CHAPTER 1
Limits and Their Properties

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Section 1.1 A Preview of Calculus

1. Precalculus: $(20 \text{ ft/sec})(15 \text{ seconds}) = 300$ feet

2. Calculus: velocity is not constant

Distance $\approx (20 \text{ ft/sec})(15 \text{ seconds}) = 300$ feet

3. Calculus required: slope of tangent line at $x = 2$ is rate of change, and equals about 0.16.

4. Precalculus: rate of change $=$ slope $= 0.08$

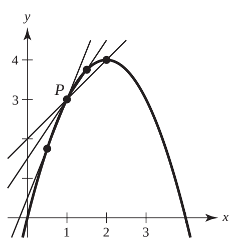
5. Precalculus: Area $= \frac{1}{2}bh = \frac{1}{2}(5)(3) = \frac{15}{2}$ sq. units

6. Calculus required: Area $= bh$

$$\approx 2(2.5)$$
$$= 5 \text{ sq. units}$$

7. $f(x) = 4x - x^2$

(a)



(b) slope $= m = \dfrac{(4x - x^2) - 3}{x - 1}$

$$= \frac{(x - 1)(3 - x)}{x - 1} = 3 - x, \quad x \neq 1$$

$x = 2$: $m = 3 - 2 = 1$

$x = 1.5$: $m = 3 - 1.5 = 1.5$

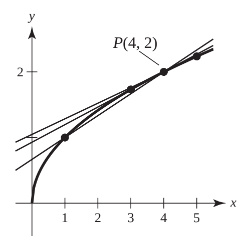
$x = 0.5$: $m = 3 - 0.5 = 2.5$

(c) At $P(1, 3)$ the slope is 2.

You can improve your approximation of the slope at $x = 1$ by considering x-values very close to 1.

8. $f(x) = \sqrt{x}$

(a)



(b) slope $= m = \dfrac{\sqrt{x} - 2}{x - 4}$

$$= \frac{\sqrt{x} - 2}{(\sqrt{x} + 2)(\sqrt{x} - 2)} = \frac{1}{\sqrt{x} + 2}, \quad x \neq 4$$

$x = 1$: $m = \dfrac{1}{\sqrt{1} + 2} = \dfrac{1}{3}$

$x = 3$: $m = \dfrac{1}{\sqrt{3} + 2} \approx 0.2679$

$x = 5$: $m = \dfrac{1}{\sqrt{5} + 2} \approx 0.2361$

(c) At $P(4, 2)$ the slope is $\dfrac{1}{\sqrt{4} + 2} = \dfrac{1}{4} = 0.25$.

You can improve your approximation of the slope at $x = 4$ by considering x-values very close to 4.

9. (a) Area $\approx 5 + \dfrac{5}{2} + \dfrac{5}{3} + \dfrac{5}{4} \approx 10.417$

Area $\approx \dfrac{1}{2}\left(5 + \dfrac{5}{1.5} + \dfrac{5}{2} + \dfrac{5}{2.5} + \dfrac{5}{3} + \dfrac{5}{3.5} + \dfrac{5}{4} + \dfrac{5}{4.5}\right) \approx 9.145$

(b) You could improve the approximation by using more rectangles.

10. (a) For the figure on the left, each rectangle has width $\frac{\pi}{4}$.

$$\begin{aligned}\text{Area} &\approx \frac{\pi}{4}\left[\sin\frac{\pi}{4} + \sin\frac{\pi}{2} + \sin\frac{3\pi}{4} + \sin\pi\right]\\ &= \frac{\pi}{4}\left[\frac{\sqrt{2}}{2} + 1 + \frac{\sqrt{2}}{2}\right]\\ &= \frac{\sqrt{2}+1}{4}\pi \approx 1.8961\end{aligned}$$

For the figure on the right, each rectangle has width $\frac{\pi}{6}$.

$$\begin{aligned}\text{Area} &\approx \frac{\pi}{6}\left[\sin\frac{\pi}{6} + \sin\frac{\pi}{3} + \sin\frac{\pi}{2} + \sin\frac{2\pi}{3} + \frac{5\pi}{6} + \sin\pi\right]\\ &= \frac{\pi}{6}\left[\frac{1}{2} + \frac{\sqrt{3}}{2} + 1 + \frac{\sqrt{3}}{2} + \frac{1}{2}\right]\\ &= \frac{\sqrt{3}+2}{6}\pi \approx 1.9541\end{aligned}$$

(b) You could obtain a more accurate approximation by using more rectangles. You will learn later that the exact area is 2.

11. (a) $D_1 = \sqrt{(5-1)^2 + (1-5)^2} = \sqrt{16+16} \approx 5.66$

(b) $D_2 = \sqrt{1 + \left(\frac{5}{2}\right)^2} + \sqrt{1 + \left(\frac{5}{2} - \frac{5}{3}\right)^2} + \sqrt{1 + \left(\frac{5}{3} - \frac{5}{4}\right)^2} + \sqrt{1 + \left(\frac{5}{4} - 1\right)^2}$

$\approx 2.693 + 1.302 + 1.083 + 1.031 \approx 6.11$

(c) Increase the number of line segments.

Section 1.2 Finding Limits Graphically and Numerically

1.

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.3448	0.3344	0.3334	0.3332	0.3322	0.3226

$\lim_{x\to 2} \frac{x-2}{x^2-x-2} \approx 0.3333$ (Actual limit is $\frac{1}{3}$.)

2.

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.2564	0.2506	0.2501	0.2499	0.2494	0.2439

$\lim_{x\to 2} \frac{x-2}{x^2-4} \approx 0.25$ (Actual limit is $\frac{1}{4}$.)

3.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.2911	0.2889	0.2887	0.2887	0.2884	0.2863

$\lim_{x\to 0} \frac{\sqrt{x+3}-\sqrt{3}}{x} \approx 0.2887$ (Actual limit is $1/(2\sqrt{3})$.)

4.

x	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9
$f(x)$	-0.2485	-0.2498	-0.2500	-0.2500	-0.2502	-0.2516

$\lim_{x\to -3} \dfrac{\sqrt{1-x}-2}{x+3} \approx -0.25$ $\left(\text{Actual limit is } -\frac{1}{4}.\right)$

5.

x	2.9	2.99	2.999	3.001	3.01	3.1
$f(x)$	-0.0641	-0.0627	-0.0625	-0.0625	-0.0623	-0.0610

$\lim_{x\to 3} \dfrac{[1/(x+1)]-(1/4)}{x-3} \approx -0.0625$ $\left(\text{Actual limit is } -\frac{1}{16}.\right)$

6.

x	3.9	3.99	3.999	4.001	4.01	4.1
$f(x)$	0.0408	0.0401	0.0400	0.0400	0.0399	0.0392

$\lim_{x\to 4} \dfrac{[x/(x+1)]-(4/5)}{x-4} \approx 0.04$ $\left(\text{Actual limit is } \frac{1}{25}.\right)$

7.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.9983	0.99998	1.0000	1.0000	0.99998	0.9983

$\lim_{x\to 0} \dfrac{\sin x}{x} \approx 1.0000$ (Actual limit is 1.) (Make sure you use radian mode.)

8.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.0500	0.0050	0.0005	-0.0005	-0.0050	-0.0500

$\lim_{x\to 0} \dfrac{\cos x - 1}{x} \approx 0.0000$ (Actual limit is 0.) (Make sure you use radian mode.)

9. $\lim_{x\to 3} (4-x) = 1$

10. $\lim_{x\to 1} (x^2+2) = 3$

11. $\lim_{x\to 2} f(x) = \lim_{x\to 2} (4-x) = 2$

12. $\lim_{x\to 1} f(x) = \lim_{x\to 1} (x^2+2) = 3$

13. $\lim_{x\to 5} \dfrac{|x-5|}{x-5}$ does not exist. For values of x to the left of 5, $|x-5|/(x-5)$ equals -1, whereas for values of x to the right of 5, $|x-5|/(x-5)$ equals 1.

14. $\lim_{x\to 3} \dfrac{1}{x-3}$ does not exist since the function increases and decreases without bound as x approaches 3.

15. $\lim_{x\to 1} \sin \pi x = 0$

16. $\lim_{x\to 0} \sec x = 1$

17. $\lim_{x\to 0} \cos(1/x)$ does not exist since the function oscillates between -1 and 1 as x approaches 0.

18. $\lim_{x\to \pi/2} \tan x$ does not exist since the function increases and decreases without bound as x approaches $\pi/2$.

19. (a) $f(1)$ exists. The black dot at $(1, 2)$ indicates that $f(1) = 2$.

(b) $\lim_{x\to 1} f(x)$ does not exist. As x approaches 1 from the left, $f(x)$ approaches 2.5, whereas as x approaches 1 from the right, $f(x)$ approaches 1.

(c) $f(4)$ does not exist. The hollow circle at $(4, 2)$ indicates that f is not defined at 4.

(d) $\lim_{x\to 4} f(x)$ exists. As x approaches 4, $f(x)$ approaches 2: $\lim_{x\to 4} f(x) = 2$.

20. (a) $f(-2)$ does not exist. The vertical dotted line indicates that f is not defined at -2.

(b) $\lim_{x\to -2} f(x)$ does not exist. As x approaches -2, the values of $f(x)$ do not approach a specific number.

(c) $f(0)$ exists. The black dot at $(0, 4)$ indicates that $f(0) = 4$.

(d) $\lim_{x\to 0} f(x)$ does not exist. As x approaches 0 from the left, $f(x)$ approaches $\frac{1}{2}$, whereas as x approaches 0 from the right, $f(x)$ approaches 4.

(e) $f(2)$ does not exist. The hollow circle at $\left(2, \frac{1}{2}\right)$ indicates that $f(2)$ is not defined.

(f) $\lim_{x\to 2} f(x)$ exists. As x approaches 2, $f(x)$ approaches $\frac{1}{2}$: $\lim_{x\to 2} f(x) = \frac{1}{2}$.

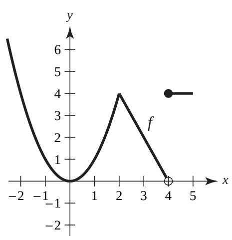
(g) $f(4)$ exists. The black dot at $(4, 2)$ indicates that $f(4) = 2$.

(h) $\lim_{x\to 4} f(x)$ does not exist. As x approaches 4, the values of $f(x)$ do not approach a specific number.

21. $\lim_{x\to c} f(x)$ exists for all $c \neq -3$. In particular, $\lim_{x\to 2} f(x) = 2$.

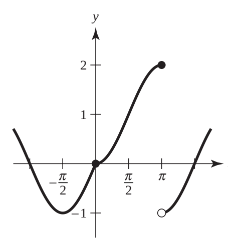
22. $\lim_{x\to c} f(x)$ exists for all $c \neq -2, 0$. In particular, $\lim_{x\to -4} f(x) = 2$.

23.



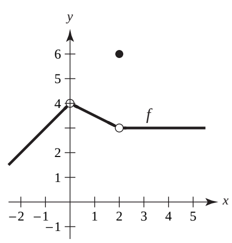
$\lim_{x\to c} f(x)$ exists for all values of $c \neq 4$.

24.

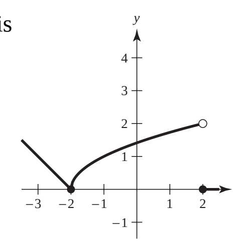


$\lim_{x\to c} f(x)$ exists for all values of $c \neq \pi$.

25. One possible answer is

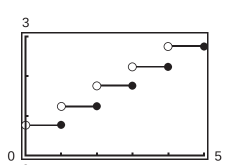


26. One possible answer is



27. $C(t) = 0.75 - 0.50[\![-(t - 1)]\!]$

(a)



(b)

t	3	3.3	3.4	3.5	3.6	3.7	4
C	1.75	2.25	2.25	2.25	2.25	2.25	2.25

$\lim_{t\to 3.5} C(t) = 2.25$

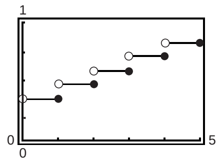
(c)

t	2	2.5	2.9	3	3.1	3.5	4
C	1.25	1.75	1.75	1.75	2.25	2.25	2.25

$\lim_{t\to 3} C(t)$ does not exist. The values of C jump from 1.75 to 2.25 at $t = 3$.

28. $C(t) = 0.35 - 0.12[\![-(t-1)]\!]$

(a)



(b)

t	3	3.3	3.4	3.5	3.6	3.7	4
$C(t)$	0.59	0.71	0.71	0.71	0.71	0.71	0.71

$\lim_{t\to 3.5} C(t) = 0.71$

(c)

t	2	2.5	2.9	3	3.1	3.5	4
$C(t)$	0.47	0.59	0.59	0.59	0.71	0.71	0.71

$\lim_{t\to 3} C(t)$ does not exist. The values of C jump from 0.59 to 0.71 at $t = 3$.

29. We need $|f(x) - 3| = |(x+1) - 3| = |x - 2| < 0.4$. Hence, take $\delta = 0.4$. If $0 < |x-2| < 0.4$, then $|x-2| = |(x+1)-3| = |f(x)-3| < 0.4$, as desired.

30. We need $|f(x) - 1| = \left|\dfrac{1}{x-1} - 1\right| = \left|\dfrac{2-x}{x-1}\right| < 0.01$. Let $\delta = \dfrac{1}{101}$. If $0 < |x-2| < \dfrac{1}{101}$, then

$$-\frac{1}{101} < x - 2 < \frac{1}{101} \Rightarrow 1 - \frac{1}{101} < x - 1 < 1 + \frac{1}{101}$$
$$\Rightarrow \frac{100}{101} < x - 1 < \frac{102}{101}$$
$$\Rightarrow |x-1| > \frac{100}{101}$$

and we have

$$|f(x) - 1| = \left|\frac{1}{x-1} - 1\right| = \left|\frac{2-x}{x-1}\right| < \frac{1/101}{100/101} = \frac{1}{100} = 0.01.$$

31. You need to find δ such that $0 < |x-1| < \delta$ implies $|f(x) - 1| = \left|\dfrac{1}{x} - 1\right| < 0.1$. That is,

$$-0.1 < \frac{1}{x} - 1 < 0.1$$
$$1 - 0.1 < \frac{1}{x} < 1 + 0.1$$
$$\frac{9}{10} < \frac{1}{x} < \frac{11}{10}$$
$$\frac{10}{9} > x > \frac{10}{11}$$
$$\frac{10}{9} - 1 > x - 1 > \frac{10}{11} - 1$$
$$\frac{1}{9} > x - 1 > -\frac{1}{11}.$$

So take $\delta = \dfrac{1}{11}$. Then $0 < |x-1| < \delta$ implies

$$-\frac{1}{11} < x - 1 < \frac{1}{11}$$
$$-\frac{1}{11} < x - 1 < \frac{1}{9}.$$

Using the first series of equivalent inequalities, you obtain

$$|f(x) - 1| = \left|\frac{1}{x} - 1\right| < 0.1.$$

32. You need to find δ such that $0 < |x - 2| < \delta$ implies $|f(x) - 3| = |x^2 - 1 - 3| = |x^2 - 4| < 0.2$. That is,

$$\begin{aligned} -0.2 &< x^2 - 4 < 0.2 \\ 4 - 0.2 &< \quad x^2 \quad < 4 + 0.2 \\ 3.8 &< \quad x^2 \quad < 4.2 \\ \sqrt{3.8} &< \quad x \quad < \sqrt{4.2} \\ \sqrt{3.8} - 2 &< x - 2 < \sqrt{4.2} - 2 \end{aligned}$$

So take $\delta = \sqrt{4.2} - 2 \approx 0.0494$.

Then $0 < |x - 2| < \delta$ implies

$$\begin{aligned} -\left(\sqrt{4.2} - 2\right) &< x - 2 < \sqrt{4.2} - 2 \\ \sqrt{3.8} - 2 &< x - 2 < \sqrt{4.2} - 2. \end{aligned}$$

Using the first series of equivalent inequalities, you obtain

$$|f(x) - 3| = |x^2 - 4| < 0.2.$$

33. $\lim_{x\to 2} (3x + 2) = 8 = L$

$$\begin{aligned} |(3x + 2) - 8| &< 0.01 \\ |3x - 6| &< 0.01 \\ 3|x - 2| &< 0.01 \\ 0 < |x - 2| &< \frac{0.01}{3} \approx 0.0033 = \delta \end{aligned}$$

Hence, if $0 < |x - 2| < \delta = \dfrac{0.01}{3}$, you have

$$\begin{aligned} 3|x - 2| &< 0.01 \\ |3x - 6| &< 0.01 \\ |(3x + 2) - 8| &< 0.01 \\ |f(x) - L| &< 0.01 \end{aligned}$$

34. $\lim_{x\to 4} \left(4 - \dfrac{x}{2}\right) = 2 = L$

$$\begin{aligned} \left|\left(4 - \frac{x}{2}\right) - 2\right| &< 0.01 \\ \left|2 - \frac{x}{2}\right| &< 0.01 \\ \left|-\frac{1}{2}(x - 4)\right| &< 0.01 \\ 0 < |x - 4| &< 0.02 = \delta \end{aligned}$$

Hence, if $0 < |x - 4| < \delta = 0.02$, you have

$$\begin{aligned} \left|-\frac{1}{2}(x - 4)\right| &< 0.01 \\ \left|2 - \frac{x}{2}\right| &< 0.01 \\ \left|\left(4 - \frac{x}{2}\right) - 2\right| &< 0.01 \\ |f(x) - L| &< 0.01 \end{aligned}$$

35. $\lim_{x\to 2} (x^2 - 3) = 1 = L$

$$\begin{aligned} |(x^2 - 3) - 1| &< 0.01 \\ |x^2 - 4| &< 0.01 \\ |(x + 2)(x - 2)| &< 0.01 \\ |x + 2|\,|x - 2| &< 0.01 \\ |x - 2| &< \frac{0.01}{|x + 2|} \end{aligned}$$

If we assume $1 < x < 3$, then $\delta = 0.01/5 = 0.002$.

Hence, if $0 < |x - 2| < \delta = 0.002$, you have

$$\begin{aligned} |x - 2| < 0.002 = \frac{1}{5}(0.01) &< \frac{1}{|x + 2|}(0.01) \\ |x + 2||x - 2| &< 0.01 \\ |x^2 - 4| &< 0.01 \\ |(x^2 - 3) - 1| &< 0.01 \\ |f(x) - L| &< 0.01 \end{aligned}$$

36. $\lim_{x\to 5} (x^2 + 4) = 29 = L$

$$\begin{aligned} |(x^2 + 4) - 29| &< 0.01 \\ |x^2 - 25| &< 0.01 \\ |(x + 5)(x - 5)| &< 0.01 \\ |x - 5| &< \frac{0.01}{|x + 5|} \end{aligned}$$

If we assume $4 < x < 6$, then $\delta = 0.01/11 \approx 0.0009$.

Hence, if $0 < |x - 5| < \delta = \dfrac{0.01}{11}$, you have

$$\begin{aligned} |x - 5| < \frac{0.01}{11} &< \frac{1}{|x + 5|}(0.01) \\ |x - 5||x + 5| &< 0.01 \\ |x^2 - 25| &< 0.01 \\ |(x^2 + 4) - 29| &< 0.01 \\ |f(x) - L| &< 0.01 \end{aligned}$$

37. $\lim_{x\to 2}(x+3) = 5$

Given $\varepsilon > 0$:

$$\begin{aligned} |(x+3)-5| &< \varepsilon \\ |x-2| &< \varepsilon = \delta \end{aligned}$$

Hence, let $\delta = \varepsilon$.

Hence, if $0 < |x-2| < \delta = \varepsilon$, you have

$$\begin{aligned} |x-2| &< \varepsilon \\ |(x+3)-5| &< \varepsilon \\ |f(x)-L| &< \varepsilon. \end{aligned}$$

38. $\lim_{x\to -3}(2x+5) = -1$

Given $\varepsilon > 0$:

$$\begin{aligned} |(2x+5)-(-1)| &< \varepsilon \\ |2x+6| &< \varepsilon \\ 2|x+3| &< \varepsilon \\ |x+3| &< \frac{\varepsilon}{2} = \delta \end{aligned}$$

Hence, let $\delta = \varepsilon/2$.

Hence, if $0 < |x+3| < \delta = \dfrac{\varepsilon}{2}$, you have

$$\begin{aligned} |x+3| &< \frac{\varepsilon}{2} \\ |2x+6| &< \varepsilon \\ |(2x+5)-(-1)| &< \varepsilon \\ |f(x)-L| &< \varepsilon. \end{aligned}$$

39. $\lim_{x\to -4}\left(\frac{1}{2}x - 1\right) = \frac{1}{2}(-4) - 1 = -3$

Given $\varepsilon > 0$:

$$\begin{aligned} \left|\left(\tfrac{1}{2}x-1\right)-(-3)\right| &< \varepsilon \\ \left|\tfrac{1}{2}x+2\right| &< \varepsilon \\ \tfrac{1}{2}|x-(-4)| &< \varepsilon \\ |x-(-4)| &< 2\varepsilon \end{aligned}$$

Hence, let $\delta = 2\varepsilon$.

Hence, if $0 < |x-(-4)| < \delta = 2\varepsilon$, you have

$$\begin{aligned} |x-(-4)| &< 2\varepsilon \\ \left|\tfrac{1}{2}x+2\right| &< \varepsilon \\ \left|\left(\tfrac{1}{2}x-1\right)+3\right| &< \varepsilon \\ |f(x)-L| &< \varepsilon. \end{aligned}$$

40. $\lim_{x\to 1}\left(\frac{2}{3}x + 9\right) = \frac{2}{3}(1) + 9 = \frac{29}{3}$

Given $\varepsilon > 0$:

$$\begin{aligned} \left|\left(\tfrac{2}{3}x+9\right)-\tfrac{29}{3}\right| &< \varepsilon \\ \left|\tfrac{2}{3}x-\tfrac{2}{3}\right| &< \varepsilon \\ \tfrac{2}{3}|x-1| &< \varepsilon \\ |x-1| &< \tfrac{3}{2}\varepsilon \end{aligned}$$

Hence, let $\delta = (3/2)\varepsilon$.

Hence, if $0 < |x-1| < \delta = \frac{3}{2}\varepsilon$, you have

$$\begin{aligned} |x-1| &< \tfrac{3}{2}\varepsilon \\ \left|\tfrac{2}{3}x-\tfrac{2}{3}\right| &< \varepsilon \\ \left|\left(\tfrac{2}{3}x+9\right)-\tfrac{29}{3}\right| &< \varepsilon \\ |f(x)-L| &< \varepsilon. \end{aligned}$$

41. $\lim_{x\to 6} 3 = 3$

Given $\varepsilon > 0$:

$$\begin{aligned} |3-3| &< \varepsilon \\ 0 &< \epsilon \end{aligned}$$

Hence, any $\delta > 0$ will work.

Hence, for any $\delta > 0$, you have

$$\begin{aligned} |3-3| &< \varepsilon \\ |f(x)-L| &< \varepsilon. \end{aligned}$$

42. $\lim_{x\to 2}(-1) = -1$

Given $\varepsilon > 0$: $|-1-(-1)| < \varepsilon$

$$0 < \varepsilon$$

Hence, any $\delta > 0$ will work.

Hence, for any $\delta > 0$, you have

$$\begin{aligned} |(-1)-(-1)| &< \varepsilon \\ |f(x)-L| &< \varepsilon. \end{aligned}$$

43. $\lim_{x\to 0} \sqrt[3]{x} = 0$

Given $\varepsilon > 0$:
$$\begin{aligned}\left|\sqrt[3]{x} - 0\right| &< \varepsilon\\ \left|\sqrt[3]{x}\right| &< \varepsilon\\ |x| &< \varepsilon^3 = \delta\end{aligned}$$

Hence, let $\delta = \varepsilon^3$.

Hence for $0|x - 0|\delta = \varepsilon^3$, you have
$$\begin{aligned}|x| &< \varepsilon^3\\ \left|\sqrt[3]{x}\right| &< \varepsilon\\ \left|\sqrt[3]{x} - 0\right| &< \varepsilon\\ |f(x) - L| &< \varepsilon.\end{aligned}$$

44. $\lim_{x\to 4} \sqrt{x} = \sqrt{4} = 2$

Given $\varepsilon > 0$:
$$\begin{aligned}\left|\sqrt{x} - 2\right| &< \varepsilon\\ \left|\sqrt{x} - 2\right|\left|\sqrt{x} + 2\right| &< \varepsilon\left|\sqrt{x} + 2\right|\\ |x - 4| &< \varepsilon\left|\sqrt{x} + 2\right|\end{aligned}$$

Assuming $1 < x < 9$, you can choose $\delta = 3\varepsilon$. Then,
$$\begin{aligned}0 < |x - 4| < \delta = 3\varepsilon \implies |x - 4| &< \varepsilon\left|\sqrt{x} + 2\right|\\ \implies \left|\sqrt{x} - 2\right| &< \varepsilon.\end{aligned}$$

45. $\lim_{x\to -2} |x - 2| = |(-2) - 2| = 4$

Given $\varepsilon > 0$:
$$\begin{aligned}\big||x - 2| - 4\big| &< \varepsilon\\ |-(x - 2) - 4| &< \varepsilon \quad (x - 2 < 0)\\ |-x - 2| = |x + 2| = |x - (-2)| &< \varepsilon\end{aligned}$$

Hence, let $\delta = \varepsilon$.

Hence for $0 < |x - (-2)| < \delta = \varepsilon$, you have
$$\begin{aligned}|x + 2| &< \varepsilon\\ |-(x + 2)| &< \varepsilon\\ |-(x - 2) - 4| &< \varepsilon\\ \big||x - 2| - 4\big| &< \varepsilon \quad (\text{because } x - 2 < 0)\\ |f(x) - L| &< \varepsilon.\end{aligned}$$

46. $\lim_{x\to 3} |x - 3| = 0$

Given $\varepsilon > 0$:
$$\begin{aligned}\big||x - 3| - 0\big| &< \varepsilon\\ |x - 3| &< \varepsilon = \delta\end{aligned}$$

Hence, let $\delta = \varepsilon$.

Hence for $0 < |x - 3| < \delta = \varepsilon$, you have
$$\begin{aligned}|x - 3| &< \varepsilon\\ \big||x - 3| - 0\big| &< \varepsilon\\ |f(x) - L| &< \varepsilon.\end{aligned}$$

47. $\lim_{x\to 1} (x^2 + 1) = 2$

Given $\varepsilon > 0$:
$$\begin{aligned}|(x^2 + 1) - 2| &< \varepsilon\\ |x^2 - 1| &< \varepsilon\\ |(x + 1)(x - 1)| &< \varepsilon\\ |x - 1| &< \frac{\varepsilon}{|x + 1|}\end{aligned}$$

If we assume $0 < x < 2$, then $\delta = \varepsilon/3$.

Hence for $0 < |x - 1| < \delta = \dfrac{\varepsilon}{3}$, you have
$$\begin{aligned}|x - 1| < \frac{1}{3}\varepsilon &< \frac{1}{|x + 1|}\varepsilon\\ |x^2 - 1| &< \varepsilon\\ |(x^2 + 1) - 2| &< \varepsilon\\ |f(x) - 2| &< \varepsilon.\end{aligned}$$

48. $\lim_{x\to -3} (x^2 + 3x) = 0$

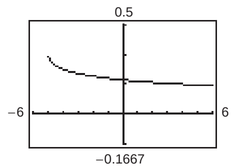
Given $\varepsilon > 0$:
$$\begin{aligned}|(x^2 + 3x) - 0| &< \varepsilon\\ |x(x + 3)| &< \varepsilon\\ |x + 3| &< \frac{\varepsilon}{|x|}\end{aligned}$$

If we assume $-4 < x < -2$, then $\delta = \varepsilon/4$.

Hence for $0 < |x - (-3)| < \delta = \dfrac{\varepsilon}{4}$, you have
$$\begin{aligned}|x + 3| < \frac{1}{4}\varepsilon &< \frac{1}{|x|}\varepsilon\\ |x(x + 3)| &< \varepsilon\\ |x^2 + 3x - 0| &< \varepsilon\\ |f(x) - L| &< \varepsilon.\end{aligned}$$

49. $f(x) = \dfrac{\sqrt{x+5} - 3}{x - 4}$

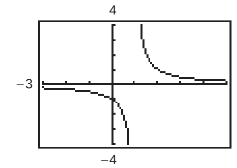
$\lim\limits_{x \to 4} f(x) = \dfrac{1}{6}$



The domain is $[-5, 4) \cup (4, \infty)$. The graphing utility does not show the hole at $\left(4, \frac{1}{6}\right)$.

50. $f(x) = \dfrac{x - 3}{x^2 - 4x + 3}$

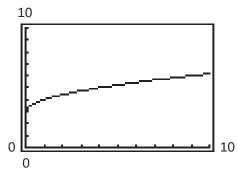
$\lim\limits_{x \to 3} f(x) = \dfrac{1}{2}$



The domain is all $x \neq 1, 3$. The graphing utility does not show the hole at $\left(3, \frac{1}{2}\right)$.

51. $f(x) = \dfrac{x - 9}{\sqrt{x} - 3}$

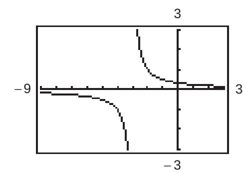
$\lim\limits_{x \to 9} f(x) = 6$



The domain is all $x \geq 0$ except $x = 9$. The graphing utility does not show the hole at $(9, 6)$.

52. $f(x) = \dfrac{x - 3}{x^2 - 9}$

$\lim\limits_{x \to 3} f(x) = \dfrac{1}{6}$



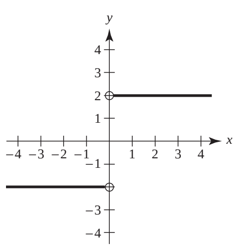
The domain is all $x \neq \pm 3$. The graphing utility does not show the hole at $\left(3, \frac{1}{6}\right)$.

53. $\lim\limits_{x \to 8} f(x) = 25$ means that the values of f approach 25 as x gets closer and closer to 8.

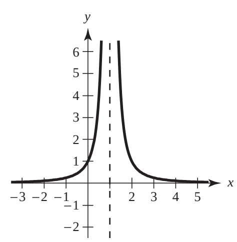
54. No. The fact that $f(2) = 4$ has no bearing on the existence of the limit of $f(x)$ as x approaches 2.

55. No. The fact that $\lim\limits_{x \to 2} f(x) = 4$ has no bearing on the value of f at 2.

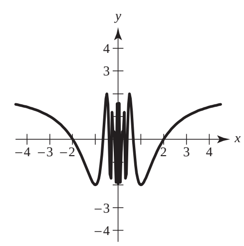
56. (i) The values of f approach different numbers as x approaches c from different sides of c:



(ii) The values of f increase without bound as x approaches c:



(iii) The values of f oscillate between two fixed numbers as x approaches c:



57. (a) $C = 2\pi r$

$$r = \frac{C}{2\pi} = \frac{6}{2\pi} = \frac{3}{\pi} \approx 0.9549 \text{ cm}$$

(b) If $C = 5.5$, $r = \dfrac{5.5}{2\pi} \approx 0.87535$ cm

If $C = 6.5$, $r = \dfrac{6.5}{2\pi} \approx 1.03451$ cm

Thus $0.87535 < r < 1.03451$

(c) $\lim\limits_{r \to 3/\pi} (2\pi r) = 6$; $\varepsilon = 0.5$; $\delta \approx 0.0796$

58. $V = \dfrac{4}{3}\pi r^3$, $V = 2.48$

(a) $2.48 = \dfrac{4}{3}\pi r^3$

$$r^3 = \frac{1.86}{\pi}$$

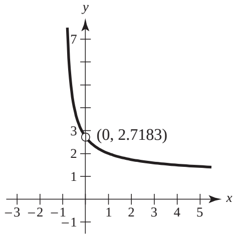
$$r \approx 0.8397 \text{ in.}$$

(b)
$$2.45 \leq V \leq 2.51$$
$$2.45 \leq \frac{4}{3}\pi r^3 \leq 2.51$$
$$0.5849 \leq r^3 \leq 0.5992$$
$$0.8363 \leq r \leq 0.8431$$

(c) For $\varepsilon = 2.51 - 2.48 = 0.03$, $\delta \approx 0.003$

59. $f(x) = (1 + x)^{1/x}$

$\lim_{x\to 0} (1 + x)^{1/x} = e \approx 2.71828$



x	$f(x)$
-0.1	2.867972
-0.01	2.731999
-0.001	2.719642
-0.0001	2.718418
-0.00001	2.718295
-0.000001	2.718283

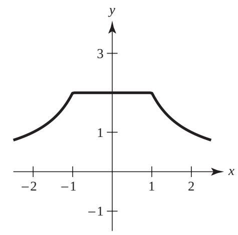
x	$f(x)$
0.1	2.593742
0.01	2.704814
0.001	2.716942
0.0001	2.718146
0.00001	2.718268
0.000001	2.718280

60. $f(x) = \dfrac{|x + 1| - |x - 1|}{x}$

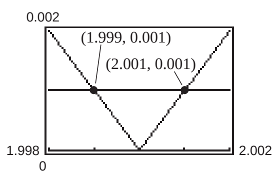
x	-1	-0.5	-0.1	0	0.1	0.5	1.0
$f(x)$	2	2	2	Undef.	2	2	2

$\lim_{x\to 0} f(x) = 2$

Note that for $-1 < x < 1$, $x \neq 0$, $f(x) = \dfrac{(x + 1) + (x - 1)}{x} = 2$.



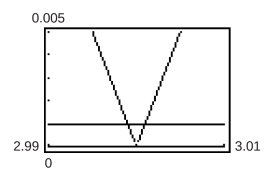
61.



Using the zoom and trace feature, $\delta = 0.001$. That is, for

$$0 < |x - 2| < 0.001, \left|\frac{x^2 - 4}{x - 2} - 4\right| < 0.001.$$

62.



From the graph, $\delta = 0.001$. Thus $(3 - \delta, 3 + \delta) = (2.999, 3.001)$.

Note: $\dfrac{x^2 - 3x}{x - 3} = x$ for $x \neq 3$.

63. False; $f(x) = (\sin x)/x$ is undefined when $x = 0$. From Exercise 7, we have

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

64. True

65. False; let

$$f(x) = \begin{cases} x^2 - 4x, & x \neq 4 \\ 10, & x = 4 \end{cases}.$$

$f(4) = 10$

$\lim_{x\to 4} f(x) = \lim_{x\to 4} (x^2 - 4x) = 0 \neq 10$

66. False; let

$$f(x) = \begin{cases} x^2 - 4x, & x \neq 4 \\ 10, & x = 4 \end{cases}.$$

$\lim_{x\to 4} f(x) = \lim_{x\to 4} (x^2 - 4x) = 0$ and $f(4) = 10 \neq 0$

67. $f(x) = \sqrt{x}$

(a) $\lim_{x\to 0.25} \sqrt{x} = 0.5$ is true.

As x approaches $0.25 = \frac{1}{4}$, $f(x) = \sqrt{x}$ approaches $\frac{1}{2} = 0.5$.

(b) $\lim_{x\to 0} \sqrt{x} = 0$ is false.

$f(x) = \sqrt{x}$ is not defined on an open interval containing 0 because the domain of f is $x \geq 0$.

68. The value of f at c has no bearing on the limit as x approaches c.

69. If $\lim_{x\to c} f(x) = L_1$ and $\lim_{x\to c} f(x) = L_2$, then for every $\varepsilon > 0$, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that $|x - c| < \delta_1 \implies |f(x) - L_1| < \varepsilon$ and $|x - c| < \delta_2 \implies |f(x) - L_2| < \varepsilon$. Let δ equal the smaller of δ_1 and δ_2. Then for $|x - c| < \delta$, we have $|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \le |L_1 - f(x)| + |f(x) - L_2| < \varepsilon + \varepsilon$. Therefore, $|L_1 - L_2| < 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that $L_1 = L_2$.

70. $f(x) = mx + b$, $m \ne 0$. Let $\varepsilon > 0$ be given. Take $\delta = \dfrac{\varepsilon}{|m|}$.

If $0 < |x - c| < \delta = \dfrac{\varepsilon}{|m|}$, then

$$|m||x - c| < \varepsilon$$
$$|mx - mc| < \varepsilon$$
$$|(mx + b) - (mc + b)| < \varepsilon$$

which shows that $\lim_{x\to c} (mx + b) = mc + b$.

71. $\lim_{x\to c} [f(x) - L] = 0$ means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if

$$0 < |x - c| < \delta,$$

then

$$|(f(x) - L) - 0| < \varepsilon.$$

This means the same as $|f(x) - L| < \varepsilon$ when

$$0 < |x - c| < \delta.$$

Thus, $\lim_{x\to c} f(x) = L$.

72. (a)
$$\begin{aligned}(3x + 1)(3x - 1)x^2 + 0.01 &= (9x^2 - 1)x^2 + \frac{1}{100}\\ &= 9x^4 - x^2 + \frac{1}{100}\\ &= \frac{1}{100}(10x^2 - 1)(90x^2 - 1)\end{aligned}$$

Thus, $(3x + 1)(3x - 1)x^2 + 0.01 > 0$ if $10x^2 - 1 < 0$ and $90x^2 - 1 < 0$.

Let $(a, b) = \left(-\dfrac{1}{\sqrt{90}}, \dfrac{1}{\sqrt{90}}\right)$.

For all $x \ne 0$ in (a, b), the graph is positive. You can verify this with a graphing utility.

(b) We are given $\lim_{x\to c} g(x) = L > 0$. Let $\varepsilon = \frac{1}{2}L$. There exists $\delta > 0$ such that $0 < |x - c| < \delta$ implies that $|g(x) - L| < \varepsilon = \dfrac{L}{2}$. That is,

$$-\frac{L}{2} < g(x) - L < \frac{L}{2}$$
$$\frac{L}{2} < \quad g(x) \quad < \frac{3L}{2}$$

For x in the interval $(c - \delta, c + \delta)$, $x \ne c$, we have $g(x) > \dfrac{L}{2} > 0$, as desired.

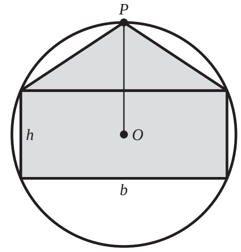
73. Answers will vary.

74. $\lim_{x\to 4} \dfrac{x^2 - x - 12}{x - 4} = 7$

n	$4 + [0.1]^n$	$f(4 + [0.1]^n)$
1	4.1	7.1
2	4.01	7.01
3	4.001	7.001
4	4.0001	7.0001

n	$4 - [0.1]^n$	$f(4 - [0.1]^n)$
1	3.9	6.9
2	3.99	6.99
3	3.999	6.999
4	3.9999	6.9999

75. The radius OP has a length equal to the altitude z of the triangle plus $\frac{h}{2}$. Thus, $z = 1 - \frac{h}{2}$.



$$\text{Area triangle} = \frac{1}{2}b\left(1 - \frac{h}{2}\right)$$

$$\text{Area rectangle} = bh$$

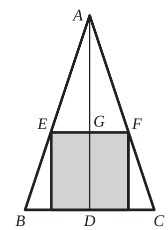
Since these are equal, $\frac{1}{2}b\left(1 - \frac{h}{2}\right) = bh$

$$1 - \frac{h}{2} = 2h$$

$$\frac{5}{2}h = 1$$

$$h = \frac{2}{5}$$

76. Consider a cross section of the cone, where EF is a diagonal of the inscribed cube. $AD = 3$, $BC = 2$. Let x be the length of a side of the cube. Then $EF = x\sqrt{2}$.



By similar triangles,

$$\frac{EF}{BC} = \frac{AG}{AD}$$

$$\frac{x\sqrt{2}}{2} = \frac{3 - x}{3}$$

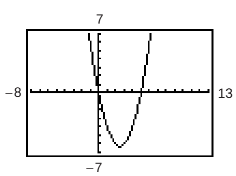
Solving for x, $3\sqrt{2}x = 6 - 2x$

$$\left(3\sqrt{2} + 2\right)x = 6$$

$$x = \frac{6}{3\sqrt{2} + 2} = \frac{9\sqrt{2} - 6}{7} \approx 0.96.$$

Section 1.3 Evaluating Limits Analytically

1.

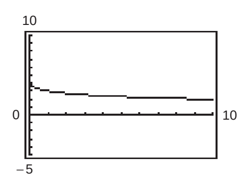


$h(x) = x^2 - 5x$

(a) $\lim_{x\to 5} h(x) = 0$

(b) $\lim_{x\to -1} h(x) = 6$

2.

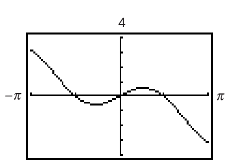


$g(x) = \dfrac{12\left(\sqrt{x} - 3\right)}{x - 9}$

(a) $\lim_{x\to 4} g(x) = 2.4$

(b) $\lim_{x\to 0} g(x) = 4$

3.



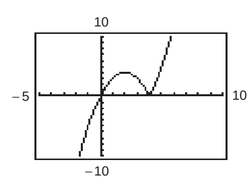
$f(x) = x \cos x$

(a) $\lim_{x\to 0} f(x) = 0$

(b) $\lim_{x\to \pi/3} f(x) \approx 0.524$

$\left(= \frac{\pi}{6}\right)$

4.



$f(t) = t|t - 4|$

(a) $\lim_{t\to 4} f(t) = 0$

(b) $\lim_{t\to -1} f(t) = -5$

5. $\lim_{x\to 2} x^4 = 2^4 = 16$

6. $\lim_{x\to -2} x^3 = (-2)^3 = -8$

7. $\lim_{x\to 0} (2x - 1) = 2(0) - 1 = -1$

8. $\lim_{x\to -3} (3x + 2) = 3(-3) + 2 = -7$

9. $\lim_{x\to -3} (x^2 + 3x) = (-3)^2 + 3(-3) = 9 - 9 = 0$

10. $\lim_{x\to 1} (-x^2 + 1) = -(1)^2 + 1 = 0$

11. $\lim_{x\to -3} (2x^2 + 4x + 1) = 2(-3)^2 + 4(-3) + 1 = 18 - 12 + 1 = 7$

12. $\lim_{x\to 1} (3x^3 - 2x^2 + 4) = 3(1)^3 - 2(1)^2 + 4 = 5$

13. $\lim_{x\to 2} \frac{1}{x} = \frac{1}{2}$

14. $\lim_{x\to -3} \frac{2}{x + 2} = \frac{2}{-3 + 2} = -2$

15. $\lim_{x\to 1} \frac{x - 3}{x^2 + 4} = \frac{1 - 3}{1^2 + 4} = \frac{-2}{5} = -\frac{2}{5}$

16. $\lim_{x\to 3} \frac{2x - 3}{x + 5} = \frac{2(3) - 3}{3 + 5} = \frac{3}{8}$

17. $\lim_{x\to 7} \frac{5x}{\sqrt{x + 2}} = \frac{5(7)}{\sqrt{7 + 2}} = \frac{35}{\sqrt{9}} = \frac{35}{3}$

18. $\lim_{x\to 3} \frac{\sqrt{x + 1}}{x - 4} = \frac{\sqrt{3 + 1}}{3 - 4} = -2$

19. $\lim_{x\to 3} \sqrt{x + 1} = \sqrt{3 + 1} = 2$

20. $\lim_{x\to 4} \sqrt[3]{x + 4} = \sqrt[3]{4 + 4} = 2$

21. $\lim_{x\to -4} (x + 3)^2 = (-4 + 3)^2 = 1$

22. $\lim_{x\to 0} (2x - 1)^3 = [2(0) - 1]^3 = -1$

23. (a) $\lim_{x\to 1} f(x) = 5 - 1 = 4$
(b) $\lim_{x\to 4} g(x) = 4^3 = 64$
(c) $\lim_{x\to 1} g(f(x)) = g(f(1)) = g(4) = 64$

24. (a) $\lim_{x\to -3} f(x) = (-3) + 7 = 4$
(b) $\lim_{x\to 4} g(x) = 4^2 = 16$
(c) $\lim_{x\to -3} g(f(x)) = g(4) = 16$

25. (a) $\lim_{x\to 1} f(x) = 4 - 1 = 3$
(b) $\lim_{x\to 3} g(x) = \sqrt{3 + 1} = 2$
(c) $\lim_{x\to 1} g(f(x)) = g(3) = 2$

26. (a) $\lim_{x\to 4} f(x) = 2(4^2) - 3(4) + 1 = 21$
(b) $\lim_{x\to 21} g(x) = \sqrt[3]{21 + 6} = 3$
(c) $\lim_{x\to 4} g(f(x)) = g(21) = 3$

27. $\lim_{x\to \pi/2} \sin x = \sin \frac{\pi}{2} = 1$

28. $\lim_{x\to \pi} \tan x = \tan \pi = 0$

29. $\lim_{x\to 2} \cos \frac{\pi x}{3} = \cos \frac{2\pi}{3} = -\frac{1}{2}$

30. $\lim_{x\to 1} \sin \frac{\pi x}{2} = \sin \frac{\pi}{2} = 1$

31. $\lim_{x\to 0} \sec 2x = \sec 0 = 1$

32. $\lim_{x\to \pi} \cos 3x = \cos 3\pi = -1$

33. $\lim_{x\to 5\pi/6} \sin x = \sin \frac{5\pi}{6} = \frac{1}{2}$

34. $\lim_{x\to 5\pi/3} \cos x = \cos \frac{5\pi}{3} = \frac{1}{2}$

35. $\lim_{x\to 3} \tan\left(\frac{\pi x}{4}\right) = \tan \frac{3\pi}{4} = -1$

36. $\lim_{x\to 7} \sec\left(\frac{\pi x}{6}\right) = \sec \frac{7\pi}{6} = \frac{-2\sqrt{3}}{3}$

37. (a) $\lim_{x\to c}[5g(x)] = 5\lim_{x\to c} g(x) = 5(3) = 15$

(b) $\lim_{x\to c}[f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = 2 + 3 = 5$

(c) $\lim_{x\to c}[f(x)g(x)] = \left[\lim_{x\to c} f(x)\right]\left[\lim_{x\to c} g(x)\right] = (2)(3) = 6$

(d) $\lim_{x\to c}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \dfrac{2}{3}$

38. (a) $\lim_{x\to c}[4f(x)] = 4\lim_{x\to c} f(x) = 4\left(\dfrac{3}{2}\right) = 6$

(b) $\lim_{x\to c}[f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = \dfrac{3}{2} + \dfrac{1}{2} = 2$

(c) $\lim_{x\to c}[f(x)g(x)] = \left[\lim_{x\to c} f(x)\right]\left[\lim_{x\to c} g(x)\right] = \left(\dfrac{3}{2}\right)\left(\dfrac{1}{2}\right) = \dfrac{3}{4}$

(d) $\lim_{x\to c}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \dfrac{3/2}{1/2} = 3$

39. (a) $\lim_{x\to c}[f(x)]^3 = \left[\lim_{x\to c} f(x)\right]^3 = (4)^3 = 64$

(b) $\lim_{x\to c}\sqrt{f(x)} = \sqrt{\lim_{x\to c} f(x)} = \sqrt{4} = 2$

(c) $\lim_{x\to c}[3f(x)] = 3\lim_{x\to c} f(x) = 3(4) = 12$

(d) $\lim_{x\to c}[f(x)]^{3/2} = \left[\lim_{x\to c} f(x)\right]^{3/2} = (4)^{3/2} = 8$

40. (a) $\lim_{x\to c}\sqrt[3]{f(x)} = \sqrt[3]{\lim_{x\to c} f(x)} = \sqrt[3]{27} = 3$

(b) $\lim_{x\to c}\dfrac{f(x)}{18} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} 18} = \dfrac{27}{18} = \dfrac{3}{2}$

(c) $\lim_{x\to c}[f(x)]^2 = \left[\lim_{x\to c} f(x)\right]^2 = (27)^2 = 729$

(d) $\lim_{x\to c}[f(x)]^{2/3} = \left[\lim_{x\to c} f(x)\right]^{2/3} = (27)^{2/3} = 9$

41. $f(x) = -2x + 1$ and $g(x) = \dfrac{-2x^2 + x}{x}$ agree except at $x = 0$.

(a) $\lim_{x\to 0} g(x) = \lim_{x\to 0} f(x) = 1$

(b) $\lim_{x\to -1} g(x) = \lim_{x\to -1} f(x) = 3$

42. $f(x) = x - 3$ and $h(x) = \dfrac{x^2 - 3x}{x}$ agree except at $x = 0$.

(a) $\lim_{x\to -2} h(x) = \lim_{x\to -2} f(x) = -5$

(b) $\lim_{x\to 0} h(x) = \lim_{x\to 0} f(x) = -3$

43. $f(x) = x(x + 1)$ and $g(x) = \dfrac{x^3 - x}{x - 1}$ agree except at $x = 1$.

(a) $\lim_{x\to 1} g(x) = \lim_{x\to 1} f(x) = 2$

(b) $\lim_{x\to -1} g(x) = \lim_{x\to -1} f(x) = 0$

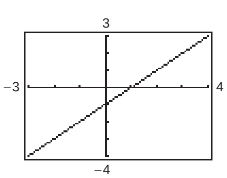
44. $g(x) = \dfrac{1}{x - 1}$ and $f(x) = \dfrac{x}{x^2 - x}$ agree except at $x = 0$.

(a) $\lim_{x\to 1} f(x)$ does not exist.

(b) $\lim_{x\to 0} f(x) = -1$

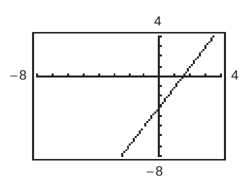
45. $f(x) = \dfrac{x^2 - 1}{x + 1}$ and $g(x) = x - 1$ agree except at $x = -1$.

$$\lim_{x\to -1} f(x) = \lim_{x\to -1} g(x) = -2$$



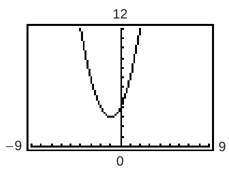
46. $f(x) = \dfrac{2x^2 - x - 3}{x + 1}$ and $g(x) = 2x - 3$ agree except at $x = -1$.

$$\lim_{x\to -1} f(x) = \lim_{x\to -1} g(x) = -5$$



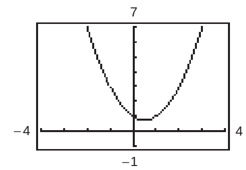
47. $f(x) = \dfrac{x^3 - 8}{x - 2}$ and $g(x) = x^2 + 2x + 4$ agree except at $x = 2$.

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} g(x) = 12$$



48. $f(x) = \dfrac{x^3 + 1}{x + 1}$ and $g(x) = x^2 - x + 1$ agree except at $x = -1$.

$$\lim_{x \to -1} f(x) = \lim_{x \to -1} g(x) = 3$$



49. $$\lim_{x \to 5} \frac{x - 5}{x^2 - 25} = \lim_{x \to 5} \frac{x - 5}{(x + 5)(x - 5)} = \lim_{x \to 5} \frac{1}{x + 5} = \frac{1}{10}$$

50. $$\lim_{x \to 2} \frac{2 - x}{x^2 - 4} = \lim_{x \to 2} \frac{-(x - 2)}{(x - 2)(x + 2)} = \lim_{x \to 2} \frac{-1}{x + 2} = -\frac{1}{4}$$

51. $$\lim_{x \to -3} \frac{x^2 + x - 6}{x^2 - 9} = \lim_{x \to -3} \frac{(x + 3)(x - 2)}{(x + 3)(x - 3)} = \lim_{x \to -3} \frac{x - 2}{x - 3} = \frac{-5}{-6} = \frac{5}{6}$$

52. $$\lim_{x \to 4} \frac{x^2 - 5x + 4}{x^2 - 2x - 8} = \lim_{x \to 4} \frac{(x - 4)(x - 1)}{(x - 4)(x + 2)} = \lim_{x \to 4} \frac{(x - 1)}{(x + 2)} = \frac{3}{6} = \frac{1}{2}$$

53. $$\lim_{x \to 0} \frac{\sqrt{x + 5} - \sqrt{5}}{x} = \lim_{x \to 0} \frac{\sqrt{x + 5} - \sqrt{5}}{x} \cdot \frac{\sqrt{x + 5} + \sqrt{5}}{\sqrt{x + 5} + \sqrt{5}}$$
$$= \lim_{x \to 0} \frac{(x + 5) - 5}{x\left(\sqrt{x + 5} + \sqrt{5}\right)} = \lim_{x \to 0} \frac{1}{\sqrt{x + 5} + \sqrt{5}} = \frac{1}{2\sqrt{5}} = \frac{\sqrt{5}}{10}$$

54. $$\lim_{x \to 0} \frac{\sqrt{2 + x} - \sqrt{2}}{x} = \lim_{x \to 0} \frac{\sqrt{2 + x} - \sqrt{2}}{x} \cdot \frac{\sqrt{2 + x} + \sqrt{2}}{\sqrt{2 + x} + \sqrt{2}}$$
$$= \lim_{x \to 0} \frac{2 + x - 2}{\left(\sqrt{2 + x} + \sqrt{2}\right)x} = \lim_{x \to 0} \frac{1}{\sqrt{2 + x} + \sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

55. $$\lim_{x \to 4} \frac{\sqrt{x + 5} - 3}{x - 4} = \lim_{x \to 4} \frac{\sqrt{x + 5} - 3}{x - 4} \cdot \frac{\sqrt{x + 5} + 3}{\sqrt{x + 5} + 3}$$
$$= \lim_{x \to 4} \frac{(x + 5) - 9}{(x - 4)\left(\sqrt{x + 5} + 3\right)} = \lim_{x \to 4} \frac{1}{\sqrt{x + 5} + 3} = \frac{1}{\sqrt{9} + 3} = \frac{1}{6}$$

56. $$\lim_{x \to 3} \frac{\sqrt{x + 1} - 2}{x - 3} = \lim_{x \to 3} \frac{\sqrt{x + 1} - 2}{x - 3} \cdot \frac{\sqrt{x + 1} + 2}{\sqrt{x + 1} + 2} = \lim_{x \to 3} \frac{x - 3}{(x - 3)\left[\sqrt{x + 1} + 2\right]} = \lim_{x \to 3} \frac{1}{\sqrt{x + 1} + 2} = \frac{1}{4}$$

57. $$\lim_{x \to 0} \frac{\frac{1}{3 + x} - \frac{1}{3}}{x} = \lim_{x \to 0} \frac{3 - (3 + x)}{(3 + x)3(x)} = \lim_{x \to 0} \frac{-x}{(3 + x)(3)(x)} = \lim_{x \to 0} \frac{-1}{(3 + x)3} = -\frac{1}{9}$$

58. $$\lim_{x \to 0} \frac{\frac{1}{x + 4} - \frac{1}{4}}{x} = \lim_{x \to 0} \frac{\frac{4 - (x + 4)}{4(x + 4)}}{x}$$
$$= \lim_{x \to 0} \frac{-1}{4(x + 4)} = -\frac{1}{16}$$

59. $$\lim_{\Delta x \to 0} \frac{2(x + \Delta x) - 2x}{\Delta x} = \lim_{\Delta x \to 0} \frac{2x + 2\Delta x - 2x}{\Delta x}$$
$$= \lim_{\Delta x \to 0} 2 = 2$$

60. $\displaystyle\lim_{\Delta x\to 0}\frac{(x+\Delta x)^2-x^2}{\Delta x}=\lim_{\Delta x\to 0}\frac{x^2+2x\Delta x+(\Delta x)^2-x^2}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta x(2x+\Delta x)}{\Delta x}=\lim_{\Delta x\to 0}(2x+\Delta x)=2x$

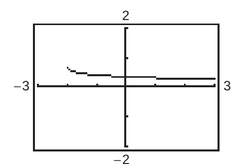
61. $\displaystyle\lim_{\Delta x\to 0}\frac{(x+\Delta x)^2-2(x+\Delta x)+1-(x^2-2x+1)}{\Delta x}=\lim_{\Delta x\to 0}\frac{x^2+2x\Delta x+(\Delta x)^2-2x-2\Delta x+1-x^2+2x-1}{\Delta x}$

$$=\lim_{\Delta x\to 0}(2x+\Delta x-2)=2x-2$$

62. $\displaystyle\lim_{\Delta x\to 0}\frac{(x+\Delta x)^3-x^3}{\Delta x}=\lim_{\Delta x\to 0}\frac{x^3+3x^2\Delta x+3x(\Delta x)^2+(\Delta x)^3-x^3}{\Delta x}$

$$=\lim_{\Delta x\to 0}\frac{\Delta x(3x^2+3x\Delta x+(\Delta x)^2)}{\Delta x}=\lim_{\Delta x\to 0}(3x^2+3x\Delta x+(\Delta x)^2)=3x^2$$

63. $\displaystyle\lim_{x\to 0}\frac{\sqrt{x+2}-\sqrt{2}}{x}\approx 0.354$

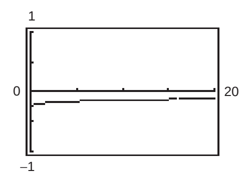


x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	0.358	0.354	0.354	?	0.354	0.353	0.349

Analytically, $\displaystyle\lim_{x\to 0}\frac{\sqrt{x+2}-\sqrt{2}}{x}=\lim_{x\to 0}\frac{\sqrt{x+2}-\sqrt{2}}{x}\cdot\frac{\sqrt{x+2}+\sqrt{2}}{\sqrt{x+2}+\sqrt{2}}$

$$=\lim_{x\to 0}\frac{x+2-2}{x\left(\sqrt{x+2}+\sqrt{2}\right)}=\lim_{x\to 0}\frac{1}{\sqrt{x+2}+\sqrt{2}}=\frac{1}{2\sqrt{2}}=\frac{\sqrt{2}}{4}\approx 0.354.$$

64. $f(x)=\dfrac{4-\sqrt{x}}{x-16}$



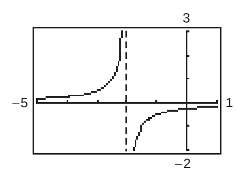
x	15.9	15.99	15.999	16	16.001	16.01	16.1
$f(x)$	-0.1252	-0.125	-0.125	?	-0.125	-0.125	-0.1248

It appears that the limit is -0.125.

Analytically, $\displaystyle\lim_{x\to 16}\frac{4-\sqrt{x}}{x-16}=\lim_{x\to 16}\frac{\left(4-\sqrt{x}\right)}{\left(\sqrt{x}+4\right)\left(\sqrt{x}-4\right)}$

$$=\lim_{x\to 16}\frac{-1}{\sqrt{x}+4}=-\frac{1}{8}.$$

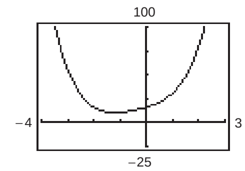
65. $\displaystyle\lim_{x\to 0}\frac{\frac{1}{2+x}-\frac{1}{2}}{x}=-\frac{1}{4}$



x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	-0.263	-0.251	-0.250	?	-0.250	-0.249	-0.238

Analytically, $\displaystyle\lim_{x\to 0}\frac{\frac{1}{2+x}-\frac{1}{2}}{x}=\lim_{x\to 0}\frac{2-(2+x)}{2(2+x)}\cdot\frac{1}{x}=\lim_{x\to 0}\frac{-x}{2(2+x)}\cdot\frac{1}{x}=\lim_{x\to 0}\frac{-1}{2(2+x)}=-\frac{1}{4}.$

66. $\lim_{x\to 2} \dfrac{x^5 - 32}{x - 2} = 80$



x	1.9	1.99	1.999	1.9999	2.0	2.0001	2.001	2.01	2.1
$f(x)$	72.39	79.20	79.92	79.99	?	80.01	80.08	80.80	88.41

Analytically, $\lim_{x\to 2} \dfrac{x^5 - 32}{x - 2} = \lim_{x\to 2} \dfrac{(x-2)(x^4 + 2x^3 + 4x^2 + 8x + 16)}{x - 2}$

$$= \lim_{x\to 2} (x^4 + 2x^3 + 4x^2 + 8x + 16) = 80.$$

(*Hint:* Use long division to factor $x^5 - 32$.)

67. $\lim_{x\to 0} \dfrac{\sin x}{5x} = \lim_{x\to 0} \left[\left(\dfrac{\sin x}{x}\right)\left(\dfrac{1}{5}\right)\right] = (1)\left(\dfrac{1}{5}\right) = \dfrac{1}{5}$

68. $\lim_{x\to 0} \dfrac{3(1 - \cos x)}{x} = \lim_{x\to 0} \left[3\left(\dfrac{1 - \cos x}{x}\right)\right] = (3)(0) = 0$

69. $\lim_{x\to 0} \dfrac{\sin x(1 - \cos x)}{2x^2} = \lim_{x\to 0} \left[\dfrac{1}{2} \cdot \dfrac{\sin x}{x} \cdot \dfrac{1 - \cos x}{x}\right]$

$$= \frac{1}{2}(1)(0) = 0$$

70. $\lim_{\theta\to 0} \dfrac{\cos \theta \tan \theta}{\theta} = \lim_{\theta\to 0} \dfrac{\sin \theta}{\theta} = 1$

71. $\lim_{x\to 0} \dfrac{\sin^2 x}{x} = \lim_{x\to 0} \left[\dfrac{\sin x}{x} \sin x\right] = (1) \sin 0 = 0$

72. $\lim_{x\to 0} \dfrac{\tan^2 x}{x} = \lim_{x\to 0} \dfrac{\sin^2 x}{x \cos^2 x} = \lim_{x\to 0} \left[\dfrac{\sin x}{x} \cdot \dfrac{\sin x}{\cos^2 x}\right]$

$$= (1)(0) = 0$$

73. $\lim_{h\to 0} \dfrac{(1 - \cos h)^2}{h} = \lim_{h\to 0} \left[\dfrac{1 - \cos h}{h}(1 - \cos h)\right]$

$$= (0)(0) = 0$$

74. $\lim_{\phi\to \pi} \phi \sec \phi = \pi(-1) = -\pi$

75. $\lim_{x\to \pi/2} \dfrac{\cos x}{\cot x} = \lim_{x\to \pi/2} \sin x = 1$

76. $\lim_{x\to \pi/4} \dfrac{1 - \tan x}{\sin x - \cos x} = \lim_{x\to \pi/4} \dfrac{\cos x - \sin x}{\sin x \cos x - \cos^2 x}$

$$= \lim_{x\to \pi/4} \frac{-(\sin x - \cos x)}{\cos x(\sin x - \cos x)}$$

$$= \lim_{x\to \pi/4} \frac{-1}{\cos x}$$

$$= \lim_{x\to \pi/4} (-\sec x)$$

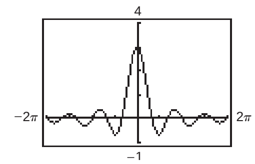
$$= -\sqrt{2}$$

77. $\lim_{t\to 0} \dfrac{\sin 3t}{2t} = \lim_{t\to 0} \left(\dfrac{\sin 3t}{3t}\right)\left(\dfrac{3}{2}\right) = (1)\left(\dfrac{3}{2}\right) = \dfrac{3}{2}$

78. $\lim_{x\to 0} \dfrac{\sin 2x}{\sin 3x} = \lim_{x\to 0} \left[2\left(\dfrac{\sin 2x}{2x}\right)\left(\dfrac{1}{3}\right)\left(\dfrac{3x}{\sin 3x}\right)\right] = 2(1)\left(\dfrac{1}{3}\right)(1) = \dfrac{2}{3}$

79. $f(t) = \dfrac{\sin 3t}{t}$

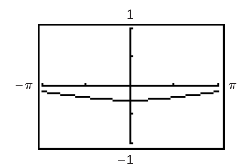
t	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(t)$	2.96	2.9996	3	?	3	2.9996	2.96



The limit appears to equal 3.

Analytically, $\lim\limits_{t\to 0} \dfrac{\sin 3t}{t} = \lim\limits_{t\to 0} 3\left(\dfrac{\sin 3t}{3t}\right) = 3(1) = 3.$

80. From the graph, $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{2x^2} \approx -0.25$



x	-1	-0.1	-0.01	0.01	0.1	1
$f(x)$	-0.2298	-0.2498	-0.25	-0.25	-0.2498	-0.2298

$\lim\limits_{x\to 0} \dfrac{\cos x - 1}{2x^2} \approx -0.25$

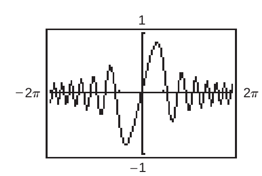
$$\text{Analytically, } \frac{\cos x - 1}{2x^2} \cdot \frac{\cos x + 1}{\cos x + 1} = \frac{\cos^2 x - 1}{2x^2(\cos x + 1)}$$

$$= \frac{-\sin^2 x}{2x^2(\cos x + 1)}$$

$$= \frac{\sin^2 x}{x^2} \cdot \frac{-1}{2(\cos x + 1)}$$

$$\lim_{x\to 0}\left[\frac{\sin^2 x}{x^2} \cdot \frac{-1}{2(\cos x + 1)}\right] = 1\left(\frac{-1}{4}\right) = -\frac{1}{4} = -0.25$$

81. $f(x) = \dfrac{\sin x^2}{x}$

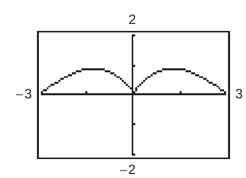


x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	-0.099998	-0.01	-0.001	?	0.001	0.01	0.099998

Analytically, $\lim\limits_{x\to 0} \dfrac{\sin x^2}{x} = \lim\limits_{x\to 0} x\left(\dfrac{\sin x^2}{x^2}\right) = 0(1) = 0.$

82. $f(x) = \dfrac{\sin x}{\sqrt[3]{x}}$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	0.215	0.0464	0.01	?	0.01	0.0464	0.215



The limit appear to equal 0.

Analytically, $\lim\limits_{x\to 0} \dfrac{\sin x}{\sqrt[3]{x}} = \lim\limits_{x\to 0} \sqrt[3]{x^2}\left(\dfrac{\sin x}{x}\right) = (0)(1) = 0.$

83. $\lim\limits_{\Delta x\to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x} = \lim\limits_{\Delta x\to 0} \dfrac{2(x + \Delta x) + 3 - (2x + 3)}{\Delta x} = \lim\limits_{\Delta x\to 0} \dfrac{2x + 2\Delta x + 3 - 2x - 3}{\Delta x} = \lim\limits_{\Delta x\to 0} \dfrac{2\Delta x}{\Delta x} = 2$

84. $$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}=\lim_{\Delta x\to 0}\frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}\cdot\frac{\sqrt{x+\Delta x}+\sqrt{x}}{\sqrt{x+\Delta x}+\sqrt{x}}$$
$$=\lim_{\Delta x\to 0}\frac{x+\Delta x-x}{\Delta x\left(\sqrt{x+\Delta x}+\sqrt{x}\right)}=\lim_{\Delta x\to 0}\frac{1}{\sqrt{x+\Delta x}+\sqrt{x}}=\frac{1}{2\sqrt{x}}$$

85. $$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\dfrac{4}{x+\Delta x}-\dfrac{4}{x}}{\Delta x}=\lim_{\Delta x\to 0}\frac{4x-4(x+\Delta x)}{(x+\Delta x)x\,\Delta x}=\lim_{\Delta x\to 0}\frac{-4}{(x+\Delta x)x}=\frac{-4}{x^2}$$

86. $$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{(x+\Delta x)^2-4(x+\Delta x)-(x^2-4x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{x^2+2x\,\Delta x+\Delta x^2-4x-4\Delta x-x^2+4x}{\Delta x}$$
$$=\lim_{\Delta x\to 0}\frac{\Delta x(2x+\Delta x-4)}{\Delta x}=\lim_{\Delta x\to 0}(2x+\Delta x-4)=2x-4$$

87. $\lim_{x\to 0}(4-x^2)\le\lim_{x\to 0}f(x)\le\lim_{x\to 0}(4+x^2)$

$$4\le\lim_{x\to 0}f(x)\le 4$$

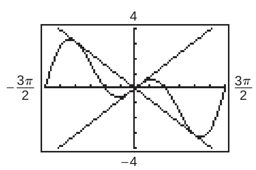
Therefore, $\lim_{x\to 0}f(x)=4$.

88. $\lim_{x\to a}\left[b-|x-a|\right]\le\lim_{x\to a}f(x)\le\lim_{x\to a}\left[b+|x-a|\right]$

$$b\le\lim_{x\to a}f(x)\le b$$

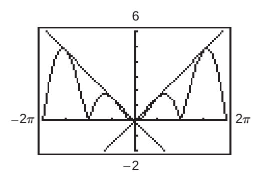
Therefore, $\lim_{x\to a}f(x)=b$.

89. $f(x)=x\cos x$



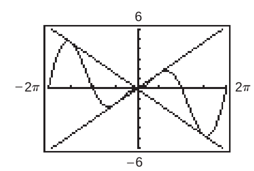
$\lim_{x\to 0}(x\cos x)=0$

90. $f(x)=|x\sin x|$



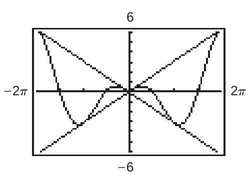
$\lim_{x\to 0}|x\sin x|=0$

91. $f(x)=|x|\sin x$



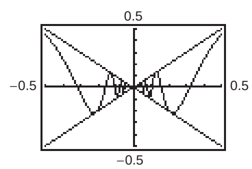
$\lim_{x\to 0}|x|\sin x=0$

92. $f(x)=|x|\cos x$



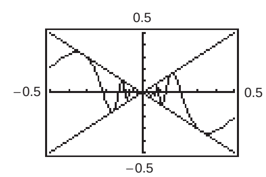
$\lim_{x\to 0}|x|\cos x=0$

93. $f(x)=x\sin\dfrac{1}{x}$



$\lim_{x\to 0}\left(x\sin\dfrac{1}{x}\right)=0$

94. $h(x)=x\cos\dfrac{1}{x}$



$\lim_{x\to 0}\left(x\cos\dfrac{1}{x}\right)=0$

95. We say that two functions f and g agree at all but one point (on an open interval) if $f(x)=g(x)$ for all x in the interval except for $x=c$, where c is in the interval.

96. $f(x)=\dfrac{x^2-1}{x-1}$ and $g(x)=x+1$ agree at all points except $x=1$.

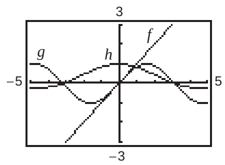
97. An indeterminant form is obtained when evaluating a limit using direct substitution produces a meaningless fractional expression such as $0/0$. That is,

$$\lim_{x\to c}\frac{f(x)}{g(x)}$$

for which $\lim_{x\to c}f(x)=\lim_{x\to c}g(x)=0$

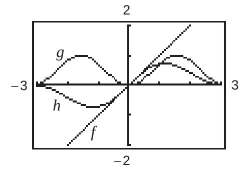
98. If a function f is squeezed between two functions h and g, $h(x)\le f(x)\le g(x)$, and h and g have the same limit L as $x\to c$, then $\lim_{x\to c}f(x)$ exists and equals L.

99. $f(x) = x,\ g(x) = \sin x,\ h(x) = \dfrac{\sin x}{x}$



When you are "close to" 0 the magnitude of f is approximately equal to the magnitude of g. Thus, $|g|/|f| \approx 1$ when x is "close to" 0.

100. $f(x) = x,\ g(x) = \sin^2 x,\ h(x) = \dfrac{\sin^2 x}{x}$



When you are "close to" 0 the magnitude of g is "smaller" than the magnitude of f and the magnitude of g is approaching zero "faster" than the magnitude of f. Thus, $|g|/|f| \approx 0$ when x is "close to" 0.

101. $s(t) = -16t^2 + 1000$

$$\lim_{t\to 5}\frac{s(5)-s(t)}{5-t} = \lim_{t\to 5}\frac{600-(-16t^2+1000)}{5-t} = \lim_{t\to 5}\frac{16(t+5)(t-5)}{-(t-5)} = \lim_{t\to 5} -16(t+5) = -160 \text{ ft/sec.}$$

Speed $= 160$ ft/sec

102. $s(t) = -16t^2 + 1000 = 0$ when $t = \sqrt{\dfrac{1000}{16}} = \dfrac{5\sqrt{10}}{2}$ seconds

$$\lim_{t\to 5\sqrt{10}/2}\frac{s\left(\frac{5\sqrt{10}}{2}\right)-s(t)}{\frac{5\sqrt{10}}{2}-t} = \lim_{t\to 5\sqrt{10}/2}\frac{0-(-16t^2+1000)}{\frac{5\sqrt{10}}{2}-t}$$

$$= \lim_{t\to 5\sqrt{10}/2}\frac{16\left(t^2-\frac{125}{2}\right)}{\frac{5\sqrt{10}}{2}-t} = \lim_{t\to 5\sqrt{10}/2}\frac{16\left(t+\frac{5\sqrt{10}}{2}\right)\left(t-\frac{5\sqrt{10}}{2}\right)}{-\left(t-\frac{5\sqrt{10}}{2}\right)}$$

$$= \lim_{t\to 5\sqrt{10}/2} -16\left(t+\frac{5\sqrt{10}}{2}\right) = -80\sqrt{10} \text{ ft/sec} \approx -253 \text{ ft/sec}$$

103. $s(t) = -4.9t^2 + 150$

$$\lim_{t\to 3}\frac{s(3)-s(t)}{3-t} = \lim_{t\to 3}\frac{-4.9(3^2)+150-(-4.9t^2+150)}{3-t} = \lim_{t\to 3}\frac{-4.9(9-t^2)}{3-t}$$

$$= \lim_{t\to 3}\frac{-4.9(3-t)(3+t)}{3-t} = \lim_{t\to 3} -4.9(3+t) = -29.4 \text{ m/sec}$$

104. $-4.9t^2 + 150 = 0$ when $t = \sqrt{\dfrac{150}{4.9}} = \sqrt{\dfrac{1500}{49}} \approx 5.53$ seconds.

The velocity at time $t = a$ is

$$\lim_{t\to a}\frac{s(a)-s(t)}{a-t} = \lim_{t\to a}\frac{(-4.9a^2+150)-(-4.9t^2+150)}{a-t} = \lim_{t\to a}\frac{-4.9(a-t)(a+t)}{a-t}$$

$$= \lim_{t\to a} -4.9(a+t) = -2a(4.9) = -9.8a \text{ m/sec.}$$

Hence, if $a = \sqrt{1500/49}$, the velocity is $-9.8\sqrt{1500/49} \approx -54.2$ m/sec.

105. Let $f(x) = 1/x$ and $g(x) = -1/x$. $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ do not exist.

$$\lim_{x\to 0}\left[f(x) + g(x)\right] = \lim_{x\to 0}\left[\frac{1}{x} + \left(-\frac{1}{x}\right)\right] = \lim_{x\to 0}\left[0\right] = 0$$

106. Suppose, on the contrary, that $\lim_{x\to c} g(x)$ exists. Then, since $\lim_{x\to c} f(x)$ exists, so would $\lim_{x\to c}\left[f(x) + g(x)\right]$, which is a contradiction. Hence, $\lim_{x\to c} g(x)$ does not exist.

107. Given $f(x) = b$, show that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - b| < \varepsilon$ whenever $|x - c| < \delta$. Since $|f(x) - b| = |b - b| = 0 < \varepsilon$ for any $\varepsilon > 0$, then any value of $\delta > 0$ will work.

108. Given $f(x) = x^n$, n is a positive integer, then

$$\begin{aligned}\lim_{x\to c} x^n &= \lim_{x\to c}\left(xx^{n-1}\right)\\ &= \left[\lim_{x\to c} x\right]\left[\lim_{x\to c} x^{n-1}\right] = c\left[\lim_{x\to c}\left(xx^{n-2}\right)\right]\\ &= c\left[\lim_{x\to c} x\right]\left[\lim_{x\to c} x^{n-2}\right] = c(c)\lim_{x\to c}\left(xx^{n-3}\right)\\ &= \cdot\cdot\cdot = c^n.\end{aligned}$$

109. If $b = 0$, then the property is true because both sides are equal to 0. If $b \neq 0$, let $\varepsilon > 0$ be given. Since $\lim_{x\to c} f(x) = L$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon/|b|$ whenever $0 < |x - c| < \delta$. Hence, wherever $0 < |x - c| < \delta$, we have

$$|b||f(x) - L| < \varepsilon \quad \text{or} \quad |bf(x) - bL| < \varepsilon$$

which implies that $\lim_{x\to c}\left[bf(x)\right] = bL$.

110. Given $\lim_{x\to c} f(x) = 0$:

For every $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - 0| < \varepsilon$ whenever $0 < |x - c| < \delta$. Now $|f(x) - 0| = |f(x)| = ||f(x)| - 0| < \varepsilon$ for $|x - c| < \delta$. Therefore, $\lim_{x\to c} |f(x)| = 0$.

111.

$$\begin{aligned} -M|f(x)| &\le f(x)g(x) \le M|f(x)|\\ \lim_{x\to c}\left(-M|f(x)|\right) &\le \lim_{x\to c} f(x)g(x) \le \lim_{x\to c}\left(M|f(x)|\right)\\ -M(0) &\le \lim_{x\to c} f(x)g(x) \le M(0)\\ 0 &\le \lim_{x\to c} f(x)g(x) \le 0\end{aligned}$$

Therefore, $\lim_{x\to c} f(x)g(x) = 0$.

112. (a) If $\lim_{x\to c} |f(x)| = 0$, then $\lim_{x\to c}\left[-|f(x)|\right] = 0$.

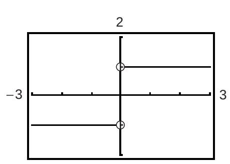
$$\begin{aligned} -|f(x)| &\le f(x) \le |f(x)|\\ \lim_{x\to c}\left[-|f(x)|\right] &\le \lim_{x\to c} f(x) \le \lim_{x\to c}|f(x)|\\ 0 &\le \lim_{x\to c} f(x) \le 0\end{aligned}$$

Therefore, $\lim_{x\to c} f(x) = 0$.

(b) Given $\lim_{x\to c} f(x) = L$:

For every $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$. Since $||f(x)| - |L|| \le |f(x) - L| < \varepsilon$ for $|x - c| < \delta$, then $\lim_{x\to c} |f(x)| = |L|$.

113. False. As x approaches 0 from the left, $\dfrac{|x|}{x} = -1$.



114. False. $\lim_{x\to\pi} \dfrac{\sin x}{x} = \dfrac{0}{\pi} = 0$

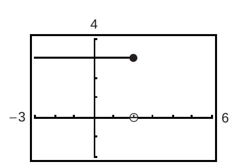
115. True

116. False. Let

$$f(x) = \begin{cases} x & x \neq 1\\ 3 & x = 1\end{cases},\quad c = 1.$$

Then $\lim_{x\to 1} f(x) = 1$ but $f(1) \neq 1$.

117. False. The limit does not exist.



118. False. Let

$$f(x) = \tfrac{1}{2}x^2 \text{ and } g(x) = x^2.$$

Then $f(x) < g(x)$ for all $x \neq 0$. But $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$.

119. Let

$$f(x) = \begin{cases} 4, & \text{if } x \geq 0 \\ -4, & \text{if } x < 0 \end{cases}$$

$$\lim_{x\to 0} |f(x)| = \lim_{x\to 0} 4 = 4.$$

$\lim_{x\to 0} f(x)$ does not exist since for $x < 0$, $f(x) = -4$ and for $x \geq 0$, $f(x) = 4$.

120. $$\lim_{x\to 0} \frac{1 - \cos x}{x} = \lim_{x\to 0} \frac{1 - \cos x}{x} \cdot \frac{1 + \cos x}{1 + \cos x}$$

$$= \lim_{x\to 0} \frac{1 - \cos^2 x}{x(1 + \cos x)} = \lim_{x\to 0} \frac{\sin^2 x}{x(1 + \cos x)}$$

$$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{1 + \cos x}$$

$$= \left[\lim_{x\to 0} \frac{\sin x}{x}\right]\left[\lim_{x\to 0} \frac{\sin x}{1 + \cos x}\right]$$

$$= (1)(0) = 0$$

121. $$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$$

$$g(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ x, & \text{if } x \text{ is irrational} \end{cases}$$

$\lim_{x\to 0} f(x)$ does not exist.

No matter how "close to" 0 x is, there are still an infinite number of rational and irrational numbers so that $\lim_{x\to 0} f(x)$ does not exist.

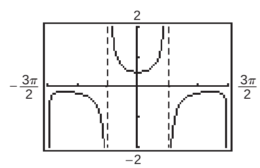
$\lim_{x\to 0} g(x) = 0$

When x is "close to" 0, both parts of the function are "close to" 0.

122. $$f(x) = \frac{\sec x - 1}{x^2}$$

(a) The domain of f is all $x \neq 0$, $\pi/2 + n\pi$.

(b)



The domain is not obvious. The hole at $x = 0$ is not apparent.

(c) $\lim_{x\to 0} f(x) = \frac{1}{2}$

(d) $$\frac{\sec x - 1}{x^2} = \frac{\sec x - 1}{x^2} \cdot \frac{\sec x + 1}{\sec x + 1} = \frac{\sec^2 x - 1}{x^2(\sec x + 1)}$$

$$= \frac{\tan^2 x}{x^2(\sec x + 1)} = \frac{1}{\cos^2 x}\left(\frac{\sin^2 x}{x^2}\right)\frac{1}{\sec x + 1}$$

Hence, $$\lim_{x\to 0} \frac{\sec x - 1}{x^2} = \lim_{x\to 0} \frac{1}{\cos^2 x}\left(\frac{\sin^2 x}{x^2}\right)\frac{1}{\sec x + 1}$$

$$= 1(1)\left(\frac{1}{2}\right) = \frac{1}{2}.$$

123. (a) $$\lim_{x\to 0} \frac{1 - \cos x}{x^2} = \lim_{x\to 0} \frac{1 - \cos x}{x^2} \cdot \frac{1 + \cos x}{1 + \cos x}$$

$$= \lim_{x\to 0} \frac{1 - \cos^2 x}{x^2(1 + \cos x)}$$

$$= \lim_{x\to 0} \frac{\sin^2 x}{x^2} \cdot \frac{1}{1 + \cos x}$$

$$= (1)\left(\frac{1}{2}\right) = \frac{1}{2}$$

(b) Thus, $\dfrac{1 - \cos x}{x^2} \approx \dfrac{1}{2} \implies 1 - \cos x \approx \dfrac{1}{2}x^2$

$\implies \cos x \approx 1 - \dfrac{1}{2}x^2$ for $x \approx 0$.

(c) $\cos(0.1) \approx 1 - \frac{1}{2}(0.1)^2 = 0.995$

(d) $\cos(0.1) \approx 0.9950$, which agrees with part (c).

124. The calculator was set in degree mode, instead of radian mode.

Section 1.4 Continuity and One-Sided Limits

1. (a) $\lim_{x\to 3^+} f(x) = 1$

(b) $\lim_{x\to 3^-} f(x) = 1$

(c) $\lim_{x\to 3} f(x) = 1$

The function is continuous at $x = 3$.

2. (a) $\lim_{x\to -2^+} f(x) = -2$

(b) $\lim_{x\to -2^-} f(x) = -2$

(c) $\lim_{x\to -2} f(x) = -2$

The function is continuous at $x = -2$.

3. (a) $\lim_{x\to 3^+} f(x) = 0$

(b) $\lim_{x\to 3^-} f(x) = 0$

(c) $\lim_{x\to 3} f(x) = 0$

The function is NOT continuous at $x = 3$.

4. (a) $\lim_{x\to -2^+} f(x) = 2$

(b) $\lim_{x\to -2^-} f(x) = 2$

(c) $\lim_{x\to -2} f(x) = 2$

The function is NOT continuous at $x = -2$.

5. (a) $\lim_{x\to 4^+} f(x) = 2$

(b) $\lim_{x\to 4^-} f(x) = -2$

(c) $\lim_{x\to 4} f(x)$ does not exist

The function is NOT continuous at $x = 4$.

6. (a) $\lim_{x\to -1^+} f(x) = 0$

(b) $\lim_{x\to -1^-} f(x) = 2$

(c) $\lim_{x\to -1} f(x)$ does not exist.

The function is NOT continuous at $x = -1$.

7. $\lim_{x\to 5^+} \dfrac{x-5}{x^2-25} = \lim_{x\to 5^+} \dfrac{1}{x+5} = \dfrac{1}{10}$

8. $\lim_{x\to 2^+} \dfrac{2-x}{x^2-4} = \lim_{x\to 2^+} -\dfrac{1}{x+2} = -\dfrac{1}{4}$

9. $\lim_{x\to -3^-} \dfrac{x}{\sqrt{x^2-9}}$ does not exist because $\dfrac{x}{\sqrt{x^2-9}}$ decreases without bound as $x \to -3^-$.

10.
$$\lim_{x\to 4^-} \frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4^-} \frac{\sqrt{x}-2}{x-4}\cdot\frac{\sqrt{x}+2}{\sqrt{x}+2}$$
$$= \lim_{x\to 4^-} \frac{x-4}{(x-4)\left(\sqrt{x}+2\right)}$$
$$= \lim_{x\to 4^-} \frac{1}{\sqrt{x}+2} = \frac{1}{4}$$

11. $\lim_{x\to 0^-} \dfrac{|x|}{x} = \lim_{x\to 0^-} \dfrac{-x}{x} = -1$

12. $\lim_{x\to 2^+} \dfrac{|x-2|}{x-2} = \lim_{x\to 2^+} \dfrac{x-2}{x-2} = 1$

13.
$$\lim_{\Delta x\to 0^-} \frac{\dfrac{1}{x+\Delta x} - \dfrac{1}{x}}{\Delta x} = \lim_{\Delta x\to 0^-} \frac{x-(x+\Delta x)}{x(x+\Delta x)}\cdot\frac{1}{\Delta x} = \lim_{\Delta x\to 0^-} \frac{-\Delta x}{x(x+\Delta x)}\cdot\frac{1}{\Delta x}$$
$$= \lim_{\Delta x\to 0^-} \frac{-1}{x(x+\Delta x)}$$
$$= \frac{-1}{x(x+0)} = -\frac{1}{x^2}$$

14.
$$\lim_{\Delta x\to 0^+} \frac{(x+\Delta x)^2 + (x+\Delta x) - (x^2+x)}{\Delta x} = \lim_{\Delta x\to 0^+} \frac{x^2 + 2x(\Delta x) + (\Delta x)^2 + x + \Delta x - x^2 - x}{\Delta x}$$
$$= \lim_{\Delta x\to 0^+} \frac{2x(\Delta x) + (\Delta x)^2 + \Delta x}{\Delta x}$$
$$= \lim_{\Delta x\to 0^+} (2x + \Delta x + 1)$$
$$= 2x + 0 + 1 = 2x + 1$$

15. $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \dfrac{x+2}{2} = \dfrac{5}{2}$

16. $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (-x^2 + 4x - 2) = 2$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x^2 - 4x + 6) = 2$

$\lim_{x\to 2} f(x) = 2$

17. $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x + 1) = 2$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^3 + 1) = 2$

$\lim_{x\to 1} f(x) = 2$

18. $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (1 - x) = 0$

19. $\lim_{x\to \pi} \cot x$ does not exist since $\lim_{x\to \pi^+} \cot x$ and $\lim_{x\to \pi^-} \cot x$ do not exist.

20. $\lim_{x\to \pi/2} \sec x$ does not exist since $\lim_{x\to (\pi/2)^+} \sec x$ and $\lim_{x\to (\pi/2)^-} \sec x$ do not exist.

21. $\lim_{x\to 4^-} (3[\![x]\!] - 5) = 3(3) - 5 = 4$

($[\![x]\!] = 3$ for $3 \le x < 4$)

22. $\lim_{x\to 2^+} (2x - [\![x]\!]) = 2(2) - 2 = 2$

23. $\lim_{x\to 3} (2 - [\![-x]\!])$ does not exist because

$\lim_{x\to 3^-} (2 - [\![-x]\!]) = 2 - (-3) = 5$

and

$\lim_{x\to 3^+} (2 - [\![-x]\!]) = 2 - (-4) = 6.$

24. $\lim_{x\to 1} \left(1 - \left[\!\!\left[-\frac{x}{2}\right]\!\!\right]\right) = 1 - (-1) = 2$

25. $f(x) = \dfrac{1}{x^2 - 4}$

has discontinuities at $x = -2$ and $x = 2$ since $f(-2)$ and $f(2)$ are not defined.

26. $f(x) = \dfrac{x^2 - 1}{x + 1}$

has a discontinuity at $x = -1$ since $f(-1)$ is not defined.

27. $f(x) = \dfrac{[\![x]\!]}{2} + x$

has discontinuities at each integer k since $\lim_{x\to k^-} f(x) \ne \lim_{x\to k^+} f(x)$.

28. $f(x) = \begin{cases} x, & x < 1 \\ 2, & x = 1 \\ 2x - 1, & x > 1 \end{cases}$ has a discontinuity at $x = 1$ since $f(1) = 2 \ne \lim_{x\to 1} f(x) = 1$.

29. $g(x) = \sqrt{25 - x^2}$ is continuous on $[-5, 5]$.

30. $f(t) = 3 - \sqrt{9 - t^2}$ is continuous on $[-3, 3]$.

31. $\lim_{x\to 0^-} f(x) = 3 = \lim_{x\to 0^+} f(x)$.

f is continuous on $[-1, 4]$.

32. $g(2)$ is not defined. g is continuous on $[-1, 2)$.

33. $f(x) = x^2 - 2x + 1$ is continuous for all real x.

34. $f(x) = \dfrac{1}{x^2 + 1}$ is continuous for all real x.

35. $f(x) = 3x - \cos x$ is continuous for all real x.

36. $f(x) = \cos \dfrac{\pi x}{2}$ is continuous for all real x.

37. $f(x) = \dfrac{x}{x^2 - x}$ is not continuous at $x = 0, 1$. Since $\dfrac{x}{x^2 - x} = \dfrac{1}{x - 1}$ for $x \ne 0$, $x = 0$ is a removable discontinuity, whereas $x = 1$ is a nonremovable discontinuity.

38. $f(x) = \dfrac{x}{x^2 - 1}$ has nonremovable discontinuities at $x = 1$ and $x = -1$ since $\lim_{x\to 1} f(x)$ and $\lim_{x\to -1} f(x)$ do not exist.

39. $f(x) = \dfrac{x}{x^2 + 1}$ is continuous for all real x.

40. $f(x) = \dfrac{x-3}{x^2-9}$

has a nonremovable discontinuity at $x = -3$ since $\lim_{x\to -3} f(x)$ does not exist, and has a removable discontinuity at $x = 3$ since

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{1}{x+3} = \frac{1}{6}.$$

41. $f(x) = \dfrac{x+2}{(x+2)(x-5)}$

has a nonremovable discontinuity at $x = 5$ since $\lim_{x\to 5} f(x)$ does not exist, and has a removable discontinuity at $x = -2$ since

$$\lim_{x\to -2} f(x) = \lim_{x\to -2} \frac{1}{x-5} = -\frac{1}{7}.$$

42. $f(x) = \dfrac{x-1}{(x+2)(x-1)}$

has a nonremovable discontinuity at $x = -2$ since $\lim_{x\to -2} f(x)$ does not exist, and has a removable discontinuity at $x = 1$ since

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{1}{x+2} = \frac{1}{3}.$$

43. $f(x) = \dfrac{|x+2|}{x+2}$

has a nonremovable discontinuity at $x = -2$ since $\lim_{x\to -2} f(x)$ does not exist.

44. $f(x) = \dfrac{|x-3|}{x-3}$ has a nonremovable discontinuity at $x = 3$ since $\lim_{x\to 3} f(x)$ does not exist.

45. $f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$

has a **possible** discontinuity at $x = 1$.

1. $f(1) = 1$

2. $\left.\begin{array}{l} \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = 1 \\ \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^2 = 1 \end{array}\right\} \lim_{x\to 1} f(x) = 1$

3. $f(1) = \lim_{x\to 1} f(x)$

f is continuous at $x = 1$, therefore, f is continuous for all real x.

46. $f(x) = \begin{cases} -2x+3, & x < 1 \\ x^2, & x \ge 1 \end{cases}$

has a **possible** discontinuity at $x = 1$.

1. $f(1) = 1^2 = 1$

2. $\left.\begin{array}{l} \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (-2x+3) = 1 \\ \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^2 = 1 \end{array}\right\} \lim_{x\to 1} f(x) = 1$

3. $f(1) = \lim_{x\to 1} f(x)$

f is continuous at $x = 1$, therefore, f is continuous for all real x.

47. $f(x) = \begin{cases} \dfrac{x}{2}+1, & x \le 2 \\ 3-x, & x > 2 \end{cases}$

has a **possible** discontinuity at $x = 2$.

1. $f(2) = \dfrac{2}{2} + 1 = 2$

2. $\left.\begin{array}{l} \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \left(\dfrac{x}{2}+1\right) = 2 \\ \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (3-x) = 1 \end{array}\right\} \lim_{x\to 2} f(x)$ does not exist.

Therefore, f has a nonremovable discontinuity at $x = 2$.

48. $f(x) = \begin{cases} -2x, & x \le 2 \\ x^2-4x+1, & x > 2 \end{cases}$

has a **possible** discontinuity at $x = 2$.

1. $f(2) = -2(2) = -4$

2. $\left.\begin{array}{l} \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (-2x) = -4 \\ \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x^2-4x+1) = -3 \end{array}\right\} \lim_{x\to 2} f(x)$ does not exist.

Therefore, f has a nonremovable discontinuity at $x = 2$.

49. $f(x) = \begin{cases} \tan \dfrac{\pi x}{4}, & |x| < 1 \\ x, & |x| \geq 1 \end{cases}$

$= \begin{cases} \tan \dfrac{\pi x}{4}, & -1 < x < 1 \\ x, & x \leq -1 \text{ or } x \geq 1 \end{cases}$

has **possible** discontinuities at $x = -1$, $x = 1$.

1. $f(-1) = -1$ $\qquad f(1) = 1$

2. $\lim\limits_{x \to -1} f(x) = -1$ $\qquad \lim\limits_{x \to 1} f(x) = 1$

3. $f(-1) = \lim\limits_{x \to -1} f(x)$ $\qquad f(1) = \lim\limits_{x \to 1} f(x)$

f is continuous at $x = \pm 1$, therefore, f is continuous for all real x.

50. $f(x) = \begin{cases} \csc \dfrac{\pi x}{6}, & |x - 3| \leq 2 \\ 2, & |x - 3| > 2 \end{cases}$

$= \begin{cases} \csc \dfrac{\pi x}{6}, & 1 \leq x \leq 5 \\ 2, & x < 1 \text{ or } x > 5 \end{cases}$

has **possible** discontinuities at $x = 1$, $x = 5$.

1. $f(1) = \csc \dfrac{\pi}{6} = 2$ $\qquad f(5) = \csc \dfrac{5\pi}{6} = 2$

2. $\lim\limits_{x \to 1} f(x) = 2$ $\qquad \lim\limits_{x \to 5} f(x) = 2$

3. $f(1) = \lim\limits_{x \to 1} f(x)$ $\qquad f(5) = \lim\limits_{x \to 5} f(x)$

f is continuous at $x = 1$ and $x = 5$, therefore, f is continuous for all real x.

51. $f(x) = \csc 2x$ has nonremovable discontinuities at integer multiples of $\pi/2$.

52. $f(x) = \tan \dfrac{\pi x}{2}$ has nonremovable discontinuities at each $2k + 1$, k is an integer.

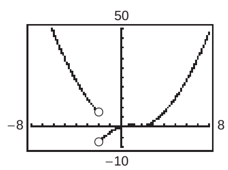
53. $f(x) = [\![x - 1]\!]$ has nonremovable discontinuities at each integer k.

54. $f(x) = 3 - [\![x]\!]$ has nonremovable discontinuities at each integer k.

55. $\lim\limits_{x \to 0^+} f(x) = 0$

$\lim\limits_{x \to 0^-} f(x) = 0$

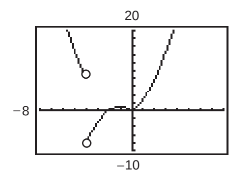
f is not continuous at $x = -2$.



56. $\lim\limits_{x \to 0^+} f(x) = 0$

$\lim\limits_{x \to 0^-} f(x) = 0$

f is not continuous at $x = -4$.



57. $f(2) = 8$

Find a so that $\lim\limits_{x \to 2^+} ax^2 = 8 \implies a = \dfrac{8}{2^2} = 2$.

58. $\lim\limits_{x \to 0^-} g(x) = \lim\limits_{x \to 0^-} \dfrac{4 \sin x}{x} = 4$

$\lim\limits_{x \to 0^+} g(x) = \lim\limits_{x \to 0^+} (a - 2x) = a$

Let $a = 4$.

59. Find a and b such that $\lim\limits_{x \to -1^+} (ax + b) = -a + b = 2$ and $\lim\limits_{x \to 3^-} (ax + b) = 3a + b = -2$.

$$\begin{aligned} a - b &= -2 \\ \underline{(+)\ 3a + b} &\underline{= -2} \\ 4a \quad &= -4 \\ a &= -1 \\ b &= 2 + (-1) = 1 \end{aligned}$$

$$f(x) = \begin{cases} 2, & x \leq -1 \\ -x + 1, & -1 < x < 3 \\ -2, & x \geq 3 \end{cases}$$

60. $\lim\limits_{x \to a} g(x) = \lim\limits_{x \to a} \dfrac{x^2 - a^2}{x - a}$

$= \lim\limits_{x \to a} (x + a) = 2a$

Find a such that $2a = 8 \implies a = 4$.

61. $f(g(x)) = (x - 1)^2$

Continuous for all real x.

62. $f(g(x)) = \dfrac{1}{\sqrt{x-1}}$

Nonremovable discontinuity at $x = 1$. Continuous for all $x > 1$.

63. $f(g(x)) = \dfrac{1}{(x^2+5)-6} = \dfrac{1}{x^2-1}$

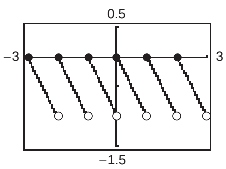
Nonremovable discontinuities at $x = \pm 1$

64. $f(g(x)) = \sin x^2$

Continuous for all real x

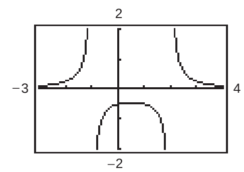
65. $y = [\![x]\!] - x$

Nonremovable discontinuity at each integer



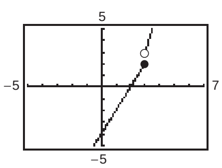
66. $h(x) = \dfrac{1}{(x+1)(x-2)}$

Nonremovable discontinuities at $x = -1$ and $x = 2$.

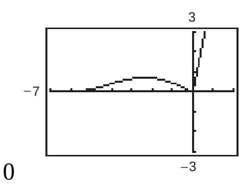


67. $f(x) = \begin{cases} 2x - 4, & x \le 3 \\ x^2 - 2x, & x > 3 \end{cases}$

Nonremovable discontinuity at $x = 3$



68. $f(x) = \begin{cases} \dfrac{\cos x - 1}{x}, & x < 0 \\ 5x, & x \ge 0 \end{cases}$



$f(0) = 5(0) = 0$

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \dfrac{(\cos x - 1)}{x} = 0$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} (5x) = 0$

Therefore, $\lim\limits_{x\to 0} f(x) = 0 = f(0)$ and f is continuous on the entire real line. ($x = 0$ was the only possible discontinuity.)

69. $f(x) = \dfrac{x}{x^2+1}$

Continuous on $(-\infty, \infty)$

70. $f(x) = x\sqrt{x+3}$

Continuous on $[-3, \infty)$

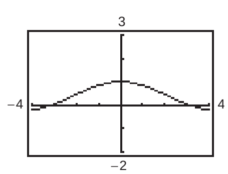
71. $f(x) = \sec \dfrac{\pi x}{4}$

Continuous on:
$\ldots, (-6, -2), (-2, 2), (2, 6), (6, 10), \ldots$

72. $f(x) = \dfrac{x+1}{\sqrt{x}}$

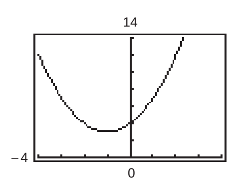
Continuous on $(0, \infty)$

73. $f(x) = \dfrac{\sin x}{x}$



The graph **appears** to be continuous on the interval $[-4, 4]$. Since $f(0)$ is not defined, we know that f has a discontinuity at $x = 0$. This discontinuity is removable so it does not show up on the graph.

74. $f(x) = \dfrac{x^3-8}{x-2}$



The graph **appears** to be continuous on the interval $[-4, 4]$. Since $f(2)$ is not defined, we know that f has a discontinuity at $x = 2$. This discontinuity is removable so it does not show up on the graph.

75. $f(x) = \frac{1}{16}x^4 - x^3 + 3$ is continuous on $[1, 2]$.

$f(1) = \frac{33}{16}$ and $f(2) = -4$. By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of c between 1 and 2.

76. $f(x) = x^3 + 3x - 2$ is continuous on $[0, 1]$.

$f(0) = -2$ and $f(1) = 2$

By the Intermediate Value Theorem, $f(x) = 0$ for at least one value of c between 0 and 1.

77. $f(x) = x^2 - 2 - \cos x$ is continuous on $[0, \pi]$.

$f(0) = -3$ and $f(\pi) = \pi^2 - 1 > 0$. By the Intermediate Value Theorem, $f(c) = 0$ for the least one value of c between 0 and π.

78. $f(x) = \dfrac{-4}{x} + \tan \dfrac{\pi x}{8}$ is continuous on $[1, 3]$.

$f(1) = -4 + \tan \dfrac{\pi}{8} < 0$ and $f(3) = -\dfrac{4}{3} + \tan \dfrac{3\pi}{8} > 0$.

By the Intermediate Value Theorem, $f(1) = 0$ for at least one value of c between 1 and 3.

79. $f(x) = x^3 + x - 1$

$f(x)$ is continuous on $[0, 1]$.

$f(0) = -1$ and $f(1) = 1$

By the Intermediate Value Theorem, $f(x) = 0$ for at least one value of c between 0 and 1. Using a graphing utility, we find that $x \approx 0.6823$.

80. $f(x) = x^3 + 3x - 2$

$f(x)$ is continuous on $[0, 1]$.

$f(0) = -2$ and $f(1) = 2$

By the Intermediate Value Theorem, $f(x) = 0$ for at least one value of c between 0 and 1. Using a graphing utility, we find that $x \approx 0.5961$.

81. $g(t) = 2 \cos t - 3t$

g is continuous on $[0, 1]$.

$g(0) = 2 > 0$ and $g(1) \approx -1.9 < 0$.

By the Intermediate Value Theorem, $g(t) = 0$ for at least one value c between 0 and 1. Using a graphing utility, we find that $t \approx 0.5636$.

82. $h(\theta) = 1 + \theta - 3 \tan \theta$

h is continuous on $[0, 1]$.

$h(0) = 1 > 0$ and $h(1) \approx -2.67 < 0$.

By the Intermediate Value Theorem, $h(\theta) = 0$ for at least one value θ between 0 and 1. Using a graphing utility, we find that $\theta \approx 0.4503$.

83. $f(x) = x^2 + x - 1$

f is continuous on $[0, 5]$.

$f(0) = -1$ and $f(5) = 29$

$-1 < 11 < 29$

The Intermediate Value Theorem applies.

$$x^2 + x - 1 = 11$$
$$x^2 + x - 12 = 0$$
$$(x + 4)(x - 3) = 0$$
$$x = -4 \quad \text{or} \quad x = 3$$

$c = 3$ ($x = -4$ is not in the interval.)

Thus, $f(3) = 11$.

84. $f(x) = x^2 - 6x + 8$

f is continuous on $[0, 3]$.

$f(0) = 8$ and $f(3) = -1$

$-1 < 0 < 8$

The Intermediate Value Theorem applies.

$$x^2 - 6x + 8 = 0$$
$$(x - 2)(x - 4) = 0$$
$$x = 2 \quad \text{or} \quad x = 4$$

$c = 2$ ($x = 4$ is not in the interval.)

Thus, $f(2) = 0$.

85. $f(x) = x^3 - x^2 + x - 2$

f is continuous on $[0, 3]$.

$f(0) = -2$ and $f(3) = 19$

$-2 < 4 < 19$

The Intermediate Value Theorem applies.

$$x^3 - x^2 + x - 2 = 4$$
$$x^3 - x^2 + x - 6 = 0$$
$$(x - 2)(x^2 + x + 3) = 0$$
$$x = 2$$

($x^2 + x + 3$ has no real solution.)

$$c = 2$$

Thus, $f(2) = 4$.

86. $f(x) = \dfrac{x^2 + x}{x - 1}$

f is continuous on $\left[\frac{5}{2}, 4\right]$. The nonremovable discontinuity, $x = 1$, lies outside the interval.

$f\left(\dfrac{5}{2}\right) = \dfrac{35}{6}$ and $f(4) = \dfrac{20}{3}$

$\dfrac{35}{6} < 6 < \dfrac{20}{3}$

The Intermediate Value Theorem applies.

$$\frac{x^2 + x}{x - 1} = 6$$
$$x^2 + x = 6x - 6$$
$$x^2 - 5x + 6 = 0$$
$$(x - 2)(x - 3) = 0$$
$$x = 2 \text{ or } x = 3$$

$c = 3$ ($x = 2$ is not in the interval.)

Thus, $f(3) = 6$.

87. (a) The limit does not exist at $x = c$.

(b) The function is not defined at $x = c$.

(c) The limit exists at $x = c$, but it is not equal to the value of the function at $x = c$.

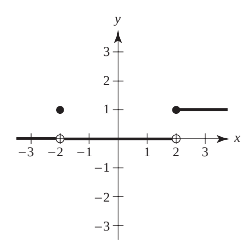
(d) The limit does not exist at $x = c$.

88. A discontinuity at $x = c$ is removable if you can define (or redefine) the function at $x = c$ in such a way that the new function is continuous at $x = c$. Answers will vary.

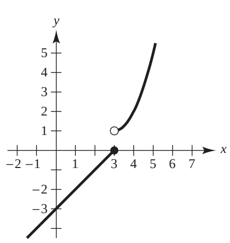
(a) $f(x) = \dfrac{|x - 2|}{x - 2}$

(b) $f(x) = \dfrac{\sin(x + 2)}{x + 2}$

(c) $f(x) = \begin{cases} 1, & \text{if } x \geq 2 \\ 0, & \text{if } -2 < x < 2 \\ 1, & \text{if } x = -2 \\ 0, & \text{if } x < -2 \end{cases}$



89.



The function is not continuous at $x = 3$ because $\lim\limits_{x \to 3^+} f(x) = 1 \neq 0 = \lim\limits_{x \to 3^-} f(x)$.

90. If f and g are continuous for all real x, then so is $f + g$ (Theorem 1.11, part 2). However, f/g might not be continuous if $g(x) = 0$. For example, let $f(x) = x$ and $g(x) = x^2 - 1$. Then f and g are continuous for all real x, but f/g is not continuous at $x = \pm 1$.

91. True

1. $f(c) = L$ is defined.

2. $\lim_{x\to c} f(x) = L$ exists.

3. $f(c) = \lim_{x\to c} f(x)$

All of the conditions for continuity are met.

92. True; if $f(x) = g(x)$, $x \neq c$, then $\lim_{x\to c} f(x) = \lim_{x\to c} g(x)$ (if they exist) and at least one of these limits then does not equal the corresponding function value at $x = c$.

93. False; a rational function can be written as $P(x)/Q(x)$ where P and Q are polynomials of degree m and n, respectively. It can have, at most, n discontinuities.

94. False; $f(1)$ is not defined and $\lim_{x\to 1} f(x)$ does not exist.

95. $\lim_{t\to 4^-} f(t) \approx 28$

$\lim_{t\to 4^+} f(t) \approx 56$

At the end of day 3, the amount of chlorine in the pool has decreased to about 28 oz. At the beginning of day 4, more chlorine was added, and the amount was about 56 oz.

96. The functions agree for integer values of x:

$$\left.\begin{aligned} g(x) &= 3 - [\![-x]\!] = 3 - (-x) = 3 + x \\ f(x) &= 3 + [\![x]\!] = 3 + x \end{aligned}\right\} \text{for } x \text{ an integer}$$

However, for non-integer values of x, the functions differ by 1.

$f(x) = 3 + [\![x]\!] = g(x) - 1 = 2 - [\![-x]\!]$.

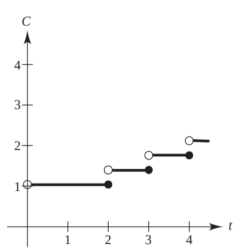
For example, $f\left(\frac{1}{2}\right) = 3 + 0 = 3$, $g\left(\frac{1}{2}\right) = 3 - (-1) = 4$.

97. $$C = \begin{cases} 1.04, & 0 < t \leq 2 \\ 1.04 + 0.36[\![t - 1]\!], & t > 2,\ t \text{ is not an integer} \\ 1.04 + 0.36(t - 2), & t > 2,\ t \text{ is an integer} \end{cases}$$

Nonremovable discontinuity at each integer greater than or equal to 2.

You can also write C as

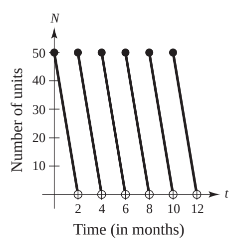
$$C = \begin{cases} 1.04, & 0 < t \leq 2 \\ 1.04 - 0.36[\![2 - t]\!], & t > 2 \end{cases}.$$



98. $N(t) = 25\left(2\left[\!\!\left[\dfrac{t + 2}{2}\right]\!\!\right] - t\right)$

t	0	1	1.8	2	3	3.8
$N(t)$	50	25	5	50	25	5

Discontinuous at every positive even integer. The company replenishes its inventory every two months.



99. Let $s(t)$ be the position function for the run up to the campsite. $s(0) = 0$ ($t = 0$ corresponds to 8:00 A.M., $s(20) = k$ (distance to campsite)). Let $r(t)$ be the position function for the run back down the mountain: $r(0) = k$, $r(10) = 0$. Let $f(t) = s(t) - r(t)$.

When $t = 0$ (8:00 A.M.), $f(0) = s(0) - r(0) = 0 - k < 0$.

When $t = 10$ (8:10 A.M.), $f(10) = s(10) - r(10) > 0$.

Since $f(0) < 0$ and $f(10) > 0$, then there must be a value t in the interval $[0, 10]$ such that $f(t) = 0$. If $f(t) = 0$, then $s(t) - r(t) = 0$, which gives us $s(t) = r(t)$. Therefore, at some time t, where $0 \leq t \leq 10$, the position functions for the run up and the run down are equal.

100. Let $V = \frac{4}{3}\pi r^3$ be the volume of a sphere of radius r. V is continuous on $[1, 5]$.

$V(1) = \frac{4}{3}\pi \approx 4.19$

$V(5) = \frac{4}{3}\pi(5^3) \approx 523.6$

Since $4.19 < 275 < 523.6$, the Intermediate Value Theorem implies that there is at least one value r between 1 and 5 such that $V(r) = 275$. (In fact, $r \approx 4.0341$.)

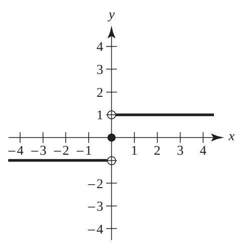
101. Suppose there exists x_1 in $[a, b]$ such that $f(x_1) > 0$ and there exists x_2 in $[a, b]$ such that $f(x_2) < 0$. Then by the Intermediate Value Theorem, $f(x)$ must equal zero for some value of x in $[x_1, x_2]$ (or $[x_2, x_1]$ if $x_2 < x_1$). Thus, f would have a zero in $[a, b]$, which is a contradiction. Therefore, $f(x) > 0$ for all x in $[a, b]$ or $f(x) < 0$ for all x in $[a, b]$.

102. Let c be any real number. Then $\lim\limits_{x \to c} f(x)$ does not exist since there are both rational and irrational numbers arbitrarily close to c. Therefore, f is not continuous at c.

103. If $x = 0$, then $f(0) = 0$ and $\lim\limits_{x \to 0} f(x) = 0$. Hence, f is continuous at $x = 0$.

If $x \neq 0$, then $\lim\limits_{t \to x} f(t) = 0$ for x rational, whereas $\lim\limits_{t \to x} f(t) = \lim\limits_{t \to x} kt = kx \neq 0$ for x irrational. Hence, f is not continuous for all $x \neq 0$.

104. $\text{sgn}(x) = \begin{cases} -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases}$

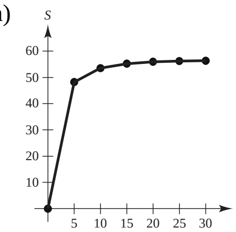


(a) $\lim\limits_{x \to 0^-} \text{sgn}(x) = -1$

(b) $\lim\limits_{x \to 0^+} \text{sgn}(x) = 1$

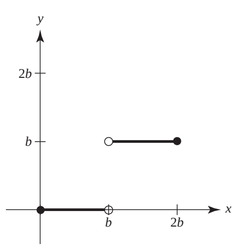
(c) $\lim\limits_{x \to 0} \text{sgn}(x)$ does not exist.

105. (a)



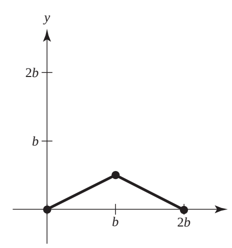
(b) There appears to be a limiting speed and a possible cause is air resistance.

106. (a) $f(x) = \begin{cases} 0 & 0 \le x < b \\ b & b < x \le 2b \end{cases}$



NOT continuous at $x = b$.

(b) $g(x) = \begin{cases} \dfrac{x}{2} & 0 \le x \le b \\ b - \dfrac{x}{2} & b < x \le 2b \end{cases}$



Continuous on $[0, 2b]$.

107. $f(x) = \begin{cases} 1 - x^2, & x \le c \\ x, & x > c \end{cases}$

f is continuous for $x < c$ and for $x > c$. At $x = c$, you need $1 - c^2 = c$. Solving $c^2 + c - 1$, you obtain

$$c = \frac{-1 \pm \sqrt{1 + 4}}{2} = \frac{-1 \pm \sqrt{5}}{2}.$$

108. Let y be a real number. If $y = 0$, then $x = 0$. If $y > 0$, then let $0 < x_0 < \pi/2$ such that $M = \tan x_0 > y$ (this is possible since the tangent function increases without bound on $[0, \pi/2)$). By the Intermediate Value Theorem, $f(x) = \tan x$ is continuous on $[0, x_0]$ and $0 < y < M$, which implies that there exists x between 0 and x_0 such that $\tan x = y$. The argument is similar if $y < 0$.

109. $f(x) = \dfrac{\sqrt{x + c^2} - c}{x}$, $c > 0$

Domain: $x + c^2 \geq 0 \implies x \geq -c^2$ and $x \neq 0$, $[-c^2, 0) \cup (0, \infty)$

$$\lim_{x\to 0} \frac{\sqrt{x + c^2} - c}{x} = \lim_{x\to 0} \frac{\sqrt{x + c^2} - c}{x} \cdot \frac{\sqrt{x + c^2} + c}{\sqrt{x + c^2} + c}$$

$$= \lim_{x\to 0} \frac{(x + c^2) - c^2}{x\left[\sqrt{x + c^2} + c\right]} = \lim_{x\to 0} \frac{1}{\sqrt{x + c^2} + c} = \frac{1}{2c}$$

Define $f(0) = 1/(2c)$ to make f continuous at $x = 0$.

110. **1.** $f(c)$ is defined.

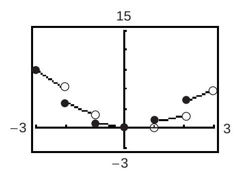
2. $\lim_{x\to c} f(x) = \lim_{\Delta x\to 0} f(c + \Delta x) = f(c)$ exists.

[Let $x = c + \Delta x$. As $x \to c$, $\Delta x \to 0$]

3. $\lim_{x\to c} f(x) = f(c)$.

Therefore, f is continuous at $x = c$.

111. $h(x) = x[\![x]\!]$



h has nonremovable discontinuities at $x = \pm 1, \pm 2, \pm 3, \ldots$.

112. (a) Define $f(x) = f_2(x) - f_1(x)$. Since f_1 and f_2 are continuous on $[a, b]$, so is f.

$f(a) = f_2(a) - f_1(a) > 0$ and $f(b) = f_2(b) - f_1(b) < 0$

By the Intermediate Value Theorem, there exists c in $[a, b]$ such that $f(c) = 0$.

$f(c) = f_2(c) - f_1(c) = 0 \implies f_1(c) = f_2(c)$

(b) Let $f_1(x) = x$ and $f_2(x) = \cos x$, continuous on $[0, \pi/2]$, $f_1(0) < f_2(0)$ and $f_1(\pi/2) > f_2(\pi/2)$.

Hence by part (a), there exists c in $[0, \pi/2]$ such that $c = \cos(c)$.

Using a graphing utility, $c \approx 0.739$.

113. The statement is true.

If $y \geq 0$ and $y \leq 1$, then $y(y - 1) \leq 0 \leq x^2$, as desired. So assume $y > 1$. There are now two cases.

Case 1: If $x \leq y - \frac{1}{2}$, then $2x + 1 \leq 2y$ and

$$\begin{aligned} y(y-1) &= y(y+1) - 2y \\ &\leq (x+1)^2 - 2y \\ &= x^2 + 2x + 1 - 2y \\ &\leq x^2 + 2y - 2y \\ &= x^2 \end{aligned}$$

Case 2: If $x \geq y - \frac{1}{2}$

$$\begin{aligned} x^2 &\geq \left(y - \tfrac{1}{2}\right)^2 \\ &= y^2 - y + \tfrac{1}{4} \\ &> y^2 - y \\ &= y(y-1) \end{aligned}$$

In both cases, $y(y - 1) \leq x^2$.

114. $P(1) = P(0^2 + 1) = P(0)^2 + 1 = 1$

$P(2) = P(1^2 + 1) = P(1)^2 + 1 = 2$

$P(5) = P(2^2 + 1) = P(2)^2 + 1 = 5$

Continuing this pattern, we see that $P(x) = x$ for infinitely many values of x.
Hence, the finite degree polynomial must be constant: $P(x) = x$ for all x.

Section 1.5 Infinite Limits

1. $\lim_{x\to -2^+} 2\left|\frac{x}{x^2-4}\right| = \infty$

$\lim_{x\to -2^-} 2\left|\frac{x}{x^2-4}\right| = \infty$

2. $\lim_{x\to -2^+} \frac{1}{x+2} = \infty$

$\lim_{x\to -2^-} \frac{1}{x+2} = -\infty$

3. $\lim_{x\to -2^+} \tan\frac{\pi x}{4} = -\infty$

$\lim_{x\to -2^-} \tan\frac{\pi x}{4} = \infty$

4. $\lim_{x\to -2^+} \sec\frac{\pi x}{4} = \infty$

$\lim_{x\to -2^-} \sec\frac{\pi x}{4} = -\infty$

5. $f(x) = \dfrac{1}{x^2-9}$

x	-3.5	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9	-2.5
$f(x)$	0.308	1.639	16.64	166.6	-166.7	-16.69	-1.695	-0.364

$\lim_{x\to -3^-} f(x) = \infty$

$\lim_{x\to -3^+} f(x) = -\infty$

6. $f(x) = \dfrac{x}{x^2-9}$

x	-3.5	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9	-2.5
$f(x)$	-1.077	-5.082	-50.08	-500.1	499.9	49.92	4.915	0.9091

$\lim_{x\to -3^-} f(x) = -\infty$

$\lim_{x\to -3^+} f(x) = \infty$

7. $f(x) = \dfrac{x^2}{x^2-9}$

x	-3.5	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9	-2.5
$f(x)$	3.769	15.75	150.8	1501	-1499	-149.3	-14.25	-2.273

$\lim_{x\to -3^-} f(x) = \infty$

$\lim_{x\to -3^+} f(x) = -\infty$

8. $f(x) = \sec\dfrac{\pi x}{6}$

x	-3.5	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9	-2.5
$f(x)$	-3.864	-19.11	-191.0	-1910	1910	191.0	19.11	3.864

$\lim_{x\to -3^-} f(x) = -\infty$

$\lim_{x\to -3^+} f(x) = \infty$

9. $\lim_{x\to 0^+} \frac{1}{x^2} = \infty = \lim_{x\to 0^-} \frac{1}{x^2}$

Therefore, $x = 0$ is a vertical asymptote.

10. $\lim_{x\to 2^+} \frac{4}{(x-2)^3} = \infty$

$\lim_{x\to 2^-} \frac{4}{(x-2)^3} = -\infty$

Therefore, $x = 2$ is a vertical asymptote.

11. $\lim_{x\to 2^+} \frac{x^2-2}{(x-2)(x+1)} = \infty$

$\lim_{x\to 2^-} \frac{x^2-2}{(x-2)(x+1)} = -\infty$

Therefore, $x = 2$ is a vertical asymptote.

$\lim_{x\to -1^+} \frac{x^2-2}{(x-2)(x+1)} = \infty$

$\lim_{x\to -1^-} \frac{x^2-2}{(x-2)(x+1)} = -\infty$

Therefore, $x = -1$ is a vertical asymptote.

12. $\lim_{x\to 0^-} \frac{2+x}{x^2(1-x)} = \lim_{x\to 0^+} \frac{2+x}{x^2(1-x)} = \infty$

Therefore, $x = 0$ is a vertical asymptote.

$\lim_{x\to 1^-} \frac{2+x}{x^2(1-x)} = \infty$

$\lim_{x\to 1^+} \frac{2+x}{x^2(1-x)} = -\infty$

Therefore, $x = 1$ is a vertical asymptote.

13. $\lim_{x\to -2^-} \frac{x^2}{x^2-4} = \infty$ and $\lim_{x\to -2^+} \frac{x^2}{x^2-4} = -\infty$

Therefore, $x = -2$ is a vertical asymptote.

$\lim_{x\to 2^-} \frac{x^2}{x^2-4} = -\infty$ and $\lim_{x\to 2^+} \frac{x^2}{x^2-4} = \infty$

Therefore, $x = 2$ is a vertical asymptote.

14. No vertical asymptote since the denominator is never zero.

15. No vertical asymptote since the denominator is never zero.

16. $\lim_{s\to -5^-} h(s) = -\infty$ and $\lim_{s\to -5^+} h(s) = \infty$.

Therefore, $s = -5$ is a vertical asymptote.

$\lim_{s\to 5^-} h(s) = -\infty$ and $\lim_{s\to 5^+} h(s) = \infty$.

Therefore, $s = 5$ is a vertical asymptote.

17. $f(x) = \tan 2x = \frac{\sin 2x}{\cos 2x}$ has vertical asymptotes at

$$x = \frac{(2n+1)\pi}{4} = \frac{\pi}{4} + \frac{n\pi}{2},\ n \text{ any integer.}$$

18. $f(x) = \sec \pi x = \frac{1}{\cos \pi x}$ has vertical asymptotes at

$$x = \frac{2n+1}{2},\ n \text{ any integer.}$$

19. $\lim_{t\to 0^+} \left(1 - \frac{4}{t^2}\right) = -\infty = \lim_{t\to 0^-} \left(1 - \frac{4}{t^2}\right)$

Therefore, $t = 0$ is a vertical asymptote.

20. $g(x) = \frac{(1/2)x^3 - x^2 - 4x}{3x^2 - 6x - 24} = \frac{1}{6}\frac{x(x^2-2x-8)}{x^2-2x-8}$

$$= \frac{1}{6}x, \quad x \neq -2, 4$$

No vertical asymptotes. The graph has holes at $x = -2$ and $x = 4$.

21. $\lim_{x\to -2^+} \dfrac{x}{(x+2)(x-1)} = \infty$

$\lim_{x\to -2^-} \dfrac{x}{(x+2)(x-1)} = -\infty$

Therefore, $x = -2$ is a vertical asymptote.

$\lim_{x\to 1^+} \dfrac{x}{(x+2)(x-1)} = \infty$

$\lim_{x\to 1^-} \dfrac{x}{(x+2)(x-1)} = -\infty$

Therefore, $x = 1$ is a vertical asymptote.

22. $f(x) = \dfrac{4(x^2+x-6)}{x(x^3-2x^2-9x+18)} = \dfrac{4(x+3)(x-2)}{x(x-2)(x^2-9)} = \dfrac{4}{x(x-3)}, x \neq -3, 2$

Vertical asymptotes at $x = 0$ and $x = 3$. The graph has holes at $x = -3$ and $x = 2$.

23. $f(x) = \dfrac{x^3+1}{x+1} = \dfrac{(x+1)(x^2-x+1)}{x+1}$

has no vertical asymptote since

$\lim_{x\to -1} f(x) = \lim_{x\to -1} (x^2-x+1) = 3.$

The graph has a hole at $x = -1$.

24. $h(x) = \dfrac{x^2-4}{x^3+2x^2+x+2} = \dfrac{(x+2)(x-2)}{(x+2)(x^2+1)}$

has no vertical asymptote since

$\lim_{x\to -2} h(x) = \lim_{x\to -2} \dfrac{x-2}{x^2+1} = -\dfrac{4}{5}.$

The graph has a hole at $x = -2$.

25. $f(x) = \dfrac{(x-5)(x+3)}{(x-5)(x^2+1)} = \dfrac{x+3}{x^2+1}, x \neq 5$

No vertical asymptotes. The graph has a hole at $x = 5$.

26. $h(t) = \dfrac{t(t-2)}{(t-2)(t+2)(t^2+4)} = \dfrac{t}{(t+2)(t^2+4)}, t \neq 2$

Vertical asymptote at $t = -2$. The graph has a hole at $t = 2$.

27. $s(t) = \dfrac{t}{\sin t}$ has vertical asymptotes at $t = n\pi$, n a nonzero integer. There is no vertical asymptote at $t = 0$ since

$\lim_{t\to 0} \dfrac{t}{\sin t} = 1.$

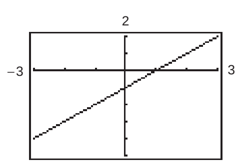
28. $g(\theta) = \dfrac{\tan\theta}{\theta} = \dfrac{\sin\theta}{\theta\cos\theta}$ has vertical asymptotes at

$\theta = \dfrac{(2n+1)\pi}{2} = \dfrac{\pi}{2} + n\pi$, n any integer.

There is no vertical asymptote at $\theta = 0$ since

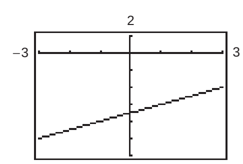
$\lim_{\theta\to 0} \dfrac{\tan\theta}{\theta} = 1.$

29. $\lim_{x\to -1} \dfrac{x^2-1}{x+1} = \lim_{x\to -1} (x-1) = -2$



Removable discontinuity at $x = -1$

30. $\lim_{x\to -1} \dfrac{x^2-6x-7}{x+1} = \lim_{x\to -1} (x-7) = -8$

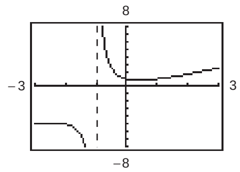


Removable discontinuity at $x = -1$

31. $\lim_{x\to -1^+} \dfrac{x^2+1}{x+1} = \infty$

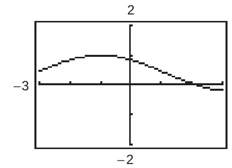
$\lim_{x\to -1^-} \dfrac{x^2+1}{x+1} = -\infty$

Vertical asymptote at $x = -1$



32. $\lim_{x\to -1} \dfrac{\sin(x+1)}{x+1} = 1$

Removable discontinuity at $x = -1$



33. $\lim_{x\to 2^+} \dfrac{x-3}{x-2} = -\infty$

34. $\lim_{x\to 1^+} \dfrac{2+x}{1-x} = -\infty$

35. $\lim_{x\to 3^+} \dfrac{x^2}{(x-3)(x+3)} = \infty$

36. $\lim_{x\to 4^-} \dfrac{x^2}{x^2+16} = \dfrac{1}{2}$

37. $\lim_{x\to -3^-} \dfrac{x^2+2x-3}{x^2+x-6} = \lim_{x\to -3^-} \dfrac{(x-1)(x+3)}{(x-2)(x+3)} = \lim_{x\to -3^-} \dfrac{x-1}{x-2} = \dfrac{4}{5}$

38. $\lim_{x\to -(1/2)^+} \dfrac{6x^2+x-1}{4x^2-4x-3} = \lim_{x\to -(1/2)^+} \dfrac{(3x-1)(2x+1)}{(2x-3)(2x+1)} = \lim_{x\to -(1/2)^+} \dfrac{3x-1}{2x-3} = \dfrac{5}{8}$

39. $\lim_{x\to 1} \dfrac{x^2-x}{(x^2+1)(x-1)} = \lim_{x\to 1} \dfrac{x}{x^2+1} = \dfrac{1}{2}$

40. $\lim_{x\to 3} \dfrac{x-2}{x^2} = \dfrac{1}{9}$

41. $\lim_{x\to 0^-} \left(1 + \dfrac{1}{x}\right) = -\infty$

42. $\lim_{x\to 0^-} \left(x^2 - \dfrac{1}{x}\right) = \infty$

43. $\lim_{x\to 0^+} \dfrac{2}{\sin x} = \infty$

44. $\lim_{x\to (\pi/2)^+} \dfrac{-2}{\cos x} = \infty$

45. $\lim_{x\to \pi} \dfrac{\sqrt{x}}{\csc x} = \lim_{x\to \pi} \left(\sqrt{x}\sin x\right) = 0$

46. $\lim_{x\to 0} \dfrac{(x+2)}{\cot x} = \lim_{x\to 0} [(x+2)\tan x] = 0$

47. $\lim_{x\to (1/2)^-} x\sec(\pi x) = \infty$ and $\lim_{x\to (1/2)^+} x\sec(\pi x) = -\infty$.

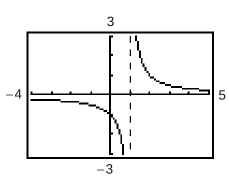
Therefore, $\lim_{x\to (1/2)} x\sec(\pi x)$ does not exist.

48. $\lim_{x\to (1/2)^-} x^2 \tan \pi x = \infty$ and $\lim_{x\to (1/2)^+} x^2 \tan \pi x = -\infty$.

Therefore, $\lim_{x\to (1/2)} x^2 \tan \pi x$ does not exist.

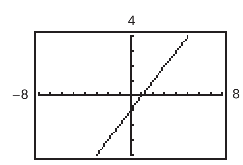
49. $f(x) = \dfrac{x^2+x+1}{x^3-1} = \dfrac{x^2+x+1}{(x-1)(x^2+x+1)}$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \dfrac{1}{x-1} = \infty$



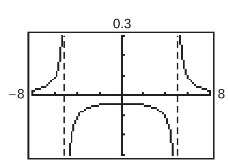
50. $f(x) = \dfrac{x^3-1}{x^2+x+1} = \dfrac{(x-1)(x^2+x+1)}{x^2+x+1}$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x-1) = 0$



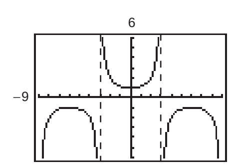
51. $f(x) = \dfrac{1}{x^2-25}$

$\lim_{x\to 5^-} f(x) = -\infty$



52. $f(x) = \sec \dfrac{\pi x}{6}$

$\lim_{x\to 3^+} f(x) = -\infty$



53. A limit in which $f(x)$ increases or decreases without bound as x approaches c is called an infinite limit. ∞ is not a number. Rather, the symbol

$$\lim_{x \to c} f(x) = \infty$$

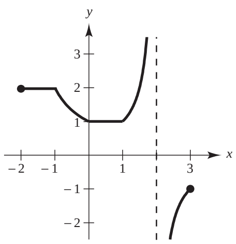
says how the limit fails to exist.

54. The line $x = c$ is a vertical asymptote if the graph of f approaches $\pm\infty$ as x approaches c.

55. One answer is $f(x) = \dfrac{x-3}{(x-6)(x+2)} = \dfrac{x-3}{x^2-4x-12}$.

56. No. For example, $f(x) = \dfrac{1}{x^2+1}$ has no vertical asymptote.

57.



58. $P = \dfrac{k}{V}$

$$\lim_{V \to 0^+} \frac{k}{V} = k(\infty) = \infty$$

(In this case we know that $k > 0$.)

59. (a) $r = 50\pi \sec^2 \dfrac{\pi}{6} = \dfrac{200\pi}{3}$ ft/sec

(b) $r = 50\pi \sec^2 \dfrac{\pi}{3} = 200\pi$ ft/sec

(c) $\displaystyle\lim_{\theta \to (\pi/2)^-} \left[50\pi \sec^2 \theta\right] = \infty$

60. $C = \dfrac{528x}{100-x}, \ 0 \le x < 100$

(a) $C(25) = \$176$ million

(b) $C(50) = \$528$ million

(c) $C(75) = \$1584$ million

(d) $\displaystyle\lim_{x \to 100^-} \frac{528}{100-x} = \infty$. Thus, it is not possible.

61. $m = \dfrac{m_0}{\sqrt{1-(v^2/c^2)}}$

$$\lim_{v \to c^-} m = \lim_{v \to c^-} \frac{m_0}{\sqrt{1-(v^2/c^2)}} = \infty$$

62. (a) $r = \dfrac{2(7)}{\sqrt{625-49}} = \dfrac{7}{12}$ ft/sec

(b) $r = \dfrac{2(15)}{\sqrt{625-225}} = \dfrac{3}{2}$ ft/sec

(c) $\displaystyle\lim_{x \to 25^-} \frac{2x}{\sqrt{625-x^2}} = \infty$

63. (a) Average speed $= \dfrac{\text{Total distance}}{\text{Total time}}$

$$50 = \frac{2d}{(d/x)+(d/y)}$$

$$50 = \frac{2xy}{y+x}$$

$$50y + 50x = 2xy$$

$$50x = 2xy - 50y$$

$$50x = 2y(x-25)$$

$$\frac{25x}{x-25} = y$$

Domain: $x > 25$

(b)

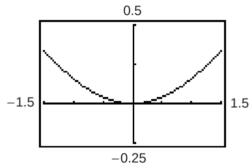
x	30	40	50	60
y	150	6.667	50	42.857

(c) $\displaystyle\lim_{x \to 25^+} \frac{25x}{x-25} = \infty$

As x gets close to 25 mph, y becomes larger and larger.

64. (a)

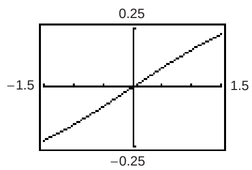
x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.0411	0.0067	0.0017	≈ 0	≈ 0	≈ 0



$$\lim_{x \to 0^+} \frac{x - \sin x}{x} = 0$$

(b)

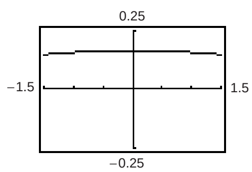
x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.0823	0.0333	0.0167	0.0017	≈ 0	≈ 0



$$\lim_{x \to 0^+} \frac{x - \sin x}{x^2} = 0$$

(c)

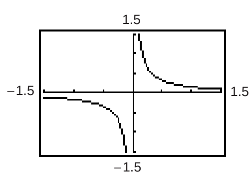
x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.1646	0.1663	0.1666	0.1667	0.1667	0.1667



$$\lim_{x \to 0^+} \frac{x - \sin x}{x^3} = 0.1667 \ \ (1/6)$$

(d)

x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.3292	0.8317	1.6658	16.67	166.7	1667.0



$$\lim_{x \to 0^+} \frac{x - \sin x}{x^4} = \infty$$

For $n > 3$, $\displaystyle\lim_{x \to 0^+} \frac{x - \sin x}{x^n} = \infty$.

65. (a) $A = \frac{1}{2}bh - \frac{1}{2}r^2\,\theta = \frac{1}{2}(10)(10 \tan \theta) - \frac{1}{2}(10)^2\theta$

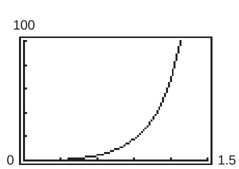
$= 50 \tan \theta - 50\,\theta$

Domain: $\left(0, \frac{\pi}{2}\right)$

(b)

θ	0.3	0.6	0.9	1.2	1.5
$f(\theta)$	0.47	4.21	18.0	68.6	630.1

(c)



(d) $\displaystyle\lim_{\theta \to \pi/2^-} A = \infty$

66. (a) Because the circumference of the motor is half that of the saw arbor, the saw makes $1700/2 = 850$ revolutions per minute.

(b) The direction of rotation is reversed.

(c) $2(20 \cot \phi) + 2(10 \cot \phi)$: straight sections. The angle subtended in each circle is

$$2\pi - \left(2\left(\frac{\pi}{2} - \phi\right)\right) = \pi + 2\phi.$$

Thus, the length of the belt around the pulleys is

$$20(\pi + 2\phi) + 10(\pi + 2\phi) = 30(\pi + 2\phi).$$

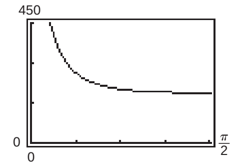
Total length $= 60 \cot \phi + 30(\pi + 2\phi)$

Domain: $\left(0, \frac{\pi}{2}\right)$

(d)

ϕ	0.3	0.6	0.9	1.2	1.5
L	306.2	217.9	195.9	189.6	188.5

(e)



(f) $\lim_{\phi \to (\pi/2)^-} L = 60\pi \approx 188.5$

(All the belts are around pulleys.)

(g) $\lim_{\phi \to 0^+} L = \infty$

67. False; for instance, let

$$f(x) = \frac{x^2 - 1}{x - 1}.$$

The graph of f has a hole at $(1, 2)$, not a vertical asymptote.

68. False; for instance, let

$$f(x) = \frac{x^2 - 1}{x - 1} \text{ or}$$

$$g(x) = \frac{x}{x^2 + 1}.$$

69. True

70. False; let

$$f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 3, & x = 0. \end{cases}$$

The graph of f has a vertical asymptote at $x = 0$, but $f(0) = 3$.

71. Let $f(x) = \frac{1}{x^2}$ and $g(x) = \frac{1}{x^4}$, and $c = 0$.

$\lim_{x \to 0} \frac{1}{x^2} = \infty$ and $\lim_{x \to 0} \frac{1}{x^4} = \infty$, but

$$\lim_{x \to 0} \left(\frac{1}{x^2} - \frac{1}{x^4}\right) = \lim_{x \to 0} \left(\frac{x^2 - 1}{x^4}\right) = -\infty \neq 0.$$

72. Given $\lim_{x \to c} f(x) = \infty$ and $\lim_{x \to c} g(x) = L$:

(2) Product:
If $L > 0$, then for $\varepsilon = L/2 > 0$ there exists $\delta_1 > 0$ such that $|g(x) - L| < L/2$ whenever $0 < |x - c| < \delta_1$. Thus, $L/2 < g(x) < 3L/2$. Since $\lim_{x \to c} f(x) = \infty$ then for $M > 0$, there exists $\delta_2 > 0$ such that $f(x) > M(2/L)$ whenever $|x - c| < \delta_2$. Let δ be the smaller of δ_1 and δ_2. Then for $0 < |x - c| < \delta$, we have $f(x)g(x) > M(2/L)(L/2) = M$. Therefore $\lim_{x \to c} f(x)g(x) = \infty$. The proof is similar for $L < 0$.

(3) Quotient: Let $\varepsilon > 0$ be given.

There exists $\delta_1 > 0$ such that $f(x) > 3L/2\varepsilon$ whenever $0 < |x - c| < \delta_1$ and there exists $\delta_2 > 0$ such that $|g(x) - L| < L/2$ whenever $0 < |x - c| < \delta_2$. This inequality gives us $L/2 < g(x) < 3L/2$. Let δ be the smaller of δ_1 and δ_2. Then for $0 < |x - c| < \delta$, we have

$$\left|\frac{g(x)}{f(x)}\right| < \frac{3L/2}{3L/2\varepsilon} = \varepsilon.$$

Therefore, $\lim_{x \to c} \frac{g(x)}{f(x)} = 0$.

73. Given $\lim_{x \to c} f(x) = \infty$, let $g(x) = 1$. then $\lim_{x \to c} \frac{g(x)}{f(x)} = 0$ by Theorem 1.15.

74. Given $\lim_{x\to c} \frac{1}{f(x)} = 0$. Suppose $\lim_{x\to c} f(x)$ exists and equals L.

Then, $\lim_{x\to c} \frac{1}{f(x)} = \frac{\lim_{x\to c} 1}{\lim_{x\to c} f(x)} = \frac{1}{L} = 0.$

This is not possible. Thus, $\lim_{x\to c} f(x)$ does not exist.

75. $f(x) = \frac{1}{x-3}$ is defined for all $x > 3$. Let $M > 0$ be given. We need $\delta > 0$ such that $f(x) = \frac{1}{x-3} > M$ whenever $3 < x < 3 + \delta$.

Equivalently, $x - 3 < \frac{1}{M}$ whenever $|x - 3| < \delta$, $x > 3$.

So take $\delta = \frac{1}{M}$. Then for $x > 3$ and $|x - 3| < \delta$,

$\frac{1}{x-3} > \frac{1}{8} = M$ and hence $f(x) > M$.

76. $f(x) = \frac{1}{x-4}$ is defined for all $x < 4$. Let $N < 0$ be given. We need $\delta > 0$ such that $f(x) = \frac{1}{x-4} < N$ whenever $4 - \delta < x < 4$. Equivalently, $x - 4 > \frac{1}{N}$ whenever $|x - 4| < \delta$, $x < 4$. Equivalently, $\frac{1}{|x-4|} < -\frac{1}{N}$ whenever $|x - 4| < \delta$, $x < 4$. So take $\delta = -\frac{1}{N}$. Note that $\delta > 0$ because $N < 0$. For $|x - 4| < \delta$ and $x < 4$, $\frac{1}{|x-4|} > \frac{1}{\delta} = -N$, and $\frac{1}{x-4} = -\frac{1}{|x-4|} < N$.

Review Exercises for Chapter 1

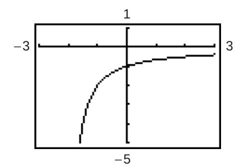
1. Calculus required. Using a graphing utility, you can estimate the length to be 8.3. Or, the length is slightly longer than the distance between the two points, approximately 8.25.

2. Precalculus. $L = \sqrt{(9-1)^2 + (3-1)^2} \approx 8.25$

3. $f(x) = \dfrac{\frac{4}{x+2} - 2}{x}$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	-1.0526	-1.0050	-1.0005	-0.9995	-0.9950	-0.9524

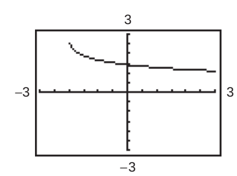
$\lim_{x\to 0} f(x) \approx -1.0$



4.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	1.432	1.416	1.414	1.414	1.413	1.397

$\lim_{x\to 0} f(x) \approx 1.414$



5. $h(x) = \frac{x^2 - 2x}{x}$

(a) $\lim_{x\to 0} h(x) = -2$

(b) $\lim_{x\to -1} h(x) = -3$

6. $g(x) = \frac{3x}{x-2}$

(a) $\lim_{x\to 2} g(x)$ does not exist.

(b) $\lim_{x\to 0} g(x) = 0$

7. $\lim_{x\to 1} (3 - x) = 3 - 1 = 2$

Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon$. Then for $0 < |x - 1| < \delta = \varepsilon$, you have

$$\begin{aligned} |x - 1| &< \varepsilon \\ |1 - x| &< \varepsilon \\ |(3 - x) - 2| &< \varepsilon \\ |f(x) - L| &< \varepsilon. \end{aligned}$$

8. $\lim_{x\to 9} \sqrt{x} = \sqrt{9} = 3$

Let $\varepsilon > 0$ be given. We need

$\left|\sqrt{x} - 3\right| < \varepsilon \implies \left|\sqrt{x} + 3\right|\left|\sqrt{x} - 3\right| < \varepsilon\left|\sqrt{x} + 3\right| \implies |x - 9| < \varepsilon\left|\sqrt{x} + 3\right|.$

Assuming $4 < x < 16$, you can choose $\delta = 5\varepsilon$.

Hence, for $0 < |x - 9| < \delta = 5\varepsilon$, you have

$$\begin{aligned} |x - 9| &< 5\varepsilon < \left|\sqrt{x} + 3\right|\varepsilon \\ \left|\sqrt{x} - 3\right| &< \varepsilon \\ |f(x) - L| &< \varepsilon. \end{aligned}$$

9. $\lim_{x\to 2} (x^2 - 3) = 1$

Let $\varepsilon > 0$ be given. We need $|x^2 - 3 - 1| < \varepsilon \implies |x^2 - 4| = |(x - 2)(x + 2)| < \varepsilon \implies |x - 2| < \dfrac{1}{|x + 2|}\varepsilon.$

Assuming, $1 < x < 3$, you can choose $\delta = \varepsilon/5$. Hence, for $0 < |x - 2| < \delta = \varepsilon/5$ you have

$$\begin{aligned} |x - 2| &< \frac{\varepsilon}{5} < \frac{1}{|x + 2|}\varepsilon \\ |x - 2||x + 2| &< \varepsilon \\ |x^2 - 4| &< \varepsilon \\ |(x^2 - 3) - 1| &< \varepsilon \\ |f(x) - L| &< \varepsilon. \end{aligned}$$

10. $\lim_{x\to 5} 9 = 9$. Let $\varepsilon > 0$ be given. δ can be any positive number. Hence, for $0 < |x - 5| < \delta$, you have

$$\begin{aligned} |9 - 9| &< \varepsilon \\ |f(x) - L| &< \varepsilon. \end{aligned}$$

11. $\lim_{t\to 4} \sqrt{t + 2} = \sqrt{4 + 2} = \sqrt{6} \approx 2.45$

12. $\lim_{y\to 4} 3|y - 1| = 3|4 - 1| = 9$

13. $\lim_{t\to -2} \dfrac{t + 2}{t^2 - 4} = \lim_{t\to -2} \dfrac{1}{t - 2} = -\dfrac{1}{4}$

14. $\lim_{t\to 3} \dfrac{t^2 - 9}{t - 3} = \lim_{t\to 3} (t + 3) = 6$

15. $$\lim_{x\to 4} \frac{\sqrt{x} - 2}{x - 4} = \lim_{x\to 4} \frac{\sqrt{x} - 2}{\left(\sqrt{x} - 2\right)\left(\sqrt{x} + 2\right)} = \lim_{x\to 4} \frac{1}{\sqrt{x} + 2} = \frac{1}{\sqrt{4} + 2} = \frac{1}{4}$$

16. $$\lim_{x\to 0} \frac{\sqrt{4 + x} - 2}{x} = \lim_{x\to 0} \frac{\sqrt{4 + x} - 2}{x} \cdot \frac{\sqrt{4 + x} + 2}{\sqrt{4 + x} + 2} = \lim_{x\to 0} \frac{1}{\sqrt{4 + x} + 2} = \frac{1}{4}$$

17. $$\lim_{x\to 0} \frac{[1/(x + 1)] - 1}{x} = \lim_{x\to 0} \frac{1 - (x + 1)}{x(x + 1)} = \lim_{x\to 0} \frac{-1}{x + 1} = -1$$

18. $$\begin{aligned} \lim_{s\to 0} \frac{\left(1/\sqrt{1 + s}\right) - 1}{s} &= \lim_{s\to 0} \left[\frac{\left(1/\sqrt{1 + s}\right) - 1}{s} \cdot \frac{\left(1/\sqrt{1 + s}\right) + 1}{\left(1/\sqrt{1 + s}\right) + 1}\right] \\ &= \lim_{s\to 0} \frac{[1/(1 + s)] - 1}{s\left[\left(1/\sqrt{1 + s}\right) + 1\right]} = \lim_{s\to 0} \frac{-1}{(1 + s)\left[\left(1/\sqrt{1 + s}\right) + 1\right]} = -\frac{1}{2} \end{aligned}$$

19. $$\begin{aligned} \lim_{x\to -5} \frac{x^3 + 125}{x + 5} &= \lim_{x\to -5} \frac{(x + 5)(x^2 - 5x + 25)}{x + 5} \\ &= \lim_{x\to -5} (x^2 - 5x + 25) = 75 \end{aligned}$$

20. $$\begin{aligned} \lim_{x\to -2} \frac{x^2 - 4}{x^3 + 8} &= \lim_{x\to -2} \frac{(x + 2)(x - 2)}{(x + 2)(x^2 - 2x + 4)} \\ &= \lim_{x\to -2} \frac{x - 2}{x^2 - 2x + 4} = -\frac{4}{12} = -\frac{1}{3} \end{aligned}$$

21. $\lim_{x\to 0}\frac{1-\cos x}{\sin x}=\lim_{x\to 0}\left(\frac{x}{\sin x}\right)\left(\frac{1-\cos x}{x}\right)=(1)(0)=0$

22. $\lim_{x\to(\pi/4)}\frac{4x}{\tan x}=\frac{4(\pi/4)}{1}=\pi$

23. $\lim_{\Delta x\to 0}\frac{\sin[(\pi/6)+\Delta x]-(1/2)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\sin(\pi/6)\cos\Delta x+\cos(\pi/6)\sin\Delta x-(1/2)}{\Delta x}$

$$=\lim_{\Delta x\to 0}\frac{1}{2}\cdot\frac{(\cos\Delta x-1)}{\Delta x}+\lim_{\Delta x\to 0}\frac{\sqrt{3}}{2}\cdot\frac{\sin\Delta x}{\Delta x}=0+\frac{\sqrt{3}}{2}(1)=\frac{\sqrt{3}}{2}$$

24. $\lim_{\Delta x\to 0}\frac{\cos(\pi+\Delta x)+1}{\Delta x}=\lim_{\Delta x\to 0}\frac{\cos\pi\cos\Delta x-\sin\pi\sin\Delta x+1}{\Delta x}$

$$=\lim_{\Delta x\to 0}\left[-\frac{(\cos\Delta x-1)}{\Delta x}\right]-\lim_{\Delta x\to 0}\left[\sin\pi\frac{\sin\Delta x}{\Delta x}\right]$$

$$=-0-(0)(1)=0$$

25. $\lim_{x\to c}[f(x)\cdot g(x)]=\left(-\frac{3}{4}\right)\left(\frac{2}{3}\right)=-\frac{1}{2}$

26. $\lim_{x\to c}[f(x)+2g(x)]=-\frac{3}{4}+2\left(\frac{2}{3}\right)=\frac{7}{12}$

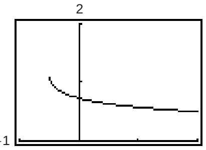
27. $f(x)=\frac{\sqrt{2x+1}-\sqrt{3}}{x-1}$

(a)

x	1.1	1.01	1.001	1.0001
$f(x)$	0.5680	0.5764	0.5773	0.5773

$\lim_{x\to 1^+}\frac{\sqrt{2x+1}-\sqrt{3}}{x-1}\approx 0.577$ (Actual limit is $\sqrt{3}/3$.)

(b)



(c) $\lim_{x\to 1^+}\frac{\sqrt{2x+1}-\sqrt{3}}{x-1}=\lim_{x\to 1^+}\frac{\sqrt{2x+1}-\sqrt{3}}{x-1}\cdot\frac{\sqrt{2x+1}+\sqrt{3}}{\sqrt{2x+1}+\sqrt{3}}$

$$=\lim_{x\to 1^+}\frac{(2x+1)-3}{(x-1)\left(\sqrt{2x+1}+\sqrt{3}\right)}$$

$$=\lim_{x\to 1^+}\frac{2}{\sqrt{2x+1}+\sqrt{3}}$$

$$=\frac{2}{2\sqrt{3}}=\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$$

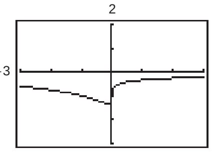
28. $f(x)=\frac{1-\sqrt[3]{x}}{x-1}$

(a)

x	1.1	1.01	1.001	1.0001
$f(x)$	−0.3228	−0.3322	−0.3332	−0.3333

$\lim_{x\to 1^+}\frac{1-\sqrt[3]{x}}{x-1}\approx -0.333$ (Actual limit is $-\frac{1}{3}$.)

(b)



(c) $\lim_{x\to 1^+}\frac{1-\sqrt[3]{x}}{x-1}=\lim_{x\to 1^+}\frac{1-\sqrt[3]{x}}{x-1}\cdot\frac{1+\sqrt[3]{x}+\left(\sqrt[3]{x}\right)^2}{1+\sqrt[3]{x}+\left(\sqrt[3]{x}\right)^2}$

$$=\lim_{x\to 1^+}\frac{1-x}{(x-1)\left[1+\sqrt[3]{x}+\left(\sqrt[3]{x}\right)^2\right]}$$

$$=\lim_{x\to 1^+}\frac{-1}{1+\sqrt[3]{x}+\left(\sqrt[3]{x}\right)^2}=-\frac{1}{3}$$

29. $\lim_{t\to a}\dfrac{s(a)-s(t)}{a-t} = \lim_{t\to 4}\dfrac{(-4.9(4)^2+200)-(-4.9t^2+200)}{4-t}$

$= \lim_{t\to 4}\dfrac{4.9(t-4)(t+4)}{4-t}$

$= \lim_{t\to 4} -4.9(t+4) = -39.2 \text{ m/sec}$

30. $s(t) = 0 \implies -4.9t^2 + 200 = 0$

$\implies t^2 \approx 40.816 \implies t \approx 6.39 \text{ sec}$

When $a = 6.39$, the velocity is approximately

$\lim_{t\to 6.39}\dfrac{s(a)-s(t)}{a-t} = \lim_{t\to 6.39} -4.9(a+t)$

$= -4.9(6.39 + 6.39)$

$= -62.6 \text{ m/sec.}$

31. $\lim_{x\to 3^-}\dfrac{|x-3|}{x-3} = \lim_{x\to 3^-}\dfrac{-(x-3)}{x-3}$

$= -1$

32. $\lim_{x\to 4}[\![x-1]\!]$ does not exist. The graph jumps from 2 to 3 at $x = 4$.

33. $\lim_{x\to 2} f(x) = 0$

34. $\lim_{x\to 1^+} g(x) = 1 + 1 = 2$

35. $\lim_{t\to 1} h(t)$ does not exist because $\lim_{t\to 1^-} h(t) = 1 + 1 = 2$ and $\lim_{t\to 1^+} h(t) = \frac{1}{2}(1+1) = 1.$

36. $\lim_{s\to -2} f(s) = 2$

37. $f(x) = [\![x+3]\!]$

$\lim_{x\to k^+}[\![x+3]\!] = k + 3$ where k is an integer.

$\lim_{x\to k^-}[\![x+3]\!] = k + 2$ where k is an integer.

Nonremovable discontinuity at each integer k
Continuous on $(k, k+1)$ for all integers k

38. $f(x) = \dfrac{3x^2 - x - 2}{x-1} = \dfrac{(3x+2)(x-1)}{x-1}$

f is continuous on $(-\infty, 1) \cup (1, \infty)$.

39. $f(x) = \dfrac{3x^2 - x - 2}{x-1} = \dfrac{(3x+2)(x-1)}{x-1}$

$\lim_{x\to 1} f(x) = \lim_{x\to 1}(3x+2) = 5$

Removable discontinuity at $x = 1$
Continuous on $(-\infty, 1) \cup (1, \infty)$

40. $f(x) = \begin{cases} 5 - x, & x \le 2 \\ 2x - 3, & x > 2 \end{cases}$

$\lim_{x\to 2^-}(5-x) = 3$

$\lim_{x\to 2^+}(2x-3) = 1$

Nonremovable discontinuity at $x = 2$
Continuous on $(-\infty, 2) \cup (2, \infty)$

41. $f(x) = \dfrac{1}{(x-2)^2}$

$\lim_{x\to 2}\dfrac{1}{(x-2)^2} = \infty$

Nonremovable discontinuity at $x = 2$
Continuous on $(-\infty, 2) \cup (2, \infty)$

42. $f(x) = \sqrt{\dfrac{x+1}{x}} = \sqrt{1 + \dfrac{1}{x}}$

$\lim_{x\to 0^+}\sqrt{1 + \dfrac{1}{x}} = \infty$

Domain: $(-\infty, -1], (0, \infty)$

Nonremovable discontinuity at $x = 0$
Continuous on $(-\infty, -1] \cup (0, \infty)$

43. $f(x) = \dfrac{3}{x+1}$

$\lim_{x\to 1^-} f(x) = -\infty$

$\lim_{x\to 1^+} f(x) = \infty$

Nonremovable discontinuity at $x = -1$
Continuous on $(-\infty, -1) \cup (-1, \infty)$

44. $f(x) = \dfrac{x+1}{2x+2}$

$\lim_{x\to -1}\dfrac{x+1}{2(x+1)} = \dfrac{1}{2}$

Removable discontinuity at $x = -1$
Continuous on $(-\infty, -1) \cup (-1, \infty)$

45. $f(x) = \csc \frac{\pi x}{2}$

Nonremovable discontinuities at each even integer.
Continuous on

$$(2k, 2k + 2)$$

for all integers k.

46. $f(x) = \tan 2x$

Nonremovable discontinuities when

$$x = \frac{(2n+1)\pi}{4}$$

Continuous on

$$\left(\frac{(2n-1)\pi}{4}, \frac{(2n+1)\pi}{4}\right)$$

for all integers n.

47. $f(2) = 5$

Find c so that $\lim_{x\to 2^+} (cx + 6) = 5$.

$$c(2) + 6 = 5$$
$$2c = -1$$
$$c = -\frac{1}{2}$$

48. $\lim_{x\to 1^+} (x + 1) = 2$

$\lim_{x\to 3^-} (x + 1) = 4$

Find b and c so that $\lim_{x\to 1^-} (x^2 + bx + c) = 2$ and $\lim_{x\to 3^+} (x^2 + bx + c) = 4$.

Consequently we get $1 + b + c = 2$ and $9 + 3b + c = 4$.

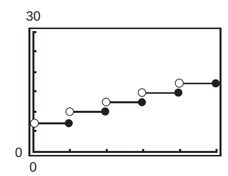
Solving simultaneously, $b = -3$ and $c = 4$.

49. f is continuous on $[1, 2]$. $f(1) = -1 < 0$ and $f(2) = 13 > 0$. Therefore by the Intermediate Value Theorem, there is at least one value c in $(1, 2)$ such that $2c^3 - 3 = 0$.

50. $C = 9.80 + 2.50[-[\![-x]\!] - 1],\ x > 0$

$= 9.80 - 2.50[[\![-x]\!] + 1]$

C has a nonremovable discontinuity at each integer.



51. $f(x) = \frac{x^2 - 4}{|x - 2|} = (x + 2)\left[\frac{x - 2}{|x - 2|}\right]$

(a) $\lim_{x\to 2^-} f(x) = -4$

(b) $\lim_{x\to 2^+} f(x) = 4$

(c) $\lim_{x\to 2} f(x)$ does not exist.

52. $f(x) = \sqrt{(x - 1)x}$

(a) Domain: $(-\infty, 0] \cup [1, \infty)$

(b) $\lim_{x\to 0^-} f(x) = 0$

(c) $\lim_{x\to 1^+} f(x) = 0$

53. $g(x) = 1 + \frac{2}{x}$

Vertical asymptote at $x = 0$

54. $h(x) = \frac{4x}{4 - x^2}$

Vertical asymptotes at $x = 2$ and $x = -2$

55. $f(x) = \frac{8}{(x - 10)^2}$

Vertical asymptote at $x = 10$

56. $f(x) = \csc \pi x$

Vertical asymptote at every integer k

57. $\lim_{x\to -2^-} \frac{2x^2 + x + 1}{x + 2} = -\infty$

58. $\lim_{x\to (1/2)^+} \frac{x}{2x - 1} = \infty$

59. $\lim_{x\to -1^+} \frac{x + 1}{x^3 + 1} = \lim_{x\to -1^+} \frac{1}{x^2 - x + 1} = \frac{1}{3}$

60. $\lim_{x\to -1^-} \frac{x + 1}{x^4 - 1} = \lim_{x\to -1^-} \frac{1}{(x^2 + 1)(x - 1)} = -\frac{1}{4}$

61. $\lim_{x\to 1^-} \frac{x^2 + 2x + 1}{x - 1} = -\infty$

62. $\lim_{x\to -1^+} \dfrac{x^2 - 2x + 1}{x + 1} = \infty$

63. $\lim_{x\to 0^+} \left(x - \dfrac{1}{x^3}\right) = -\infty$

64. $\lim_{x\to 2^-} \dfrac{1}{\sqrt[3]{x^2 - 4}} = -\infty$

65. $\lim_{x\to 0^+} \dfrac{\sin 4x}{5x} = \lim_{x\to 0^+} \left[\dfrac{4}{5}\left(\dfrac{\sin 4x}{4x}\right)\right] = \dfrac{4}{5}$

66. $\lim_{x\to 0^+} \dfrac{\sec x}{x} = \infty$

67. $\lim_{x\to 0^+} \dfrac{\csc 2x}{x} = \lim_{x\to 0^+} \dfrac{1}{x \sin 2x} = \infty$

68. $\lim_{x\to 0^-} \dfrac{\cos^2 x}{x} = -\infty$

69. $C = \dfrac{80{,}000p}{100 - p},\ 0 \le p < 100$

(a) $C(15) \approx \$14{,}117.65$ (b) $C(50) = \$80.000$

(c) $C(90) = \$720{,}000$ (d) $\lim_{p\to 100^-} \dfrac{80{,}000p}{100 - p} = \infty$

70. $f(x) = \dfrac{\tan 2x}{x}$

(a)

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	2.0271	2.0003	2.0000	2.0000	2.0003	2.0271

$\lim_{x\to 0} \dfrac{\tan 2x}{x} = 2$

(b) Yes, define

$$f(x) = \begin{cases} \dfrac{\tan 2x}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}.$$

Now $f(x)$ is continuous at $x = 0$.

Problem Solving for Chapter 1

1. (a) Perimeter $\triangle PAO = \sqrt{x^2 + (y-1)^2} + \sqrt{x^2 + y^2} + 1$

$= \sqrt{x^2 + (x^2-1)^2} + \sqrt{x^2 + x^4} + 1$

Perimeter $\triangle PBO = \sqrt{(x-1)^2 + y^2} + \sqrt{x^2 + y^2} + 1$

$= \sqrt{(x-1)^2 + x^4} + \sqrt{x^2 + x^4} + 1$

(b) $r(x) = \dfrac{\sqrt{x^2 + (x^2-1)^2} + \sqrt{x^2 + x^4} + 1}{\sqrt{(x-1)^2 + x^4} + \sqrt{x^2 + x^4} + 1}$

x	4	2	1	0.1	0.01
Perimeter $\triangle PAO$	33.02	9.08	3.41	2.10	2.01
Perimeter $\triangle PBO$	33.77	9.60	3.41	2.00	2.00
$r(x)$	0.98	0.95	1	1.05	1.005

(c) $\lim_{x\to 0^+} r(x) = \dfrac{1 + 0 + 1}{1 + 0 + 1} = \dfrac{2}{2} = 1$

2. (a) Area $\triangle PAO = \dfrac{1}{2}bh = \dfrac{1}{2}(1)(x) = \dfrac{x}{2}$

Area $\triangle PBO = \dfrac{1}{2}bh = \dfrac{1}{2}(1)(y) = \dfrac{y}{2} = \dfrac{x^2}{2}$

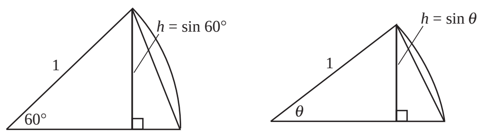
(b) $a(x) = \dfrac{\text{Area } \triangle PBO}{\text{Area } \triangle PAO} = \dfrac{x^2/2}{x/2} = x$

x	4	2	1	0.1	0.01
Area $\triangle PAO$	2	1	1/2	1/20	1/200
Area $\triangle PBO$	8	2	1/2	1/200	1/20,000
$a(x)$	4	2	1	1/10	1/100

(c) $\lim_{x\to 0^+} a(x) = \lim_{x\to 0^+} x = 0$

3. (a) There are 6 triangles, each with a central angle of $60° = \pi/3$. Hence,

$$\text{Area hexagon} = 6\left[\frac{1}{2}bh\right] = 6\left[\frac{1}{2}(1)\sin\frac{\pi}{3}\right]$$
$$= \frac{3\sqrt{3}}{2} \approx 2.598.$$



Error: $\pi - \dfrac{3\sqrt{3}}{2} \approx 0.5435$

(b) There are n triangles, each with central angle of $\theta = 2\pi/n$. Hence,

$$A_n = n\left[\frac{1}{2}bh\right] = n\left[\frac{1}{2}(1)\sin\frac{2\pi}{n}\right] = \frac{n\sin(2\pi/n)}{2}.$$

(c)

n	6	12	24	48	96
A_n	2.598	3	3.106	3.133	3.139

(d) As n gets larger and larger, $2\pi/n$ approaches 0.

Letting $x = 2\pi/n$,

$$A_n = \frac{\sin(2\pi/n)}{2/n} = \frac{\sin(2\pi/n)}{(2\pi/n)}\pi = \frac{\sin x}{x}\pi$$

which approaches $(1)\pi = \pi$.

4. (a) Slope $= \dfrac{4-0}{3-0} = \dfrac{4}{3}$

(b) Slope $= -\dfrac{3}{4}$ Tangent line: $y - 4 = -\dfrac{3}{4}(x-3)$

$$y = -\frac{3}{4}x + \frac{25}{4}$$

(c) Let $Q = (x, y) = \left(x, \sqrt{25-x^2}\right)$

$$m_x = \frac{\sqrt{25-x^2}-4}{x-3}$$

(d)
$$\lim_{x\to3} m_x = \lim_{x\to3}\frac{\sqrt{25-x^2}-4}{x-3}\cdot\frac{\sqrt{25-x^2}+4}{\sqrt{25-x^2}+4}$$
$$= \lim_{x\to3}\frac{25-x^2-16}{(x-3)\left(\sqrt{25-x^2}+4\right)}$$
$$= \lim_{x\to3}\frac{(3-x)(3+x)}{(x-3)\left(\sqrt{25-x^2}+4\right)}$$
$$= \lim_{x\to3}\frac{-(3+x)}{\sqrt{25-x^2}+4} = \frac{-6}{4+4} = -\frac{3}{4}$$

This is the slope of the tangent line at P.

5. (a) Slope $= -\dfrac{12}{5}$

(b) Slope of tangent line is $\dfrac{5}{12}$.

$$y + 12 = \frac{5}{12}(x-5)$$
$$y = \frac{5}{12}x - \frac{169}{12} \quad \text{Tangent line}$$

(c) $Q = (x, y) = \left(x, -\sqrt{169-x^2}\right)$

$$m_x = \frac{-\sqrt{169-x^2}+12}{x-5}$$

(d)
$$\lim_{x\to5} m_x = \lim_{x\to5}\frac{12-\sqrt{169-x^2}}{x-5}\cdot\frac{12+\sqrt{169-x^2}}{12+\sqrt{169-x^2}}$$
$$= \lim_{x\to5}\frac{144-(169-x^2)}{(x-5)\left(12+\sqrt{169-x^2}\right)}$$
$$= \lim_{x\to5}\frac{x^2-25}{(x-5)\left(12+\sqrt{169-x^2}\right)}$$
$$= \lim_{x\to5}\frac{(x+5)}{12+\sqrt{169-x^2}} = \frac{10}{12+12} = \frac{5}{12}$$

This is the same slope as part (b).

6.
$$\frac{\sqrt{a+bx}-\sqrt{3}}{x} = \frac{\sqrt{a+bx}-\sqrt{3}}{x}\cdot\frac{\sqrt{a+bx}+\sqrt{3}}{\sqrt{a+bx}+\sqrt{3}}$$
$$= \frac{(a+bx)-3}{x\left(\sqrt{a+bx}+\sqrt{3}\right)}$$

Letting $a = 3$ simplifies the numerator.

Thus,

$$\lim_{x\to0}\frac{\sqrt{3+bx}-\sqrt{3}}{x} = \lim_{x\to0}\frac{bx}{x\left(\sqrt{3+bx}+\sqrt{3}\right)}$$
$$= \lim_{x\to0}\frac{b}{\sqrt{3+bx}+\sqrt{3}}.$$

Setting $\dfrac{b}{\sqrt{3}+\sqrt{3}} = \sqrt{3}$, you obtain $b = 6$.

Thus, $a = 3$ and $b = 6$.

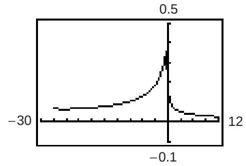
7. (a) $3 + x^{1/3} \geq 0$

$x^{1/3} \geq -3$

$x \geq -27$

Domain: $x \geq -27, x \neq 1$

(b)



(c) $\displaystyle\lim_{x\to-27^+} f(x) = \frac{\sqrt{3+(-27)^{1/3}}-2}{-27-1}$

$\displaystyle = \frac{-2}{-28} = \frac{1}{14} \approx 0.0714$

(d) $\displaystyle\lim_{x\to1} f(x) = \lim_{x\to1} \frac{\sqrt{3+x^{1/3}}-2}{x-1} \cdot \frac{\sqrt{3+x^{1/3}}+2}{\sqrt{3+x^{1/3}}+2}$

$\displaystyle = \lim_{x\to1} \frac{3+x^{1/3}-4}{(x-1)\left(\sqrt{3+x^{1/3}}+2\right)}$

$\displaystyle = \lim_{x\to1} \frac{x^{1/3}-1}{(x^{1/3}-1)(x^{2/3}+x^{1/3}+1)\left(\sqrt{3+x^{1/3}}+2\right)}$

$\displaystyle = \lim_{x\to1} \frac{1}{(x^{2/3}+x^{1/3}+1)\left(\sqrt{3+x^{1/3}}+2\right)}$

$\displaystyle = \frac{1}{(1+1+1)(2+2)} = \frac{1}{12}$

8. $\displaystyle\lim_{x\to0^-} f(x) = \lim_{x\to0^-} (a^2-2) = a^2-2$

$\displaystyle\lim_{x\to0^+} f(x) = \lim_{x\to0^+} \frac{ax}{\tan x} = a \left(\text{because } \lim_{x\to0} \frac{\tan x}{x} = 1\right)$

Thus,

$a^2 - 2 = a$

$a^2 - a - 2 = 0$

$(a-2)(a+1) = 0$

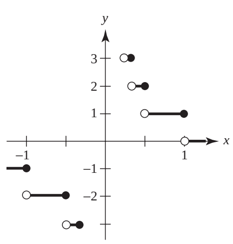
$a = -1, 2$

9. (a) $\displaystyle\lim_{x\to2} f(x) = 3$: g_1, g_4

(b) f continuous at 2: g_1

(c) $\displaystyle\lim_{x\to2^-} f(x) = 3$: g_1, g_3, g_4

10.



(a) $f\left(\frac{1}{4}\right) = [\![4]\!] = 4$

$f(3) = \left[\!\left[\frac{1}{3}\right]\!\right] = 0$

$f(1) = [\![1]\!] = 1$

(b) $\displaystyle\lim_{x\to1^-} f(x) = 1$

$\displaystyle\lim_{x\to1^+} f(x) = 0$

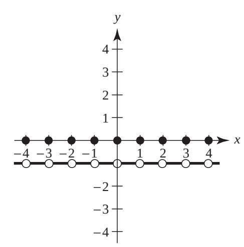
$\displaystyle\lim_{x\to0^-} f(x) = -\infty$

$\displaystyle\lim_{x\to0^+} f(x) = \infty$

(c) f is continuous for all real numbers except

$x = 0, \pm1, \pm\frac{1}{2}, \pm\frac{1}{3}, \ldots$

11.



(a) $f(1) = [\![1]\!] + [\![-1]\!] = 1 + (-1) = 0$

$f(0) = 0$

$f\left(\frac{1}{2}\right) = 0 + (-1) = -1$

$f(-2.7) = -3 + 2 = -1$

(b) $\displaystyle\lim_{x\to1^-} f(x) = -1$

$\displaystyle\lim_{x\to1^+} f(x) = -1$

$\displaystyle\lim_{x\to1/2} f(x) = -1$

(c) f is continuous for all real numbers except

$x = 0, \pm1, \pm2, \pm3, \ldots$

12. (a) $$v^2 = \frac{192{,}000}{r} + v_0^2 - 48$$

$$\frac{192{,}000}{r} = v^2 - v_0^2 + 48$$

$$r = \frac{192{,}000}{v^2 - v_0^2 + 48}$$

$$\lim_{v \to 0} r = \frac{192{,}000}{48 - v_0^2}$$

Let $v_0 = \sqrt{48} = 4\sqrt{3}$ mi/sec.

(b) $$v^2 = \frac{1920}{r} + v_0^2 - 2.17$$

$$\frac{1920}{r} = v^2 - v_0^2 + 2.17$$

$$r = \frac{1920}{v^2 - v_0^2 + 2.17}$$

$$\lim_{v \to 0} r = \frac{1920}{2.17 - v_0^2}$$

Let $v_0 = \sqrt{2.17}$ mi/sec $\quad (\approx 1.47$ mi/sec).

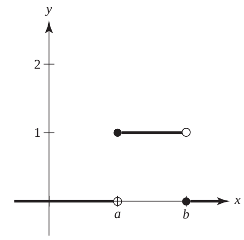
(c) $$r = \frac{10{,}600}{v^2 - v_0^2 + 6.99}$$

$$\lim_{v \to 0} r = \frac{10{,}600}{6.99 - v_0^2}$$

Let $v_0 = \sqrt{6.99} \approx 2.64$ mi/sec.

Since this is smaller than the escape velocity for Earth, the mass is less.

13. (a)



(b) (i) $\lim_{x \to a^+} P_{a,b}(x) = 1$

(ii) $\lim_{x \to a^-} P_{a,b}(x) = 0$

(iii) $\lim_{x \to b^+} P_{a,b}(x) = 0$

(iv) $\lim_{x \to b^-} P_{a,b}(x) = 1$

(c) $P_{a,b}$ is continuous for all positive real numbers except $x = a, b$.

(d) The area under the graph of U, and above the x-axis, is 1.

14. Let $a \neq 0$ and let $\varepsilon > 0$ be given. There exists $\delta_1 > 0$ such that if $0 < |x - 0| < \delta_1$ then $|f(x) - L| < \varepsilon$. Let $\delta = \delta_1/|a|$. Then for $0 < |x - 0| < \delta = \delta_1/|a|$, you have

$$|x| < \frac{\delta_1}{|a|}$$

$$|ax| < \delta_1$$

$$|f(ax) - L| < \varepsilon.$$

As a counterexample, let $a = 0$ and $f(x) = \begin{cases} 1, & x \neq 0 \\ 2, & x = 0 \end{cases}$.

Then $\lim_{x \to 0} f(x) = 1 = L$, but $\lim_{x \to 0} f(ax) = \lim_{x \to 0} f(0) = \lim_{x \to 0} 2 = 2$.