] $3 + 8 + 13 + 18 + 23$ • $n = 5$, $d = 5$, $a_n = a_1 + (n-1)d = 3 + (n-1)5 = 5n - 2$.

The sum in terms of summation notation is $\sum_{n=1}^{5} a_n = \sum_{n=1}^{5} (5n - 2)$ or $\sum_{n=0}^{4} (3 + 5n)$.

[31] $4 + 11 + 18 + \cdots + 466$ • From Exercise 29, the general term is $7n - 3$ with n starting at 1. To find the largest value of n, set the general term equal to the largest value of the sum and solve for n. $7n - 3 = 466$ $\Rightarrow$ $7n = 469$ $\Rightarrow$ $n = 67$.

$$\sum_{n=1}^{67} (7n - 3) \quad \text{or} \quad \sum_{n=0}^{66} (4 + 7n)$$

[32] $3+8+13+\cdots+463$ • From Exercise 30, the general term is $5n-2$ with n starting at 1. $5n-2=463 \Rightarrow n=93$, the largest value of n.

$$\sum_{n=1}^{93}(5n-2) \quad \text{or} \quad \sum_{n=0}^{92}(3+5n)$$

[33] $\frac{3}{7}+\frac{6}{11}+\frac{9}{15}+\frac{12}{19}+\frac{15}{23}+\frac{18}{27}$ • The numerators increase by 3, the denominators increase by 4. The general terms are $3+(n-1)3=3n$ and $7+(n-1)4=4n+3$.

$$\sum_{n=1}^{6}\frac{3n}{4n+3}$$

[34] $\frac{5}{13}+\frac{10}{11}+\frac{15}{9}+\frac{20}{7}$ • The numerators increase by 5, the denominators decrease by 2. The general terms are $5+(n-1)5=5n$ and $13+(n-1)(-2)=15-2n$. $\sum_{n=1}^{4}\frac{5n}{15-2n}$

[35] $8+19+30+\cdots+16{,}805$ • $d=19-8=11$ and $a_n=a_1+(n-1)d=$ $8+(n-1)11=11n-3$. Find the largest value of n: $11n-3=16{,}805 \Rightarrow$ $11n=16{,}808 \Rightarrow n=1528$. Thus, the summation notation is $\sum_{n=1}^{1528}(11n-3)$.

Now find the sum: $S_{1528}=\frac{1528}{2}(8+16{,}805)=12{,}845{,}132$.

[36] $2+11+20+\cdots+16{,}058$ • $d=11-2=9$ and $a_n=a_1+(n-1)d=$ $2+(n-1)9=9n-7$. Find the largest value of n: $9n-7=16{,}058 \Rightarrow$ $9n=16{,}065 \Rightarrow n=1785$. Thus, the summation notation is $\sum_{n=1}^{1785}(9n-7)$.

Now find the sum: $S_{1785}=\frac{1785}{2}(2+16{,}058)=14{,}333{,}550$.

[37] $S_n=\frac{n}{2}[2a_1+(n-1)(d)] \Rightarrow 21=\frac{n}{2}\left[2(-2)+(n-1)(\frac{1}{4})\right] \Rightarrow$

$$42=n(\tfrac{1}{4}n-\tfrac{17}{4}) \Rightarrow 168=n^2-17n \Rightarrow (n-24)(n+7)=0 \Rightarrow n=24$$

[38] $S_n=\frac{n}{2}[2a_1+(n-1)(d)] \Rightarrow 21=\frac{n}{2}\left[2(-1)+(n-1)(\frac{1}{5})\right] \Rightarrow$

$$42=n(\tfrac{1}{5}n-\tfrac{11}{5}) \Rightarrow 210=n^2-11n \Rightarrow (n-21)(n+10)=0 \Rightarrow n=21$$

[39] $S_n=\frac{n}{2}[2a_1+(n-1)(d)] \Rightarrow -36=\frac{n}{2}\left[2(-\frac{29}{6})+(n-1)(\frac{1}{3})\right] \Rightarrow$

$-72=n(-\frac{29}{3}+\frac{1}{3}n-\frac{1}{3}) \Rightarrow -216=n(n-30) \Rightarrow$

$n^2-30n+216=0 \Rightarrow (n-12)(n-18)=0 \Rightarrow n=12,\ 18.$

There are two sequences that satisfy the given conditions.

[40] $a_6=a_1+5d \Rightarrow -3=a_1+1 \Rightarrow a_1=-4$;

$S_n=\frac{n}{2}[2a_1+(n-1)(d)] \Rightarrow -33=\frac{n}{2}[2(-4)+(n-1)(0.2)] \Rightarrow$

$-66=n(0.2n-8.2) \Rightarrow 2n^2-82n+660=0 \Rightarrow n^2-41n+330=0 \Rightarrow$

$(n-11)(n-30)=0 \Rightarrow n=11,\ 30.$

There are two sequences that satisfy the given conditions.

$\boxed{41}$ Five arithmetic means $\Rightarrow$ $6d = 10 - 2$ $\Rightarrow$ $d = \frac{4}{3}$.

The terms are 2, $\frac{10}{3}$, $\frac{14}{3}$, 6, $\frac{22}{3}$, $\frac{26}{3}$, 10.

$\boxed{42}$ Three arithmetic means $\Rightarrow$ $4d = -5 - 3$ $\Rightarrow$ $d = -2$.

The terms are 3, 1, -1, -3, -5.

$\boxed{43}$ (a) The first integer greater than 32 that is divisible by 6 is 36 $\{6 \cdot 6\}$

and the last integer less than 395 that is divisible by 6 is 390 $\{65 \cdot 6\}$.

The number of terms is then $65 - 6 + 1 = 60$.

(b) The sum is $S_{60} = \frac{60}{2}(36 + 390) = 12{,}780$.

$\boxed{44}$ (a) The first integer greater than -500 that is divisible by 33 is -495 $\{-15 \cdot 33\}$.

There are 15 negative integers greater than -500 that are divisible by 33.

(b) The sum is $S_{15} = \frac{15}{2}[-495 + (-33)] = -3960$.

$\boxed{45}$ There are $(24 - 10 + 1) = 15$ layers. Model this problem as an arithmetic sequence

with $a_1 = 10$ and $a_{15} = 24$. $S_{15} = \frac{15}{2}(10 + 24) = 255$.

$\boxed{46}$ Model this problem as an arithmetic sequence with $a_1 = 30$ and $d = 2$.

$S_{10} = \frac{10}{2}[2(30) + (10 - 1)(2)] = 390$. The last ten rows $\{$11th to 20th$\}$ each have

$10(50) = 500$ seats so that the total is 890 seats.

$\boxed{47}$ This is similar to inserting 9 arithmetic means between 4 and 24.

$10d = 20$ $\Rightarrow$ $d = 2$. The circumference of each ring is πD with $D = 4, 6, 8, \ldots, 24$.

$S_{11} = \frac{11}{2}(4\pi + 24\pi) = 154\pi$ ft.

$\boxed{48}$ The sequence of feet traveled each second is 4, 9, 14, ...

$S_{11} = \frac{11}{2}[2(4) + (11 - 1)(5)] = 319$ ft.

$\boxed{49}$ $n = 5$, $S_5 = 5000$, $d = -100$ $\Rightarrow$

$5000 = \frac{5}{2}[2a_1 + 4(-100)]$ $\Rightarrow$ $2000 = 2a_1 - 400$ $\Rightarrow$ $a_1 = \$1200$

$\boxed{50}$ $n = 10$, $S_{10} = 46{,}000$, $a_1 = 1000$ $\Rightarrow$ $46{,}000 = \frac{10}{2}[2(1000) + 9d]$ $\Rightarrow$

$7200 = 9d$ $\Rightarrow$ $d = 800$. The bonuses $\{$from 10th to 1st$\}$ are:

\$1000, \$1800, \$2600, \$3400, \$4200, \$5000, \$5800, \$6600, \$7400, and \$8200.

$\boxed{51}$ The sequence 16, 48, 80, 112, ..., is an arithmetic sequence with $a_1 = 16$ and

$d = 48 - 16 = 32$. The total distance traveled in n seconds is

$a_1 + a_2 + \cdots + a_n = \frac{n}{2}[2a_1 + (n-1)d] = \frac{n}{2}[2(16) + (n-1)(32)] = \frac{n}{2}(32n) = 16n^2$.

$\boxed{52}$ Let f be the linear function $f(n) = a(n) + b$.

The difference between the $(n+1)$st term and the nth term is

$f(n+1) - f(n) = [a(n+1) + b] - [a(n) + b] = an + a + b - an - b = a$.

$\therefore$ successive terms differ by the same real number and the sequence is arithmetic.

[53] If the nth term is $\frac{1}{x_n}$ and $x_{n+1} = \dfrac{x_n}{1+x_n}$, then the $(n+1)$st term is

$\dfrac{1}{x_{n+1}} = \dfrac{1}{\dfrac{x_n}{1+x_n}} = 1 + \dfrac{1}{x_n}$, which is 1 greater than the nth term and therefore the sequence is arithmetic.

[54] The sequence of the lengths is 2–1 inch lengths, 2–2 in, 2–3 in, ..., 2–16 in.

This sequence is the same as 2, 4, 6, ..., 32, and has sum $S_{16} = \frac{16}{2}(2+32) = 272$ in.

If the width is 32, then the sequence is 2, 4, 6, ..., 64,

and the sum is $S_{32} = \frac{32}{2}(2+64) = 1056$ in.

[55] (a) $T_8 = 1 + 2 + \cdots + 8 = 36$. $A_1 = \frac{8-1+1}{36} = \frac{8}{36}$.

$A_2 = \frac{7}{36}$, $A_3 = \frac{6}{36}$, $A_4 = \frac{5}{36}$, $A_5 = \frac{4}{36}$, $A_6 = \frac{3}{36}$, $A_7 = \frac{2}{36}$, $A_8 = \frac{1}{36}$.

(b) $d = A_{k+1} - A_k = -\frac{1}{36}$ for $k = 1, 2, \ldots, 7$. $S_8 = \sum\limits_{k=1}^{8} A_k = \frac{8}{36} + \frac{7}{36} + \cdots + \frac{1}{36} = 1$.

(c) $\$1000\left(\frac{8}{36} + \frac{7}{36} + \frac{6}{36} + \frac{5}{36}\right) \approx \$722.22$

[56] (a) $A_1 = \dfrac{n-1+1}{T_n} = \dfrac{n}{T_n}$. $A_2, A_3, \ldots, A_n = \dfrac{n-1}{T_n}, \dfrac{n-2}{T_n}, \ldots, \dfrac{1}{T_n}$.

(b) $d = A_{k+1} - A_k = \dfrac{n-(k+1)+1}{T_n} - \dfrac{n-k+1}{T_n} = -\dfrac{1}{T_n}$.

$$S_n = \sum_{k=1}^{n} A_k = \frac{1}{T_n} + \frac{2}{T_n} + \cdots + \frac{n}{T_n} = \frac{1+2+\cdots+n}{T_n} = \frac{1+2+\cdots+n}{1+2+\cdots+n} = 1.$$

10.3 Exercises

[1] To show that the given sequence, $5, -\frac{5}{4}, \frac{5}{16}, \ldots, 5(-\frac{1}{4})^{n-1}, \ldots$, is geometric, we must show that $\dfrac{a_{k+1}}{a_k}$ is equal to some constant, which is the common ratio.

$$a_n = 5(-\tfrac{1}{4})^{n-1} \Rightarrow \frac{a_{k+1}}{a_k} = \frac{5(-\frac{1}{4})^{(k+1)-1}}{5(-\frac{1}{4})^{k-1}} = -\frac{1}{4}$$

[2] $\frac{1}{7}, \frac{3}{7}, \frac{9}{7}, \ldots, \frac{1}{7}(3)^{n-1}, \ldots$ • $a_n = \frac{1}{7}(3)^{n-1} \Rightarrow \dfrac{a_{k+1}}{a_k} = \dfrac{\frac{1}{7}(3)^{(k+1)-1}}{\frac{1}{7}(3)^{k-1}} = 3$

[3] $r = \frac{4}{8} = \frac{1}{2}$; $a_n = 8(\frac{1}{2})^{n-1} = 2^3(2^{-1})^{n-1} = 2^{4-n}$; $a_5 = 2^{-1} = \frac{1}{2}$; $a_8 = 2^{-4} = \frac{1}{16}$

[4] $r = \frac{1.2}{4} = 0.3$; $a_n = 4(0.3)^{n-1}$; $a_5 = 4(0.3)^4 = 0.0324$; $a_8 = 4(0.3)^7 = 0.0008748$

[5] $r = \frac{-30}{300} = -0.1$; $a_n = 300(-0.1)^{n-1}$;

$a_5 = 300(-0.1)^4 = 0.03$; $a_8 = 300(-0.1)^7 = -0.00003$

[6] $r = \dfrac{-\sqrt{3}}{1} = -\sqrt{3}$; $a_n = 1(-\sqrt{3})^{n-1}$; $a_5 = (-\sqrt{3})^4 = 9$; $a_8 = (-\sqrt{3})^7 = -27\sqrt{3}$

[7] $r = \frac{25}{5} = 5$; $a_n = 5(5)^{n-1} = 5^n$; $a_5 = 5^5 = 3125$; $a_8 = 5^8 = 390{,}625$

[8] $r = \frac{6}{2} = 3$; $a_n = 2(3)^{n-1}$; $a_5 = 2 \cdot 3^4 = 162$; $a_8 = 2 \cdot 3^7 = 4374$

[9] $r = \frac{-6}{4} = -1.5$; $a_n = 4(-1.5)^{n-1}$; $a_5 = 4(-1.5)^4 = 20.25$; $a_8 = 4(-1.5)^7 = -68.34375$

[10] $r = \frac{-54}{162} = -\frac{1}{3}$; $a_n = 162(-\frac{1}{3})^{n-1}$; $a_5 = 162(-\frac{1}{3})^4 = 2$; $a_8 = 162(-\frac{1}{3})^7 = -\frac{2}{27}$

[11] $r = \dfrac{-x^2}{1} = -x^2$; $a_n = 1(-x^2)^{n-1} = (-1)^{n-1}x^{2n-2}$; $a_5 = x^8$; $a_8 = -x^{14}$

[12] $r = \dfrac{-\frac{x}{3}}{1} = -\frac{x}{3}$; $a_n = 1\left(-\frac{x}{3}\right)^{n-1} = (-1)^{n-1}\left(\frac{x}{3}\right)^{n-1}$; $a_5 = \frac{x^4}{81}$; $a_8 = -\frac{x^7}{2187}$

[13] $r = \dfrac{2^{x+1}}{2} = 2^x$;

$$a_n = 2(2^x)^{n-1} = 2^{\left[x(n-1)\right]} \cdot 2^1 = 2^{(n-1)x+1};\ a_5 = 2^{4x+1};\ a_8 = 2^{7x+1}$$

[14] $r = \dfrac{10^{2x-1}}{10} = 10^{2x-2}$; $a_n = 10(10^{2x-2})^{n-1} = 10^{2(n-1)x+(3-2n)}$;

$$a_5 = 10^{8x-7};\ a_8 = 10^{14x-13}$$

[15] $\dfrac{a_6}{a_4} = \dfrac{9}{3} = 3$ and $\dfrac{a_6}{a_4} = \dfrac{a_1r^5}{a_1r^3} = r^2$. Hence, $r^2 = 3$ and $r = \pm\sqrt{3}$.

[16] $\dfrac{a_7}{a_3} = \dfrac{1/4}{4} = \dfrac{1}{16}$ and $\dfrac{a_7}{a_3} = \dfrac{a_1r^6}{a_1r^2} = r^4$. Hence, $r^4 = \frac{1}{16}$ and $r = \pm\frac{1}{2}$.

[17] $r = \frac{6}{4} = \frac{3}{2}$; $a_6 = 4(\frac{3}{2})^5 = \frac{243}{8}$

[18] $r = \dfrac{-\sqrt{2}}{2} = -\dfrac{1}{\sqrt{2}}$; $a_1 = \dfrac{a_2}{r} = -2\sqrt{2}$; $a_7 = -2\sqrt{2}\left(-\dfrac{1}{\sqrt{2}}\right)^6 = -\dfrac{\sqrt{2}}{4}$

[19] $a_4 = 4$ and $a_7 = 12 \;\Rightarrow\; r^3 = \frac{12}{4} = 3 \;\Rightarrow\; r = \sqrt[3]{3}$. $a_{10} = a_7r^3 = 12(3) = 36$.

[20] $a_2 = 3$ and $a_5 = -81 \;\Rightarrow\; r^3 = \frac{-81}{3} = -27 \;\Rightarrow\; r = -3$.

$$a_9 = a_5r^4 = (-81)(-3)^4 = -6561.$$

[21] $\displaystyle\sum_{k=1}^{10} 3^k = 3 \cdot \frac{1-3^{10}}{1-3} = 3 \cdot \frac{-59{,}048}{-2} = 88{,}572$

[22] $\displaystyle\sum_{k=1}^{9} (-\sqrt{5})^k = -\sqrt{5} \cdot \frac{1-(-\sqrt{5})^9}{1-(-\sqrt{5})} = \frac{(-\sqrt{5})(1+625\sqrt{5})}{1+\sqrt{5}} \cdot \frac{1-\sqrt{5}}{1-\sqrt{5}} = \frac{3124\sqrt{5}-3120}{-4}$

$$= 780 - 781\sqrt{5}$$

[23] $\displaystyle\sum_{k=0}^{9} (-\tfrac{1}{2})^{k+1} = \sum_{k=1}^{10} (-\tfrac{1}{2})^k = -\frac{1}{2} \cdot \frac{1-(-\frac{1}{2})^{10}}{1-(-\frac{1}{2})} = -\frac{1}{2} \cdot \frac{\frac{1023}{1024}}{\frac{3}{2}} = -\frac{1023}{3072} = -\frac{341}{1024}$

[24] $\displaystyle\sum_{k=1}^{7} (3^{-k}) = \sum_{k=1}^{7} (\tfrac{1}{3})^k = \frac{1}{3} \cdot \frac{1-(\frac{1}{3})^7}{1-\frac{1}{3}} = \frac{1}{3} \cdot \frac{\frac{2186}{2187}}{\frac{2}{3}} = \frac{1093}{2187}$

[25] $\sum_{k=16}^{26}(2^{k-14}+5j)=\sum_{k=16}^{26}2^{k-14}+5\sum_{k=16}^{26}j=\text{SUM1}+5(\text{SUM2})$.

To evaluate SUM1, we use $a_1=2^{16-14}=4$, $r=2$, and $n=26-16+1=11$ to get:

$$S_{11}=4\cdot\frac{1-2^{11}}{1-2}=4\cdot\frac{-2047}{-1}=8188.$$

SUM2 is just $11\cdot j$ because j is a constant (k is the index).

Thus, $\text{SUM1}+5(\text{SUM2})=8188+5(11j)=8188+55j$.

[26] $\sum_{k=8}^{14}(3^{k-7}+2j^2)=\sum_{k=8}^{14}3^{k-7}+2\sum_{k=8}^{14}j^2=\text{SUM1}+2(\text{SUM2})$.

To evaluate SUM1, we use $a_1=3^{8-7}=3$, $r=3$, and $n=14-8+1=7$ to get:

$$S_7=3\cdot\frac{1-3^7}{1-3}=3\cdot\frac{-2186}{-2}=3279.$$

SUM2 is just $7\cdot j^2$ because j is a constant (k is the index).

Thus, $\text{SUM1}+2(\text{SUM2})=3279+2(7j^2)=3279+14j^2$.

[27] $2+4+8+16+32+64+128=2^1+2^2+2^3+2^4+2^5+2^6+2^7=\sum_{n=1}^{7}2^n$

[28] $2-4+8-16+32-64=2^1-2^2+2^3-2^4+2^5-2^6$.

The terms have alternating signs and are doubling. $\sum_{n=1}^{6}(-1)^{n+1}(2)^n$

[29] $\frac{1}{4}-\frac{1}{12}+\frac{1}{36}-\frac{1}{108}=\frac{1}{4}-\frac{1}{4}\cdot\frac{1}{3^1}+\frac{1}{4}\cdot\frac{1}{3^2}-\frac{1}{4}\cdot\frac{1}{3^3}=\sum_{n=1}^{4}(-1)^{n+1}\frac{1}{4}(\frac{1}{3})^{n-1}$

[30] $3+\frac{3}{5}+\frac{3}{25}+\frac{3}{125}+\frac{3}{625}=3+3(\frac{1}{5})^1+3(\frac{1}{5})^2+3(\frac{1}{5})^3+3(\frac{1}{5})^4=\sum_{n=1}^{5}3(\frac{1}{5})^{n-1}$

[31] $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots$ • $a_1=1$, $r=-\frac{1}{2}$, $S=\dfrac{1}{1+\frac{1}{2}}=\frac{2}{3}$

[32] $2+\frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\cdots$ • $a_1=2$, $r=\frac{1}{3}$, $S=\dfrac{2}{1-\frac{1}{3}}=3$

[33] $1.5+0.015+0.00015+\cdots$ • $a_1=1.5$, $r=0.01$, $S=\dfrac{1.5}{1-0.01}=\frac{50}{33}$

[34] $1-0.1+0.01-0.001+\cdots$ • $a_1=1$, $r=-0.1$, $S=\dfrac{1}{1+0.1}=\frac{10}{11}$

[35] $\sqrt{2}-2+\sqrt{8}-4+\cdots$ • ★ Since $|r|=\sqrt{2}>1$, the sum does not exist.

[36] $1+\frac{3}{2}+\frac{9}{4}+\frac{27}{8}+\cdots$ • ★ Since $|r|=\frac{3}{2}>1$, the sum does not exist.

[37] $256+192+144+108+\cdots$ • $a_1=256$, $r=\frac{192}{256}=\frac{3}{4}$, $S=\dfrac{256}{1-\frac{3}{4}}=1024$

[38] $250-100+40-16+\cdots$ • $a_1=250$, $r=-\frac{2}{5}$, $S=\dfrac{250}{1+\frac{2}{5}}=\frac{1250}{7}$

[39] $0.\overline{23}$ • $a_1=0.23$, $r=0.01$, $S=\dfrac{0.23}{1-0.01}=\frac{23}{99}$

[40] $0.0\overline{71}$ • $a_1=0.071$, $r=0.01$, $S=\dfrac{0.071}{1-0.01}=\frac{71}{990}$

[41] $2.4\overline{17}$ • $a_1 = 0.017$, $r = 0.01$, $S = \dfrac{0.017}{1-0.01} = \frac{17}{990}$; $2.4\overline{17} = 2.4 + \frac{17}{990} = \frac{2393}{990}$

[42] $10.\overline{5}$ • $a_1 = 0.5$, $r = 0.1$, $S = \dfrac{0.5}{1-0.1} = \frac{5}{9}$; $10.\overline{5} = 10 + \frac{5}{9} = \frac{95}{9}$

[43] $5.\overline{146}$ • $a_1 = 0.146$, $r = 0.001$, $S = \dfrac{0.146}{1-0.001} = \frac{146}{999}$; $5.\overline{146} = 5 + \frac{146}{999} = \frac{5141}{999}$

[44] $3.2\overline{394}$ • $a_1 = 0.0394$, $r = 0.001$, $S = \dfrac{0.0394}{1-0.001} = \frac{394}{9990}$;

$$3.2\overline{394} = 3.2 + \frac{394}{9990} = \frac{32{,}362}{9990} = \frac{16{,}181}{4995}$$

[45] $1.\overline{6124}$ • $a_1 = 0.6124$, $r = 0.0001$, $S = \dfrac{0.6124}{1-0.0001} = \frac{6124}{9999}$;

$$1.\overline{6124} = 1 + \frac{6124}{9999} = \frac{16{,}123}{9999}$$

[46] $123.61\overline{83}$ • $a_1 = 0.0083$, $r = 0.01$, $S = \dfrac{0.0083}{1-0.01} = \frac{83}{9900}$;

$$123.61\overline{83} = 123.61 + \frac{83}{9900} = \frac{1{,}223{,}822}{9900} = \frac{611{,}911}{4950}$$

[47] The geometric mean of 12 and 48 is $\sqrt{12 \cdot 48} = \sqrt{576} = 24$.

[48] The geometric mean of 20 and 25 is $\sqrt{20 \cdot 25} = \sqrt{500} = 10\sqrt{5}$.

[49] 2 geometric means $\Rightarrow$ $4 \cdot r^{2+1} = 500$ $\Rightarrow$ $r^3 = \frac{500}{4} = 125$ $\Rightarrow$ $r = 5$.

The terms are 4, 20, 100, and 500.

[50] 3 geometric means $\Rightarrow$ $2 \cdot r^{3+1} = 512$ $\Rightarrow$ $r^4 = \frac{512}{2} = 256$ $\Rightarrow$ $r = 4\ \{r > 0\}$.

The terms are 2, 8, 32, 128, and 512.

[51] Let $a_1 = x$ and $r = \frac{1}{2}$. $a_{11} = x(\frac{1}{2})^{10} = \frac{1}{1024}x$.

This is $(\frac{1}{1024} \cdot 100)\%$ or $\frac{25}{256}\%$ or approximately 0.1% of x.

[52] Let $a_1 = 20{,}000$ and $r = 1 - \frac{1}{4} = \frac{3}{4}$ {since the value at the end of the year is 75% of its value at the beginning of the year}. $a_7 = 20{,}000(\frac{3}{4})^6 \approx \$3559.57$

[53] Let $a_1 = 10{,}000$ and $r = 1.2$, that is, 120% every hour.

(a) $N(1) = a_2 = 10{,}000(1.2)^1$, $N(2) = a_3 = 10{,}000(1.2)^2$, ...,

$$N(t) = a_{t+1} = 10{,}000(1.2)^t$$

(b) $N(10) = a_{11} = 10{,}000(1.2)^{10} \approx 61{,}917$.

[54] The sequence of terms is P, $P + P \cdot \frac{r}{4} = P\left(1 + \frac{r}{4}\right)^1$,

$$P\left(1 + \frac{r}{4}\right) + P\left(1 + \frac{r}{4}\right) \cdot \frac{r}{4} = P\left(1 + \frac{r}{4}\right)\left(1 + \frac{r}{4}\right) = P\left(1 + \frac{r}{4}\right)^2, \ldots.$$

In n years, there will be $4n$ compounding periods. $a_{4n} = P\left(1 + \frac{r}{4}\right)^{4n}$

[55] $\text{Distance}_{\text{total}} = \text{Distance}_{\text{down}} + \text{Distance}_{\text{up}} = 60 + 2 \cdot \text{Distance}_{\text{up}}$

$$= 60 + 2\left[60(\tfrac{2}{3}) + 60(\tfrac{2}{3})^2 + \cdots\right] = 60 + 2\left(\frac{60(\frac{2}{3})}{1 - \frac{2}{3}}\right) = 60 + 2(120) = 300 \text{ ft.}$$

[56] The pendulum travels $24 + 24(\frac{5}{6}) + 24(\frac{5}{6})^2 + \cdots = \dfrac{24}{1 - \frac{5}{6}} = 144$ cm.

[57] Spending $= 2{,}000{,}000(0.60) + 2{,}000{,}000(0.60)^2 + \cdots = \dfrac{1{,}200{,}000}{1-0.60} = \$3{,}000{,}000$

[58] (a) The number of flies after n days for one group of flies is $a_n = N(0.9)^{n-1}$.

Since N flies are released each day, the number of flies on the nth day is

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} N(0.9)^{k-1} = N + (0.9)N + (0.9)^2N + \cdots + (0.9)^{n-1}N.$$

(b) $r = 0.9 < 1$. For a long-range goal, the sum of an infinite geometric series may be used with $S = 20{,}000$ and $a_1 = N$. $S = \dfrac{a_1}{1-r} \Rightarrow 20{,}000 = \dfrac{N}{1-0.9} \Rightarrow$

$N = 2000$ flies per day. Alternatively, since 10% of the flies *do not* survive a given day, we need to replace 10% of 20,000, or 2000 flies per day.

[59] (a) A half-life of 2 hours means there will be $(\frac{1}{2})(\frac{1}{2}D) = \frac{1}{4}D$ after 4 hours. The amount remaining after n doses {not hours} for a given dose is $a_n = D(\frac{1}{4})^{n-1}$.

Since D mg are administered every 4 hours, the amount of the drug in the bloodstream after n doses is $\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} D(\frac{1}{4})^{k-1} = D + \frac{1}{4}D + \cdots + (\frac{1}{4})^{n-1}D$.

Since $r = \frac{1}{4} < 1$, S_n may be approximated by

$$S = \frac{a_1}{1-r} \text{ for large } n. \quad S = \frac{D}{1-\frac{1}{4}} = \tfrac{4}{3}D.$$

(b) $\frac{4}{3}D \le 500 \Rightarrow D \le 375$ mg.

[60] From the figure we see that 2 prior generations yields 4 grandparents, 3 prior generations yields 8 grandparents, and in general n prior generations yields 2^n grandparents for $n \ge 2$. Going back 10 generations, there would be 2^{10} {1024}

grandparents and the total would be $\sum_{k=2}^{10} 2^k = \sum_{k=1}^{9} 2^{k+1} = 4 \cdot \dfrac{1-2^9}{1-2} = 2044$.

[61] (a) From the figure we see that $(\frac{1}{4}a_k)^2 + (\frac{3}{4}a_k)^2 = (a_{k+1})^2 \Rightarrow$

$$\tfrac{10}{16}a_k^2 = a_{k+1}^2 \Rightarrow a_{k+1} = \tfrac{1}{4}\sqrt{10}\,a_k.$$

(b) From part (a), $a_n = a_1(\frac{1}{4}\sqrt{10})^{n-1}$.

$A_{k+1} = a_{k+1}^2 = \frac{10}{16}a_k^2 = \frac{5}{8}A_k$, hence $A_n = (\frac{5}{8})^{n-1}A_1$.

$P_{k+1} = 4a_{k+1} = 4 \cdot \frac{1}{4}\sqrt{10}\,a_k = \sqrt{10}\,a_k = \sqrt{10}(\frac{1}{4}P_k)$, hence $P_n = (\frac{1}{4}\sqrt{10})^{n-1}P_1$.

(c) $\sum_{n=1}^{\infty} P_n$ is an infinite geometric series with first term P_1 and $r = \frac{1}{4}\sqrt{10}$.

$$S = \frac{P_1}{1-\frac{1}{4}\sqrt{10}} = \frac{4P_1}{4-\sqrt{10}} = \frac{16a_1}{4-\sqrt{10}}.$$

[62] (a) Let s_n denote the length of a side of the nth square and $\frac{1}{2}s_n$ the length of the radius of the inscribed circle. Now $C_n = \pi(\frac{1}{2}s_n)^2 = \frac{\pi}{4}s_n^2 = \frac{\pi}{4}S_n$, i.e., $S_n = \frac{4}{\pi}C_n$. Let r_n be the radius of the nth circle. The inscribed square will have a side of length $s_{n+1} = \sqrt{r_n^2 + r_n^2} = \sqrt{2}\,r_n$.

Thus, $C_n = \pi r_n^2$ and $S_{n+1} = s_{n+1}^2 = 2r_n^2 \Rightarrow C_n = \frac{\pi}{2}S_{n+1}$.

(b) The shaded region has area $(S_1 - C_1) + (S_2 - C_2) + (S_3 - C_3) + \cdots =$

$\sum_{n=1}^{\infty} S_n - \sum_{n=1}^{\infty} C_n = \sum_{n=1}^{\infty} S_n - \sum_{n=1}^{\infty} \frac{\pi}{4} S_n = \frac{4-\pi}{4} \sum_{n=1}^{\infty} S_n$. From part (a),

$S_{n+1} = \frac{2}{\pi}(\frac{\pi}{4} S_n) = \frac{1}{2} S_n$. Hence, $\sum_{n=1}^{\infty} S_n = S_1 + \frac{1}{2} S_1 + \frac{1}{4} S_1 + \cdots = \dfrac{S_1}{1 - \frac{1}{2}} =$

$2S_1$. Thus, the area is $\frac{4-\pi}{4}(2S_1) = \frac{4-\pi}{2} S_1$ or approximately 43% of S_1.

$\boxed{63}$ (a) The sequence is 1, 3, 9, 27, 81, …. Thus, $a_k = 3^{k-1}$ for $k = 1, 2, 3, \ldots$.

(b) $a_{15} = 3^{14} = 4{,}782{,}969$

(c) The area of the triangle removed first is $\frac{1}{4}$. During the next step, 3 triangles with an area of $\frac{1}{16}$ are removed. Then, 9 triangles with an area of $\frac{1}{64}$ are removed. At each step the number of triangles increase by a factor of 3, while the area of each triangle decreases by a factor of 4. Thus, $b_k = \dfrac{3^{k-1}}{4^k} = \frac{1}{4}\left(\frac{3}{4}\right)^{k-1}$.

(d) $b_7 = \frac{1}{4}\left(\frac{3}{4}\right)^6 = \dfrac{729}{16{,}384} \approx 0.0445 = 4.45\%$.

$\boxed{64}$ (a) From the previous exercise, $\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} 3^{k-1}$.

(b) $\sum_{k=1}^{12} 3^{k-1} = 3^0 + 3^1 + 3^2 + 3^3 + \cdots + 3^{11} = 1 \cdot \dfrac{1 - 3^{12}}{1-3} = 265{,}720$

(c) From the previous exercise, $\sum_{k=1}^{n} b_k = \sum_{k=1}^{n} \frac{1}{4}\left(\frac{3}{4}\right)^{k-1}$.

(d) $\sum_{k=1}^{12} \frac{1}{4}\left(\frac{3}{4}\right)^{k-1} = \frac{1}{4} + \frac{1}{4}\left(\frac{3}{4}\right) + \frac{1}{4}\left(\frac{3}{4}\right)^2 + \frac{1}{4}\left(\frac{3}{4}\right)^3 + \cdots + \frac{1}{4}\left(\frac{3}{4}\right)^{11} = \left(\frac{1}{4}\right)\dfrac{1 - (\frac{3}{4})^{12}}{1 - \frac{3}{4}} \approx$ $0.96832 \approx 97\%$

$\boxed{65}$ Let $a_k = 100\left(1 + \frac{0.06}{12}\right)^k = 100(1.005)^k$, where k represents the number of compounding periods for each deposit. For the first deposit, $k = 18 \cdot 12 = 216$. For the last deposit, $k = 1$.

$$\begin{aligned} S_{216} &= a_1 + a_2 + \cdots + a_{216} \\ &= 100(1.005)^1 + 100(1.005)^2 + \cdots + 100(1.005)^{216} \\ &= 100(1.005)\left(\frac{1 - (1.005)^{216}}{1 - (1.005)}\right) \approx \$38{,}929.00 \end{aligned}$$

$\boxed{66}$ $A = P(1 + \frac{r}{12})^1 + P(1 + \frac{r}{12})^2 + \cdots + P(1 + \frac{r}{12})^n$

$$= P(1 + \tfrac{r}{12})\left(\frac{1 - (1 + \frac{r}{12})^n}{1 - (1 + \frac{r}{12})}\right) = P(1 + \tfrac{r}{12})\left(\frac{1 - (1 + \frac{r}{12})^n}{-\frac{r}{12}}\right) = P(\tfrac{12}{r} + 1)[(1 + \tfrac{r}{12})^n - 1]$$

$\boxed{67}$ $A = 100\left(\frac{12}{0.08} + 1\right)\left[\left(1 + \frac{0.08}{12}\right)^{60} - 1\right] \approx \$7396.67$

[68] First, solve for n. $A = P\left(\frac{12}{r}+1\right)\left[\left(1+\frac{r}{12}\right)^n - 1\right] \Rightarrow$

$$\left(1+\frac{r}{12}\right)^n - 1 = \frac{Ar}{P(12+r)} \Rightarrow \left(1+\frac{r}{12}\right)^n = \frac{Ar}{P(12+r)} + 1 \Rightarrow$$

$$n \ln\left(1+\frac{r}{12}\right) = \ln\left(\frac{Ar}{P(12+r)}+1\right) \Rightarrow n = \ln\left(\frac{Ar}{P(12+r)}+1\right) \Big/ \ln\left(1+\frac{r}{12}\right)$$

(a) $A = 100{,}000$, $r = 0.10$, and $P = 100 \Rightarrow n \approx 268.25$ mo, or, 22.35 yr.

(b) $A = 100{,}000$, $r = 0.10$, and $P = 200 \Rightarrow n \approx 197.08$ mo, or, 16.42 yr.

[69] (a) $A_1 = \frac{2}{5}\left(1-\frac{2}{5}\right)^{1-1} = \frac{2}{5}\left(\frac{3}{5}\right)^0 = \frac{2}{5}$.

$$A_2 = \frac{2}{5}\left(\frac{3}{5}\right)^1 = \frac{6}{25},\ A_3 = \frac{2}{5}\left(\frac{3}{5}\right)^2 = \frac{18}{125},\ A_4 = \frac{2}{5}\left(\frac{3}{5}\right)^3 = \frac{54}{625},\ A_5 = \frac{2}{5}\left(\frac{3}{5}\right)^4 = \frac{162}{3125}.$$

(b) $r = A_{k+1}/A_k = \frac{3}{5}$ for $k = 1, 2, 3, 4$.

$$S_5 = \sum_{k=1}^{5} A_k = A_1 + A_2 + A_3 + A_4 + A_5 = \frac{2}{5}\cdot\frac{1-\left(\frac{3}{5}\right)^5}{1-\frac{3}{5}} = 1 - \left(\tfrac{3}{5}\right)^5 = \frac{2882}{3125} = 0.92224.$$

(c) $\$25{,}000\left(\frac{2}{5}+\frac{6}{25}\right) = \$25{,}000\left(\frac{16}{25}\right) = \$16{,}000$

[70] (a) $A_1 = \frac{2}{n}\left(1-\frac{2}{n}\right)^{1-1} = \frac{2}{n}\left(1-\frac{2}{n}\right)^0 = \frac{2}{n}$.

$$A_2, A_3, \ldots, A_n = \frac{2}{n}\left(1-\frac{2}{n}\right)^1, \frac{2}{n}\left(1-\frac{2}{n}\right)^2, \ldots, \frac{2}{n}\left(1-\frac{2}{n}\right)^{n-1}.$$

(b) $$r = \frac{A_{k+1}}{A_k} = \frac{\frac{2}{n}\left(1-\frac{2}{n}\right)^{(k+1)-1}}{\frac{2}{n}\left(1-\frac{2}{n}\right)^{k-1}} = \left(1-\frac{2}{n}\right)^{k-(k-1)} = 1 - \frac{2}{n}.$$

$$S_n = \sum_{k=1}^{n} A_k = A_1 \frac{1-r^n}{1-r} = \frac{2}{n}\cdot\left[\frac{1-\left(1-\frac{2}{n}\right)^n}{1-\left(1-\frac{2}{n}\right)}\right] = 1 - \left(1-\frac{2}{n}\right)^n.$$

10.4 Exercises

[1] (1) P_1 is true, since $2(1) = 1(1+1) = 2$.

(2) Assume P_k is true:

$2 + 4 + 6 + \cdots + 2k = k(k+1)$. Hence,

$$\begin{aligned} 2 + 4 + 6 + \cdots + 2k + 2(k+1) &= k(k+1) + 2(k+1) \\ &= (k+1)(k+2) \\ &= (k+1)(k+1+1). \end{aligned}$$

Thus, P_{k+1} is true, and the proof is complete.

$\boxed{2}$ (1) P_1 is true, since $3(1)-2=\dfrac{1[3(1)-1]}{2}=1$.

(2) Assume P_k is true:

$1+4+7+\cdots+(3k-2)=\dfrac{k(3k-1)}{2}$. Hence,

$$\begin{aligned}1+4+7+\cdots+(3k-2)+3(k+1)-2 &= \frac{k(3k-1)}{2}+3(k+1)-2\\ &= \frac{3k^2+5k+2}{2}\\ &= \frac{(k+1)(3k+2)}{2}\\ &= \frac{(k+1)[3(k+1)-1]}{2}.\end{aligned}$$

Thus, P_{k+1} is true, and the proof is complete.

$\boxed{3}$ (1) P_1 is true, since $2(1)-1=(1)^2=1$.

(2) Assume P_k is true:

$1+3+5+\cdots+(2k-1)=k^2$. Hence,

$$\begin{aligned}1+3+5+\cdots+(2k-1)+2(k+1)-1 &= k^2+2(k+1)-1\\ &= k^2+2k+1\\ &= (k+1)^2.\end{aligned}$$

Thus, P_{k+1} is true, and the proof is complete.

$\boxed{4}$ (1) P_1 is true, since $6(1)-3=3(1)^2=3$.

(2) Assume P_k is true:

$3+9+15+\cdots+(6k-3)=3k^2$. Hence,

$$\begin{aligned}3+9+15+\cdots+(6k-3)+6(k+1)-3 &= 3k^2+6(k+1)-3\\ &= 3k^2+6k+3\\ &= 3(k^2+2k+1)\\ &= 3(k+1)^2.\end{aligned}$$

Thus, P_{k+1} is true, and the proof is complete.

$\boxed{5}$ (1) P_1 is true, since $5(1)-3=\frac{1}{2}(1)[5(1)-1]=2$.

(2) Assume P_k is true:

$2+7+12+\cdots+(5k-3)=\frac{1}{2}k(5k-1)$. Hence,

$$\begin{aligned}2+7+12+\cdots+(5k-3)+5(k+1)-3 &= \tfrac{1}{2}k(5k-1)+5(k+1)-3\\ &= \tfrac{5}{2}k^2+\tfrac{9}{2}k+2\\ &= \tfrac{1}{2}(5k^2+9k+4)\\ &= \tfrac{1}{2}(k+1)(5k+4)\\ &= \tfrac{1}{2}(k+1)[5(k+1)-1].\end{aligned}$$

Thus, P_{k+1} is true, and the proof is complete.

[6] (1) P_1 is true, since $2 \cdot 3^{1-1} = 3^1 - 1 = 2$.

(2) Assume P_k is true:

$2 + 6 + 18 + \cdots + 2 \cdot 3^{k-1} = 3^k - 1$. Hence,

$$\begin{aligned} 2 + 6 + 18 + \cdots + 2 \cdot 3^{k-1} + 2 \cdot 3^k &= 3^k - 1 + 2 \cdot 3^k \\ &= 1 \cdot 3^k + 2 \cdot 3^k - 1 \\ &= 3^1 \cdot 3^k - 1 \\ &= 3^{k+1} - 1. \end{aligned}$$

Thus, P_{k+1} is true, and the proof is complete.

[7] (1) P_1 is true, since $1 \cdot 2^{1-1} = 1 + (1-1) \cdot 2^1 = 1$.

(2) Assume P_k is true:

$1 + 2 \cdot 2 + 3 \cdot 2^2 + \cdots + k \cdot 2^{k-1} = 1 + (k-1) \cdot 2^k$. Hence,

$$\begin{aligned} 1 + 2 \cdot 2 + 3 \cdot 2^2 + \cdots + k \cdot 2^{k-1} + (k+1) \cdot 2^k &= 1 + (k-1) \cdot 2^k + (k+1) \cdot 2^k \\ &= 1 + k \cdot 2^k - 2^k + k \cdot 2^k + 2^k \\ &= 1 + k \cdot 2^1 \cdot 2^k \\ &= 1 + \big[(k+1) - 1\big] \cdot 2^{k+1}. \end{aligned}$$

Thus, P_{k+1} is true, and the proof is complete.

[8] (1) P_1 is true, since $(-1)^1 = \dfrac{(-1)^1 - 1}{2} = -1$.

(2) Assume P_k is true:

$(-1)^1 + (-1)^2 + (-1)^3 + \cdots + (-1)^k = \dfrac{(-1)^k - 1}{2}$. Hence,

$$\begin{aligned} (-1)^1 + (-1)^2 + (-1)^3 + \cdots + (-1)^k + (-1)^{k+1} &= \frac{(-1)^k - 1}{2} + (-1)^{k+1} \\ &= \frac{1(-1)^k}{2} - \frac{1}{2} - \frac{2(-1)^k}{2} \\ &= \frac{(-1)^k \cdot (-1) - 1}{2} \\ &= \frac{(-1)^{k+1} - 1}{2}. \end{aligned}$$

Thus, P_{k+1} is true, and the proof is complete.

[9] (1) P_1 is true, since $(1)^1 = \dfrac{1(1+1)\big[2(1)+1\big]}{6} = 1$.

(2) Assume P_k is true:

$1^2 + 2^2 + 3^2 + \cdots + k^2 = \dfrac{k(k+1)(2k+1)}{6}$. Hence,

$$\begin{aligned} 1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= (k+1)\left[\frac{k(2k+1)}{6} + \frac{6(k+1)}{6}\right] \end{aligned}$$

(continued)

$$= \frac{(k+1)(2k^2+7k+6)}{6}$$
$$= \frac{(k+1)(k+2)(2k+3)}{6}.$$

Thus, P_{k+1} is true, and the proof is complete.

$\boxed{10}$ (1) P_1 is true, since $(1)^3 = \left[\frac{1(1+1)}{2}\right]^2 = 1$.

(2) Assume P_k is true:

$1^3 + 2^3 + 3^3 + \cdots + k^3 = \left[\frac{k(k+1)}{2}\right]^2$. Hence,

$$\begin{aligned} 1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 &= \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3 \\ &= \frac{(k+1)^2}{2^2}\left[k^2 + 4(k+1)\right] \\ &= \frac{(k+1)^2}{2^2}(k+2)^2 \\ &= \left[\frac{(k+1)[(k+1)+1]}{2}\right]^2. \end{aligned}$$

Thus, P_{k+1} is true, and the proof is complete.

$\boxed{11}$ (1) P_1 is true, since $\frac{1}{1(1+1)} = \frac{1}{1+1} = \frac{1}{2}$.

(2) Assume P_k is true:

$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$. Hence,

$$\begin{aligned} \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\ &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\ &= \frac{k(k+2)+1}{(k+1)(k+2)} \\ &= \frac{k^2+2k+1}{(k+1)(k+2)} \\ &= \frac{k+1}{(k+1)+1}. \end{aligned}$$

Thus, P_{k+1} is true, and the proof is complete.

$\boxed{12}$ (1) P_1 is true, since $\frac{1}{1(1+1)(1+2)} = \frac{1(1+3)}{4(1+1)(1+2)} = \frac{1}{6}$.

(2) Assume P_k is true:

$\frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \frac{1}{3\cdot 4\cdot 5} + \cdots + \frac{1}{k(k+1)(k+2)} = \frac{k(k+3)}{4(k+1)(k+2)}$. Hence,

$$\frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \frac{1}{3\cdot 4\cdot 5} + \cdots + \frac{1}{k(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$$
$$= \frac{k(k+3)}{4(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$$

(continued)

$$= \frac{k(k+3)^2+4}{4(k+1)(k+2)(k+3)}$$
$$= \frac{k(k^2+6k+9)+4}{4(k+1)(k+2)(k+3)}$$
$$= \frac{k^3+6k^2+9k+4}{4(k+1)(k+2)(k+3)}$$
$$= \frac{(k+1)(k^2+5k+4)}{4(k+1)(k+2)(k+3)}$$
$$= \frac{(k+1)(k+4)}{4(k+2)(k+3)}.$$

Thus, P_{k+1} is true, and the proof is complete.

$\boxed{13}$ (1) P_1 is true, since $3^1 = \frac{3}{2}(3^1 - 1) = 3$.

(2) Assume P_k is true:

$3 + 3^2 + 3^3 + \cdots + 3^k = \frac{3}{2}(3^k - 1)$. Hence,

$$\begin{aligned} 3 + 3^2 + 3^3 + \cdots + 3^k + 3^{k+1} &= \tfrac{3}{2}(3^k - 1) + 3^{k+1} \\ &= \tfrac{3}{2}\cdot 3^k - \tfrac{3}{2} + 3\cdot 3^k \\ &= \tfrac{9}{2}\cdot 3^k - \tfrac{3}{2} \\ &= \tfrac{3}{2}(3\cdot 3^k - 1) \\ &= \tfrac{3}{2}(3^{k+1} - 1). \end{aligned}$$

Thus, P_{k+1} is true, and the proof is complete.

$\boxed{14}$ (1) P_1 is true, since $[2(1) - 1]^3 = (1)^2(2\cdot 1^2 - 1) = 1$.

(2) Assume P_k is true:

$1^3 + 3^3 + 5^3 + \cdots + (2k-1)^3 = k^2(2k^2 - 1)$. Hence,

$$\begin{aligned} 1^3 + 3^3 + 5^3 + \cdots + (2k-1)^3 + [2(k+1) - 1]^3 &= k^2(2k^2 - 1) + [2(k+1) - 1]^3 \\ &= k^2(2k^2 - 1) + [2(k+1) - 1]^3 \\ &= 2k^4 - k^2 + (2k+1)^3 \\ &= 2k^4 + 8k^3 + 11k^2 + 6k + 1 \\ &= (k+1)^2(2k^2 + 4k + 1) \\ &= (k+1)^2[2(k+1)^2 - 1]. \end{aligned}$$

Thus, P_{k+1} is true, and the proof is complete.

$\boxed{15}$ (1) P_1 is true, since $1 < 2^1$.

(2) Assume P_k is true: $k < 2^k$. Now $k + 1 < k + k = 2(k)$ for $k > 1$.

From P_k, we see that $2(k) < 2(2^k) = 2^{k+1}$ and conclude that $k + 1 < 2^{k+1}$.

Thus, P_{k+1} is true, and the proof is complete.

[16] (1) P_1 is true, since $1+2(1) \le 3^1$.

(2) Assume P_k is true: $1+2k \le 3^k$.

$1+2(k+1) = 2k+3 < 6k+3$ which is $3(1+2k)$. From P_k, we see that $3(1+2k) < 3(3^k) = 3^{k+1}$ and conclude that $1+2(k+1) \le 3^{k+1}$.

Thus, P_{k+1} is true, and the proof is complete.

[17] (1) P_1 is true, since $1 < \frac{1}{8}[2(1)+1]^2 = \frac{9}{8}$.

(2) Assume P_k is true: $1+2+3+\cdots+k < \frac{1}{8}(2k+1)^2$. Hence,

$$\begin{aligned} 1+2+3+\cdots+k+(k+1) &< \tfrac{1}{8}(2k+1)^2+(k+1) \\ &= \tfrac{1}{2}k^2+\tfrac{3}{2}k+\tfrac{9}{8} \\ &= \tfrac{1}{8}(4k^2+12k+9) \\ &= \tfrac{1}{8}(2k+3)^2 \\ &= \tfrac{1}{8}[2(k+1)+1]^2. \end{aligned}$$

Thus, P_{k+1} is true, and the proof is complete.

[18] (1) If $0<a<b$, then $a^2b < ab^2$ {multiply by ab} and $\frac{a^2}{b^2} < \frac{a}{b}$ {divide by b^3}.

This is P_1: $\left(\frac{a}{b}\right)^2 < \left(\frac{a}{b}\right)^1$.

(2) Assume P_k is true: $\left(\frac{a}{b}\right)^{k+1} < \left(\frac{a}{b}\right)^k$. Hence, $a^{k+1}b^k < a^kb^{k+1}$ $\Rightarrow$

$a^{k+2}b^{k+1} < a^{k+1}b^{k+2}$ {multiply by ab} $\Rightarrow$

$\frac{a^{k+2}}{b^{k+2}} < \frac{a^{k+1}}{b^{k+1}}$ {divide by b^{2k+3}}.

This is P_{k+1}: $\left(\frac{a}{b}\right)^{k+2} < \left(\frac{a}{b}\right)^{k+1}$.

Thus, P_{k+1} is true, and the proof is complete.

[19] (1) For $n=1$, $n^3-n+3=3$ and 3 is a factor of 3.

(2) Assume 3 is a factor of k^3-k+3. The $(k+1)$st term is

$$\begin{aligned} (k+1)^3-(k+1)+3 &= k^3+3k^2+2k+3 \\ &= (k^3-k+3)+3k^2+3k \\ &= (k^3-k+3)+3(k^2+k). \end{aligned}$$

By the induction hypothesis, 3 is a factor of k^3-k+3 and 3 is a factor of $3(k^2+k)$, so 3 is a factor of the $(k+1)$st term. Thus, P_{k+1} is true, and the proof is complete.

$\boxed{20}$ (1) For $n=1$, $n^2+n=2$ and 2 is a factor of 2.

(2) Assume 2 is a factor of k^2+k. The $(k+1)$st term is

$$\begin{aligned}(k+1)^2+(k+1) &= k^2+3k+2 \\ &= (k^2+k)+2k+2 \\ &= (k^2+k)+2(k+1).\end{aligned}$$

By the induction hypothesis, 2 is a factor of k^2+k and 2 is a factor of $2(k+1)$, so 2 is a factor of the $(k+1)$st term. Thus, P_{k+1} is true, and the proof is complete.

$\boxed{21}$ (1) For $n=1$, $5^n-1=4$ and 4 is a factor of 4.

(2) Assume 4 is a factor of 5^k-1. The $(k+1)$st term is

$$\begin{aligned}5^{k+1}-1 &= 5\cdot 5^k-1 \\ &= 5\cdot 5^k-5+4 \\ &= 5(5^k-1)+4.\end{aligned}$$

By the induction hypothesis, 4 is a factor of 5^k-1 and 4 is a factor of 4, so 4 is a factor of the $(k+1)$st term. Thus, P_{k+1} is true, and the proof is complete.

$\boxed{22}$ (1) For $n=1$, $10^{n+1}+3\cdot 10^n+5=135$ and 9 is a factor of 135.

(2) Assume 9 is a factor of $10^{k+1}+3\cdot 10^k+5$. The $(k+1)$st term is

$$\begin{aligned}10^{k+2}+3\cdot 10^{k+1}+5 &= 10\cdot 10^{k+1}+10\cdot 3\cdot 10^k+5 \\ &= 10^{k+1}+9\cdot 10^{k+1}+3\cdot 10^k+9\cdot 3\cdot 10^k+5 \\ &= (10^{k+1}+3\cdot 10^k+5)+9(10^{k+1}+3\cdot 10^k).\end{aligned}$$

By the induction hypothesis, 9 is a factor of $10^{k+1}+3\cdot 10^k+5$ and 9 is a factor of $9(10^{k+1}+3\cdot 10^k)$, so 9 is a factor of the $(k+1)$st term. Thus, P_{k+1} is true and the proof is complete.

$\boxed{23}$ (1) If $a>1$, then $a^1=a>1$, so P_1 is true.

(2) Assume P_k is true: $a^k>1$.

Multiply both sides by a to obtain $a^{k+1}>a$, but since $a>1$, we have $a^{k+1}>1$.

Thus, P_{k+1} is true, and the proof is complete.

$\boxed{24}$ (1) For $n=1$, $ar^{1-1}=a$ and $\dfrac{a(1-r^1)}{1-r}=a$, so P_1 is true.

(2) Assume P_k is true:

$a+ar+ar^2+\cdots+ar^{k-1}=\dfrac{a(1-r^k)}{1-r}$. Hence,

$$\begin{aligned}a+ar+ar^2+\cdots+ar^{k-1}+ar^k &= \frac{a(1-r^k)}{1-r}+ar^k \\ &= a\left(\frac{1-r^k}{1-r}+\frac{r^k(1-r)}{1-r}\right)\end{aligned}$$

(continued)

$$= a\left(\frac{1-r^k}{1-r} + \frac{r^k(1-r)}{1-r}\right) \qquad \text{(repeated)}$$

$$= a\left(\frac{1-r^k+r^k-r^{k+1}}{1-r}\right)$$

$$= \frac{a(1-r^{k+1})}{1-r}.$$

Thus, P_{k+1} is true, and the proof is complete.

[25] (1) For $n = 1$, $a - b$ is a factor of $a^1 - b^1$.

(2) Assume $a - b$ is a factor of $a^k - b^k$. Following the hint for the $(k+1)$st term, $a^{k+1} - b^{k+1} = a^k \cdot a - b \cdot a^k + b \cdot a^k - b^k \cdot b = a^k(a-b) + (a^k - b^k)b$. Since $(a-b)$ is a factor of $a^k(a-b)$ and since by the induction hypothesis $a-b$ is a factor of $(a^k - b^k)$, it follows that $a - b$ is a factor of the $(k+1)$st term. Thus, P_{k+1} is true, and the proof is complete.

[26] (1) For $n = 1$, $a + b$ is a factor of $a^{2(1)-1} + b^{2(1)-1} = a + b$.

(2) Assume $a + b$ is a factor of $a^{2k-1} + b^{2k-1}$. The $(k+1)$st term is

$$\begin{aligned} a^{2k+1} + b^{2k+1} &= a^{2k-1} \cdot a^2 - a^{2k-1} \cdot b^2 + a^{2k-1} \cdot b^2 + b^{2k-1} \cdot b^2 \\ &= a^{2k-1}(a^2 - b^2) + b^2(a^{2k-1} + b^{2k-1}). \end{aligned}$$

Since $(a+b)$ is a factor of $a^{2k-1}(a^2-b^2)$ $\{a^2 - b^2 = (a+b)(a-b)\}$ and since by the induction hypothesis, $a + b$ is a factor of $b^2(a^{2k-1} + b^{2k-1})$, it follows that $a + b$ is a factor of the $(k+1)$st term. Thus, P_{k+1} is true, and the proof is complete.

Note: For Exercises 27–32 in this section and Exercises 49–50 in the Chapter Review Exercises, there are several ways to find j. Possibilities include: solve the inequality, sketch the graphs of functions representing each side, and trial and error. Trial and error may be the easiest to use.

[27] For j: $n^2 \geq n + 12 \Rightarrow n^2 - n - 12 \geq 0 \Rightarrow (n-4)(n+3) \geq 0 \Rightarrow n \geq 4$ $\{n > 0\}$

(1) P_4 is true, since $4 + 12 \leq 4^2$.

(2) Assume P_k is true: $k + 12 \leq k^2$. Hence,

$(k+1) + 12 = (k+12) + 1 \leq (k^2) + 1 < k^2 + 2k + 1 = (k+1)^2$.

Thus, P_{k+1} is true, and the proof is complete.

[28] For j: By trial and error, $j = 3$.

(1) P_3 is true, since $3^2 + 18 \leq 3^3$.

(2) Assume P_k is true: $k^2 + 18 \leq k^3$. Hence,

$$\begin{aligned} (k+1)^2 + 18 &= (k^2 + 18) + 2k + 1 \\ &\leq (k^3) + 2k + 1 < k^3 + 3k^2 + 3k + 1 = (k+1)^3. \end{aligned}$$

Thus, P_{k+1} is true, and the proof is complete.

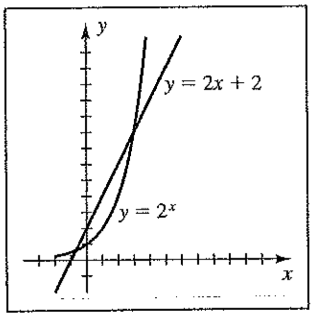


Figure 31

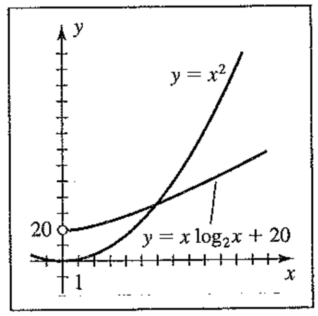


Figure 32

32 For j: Since $n^2 < 20$ if $n = 4$, a reasonable first guess would be $j = 5$.

By trial and error, $j = 6$.

(1) P_6 is true, since $6 \log_2 6 + 20 \leq 6^2$.

(2) Assume P_k is true: $k \log_2 k + 20 \leq k^2$.

$$\begin{aligned}(k+1)\log_2(k+1) + 20 &= k \log_2(k+1) + \log_2(k+1) + 20 \\ &< k \log_2 2k + \log_2 2k + 20 \\ &= k \log_2 k + k + 1 + \log_2 k + 20 \\ &\leq k^2 + k + 1 + \log_2 k \\ &< k^2 + 2k + 1 = (k+1)^2.\end{aligned}$$

Thus, P_{k+1} is true, and the proof is complete.

33 Following the hint in the text:

$$\begin{aligned}\sum_{k=1}^{n}(k^2+3k+5) &= \sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 5 \\ &= \frac{n(n+1)(2n+1)}{6} + 3\left[\frac{n(n+1)}{2}\right] + 5n \\ &= \frac{n(n+1)(2n+1) + 9n(n+1) + 30n}{6} \\ &= \frac{2n^3+12n^2+40n}{6} = \frac{n^3+6n^2+20n}{3}\end{aligned}$$

34 $$\begin{aligned}\sum_{k=1}^{n}(3k^2-2k+1) &= 3\sum_{k=1}^{n} k^2 - 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 \\ &= 3\left[\frac{n(n+1)(2n+1)}{6}\right] - 2\left[\frac{n(n+1)}{2}\right] + n \\ &= \frac{2n^3+n^2+n}{2}\end{aligned}$$

35 $$\begin{aligned}\sum_{k=1}^{n}(2k-3)^2 &= \sum_{k=1}^{n}(4k^2-12k+9) \\ &= 4\sum_{k=1}^{n} k^2 - 12\sum_{k=1}^{n} k + \sum_{k=1}^{n} 9 \\ &= 4\left[\frac{n(n+1)(2n+1)}{6}\right] - 12\left[\frac{n(n+1)}{2}\right] + 9n \\ &= \frac{4n^3-12n^2+11n}{3}\end{aligned}$$

[29] For j: By sketching $y = 5 + \log_2 x$ and $y = x$, we see that the solution for $x > 1$ must be larger than 5. By trial and error, $j = 8$.

(1) P_8 is true, since $5 + \log_2 8 \leq 8$.

(2) Assume P_k is true: $5 + \log_2 k \leq k$. Hence,

$$\begin{aligned} 5 + \log_2 (k+1) &< 5 + \log_2 (k+k) \\ &= 5 + \log_2 2k \\ &= 5 + \log_2 2 + \log_2 k \\ &= (5 + \log_2 k) + 1 \\ &\leq k + 1. \end{aligned}$$

Thus, P_{k+1} is true, and the proof is complete.

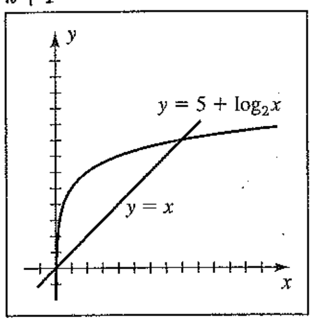


Figure 29

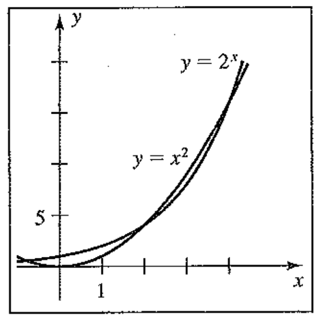


Figure 30

[30] For j: By sketching $y = x^2$ and $y = 2^x$, we see that there are three intersection points, the largest being 4. Discussion Exercise 5 in Chapter 5 also dealt with this type of problem.

(1) P_4 is true, since $4^2 \leq 2^4$.

(2) Assume P_k is true: $k^2 \leq 2^k$. Hence,

$$(k+1)^2 = k^2 + 2k + 1 = k(k + 2 + \tfrac{1}{k}) < k(k+k) = 2k^2 \leq 2 \cdot 2^k = 2^{k+1}.$$

Thus, P_{k+1} is true, and the proof is complete.

[31] For j: By sketching $y = 2x + 2$ and $y = 2^x$, we see there is one positive solution. By trial and error, $j = 3$. See *Figure 31*.

(1) P_3 is true, since $2(3) + 2 \leq 2^3$.

(2) Assume P_k is true: $2k + 2 \leq 2^k$. Hence,

$$2(k+1) + 2 = (2k+2) + 2 \leq 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}.$$

Thus, P_{k+1} is true, and the proof is complete.

$\boxed{36}$ $\displaystyle\sum_{k=1}^{n}(k^3+2k^2-k+4) = \sum_{k=1}^{n}k^3+2\sum_{k=1}^{n}k^2-\sum_{k=1}^{n}k+\sum_{k=1}^{n}4$

$$= \left[\frac{n(n+1)}{2}\right]^2+2\left[\frac{n(n+1)(2n+1)}{6}\right]-\left[\frac{n(n+1)}{2}\right]+4n$$

$$= \frac{3n^4+14n^3+9n^2+46n}{12}$$

$\boxed{37}$ (a) $n=1 \Rightarrow a(1)^3+b(1)^2+c(1)=1^2 \qquad \Rightarrow a+b+c=1$

$n=2 \Rightarrow a(2)^3+b(2)^2+c(2)=1^2+2^2 \qquad \Rightarrow 8a+4b+2c=5$

$n=3 \Rightarrow a(3)^3+b(3)^2+c(3)=1^2+2^2+3^2 \qquad \Rightarrow 27a+9b+3c=14$

$$AX=B \Rightarrow \begin{bmatrix}1&1&1\\8&4&2\\27&9&3\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}1\\5\\14\end{bmatrix} \Rightarrow X=A^{-1}B=\begin{bmatrix}1/3\\1/2\\1/6\end{bmatrix}.$$

(b) $a=\frac{1}{3},\ b=\frac{1}{2},\ c=\frac{1}{6} \Rightarrow$

$1^2+2^2+3^2+\cdots+n^2=\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n=\dfrac{n(n+1)(2n+1)}{6}$,

which is the formula found in Exercise 9. This method does not verify the formula for all n but only for $n = 1, 2, 3$. Mathematical induction should be used to verify the formula for all n as in Exercise 9.

$\boxed{38}$ (a) $n=1 \Rightarrow a(1)^4+b(1)^3+c(1)^2+d(1)=1^3 \Rightarrow a+b+c+d=1$

$n=2 \Rightarrow a(2)^4+b(2)^3+c(2)^2+d(2)=1^3+2^3 \Rightarrow 16a+8b+4c+2d=9$

$n=3 \Rightarrow a(3)^4+b(3)^3+c(3)^2+d(3)=1^3+2^3+3^3 \Rightarrow$

$81a+27b+9c+3d=36$

$n=4 \Rightarrow a(4)^4+b(4)^3+c(4)^2+d(4)=1^3+2^3+3^3+4^3 \Rightarrow$

$256a+64b+16c+4d=100$

$$AX=B \Rightarrow \begin{bmatrix}1&1&1&1\\16&8&4&2\\81&27&9&3\\256&64&16&4\end{bmatrix}\begin{bmatrix}a\\b\\c\\d\end{bmatrix}=\begin{bmatrix}1\\9\\36\\100\end{bmatrix} \Rightarrow X=A^{-1}B=\begin{bmatrix}1/4\\1/2\\1/4\\0\end{bmatrix}$$

(b) $a=\frac{1}{4},\ b=\frac{1}{2},\ c=\frac{1}{4},\ d=0 \Rightarrow$

$1^3+2^3+3^3+\cdots+n^3=\frac{1}{4}n^4+\frac{1}{2}n^3+\frac{1}{4}n^2=\left[\dfrac{n(n+1)}{2}\right]^2$, which is the formula found in Exercise 10. This method does not verify the formula for all n but only for $n = 1, 2, 3, 4$. Mathematical induction should be used to verify the formula for all n as in Exercise 10.

Note: The solutions for Exercises 39–42 are on page 584 at the end of the chapter.

10.5 Exercises

$\boxed{1}$ $2!6! = 2 \cdot 720 = 1440$

$\boxed{2}$ $3!4! = 6 \cdot 24 = 144$

$\boxed{3}$ $7!0! = 5040 \cdot 1 = 5040$

$\boxed{4}$ $5!0! = 120 \cdot 1 = 120$

$\boxed{5}$ $\frac{8!}{5!} = \frac{8 \cdot 7 \cdot 6 \cdot 5!}{5!} = 8 \cdot 7 \cdot 6 = 336$

$\boxed{6}$ $\frac{6!}{3!} = 6 \cdot 5 \cdot 4 = 120$

$\boxed{7}$ $\binom{5}{5} = \frac{5!}{5!\,0!} = 1$

$\boxed{8}$ $\binom{7}{0} = \frac{7!}{0!\,7!} = 1$

$\boxed{9}$ $\binom{7}{5} = \frac{7!}{5!\,2!} = \frac{7 \cdot 6}{2} = 21$

$\boxed{10}$ $\binom{8}{4} = \frac{8!}{4!\,4!} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2} = 70$

$\boxed{11}$ $\binom{13}{4} = \frac{13!}{4!\,9!} = \frac{13 \cdot 12 \cdot 11 \cdot 10}{4 \cdot 3 \cdot 2} = 715$

$\boxed{12}$ $\binom{52}{2} = \frac{52!}{2!\,50!} = \frac{52 \cdot 51}{2} = 1326$

$\boxed{13}$ $\frac{n!}{(n-2)!} = \frac{n(n-1)(n-2)!}{(n-2)!} = n(n-1)$

$\boxed{14}$ $\frac{(n+1)!}{(n-1)!} = \frac{(n+1)(n)(n-1)!}{(n-1)!} = (n+1)n$

$\boxed{15}$ $\frac{(2n+2)!}{(2n)!} = \frac{(2n+2)(2n+1)(2n)!}{(2n)!} = (2n+2)(2n+1)$

$\boxed{16}$ $\frac{(3n+1)!}{(3n-1)!} = \frac{(3n+1)(3n)(3n-1)!}{(3n-1)!} = (3n+1)(3n)$

$\boxed{17}$
$$\begin{aligned}(4x-y)^3 &= \binom{3}{0}(4x)^3(-y)^0 + \binom{3}{1}(4x)^2(-y)^1 + \binom{3}{2}(4x)^1(-y)^2 + \binom{3}{3}(4x)^0(-y)^3 \\ &= (1)(64x^3)(1) - (3)(16x^2)(y) + (3)(4x)(y^2) - (1)(1)(y^3) \\ &= 64x^3 - 48x^2y + 12xy^2 - y^3\end{aligned}$$

$\boxed{18}$
$$\begin{aligned}(x^2+2y)^3 &= \binom{3}{0}(x^2)^3(2y)^0 + \binom{3}{1}(x^2)^2(2y)^1 + \binom{3}{2}(x^2)^1(2y)^2 + \binom{3}{3}(x^2)^0(2y)^3 \\ &= (1)(x^6)(1) + (3)(x^4)(2y) + (3)(x^2)(4y^2) + (1)(1)(8y^3) \\ &= x^6 + 6x^4y + 12x^2y^2 + 8y^3\end{aligned}$$

$\boxed{19}$
$$\begin{aligned}(x+y)^6 &= x^6 + \binom{6}{1}x^5y^1 + \binom{6}{2}x^4y^2 + \binom{6}{3}x^3y^3 + \binom{6}{4}x^2y^4 + \binom{6}{5}x^1y^5 + y^6 \\ &= x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6\end{aligned}$$

$\boxed{20}$ $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

$\boxed{21}$ $(x-y)^7 = x^7 - 7x^6y + 21x^5y^2 - 35x^4y^3 + 35x^3y^4 - 21x^2y^5 + 7xy^6 - y^7$

$\boxed{22}$ $(x-y)^5 = x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$

$\boxed{23}$
$$\begin{aligned}(3t-5s)^4 &= (3t)^4 + \binom{4}{1}(3t)^3(-5s)^1 + \binom{4}{2}(3t)^2(-5s)^2 + \binom{4}{3}(3t)^1(-5s)^3 + (-5s)^4 \\ &= 81t^4 - 540t^3s + 1350t^2s^2 - 1500ts^3 + 625s^4\end{aligned}$$

$\boxed{24}$ $(2t-s)^5$
$$\begin{aligned}&= (2t)^5 + \binom{5}{1}(2t)^4(-s)^1 + \binom{5}{2}(2t)^3(-s)^2 + \binom{5}{3}(2t)^2(-s)^3 + \binom{5}{4}(2t)^1(-s)^4 + (-s)^5 \\ &= 32t^5 - 80t^4s + 80t^3s^2 - 40t^2s^3 + 10ts^4 - s^5\end{aligned}$$

$\boxed{25}$ $(\frac{1}{3}x + y^2)^5 = \frac{1}{243}x^5 + \frac{5}{81}x^4y^2 + \frac{10}{27}x^3y^4 + \frac{10}{9}x^2y^6 + \frac{5}{3}xy^8 + y^{10}$

$\boxed{26}$ $(\frac{1}{2}x + y^3)^4 = \frac{1}{16}x^4 + \frac{1}{2}x^3y^3 + \frac{3}{2}x^2y^6 + 2xy^9 + y^{12}$

$\boxed{27}$ $\left(\dfrac{1}{x^2} + 3x\right)^6 = (x^{-2} + 3x)^6 =$

$$x^{-12} + 18x^{-9} + 135x^{-6} + 540x^{-3} + 1215 + 1458x^3 + 729x^6$$

$\boxed{28}$ $\left(\dfrac{1}{x^3} - 2x\right)^5 = (x^{-3} - 2x)^5 = x^{-15} - 10x^{-11} + 40x^{-7} - 80x^{-3} + 80x - 32x^5$

$\boxed{29}$ $\left(\sqrt{x} - \dfrac{1}{\sqrt{x}}\right)^5 = (x^{1/2} - x^{-1/2})^5 = x^{5/2} - 5x^{3/2} + 10x^{1/2} - 10x^{-1/2} + 5x^{-3/2} - x^{-5/2}$

$\boxed{30}$ $\left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)^5 = (x^{1/2} + x^{-1/2})^5 = x^{5/2} + 5x^{3/2} + 10x^{1/2} + 10x^{-1/2} + 5x^{-3/2} + x^{-5/2}$

$\boxed{31}$ $(3c^{2/5} + c^{4/5})^{25}$; first three terms $= \displaystyle\sum_{k=0}^{2} \binom{25}{k} (3c^{2/5})^{25-k} (c^{4/5})^k =$

$$3^{25}c^{10} + 25 \cdot 3^{24}c^{52/5} + 300 \cdot 3^{23}c^{54/5}$$

$\boxed{32}$ $(x^3 + 5x^{-2})^{20}$; first three terms $= \displaystyle\sum_{k=0}^{2} \binom{20}{k} (x^3)^{20-k} (5x^{-2})^k =$

$$x^{60} + 100x^{55} + 4750x^{50}$$

$\boxed{33}$ $(4z^{-1} - 3z)^{15}$; last three terms $= \displaystyle\sum_{k=13}^{15} \binom{15}{k} (4z^{-1})^{15-k} (-3z)^k =$

$$-1680 \cdot 3^{13}z^{11} + 60 \cdot 3^{14}z^{13} - 3^{15}z^{15}$$

$\boxed{34}$ $(s - 2t^3)^{12}$; last three terms $= \displaystyle\sum_{k=10}^{12} \binom{12}{k} (s)^{12-k} (-2t^3)^k =$

$$67{,}584s^2t^{30} - 24{,}576st^{33} + 4096t^{36}$$

Note: For the following exercises, the general formula for the

<u>$(k+1)$st term of the expansion of $(a+b)^n$</u> is $\boxed{\binom{n}{k}(a)^{n-k}(b)^k}$.

$\boxed{35}$ $\left(\dfrac{3}{c} + \dfrac{c^2}{4}\right)^7$; sixth term $\{k = 5\} = \binom{7}{5}\left(\dfrac{3}{c}\right)^2\left(\dfrac{c^2}{4}\right)^5 = 21\left(\dfrac{9}{c^2}\right)\left(\dfrac{c^{10}}{1024}\right) = \dfrac{189}{1024}c^8$

$\boxed{36}$ $(3x^2 - \sqrt{y})^9$; fifth term $= \binom{9}{4}(3x^2)^5(-\sqrt{y})^4 = 126(243x^{10})(y^2) = 30{,}618x^{10}y^2$

$\boxed{37}$ $(\frac{1}{3}u + 4v)^8$; seventh term $= \binom{8}{6}(\frac{1}{3}u)^2(4v)^6 = 28\left(\dfrac{u^2}{9}\right)(4096v^6) = \dfrac{114{,}688}{9}u^2v^6$

$\boxed{38}$ $(3x^2 - y^3)^{10}$; fourth term $= \binom{10}{3}(3x^2)^7(-y^3)^3 = 120(3^7x^{14})(-y^9) = -120 \cdot 3^7x^{14}y^9$

$\boxed{39}$ $(x^{1/2} + y^{1/2})^8$; middle term $\{$5th term$\} = \binom{8}{4}(x^{1/2})^4(y^{1/2})^4 = 70x^2y^2$

$\boxed{40}$ $(rs^2 + t)^7$; two middle terms $\{$4th and 5th terms$\} =$

$$\binom{7}{3}(rs^2)^4(t)^3 \text{ and } \binom{7}{4}(rs^2)^3(t)^4 = 35r^4s^8t^3 \text{ and } 35r^3s^6t^4$$

$\boxed{41}$ $(2y + x^2)^8$; term that contains x^{10} •

Consider only the variable x in the expansion: $(x^2)^k = x^{10} \Rightarrow 2k = 10 \Rightarrow k = 5$;

$$\text{6th term} = \binom{8}{5}(2y)^3(x^2)^5 = 448y^3x^{10}$$

[42] $(x^2 - 2y^3)^5$; term that contains y^6 •

Consider only the variable y in the expansion: $(y^3)^k = y^6 \Rightarrow 3k = 6 \Rightarrow k = 2$;

3rd term $= \binom{5}{2}(x^2)^3(-2y^3)^2 = 40x^6y^6$

[43] $(3y^3 - 2x^2)^4$; term that contains y^9 •

Consider only the variable y in the expansion:

$(y^3)^{4-k} = y^9 \Rightarrow 12 - 3k = 9 \Rightarrow k = 1$; 2nd term $= \binom{4}{1}(3y^3)^3(-2x^2)^1 = -216y^9x^2$

[44] $(\sqrt{c} + \sqrt{d})^8$; term that contains c^2 •

Consider only the variable c in the expansion:

$(c^{1/2})^{8-k} = c^2 \Rightarrow 4 - \frac{1}{2}k = 2 \Rightarrow k = 4$; 5th term $= \binom{8}{4}(\sqrt{c})^4(\sqrt{d})^4 = 70c^2d^2$

[45] $\left(3x - \frac{1}{4x}\right)^6$; term that does not contain x •

Consider only the variable x in the expansion:

$x^{6-k}(x^{-1})^k = x^0 \Rightarrow x^{6-2k} = x^0 \Rightarrow k = 3$; 4th term $= \binom{6}{3}(3x)^3\left(-\frac{1}{4x}\right)^3 = -\frac{135}{16}$

[46] $(xy - 3y^{-3})^8$; term that does not contain y • Consider only the variable y in the expansion: $y^{8-k}(y^{-3})^k = y^0 \Rightarrow y^{8-4k} = y^0 \Rightarrow k = 2$;

3rd term $= \binom{8}{2}(xy)^6(-3y^{-3})^2 = 252x^6$

[47] $\sum_{k=0}^{2} \binom{10}{k}(1)^{10-k}(0.2)^k = 1 + 2 + 1.8 = 4.8$; calculator result for $(1.2)^{10} \approx 6.19$

[48] $\sum_{k=0}^{2} \binom{4}{k}(1)^{4-k}(-0.1)^k = 1 - 0.4 + 0.06 = 0.66$; calculator result for $(0.9)^4 = 0.6561$

[49] $\frac{(x+h)^4 - x^4}{h} = \frac{(x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4) - x^4}{h} = \frac{h(4x^3 + 6x^2h + 4xh^2 + h^3)}{h} =$

$4x^3 + 6x^2h + 4xh^2 + h^3$

[50] $\frac{(x+h)^5 - x^5}{h} = \frac{(x^5 + 5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5) - x^5}{h} =$

$\frac{h(5x^4 + 10x^3h + 10x^2h^2 + 5xh^3 + h^4)}{h} = 5x^4 + 10x^3h + 10x^2h^2 + 5xh^3 + h^4$

[51] $\binom{n}{1} = \frac{n!}{(n-1)!\,1!} = n$ and $\binom{n}{n-1} = \frac{n!}{[n-(n-1)]!\,(n-1)!} = \frac{n!}{1!\,(n-1)!} = n$

[52] $\binom{n}{0} = \frac{n!}{(n-0)!\,0!} = \frac{n!}{n!} = 1$ and $\binom{n}{n} = \frac{n!}{(n-n)!\,n!} = \frac{n!}{0!\,n!} = 1$

10.6 Exercises

[1] $P(7, 3) = \frac{7!}{4!} = 7 \cdot 6 \cdot 5 = 210$

[2] $P(8, 5) = \frac{8!}{3!} = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 6720$

[3] $P(9, 6) = \frac{9!}{3!} = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 60{,}480$

[4] $P(5, 3) = \frac{5!}{2!} = 5 \cdot 4 \cdot 3 = 60$

[5] $P(5, 5) = \frac{5!}{0!} = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

[6] $P(4, 4) = \frac{4!}{0!} = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

$\boxed{7}$ $P(6,\ 1) = \frac{6!}{5!} = 6$

$\boxed{8}$ $P(5,\ 1) = \frac{5!}{4!} = 5$

$\boxed{9}$ $P(n,\ 0) = \frac{n!}{(n-0)!} = \frac{n!}{n!} = 1$

$\boxed{10}$ $P(n,\ 1) = \frac{n!}{(n-1)!} = \frac{n(n-1)!}{(n-1)!} = n$

$\boxed{11}$ $P(n,\ n-1) = \frac{n!}{[n-(n-1)]!} = \frac{n!}{1!} = \frac{n!}{1} = n!$

$\boxed{12}$ $P(n,\ 2) = \frac{n!}{(n-2)!} = \frac{n(n-1)(n-2)!}{(n-2)!} = n(n-1)$

$\boxed{13}$ (a) $5 \cdot 4 \cdot 3 = 60$ (b) $5 \cdot 5 \cdot 5 = 125$

$\boxed{14}$ (a) $5 \cdot 4 \cdot 3 \cdot 2 = 120$ (b) $5 \cdot 5 \cdot 5 \cdot 5 = 625$

$\boxed{15}$ There are 4 one digit numbers; $4 \cdot 3 = 12$ two digit numbers;

$4 \cdot 3 \cdot 2 = 24$ three digit numbers; $4 \cdot 3 \cdot 2 \cdot 1 = 24$ four digit numbers.

Total is $4 + 12 + 24 + 24 = 64$.

$\boxed{16}$ As in Exercise 15, 4; 4^2; 4^3; 4^4. Total is $4 + 16 + 64 + 256 = 340$.

$\boxed{17}$ $P(8,\ 3) = \frac{8!}{5!} = 8 \cdot 7 \cdot 6 = 336$

$\boxed{18}$ $P(12,\ 3) = \frac{12!}{9!} = 12 \cdot 11 \cdot 10 = 1320$

$\boxed{19}$ By the fundamental counting principle, $4 \cdot 6 = 24$.

$\boxed{20}$ By the fundamental counting principle, $4 \cdot 6 \cdot 3 = 72$.

$\boxed{21}$ (a) $26 \cdot 9 \cdot 10^4 = 2{,}340{,}000$ (b) $24 \cdot 9 \cdot 10^4 = 2{,}160{,}000$

$\boxed{22}$ $6 \cdot 6 = 36$ ways (a) 2 & 1 or 1 & 2, 2 ways to equal 3

(b) 4 & 1 twice, 3 & 2 twice, 4 ways to equal 5

(c) 6 & 1 twice, 5 & 2 twice, 4 & 3 twice, 6 ways to equal 7

(d) 5 & 4 twice, 6 & 3 twice, 4 ways to equal 9

(e) 6 & 5 twice, 2 ways to equal 11

$\boxed{23}$ (a) $P(10,\ 6) = \frac{10!}{4!} = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 151{,}200$

(b) $\boxed{\text{Boy}}$-$\underline{\text{girl}}$ seatings: $\boxed{6} \cdot \underline{\ 4\ } \cdot \boxed{5} \cdot \underline{\ 3\ } \cdot \boxed{4} \cdot \underline{\ 2\ } = 2880$.

$\underline{\text{Girl}}$-$\boxed{\text{boy}}$ seatings: $\underline{\ 4\ } \cdot \boxed{6} \cdot \underline{\ 3\ } \cdot \boxed{5} \cdot \underline{\ 2\ } \cdot \boxed{4} = 2880$. Total $= 2880 + 2880 = 5760$

$\boxed{24}$ Picking (in order) the <u>M</u>ath, <u>E</u>nglish, and <u>H</u>istory class, we obtain:

<u>M @ 8</u>, E-9, H-11, 2, 3; E-10, H-11, 2, 3; E-1, H-11, 2, 3; E-2, H-11, 3. {11 ways}

<u>M @ 10</u>, E-9, H-8, 11, 2, 3; E-1, H-8, 11, 2, 3; E-2, H-8, 11, 3. {11 ways}

<u>M @ 11</u>, E-9, H-8, 2, 3; E-10, H-8, 2, 3; E-1, H-8, 2, 3; E-2, H-8, 3. {11 ways}

<u>M @ 2</u>, E-9, H-8, 11, 3; E-10, H-8, 11, 3; E-1, H-8, 11, 3. {9 ways, 42 total}

$\boxed{25}$ 2 times itself 10 times $= 2^{10} = 1024$

$\boxed{26}$ 5 times itself 6 times $= 5^6 = 15{,}625$

$\boxed{27}$ $P(8,\ 8) = \frac{8!}{0!} = 8! = 40{,}320$

$\boxed{28}$ $P(10,\ 10) = \frac{10!}{0!} = 10! = 3{,}628{,}800$

$\boxed{29}$ $P(6,\ 3) = \frac{6!}{3!} = 6 \cdot 5 \cdot 4 = 120$

[30] $P(12, 5) = \frac{12!}{7!} = 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 = 95{,}040$

[31] (a) The number of choices for each letter are: $\underline{2} \cdot \underline{25} \cdot \underline{24} \cdot \underline{23} = 27{,}600$

(b) The number of choices for each letter are: $\underline{2} \cdot \underline{26} \cdot \underline{26} \cdot \underline{26} = 35{,}152$

[32] (a) $P(24, 3) = \frac{24!}{21!} = 24 \cdot 23 \cdot 22 = 12{,}144$ (b) $24 \cdot 24 \cdot 24 = 13{,}824$

[33] $9 \cdot 10^6 = 9{,}000{,}000$

[34] There are 7 spots to fill. $P(7, 7) = \frac{7!}{0!} = 7! = 5040$

[35] $P(4, 4) = \frac{4!}{0!} = 4! = 24$

[36] Suppose the "2" is repeated. There are 4! ways to arrange 2_a, 2_b, 7, and 9. Since 2_a, 2_b, 7, 9 and 2_b, 2_a, 7, 9 are not distinguishable permutations, we divide 4! by 2. Since there are 3 numbers to repeat, the total number of trials is $3 \cdot \frac{4!}{2} = 36$.

[37] There are 3! ways to choose the couples and 2 ways for each couple to sit. $3! \cdot 2^3 = 48$

[38] $P(10, 3) = \frac{10!}{7!} = 10 \cdot 9 \cdot 8 = 720$

[39] There are 16 "fixins" and each one can be "on" or "off" your sandwich (or hot dog or salad). With just the "fixins," we have $\underbrace{2 \cdot 2 \cdot 2 \cdot \cdots \cdot 2}_{16 \text{ 2's}} = 2^{16}$, but we subtract 1 because that would be the one choice that would have no "fixins." Now we have 17 sandwiches, hot dogs, or salads that can be combined with any of the choices of "fixins," so we use the fundamental counting principle to obtain $(2^{16} - 1) \cdot 17 = 1{,}114{,}095$ different lunches.

[40] (a) $52! \approx 8.07 \times 10^{67}$

(b) The aces can be arranged in 4! ways and the other 48 cards can be arranged in 48! ways. Total number of arrangements $= 4! \cdot 48! \approx 2.98 \times 10^{62}$

[41] (a) There are 9 choices for the first digit, 10 for the second, 10 for the third, and 1 for the fourth and fifth. $9 \cdot 10 \cdot 10 \cdot 1 \cdot 1 = 900$

(b) If n is even, we need to select the first $\frac{n}{2}$ digits. $9 \cdot 10^{(n/2) - 1}$

If n is odd, we need to select the first $\frac{n+1}{2}$ digits. $9 \cdot 10^{(n-1)/2}$

[42] There are 10 choices for the first square and 9 choices for each successive square. The number of ways of coloring the strip is $10 \cdot 9^5 = 590{,}490$.

[43] (a) There is a horizontal asymptote of $y = 1$.

(b) $\dfrac{n!\,e^n}{n^n\sqrt{2\pi n}} \approx 1 \;\Rightarrow\; n! \approx \dfrac{n^n\sqrt{2\pi n}}{e^n}$.

Example: $50! \approx \dfrac{50^{50}\sqrt{2\pi(50)}}{e^{50}} \approx 3.0363 \times 10^{64}$.

The actual value is closer to 3.0414×10^{64}.

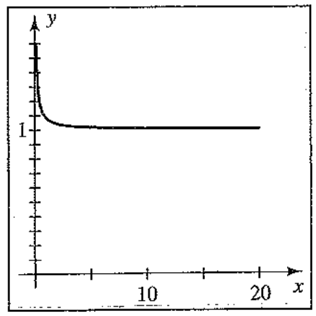


Figure 43

[44] (a) Calculators such as the TI-83 Plus give an error message since they can only deal with numbers the size of 10^n, where n is a 2-digit number. Looking ahead to part (b), we see that there is a 3-digit number in the exponent. Calculators such as the TI-86 fail to calculate $P(800,\,400)$, which is about 1.2×10^{1108}.

(b) $P(150,\,50) = 10^r \;\Rightarrow\; \dfrac{150!}{(150-50)!} = 10^r \;\Rightarrow\; r = \log\dfrac{150!}{100!} = \log 150! - \log 100! \approx$

$$\frac{150\ln 150 - 150}{\ln 10} - \frac{100\ln 100 - 100}{\ln 10} \approx 104.7. \text{ Thus, } P(150,\,50) \approx 10^{104.7}.$$

10.7 Exercises

[1] $C(7,\,3) = \dfrac{7!}{4!\,3!} = 35$

[2] $C(8,\,4) = \dfrac{8!}{4!\,4!} = 70$

[3] $C(9,\,8) = \dfrac{9!}{1!\,8!} = 9$

[4] $C(6,\,2) = \dfrac{6!}{4!\,2!} = 15$

[5] $C(n,\,n-1) = \dfrac{n!}{[n-(n-1)]!\,(n-1)!} = \dfrac{n!}{1!\,(n-1)!} = n$

[6] $C(n,\,1) = \dfrac{n!}{(n-1)!\,1!} = n$

[7] $C(7,\,0) = \dfrac{7!}{7!\,0!} = 1$

[8] $C(5,\,5) = \dfrac{5!}{5!\,0!} = 1$

[9] $\dfrac{(5+3+2+2)!}{5!\,3!\,2!\,2!} = \dfrac{12!}{5!\,3!\,2!\,2!} = 166{,}320$

[10] $\dfrac{(3+3+3+3)!}{3!\,3!\,3!\,3!} = \dfrac{12!}{(3!)^4} = 369{,}600$

[11] There are 3 e's, 2 o's, and 2 k's. $\dfrac{10!}{3!\,2!\,2!\,1!\,1!\,1!} = 151{,}200$

[12] $\dfrac{4!}{2!\,1!\,1!} = 12;$

moon, mono, mnoo, nmoo, nomo, noom, oomn, oonm, omon, omno, onom, onmo

[13] There are $C(10,\,5)$ ways to pick the first team.

The second team is determined once the first team is selected. $C(10,\,5) = 252$

[14] (a) $C(10,\,6) = 210$ (b) The student needs to answer 4 of the last 8. $C(8,\,4) = 70$

[15] Two points determine a unique line. $C(8,\,2) = 28$

[16] Three points determine a unique triangle. $C(8,\,3) = 56$

[17] There are 3! ways to order the categories. $(5! \cdot 4! \cdot 8!) \cdot 3! = 696{,}729{,}600$

$\boxed{18}$ (a) $C(12,\ 5) = 792$ (b) $C(2,\ 1) \cdot C(10,\ 4) = 420$

$\boxed{19}$ Pick the center, $C(3,\ 1)$; two guards, $C(10,\ 2)$;

two tackles from the 8 remaining linemen, $C(8,\ 2)$; two ends, $C(4,\ 2)$;

two halfbacks, $C(6,\ 2)$; the quarterback, $C(3,\ 1)$; and the fullback, $C(4,\ 1)$.

$3 \cdot C(10,\ 2) \cdot C(8,\ 2) \cdot C(4,\ 2) \cdot C(6,\ 2) \cdot 3 \cdot 4 = 4{,}082{,}400$

$\boxed{20}$ There would be 7! orderings if the keys were in a row.

Since the keys are on a ring, any unique ordering can be shifted to 7 different positions and would be counted as only 1 ordering. $\frac{7!}{7} = 6! = 720$

$\boxed{21}$ There are $C(12,\ 3)$ ways to pick the men and $C(8,\ 2)$ ways to pick the women.

$C(12,\ 3) \cdot C(8,\ 2) = 6160$

$\boxed{22}$ If we thought of 6 positions for birth order, the girls could be selected $C(6,\ 3)$ ways to be put in those positions and the boys would fill the remaining positions.

$C(6,\ 3) = 20$

$\boxed{23}$ We need 3 U's out of 8 moves. $C(8,\ 3) = 56$

$\boxed{24}$ We need 6 U's out of 15 moves. $C(15,\ 6) = 5005$

$\boxed{25}$ (a) $C(49,\ 6) = 13{,}983{,}816$ (b) $C(24,\ 6) = 134{,}596$

$\boxed{26}$ There are $C(10,\ 2) = 45$ ways to pick the two faculty members to share an office.

There are 9! ways to pick the offices. $C(10,\ 2) \cdot 9! = 16{,}329{,}600$

$\boxed{27}$ Let n denote the number of players. $C(n,\ 2) = 45 \Rightarrow \frac{n!}{(n-2)!\,2!} = 45 \Rightarrow$

$n(n-1) = 90 \Rightarrow n^2 - n - 90 = 0 \Rightarrow (n-10)(n+9) = 0 \Rightarrow \{n > 0\}\ n = 10.$

$\boxed{28}$ (a) There are 2 answers for each question. $2^{20} = 1{,}048{,}576$

(b) Select the 10 questions to be answered correctly. $C(20,\ 10) = 184{,}756$

$\boxed{29}$ Each team must win 3 of the first 6 games for the series to be extended to a

7th game. $C(6,\ 3) = 20$

$\boxed{30}$ (a) Select 3 of the vertices to form a triangle. $C(8,\ 3) = 56$

(b) Select 4 of the vertices to form a quadrilateral. $C(8,\ 4) = 70$

$\boxed{31}$ They may have computed $C(31,\ 3)$, which is 4495.

$\boxed{32}$ Consider each condiment as either being *on* or *off*. Hence, there are 2 choices for each condiment and $2^8 = 256$ possible combinations. Alternatively, we could calculate $\sum_{k=0}^{8} \binom{8}{k}$, which is also 256.

$\boxed{33}$ (a) The amounts received are the same, so the order of selection *is not* important,

and we use a combination. $C(1000,\ 30) = \frac{1000!}{970!\,30!} \approx 2.43 \times 10^{57}$

(b) The amounts received are different, so the order of selection *is* important,

and we use a permutation. $P(1000,\ 30) = \frac{1000!}{970!} \approx 6.44 \times 10^{89}$

[34] (a) The order of selection *is not* important, so we have $C(12, 4) = 495$.

(b) The order of selection *is* important, so we have $P(12, 4) = 11{,}880$.

[35] We want to select 3 of the 4 kings and 2 of the remaining 48 cards. The order of selection is not important, so we use combinations. Any of the groups of 3 kings can be selected with any pair of the remaining 48 cards, so by the fundamental counting principle the total number of hands is $C(4, 3) \cdot C(48, 2) = 4 \cdot 1128 = 4512$.

[36] We want 7 of the 13 spades and 6 of the remaining 39 cards.

$$C(13, 7) \cdot C(39, 6) = 1716 \cdot 3{,}262{,}623 = 5{,}598{,}661{,}068$$

[37] (a) $S_1 = \binom{1}{1} + \binom{1}{3} + \binom{1}{5} + \cdots = 1 + 0 + 0 + \cdots = 1$.

$S_2 = \binom{2}{1} + \binom{2}{3} + \binom{2}{5} + \cdots = 2 + 0 + 0 + \cdots = 2$.

$S_3 = 3 + 1 + 0 + \cdots = 4$. $S_4 = 4 + 4 + 0 + \cdots = 8$. $S_5 = 16$, $S_6 = 32$,

$S_7 = 64$, $S_8 = 128$, $S_9 = 256$, $S_{10} = 512$.

(b) It appears that $S_n = 2^{n-1}$.

[38] (a) $S_1 = 1$ and $S_n = 0$ for $n = 2, 3, 4, \ldots, 10$.

(b) $S_1 = 1$ and $S_n = 0$ for $n > 1$.

[39] (a) Graph the values of $C(10, 1)$, $C(10, 2)$, $C(10, 3)$, $\ldots$, $C(10, 10)$.

(b) The maximum value of $C(10, r)$ is 252 and occurs at $r = 5$.

[0, 10] by [0, 300, 50]

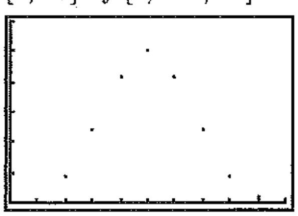


Figure 39

[0, 13] by [0, 2000, 500]

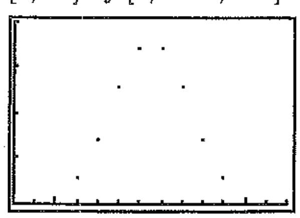


Figure 40

[40] (a) Graph the values of $C(13, 1)$, $C(13, 2)$, $C(13, 3)$, $\ldots$, $C(13, 13)$.

(b) The maximum value of $C(13, r)$ is 1716 and occurs at $r = 6, 7$.

[41] (a) Graph the values of $C(19, 1)$, $C(19, 2)$, $C(19, 3)$, $\ldots$, $C(19, 19)$.

(b) The maximum value of $C(19, r)$ is 92,378 and occurs at $r = 9, 10$.

[0, 19] by [0, 1E5, 1E4]

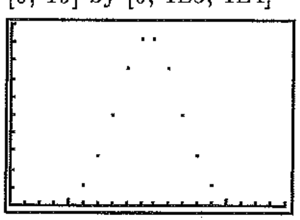


Figure 41

[0, 20] by [0, 2E5, 5E4]

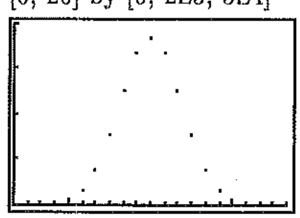


Figure 42

[42] (a) Graph the values of $C(20, 1)$, $C(20, 2)$, $C(20, 3)$, $\ldots$, $C(20, 20)$.

(b) The maximum value of $C(20, r)$ is 184,756 and occurs at $r = 10$.

10.8 Exercises

[1] (a) The probability is $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{4}{52} = \dfrac{1}{13}$.

The odds are $n(E)$ to $n(E')$, which are 4 to 48, or 1 to 12.

(b) $\frac{4}{52} + \frac{4}{52} = \frac{8}{52} = \frac{2}{13}$; $O(E)$ are 8 to 44, or 2 to 11

(c) $\frac{4}{52} + \frac{4}{52} + \frac{4}{52} = \frac{12}{52} = \frac{3}{13}$; $O(E)$ are 12 to 40, or 3 to 10

[2] (a) $\frac{13}{52} = \frac{1}{4}$; $O(E)$ are 13 to 39, or 1 to 3

(b) $\frac{13}{52} + \frac{13}{52} = \frac{26}{52} = \frac{1}{2}$; $O(E)$ are 26 to 26, or 1 to 1

(c) $\frac{13}{52} + \frac{13}{52} + \frac{13}{52} = \frac{39}{52} = \frac{3}{4}$; $O(E)$ are 39 to 13, or 3 to 1

[3] (a) $\frac{1}{6}$; $O(E)$ are 1 to 5 (b) $\frac{1}{6}$; $O(E)$ are 1 to 5

(c) $\frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$; $O(E)$ are 2 to 4, or 1 to 2

[4] (a) $\frac{3}{6}$; $O(E)$ are 3 to 3, or 1 to 1

(b) $\frac{1}{6}$; $O(E)$ are 1 to 5

(c) $\frac{3}{6} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$; $O(E)$ are 4 to 2, or 2 to 1

[5] $n(S) = 5 + 6 + 4 = 15$ (a) $\frac{5}{15} = \frac{1}{3}$; $O(E)$ are 5 to 10, or 1 to 2

(b) $\frac{6}{15} = \frac{2}{5}$; $O(E)$ are 6 to 9, or 2 to 3

(c) $\frac{5}{15} + \frac{4}{15} = \frac{9}{15} = \frac{3}{5}$; $O(E)$ are 9 to 6, or 3 to 2

[6] $n(S) = 5 + 6 + 4 = 15$ (a) $\frac{4}{15}$; $O(E)$ are 4 to 11

(b) $\frac{6}{15} + \frac{4}{15} = \frac{10}{15} = \frac{2}{3}$; $O(E)$ are 10 to 5, or 2 to 1

(c) $1 - \frac{6}{15} = \frac{9}{15} = \frac{3}{5}$; $O(E)$ are 9 to 6, or 3 to 2

[7] *Note:* The following table summarizes the results for the sum of two dice being tossed. Notice the symmetry about the sum of 7 in the # of ways to obtain.

Sum of two dice	2	3	4	5	6	7	8	9	10	11	12
# of ways to obtain	1	2	3	4	5	6	5	4	3	2	1

(a) $\frac{2}{36} = \frac{1}{18}$; $O(E)$ are 2 to 34, or 1 to 17

(b) $\frac{5}{36}$; $O(E)$ are 5 to 31

(c) $\frac{2}{36} + \frac{5}{36} = \frac{7}{36}$; $O(E)$ are 7 to 29

[8] (a) $P(10) + P(11) + P(12) = \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{6}{36} = \frac{1}{6}$; $O(E)$ are 6 to 30, or 1 to 5

(b) $P(3) + P(5) + P(7) + P(9) + P(11) = \frac{2}{36} + \frac{4}{36} + \frac{6}{36} + \frac{4}{36} + \frac{2}{36} = \frac{18}{36} = \frac{1}{2}$;

$O(E)$ are 18 to 18, or 1 to 1

[9] There are 6 ways to make a sum of 5 (3 with 1, 1, 3 and 3 with 1, 2, 2). $\frac{6}{216} = \frac{1}{36}$

[10] We must have a 6, not a 6, and not a 6.

There are $3(1 \cdot 5 \cdot 5) = 75$ ways this can occur. $\frac{75}{216} = \frac{25}{72}$

$\boxed{11}$ There are 3 ways to pick one tail in 3 coins. $\frac{3}{2^3} = \frac{3}{8}$

$\boxed{12}$ There are $C(4,\ 2)$ ways to obtain two heads from 4 coins. $\frac{C(4,\ 2)}{2^4} = \frac{6}{16} = \frac{3}{8}$

$\boxed{13}$ If $P(E) = \frac{5}{7}$, then $O(E)$ are 5 to 2 and $O(E')$ are 2 to 5.

$\boxed{14}$ If $P(E) = 0.4 = \frac{4}{10} = \frac{2}{5}$, then $O(E)$ are 2 to 3 and $O(E')$ are 3 to 2.

$\boxed{15}$ If $O(E)$ are 9 to 5, then $O(E')$ are 5 to 9 and $P(E) = \frac{9}{5+9} = \frac{9}{14}$.

$\boxed{16}$ If $O(E')$ are 7 to 3, then $O(E)$ are 3 to 7 and $P(E) = \frac{3}{3+7} = \frac{3}{10}$.

$\boxed{17}$ $P(E) \approx 0.659 = \frac{659}{1000}$, so $n(E) = 659$ and $n(E') = 1000 - 659 = 341$ and

$O(E)$ are 659 to 341. Divide both numbers by 341 to get odds of 1.93 to 1.

$\boxed{18}$ $P(E) \approx 0.822 = \frac{822}{1000}$, so $n(E) = 822$ and $n(E') = 1000 - 822 = 178$ and

$O(E)$ are 822 to 178. Divide both numbers by 178 to get odds of 4.62 to 1.

Note: For Exercises 19–24, there are $C(52,\ 5) = 2{,}598{,}960$ ways to draw 5 cards.

$\boxed{19}$ There are 13 denominations to pick from and any one of them could be combined

with any one of the remaining 48 cards. $\frac{48 \cdot 13}{C(52,\ 5)} = \frac{1}{4165} \approx 0.00024$

$\boxed{20}$ Pick the 3 aces from 4, $C(4,\ 3)$, and the 2 kings from 4, $C(4,\ 2)$.

$$\frac{C(4,\ 3) \cdot C(4,\ 2)}{C(52,\ 5)} = \frac{1}{108{,}290} \approx 0.00000923$$

$\boxed{21}$ Pick 4 of the 13 diamonds and 1 of the 13 spades.

$$\frac{C(13,\ 4) \cdot C(13,\ 1)}{C(52,\ 5)} = \frac{143}{39{,}984} \approx 0.00358$$

$\boxed{22}$ Pick 5 of the 12 face cards. $\frac{C(12,\ 5)}{C(52,\ 5)} = \frac{33}{108{,}290} \approx 0.000305$

$\boxed{23}$ Pick 5 of the 13 cards in one suit. There are 4 suits. $\frac{C(13,\ 5) \cdot 4}{C(52,\ 5)} = \frac{33}{16{,}660} \approx 0.00198$

$\boxed{24}$ There are 4 of these hands, one in each suit. $\frac{4}{C(52,\ 5)} = \frac{1}{649{,}740} \approx 0.00000154$

$\boxed{25}$ Let E_1 be the event that the number is odd, E_2 that the number is prime.

$E_1 = \{1, 3, 5\}$ and $E_2 = \{2, 3, 5\}$.

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) = \tfrac{3}{6} + \tfrac{3}{6} - \tfrac{2}{6} = \tfrac{4}{6} = \tfrac{2}{3}.$$

$\boxed{26}$ Let E_1 be the event that the card is red, E_2 that the card is a face card.

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) = \tfrac{26}{52} + \tfrac{12}{52} - \tfrac{6}{52} = \tfrac{32}{52} = \tfrac{8}{13}.$$

$\boxed{27}$ $1 - 0.326 = 0.674$. $(0.674)^4 \approx 0.2064$

$\boxed{28}$ $P(\text{at least } 1) = 1 - P(\text{none}) = 1 - (0.1)^2 = 0.99$

$\boxed{29}$ (a) $P(E_2) = P(2) + P(3) + P(4) = 0.10 + 0.15 + 0.20 = 0.45$

(b) $P(E_1 \cap E_2) = P(2) = 0.10$

(c) $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) = 0.35 + 0.45 - 0.10 = 0.70$

(d) $P(E_2 \cup E_3') = P(E_2) + P(E_3') - P(E_2 \cap E_3')$. $E_3' = \{1, 2, 3, 5\}$ and

$$E_2 \cap E_3' = \{2, 3\} \Rightarrow P(E_2 \cup E_3') = 0.45 + 0.75 - 0.25 = 0.95.$$

30 (a) $P(E_2) = P(3) + P(4) = 0.15 + 0.20 = 0.35$

(b) $P(E_1 \cap E_2) = P(3) = 0.15$

(c) $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) = 0.55 + 0.35 - 0.15 = 0.75$

(d) $P(E_2 \cup E_3') = P(E_2) + P(E_3') - P(E_2 \cap E_3')$. $E_3' = \{1, 2, 3\}$ and

$$E_2 \cap E_3' = \{3\} \Rightarrow P(E_2 \cup E_3') = 0.35 + 0.50 - 0.15 = 0.70.$$

Note: For Exercises 31–32, there are $C(60, 5) = 5{,}461{,}512$ ways to draw 5 chips.

31 (a) We want 5 blue and 0 non-blue. $\dfrac{C(20, 5) \cdot C(40, 0)}{C(60, 5)} = \dfrac{34}{11{,}977} \approx 0.0028.$

(b) $P(\text{at least 1 green}) = 1 - P(\text{no green}) =$

$$1 - \frac{C(30, 0) \cdot C(30, 5)}{C(60, 5)} = 1 - \frac{117}{4484} = \frac{4367}{4484} \approx 0.9739.$$

(c) $P(\text{at most 1 red}) = P(0 \text{ red}) + P(1 \text{ red}) =$

$$\frac{C(10, 0) \cdot C(50, 5)}{C(60, 5)} + \frac{C(10, 1) \cdot C(50, 4)}{C(60, 5)} = \frac{26{,}320}{32{,}509} \approx 0.8096.$$

32 (a) We want 4 green and 1 non-green. $\dfrac{C(30, 4) \cdot C(30, 1)}{C(60, 5)} = \dfrac{675}{4484} \approx 0.1505.$

(b) This event is the complement of part (c) in the previous exercise.

$$P(\text{at least 2 red}) = 1 - P(\text{at most 1 red}) \approx 1 - 0.8096 = 0.1904.$$

(c) $P(\text{at most 2 blue}) = P(0 \text{ blue}) + P(1 \text{ blue}) + P(2 \text{ blue}) =$

$$\frac{C(20, 0) \cdot C(40, 5)}{C(60, 5)} + \frac{C(20, 1) \cdot C(40, 4)}{C(60, 5)} + \frac{C(20, 2) \cdot C(40, 3)}{C(60, 5)} = \frac{181{,}792}{227{,}563} \approx 0.7989.$$

33 (a) $\dfrac{C(8, 8)}{2^8} = \dfrac{1}{256} \approx 0.00391$ (b) $\dfrac{C(8, 7)}{2^8} = \dfrac{1}{32} = 0.03125$

(c) $\dfrac{C(8, 6)}{2^8} = \dfrac{7}{64} = 0.109375$ (d) $\dfrac{C(8, 6) + C(8, 7) + C(8, 8)}{2^8} = \dfrac{37}{256} \approx 0.14453$

34 $\dfrac{C(8, 3) \cdot C(14, 3)}{C(22, 6)} = \dfrac{2912}{10{,}659} \approx 0.2732$

35 $1 - P(\text{no aces}) = 1 - \dfrac{C(48, 5)}{C(52, 5)} = 1 - \dfrac{35{,}673}{54{,}145} = \dfrac{18{,}472}{54{,}145} \approx 0.34116$

36 $P(\text{at least 1 heart}) = 1 - P(\text{no hearts}) =$

$$1 - \frac{C(13, 0) \cdot C(39, 5)}{C(52, 5)} = 1 - \frac{27{,}417}{123{,}760} = 1 - \frac{2109}{9520} = \frac{7411}{9520} \approx 0.7785$$

37 (a) We may use ordered pairs to represent the outcomes of the sample space S of the experiment. A representative outcome is (nine of clubs, 3). The number of outcomes in the sample space S is $n(S) = 52 \cdot 6 = 312$.

(b) For each integer k, where $2 \leq k \leq 6$, there are 4 ways to obtain an outcome of the form $(k,\ k)$ since there are 4 suits. Because there are 5 values of k, $n(E_1) = 5 \cdot 4 = 20$. $n(E_1') = n(S) - n(E_1) = 312 - 20 = 292$.

$$P(E_1) = \frac{n(E_1)}{n(S)} = \frac{20}{312} = \frac{5}{78}.$$

(c) No, if E_2 or E_3 occurs, then the other event may occur.

Yes, the occurrence of either E_2 or E_3 has no effect on the other event.

$$P(E_2) = \frac{n(E_2)}{n(S)} = \frac{12 \cdot 6}{312} = \frac{72}{312} = \frac{3}{13}. \qquad P(E_3) = \frac{n(E_3)}{n(S)} = \frac{52 \cdot 3}{312} = \frac{156}{312} = \frac{1}{2}.$$

Since E_2 and E_3 are indep., $P(E_2 \cap E_3) = P(E_2) \cdot P(E_3) = \frac{3}{13} \cdot \frac{1}{2} = \frac{3}{26} = \frac{36}{312}$.

$P(E_2 \cup E_3) = P(E_2) + P(E_3) - P(E_2 \cap E_3) = \frac{72}{312} + \frac{156}{312} - \frac{36}{312} = \frac{192}{312} = \frac{8}{13}$.

(d) Yes, if E_1 or E_2 occurs, then the other event cannot occur. No, the occurrence of either E_1 or E_2 influences the occurrence of the other event.

Remember, (non-empty) *mutually exclusive events* ***cannot*** *be independent events.*

Since E_1 and E_2 are mutually exclusive, $P(E_1 \cap E_2) = 0$ and

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) = \frac{20}{312} + \frac{72}{312} = \frac{92}{312} = \frac{23}{78}.$$

38 (a) A representative outcome of the sample space S of the experiment is C7.

The number of outcomes in the sample space S is $n(S) = 26 \cdot 10 = 260$.

(b) The digits 1, 2, 3, 4, 5, and 6 can be matched with 3 letters each. For example, A1, K1, and U1 are the outcomes with 1 as the selected digit. The digits 7, 8, 9, and 0 can be matched with 2 letters each. Thus, $n(E_1) = 6 \cdot 3 + 4 \cdot 2 = 26$. This answer makes sense because each letter can be paired with exactly one digit.

$$n(E_1') = n(S) - n(E_1) = 260 - 26 = 234. \qquad P(E_1) = \frac{n(E_1)}{n(S)} = \frac{26}{260} = \frac{1}{10}.$$

(c) No, if E_2 or E_3 occurs, then the other event may occur.

Yes, the occurrence of either E_2 or E_3 has no effect on the other event.

$$P(E_2) = \frac{n(E_2)}{n(S)} = \frac{5 \cdot 10}{260} = \frac{50}{260} = \frac{5}{26}. \qquad P(E_3) = \frac{n(E_3)}{n(S)} = \frac{26 \cdot 4}{260} = \frac{104}{260} = \frac{2}{5}.$$

Since E_2 and E_3 are indep., $P(E_2 \cap E_3) = P(E_2) \cdot P(E_3) = \frac{5}{26} \cdot \frac{2}{5} = \frac{1}{13} = \frac{20}{260}$.

$P(E_2 \cup E_3) = P(E_2) + P(E_3) - P(E_2 \cap E_3) = \frac{50}{260} + \frac{104}{260} - \frac{20}{260} = \frac{134}{260} = \frac{67}{130}$.

(d) The numerical values of the five vowels are: A-1, E-5, I-9, O-15, U-21. Since if E_2 or E_4 occurs, the other event cannot occur, the events are mutually exclusive. No, the occurrence of either E_2 or E_4 influences the occurrence of the other event.

Remember, (non-empty) *mutually exclusive events* ***cannot*** *be independent events.*

Since E_2 and E_4 are mutually exclusive, $P(E_2 \cap E_4) = 0$. (continued)

$P(E_4) = \dfrac{n(E_4)}{n(S)} = \dfrac{13 \cdot 10}{260} = \dfrac{130}{260} = \frac{1}{2}$. Since E_2 and E_4 are mutually exclusive,

$$P(E_2 \cup E_4) = P(E_2) + P(E_4) = \tfrac{50}{260} + \tfrac{130}{260} = \tfrac{180}{260} = \tfrac{9}{13}.$$

[39] Let k denote the sum.

$$P(k > 5) = 1 - P(k \leq 5) = 1 - \left(\tfrac{1}{36} + \tfrac{2}{36} + \tfrac{3}{36} + \tfrac{4}{36}\right) = 1 - \tfrac{10}{36} = \tfrac{26}{36} = \tfrac{13}{18}.$$

[40] There is 1 way to obtain a sum of 18 (6, 6, 6), 3 ways for a sum of 17 (6, 6, 5; 6, 5, 6; 5, 6, 6) , and 6 ways for a sum of 16 (3 with 6, 5, 5 and 3 with 6, 6, 4). Let k denote the sum.

$$P(k < 16) = 1 - P(k \geq 16) = 1 - \left(\tfrac{6}{216} + \tfrac{3}{216} + \tfrac{1}{216}\right) = 1 - \tfrac{10}{216} = \tfrac{206}{216} = \tfrac{103}{108} \approx 0.9537.$$

[41] (a) $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{32} = 0.03125$ (b) $1 - \frac{1}{32} = \frac{31}{32} = 0.96875$

[42] $\frac{5}{20} \cdot \frac{4}{20} \cdot \frac{2}{20} = \frac{40}{8000} = \frac{1}{200} = 0.005$

[43] (a) $\dfrac{C(4,\ 4)}{4!} = \dfrac{1}{24} \approx 0.04167$ (b) $\dfrac{C(4,\ 2)}{4!} = \dfrac{1}{4} = 0.25$

[44] (a) There is 1 chance in 216 that all three dice show the same value and there are six different values. $6 \cdot \frac{1}{216} = \frac{6}{216} = \frac{1}{36} = 0.02\overline{7}$

(b) After the first value is chosen, the second value could be one of 5 others, and third could be one of 4 others. $\dfrac{P(6,\ 3)}{6^3} = \dfrac{120}{216} = \dfrac{5}{9} = 0.\overline{5}$

(c) Same number of dots: $6\left(\dfrac{1}{6^n}\right) = \dfrac{1}{6^{n-1}}$ for $n \geq 1$

Dots all different: $\dfrac{P(6,\ n)}{6^n}$ for $n \leq 6$ and 0 for $n > 6$

[45] (a) The 3, 4, or 5 on the left die would need to combine with a 4, 3, or 2 on the right die to sum to 7, but the right die only has 1, 5, or 6. The probability is 0.

(b) To obtain 8, we would need a 3 on the left die and a 5 on the right.

$$\tfrac{1}{3} \cdot \tfrac{1}{3} = \tfrac{1}{9} = 0.\overline{1}$$

[46] There is 1 way to obtain a sum of 3, 3 ways for a sum of 4 (1, 1, 2; 1, 2, 1; 2, 1, 1), 6 ways for a sum of 5 (3 with 1, 1, 3 and 3 with 1, 2, 2), and 10 ways for a sum of 6 (3 with 1, 1, 4; 6 with 1, 2, 3; 1 with 2, 2, 2). $\frac{1}{729} + \frac{3}{729} + \frac{6}{729} + \frac{10}{729} = \frac{20}{729} \approx 0.0274$

[47] (a) $P = \dfrac{179{,}820 + 151{,}322}{418{,}890} = \dfrac{331{,}142}{418{,}890} \approx 0.791$

(b) $P = \dfrac{418{,}890 - 84{,}475}{418{,}890} = \dfrac{334{,}415}{418{,}890} \approx 0.798$

[48] (a) $P = \dfrac{60.4 + 18.3}{8.2 + 60.4 + 18.3} = \dfrac{78.7}{86.9} \approx 0.906$

(b) $P = \dfrac{8.2}{86.9} \approx 0.094$, or by using the complement of part (a), $1 - 0.906 = 0.094$.

[49] Start the tree diagram by making branches for those with cancer (C, 2%) and those without cancer (C', 98%). On the C branch, make a branch for those exposed to high levels of arsenic (E, 70%) and a branch for those not exposed (E', 30%). On the C' branch, make a branch for those exposed to high levels of arsenic (E, 10%) and a branch for those not exposed (E', 90%). Find the products: $0.02 \cdot 0.70 = \mathbf{0.014}$, $0.02 \cdot 0.30 = \mathbf{0.006}$, $0.98 \cdot 0.10 = \mathbf{0.098}$, and $0.98 \cdot 0.90 = \mathbf{0.882}$. Note that the sum of the boldfaced numbers equals 1. The fraction of people that have been exposed to high levels of arsenic and have cancer is

$$\frac{\text{exposed and have cancer}}{\text{all those exposed}} = \frac{C \text{ and } E}{C \text{ and } E + C' \text{ and } E} = \frac{0.014}{0.014 + 0.098} = \frac{0.014}{0.112} = 0.125, \text{ so}$$

the percentage is 12.5%.

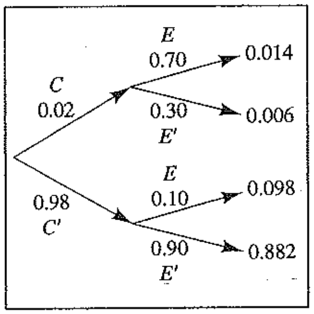


Figure 49

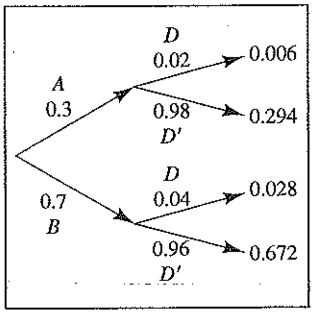


Figure 50

[50] From the figure, the fraction of defective chips that are from supplier B is

$$\frac{\text{from } B \text{ and defective}}{\text{all the defective chips}} = \frac{B \text{ and } D}{A \text{ and } D + B \text{ and } D} = \frac{0.028}{0.006 + 0.028} = \frac{0.028}{0.034} \approx 0.824, \text{ so the}$$

percentage is about 82.4%.

[51] (a) The ball must take 4 "lefts." $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16} = 0.0625$

(b) We need two "lefts." $\dfrac{C(4,\ 2)}{2^4} = \dfrac{6}{16} = \dfrac{3}{8} = 0.375$

[52] (a) There are 18 black slots. $\frac{18}{38} = \frac{9}{19} \approx 0.4737$ (b) $\left(\frac{18}{38}\right)^2 = \frac{81}{361} \approx 0.2244$

[53] For one ticket, $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{C(6,\ 6)}{C(54,\ 6)} = \dfrac{1}{25{,}827{,}165}$.

For two tickets, $P(E) = \dfrac{2 \times 1}{25{,}827{,}165}$, or about 1 chance in 13 million.

[54] $P(\text{match } 5) = \dfrac{C(6,\ 5) \times C(48,\ 1)}{C(54,\ 6)} = \dfrac{6 \times 48}{25{,}827{,}165} = \dfrac{288}{25{,}827{,}165}$.

$P(\text{match } 4) = \dfrac{C(6,\ 4) \times C(48,\ 2)}{C(54,\ 6)} = \dfrac{15 \times 1128}{25{,}827{,}165} = \dfrac{16{,}920}{25{,}827{,}165}$.

$P(\text{win}) = P(\text{match 4, 5, or 6}) = \dfrac{288}{25{,}827{,}165} + \dfrac{16{,}920}{25{,}827{,}165} + \dfrac{1}{25{,}827{,}165} = \dfrac{17{,}209}{25{,}827{,}165} \approx$

0.000666 (about 1 chance in 1500).

55 The probability that the first bulb is not defective is $\frac{195}{200}$ since 195 of the 200 bulbs are not defective. If the first bulb is not replaced and not defective, then there are 199 bulbs left, and 194 of them are not defective. The probability that the second bulb is not defective is then $\frac{194}{199}$. Thus, the probability that both bulbs are not defective is $\frac{195}{200} \times \frac{194}{199} = \frac{37{,}830}{39{,}800} \approx 0.9505$. The event that either light bulb is defective is the complement of the event that neither bulb is defective. The probability that the sample will be rejected is $1 - \frac{37{,}830}{39{,}800} = \frac{1970}{39{,}800} \approx 0.0495$.

56 (a) Let E_1 denote the event that the man is alive 10 years from now and E_2 that the woman is alive in 10 years. Since their life expectancies are unrelated, they are independent events. Thus, $P(E_1 \cap E_2) = P(E_1) \cdot P(E_2) = 0.74 \times 0.94 = 0.6956$.

(b) The probability that the man will be dead in 10 years is $1 - 0.74 = 0.26$ and the probability for the woman is $1 - 0.94 = 0.06$. Since their life expectancies are independent events, the probability that both of them will be dead in 10 years is $0.26 \times 0.06 = 0.0156$.

(c) Using the complement of part (b), we have $1 - 0.0156 = 0.9844$.

57 (a) $P(7 \text{ or } 11) = P(7) + P(11) = \frac{6}{36} + \frac{2}{36} = \frac{8}{36}$

(b) To win with a 4 on the first roll, we must first get a 4, and then get another 4 before a 7. The probability of getting a 4 is $\frac{3}{36}$. The probability of getting another 4 before a 7 is $\frac{3}{3+6}$ since there are 3 ways to get a 4, 6 ways to get a 7, and numbers other than 4 and 7 are immaterial.

Thus, $P(\text{winning with } 4) = \frac{3}{36} \cdot \frac{3}{3+6} = \frac{1}{36}$.

(c) Let $P(k)$ denote the probability of winning a pass line bet with the number k. We first note that $P(4) = P(10)$, $P(5) = P(9)$, and $P(6) = P(8)$.

$$\begin{aligned} P(\text{winning}) &= 2 \cdot P(4) + 2 \cdot P(5) + 2 \cdot P(6) + P(7) + P(11) \\ &= 2 \cdot \tfrac{3}{36} \cdot \tfrac{3}{3+6} + 2 \cdot \tfrac{4}{36} \cdot \tfrac{4}{4+6} + 2 \cdot \tfrac{5}{36} \cdot \tfrac{5}{5+6} + \tfrac{6}{36} + \tfrac{2}{36} \\ &= 2 \cdot \tfrac{1}{36} + 2 \cdot \tfrac{2}{45} + 2 \cdot \tfrac{25}{396} + \tfrac{1}{6} + \tfrac{1}{18} = \tfrac{488}{990} = \tfrac{244}{495} \approx 0.4929 \end{aligned}$$

58 Using the same notation as in Exercise 57, and the additional facts that $P(2) = P(12)$ and $P(3) = P(11)$,

$$\begin{aligned} P(\text{winning}) &= 2 \cdot P(2) + 2 \cdot P(3) + 2 \cdot P(4) + 2 \cdot P(5) + 2 \cdot P(6) + P(7) \\ &= 2 \cdot \tfrac{1}{36} \cdot \tfrac{1}{1+6} + 2 \cdot \tfrac{2}{36} \cdot \tfrac{2}{2+6} + 2 \cdot \tfrac{3}{36} \cdot \tfrac{3}{3+6} + 2 \cdot \tfrac{4}{36} \cdot \tfrac{4}{4+6} + 2 \cdot \tfrac{5}{36} \cdot \tfrac{5}{5+6} + \tfrac{6}{36} \\ &= \tfrac{2}{36}\left(\tfrac{1}{7} + \tfrac{4}{8} + \tfrac{9}{9} + \tfrac{16}{10} + \tfrac{25}{11} + 3\right) = \tfrac{1}{18} \cdot \tfrac{6557}{770} = \tfrac{6557}{13{,}860} \approx 0.4731. \end{aligned}$$

Note that this probability is almost 2% less than the one for the game of craps.

$\boxed{59}$ (a) $p = P((S_1 \cap S_2) \cup (S_3 \cap S_4))$

$= P(S_1 \cap S_2) + P(S_3 \cap S_4) - P((S_1 \cap S_2) \cap (S_3 \cap S_4))$

$= P(S_1) \cdot P(S_2) + P(S_3) \cdot P(S_4) - P(S_1 \cap S_2) \cdot P(S_3 \cap S_4)$

$= P(S_1) \cdot P(S_2) + P(S_3) \cdot P(S_4) - P(S_1) \cdot P(S_2) \cdot P(S_3) \cdot P(S_4)$

Let $P(S_k) = x$. Then $p = x \cdot x + x \cdot x - x \cdot x \cdot x \cdot x = -x^4 + 2x^2$.

$x = 0.9 \Rightarrow p = 0.9639.$

[−2.25, 2.25, 0.5] by [−2, 1, 0.5]

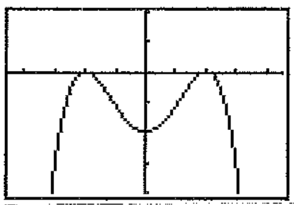


Figure 59(a)

[0.96, 1.05, 0.5] by [−0.03, 0.04, 0.5]

Figure 59(b)

(b) $p = 0.99 \Rightarrow -x^4 + 2x^2 = 0.99$. The graph of $y = -x^4 + 2x^2 - 0.99$ is shown in *Figure 59(a)*. The region near $x = 1$ is enlarged in *Figure 59(b)* to show that the graph is above the x-axis for some values of x. The approximate x-intercepts are ± 0.95, ± 1.05. Since $0 \le P(S_k) \le 1$, $P(S_k) = 0.95$.

$\boxed{60}$ (a) $p = P((S_1 \cup S_2) \cup (S_3 \cap S_4))$

$= P(S_1 \cup S_2) + P(S_3 \cap S_4) - P((S_1 \cup S_2) \cap (S_3 \cap S_4))$

$= P(S_1 \cup S_2) + P(S_3 \cap S_4) - P(S_1 \cup S_2) \cdot P(S_3 \cap S_4)$

Let $P(S_k) = x$. Then $P(S_1 \cup S_2) = P(S_1) + P(S_2) - P(S_1 \cap S_2) =$

$P(S_1) + P(S_2) - P(S_1) \cdot P(S_2) = x + x - x \cdot x = 2x - x^2.$

Also, $P(S_3 \cap S_4) = P(S_3) \cdot P(S_4) = x \cdot x = x^2$.

Thus, $p = (2x - x^2) + x^2 - (2x - x^2)x^2 = x^4 - 2x^3 + 2x$. $x = 0.9 \Rightarrow p = 0.9981$.

(b) $p = 0.99 \Rightarrow x^4 - 2x^3 + 2x = 0.99$. Graph $y = x^4 - 2x^3 + 2x - 0.99$. The approximate x-intercepts are -1.00 and 0.82. Since $0 \le P(S_k) \le 1$, $P(S_k) = 0.82$. Note that lowering the individual probability from 0.90 to approximately 0.82 *only* lowers the system probability from 0.9981 to approximately 0.99.

[−4.5, 4.5] by [−3, 3]

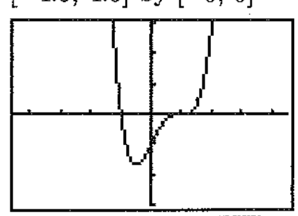


Figure 60

$\boxed{61}$ (a) The number of ways that n people can all have a different birthday is $P(365, n)$. The number of outcomes in the sample space is 365^n.

Thus, $p = \dfrac{P(365,\ n)}{365^n} = \dfrac{365!}{365^n(365-n)!}$.

(b) $n = 32 \Rightarrow p = \dfrac{365!}{365^{32}\,333!} \Rightarrow \ln p = \ln \dfrac{365!}{365^{32}\,333!}$

$= \ln 365! - \ln 365^{32} - \ln 333!$

$\approx (365 \ln 365 - 365) - (32 \ln 365) - (333 \ln 333 - 333) \approx -1.45.$

Thus, $p \approx e^{-1.45} \approx 0.24.$

The probability that two or more people have the same birthday is $1 - p \approx 0.76.$

[62] $p = \dfrac{365!}{365^n (365 - n)!} \Rightarrow \ln p = \ln 365! - n \ln 365 - \ln (365 - n)! \Rightarrow$

$\ln p \approx (365 \ln 365 - 365) - n \ln 365 - \left[(365 - n) \ln (365 - n) - (365 - n)\right].$

To find the value of n such that $\ln p = \ln 0.5$, we graph

$y = (365 \ln 365 - 365) - x \ln 365 - \left[(365 - x) \ln (365 - x) - (365 - x)\right] - \ln 0.5$

and estimate the positive x-intercept.

From the graph, we see that this occurs at $x \approx 22.26$. Thus, if $n = 23$, the probability of everyone having a different birthday is less than $\frac{1}{2}$. The following (n, p) pairs may be of interest: (30, 0.294), (40, 0.109), (50, 0.030), (60, 0.006).

[0, 50, 10] by [−1, 1, 0.5]

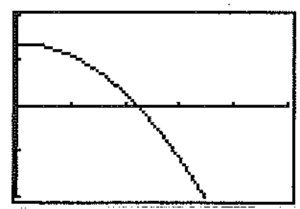


Figure 62

[63] From Exercise 57(c), the payoff amount of \$2 has a probability of $\frac{244}{495}$.

Hence, $EV = 2 \cdot \frac{244}{495} = \frac{488}{495} \approx \$0.986$, or about \$0.99.

[64] From Exercise 52(a), the payoff amount of \$2 has a probability of $\frac{18}{38}$.

Hence, $EV = 2 \cdot \frac{18}{38} = \frac{18}{19} \approx \$0.947$, or about \$0.95.

[65] $EV = 1{,}000{,}000 \cdot \frac{1}{20{,}000{,}000} + 100{,}000 \cdot \frac{10}{20{,}000{,}000} + 10{,}000 \cdot \frac{100}{20{,}000{,}000} + 1000 \cdot \frac{1000}{20{,}000{,}000}$

$= \$0.20$ {less than the cost of a first class stamp}

[66] $EV = 1000 \cdot \frac{1}{80} + 500 \cdot \frac{1}{80} + 300 \cdot \frac{1}{80} + 200 \cdot \frac{1}{80} + 100 \cdot 6 \cdot \frac{1}{80} = \$32.50$

Chapter 10 Review Exercises

[1] $a_n = \dfrac{5n}{3 - 2n^2}$ ★ $5, -2, -1, -\frac{20}{29}; -\frac{7}{19}$

[2] $a_n = (-1)^{n+1} - (0.1)^n$ ★ $0.9, -1.01, 0.999, -1.0001; 0.9999999$

[3] $a_n = 1 + (-\frac{1}{2})^{n-1}$ ★ $2, \frac{1}{2}, \frac{5}{4}, \frac{7}{8}; \frac{65}{64}$

[4] $a_n = \dfrac{2^n}{(n+1)(n+2)(n+3)}$ ★ $\frac{1}{12}, \frac{1}{15}, \frac{1}{15}, \frac{8}{105}; \frac{8}{45}$

[5] $a_1 = 10,\ a_{k+1} = 1 + (1/a_k)$ ★ $10, \frac{11}{10}, \frac{21}{11}, \frac{32}{21}, \frac{53}{32}$

[6] $a_1 = 2,\ a_{k+1} = a_k!$ ★ $2, 2, 2, 2, 2$

[7] $a_1 = 9,\ a_{k+1} = \sqrt{a_k}$ ★ $9, 3, \sqrt{3}, \sqrt[4]{3}, \sqrt[8]{3}$

[8] $a_1 = 1,\ a_{k+1} = (1 + a_k)^{-1}$ ★ $1, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}$

[9] $\sum_{k=1}^{5} (k^2 + 4) = 5 + 8 + 13 + 20 + 29 = 75$

[10] $\sum_{k=2}^{6} \frac{2k-8}{k-1} = (-4) + (-1) + 0 + \frac{1}{2} + \frac{4}{5} = -\frac{37}{10}$

[11] $\sum_{k=7}^{100} 10 = (100 - 7 + 1)(10) = 94(10) = 940$

[12] $\sum_{k=1}^{4} (2^k - 10) = (-8) + (-6) + (-2) + 6 = -10$

[13] $3 + 6 + 9 + 12 + 15 = \sum_{n=1}^{5} 3n$

[14] $4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \sum_{n=1}^{6} 4(\frac{1}{2})^{n-1} = \sum_{n=1}^{6} 2^2(2^{-1})^{n-1} = \sum_{n=1}^{6} 2^2\, 2^{1-n} = \sum_{n=1}^{6} 2^{3-n}$

[15] $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{99 \cdot 100} = \sum_{n=1}^{99} \frac{1}{n(n+1)}$

[16] $\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \cdots + \frac{1}{98 \cdot 99 \cdot 100} = \sum_{n=1}^{98} \frac{1}{n(n+1)(n+2)}$

[17] $\frac{1}{2} + \frac{2}{5} + \frac{3}{8} + \frac{4}{11}$. The numerators increase by 1, the denominators by 3. $\sum_{n=1}^{4} \frac{n}{3n-1}$

[18] $\frac{1}{4} + \frac{2}{9} + \frac{3}{14} + \frac{4}{19}$. The numerators increase by 1, the denominators by 5. $\sum_{n=1}^{4} \frac{n}{5n-1}$

[19] $100 - 95 + 90 - 85 + 80 = \sum_{n=1}^{5} (-1)^{n+1}(105 - 5n)$

[20] $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7}$. The terms have alternating signs,

the numerator is 1, and the denominators increase by 1. $\sum_{n=1}^{7} (-1)^{n-1}\frac{1}{n}$

[21] $a_0 + a_1x^4 + a_2x^8 + \cdots + a_{25}x^{100} = \sum_{n=0}^{25} a_n x^{4n}$

[22] $a_0 + a_1x^3 + a_2x^6 + \cdots + a_{20}x^{60} = \sum_{n=0}^{20} a_n x^{3n}$

[23] $1 - \frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{6} + \cdots + (-1)^n \frac{x^{2n}}{2n}$. The pattern begins with the second term

and the general term is listed. $1 + \sum_{k=1}^{n} (-1)^k \frac{x^{2k}}{2k}$

[24] $1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n}$.

The pattern begins with the second term and the general term is listed. $1 + \sum_{k=1}^{n} \frac{x^k}{k}$

[25] $d = 3 - (4 + \sqrt{3}) = -1 - \sqrt{3};\ a_{10} = (4 + \sqrt{3}) + (9)(-1 - \sqrt{3}) = -5 - 8\sqrt{3};$

$S_{10} = \frac{10}{2}\left[(4 + \sqrt{3}) + (-5 - 8\sqrt{3})\right] = -5 - 35\sqrt{3}.$

[26] $a_4 = a_1 + 3d \Rightarrow 9 = a_1 - 15 \Rightarrow a_1 = 24;\ S_8 = \frac{8}{2}[2(24) + (8-1)(-5)] = 52$

$\boxed{27}$ $a_5 = 5$ and $a_{13} = 77 \Rightarrow 8d = 72 \Rightarrow d = 9$;

$$a_5 = a_1 + 4d \Rightarrow 5 = a_1 + 36 \Rightarrow a_1 = -31;\ a_{10} = -31 + 9(9) = 50$$

$\boxed{28}$ $S_n = \frac{n}{2}[2a_1 + (n-1)(d)] \Rightarrow 342 = \frac{n}{2}[2(1) + (n-1)(5)] \Rightarrow 684 = n(5n-3) \Rightarrow$

$$684 = 5n^2 - 3n \Rightarrow 0 = 5n^2 - 3n - 684 \Rightarrow (n-12)(5n+57) = 0 \Rightarrow n = 12$$

$\boxed{29}$ Four arithmetic means $\Rightarrow 5d = -10 - 20 \Rightarrow d = -6$.

The terms are 20, 14, 8, 2, -4, and -10.

$\boxed{30}$ $r = \dfrac{\frac{1}{4}}{\frac{1}{8}} = 2;\ a_n = \frac{1}{8}(2)^{n-1} = 2^{-3} \cdot 2^{n-1} = 2^{n-4} \Rightarrow a_{10} = 2^6 = 64$

$\boxed{31}$ $r = \dfrac{-0.3}{3} = -0.1;\ a_8 = a_3 r^5 = 3(-0.1)^5 = -0.00003$

$\boxed{32}$ $a_7 = a_3 r^4 \Rightarrow 625 = 16r^4 \Rightarrow r^4 = \frac{625}{16} \Rightarrow r^2 = \frac{25}{4} \Rightarrow r = \pm\frac{5}{2}$.

Since $a_8 = a_7 r$, $a_8 = 625(\pm\frac{5}{2}) = \pm 1562.5$.

$\boxed{33}$ The geometric mean of 4 and 8 is $\sqrt{4 \cdot 8} = \sqrt{32} = 4\sqrt{2}$.

$\boxed{34}$ $a_8 = a_1 r^7 \Rightarrow 100 = a_1(-\frac{3}{2})^7 \Rightarrow a_1 = 100(-\frac{2}{3})^7 = -\frac{12{,}800}{2187}$

$\boxed{35}$ $402 = \frac{12}{2}(a_1 + 50) \Rightarrow a_1 = 17.\ a_{12} = a_1 + 11d \Rightarrow d = (50-17)/11 = 3$.

$\boxed{36}$ $a_5 = a_1 r^4 \Rightarrow \frac{1}{16} = a_1(\frac{3}{2})^4 \Rightarrow a_1 = \frac{1}{81}$.

$$S_5 = \frac{1}{81} \cdot \frac{1-(\frac{3}{2})^5}{1-\frac{3}{2}} = \frac{1}{81} \cdot \frac{1-\frac{243}{32}}{-\frac{1}{2}} = \frac{1}{81} \cdot \frac{-\frac{211}{32}}{-\frac{1}{2}} = \frac{211}{1296}.$$

$\boxed{37}$ The sequence of terms is arithmetic. $S_{15} = \frac{15}{2}(3 + 73) = 570$.

$\boxed{38}$ The sequence of terms is arithmetic. $S_{10} = \frac{10}{2}(5.5 + 1) = 32.5$.

$\boxed{39}$ $\displaystyle\sum_{k=1}^{10} (2^k - \tfrac{1}{2}) = \sum_{k=1}^{10} 2^k - \sum_{k=1}^{10} \tfrac{1}{2} = 2 \cdot \frac{1-2^{10}}{1-2} - 10(\tfrac{1}{2}) = 2046 - 5 = 2041$

$\boxed{40}$ $\displaystyle\sum_{k=1}^{8} (\tfrac{1}{2} - 2^k) = \sum_{k=1}^{8} \tfrac{1}{2} - \sum_{k=1}^{8} 2^k = 8(\tfrac{1}{2}) - 2 \cdot \frac{1-2^8}{1-2} = 4 - 510 = -506$

$\boxed{41}$ $a_1 = 1,\ r = -\frac{2}{5} \Rightarrow S = \dfrac{1}{1-(-\frac{2}{5})} = \dfrac{1}{\frac{7}{5}} = \frac{5}{7}$.

$\boxed{42}$ $a_1 = 0.274,\ r = 0.001 \Rightarrow S = \dfrac{0.274}{1-0.001} = \dfrac{274}{999}.\ 6.\overline{274} = 6 + \dfrac{274}{999} = \dfrac{6268}{999}$

$\boxed{43}$ (1) P_1 is true, since $3(1) - 1 = \dfrac{1[3(1)+1]}{2} = 2$.

(2) Assume P_k is true: $2 + 5 + 8 + \cdots + (3k-1) = \dfrac{k(3k+1)}{2}$. Hence,

$$\begin{aligned} 2 + 5 + 8 + \cdots + (3k-1) + 3(k+1) - 1 &= \frac{k(3k+1)}{2} + 3(k+1) - 1 \\ &= \frac{3k^2 + k + 6k + 4}{2} \\ &= \frac{3k^2 + 7k + 4}{2} \end{aligned}$$

(continued)

$$= \frac{(k+1)(3k+4)}{2}$$

$$= \frac{(k+1)[3(k+1)+1]}{2}.$$

Thus, P_{k+1} is true, and the proof is complete.

44 (1) P_1 is true, since $[2(1)]^2 = \frac{[2(1)][2(1)+1][1+1]}{3} = 4$.

(2) Assume P_k is true: $2^2 + 4^2 + 6^2 + \cdots + (2k)^2 = \frac{(2k)(2k+1)(k+1)}{3}$. Hence,

$$2^2 + 4^2 + 6^2 + \cdots + (2k)^2 + [2(k+1)]^2 = \frac{(2k)(2k+1)(k+1)}{3} + [2(k+1)]^2$$

$$= (k+1)\left(\frac{4k^2+2k}{3} + \frac{12(k+1)}{3}\right)$$

$$= \frac{(k+1)(4k^2+14k+12)}{3}$$

$$= \frac{2(k+1)(2k+3)(k+2)}{3}.$$

Thus, P_{k+1} is true, and the proof is complete.

45 (1) P_1 is true, since $\frac{1}{[2(1)-1][2(1)+1]} = \frac{1}{2(1)+1} = \frac{1}{3}$.

(2) Assume P_k is true:

$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}$. Hence,

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)(2k+3)}$$

$$= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$$

$$= \frac{k(2k+3)+1}{(2k+1)(2k+3)}$$

$$= \frac{2k^2+3k+1}{(2k+1)(2k+3)}$$

$$= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$$

$$= \frac{k+1}{2(k+1)+1}.$$

Thus, P_{k+1} is true, and the proof is complete.

46 (1) P_1 is true, since $1(1+1) = \frac{(1)\ (1+1)\ (1+2)}{3} = 2$.

(2) Assume P_k is true:

$1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + k(k+1) = \frac{k(k+1)(k+2)}{3}$. Hence,

$1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + k(k+1) + (k+1)(k+2)$

(continued)

$$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$$
$$= (k+1)(k+2)\left(\frac{k}{3}+1\right)$$
$$= \frac{(k+1)(k+2)(k+3)}{3}.$$

Thus, P_{k+1} is true, and the proof is complete.

[47] (1) For $n = 1$, $n^3 + 2n = 3$ and 3 is a factor of 3.

(2) Assume 3 is a factor of $k^3 + 2k$. The $(k+1)$st term is

$$\begin{aligned}(k+1)^3 + 2(k+1) &= k^3 + 3k^2 + 5k + 3 \\ &= (k^3 + 2k) + (3k^2 + 3k + 3) \\ &= (k^3 + 2k) + 3(k^2 + k + 1).\end{aligned}$$

By the induction hypothesis, 3 is a factor of $k^3 + 2k$ and 3 is a factor of $3(k^2 + k + 1)$, so 3 is a factor of the $(k+1)$st term. Thus, P_{k+1} is true, and the proof is complete.

[48] (1) P_5 is true, since $5^2 + 3 < 2^5$.

(2) Assume P_k is true: $k^2 + 3 < 2^k$. Hence, $(k+1)^2 + 3 = k^2 + 2k + 4 =$

$$(k^2 + 3) + (k+1) < 2^k + (k+1) < 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}.$$

Thus, P_{k+1} is true, and the proof is complete.

[49] For j: Examining the pattern formed by letting $n = 1, 2, 3, 4$ leads us to the conclusion that $j = 4$.

(1) P_4 is true, since $2^4 \le 4!$.

(2) Assume P_k is true: $2^k \le k!$. Hence,

$2^{k+1} = 2 \cdot 2^k \le 2 \cdot k! < (k+1) \cdot k! = (k+1)!$.

Thus, P_{k+1} is true, and the proof is complete.

[50] For j: $10^n \le n^n \;\Rightarrow\; \left(\frac{n}{10}\right)^n \ge 1$. This is true if $\frac{n}{10} \ge 1$ or $n \ge 10$. Thus, $j = 10$.

(1) P_{10} is true, since $10^{10} \le 10^{10}$.

(2) Assume P_k is true: $10^k \le k^k$. Hence,

$10^{k+1} = 10 \cdot 10^k \le 10 \cdot k^k < (k+1) \cdot k^k < (k+1) \cdot (k+1)^k = (k+1)^{k+1}$.

Thus, P_{k+1} is true, and the proof is complete.

[51] $(x^2 - 3y)^6 = x^{12} - 18x^{10}y + 135x^8y^2 - 540x^6y^3 + 1215x^4y^4 - 1458x^2y^5 + 729y^6$

[52] $$\begin{aligned}(2x + y^3)^4 &= (1)(2x)^4 + (4)(2x)^3(y^3)^1 + (6)(2x)^2(y^3)^2 + (4)(2x)(y^3)^3 + (1)(y^3)^4 \\ &= 16x^4 + 32x^3y^3 + 24x^2y^6 + 8xy^9 + y^{12}\end{aligned}$$

[53] $(x^{2/5} + 2x^{-3/5})^{20}$; first three terms $= x^8 + 40x^7 + 760x^6$

[54] $(y^3 - \frac{1}{2}c^2)^9$; sixth term $\{ k = 5 \} = \binom{9}{5}(y^3)^4(-\frac{1}{2}c^2)^5 = -\frac{63}{16}y^{12}c^{10}$

[55] $(4x^2 - y)^7$; term that contains x^{10} •

Consider only the variable x in the expansion: $(x^2)^{7-k} = x^{10} \Rightarrow$

$14 - 2k = 10 \Rightarrow k = 2$; $(k+1)$st term = 3rd term $= \binom{7}{2}(4x^2)^5(-y)^2 = 21{,}504x^{10}y^2$

[56] $(2c^3 + 5c^{-2})^{10}$; term that does not contain c •

$(c^3)^{10-k}(c^{-2})^k = c^0 \Rightarrow 30 - 3k - 2k = 0 \Rightarrow 30 = 5k \Rightarrow k = 6$;

$(k+1)$st term = 7th term $= \binom{10}{6}(2c^3)^4(5c^{-2})^6 = 210 \cdot 16c^{12} \cdot 15{,}625c^{-12} = 52{,}500{,}000$

[57] (a) $S_5 = 10 \Rightarrow 10 = \frac{5}{2}(2a_1 + 4d) \Rightarrow 4 = 2a_1 + 4d \Rightarrow 4d = 4 - 2a_1 \Rightarrow$

$d = 1 - \frac{1}{2}a_1$. Since a_1 is positive, $1 - \frac{1}{2}a_1$ is less than 1 ft.

(b) $a_1 = \frac{1}{2} \Rightarrow d = 1 - \frac{1}{2}(\frac{1}{2}) = \frac{3}{4}$.

The lengths of the other four pieces are $1\frac{1}{4}$, 2, $2\frac{3}{4}$, and $3\frac{1}{2}$ ft.

[58] $n = 16 \Rightarrow a_{16} = a_1 + 15d \Rightarrow 16 = 20 + 15d \Rightarrow d = -\frac{4}{15}$.

$$S_{16} = \tfrac{16}{2}\left[2(20) + 15(-\tfrac{4}{15})\right] = 8(40 - 4) = 288 \text{ in. or } 24 \text{ ft.}$$

[59] If $s_1 = 1$, then $s_2 = f$, $s_3 = f^2$, ...

The sum of the s_k's is $2(1 + f + f^2 + \cdots) = 2\left(\frac{1}{1-f}\right) = \frac{2}{1-f}$.

[60] $P(10, 10) = \frac{10!}{0!} = 10! = 3{,}628{,}800$

[61] (a) $P(52, 13) \approx 3.954 \times 10^{21}$

(b) $P(13, 5) \cdot P(13, 3) \cdot P(13, 3) \cdot P(13, 2) \approx 7.094 \times 10^{13}$

[62] (a) $P(6, 4) = 360$ (b) $6^4 = 1296$

[63] (a) $C(12, 8) = 495$ (b) $C(9, 5) = 126$

[64] $\frac{(6+5+4+2)!}{6!\,5!\,4!\,2!} = \frac{17!}{6!\,5!\,4!\,2!} = 85{,}765{,}680$

[65] If $O(E)$ are 8 to 5, then $O(E')$ are 5 to 8 and $P(E) = \frac{8}{5+8} = \frac{8}{13}$.

[66] There is one way to get all heads and one way to get all tails. (a) $\frac{2}{4} = \frac{1}{2}$ (b) $\frac{2}{8} = \frac{1}{4}$

[67] (a) We need 4 of the 26 cards of one color. There are 2 colors.

$$\frac{P(26, 4) \cdot 2}{P(52, 4)} = \frac{92}{833} \approx 0.1104$$

(b) We need R-B-R-B. $\frac{26^2 \cdot 25^2}{P(52, 4)}$ or $\frac{26}{52} \cdot \frac{26}{51} \cdot \frac{25}{50} \cdot \frac{25}{49} = \frac{325}{4998} \approx 0.0650$

[68] (a) $\frac{1}{1000}$ (b) $\frac{10}{1000} = \frac{1}{100}$ (c) $\frac{50}{1000} = \frac{1}{20}$

[69] $P(1 \text{ head}) = \frac{C(4, 1)}{2^4} = \frac{4}{16} = \frac{1}{4} = 0.25$. Odds are 4 to 12, or 1 to 3.

[70] (a) $\frac{C(6, 4) + C(6, 5) + C(6, 6)}{2^6} = \frac{15 + 6 + 1}{64} = \frac{22}{64} = \frac{11}{32}$ (b) $1 - \frac{22}{64} = \frac{42}{64}$

$\boxed{71}$ (a) $\frac{1}{6} \cdot \frac{1}{52} = \frac{1}{312} \approx 0.0032$

(b) $\frac{1}{6} + \frac{1}{52} - \frac{1}{312} = \frac{52 + 6 - 1}{312} = \frac{57}{312} = \frac{19}{104} \approx 0.1827$

$\boxed{72}$ $P(O \cup F) = P(O) + P(F) - P(O \cap F) = \frac{1000}{5000} + \frac{2000}{5000} - \frac{0.40(2000)}{5000} = \frac{2200}{5000} = 0.44$

$\boxed{73}$ There are 2 ways to obtain 10 (6, 4 and 4, 6), 2 ways for 11 (6, 5 and 5, 6), 1 way for 12, 16, 20, and 24 (double 3's, 4's, 5's, and 6's). $\frac{2+2+1+1+1+1}{36} = \frac{8}{36} = \frac{2}{9} = 0.\overline{2}$

$\boxed{74}$ The two teams, A and B, can play as few as 4 games or as many as 7 games.

$P(\text{team A wins in 4 games}) = (\frac{1}{2})^4 = \frac{1}{16} = 0.0625.$

$P(\text{team A wins in 5 games})$

$= P(\text{team A wins 3 of the first 4 games \{losing 1 game\} } \textit{and} \text{ then wins game 5})$

$= \binom{4}{3}(\frac{1}{2})^3 \cdot (\frac{1}{2})^1 \cdot \frac{1}{2} = \binom{4}{3}(\frac{1}{2})^5 = \frac{4}{32} = 0.125.$ In a similar fashion,

$$P(\text{team A wins in 6 games}) = \binom{5}{3}(\tfrac{1}{2})^6 = \tfrac{10}{64} = \tfrac{5}{32} = 0.15625$$

and $$P(\text{team A wins in 7 games}) = \binom{6}{3}(\tfrac{1}{2})^7 = \tfrac{20}{128} = \tfrac{5}{32} = 0.15625.$$

Since the probabilities for team B winning the series are the same, the expected number of games is $4(2 \cdot \frac{1}{16}) + 5(2 \cdot \frac{4}{32}) + 6(2 \cdot \frac{5}{32}) + 7(2 \cdot \frac{5}{32}) = 5\frac{13}{16} = 5.8125.$

Chapter 10 Discussion Exercises

$\boxed{1}$ The desired sequence starts with $2n$ for $n = 1, 2, 3, 4$. To obtain the fifth term, i.e., a, add $\frac{(n-1)(n-2)(n-3)(n-4)(a-10)}{4 \cdot 3 \cdot 2 \cdot 1}$, which is 0 for $n = 1, 2, 3, 4$ and $(a-10)$ for $n = 5$. $a_n = 2n + \frac{(n-1)(n-2)(n-3)(n-4)(a-10)}{24}$

Another possibility is $a_n = \begin{cases} 2n & \text{if } 1 \leq n \leq 4 \\ (n-4)a & \text{if } n \geq 5 \end{cases}$

$\boxed{2}$ Examining the graphs of $y_1 = x$ and $y_2 = (\ln x)^3$, we see that $y_1 = y_2 \Rightarrow x \approx 6.4$, 93.35. For large values of x, $y_1 \boxed{\geq} y_2$. The value of j is 94.

$\boxed{3}$ (a) Following the pattern from Exercises 37–38 in Section 10.4, write

$1^4 + 2^4 + 3^4 + \cdots + n^4 = an^5 + bn^4 + cn^3 + dn^2 + en$. Then, it follows that:

$n = 1 \;\Rightarrow\; a(1)^5 + b(1)^4 + c(1)^3 + d(1)^2 + e(1) = 1^4 \;\Rightarrow\; a + b + c + d + e = 1,$

$n = 2 \;\Rightarrow\; a(2)^5 + b(2)^4 + c(2)^3 + d(2)^2 + e(2) = 1^4 + 2^4 \;\Rightarrow$

$32a + 16b + 8c + 4d + 2e = 17,$

$n = 3 \;\Rightarrow\; a(3)^5 + b(3)^4 + c(3)^3 + d(3)^2 + e(3) = 1^4 + 2^4 + 3^4 \;\Rightarrow$

$243a + 81b + 27c + 9d + 3e = 98,$

$n = 4 \;\Rightarrow\; a(4)^5 + b(4)^4 + c(4)^3 + d(4)^2 + e(4) = 1^4 + 2^4 + 3^4 + 4^4 \;\Rightarrow$

$1024a + 256b + 64c + 16d + 4e = 354,$

$n = 5 \;\Rightarrow\; a(5)^5 + b(5)^4 + c(5)^3 + d(5)^2 + e(5) = 1^4 + 2^4 + 3^4 + 4^4 + 5^4 \;\Rightarrow$

$3125a + 625b + 125c + 25d + 5e = 979.$

$AX = B \;\Rightarrow$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 32 & 16 & 8 & 4 & 2 \\ 243 & 81 & 27 & 9 & 3 \\ 1024 & 256 & 64 & 16 & 4 \\ 3125 & 625 & 125 & 25 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 1 \\ 17 \\ 98 \\ 354 \\ 979 \end{bmatrix} \;\Rightarrow\; X = A^{-1}B = \begin{bmatrix} 1/5 \\ 1/2 \\ 1/3 \\ 0 \\ -1/30 \end{bmatrix}$$

Thus, $1^4 + 2^4 + 3^4 + \cdots + n^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$.

This formula must be verified.

(b) Let P_n be the statement that $1^4 + 2^4 + 3^4 + \cdots + n^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$.

(1) $1^4 = \frac{1}{5} + \frac{1}{2} + \frac{1}{3} - \frac{1}{30} = 1$ and P_1 is true.

(2) Assume that P_k is true. We must show that P_{k+1} is true.

$P_k \;\Rightarrow\; 1^4 + 2^4 + 3^4 + \cdots + k^4 = \frac{1}{5}k^5 + \frac{1}{2}k^4 + \frac{1}{3}k^3 - \frac{1}{30}k \;\Rightarrow$

$1^4 + 2^4 + 3^4 + \cdots + k^4 + (k+1)^4 = \frac{1}{5}k^5 + \frac{1}{2}k^4 + \frac{1}{3}k^3 - \frac{1}{30}k + (k+1)^4 \;\Rightarrow$

$1^4 + 2^4 + 3^4 + \cdots + k^4 + (k+1)^4 =$

$\frac{1}{5}k^5 + \frac{1}{2}k^4 + \frac{1}{3}k^3 - \frac{1}{30}k + (k^4 + 4k^3 + 6k^2 + 4k + 1) \;\Rightarrow$

$1^4 + 2^4 + 3^4 + \cdots + k^4 + (k+1)^4 = \frac{1}{5}k^5 + \frac{3}{2}k^4 + \frac{13}{3}k^3 + 6k^2 + \frac{119}{30}k + 1.$

$P_{k+1} \;\Rightarrow\; 1^4 + 2^4 + 3^4 + \cdots + k^4 + (k+1)^4$

$= \frac{1}{5}(k+1)^5 + \frac{1}{2}(k+1)^4 + \frac{1}{3}(k+1)^3 - \frac{1}{30}(k+1)$

$= \frac{1}{5}(k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1) + \frac{1}{2}(k^4 + 4k^3 + 6k^2 + 4k + 1) +$

$\frac{1}{3}(k^3 + 3k^2 + 3k + 1) - \frac{1}{30}(k+1)$

$= \frac{1}{5}k^5 + \frac{3}{2}k^4 + \frac{13}{3}k^3 + 6k^2 + \frac{119}{30}k + 1$. Thus, the formula is true for all n.

[4] (a) Following the pattern of setting the sum equal to a polynomial of degree one higher than the power in the exercise we write

$2^3+4^3+6^3+\cdots+(2n)^3 = an^4+bn^3+cn^2+dn$. Then, it follows that:

$n=1 \;\Rightarrow\; a(1)^4+b(1)^3+c(1)^2+d(1)=2^3 \;\Rightarrow\; a+b+c+d=8$

$n=2 \;\Rightarrow\; a(2)^4+b(2)^3+c(2)^2+d(2)=2^3+4^3 \;\Rightarrow\; 16a+8b+4c+2d=72$

$n=3 \;\Rightarrow\; a(3)^4+b(3)^3+c(3)^2+d(3)=2^3+4^3+6^3 \;\Rightarrow$

$81a+27b+9c+3d=288$

$n=4 \;\Rightarrow\; a(4)^4+b(4)^3+c(4)^2+d(4)=2^3+4^3+6^3+8^3 \;\Rightarrow$

$256a+64b+16c+4d=800$

$$AX=B \;\Rightarrow\; \begin{bmatrix} 1 & 1 & 1 & 1 \\ 16 & 8 & 4 & 2 \\ 81 & 27 & 9 & 3 \\ 256 & 64 & 16 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}=\begin{bmatrix} 8 \\ 72 \\ 288 \\ 800 \end{bmatrix} \;\Rightarrow\; X=A^{-1}B=\begin{bmatrix} 2 \\ 4 \\ 2 \\ 0 \end{bmatrix}$$

$a=2,\ b=4,\ c=2,\ d=0 \;\Rightarrow\; 2^3+4^3+6^3+\cdots+(2n)^3=2n^4+4n^3+2n^2$.

(b) Let P_n be the statement that $2^3+4^3+6^3+\cdots+(2n)^3=2n^4+4n^3+2n^2$

(1) $2^3=2+4+2=8$ and P_1 is true.

(2) Assume that P_k is true. We must show that P_{k+1} is true.

$P_k \;\Rightarrow\; 2^3+4^3+6^3+\cdots+(2k)^3=2k^4+4k^3+2k^2 \;\Rightarrow$

$2^3+4^3+6^3+\cdots+(2k)^3+(2k+2)^3=2k^4+4k^3+2k^2+(2k+2)^3=$

$2k^4+12k^3+26k^2+24k+8$

$P_{k+1} \;\Rightarrow\; 2^3+4^3+6^3+\cdots+(2k)^3+(2k+2)^3$

$=2(k+1)^4+4(k+1)^3+2(k+1)^2$

$=2(k^4+4k^3+6k^2+4k+1)+4(k^3+3k^2+3k+1)+2(k^2+2k+1)$

$=2k^4+12k^3+26k^2+24k+8$. Thus, the formula is true for all n.

[5] Examine the number of digits in the exponent of the value in scientific notation. The TI-82/83 can compute 69!, but not 70!, since 70! is larger than a 2-digit exponent. The TI-85/86 can compute 449!, but not 450!, since 450! is larger than a 3-digit exponent.

[6] The $(k+1)$st coefficient ($k=0,\ 1,\ 2,\ \ldots,\ n$) of the expansion of $(a+b)^n$, namely $\binom{n}{k}$, is the same as the number of k-element subsets of an n-element set.

[7] $\text{Time}_{\text{total}} = \text{Time}_{\text{down}} + \text{Time}_{\text{up}}$ {note $h = 10$ and $d = 10 \cdot \frac{1}{2}$ for the first rebound}

$$= \left[\frac{\sqrt{10}}{4} + \frac{\sqrt{10 \cdot \frac{1}{2}}}{4} + \frac{\sqrt{10 \cdot (\frac{1}{2})^2}}{4} + \cdots\right] + \left[\frac{\sqrt{10 \cdot \frac{1}{2}}}{4} + \frac{\sqrt{10 \cdot (\frac{1}{2})^2}}{4} + \cdots\right]$$

$= \dfrac{\sqrt{10}}{4} + 2\left[\dfrac{\sqrt{10 \cdot \frac{1}{2}}}{4} + \dfrac{\sqrt{10 \cdot (\frac{1}{2})^2}}{4} + \cdots\right]$. The ratio of this geometric sequence is $\sqrt{\frac{1}{2}} < 1$, so we can find its infinite sum using $S = a_1/(1 - r)$. Thus, the total time is

$$\frac{\sqrt{10}}{4} + 2 \cdot \frac{\sqrt{5}/4}{1 - \sqrt{\frac{1}{2}}} \approx 4.61 \text{ seconds.}$$

[8] There are $8 \times 36 = 288$ contestants daily and $288 \times 30 = 8640$ contestants for the month. The probability that a contestant wins any particular prize for the daily tournament and the monthly tournament is $p_1 = \frac{1}{288}$ and $p_2 = \frac{1}{8640}$, respectively. If the game is to be fair, then the total expected value should equal the entry fee.

$EV_1(\text{daily}) = 250p_1 + 100p_1 + 50p_1 = 400p_1$.

$$\begin{aligned} EV_2(\text{month}) = {} & 4000p_2 + 2000p_2 + 1500p_2 + 1000p_2 + 800p_2 + \\ & 600p_2 + 500p_2 + 400p_2 + 300p_2 + 200p_2 + \\ & 100(40p_2) + 75(50p_2) + 50(200p_2) + 25(200p_2) = 34{,}050p_2. \end{aligned}$$

$$\begin{aligned} \text{Thus, } EV_{\text{total}} &= EV_1 + EV_2 \\ &= 400p_1 + 34{,}050p_2 \\ &= 400 \cdot \tfrac{1}{288} + 34{,}050 \cdot \tfrac{1}{8640} = \tfrac{46{,}050}{8640} = \tfrac{1535}{288} \approx \$5.33. \end{aligned}$$

[9] To the nearest penny with $r = 1.1008163$ (found by trial and error), we have the places 1st–10th:

$0.01:	237.37	215.63	195.89	177.95	161.65	146.85	133.40	121.18	110.08	100
$1.00:	237.00	216.00	196.00	178.00	162.00	147.00	133.00	121.00	110.00	100
$5.00:	240.00	215.00	195.00	180.00	160.00	145.00	135.00	120.00	110.00	100
$10.00:	240.00	220.00	200.00	180.00	160.00	140.00	130.00	120.00	110.00	100

If the amounts are to be realistic, the amounts may not be rounded to the *nearest* amount—i.e, $134 may be rounded to $140 rather than $130.

[10] Since we can have 0 to 5 toppings on a pizza, the number of ways to order one pizza is $\sum_{k=0}^{5} \binom{n}{k}$. Because there are two pizzas, we have $\left[\sum_{k=0}^{5} \binom{n}{k}\right]^2 = 1{,}048{,}576 \Rightarrow$ $\sum_{k=0}^{5} \binom{n}{k} = 1024$. By trial and error, we find that $n = 11$. On the TI graphing calculators, store 11 in N and use "sum(seq(N nCr R,R,0,5,1))" to obtain 1024.

$\boxed{11}$ First, calculate the probabilities for all prizes: $n(S) = C(49,\ 5) \cdot 42 = 80{,}089{,}128$

$$\begin{aligned}
n(5\,\mathrm{W},\ 1\,\mathrm{R}) &= C(5,\ 5) \cdot C(44,\ 0) \cdot 1 & \Rightarrow P(5\,\mathrm{W},\ 1\,\mathrm{R}) &= 1/80{,}089{,}128\ (p_1)\\
n(5\,\mathrm{W},\ 0\,\mathrm{R}) &= C(5,\ 5) \cdot C(44,\ 0) \cdot 41 & \Rightarrow P(5\,\mathrm{W},\ 0\,\mathrm{R}) &= 41/80{,}089{,}128\ (p_2)\\
n(4\,\mathrm{W},\ 1\,\mathrm{R}) &= C(5,\ 4) \cdot C(44,\ 1) \cdot 1 & \Rightarrow P(4\,\mathrm{W},\ 1\,\mathrm{R}) &= 220/80{,}089{,}128\ (p_3)\\
n(4\,\mathrm{W},\ 0\,\mathrm{R}) &= C(5,\ 4) \cdot C(44,\ 1) \cdot 41 & \Rightarrow P(4\,\mathrm{W},\ 0\,\mathrm{R}) &= 9020/80{,}089{,}128\ (p_4)\\
n(3\,\mathrm{W},\ 1\,\mathrm{R}) &= C(5,\ 3) \cdot C(44,\ 2) \cdot 1 & \Rightarrow P(3\,\mathrm{W},\ 1\,\mathrm{R}) &= 9460/80{,}089{,}128\ (p_5)\\
n(3\,\mathrm{W},\ 0\,\mathrm{R}) &= C(5,\ 3) \cdot C(44,\ 2) \cdot 41 & \Rightarrow P(3\,\mathrm{W},\ 0\,\mathrm{R}) &= 387{,}860/80{,}089{,}128\ (p_6)\\
n(2\,\mathrm{W},\ 1\,\mathrm{R}) &= C(5,\ 2) \cdot C(44,\ 3) \cdot 1 & \Rightarrow P(2\,\mathrm{W},\ 1\,\mathrm{R}) &= 132{,}440/80{,}089{,}128\ (p_7)\\
n(1\,\mathrm{W},\ 1\,\mathrm{R}) &= C(5,\ 1) \cdot C(44,\ 4) \cdot 1 & \Rightarrow P(1\,\mathrm{W},\ 1\,\mathrm{R}) &= 678{,}755/80{,}089{,}128\ (p_8)\\
n(0\,\mathrm{W},\ 1\,\mathrm{R}) &= C(5,\ 0) \cdot C(44,\ 5) \cdot 1 & \Rightarrow P(0\,\mathrm{W},\ 1\,\mathrm{R}) &= 1{,}086{,}008/80{,}089{,}128\ (p_9)
\end{aligned}$$

(a) The probability of winning the jackpot is $\frac{1}{80{,}089{,}128}$.

(b) The probability of winning any prize is the sum of all the probabilities, that is, $\frac{2{,}303{,}805}{80{,}089{,}128}$. This gives us odds of 77,785,323 to 2,303,805 (or about 34 to 1) for *not* winning any prize.

(c) The expected value of the game without the jackpot is

$$\sum_{k=2}^{9} a_k p_k = (\$100{,}000)p_2 + (\$5000)p_3 + (\$100)p_4 + (\$100)p_5 + (\$7)p_6 + (\$7)p_7 + (\$4)p_8 + (\$3)p_9 = \frac{16{,}663{,}144}{80{,}089{,}128} \approx 0.21$$

(d) Assuming that one ticket costs \$1, we must have an expected value of 1 to be a fair game. Since the probability of winning the jackpot is $\frac{1}{80{,}089{,}128}$, we must multiply that probability by \$63,425,984 $\{80{,}089{,}128 - 16{,}663{,}144\}$ so that the numerator of the expected value in part (c) is equal to the denominator.

$\boxed{12}$ $P(M) = 0.20 = \frac{1}{5}$, so $O(M)$ are 1 to 4. Since the odds are three times more favorable for a female applicant, $O(F)$ are 3 to 4. Hence, $P(F) = \frac{3}{7} \approx 0.43$.

$\boxed{13}$

$$\begin{aligned}
(a+b)^n &= \sum_{k=0}^{n} \binom{n}{k} (a)^{n-k} (b)^k && \text{given}\\
(0+1)^n &= \sum_{k=0}^{n} \binom{n}{k} (0)^{n-k} (1)^k && \text{let } a = 0 \text{ and } b = 1\\
1^n &= 0^n \cdot 1^0 + 0^{n-1} \cdot 1^1 + \cdots + 0^0 \cdot 1^n && \text{expand}\\
1 &= 0 + 0 + \cdots + 0^0 && \text{simplify}\\
1 &= 0^0 && \text{simplify}
\end{aligned}$$

So we can use the result $0^0 = 1$, which is consistent with results in mathematics courses at a higher level.

$\boxed{14}$ $S = \sum_{n=0}^{\infty} (-1)^n \frac{3^{3/2}}{2^{3n+2}}\left(\frac{2}{3n+1} + \frac{1}{3n+2}\right)$

n	a_n	s_n
0	3.247 595 264 191	3.247 595 264
1	− 0.113 665 834 247	3.133 929 430
2	0.008 336 461 058	3.142 265 891
3	− 0.000 738 089 833	3.141 527 801
4	0.000 071 445 423	3.141 599 247
5	− 0.000 007 287 408	3.141 591 959
6	0.000 000 769 397	3.141 592 729
7	− 0.000 000 083 244	3.141 592 645
8	0.000 000 009 172	3.141 592 655
9	− 0.000 000 000 103	3.141 592 653
10	0.000 000 000 116	3.141 592 654

As $n \to \infty$, $s_n \to \pi$. The infinite sum, S, is equal to π (*not* approximately equal to π).

$\boxed{15}$ (a) The pattern has a 1 in the denominator, an n in the numerator, and the next terms of any row of Pascal's triangle oscillate between denominator and numerator. The signs are in pairs (two positive, two negative, etc.). The power of $\tan x$ increases one with each term. Thus,

$$\tan 5x = \frac{5\tan x - 10\tan^3 x + \tan^5 x}{1 - 10\tan^2 x + 5\tan^4 x}.$$

(b) The coefficients listed in the text are in the form 1–2–1. The next identities are:

$$\begin{aligned}
\cos 3x &= 1\cos^3 x \qquad\qquad - 3\cos x\,\sin^2 x \\
\sin 3x &= \qquad\qquad 3\cos^2 x\,\sin x \qquad\qquad - 1\sin^3 x
\end{aligned}$$

$$\begin{aligned}
\cos 4x &= 1\cos^4 x \qquad\qquad - 6\cos^2 x\,\sin^2 x \qquad\qquad + 1\sin^4 x \\
\sin 4x &= \qquad\qquad 4\cos^3 x\,\sin x \qquad\qquad - 4\cos x\,\sin^3 x
\end{aligned}$$

Notice the pattern of coefficients: for $\cos 3x$ and $\sin 3x$ we have 1–3–3–1; for $\cos 4x$ and $\sin 4x$ we have 1–4–6–4–1. Since these are rows in Pascal's triangle, we predict the following pattern for $\cos 5x$ and $\sin 5x$ (1–5–10–10–5–1):

$$\begin{aligned}
\cos 5x &= 1\cos^5 x \qquad\qquad - 10\cos^3 x\,\sin^2 x \qquad\qquad + 5\cos x\,\sin^4 x \\
\sin 5x &= \qquad\qquad 5\cos^4 x\,\sin x \qquad\qquad - 10\cos^2 x\,\sin^3 x \qquad\qquad + 1\sin^5 x
\end{aligned}$$

Note: The next four solutions are for Exercises 39–42 in Section 10.4.

39 (1) For $n = 1$, $\sin(\theta + 1\pi) = \sin\theta\cos\pi + \cos\theta\sin\pi = -\sin\theta = (-1)^1\sin\theta$.

(2) Assume P_k is true: $\sin(\theta + k\pi) = (-1)^k \sin\theta$. Hence,

$$\begin{aligned}\sin[\theta + (k+1)\pi] &= \sin[(\theta + k\pi) + \pi] \\ &= \sin(\theta + k\pi)\cos\pi + \cos(\theta + k\pi)\sin\pi \\ &= \left[(-1)^k\sin\theta\right]\cdot(-1) + \cos(\theta + k\pi)\cdot(0) \\ &= (-1)^{k+1}\sin\theta.\end{aligned}$$

Thus, P_{k+1} is true, and the proof is complete.

40 (1) For $n = 1$, $\cos(\theta + 1\pi) = \cos\theta\cos\pi - \sin\theta\sin\pi = -\cos\theta = (-1)^1\cos\theta$.

(2) Assume P_k is true: $\cos(\theta + k\pi) = (-1)^k\cos\theta$. Hence,

$$\begin{aligned}\cos[\theta + (k+1)\pi] &= \cos[(\theta + k\pi) + \pi] \\ &= \cos(\theta + k\pi)\cos\pi - \sin(\theta + k\pi)\sin\pi \\ &= \left[(-1)^k\cos\theta\right]\cdot(-1) - \sin(\theta + k\pi)\cdot(0) \\ &= (-1)^{k+1}\cos\theta.\end{aligned}$$

Thus, P_{k+1} is true, and the proof is complete.

41 (1) For $n = 1$, $[r(\cos\theta + i\sin\theta)]^1 = r^1[\cos(1\theta) + i\sin(1\theta)]$.

(2) Assume P_k is true: $[r(\cos\theta + i\sin\theta)]^k = r^k(\cos k\theta + i\sin k\theta)$. Hence,

$$\begin{aligned}[r(\cos\theta + i\sin\theta)]^{k+1} &= [r(\cos\theta + i\sin\theta)]^k[r(\cos\theta + i\sin\theta)] \\ &= r^k[\cos k\theta + i\sin k\theta][r(\cos\theta + i\sin\theta)] \\ &= r^{k+1}[(\cos k\theta\cos\theta - \sin k\theta\sin\theta) \\ &\qquad + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)]\end{aligned}$$

{Use the addition formulas for the sine and cosine.}

$$= r^{k+1}[\cos(k+1)\theta + i\sin(k+1)\theta].$$

Thus, P_{k+1} is true, and the proof is complete.

42 (1) For $n = 3$, $(n-2)\cdot 180° = 180°$, which is true for any triangle.

(2) Assume the sum of the interior angles of a polygon of k sides is $(k-2)\cdot 180°$. Now any $(k+1)$-sided polygon can be dissected into a k-sided polygon and a triangle by drawing a line from vertex (i) to vertex $(i+2)$. Its angles add up to $(k-2)\cdot 180°$ {since it is k-sided, by hypothesis} $+ 180°$ {for the triangle}, which is $(k-1)\cdot 180°$. Thus, P_{k+1} is true, and the proof is complete.