# Chapter 8: Applications of Trigonometry

## 8.1 Exercises

1 $\beta = 180° - \alpha - \gamma = 180° - 41° - 77° = 62°$.

$\frac{b}{\sin\beta} = \frac{a}{\sin\alpha} \Rightarrow b = \frac{a\sin\beta}{\sin\alpha} = \frac{10.5\sin 62°}{\sin 41°} \approx 14.1.$

$\frac{c}{\sin\gamma} = \frac{a}{\sin\alpha} \Rightarrow c = \frac{a\sin\gamma}{\sin\alpha} = \frac{10.5\sin 77°}{\sin 41°} \approx 15.6.$

2 $\alpha = 180° - \beta - \gamma = 180° - 20° - 31° = 129°$.

$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} \Rightarrow a = \frac{b\sin\alpha}{\sin\beta} = \frac{210\sin 129°}{\sin 20°} \approx 477$, or 480 to 2 significant figures.

$\frac{c}{\sin\gamma} = \frac{b}{\sin\beta} \Rightarrow c = \frac{b\sin\gamma}{\sin\beta} = \frac{210\sin 31°}{\sin 20°} \approx 316$, or 320.

3 $\gamma = 180° - \alpha - \beta = 180° - 27°40' - 52°10' = 100°10'$.

$\frac{b}{\sin\beta} = \frac{a}{\sin\alpha} \Rightarrow b = \frac{a\sin\beta}{\sin\alpha} = \frac{32.4\sin 52°10'}{\sin 27°40'} \approx 55.1.$

$\frac{c}{\sin\gamma} = \frac{a}{\sin\alpha} \Rightarrow c = \frac{a\sin\gamma}{\sin\alpha} = \frac{32.4\sin 100°10'}{\sin 27°40'} \approx 68.7.$

4 $\alpha = 180° - \beta - \gamma = 180° - 50°50' - 70°30' = 58°40'$.

$\frac{a}{\sin\alpha} = \frac{c}{\sin\gamma} \Rightarrow a = \frac{c\sin\alpha}{\sin\gamma} = \frac{537\sin 58°40'}{\sin 70°30'} \approx 487.$

$\frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \Rightarrow b = \frac{c\sin\beta}{\sin\gamma} = \frac{537\sin 50°50'}{\sin 70°30'} \approx 442.$

5 $\beta = 180° - \alpha - \gamma = 180° - 42°10' - 61°20' = 76°30'$.

$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} \Rightarrow a = \frac{b\sin\alpha}{\sin\beta} = \frac{19.7\sin 42°10'}{\sin 76°30'} \approx 13.6.$

$\frac{c}{\sin\gamma} = \frac{b}{\sin\beta} \Rightarrow c = \frac{b\sin\gamma}{\sin\beta} = \frac{19.7\sin 61°20'}{\sin 76°30'} \approx 17.8.$

6 $\beta = 180° - \alpha - \gamma = 180° - 103.45° - 27.19° = 49.36°$.

$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} \Rightarrow a = \frac{b\sin\alpha}{\sin\beta} = \frac{38.84\sin 103.45°}{\sin 49.36°} \approx 49.78.$

$\frac{c}{\sin\gamma} = \frac{b}{\sin\beta} \Rightarrow c = \frac{b\sin\gamma}{\sin\beta} = \frac{38.84\sin 27.19°}{\sin 49.36°} \approx 23.39.$

7 $\frac{\sin\beta}{b} = \frac{\sin\gamma}{c} \Rightarrow \beta = \sin^{-1}\left(\frac{b\sin\gamma}{c}\right) = \sin^{-1}\left(\frac{12\sin 81°}{11}\right) \approx \sin^{-1}(1.0775).$

Since 1.0775 is not in the domain of the inverse sine function, which is $[-1, 1]$,

*no triangle exists.*

8 $\frac{\sin\gamma}{c} = \frac{\sin\alpha}{a} \Rightarrow \gamma = \sin^{-1}\left(\frac{c\sin\alpha}{a}\right) = \sin^{-1}\left(\frac{574.3\sin 32.32°}{263.6}\right) \approx \sin^{-1}(1.1648).$

Since 1.1648 is not in the domain of the inverse sine function, which is $[-1, 1]$,

*no triangle exists.*

[9] $\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c} \Rightarrow \alpha = \sin^{-1}\left(\frac{a\sin\gamma}{c}\right) = \sin^{-1}\left(\frac{140\sin 53°20'}{115}\right) \Rightarrow$

$\alpha \approx \sin^{-1}(0.9765) \approx 77°30'$ or $102°30'$ {rounded to the nearest 10 minutes}.

There are two triangles possible since in either case $\alpha + \gamma < 180°$.

$\beta = (180° - \gamma) - \alpha \approx (180° - 53°20') - (77°30' \text{ or } 102°30') = 49°10' \text{ or } 24°10'$.

$$\frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \Rightarrow b = \frac{c\sin\beta}{\sin\gamma} \approx \frac{115\sin(49°10' \text{ or } 24°10')}{\sin 53°20'} \approx 108 \text{ or } 58.7.$$

[10] $\frac{\sin\gamma}{c} = \frac{\sin\alpha}{a} \Rightarrow \gamma = \sin^{-1}\left(\frac{c\sin\alpha}{a}\right) = \sin^{-1}\left(\frac{52.8\sin 27°30'}{28.1}\right) \Rightarrow$

$\gamma \approx \sin^{-1}(0.8676) \approx 60°10'$ or $119°50'$ {rounded to the nearest 10 minutes}.

There are two triangles possible since in either case $\alpha + \gamma < 180°$.

$\beta = (180° - \alpha) - \gamma \approx (180° - 27°30') - (60°10' \text{ or } 119°50') = 92°20' \text{ or } 32°40'$.

$$\frac{b}{\sin\beta} = \frac{a}{\sin\alpha} \Rightarrow b = \frac{a\sin\beta}{\sin\alpha} \approx \frac{28.1\sin(92°20' \text{ or } 32°40')}{\sin 27°30'} \approx 60.8 \text{ or } 32.8.$$

[11] $\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c} \Rightarrow \alpha = \sin^{-1}\left(\frac{a\sin\gamma}{c}\right) = \sin^{-1}\left(\frac{131.08\sin 47.74°}{97.84}\right) \approx \sin^{-1}(0.9915) \approx$

$82.54°$ or $97.46°$. There are two triangles possible since in either case $\alpha + \gamma < 180°$.

$\beta = (180° - \gamma) - \alpha \approx (180° - 47.74°) - (82.54° \text{ or } 97.46°) = 49.72° \text{ or } 34.80°$.

$$\frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \Rightarrow b = \frac{c\sin\beta}{\sin\gamma} \approx \frac{97.84\sin(49.72° \text{ or } 34.80°)}{\sin 47.74°} \approx 100.85 \text{ or } 75.45.$$

[12] $\frac{\sin\beta}{b} = \frac{\sin\alpha}{a} \Rightarrow \beta = \sin^{-1}\left(\frac{b\sin\alpha}{a}\right) = \sin^{-1}\left(\frac{6.12\sin 42.17°}{5.01}\right) \approx \sin^{-1}(0.8201) \approx$

$55.09°$ or $124.91°$. There are two triangles possible since in either case $\alpha + \beta < 180°$.

$\gamma = (180° - \alpha) - \beta \approx (180° - 42.17°) - (55.09° \text{ or } 124.91°) = 82.74° \text{ or } 12.92°$.

$$\frac{c}{\sin\gamma} = \frac{a}{\sin\alpha} \Rightarrow c = \frac{a\sin\gamma}{\sin\alpha} \approx \frac{5.01\sin(82.74° \text{ or } 12.92°)}{\sin 42.17°} \approx 7.40 \text{ or } 1.67.$$

[13] $\frac{\sin\beta}{b} = \frac{\sin\alpha}{a} \Rightarrow \beta = \sin^{-1}\left(\frac{b\sin\alpha}{a}\right) = \sin^{-1}\left(\frac{18.9\sin 65°10'}{21.3}\right) \approx \sin^{-1}(0.8053) \approx$

$53°40'$ or $126°20'$ {rounded to the nearest 10 minutes}. Reject $126°20'$ because then $\alpha + \beta \geq 180°$. $\gamma = 180° - \alpha - \beta \approx 180° - 65°10' - 53°40' = 61°10'$.

$$\frac{c}{\sin\gamma} = \frac{a}{\sin\alpha} \Rightarrow c = \frac{a\sin\gamma}{\sin\alpha} \approx \frac{21.3\sin 61°10'}{\sin 65°10'} \approx 20.6.$$

[14] $\frac{\sin\gamma}{c} = \frac{\sin\beta}{b} \Rightarrow \gamma = \sin^{-1}\left(\frac{c\sin\beta}{b}\right) = \sin^{-1}\left(\frac{195\sin 113°10'}{248}\right) \approx \sin^{-1}(0.7229) \approx$

$46°20'$ or $133°40'$ {rounded to the nearest 10 minutes}. Reject $133°40'$ because then $\beta + \gamma \geq 180°$. $\alpha = 180° - \beta - \gamma \approx 180° - 113°10' - 46°20' = 20°30'$.

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} \Rightarrow a = \frac{b\sin\alpha}{\sin\beta} \approx \frac{248\sin 20°30'}{\sin 113°10'} \approx 94.5.$$

$\boxed{15}$ $\dfrac{\sin\gamma}{c} = \dfrac{\sin\beta}{b} \Rightarrow \gamma = \sin^{-1}\left(\dfrac{c\sin\beta}{b}\right) = \sin^{-1}\left(\dfrac{0.178\sin 121.624^\circ}{0.283}\right) \Rightarrow$

$\gamma \approx \sin^{-1}(0.5356) \approx 32.383^\circ$ or $147.617^\circ$. Reject $147.617^\circ$ because then $\beta + \gamma \geq 180^\circ$.

$\alpha = 180^\circ - \beta - \gamma \approx 180^\circ - 121.624^\circ - 32.383^\circ = 25.993^\circ$.

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} \Rightarrow a = \frac{b\sin\alpha}{\sin\beta} \approx \frac{0.283\sin 25.993^\circ}{\sin 121.624^\circ} \approx 0.146.$$

$\boxed{16}$ $\dfrac{\sin\alpha}{a} = \dfrac{\sin\gamma}{c} \Rightarrow \alpha = \sin^{-1}\left(\dfrac{a\sin\gamma}{c}\right) = \sin^{-1}\left(\dfrac{17.31\sin 73.01^\circ}{20.24}\right) \Rightarrow$

$\alpha \approx \sin^{-1}(0.8179) \approx 54.88^\circ$ or $125.12^\circ$. Reject $125.12^\circ$ because then $\alpha + \gamma \geq 180^\circ$.

$\beta = 180^\circ - \gamma - \alpha \approx 180^\circ - 73.01^\circ - 54.88^\circ = 52.11^\circ$.

$$\frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \Rightarrow b = \frac{c\sin\beta}{\sin\gamma} \approx \frac{20.24\sin 52.11^\circ}{\sin 73.01^\circ} \approx 16.70.$$

$\boxed{17}$ $\angle ABC = 180^\circ - 54^\circ 10' - 63^\circ 20' = 62^\circ 30'$. $\dfrac{\overline{AB}}{\sin 54^\circ 10'} = \dfrac{240}{\sin 62^\circ 30'} \Rightarrow \overline{AB} \approx 219.36$ yd

$\boxed{18}$ $\dfrac{\sin\angle ABC}{375} = \dfrac{\sin 49^\circ 30'}{530} \Rightarrow \angle ABC \approx 32^\circ 30'$.

$\angle ACB \approx 180^\circ - 49^\circ 30' - 32^\circ 30' = 98^\circ$.

$\dfrac{\overline{AB}}{\sin 98^\circ} = \dfrac{530}{\sin 49^\circ 30'} \Rightarrow \overline{AB} \approx 690$ yards.

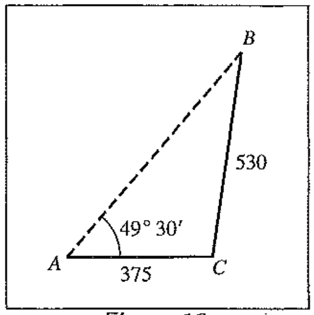


*Figure 18*

$\boxed{19}$ (a) $\angle ABP = 180^\circ - 65^\circ = 115^\circ$. $\angle APB = 180^\circ - 21^\circ - 115^\circ = 44^\circ$.

$$\frac{\overline{AP}}{\sin 115^\circ} = \frac{1.2}{\sin 44^\circ} \Rightarrow \overline{AP} \approx 1.57, \text{ or } 1.6 \text{ mi.}$$

(b) $\sin 21^\circ = \dfrac{\text{height of } P}{\overline{AP}} \Rightarrow$ height of $P = \dfrac{1.2\sin 115^\circ \sin 21^\circ}{\sin 44^\circ}$ {from part (a)} $\Rightarrow$

height of $P \approx 0.56$, or 0.6 mi.

$\boxed{20}$ The angle between the road and the dashed line to the sun is $57^\circ - 15^\circ = 42^\circ$.

The angle between the road and the pole is $90^\circ + 15^\circ = 105^\circ$.

Hence, the angle between the pole and the dashed line is $180^\circ - 42^\circ - 105^\circ = 33^\circ$.

Let $l$ denote the length of the pole. $\dfrac{l}{\sin 42^\circ} = \dfrac{75}{\sin 33^\circ} \Rightarrow l = \dfrac{75\sin 42^\circ}{\sin 33^\circ} \approx 92.14$ ft.

$\boxed{21}$ Let $C$ denote the base of the balloon and $P$ its projection on the ground.

$\angle ACB = 180^\circ - 24^\circ 10' - 47^\circ 40' = 108^\circ 10'$. $\dfrac{\overline{AC}}{\sin 47^\circ 40'} = \dfrac{8.4}{\sin 108^\circ 10'} \Rightarrow \overline{AC} \approx 6.5$ mi.

$$\sin 24^\circ 10' = \frac{\overline{PC}}{\overline{AC}} \Rightarrow \overline{PC} = \frac{8.4\sin 47^\circ 40' \sin 24^\circ 10'}{\sin 108^\circ 10'} \approx 2.7 \text{ mi.}$$

$\boxed{22}$ The angle between the panel and the roof is $45^\circ - 25^\circ = 20^\circ$.

The angle between the brace and the roof is $90^\circ + 25^\circ = 115^\circ$.

$$\frac{d}{\sin 20^\circ} = \frac{10}{\sin 115^\circ} \Rightarrow d = \frac{10 \sin 20^\circ}{\sin 115^\circ} \approx 3.77 \text{ ft.}$$

$\boxed{23}$ $\angle APQ = 57^\circ - 22^\circ = 35^\circ$. $\angle AQP = 180^\circ - (63^\circ - 22^\circ) = 139^\circ$.

$$\angle PAQ = 180^\circ - 139^\circ - 35^\circ = 6^\circ. \quad \frac{\overline{AP}}{\sin 139^\circ} = \frac{100}{\sin 6^\circ} \Rightarrow \overline{AP} = \frac{100 \sin 139^\circ}{\sin 6^\circ} \approx 628 \text{ m.}$$

$\boxed{24}$ Let $\gamma = \angle BCA$. $\angle BAC = 63^\circ - 38^\circ = 25^\circ$. $\frac{\sin \gamma}{239} = \frac{\sin 25^\circ}{374} \Rightarrow \gamma \approx 15^\circ 40'$.

$$\angle ABC \approx 180^\circ - 15^\circ 40' - 25^\circ = 139^\circ 20'. \quad \frac{\overline{AC}}{\sin 139^\circ 20'} = \frac{374}{\sin 25^\circ} \Rightarrow \overline{AC} \approx 577 \text{ yards.}$$

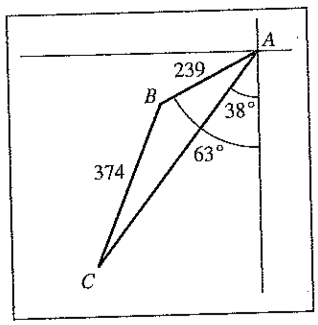



Figure 24

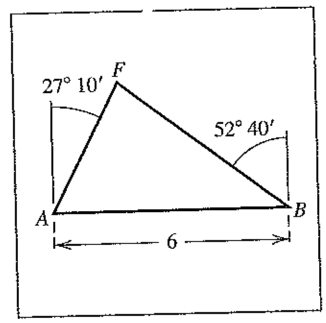



Figure 25

$\boxed{25}$ $\angle FAB = 90^\circ - 27^\circ 10' = 62^\circ 50'$. $\angle FBA = 90^\circ - 52^\circ 40' = 37^\circ 20'$.

$$\angle AFB = 180^\circ - 62^\circ 50' - 37^\circ 20' = 79^\circ 50'. \quad \frac{\overline{AF}}{\sin 37^\circ 20'} = \frac{6}{\sin 79^\circ 50'} \Rightarrow \overline{AF} \approx 3.70 \text{ mi.}$$

$$\frac{\overline{BF}}{\sin 62^\circ 50'} = \frac{6}{\sin 79^\circ 50'} \Rightarrow \overline{BF} \approx 5.42 \text{ mi.}$$

$\boxed{26}$ (a) Label the base of the tower $A$, the top of the tower $B$, and the observation point $C$. Let $\alpha = \angle BAC$ and $\beta = \angle ABC$. $\frac{\sin \beta}{150} = \frac{\sin 53.3^\circ}{179} \Rightarrow \beta \approx 42.2^\circ$.

$$\alpha \approx 180^\circ - 42.2^\circ - 53.3^\circ = 84.5^\circ. \quad \theta = 90^\circ - \alpha \approx 5.5^\circ.$$

(b) $\sin \theta = \frac{d}{179} \Rightarrow d \approx 17.2$ ft.

$\boxed{27}$ Let $A$ denote the base of the hill, $B$ the base of the cathedral, and $C$ the top of the spire. The angle at the base of the hill is $180^\circ - 48^\circ = 132^\circ$. The angle at the top of the spire is $180^\circ - 132^\circ - 41^\circ = 7^\circ$. $\frac{\overline{AC}}{\sin 41^\circ} = \frac{200}{\sin 7^\circ} \Rightarrow \overline{AC} = \frac{200 \sin 41^\circ}{\sin 7^\circ} \approx 1077$ ft.

$\angle BAC = 48^\circ - 32^\circ = 16^\circ$. $\angle ACB = 90^\circ - 48^\circ = 42^\circ$. $\angle ABC = 180^\circ - 42^\circ - 16^\circ = 122^\circ$.

$$\frac{\overline{BC}}{\sin 16^\circ} = \frac{\overline{AC}}{\sin 122^\circ} \Rightarrow \overline{BC} = \frac{200 \sin 41^\circ \sin 16^\circ}{\sin 7^\circ \sin 122^\circ} \approx 350 \text{ ft.}$$

$\boxed{28}$ (a) Let $A$ denote the shorter mountain peak, $B$ the higher mountain peak, and $C$ the helicopter. $\angle BAC = 90^\circ - 18^\circ = 72^\circ$. $\angle ACB = 90^\circ - 43^\circ = 47^\circ$.

$\angle ABC = 180^\circ - 72^\circ - 47^\circ = 61^\circ$. $\dfrac{\overline{AB}}{\sin 47^\circ} = \dfrac{1000}{\sin 61^\circ} \Rightarrow \overline{AB} = \dfrac{1000 \sin 47^\circ}{\sin 61^\circ} \approx 836$ ft.

(b) Let $h$ denote the height that $B$ is above the 5210 foot level.

$$\sin 18^\circ = \frac{h}{\overline{AB}} \Rightarrow h \approx 258 \text{ ft, so the height of } B \text{ is } 5210 + 258 = 5468 \text{ ft.}$$

$\boxed{29}$ (a) In the triangle that forms the base, the third angle is $180^\circ - 103^\circ - 52^\circ = 25^\circ$.

Let $l$ denote the length of the dashed line. $\dfrac{l}{\sin 103^\circ} = \dfrac{12.0}{\sin 25^\circ} \Rightarrow l \approx 27.7$ units.

Now $\tan 34^\circ = \dfrac{h}{l} \Rightarrow h \approx 18.7$ units.

(b) Draw a line from the $103^\circ$ angle that is perpendicular to $l$ and call it $d$.

$\sin 52^\circ = \dfrac{d}{12} \Rightarrow d \approx 9.5$ units. The area of the triangular base is $B = \frac{1}{2}ld$.

The volume $V$ is $\frac{1}{3}(\frac{1}{2}ld)h = 288 \sin 52^\circ \sin^2 103^\circ \tan 34^\circ \csc^2 25^\circ \approx 814$ cubic units.

$\boxed{30}$ (a) $\angle CBA = 180^\circ - 153^\circ = 27^\circ$. $\dfrac{\sin \angle BAC}{35.9} = \dfrac{\sin 27^\circ}{16.7} \Rightarrow \angle BAC \approx 77.4^\circ$.

$\phi = 180^\circ - \angle BAC \approx 102.6^\circ$.

(b) Let $h$ denote the perpendicular distance from $\overline{BA}$ to $C$.

$\sin \angle BAC = \dfrac{h}{16.7} \Rightarrow h \approx 16.30$ ft. The wing span $CC'$ is $2h + 4.80 \approx 37.4$ ft.

(c) $\angle BCA \approx 180^\circ - 27^\circ - 77.4^\circ = 75.6^\circ$. $\dfrac{\overline{BA}}{\sin 75.6^\circ} = \dfrac{16.7}{\sin 27^\circ} \Rightarrow \overline{BA} \approx 35.6$ ft.

The area of $\triangle ABC$ is then $\frac{1}{2}(\overline{BA})h \approx 290.3$ ft$^2$.

$\boxed{31}$ Draw a line through $P$ perpendicular to the $x$-axis. Locate points $A$ and $B$ on this line so that $\angle PAQ = \angle PBR = 90^\circ$. $\overline{AP} = 5127.5 - 3452.8 = 1674.7$,

$\overline{AQ} = 3145.8 - 1487.7 = 1658.1$, and $\tan \angle APQ = \frac{1658.1}{1674.7} \Rightarrow \angle APQ \approx 44^\circ 43'$.

Thus, $\angle BPR \approx 180^\circ - 55^\circ 50' - 44^\circ 43' = 79^\circ 27'$.

By the distance formula, $\overline{PQ} \approx \sqrt{(1674.7)^2 + (1658.1)^2} \approx 2356.7$.

Now $\dfrac{\overline{PR}}{\sin 65^\circ 22'} = \dfrac{\overline{PQ}}{\sin(180^\circ - 55^\circ 50' - 65^\circ 22')} \Rightarrow \overline{PR} = \dfrac{2356.7 \sin 65^\circ 22'}{\sin 58^\circ 48'} \approx 2504.5$.

$\sin \angle BPR = \dfrac{\overline{BR}}{\overline{PR}} \Rightarrow \overline{BR} \approx (2504.5)(\sin 79^\circ 27') \approx 2462.2$. $\cos \angle BPR = \dfrac{\overline{BP}}{\overline{PR}} \Rightarrow$

$\overline{BP} \approx (2504.5)(\cos 79^\circ 27') \approx 458.6$. Using the coordinates of $P$, we see that

$$R(x, y) \approx (1487.7 + 2462.2,\ 3452.8 - 458.6) = (3949.9,\ 2994.2).$$

## 8.2 Exercises

$\boxed{1}$ (d–f) The sides correspond to the three largest values and the angles correspond to the three smallest values. The sides $x$, $y$, and $z$ must be 13.45, 12.60, and 10, respectively, since it appears that $x > y > z$. Answers: (d) **E** (e) **A** (f) **C**

(a–c) The angles $\beta$ (opposite $x$), $\alpha$ (opposite $y$), and $\gamma$ (opposite $z$) must be 1.26, 1.10, and 0.79, respectively, since the largest angle is opposite the largest side, etc. Answers: (a) **B** (b) **F** (c) **D**

$\boxed{2}$ (d–f) The sides correspond to the three largest values and the angles correspond to the three smallest values. The sides $x$, $y$, and $z$ must be 8.24, 6.72, and 3, respectively, since it appears that $x > y > z$. Answers: (d) **C** (e) **E** (f) **A**

(a–c) The angles $\beta$ (opposite $x$), $\alpha$ (opposite $y$), and $\gamma$ (opposite $z$) must be 1.92, 0.87, and 0.35, respectively, since the largest angle is opposite the largest side, etc. Answers: (a) **B** (b) **D** (c) **F**

$\boxed{3}$ (a) Given: side $c$, side $a$, angle $\gamma$ (opposite side $c$). We are given two sides and an angle opposite one of them (SSA), so use the law of sines to find $\alpha$. $\left(\dfrac{\sin\alpha}{a} = \dfrac{\sin\gamma}{c}\right)$

(b) Given: side $c$, side $b$, angle $\alpha$ (between sides $c$ and $b$). We are given two sides and the angle between them (SAS), so use the law of cosines to find $a$. $\left(a^2 = b^2 + c^2 - 2bc\cos\alpha\right)$

(c) Given: the three sides $a$, $b$, and $c$ (SSS), use the law of cosines to find any angle. $\left(\text{to find } \alpha, \text{ use } a^2 = b^2 + c^2 - 2bc\cos\alpha\right)$

(d) Given: the three angles $\alpha$, $\beta$, and $\gamma$. There is *not enough information given* to find any side.

(e) Given: angle $\alpha$, angle $\beta$, and side $c$. We are given the two angles $\alpha$ and $\beta$, so we can easily find the third angle $\gamma$ using $\alpha + \beta + \gamma = 180^\circ$.

(f) Given: angle $\beta$, angle $\gamma$, and side $b$. We are given two angles and a side (AAS or ASA), so we could use the law of sines to find $c$. $\left(\dfrac{c}{\sin\gamma} = \dfrac{b}{\sin\beta}\right)$ Also, since we are given the two angles $\beta$ and $\gamma$, we could easily find the third angle $\alpha$ using $\alpha + \beta + \gamma = 180^\circ$.

$\boxed{4}$ (a) Given: side $a$, side $b$, angle $\gamma$ (between sides $a$ and $b$). We are given two sides and the angle between them (SAS), so use the law of cosines to find $c$. $\left(c^2 = a^2 + b^2 - 2ab\cos\gamma\right)$

(b) Given: side $c$, side $b$, angle $\beta$ (opposite side $b$). We are given two sides and an angle opposite one of them (SSA), so use the law of sines to find $\gamma$. $\left(\dfrac{\sin\gamma}{c}=\dfrac{\sin\beta}{b}\right)$

(c) Given: angle $\alpha$, angle $\gamma$, and side $b$. We are given the two angles $\alpha$ and $\gamma$, so we can easily find the third angle $\beta$ using $\alpha+\beta+\gamma=180°$.

(d) Given: angle $\alpha$, angle $\gamma$, and side $c$. We are given two angles and a side (AAS or ASA), so we could use the law of sines to find $a$. $\left(\dfrac{a}{\sin\alpha}=\dfrac{c}{\sin\gamma}\right)$ Also, since we are given the two angles $\alpha$ and $\gamma$, we could easily find the third angle $\beta$ using $\alpha+\beta+\gamma=180°$.

(e) Given: the three angles $\alpha$, $\beta$, and $\gamma$. There is *not enough information given* to find any side.

(f) Given: the three sides $a$, $b$, and $c$ (SSS), use the law of cosines to find any angle. (to find $\alpha$, use $a^2=b^2+c^2-2bc\cos\alpha$)

*Note:* These formulas will be used to solve problems that involve the law of cosines.

(1) $a^2=b^2+c^2-2bc\cos\alpha \;\Rightarrow\; a=\sqrt{b^2+c^2-2bc\cos\alpha}$

(Similar formulas are used for $b$ and $c$.)

(2) $a^2=b^2+c^2-2bc\cos\alpha \;\Rightarrow\; 2bc\cos\alpha=b^2+c^2-a^2 \;\Rightarrow$

$$\cos\alpha=\left(\frac{b^2+c^2-a^2}{2bc}\right) \;\Rightarrow\; \alpha=\cos^{-1}\left(\frac{b^2+c^2-a^2}{2bc}\right)$$

(Similar formulas are used for $\beta$ and $\gamma$.)

$\boxed{5}$ $a=\sqrt{b^2+c^2-2bc\cos\alpha}=\sqrt{20^2+30^2-2(20)(30)\cos 60°}=\sqrt{700}\approx 26.$

$$\beta=\cos^{-1}\left(\frac{a^2+c^2-b^2}{2ac}\right)=\cos^{-1}\left(\frac{700+30^2-20^2}{2(\sqrt{700})(30)}\right)\approx\cos^{-1}(0.7559)\approx 41°.$$

$$\gamma=180°-\alpha-\beta\approx 180°-60°-41°=79°.$$

$\boxed{6}$ $c=\sqrt{a^2+b^2-2ab\cos\gamma}=\sqrt{325-150\sqrt{2}}\approx 10.6.$

$$\alpha=\cos^{-1}\left(\frac{b^2+c^2-a^2}{2bc}\right)\approx\cos^{-1}(-0.0571)\approx 93°20'.$$

$$\beta=180°-\alpha-\gamma\approx 180°-93°20'-45°=41°40'.$$

$\boxed{7}$ $b=\sqrt{a^2+c^2-2ac\cos\beta}=\sqrt{23{,}400+4500\sqrt{3}}\approx 177,$ or 180.

$$\alpha=\cos^{-1}\left(\frac{b^2+c^2-a^2}{2bc}\right)\approx\cos^{-1}(0.9054)\approx 25°10', \text{ or } 25°.$$

$$\gamma=180°-\alpha-\beta\approx 180°-25°10'-150°=4°50', \text{ or } 5°.$$

$\boxed{8}$ $b=\sqrt{a^2+c^2-2ac\cos\beta}\approx\sqrt{7086.74}\approx 84.2.$

$$\alpha=\cos^{-1}\left(\frac{b^2+c^2-a^2}{2bc}\right)\approx\cos^{-1}(-0.1214)\approx 97°00'.$$

$$\gamma=180°-\alpha-\beta\approx 180°-97°-73°50'=9°10'.$$

[9] $c = \sqrt{a^2 + b^2 - 2ab\cos\gamma} \approx \sqrt{7.58} \approx 2.75.$

$$\alpha = \cos^{-1}\left(\frac{b^2 + c^2 - a^2}{2bc}\right) \approx \cos^{-1}(0.9324) \approx 21^\circ 10'.$$

$$\beta = 180^\circ - \alpha - \gamma \approx 180^\circ - 21^\circ 10' - 115^\circ 10' = 43^\circ 40'.$$

[10] $a = \sqrt{b^2 + c^2 - 2bc\cos\alpha} \approx \sqrt{4367} \approx 66.1.$

$$\beta = \cos^{-1}\left(\frac{a^2 + c^2 - b^2}{2ac}\right) \approx \cos^{-1}(-0.9051) \approx 154^\circ 50'.$$

$$\gamma = 180^\circ - \alpha - \beta \approx 180^\circ - 23^\circ 40' - 154^\circ 50' = 1^\circ 30'.$$

[11] $\alpha = \cos^{-1}\left(\frac{b^2 + c^2 - a^2}{2bc}\right) = \cos^{-1}(0.875) \approx 29^\circ.$

$$\beta = \cos^{-1}\left(\frac{a^2 + c^2 - b^2}{2ac}\right) = \cos^{-1}(0.6875) \approx 47^\circ.$$

$$\gamma = 180^\circ - \alpha - \beta \approx 180^\circ - 29^\circ - 47^\circ = 104^\circ.$$

[12] $\alpha = \cos^{-1}\left(\frac{b^2 + c^2 - a^2}{2bc}\right) \approx \cos^{-1}(0.7472) \approx 41^\circ 40'$, or $42^\circ$.

$$\beta = \cos^{-1}\left(\frac{a^2 + c^2 - b^2}{2ac}\right) \approx \cos^{-1}(0.0792) \approx 85^\circ 30', \text{ or } 85^\circ.$$

$$\gamma = 180^\circ - \alpha - \beta \approx 180^\circ - 41^\circ 40' - 85^\circ 30' = 52^\circ 50', \text{ or } 53^\circ.$$

[13] $\alpha = \cos^{-1}\left(\frac{b^2 + c^2 - a^2}{2bc}\right) \approx \cos^{-1}(0.9766) \approx 12^\circ 30'.$

$$\beta = \cos^{-1}\left(\frac{a^2 + c^2 - b^2}{2ac}\right) = \cos^{-1}(-0.725) \approx 136^\circ 30'.$$

$$\gamma = 180^\circ - \alpha - \beta \approx 180^\circ - 12^\circ 30' - 136^\circ 30' = 31^\circ 00'.$$

[14] $\alpha = \cos^{-1}\left(\frac{b^2 + c^2 - a^2}{2bc}\right) = \cos^{-1}(0.25) \approx 75^\circ 30'.$ $\beta = \alpha$ since $b = a$.

$$\gamma = 180^\circ - \alpha - \beta \approx 180^\circ - 75^\circ 30' - 75^\circ 30' = 29^\circ 00'.$$

[15] Third side $= \sqrt{175^2 + 150^2 - 2(175)(150)\cos 73^\circ 40'} \approx 196$ feet.

[16] $\overline{AB} = \sqrt{420^2 + 540^2 - 2(420)(540)\cos 63^\circ 10'} \approx 513$ yards.

[17] 20 minutes $= \frac{1}{3}$ hour $\Rightarrow$

the cars have traveled $60(\frac{1}{3}) = 20$ miles and $45(\frac{1}{3}) = 15$ miles, respectively.

The distance $d$ apart is $d = \sqrt{20^2 + 15^2 - 2(20)(15)\cos 84^\circ} \approx 24$ miles.

[18] The smallest angle $\alpha$ between the sides is the angle opposite the shortest side (180 ft).

$$\alpha = \cos^{-1}\left(\frac{420^2 + 350^2 - 180^2}{2(420)(350)}\right) \approx \cos^{-1}(0.9065) \approx 25^\circ.$$

[19] The first ship travels $(24)(2) = 48$ miles in two hours. The second ship travels $(18)(1\frac{1}{2}) = 27$ miles in $1\frac{1}{2}$ hours. The angle between the paths is $20^\circ + 35^\circ = 55^\circ$.

$\overline{AB} = \sqrt{27^2 + 48^2 - 2(27)(48)\cos 55^\circ} \approx 39$ miles. See *Figure 19.*

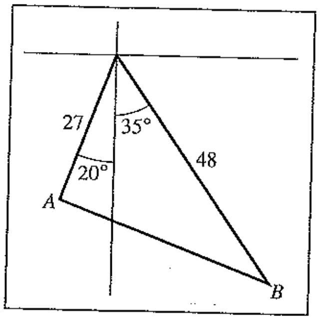



Figure 19

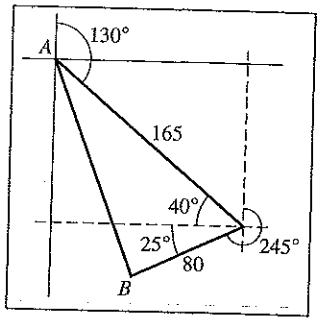



Figure 20

[20] The angle between the two paths is $40^\circ + 25^\circ = 65^\circ$.

$$\overline{AB} = \sqrt{165^2 + 80^2 - 2(165)(80)\cos 65^\circ} \approx 150 \text{ miles.}$$

[21] $\angle ABC = 40^\circ + 20^\circ = 60^\circ$.

$\overline{AB} = \left(\dfrac{1 \text{ mile}}{8 \text{ min}} \cdot 20 \text{ min}\right) = 2.5$ miles and $\overline{BC} = \left(\dfrac{1 \text{ mile}}{8 \text{ min}} \cdot 16 \text{ min}\right) = 2$ miles.

$$\overline{AC} = \sqrt{2.5^2 + 2^2 - 2(2)(2.5)\cos 60^\circ} = \sqrt{5.25} \approx 2.3 \text{ miles.}$$

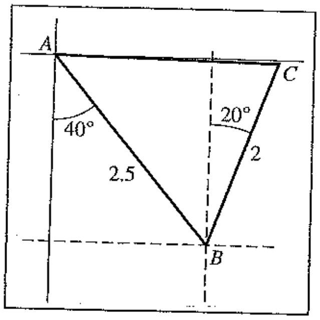



Figure 21

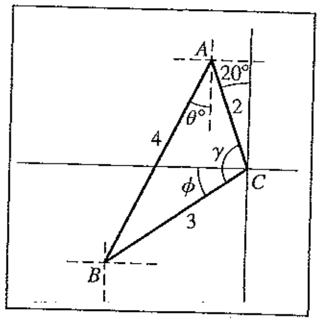



Figure 23

[22] $PQ = \sqrt{300^2 + 438^2 - 2(300)(438)\cos 37^\circ 40'} \approx 271.7$, or 272 feet

[23] $\gamma = \cos^{-1}\left(\dfrac{2^2 + 3^2 - 4^2}{2 \cdot 2 \cdot 3}\right) = \cos^{-1}(-0.25) \approx 104^\circ 29'$. $\phi \approx 104^\circ 29' - 70^\circ = 34^\circ 29'$.

The direction that the third side was traversed is approximately

$$\text{N}(90^\circ - 34^\circ 29')\text{E} = \text{N}55^\circ 31'\text{E}.$$

[24] The length of the diagonal in the base is $\sqrt{6^2 + 8^2} = \sqrt{100} = 10$ inches.

The length of the diagonal of the $6'' \times 4''$ side is $\sqrt{6^2 + 4^2} = \sqrt{52}$ inches.

The length of the diagonal of the $8'' \times 4''$ side is $\sqrt{8^2 + 4^2} = \sqrt{80}$ inches, and it is the side opposite angle $\theta$. $\theta = \cos^{-1}\left(\dfrac{100 + 52 - 80}{2(10)\sqrt{52}}\right) \approx \cos^{-1}(0.4992) \approx 60.05^\circ$, or $60^\circ$.

[25] Let $H$ denote home plate, $M$ the mound, $F$ first base, $S$ second base, and $T$ third base. $\overline{HS} = \sqrt{90^2 + 90^2} = 90\sqrt{2} \approx 127.3$ ft. $\overline{MS} = 90\sqrt{2} - 60.5 \approx 66.8$ ft. $\angle MHF = 45^\circ$ so $\overline{MF} = \sqrt{60.5^2 + 90^2 - 2(60.5)(90)\cos 45^\circ} \approx 63.7$ ft.

$\overline{MT} = \overline{MF}$ by the symmetry of the field.

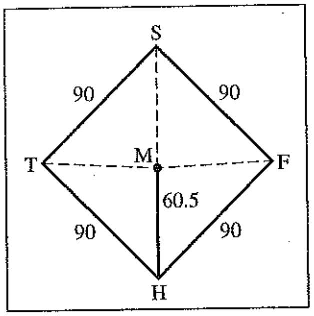



*Figure 25*

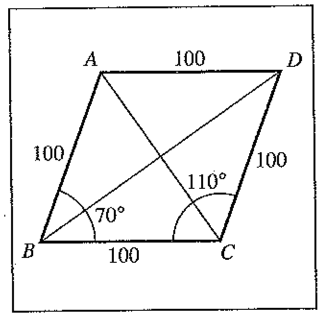



*Figure 26*

[26] $\overline{AC} = \sqrt{100^2 + 100^2 - 2(100)(100)\cos 70^\circ} \approx 114.7$.

$\overline{BD} = \sqrt{100^2 + 100^2 - 2(100)(100)\cos 110^\circ} \approx 163.8$.

[27] $\angle RTP = 21^\circ$ and $\angle RSP = 37^\circ$. $\sin\angle RSP = \dfrac{10{,}000}{\overline{SP}} \Rightarrow \overline{SP} = 10{,}000\csc 37^\circ \approx$ 16,616 ft. $\sin\angle RTP = \dfrac{10{,}000}{\overline{TP}} \Rightarrow \overline{TP} = 10{,}000\csc 21^\circ \approx 27{,}904$ ft.

$\overline{ST} = \sqrt{\overline{SP}^2 + \overline{TP}^2 - 2(\overline{SP})(\overline{TP})\cos 110^\circ} \approx 37{,}039$ ft $\approx 7$ miles.

[28] (a) The angle between the ship's path and its intended path is 14°. Let $d$ denote the distance from $P$ to the port. $d = \sqrt{80^2 + 150^2 - 2(80)(150)\cos 14^\circ} \approx 74.9$ mi.

(b) $\angle P = \cos^{-1}\left(\dfrac{80^2 + d^2 - 150^2}{2(80)d}\right) \approx \cos^{-1}(-0.8749) \approx 151^\circ$.

From *Figure 28*, $\angle P = 33^\circ + 90^\circ + 28^\circ$ $\{= 151^\circ\}$.

The angle that should then be taken is N$(90^\circ - 28^\circ)$E, or N62°E.

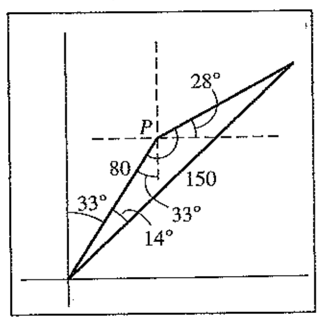



*Figure 28*

[29] Let $d = \overline{ES}$. $d^2 = R^2 + R^2 - 2RR\cos\theta \Rightarrow d^2 = 2R^2(1 - \cos\theta) \Rightarrow$

$d^2 = 4R^2\left(\dfrac{1 - \cos\theta}{2}\right) \Rightarrow d = 2R\sqrt{\dfrac{1 - \cos\theta}{2}} \Rightarrow d = 2R\sin\dfrac{\theta}{2}$.

Since $d = vt$, $t = \dfrac{d}{v} = \dfrac{2R}{v}\sin\dfrac{\theta}{2}$.

[30] (a) In $\triangle CBD$,

$$\angle CBD = \cos^{-1}\left(\frac{\overline{BC}^2 + \overline{BD}^2 - \overline{CD}^2}{2(\overline{BC})(\overline{BD})}\right) = \cos^{-1}\left(\frac{184^2 + 102^2 - 236^2}{2(184)(102)}\right) \approx 107.74^\circ.$$

In $\triangle BCE$,

$$\angle BCE = \cos^{-1}\left(\frac{\overline{BC}^2+\overline{CE}^2-\overline{BE}^2}{2(\overline{BC})(\overline{CE})}\right) = \cos^{-1}\left(\frac{184^2+80^2-218^2}{2(184)(80)}\right) \approx 104.29^\circ.$$

In $\triangle ABC$, $\angle BCA = 180^\circ - \angle BCE \approx 75.71^\circ$, $\angle CBA = 180^\circ - \angle CBD \approx 72.26^\circ$, and $\angle BAC = 180^\circ - \angle BCA - \angle CBA \approx 32.03^\circ$. We now use the law of sines to find $\overline{AB}$ and $\overline{AC}$. $\dfrac{\overline{AB}}{\sin\angle BCA} = \dfrac{\overline{BC}}{\sin\angle BAC} \Rightarrow \overline{AB} = \dfrac{184\sin 75.71^\circ}{\sin 32.03^\circ} \approx 336.2$ ft.

$$\frac{\overline{AC}}{\sin\angle CBA} = \frac{\overline{BC}}{\sin\angle BAC} \Rightarrow \overline{AC} = \frac{184\sin 72.26^\circ}{\sin 32.03^\circ} \approx 330.4 \text{ ft.}$$

(b) Let $h$ denote the perpendicular distance from $A$ to $\overline{BC}$.

$$\sin\angle BCA = \frac{h}{\overline{AC}} \Rightarrow h \approx (330.4)(\sin 75.71^\circ) \approx 320.2 \text{ ft.}$$

$\boxed{31}$ (a) $\angle BCP = \frac{1}{2}(\angle BCD) = \frac{1}{2}(72^\circ) = 36^\circ$. $\triangle BPC$ is isosceles so $\angle BPC = \angle PBC$ and $2\angle BPC = 180^\circ - 36^\circ \Rightarrow \underline{\angle BPC = 72^\circ}$.

$\underline{\angle APB} = 180^\circ - \angle BPC = 180^\circ - 72^\circ = \underline{108^\circ}$.

$\underline{\angle ABP} = 180^\circ - \angle APB - \angle BAP = 180^\circ - 108^\circ - 36^\circ = \underline{36^\circ}$.

(b) $\overline{BP} = \sqrt{\overline{BC}^2 + \overline{PC}^2 - 2(\overline{BC})(\overline{PC})\cos 36^\circ} = \sqrt{1^2+1^2-2(1)(1)\cos 36^\circ} \approx 0.62$.

(c) $\text{Area}_{\text{kite}} = 2(\text{Area of } \triangle BPC) = 2\cdot\frac{1}{2}(\overline{CB})(\overline{CP})\sin\angle BCP = \sin 36^\circ \approx 0.59$.

$\text{Area}_{\text{dart}} = 2(\text{Area of } \triangle ABP) = 2\cdot\frac{1}{2}(\overline{AB})(\overline{BP})\sin\angle ABP = \overline{BP}\sin 36^\circ \approx 0.36$.

{ $\overline{BP}$ was found in part (b) }

$\boxed{32}$ Note that $\triangle TPQ$ is similar to $\triangle THB$. $\dfrac{\overline{TQ}}{\overline{TB}} = \dfrac{\overline{PQ}}{\overline{HB}} \Rightarrow \dfrac{\overline{TQ}}{42} = \dfrac{24}{32} \Rightarrow$

$\overline{TQ} = 31.5$ in. $\overline{TP} = \sqrt{\overline{TQ}^2 + \overline{PQ}^2 - 2(\overline{TQ})(\overline{PQ})\cos\angle TQP}$

$$= \sqrt{31.5^2 + 24^2 - 2(31.5)(24)\cos(90^\circ - 26^\circ)} \approx 30.1 \text{ in.}$$

*Note:* Exer. 33–40: $\mathcal{A}$ (the area) is measured in square units.

$\boxed{33}$ Since $\alpha$ is the angle between sides $b$ and $c$, we may apply the area of a triangle formula listed in this section. $\mathcal{A} = \frac{1}{2}bc\sin\alpha = \frac{1}{2}(20)(30)\sin 60^\circ = 300(\sqrt{3}/2) \approx 260$.

$\boxed{34}$ $\mathcal{A} = \frac{1}{2}ab\sin\gamma = \frac{1}{2}(15.0)(10.0)\sin 45^\circ = 75(\sqrt{2}/2) \approx 53.0$.

$\boxed{35}$ $\gamma = 180^\circ - \alpha - \beta = 180^\circ - 40.3^\circ - 62.9^\circ = 76.8^\circ$.

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} \Rightarrow a = \frac{b\sin\alpha}{\sin\beta} = \frac{5.63\sin 40.3^\circ}{\sin 62.9^\circ}. \quad \mathcal{A} = \tfrac{1}{2}ab\sin\gamma \approx 11.21.$$

$\boxed{36}$ $\beta = 180^\circ - \alpha - \gamma = 180^\circ - 35.7^\circ - 105.2^\circ = 39.1^\circ$.

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} \Rightarrow a = \frac{b\sin\alpha}{\sin\beta} = \frac{17.2\sin 35.7^\circ}{\sin 39.1^\circ}. \quad \mathcal{A} = \tfrac{1}{2}ab\sin\gamma \approx 132.1.$$

$\boxed{37}$ $\dfrac{\sin\beta}{b} = \dfrac{\sin\alpha}{a} \Rightarrow \sin\beta = \dfrac{b\sin\alpha}{a} = \dfrac{3.4\sin 80.1^\circ}{8.0} \approx 0.4187 \Rightarrow \beta \approx 24.8^\circ$ or $155.2^\circ$.

Reject $155.2^\circ$ because then $\alpha + \beta = 235.3^\circ \geq 180^\circ$. $\gamma \approx 180^\circ - 80.1^\circ - 24.8^\circ = 75.1^\circ$.

$$\mathcal{A} = \tfrac{1}{2}ab\sin\gamma = \tfrac{1}{2}(8.0)(3.4)\sin 75.1^\circ \approx 13.1.$$

$\boxed{38}$ $\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c} \Rightarrow \sin\alpha = \frac{a\sin\gamma}{c} = \frac{14.6\sin 32.1^\circ}{15.8} \approx 0.4910 \Rightarrow \alpha \approx 29.4^\circ$ or $150.6^\circ$.

Reject $150.6^\circ$ because then $\alpha + \gamma = 182.7^\circ \geq 180^\circ$. $\beta \approx 180^\circ - 29.4^\circ - 32.1^\circ = 118.5^\circ$.

$\mathcal{A} = \frac{1}{2}ac\sin\beta = \frac{1}{2}(14.6)(15.8)\sin 118.5^\circ \approx 101.4$.

$\boxed{39}$ $s = \frac{1}{2}(a+b+c) = \frac{1}{2}(25.0 + 80.0 + 60.0) = 82.5$.

$\mathcal{A} = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{(82.5)(57.5)(2.5)(22.5)} \approx 516.56$, or $517.0$.

$\boxed{40}$ $s = \frac{1}{2}(a+b+c) = \frac{1}{2}(20.0 + 20.0 + 10.0) = 25$.

$\mathcal{A} = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{(25)(5)(5)(15)} \approx 96.8$.

$\boxed{41}$ $s = \frac{1}{2}(a+b+c) = \frac{1}{2}(115 + 140 + 200) = 227.5$. $\mathcal{A} = \sqrt{s(s-a)(s-b)(s-c)} =$

$\sqrt{(227.5)(112.5)(87.5)(27.5)} \approx 7847.6 \text{ yd}^2$, or $\mathcal{A}/4840 \approx 1.62$ acres.

$\boxed{42}$ $s = \frac{1}{2}(a+b+c) = \frac{1}{2}(320 + 350 + 500) = 585$. $\mathcal{A} = \sqrt{s(s-a)(s-b)(s-c)} =$

$\sqrt{(585)(265)(235)(85)} \approx 55{,}647.3 \text{ yd}^2$, or $\mathcal{A}/4840 \approx 11.5$ acres.

$\boxed{43}$ The area of the parallelogram is twice the area of the triangle formed by the two sides and the included angle. $\mathcal{A} = 2(\frac{1}{2})(12.0)(16.0)\sin 40^\circ \approx 123.4 \text{ ft}^2$.

$\boxed{44}$ As in Exercise 43, $\mathcal{A} = 2(\frac{1}{2})(40.3)(52.6)\sin 100^\circ \approx 2087.6 \text{ ft}^2$.

## *8.3 Exercises*

$\boxed{1}$ $\mathbf{a} + \mathbf{b} = \langle 2, -3\rangle + \langle 1, 4\rangle = \langle 2+1, -3+4\rangle = \langle 3, 1\rangle$.

$\mathbf{a} - \mathbf{b} = \langle 2, -3\rangle - \langle 1, 4\rangle = \langle 2-1, -3-4\rangle = \langle 1, -7\rangle$.

$4\mathbf{a} + 5\mathbf{b} = 4\langle 2, -3\rangle + 5\langle 1, 4\rangle = \langle 8, -12\rangle + \langle 5, 20\rangle = \langle 8+5, -12+20\rangle = \langle 13, 8\rangle$

$4\mathbf{a} - 5\mathbf{b} = 4\langle 2, -3\rangle - 5\langle 1, 4\rangle = \langle 8, -12\rangle - \langle 5, 20\rangle = \langle 8-5, -12-20\rangle = \langle 3, -32\rangle$

$\|\mathbf{a}\| = \sqrt{(2)^2 + (-3)^2} = \sqrt{4+9} = \sqrt{13}$

*Note:* For Exercises 2–6, the answers are given in the following order:

| | $\mathbf{a}+\mathbf{b}$ | $\mathbf{a}-\mathbf{b}$ | $4\mathbf{a}+5\mathbf{b}$ | $4\mathbf{a}-5\mathbf{b}$ | $\|\mathbf{a}\|$ |
|---|---|---|---|---|---|
| $\boxed{2}$ | $\langle 0, 9\rangle$, | $\langle -4, 3\rangle$, | $\langle 2, 39\rangle$, | $\langle -18, 9\rangle$, | $\sqrt{40}$ |
| $\boxed{3}$ | $\langle -15, 6\rangle$, | $\langle 1, -2\rangle$, | $\langle -68, 28\rangle$, | $\langle 12, -12\rangle$, | $\sqrt{53}$ |
| $\boxed{4}$ | $\langle 4, -8\rangle$, | $\langle 16, -8\rangle$, | $\langle 10, -32\rangle$, | $\langle 70, -32\rangle$, | $\sqrt{164}$ |
| $\boxed{5}$ | $4\mathbf{i} - 3\mathbf{j}$, | $-2\mathbf{i} + 7\mathbf{j}$, | $19\mathbf{i} - 17\mathbf{j}$, | $-11\mathbf{i} + 33\mathbf{j}$, | $\sqrt{5}$ |
| $\boxed{6}$ | $-6\mathbf{i} + 2\mathbf{j}$, | $0\mathbf{i} + 0\mathbf{j}$, | $-27\mathbf{i} + 9\mathbf{j}$, | $3\mathbf{i} - \mathbf{j}$, | $\sqrt{10}$ |

$\boxed{7}$ $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j}$ and $\mathbf{b} = -\mathbf{i} + 5\mathbf{j}$ $\Rightarrow$ $\mathbf{a} + \mathbf{b} = 2\mathbf{i} + 7\mathbf{j}$, $2\mathbf{a} = 6\mathbf{i} + 4\mathbf{j}$, and $-3\mathbf{b} = 3\mathbf{i} - 15\mathbf{j}$.

Terminal points of the vectors are $(3, 2)$, $(-1, 5)$, $(2, 7)$, $(6, 4)$, and $(3, -15)$.

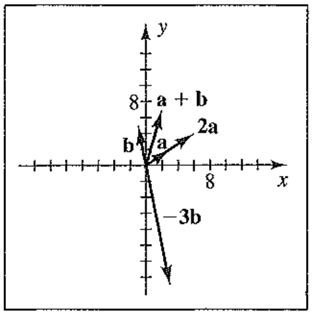



*Figure 7*

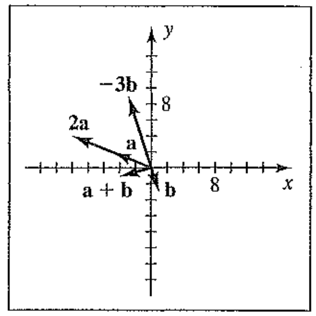



*Figure 8*

$\boxed{8}$ $\mathbf{a} = -5\mathbf{i} + 2\mathbf{j}$ and $\mathbf{b} = \mathbf{i} - 3\mathbf{j}$ $\Rightarrow$ $\mathbf{a} + \mathbf{b} = -4\mathbf{i} - \mathbf{j}$, $2\mathbf{a} = -10\mathbf{i} + 4\mathbf{j}$, and $-3\mathbf{b} = -3\mathbf{i} + 9\mathbf{j}$.

Terminal points of the vectors are $(-5, 2)$, $(1, -3)$, $(-4, -1)$, $(-10, 4)$, and $(-3, 9)$.

$\boxed{9}$ $\mathbf{a} = \langle -4, 6\rangle$ and $\mathbf{b} = \langle -2, 3\rangle$ $\Rightarrow$ $\mathbf{a} + \mathbf{b} = \langle -6, 9\rangle$, $2\mathbf{a} = \langle -8, 12\rangle$, and $-3\mathbf{b} = \langle 6, -9\rangle$.

Terminal points of the vectors are $(-4, 6)$, $(-2, 3)$, $(-6, 9)$, $(-8, 12)$, and $(6, -9)$.

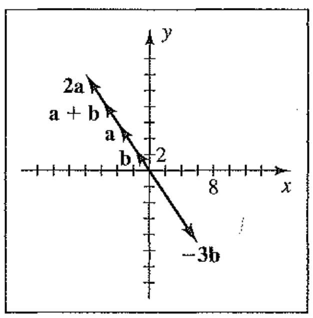



*Figure 9*

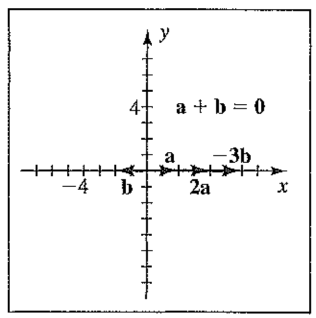



*Figure 10*

$\boxed{10}$ $\mathbf{a} = \langle 2, 0\rangle$ and $\mathbf{b} = \langle -2, 0\rangle$ $\Rightarrow$ $\mathbf{a} + \mathbf{b} = \langle 0, 0\rangle$, $2\mathbf{a} = \langle 4, 0\rangle$, and $-3\mathbf{b} = \langle 6, 0\rangle$.

Terminal points of the vectors are $(2, 0)$, $(-2, 0)$, $(0, 0)$, $(4, 0)$, and $(6, 0)$.

$\boxed{11}$ $\mathbf{a} + \mathbf{b} = \langle 2, 0\rangle + \langle -1, 0\rangle = \langle 1, 0\rangle = -\langle -1, 0\rangle = -\mathbf{b}$

$\boxed{12}$ $\mathbf{c} - \mathbf{d} = \langle 0, 2\rangle - \langle 0, -1\rangle = \langle 0, 3\rangle = -3\langle 0, -1\rangle = -3\mathbf{d}$

$\boxed{13}$ $\mathbf{b} + \mathbf{e} = \langle -1, 0\rangle + \langle 2, 2\rangle = \langle 1, 2\rangle = \mathbf{f}$

$\boxed{14}$ $\mathbf{f} - \mathbf{b} = \langle 1, 2\rangle - \langle -1, 0\rangle = \langle 2, 2\rangle = \mathbf{e}$

$\boxed{15}$ $\mathbf{b} + \mathbf{d} = \langle -1, 0\rangle + \langle 0, -1\rangle = \langle -1, -1\rangle = -\frac{1}{2}\langle 2, 2\rangle = -\frac{1}{2}\mathbf{e}$

$\boxed{16}$ $\mathbf{e} + \mathbf{c} = \langle 2, 2\rangle + \langle 0, 2\rangle = \langle 2, 4\rangle = 2\langle 1, 2\rangle = 2\mathbf{f}$

$\boxed{17}$
$$\begin{aligned}
\mathbf{a} + (\mathbf{b} + \mathbf{c}) &= \langle a_1, a_2\rangle + (\langle b_1, b_2\rangle + \langle c_1, c_2\rangle) \\
&= \langle a_1, a_2\rangle + \langle b_1 + c_1, b_2 + c_2\rangle \\
&= \langle a_1 + b_1 + c_1, a_2 + b_2 + c_2\rangle \\
&= \langle a_1 + b_1, a_2 + b_2\rangle + \langle c_1, c_2\rangle \\
&= (\langle a_1, a_2\rangle + \langle b_1, b_2\rangle) + \langle c_1, c_2\rangle = (\mathbf{a} + \mathbf{b}) + \mathbf{c}
\end{aligned}$$

$\boxed{18}$ $\mathbf{a}+\mathbf{0}=\langle a_1,\, a_2\rangle+\langle 0,\, 0\rangle=\langle a_1+0,\, a_2+0\rangle=\langle a_1,\, a_2\rangle=\mathbf{a}$

$\boxed{19}$
$$\begin{aligned}\mathbf{a}+(-\mathbf{a})&=\langle a_1,\, a_2\rangle+(-\langle a_1,\, a_2\rangle)\\&=\langle a_1,\, a_2\rangle+\langle -a_1,\, -a_2\rangle\\&=\langle a_1-a_1,\, a_2-a_2\rangle\\&=\langle 0,\, 0\rangle=\mathbf{0}\end{aligned}$$

$\boxed{20}$
$$\begin{aligned}(m+n)\mathbf{a}&=(m+n)\langle a_1,\, a_2\rangle\\&=\langle (m+n)a_1,\, (m+n)a_2\rangle\\&=\langle ma_1+na_1,\, ma_2+na_2\rangle\\&=\langle ma_1,\, ma_2\rangle+\langle na_1,\, na_2\rangle\\&=m\langle a_1,\, a_2\rangle+n\langle a_1,\, a_2\rangle=m\mathbf{a}+n\mathbf{a}\end{aligned}$$

$\boxed{21}$
$$\begin{aligned}(mn)\mathbf{a}&=(mn)\langle a_1,\, a_2\rangle\\&=\langle (mn)\,a_1,\, (mn)\,a_2\rangle\\&=\langle mna_1,\, mna_2\rangle\\&=m\langle na_1,\, na_2\rangle && \text{or } n\langle ma_1,\, ma_2\rangle\\&=m(n\langle a_1,\, a_2\rangle) && \text{or } n(m\langle a_1,\, a_2\rangle)\\&=m(n\mathbf{a}) && \text{or } n(m\mathbf{a})\end{aligned}$$

$\boxed{22}$ $1\mathbf{a}=1\langle a_1,\, a_2\rangle=\langle 1a_1,\, 1a_2\rangle=\langle a_1,\, a_2\rangle=\mathbf{a}$

$\boxed{23}$ $0\mathbf{a}=0\,\langle a_1,\, a_2\rangle=\langle 0a_1,\, 0a_2\rangle=\langle 0,\, 0\rangle=\mathbf{0}$.

Also, $m\mathbf{0}=m\langle 0,\, 0\rangle=\langle m0,\, m0\rangle=\langle 0,\, 0\rangle=\mathbf{0}$.

$\boxed{24}$
$$\begin{aligned}(-m)\mathbf{a}&=(-m)\langle a_1,\, a_2\rangle\\&=\langle (-m)a_1,\, (-m)a_2\rangle\\&=\langle -(ma_1),\, -(ma_2)\rangle\\&=-(\langle ma_1,\, ma_2\rangle)\\&=-(m\langle a_1,\, a_2\rangle)\\&=-m\langle a_1,\, a_2\rangle=-m\mathbf{a}\end{aligned}$$

$\boxed{25}$
$$\begin{aligned}-(\mathbf{a}+\mathbf{b})&=-(\langle a_1,\, a_2\rangle+\langle b_1,\, b_2\rangle)\\&=-(\langle a_1+b_1,\, a_2+b_2\rangle)\\&=\langle -(a_1+b_1),\, -(a_2+b_2)\rangle\\&=\langle -a_1-b_1,\, -a_2-b_2\rangle\\&=\langle -a_1,\, -a_2\rangle+\langle -b_1,\, -b_2\rangle\\&=-\mathbf{a}+(-\mathbf{b})=-\mathbf{a}-\mathbf{b}\end{aligned}$$

$\boxed{26}$ $m(\mathbf{a}-\mathbf{b}) = m(\langle a_1,\, a_2\rangle - \langle b_1,\, b_2\rangle)$

$= m\langle a_1 - b_1,\, a_2 - b_2\rangle$

$= \langle m(a_1 - b_1),\, m(a_2 - b_2)\rangle$

$= \langle ma_1 - mb_1,\, ma_2 - mb_2\rangle$

$= \langle ma_1,\, ma_2\rangle + \langle -mb_1,\, -mb_2\rangle$

$= m\langle a_1,\, a_2\rangle + (-m)\langle b_1,\, b_2\rangle = m\mathbf{a} - m\mathbf{b}$

$\boxed{27}$ $\|2\mathbf{v}\| = \|2\langle a,\, b\rangle\| = \|\langle 2a,\, 2b\rangle\| = \sqrt{(2a)^2+(2b)^2} = \sqrt{4a^2+4b^2} =$

$2\sqrt{a^2+b^2} = 2\|\langle a,\, b\rangle\| = 2\|\mathbf{v}\|$

$\boxed{28}$ $\|k\mathbf{v}\| = \|k\langle a,\, b\rangle\| = \|\langle ka,\, kb\rangle\| = \sqrt{(ka)^2+(kb)^2} = \sqrt{k^2a^2+k^2b^2} =$

$\sqrt{k^2}\sqrt{a^2+b^2} = |\,k\,|\,\|\langle a,\, b\rangle\| = |\,k\,|\,\|\mathbf{v}\|$

$\boxed{29}$ $\|\mathbf{a}\| = \sqrt{3^2+(-3)^2} = \sqrt{18} = 3\sqrt{2}$. $\tan\theta = \frac{-3}{3} = -1$ and $\theta$ in QIV $\Rightarrow$ $\theta = \frac{7\pi}{4}$.

$\boxed{30}$ $\|\mathbf{a}\| = \sqrt{(-2)^2+(-2\sqrt{3})^2} = \sqrt{16} = 4$. $\tan\theta = \frac{-2\sqrt{3}}{-2} = \sqrt{3}$ and $\theta$ in QIII $\Rightarrow$

$\theta = \frac{4\pi}{3}$.

$\boxed{31}$ $\|\mathbf{a}\| = 5$. The terminal side of $\theta$ is on the negative $x$-axis $\Rightarrow$ $\theta = \pi$.

$\boxed{32}$ $\|\mathbf{a}\| = 10$. The terminal side of $\theta$ is on the positive $y$-axis $\Rightarrow$ $\theta = \frac{\pi}{2}$.

$\boxed{33}$ $\|\mathbf{a}\| = \sqrt{41}$. $\tan\theta = \frac{5}{-4}$ and $\theta$ in QII $\Rightarrow$ $\theta = \tan^{-1}(-\frac{5}{4}) + \pi$.

$\boxed{34}$ $\|\mathbf{a}\| = 10\sqrt{2}$. $\tan\theta = \frac{-10}{10} = -1$ and $\theta$ in QIV $\Rightarrow$ $\theta = \frac{7\pi}{4}$.

$\boxed{35}$ $\|\mathbf{a}\| = 18$. The terminal side of $\theta$ is on the negative $y$-axis $\Rightarrow$ $\theta = \frac{3\pi}{2}$.

$\boxed{36}$ $\|\mathbf{a}\| = \sqrt{13}$. $\tan\theta = \frac{-3}{2}$ and $\theta$ in QIV $\Rightarrow$ $\theta = \tan^{-1}(-\frac{3}{2}) + 2\pi$.

*Note:* Exercises 37–42: Each resultant force is found by completing the parallelogram and then applying the law of cosines.

$\boxed{37}$ $\|\mathbf{r}\| = \sqrt{40^2+70^2-2(40)(70)\cos 135^\circ} = \sqrt{6500+2800\sqrt{2}} \approx 102.3$, or 102 lb.

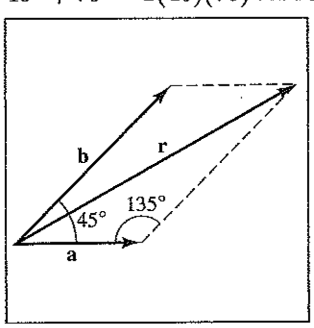



*Figure 37*

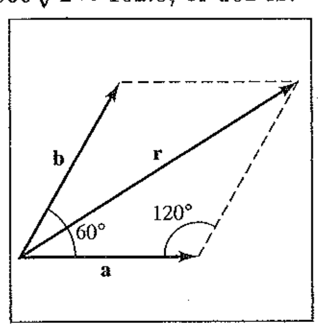



*Figure 38*

$\boxed{38}$ $\|\mathbf{r}\| = \sqrt{5.5^2+6.2^2-2(5.5)(6.2)\cos 120^\circ} = \sqrt{102.79} \approx 10.1$ lb.

$\boxed{39}$ $\|\mathbf{r}\| = \sqrt{2^2 + 8^2 - 2(2)(8)\cos 60^\circ} = \sqrt{52} \approx 7.2$ lb.

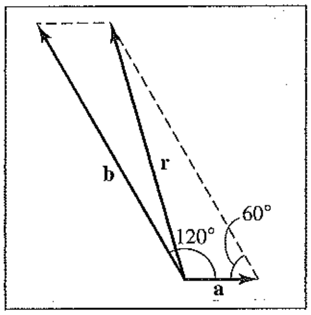



*Figure 39*

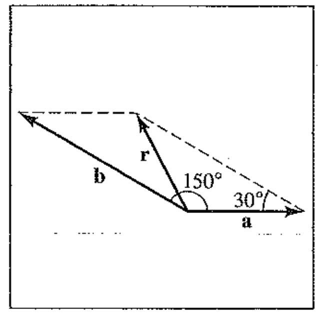



*Figure 40*

$\boxed{40}$ $\|\mathbf{r}\| = \sqrt{30^2 + 50^2 - 2(30)(50)\cos 30^\circ} = \sqrt{3400 - 1500\sqrt{3}} \approx 28.3$, or 28 lb.

$\boxed{41}$ $\|\mathbf{r}\| = \sqrt{90^2 + 60^2 - 2(90)(60)\cos 70^\circ} \approx 89.48$, or 89 lb.

Using the law of cosines, $\alpha = \cos^{-1}\left(\dfrac{90^2 + \|\mathbf{r}\|^2 - 60^2}{2(90)(\|\mathbf{r}\|)}\right) \approx \cos^{-1}(0.7765) \approx 39^\circ$,

which is 24° under the negative $x$-axis. This angle is 204°, or S66°W.

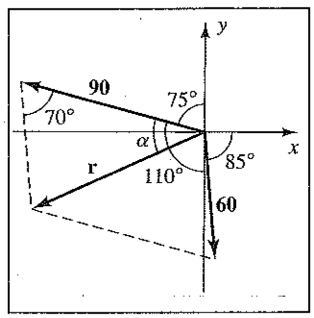



*Figure 41*

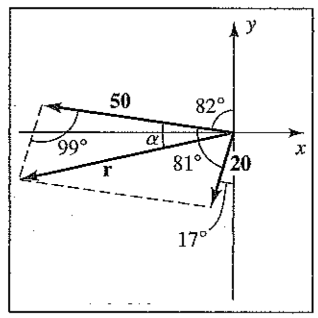



*Figure 42*

$\boxed{42}$ $\|\mathbf{r}\| = \sqrt{50^2 + 20^2 - 2(50)(20)\cos 99^\circ} \approx 56.68$, or 57 lb.

Using the law of cosines, $\alpha = \cos^{-1}\left(\dfrac{50^2 + \|\mathbf{r}\|^2 - 20^2}{2(50)(\|\mathbf{r}\|)}\right) \approx \cos^{-1}(0.9373) \approx 20^\circ$,

which is 12° under the negative $x$-axis. This angle is 192°, or S78°W.

*Note:* Exercises 43–44: We will use a component approach for these problems.

$\boxed{43}$ $(\mathbf{a}) = \langle 6\cos 110^\circ, 6\sin 110^\circ\rangle \approx \langle -2.05, 5.64\rangle$.

$(\mathbf{b}) = \langle 2\cos 215^\circ, 2\sin 215^\circ\rangle \approx \langle -1.64, -1.15\rangle$.

$\mathbf{a} + \mathbf{b} \approx \langle -3.69, 4.49\rangle$ and $\|\mathbf{a} + \mathbf{b}\| \approx 5.8$ lb.

$\tan\theta \approx \frac{4.49}{-3.69} \Rightarrow \theta \approx 129^\circ$ since $\theta$ is in QII.

$\boxed{44}$ $(\mathbf{a}) = \langle 70\cos 320^\circ, 70\sin 320^\circ\rangle \approx \langle 53.62, -45.00\rangle$.

$(\mathbf{b}) = \langle 40\cos 30^\circ, 40\sin 30^\circ\rangle \approx \langle 34.64, 20\rangle$.

$\mathbf{a} + \mathbf{b} \approx \langle 88.26, -25.00\rangle$ and $\|\mathbf{a} + \mathbf{b}\| \approx 91.73$, or 92 lb.

$\tan\theta \approx \frac{-25.00}{88.26} \Rightarrow \theta \approx 344^\circ$ since $\theta$ is in QIV.

$\boxed{45}$ Horizontal $= 50\cos 35^\circ \approx 40.96$. Vertical $= 50\sin 35^\circ \approx 28.68$.

$\boxed{46}$ Horizontal $= 20\cos 40^\circ \approx 15.32$. Vertical $= 20\sin 40^\circ \approx 12.86$.

$\boxed{47}$ Horizontal $= 20\cos 108^\circ \approx -6.18$. Vertical $= 20\sin 108^\circ \approx 19.02$.

$\boxed{48}$ Horizontal $= 160\cos 7.5^\circ \approx 158.63$. Vertical $= 160\sin 7.5^\circ \approx 20.88$.

*Note*: In Exercises 49–52, let $\mathbf{u}$ denote the unit vector in the direction of $\mathbf{a}$.

$\boxed{49}$ (a) The unit vector in the direction of $\mathbf{a}$ is $\frac{1}{\|\mathbf{a}\|}\mathbf{a}$.

$$\mathbf{a} = -8\mathbf{i} + 15\mathbf{j} \Rightarrow \|\mathbf{a}\| = \sqrt{(-8)^2 + 15^2} = \sqrt{64 + 225} = \sqrt{289} = 17.$$

$$\mathbf{u} = \frac{1}{\|\mathbf{a}\|}\mathbf{a} = \tfrac{1}{17}(-8\mathbf{i} + 15\mathbf{j}) = -\tfrac{8}{17}\mathbf{i} + \tfrac{15}{17}\mathbf{j}.$$

(b) The unit vector $-\mathbf{u}$ has the opposite direction of $\mathbf{u}$ and hence,

the opposite direction of $\mathbf{a}$. $-\mathbf{u} = -(-\tfrac{8}{17}\mathbf{i} + \tfrac{15}{17}\mathbf{j}) = \tfrac{8}{17}\mathbf{i} - \tfrac{15}{17}\mathbf{j}$.

$\boxed{50}$ (a) $\mathbf{a} = 5\mathbf{i} - 3\mathbf{j} \Rightarrow \|\mathbf{a}\| = \sqrt{5^2 + (-3)^2} = \sqrt{34}$; $\mathbf{u} = \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{5}{\sqrt{34}}\mathbf{i} - \frac{3}{\sqrt{34}}\mathbf{j}$

(b) $-\mathbf{u} = -\left(\frac{5}{\sqrt{34}}\mathbf{i} - \frac{3}{\sqrt{34}}\mathbf{j}\right) = -\frac{5}{\sqrt{34}}\mathbf{i} + \frac{3}{\sqrt{34}}\mathbf{j}$

$\boxed{51}$ (a) As in Exercise 49, $\mathbf{a} = \langle 2, -5\rangle \Rightarrow$

$$\|\mathbf{a}\| = \sqrt{2^2 + (-5)^2} = \sqrt{29} \text{ and } \mathbf{u} = \frac{\mathbf{a}}{\|\mathbf{a}\|} = \left\langle \frac{2}{\sqrt{29}}, -\frac{5}{\sqrt{29}}\right\rangle.$$

(b) $-\mathbf{u} = -\left\langle \frac{2}{\sqrt{29}}, -\frac{5}{\sqrt{29}}\right\rangle = \left\langle -\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\right\rangle$

$\boxed{52}$ (a) $\mathbf{a} = \langle 0, 6\rangle \Rightarrow \|\mathbf{a}\| = \sqrt{0^2 + 6^2} = 6$; $\mathbf{u} = \frac{\mathbf{a}}{\|\mathbf{a}\|} = \langle \frac{0}{6}, \frac{6}{6}\rangle = \langle 0, 1\rangle$

(b) $-\mathbf{u} = -\langle 0, 1\rangle = \langle 0, -1\rangle$

$\boxed{53}$ (a) $2\mathbf{a}$ has twice the magnitude of $\mathbf{a}$ and the same direction as $\mathbf{a}$.

Hence, $2\langle -6, 3\rangle = \langle -12, 6\rangle$.

(b) As in part (a), $\frac{1}{2}\langle -6, 3\rangle = \langle -3, \frac{3}{2}\rangle$.

$\boxed{54}$ (a) $-3(8\mathbf{i} - 5\mathbf{j}) = -24\mathbf{i} + 15\mathbf{j}$ (b) $-\frac{1}{3}(8\mathbf{i} - 5\mathbf{j}) = -\frac{8}{3}\mathbf{i} + \frac{5}{3}\mathbf{j}$

*Note*: In Exercises 55–56, let $\mathbf{v}$ denote the desired vector.

$\boxed{55}$ The unit vector $\frac{\mathbf{a}}{\|\mathbf{a}\|}$ has the same direction as $\mathbf{a}$. The vector $-6\left(\frac{\mathbf{a}}{\|\mathbf{a}\|}\right)$ will have a magnitude of 6 and the opposite direction of $\mathbf{a}$. $\mathbf{a} = 4\mathbf{i} - 7\mathbf{j} \Rightarrow$

$$\|\mathbf{a}\| = \sqrt{16 + 49} = \sqrt{65}. \text{ Thus, } -6\left(\frac{\mathbf{a}}{\|\mathbf{a}\|}\right) = -6\left(\frac{4}{\sqrt{65}}\mathbf{i} - \frac{7}{\sqrt{65}}\mathbf{j}\right) = -\frac{24}{\sqrt{65}}\mathbf{i} + \frac{42}{\sqrt{65}}\mathbf{j}.$$

$\boxed{56}$ $\mathbf{a} = \langle 2, -5\rangle \Rightarrow \|\mathbf{a}\| = \sqrt{29}$. $\mathbf{v} = -4\left(\frac{\mathbf{a}}{\|\mathbf{a}\|}\right) = \left\langle -\frac{8}{\sqrt{29}}, \frac{20}{\sqrt{29}}\right\rangle$.

$\boxed{57}$ (a) $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \langle 4, 3\rangle + \langle -2, -3\rangle + \langle 5, 2\rangle = \langle 7, 2\rangle$.

(b) $\mathbf{F} + \mathbf{G} = \mathbf{0} \Rightarrow \mathbf{G} = -\mathbf{F} = \langle -7, -2\rangle$.

$\boxed{58}$ (a) $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \langle -3, -1\rangle + \langle 0, -3\rangle + \langle 3, 4\rangle = \langle 0, 0\rangle$.

(b) No additional force is needed since the system is in equilibrium.

$\boxed{59}$ (a) $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = \langle 6\cos 130^\circ,\ 6\sin 130^\circ \rangle + \langle 4\cos(-120^\circ),\ 4\sin(-120^\circ) \rangle \approx \langle -5.86,\ 1.13 \rangle$.

(b) $\mathbf{F} + \mathbf{G} = \mathbf{0} \ \Rightarrow\ \mathbf{G} = -\mathbf{F} \approx \langle 5.86,\ -1.13 \rangle$.

$\boxed{60}$ (a) $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$

$= \langle 8\cos 50^\circ,\ 8\sin 50^\circ \rangle + \langle 7\cos 130^\circ,\ 7\sin 130^\circ \rangle + \langle 5\cos 200^\circ,\ 5\sin 200^\circ \rangle$

$\approx \langle -4.06,\ 9.78 \rangle$.

(b) $\mathbf{F} + \mathbf{G} = \mathbf{0} \ \Rightarrow\ \mathbf{G} = -\mathbf{F} \approx \langle 4.06,\ -9.78 \rangle$.

$\boxed{61}$ The vertical components of the forces must add up to zero for the large ship to move along $l$. The vertical component of the smaller tug is $3200\sin(-30^\circ) = -1600$.

The vertical component of the larger tug is $4000\sin\theta$.

$$4000\sin\theta = 1600 \ \Rightarrow\ \theta = \sin^{-1}\left(\tfrac{1600}{4000}\right) = \sin^{-1}(0.4) \approx 23.6^\circ.$$

$\boxed{62}$ (a) Consider the force of 160 pounds to be the resultant vector of two vectors whose initial point is at the astronaut's feet, one along the positive $x$-axis and the other along the negative $y$-axis. The angle formed by the resultant vector and the positive $x$-axis is the complement of $\theta$, $90^\circ - \theta$.

Now $\quad \cos(90^\circ - \theta) = \dfrac{x\text{-component}}{160} \ \Rightarrow\ x\text{-component} = 160\sin\theta$

and $\quad \sin(90^\circ - \theta) = \dfrac{y\text{-component}}{160} \ \Rightarrow\ y\text{-component} = 160\cos\theta$.

(b) $27 = 160\cos\theta \ \Rightarrow\ \theta = \cos^{-1}\left(\tfrac{27}{160}\right) \approx 80.28^\circ$ on the moon.

$60 = 160\cos\theta \ \Rightarrow\ \theta = \cos^{-1}\left(\tfrac{60}{160}\right) = \cos^{-1}\left(\tfrac{3}{8}\right) \approx 67.98^\circ$ on Mars.

*Note:* Exercises 63–68: Measure angles from the positive $x$-axis.

$\boxed{63}$ $\mathbf{p} = \langle 200\cos 40^\circ,\ 200\sin 40^\circ \rangle \approx \langle 153.21,\ 128.56 \rangle$. $\mathbf{w} = \langle 40\cos 0^\circ,\ 40\sin 0^\circ \rangle = \langle 40,\ 0 \rangle$.

$\mathbf{p} + \mathbf{w} \approx \langle 193.21,\ 128.56 \rangle$ and $\|\mathbf{p} + \mathbf{w}\| \approx 232.07$, or 232 mi/hr.

$\tan\theta \approx \frac{128.56}{193.21} \ \Rightarrow\ \theta \approx 34^\circ$. The true course is then N$(90^\circ - 34^\circ)$E, or N56°E.

$\boxed{64}$ $\mathbf{p} = \langle 500\cos 310^\circ,\ 500\sin 310^\circ \rangle \approx \langle 321.39,\ -383.02 \rangle$.

$\mathbf{w} = \langle 30\cos 25^\circ,\ 30\sin 25^\circ \rangle \approx \langle 27.19,\ 12.68 \rangle$.

$\mathbf{p} + \mathbf{w} \approx \langle 348.58,\ -370.34 \rangle$ and $\|\mathbf{p} + \mathbf{w}\| \approx 508.59$, or 509 mi/hr.

$\tan\theta \approx \frac{-370.34}{348.58} \ \Rightarrow\ \theta \approx -46.7^\circ$, or $-47^\circ$, or, equivalently, $137^\circ$.

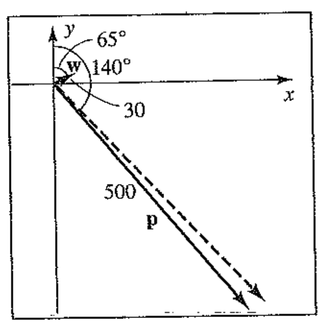



*Figure 64*

$\boxed{65}$ $\mathbf{w} = \langle 50\cos 90°,\ 50\sin 90° \rangle = \langle 0,\ 50 \rangle$.

$\mathbf{r} = \langle 400\cos 200°,\ 400\sin 200° \rangle \approx \langle -375.88,\ -136.81 \rangle$, where $\mathbf{r}$ is the desired resultant of $\mathbf{p}+\mathbf{w}$. Since $\mathbf{r} = \mathbf{p}+\mathbf{w}$, $\mathbf{p} = \mathbf{r}-\mathbf{w} \approx \langle -375.88,\ -186.81 \rangle$. $\|\mathbf{p}\| \approx 419.74$, or 420 mi/hr.

$\tan\theta \approx \frac{-186.81}{-375.88}$ and $\theta$ is in QIII $\Rightarrow$ $\theta \approx 206°$ from the positive $x$-axis,

or 244° using the directional form.

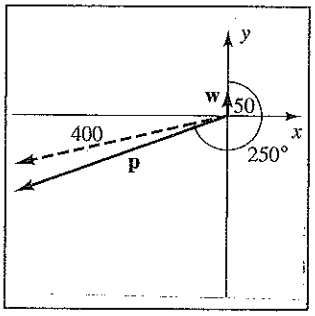


Figure 65

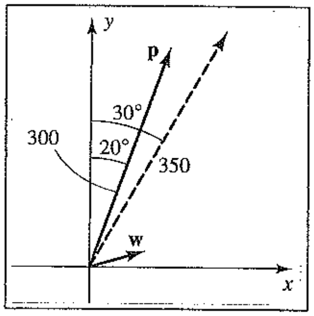


Figure 66

$\boxed{66}$ $\mathbf{p} = \langle 300\cos 70°,\ 300\sin 70° \rangle \approx \langle 102.61,\ 281.91 \rangle$.

$\mathbf{r} = \langle 350\cos 60°,\ 350\sin 60° \rangle \approx \langle 175,\ 303.11 \rangle$.

$\mathbf{w} = \mathbf{r}-\mathbf{p} \approx \langle 72.39,\ 21.20 \rangle$ and $\|\mathbf{w}\| \approx 75.43$, or 75 mi/hr.

$\tan\theta \approx \frac{21.20}{72.39}$ $\Rightarrow$ $\theta \approx 16°$, or in the direction of 74°.

$\boxed{67}$ Let the vectors $\mathbf{c}$, $\mathbf{b}$, and $\mathbf{r}$ denote the current, the boat, and the resultant, respectively. $\mathbf{c} = \langle 1.5\cos 0°,\ 1.5\sin 0° \rangle = \langle 1.5,\ 0 \rangle$. $\mathbf{r} = \langle s\cos 90°,\ s\sin 90° \rangle = \langle 0,\ s \rangle$, where $s$ is the resulting speed. $\mathbf{b} = \langle 4\cos\theta,\ 4\sin\theta \rangle$. Also, $\mathbf{b} = \mathbf{r}-\mathbf{c} = \langle -1.5,\ s \rangle$.

$4\cos\theta = -1.5$ $\Rightarrow$ $\theta \approx 112°$, or N22°W.

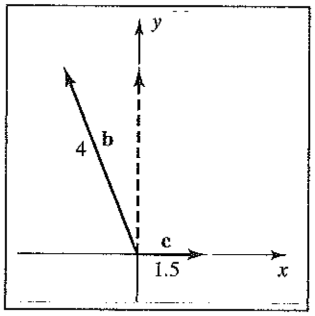


Figure 67

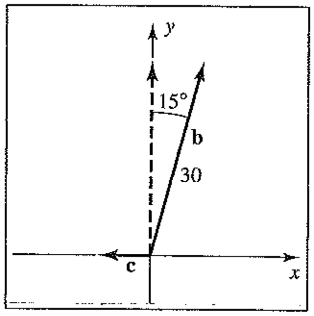


Figure 68

$\boxed{68}$ Let the vectors $\mathbf{c}$, $\mathbf{b}$, and $\mathbf{r}$ denote the current, the boat, and the resultant, respectively. Let $s$ denote the rate of the current and $t$ the resulting speed.

$\mathbf{b} = \langle 30\cos 75°,\ 30\sin 75° \rangle \approx \langle 7.76,\ 28.98 \rangle$.

$\mathbf{c} = \langle s\cos 180°,\ s\sin 180° \rangle = \langle -s,\ 0 \rangle$. $\mathbf{r} = \langle t\cos 90°,\ t\sin 90° \rangle = \langle 0,\ t \rangle$.

Since $\mathbf{c} = \mathbf{r}-\mathbf{b}$, we have $-s = 0 - 7.76$ $\Rightarrow$ $s = 7.76$, or 8 mi/hr.

[69] From the figure, we see that

$\mathbf{v}_1 = \|\mathbf{v}_1\|\sin\theta_1\,\mathbf{i} - \|\mathbf{v}_1\|\cos\theta_1\,\mathbf{j} = 8.2(\frac{1}{2})\,\mathbf{i} - 8.2(\sqrt{3}/2)\,\mathbf{j} = 4.1\,\mathbf{i} - 4.1\sqrt{3}\,\mathbf{j} \approx 4.1\,\mathbf{i} - 7.10\,\mathbf{j}.$

$\dfrac{\|\mathbf{v}_1\|}{\|\mathbf{v}_2\|} = \dfrac{\tan\theta_1}{\tan\theta_2} \Rightarrow \tan\theta_2 = \dfrac{\|\mathbf{v}_2\|}{\|\mathbf{v}_1\|}\tan\theta_1 = \dfrac{3.8}{8.2} \times \dfrac{1}{\sqrt{3}} \Rightarrow \tan\theta_2 \approx 0.2676 \Rightarrow$

$\theta_2 \approx 14.98^\circ$. It follows that $\mathbf{v}_2 = \|\mathbf{v}_2\|\sin\theta_2\,\mathbf{i} - \|\mathbf{v}_2\|\cos\theta_2\,\mathbf{j} \approx 0.98\,\mathbf{i} - 3.67\,\mathbf{j}.$

[70] Since $\theta_1$ is an acute angle, $\mathbf{v}_1 = 20\,\mathbf{i} - 82\,\mathbf{j} \Rightarrow \tan\theta_1 = \frac{20}{82} \Rightarrow \theta_1 \approx 13.71^\circ.$

$\|\mathbf{v}_1\| = \sqrt{20^2 + (-82)^2} = \sqrt{7124} \approx 84.40$ cm/day.

$\dfrac{\|\mathbf{v}_1\|}{\|\mathbf{v}_2\|} = \dfrac{\tan\theta_1}{\tan\theta_2} \Rightarrow \tan\theta_2 = \dfrac{\|\mathbf{v}_2\|}{\|\mathbf{v}_1\|}\tan\theta_1 \approx \dfrac{725}{\sqrt{7124}}(\frac{20}{82}) \approx 2.095 \Rightarrow \theta_2 \approx 64.48^\circ.$

[71] (a) $\mathbf{a} = 15\cos 40^\circ\,\mathbf{i} + 15\sin 40^\circ\,\mathbf{j} \approx 11.49\,\mathbf{i} + 9.64\,\mathbf{j}.$

$\mathbf{b} = 17\cos 40^\circ\,\mathbf{i} + 17\sin 40^\circ\,\mathbf{j} \approx 13.02\,\mathbf{i} + 10.93\,\mathbf{j}.$

$\overrightarrow{PR} = \mathbf{a} + \mathbf{b} \approx 24.51\,\mathbf{i} + 20.57\,\mathbf{j} \Rightarrow R \approx (24.51,\ 20.57).$

(b) $\mathbf{c} = 15\cos(40^\circ + 85^\circ)\,\mathbf{i} + 15\sin 125^\circ\,\mathbf{j} \approx -8.60\,\mathbf{i} + 12.29\,\mathbf{j}.$

$\mathbf{d} = 17\cos(40^\circ + 85^\circ + 35^\circ)\,\mathbf{i} + 17\sin 160^\circ\,\mathbf{j} \approx -15.97\,\mathbf{i} + 5.81\,\mathbf{j}.$

$\overrightarrow{PR} = \mathbf{c} + \mathbf{d} \approx -24.57\,\mathbf{i} + 18.10\,\mathbf{j} \Rightarrow R \approx (-24.57,\ 18.10).$

[72] (a) $\mathbf{a} = 15\cos(-50^\circ)\,\mathbf{i} + 15\sin(-50^\circ)\,\mathbf{j} \approx 9.64\,\mathbf{i} - 11.49\,\mathbf{j}.$

$\mathbf{b} = 10\cos(-50^\circ)\,\mathbf{i} + 10\sin(-50^\circ)\,\mathbf{j} \approx 6.43\,\mathbf{i} - 7.66\,\mathbf{j}.$

$\mathbf{c} = 7\cos(-50^\circ)\,\mathbf{i} + 7\sin(-50^\circ)\,\mathbf{j} \approx 4.50\,\mathbf{i} - 5.36\,\mathbf{j}.$

$\overrightarrow{PR} = \mathbf{a} + \mathbf{b} + \mathbf{c} \approx 20.57\,\mathbf{i} - 24.51\,\mathbf{j} \Rightarrow R \approx (20.57,\ -24.51).$

(b) $\mathbf{d} = 15\cos(-50^\circ + 75^\circ)\,\mathbf{i} + 15\sin 25^\circ\,\mathbf{j} \approx 13.59\,\mathbf{i} + 6.34\,\mathbf{j}.$

$\mathbf{e} = 10\cos(-50^\circ + 75^\circ - 80^\circ)\,\mathbf{i} + 10\sin(-55^\circ)\,\mathbf{j} \approx 5.74\,\mathbf{i} - 8.19\,\mathbf{j}.$

$\mathbf{f} = 7\cos(-50^\circ + 75^\circ - 80^\circ + 40^\circ)\,\mathbf{i} + 7\sin(-15^\circ)\,\mathbf{j} \approx 6.76\,\mathbf{i} - 1.81\,\mathbf{j}.$

$\overrightarrow{PR} = \mathbf{d} + \mathbf{e} + \mathbf{f} \approx 26.09\,\mathbf{i} - 3.66\,\mathbf{j} \Rightarrow R \approx (26.09,\ -3.66).$

[73] Break the force into a horizontal and a vertical component. The people had to contribute a force equal to the vertical component up the ramp. The vertical component is $99{,}000\sin 9^\circ \approx 15{,}487$ lb. $\frac{15{,}487}{550} \approx 28.2$ lb/person. (Since friction was ignored, the actual force would have been greater.)

## *8.4 Exercises*

[1] (a) $\langle -2,\ 5\rangle \cdot \langle 3,\ 6\rangle = (-2)(3) + (5)(6) = -6 + 30 = 24$

(b) $\theta = \cos^{-1}\left(\dfrac{\langle -2,\ 5\rangle \cdot \langle 3,\ 6\rangle}{\|\langle -2,\ 5\rangle\|\,\|\langle 3,\ 6\rangle\|}\right) = \cos^{-1}\left(\dfrac{24}{\sqrt{29}\,\sqrt{45}}\right) \approx 48^\circ 22'$

$\boxed{2}$ (a) $\langle 4, -7\rangle \cdot \langle -2, 3\rangle = (4)(-2) + (-7)(3) = -8 - 21 = -29$

(b) $\theta = \cos^{-1}\left(\dfrac{\langle 4, -7\rangle \cdot \langle -2, 3\rangle}{\|\langle 4, -7\rangle\|\,\|\langle -2, 3\rangle\|}\right) = \cos^{-1}\left(\dfrac{-29}{\sqrt{65}\sqrt{13}}\right) \approx 176°3'$

$\boxed{3}$ (a) $(4\mathbf{i} - \mathbf{j}) \cdot (-3\mathbf{i} + 2\mathbf{j}) = (4)(-3) + (-1)(2) = -12 - 2 = -14$

(b) $\theta = \cos^{-1}\left(\dfrac{(4\mathbf{i} - \mathbf{j}) \cdot (-3\mathbf{i} + 2\mathbf{j})}{\|4\mathbf{i} - \mathbf{j}\|\,\|-3\mathbf{i} + 2\mathbf{j}\|}\right) = \cos^{-1}\left(\dfrac{-14}{\sqrt{17}\sqrt{13}}\right) \approx 160°21'$

$\boxed{4}$ (a) $(8\mathbf{i} - 3\mathbf{j}) \cdot (2\mathbf{i} - 7\mathbf{j}) = (8)(2) + (-3)(-7) = 16 + 21 = 37$

(b) $\theta = \cos^{-1}\left(\dfrac{(8\mathbf{i} - 3\mathbf{j}) \cdot (2\mathbf{i} - 7\mathbf{j})}{\|8\mathbf{i} - 3\mathbf{j}\|\,\|2\mathbf{i} - 7\mathbf{j}\|}\right) = \cos^{-1}\left(\dfrac{37}{\sqrt{73}\sqrt{53}}\right) \approx 53°30'$

$\boxed{5}$ (a) $(9\mathbf{i}) \cdot (5\mathbf{i} + 4\mathbf{j}) = (9)(5) + (0)(4) = 45 + 0 = 45$

(b) $\theta = \cos^{-1}\left(\dfrac{(9\mathbf{i}) \cdot (5\mathbf{i} + 4\mathbf{j})}{\|9\mathbf{i}\|\,\|5\mathbf{i} + 4\mathbf{j}\|}\right) = \cos^{-1}\left(\dfrac{45}{\sqrt{81}\sqrt{41}}\right) \approx 38°40'$

$\boxed{6}$ (a) $(6\mathbf{j}) \cdot (-4\mathbf{i}) = (0)(-4) + (6)(0) = 0 + 0 = 0$

(b) $\theta = \cos^{-1}\left(\dfrac{(6\mathbf{j}) \cdot (-4\mathbf{i})}{\|6\mathbf{j}\|\,\|-4\mathbf{i}\|}\right) = \cos^{-1}\left(\dfrac{0}{\sqrt{36}\sqrt{16}}\right) = \cos^{-1}(0) = 90°$

$\boxed{7}$ (a) $\langle 10, 7\rangle \cdot \langle -2, -\frac{7}{5}\rangle = (10)(-2) + (7)(-\frac{7}{5}) = -\frac{149}{5}$

(b) $\theta = \cos^{-1}\left(\dfrac{\langle 10, 7\rangle \cdot \langle -2, -\frac{7}{5}\rangle}{\|\langle 10, 7\rangle\|\,\|\langle -2, -\frac{7}{5}\rangle\|}\right) = \cos^{-1}\left(\dfrac{-149/5}{\sqrt{149}\sqrt{149/25}}\right) = \cos^{-1}(-1) = 180°$

$\boxed{8}$ (a) $\langle -3, 6\rangle \cdot \langle -1, 2\rangle = (-3)(-1) + (6)(2) = 15$

(b) $\theta = \cos^{-1}\left(\dfrac{\langle -3, 6\rangle \cdot \langle -1, 2\rangle}{\|\langle -3, 6\rangle\|\,\|\langle -1, 2\rangle\|}\right) = \cos^{-1}\left(\dfrac{15}{\sqrt{45}\sqrt{5}}\right) = \cos^{-1}(1) = 0°$

$\boxed{9}$ $\langle 4, -1\rangle \cdot \langle 2, 8\rangle = 8 - 8 = 0 \Rightarrow$ vectors are orthogonal.

$\boxed{10}$ $\langle 3, 6\rangle \cdot \langle 4, -2\rangle = 12 - 12 = 0 \Rightarrow$ vectors are orthogonal.

$\boxed{11}$ $(-4\mathbf{j}) \cdot (-7\mathbf{i}) = 0 + 0 = 0 \Rightarrow$ vectors are orthogonal.

$\boxed{12}$ $(8\mathbf{i} - 4\mathbf{j}) \cdot (-6\mathbf{i} - 12\mathbf{j}) = -48 + 48 = 0 \Rightarrow$ vectors are orthogonal.

$\boxed{13}$ $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \dfrac{(3)(-\frac{12}{7}) + (-5)(\frac{20}{7})}{\sqrt{9 + 25}\sqrt{\frac{144}{49} + \frac{400}{49}}} = \dfrac{-\frac{136}{7}}{\sqrt{\frac{18{,}496}{49}}} = \dfrac{-\frac{136}{7}}{\frac{136}{7}} = -1 \Rightarrow$

$\theta = \cos^{-1}(-1) = \pi$. $\mathbf{b} = m\mathbf{a} \Rightarrow -\frac{12}{7}\mathbf{i} + \frac{20}{7}\mathbf{j} = 3m\mathbf{i} - 5m\mathbf{j} \Rightarrow$

$3m = -\frac{12}{7}$ and $-5m = \frac{20}{7} \Rightarrow m = -\frac{4}{7} < 0 \Rightarrow$ **a** and **b** have the opposite direction.

$\boxed{14}$ $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \dfrac{(-\frac{5}{2})(-10) + (6)(24)}{\sqrt{\frac{25}{4} + 36}\sqrt{100 + 576}} = \dfrac{169}{\sqrt{\frac{169}{4}} \cdot 26} = 1 \Rightarrow \theta = \cos^{-1} 1 = 0.$

$\mathbf{b} = m\mathbf{a} \Rightarrow -10\mathbf{i} + 24\mathbf{j} = -\frac{5}{2}m\mathbf{i} + 6m\mathbf{j} \Rightarrow -\frac{5}{2}m = -10$ and $6m = 24 \Rightarrow$

$m = 4 > 0 \Rightarrow$ **a** and **b** have the same direction.

[15] $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \dfrac{(\frac{2}{3})(8)+(\frac{1}{2})(6)}{\sqrt{\frac{4}{9}+\frac{1}{4}}\sqrt{64+36}} = \dfrac{\frac{25}{3}}{\sqrt{\frac{25}{36}}\cdot 10} = 1 \Rightarrow \theta = \cos^{-1}1 = 0.$

$\mathbf{b} = m\mathbf{a} \Rightarrow 8\mathbf{i}+6\mathbf{j} = \frac{2}{3}m\,\mathbf{i}+\frac{1}{2}m\,\mathbf{j} \Rightarrow 8 = \frac{2}{3}m$ and $6 = \frac{1}{2}m \Rightarrow m = 12 > 0 \Rightarrow$

**a** and **b** have the same direction.

[16] $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \dfrac{(6)(-4)+(18)(-12)}{\sqrt{36+324}\sqrt{16+144}} = \dfrac{-240}{\sqrt{57{,}600}} = -1 \Rightarrow \theta = \cos^{-1}(-1) = \pi.$

$\mathbf{b} = m\mathbf{a} \Rightarrow -4\mathbf{i}-12\mathbf{j} = 6m\,\mathbf{i}+18m\,\mathbf{j} \Rightarrow 6m = -4$ and $18m = -12 \Rightarrow$

$m = -\frac{2}{3} < 0 \Rightarrow$ **a** and **b** have the opposite direction.

[17] We need to have the dot product of the two vectors equal 0.

$(3\mathbf{i}-2\mathbf{j})\cdot(4\mathbf{i}+5m\mathbf{j}) = 0 \Rightarrow 12-10m = 0 \Rightarrow m = \frac{6}{5}.$

[18] $(4m\mathbf{i}+\mathbf{j})\cdot(9m\mathbf{i}-25\mathbf{j}) = 0 \Rightarrow 36m^2-25 = 0 \Rightarrow m^2 = \frac{25}{36} \Rightarrow m = \pm\frac{5}{6}.$

[19] $(9\mathbf{i}-16m\mathbf{j})\cdot(\mathbf{i}+4m\mathbf{j}) = 0 \Rightarrow 9-64m^2 = 0 \Rightarrow m^2 = \frac{9}{64} \Rightarrow m = \pm\frac{3}{8}.$

[20] $(5m\mathbf{i}+3\mathbf{j})\cdot(2\mathbf{i}+7\mathbf{j}) = 0 \Rightarrow 10m+21 = 0 \Rightarrow m = -\frac{21}{10}.$

[21] (a) $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \langle 2, -3\rangle\cdot(\langle 3, 4\rangle+\langle -1, 5\rangle) = \langle 2, -3\rangle\cdot\langle 2, 9\rangle = 4-27 = -23$

(b) $\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c} = \langle 2, -3\rangle\cdot\langle 3, 4\rangle+\langle 2, -3\rangle\cdot\langle -1, 5\rangle = (6-12)+(-2-15) = -23$

[22] (a) $\mathbf{b}\cdot(\mathbf{a}-\mathbf{c}) = \langle 3, 4\rangle\cdot(\langle 2, -3\rangle-\langle -1, 5\rangle) = \langle 3, 4\rangle\cdot\langle 3, -8\rangle = 9-32 = -23$

(b) $\mathbf{b}\cdot\mathbf{a}-\mathbf{b}\cdot\mathbf{c} = \langle 3, 4\rangle\cdot\langle 2, -3\rangle-\langle 3, 4\rangle\cdot\langle -1, 5\rangle = (6-12)-(-3+20) = -23$

[23] $$\begin{aligned}(2\mathbf{a}+\mathbf{b})\cdot(3\mathbf{c}) &= (2\langle 2, -3\rangle+\langle 3, 4\rangle)\cdot(3\langle -1, 5\rangle)\\ &= (\langle 4, -6\rangle+\langle 3, 4\rangle)\cdot\langle -3, 15\rangle\\ &= \langle 7, -2\rangle\cdot\langle -3, 15\rangle = -21-30 = -51\end{aligned}$$

[24] $$\begin{aligned}(\mathbf{a}-\mathbf{b})\cdot(\mathbf{b}+\mathbf{c}) &= (\langle 2, -3\rangle-\langle 3, 4\rangle)\cdot(\langle 3, 4\rangle+\langle -1, 5\rangle)\\ &= \langle -1, -7\rangle\cdot\langle 2, 9\rangle = -2-63 = -65\end{aligned}$$

[25] $\text{comp}_{\mathbf{c}}\mathbf{b} = \dfrac{\mathbf{b}\cdot\mathbf{c}}{\|\mathbf{c}\|} = \dfrac{\langle 3, 4\rangle\cdot\langle -1, 5\rangle}{\|\langle -1, 5\rangle\|} = \dfrac{17}{\sqrt{26}} \approx 3.33$

[26] $\text{comp}_{\mathbf{b}}\mathbf{c} = \dfrac{\mathbf{c}\cdot\mathbf{b}}{\|\mathbf{b}\|} = \dfrac{\langle -1, 5\rangle\cdot\langle 3, 4\rangle}{\|\langle 3, 4\rangle\|} = \dfrac{17}{5} = 3.4$

[27] $\text{comp}_{\mathbf{b}}(\mathbf{a}+\mathbf{c}) = \dfrac{(\mathbf{a}+\mathbf{c})\cdot\mathbf{b}}{\|\mathbf{b}\|} = \dfrac{(\langle 2, -3\rangle+\langle -1, 5\rangle)\cdot\langle 3, 4\rangle}{\|\langle 3, 4\rangle\|} = \dfrac{\langle 1, 2\rangle\cdot\langle 3, 4\rangle}{5} = \dfrac{11}{5} = 2.2$

[28] $\text{comp}_{\mathbf{c}}\mathbf{c} = \dfrac{\mathbf{c}\cdot\mathbf{c}}{\|\mathbf{c}\|} = \dfrac{\langle -1, 5\rangle\cdot\langle -1, 5\rangle}{\|\langle -1, 5\rangle\|} = \dfrac{26}{\sqrt{26}} = \sqrt{26} \approx 5.10.$ Note that, in general,

the component of a vector along itself is just the magnitude of the vector.

[29] $\mathbf{c}\cdot\overrightarrow{PQ} = \langle 3, 4\rangle\cdot\langle 5, -2\rangle = 15-8 = 7.$

[30] $\mathbf{c}\cdot\overrightarrow{PQ} = \langle -10, 12\rangle\cdot\langle 4, 7\rangle = -40+84 = 44.$

$\boxed{31}$ We want a vector with initial point at the origin and terminal point located so that this vector has the same magnitude and direction as $\overrightarrow{PQ}$. Following the hint in the text, $\mathbf{b} = \overrightarrow{PQ} \Rightarrow \langle b_1, b_2 \rangle = \langle 4-2, 3-(-1) \rangle \Rightarrow \langle b_1, b_2 \rangle = \langle 2, 4 \rangle$.

$\mathbf{c} \cdot \mathbf{b} = \langle 6, 4 \rangle \cdot \langle 2, 4 \rangle = 12 + 16 = 28.$

$\boxed{32}$ As in Exercise 31, $\mathbf{c} \cdot \overrightarrow{PQ} = \langle -1, 7 \rangle \cdot \langle 6-(-2), 1-5 \rangle = -36$.

$\boxed{33}$ The force is described by the vector $\langle 0, 4 \rangle$.

The work done is $\langle 0, 4 \rangle \cdot \langle 8, 3 \rangle = 0 + 12 = 12$.

$\boxed{34}$ The force is described by the vector $\langle -10, 0 \rangle$.

The work done is $\langle -10, 0 \rangle \cdot \langle 1-0, 0-1 \rangle = -10 + 0 = -10$.

$\boxed{35}$ $\mathbf{a} \cdot \mathbf{a} = \langle a_1, a_2 \rangle \cdot \langle a_1, a_2 \rangle = a_1^2 + a_2^2 = \left(\sqrt{a_1^2 + a_2^2}\right)^2 = \|\mathbf{a}\|^2$

$\boxed{36}$ $\mathbf{a} \cdot \mathbf{b} = \langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1 b_1 + a_2 b_2 = b_1 a_1 + b_2 a_2 = \langle b_1, b_2 \rangle \cdot \langle a_1, a_2 \rangle = \mathbf{b} \cdot \mathbf{a}$

$\boxed{37}$
$$\begin{aligned}(m\mathbf{a}) \cdot \mathbf{b} &= (m\langle a_1, a_2 \rangle) \cdot \langle b_1, b_2 \rangle \\ &= \langle ma_1, ma_2 \rangle \cdot \langle b_1, b_2 \rangle \\ &= ma_1 b_1 + ma_2 b_2 \\ &= m(a_1 b_1 + a_2 b_2) = m(\mathbf{a} \cdot \mathbf{b})\end{aligned}$$

$\boxed{38}$
$$\begin{aligned}m(\mathbf{a} \cdot \mathbf{b}) &= m(\langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle) \\ &= m(a_1 b_1 + a_2 b_2) \\ &= ma_1 b_1 + ma_2 b_2 \\ &= a_1(mb_1) + a_2(mb_2) \\ &= \langle a_1, a_2 \rangle \cdot \langle mb_1, mb_2 \rangle \\ &= \mathbf{a} \cdot (m\langle b_1, b_2 \rangle) = \mathbf{a} \cdot (m\mathbf{b})\end{aligned}$$

$\boxed{39}$ $\mathbf{0} \cdot \mathbf{a} = \langle 0, 0 \rangle \cdot \langle a_1, a_2 \rangle = 0(a_1) + 0(a_2) = 0 + 0 = 0$

$\boxed{40}$
$$\begin{aligned}(\mathbf{a}+\mathbf{b}) \cdot (\mathbf{a}-\mathbf{b}) &= (\langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle) \cdot (\langle a_1, a_2 \rangle - \langle b_1, b_2 \rangle) \\ &= \langle a_1 + b_1, a_2 + b_2 \rangle \cdot \langle a_1 - b_1, a_2 - b_2 \rangle \\ &= (a_1 + b_1)(a_1 - b_1) + (a_2 + b_2)(a_2 - b_2) \\ &= a_1^2 - b_1^2 + a_2^2 - b_2^2 \\ &= (a_1^2 + a_2^2) - (b_1^2 + b_2^2) \\ &= (\langle a_1, a_2 \rangle \cdot \langle a_1, a_2 \rangle) - (\langle b_1, b_2 \rangle \cdot \langle b_1, b_2 \rangle) = \mathbf{a} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{b}\end{aligned}$$

$\boxed{41}$ Using the horizontal and vertical components of a vector from Section 8.3, we have the force vector as $\langle 20\cos 30^\circ, 20\sin 30^\circ \rangle = \langle 10\sqrt{3}, 10 \rangle$.

The distance (direction vector) can be described by the vector $\langle 100, 0 \rangle$.

The work done is $\langle 10\sqrt{3}, 10 \rangle \cdot \langle 100, 0 \rangle = 1000\sqrt{3} \approx 1732$ ft-lb.

[42] The force vector is now $\langle 20\cos 60^\circ,\ 20\sin 60^\circ\rangle = \langle 10,\ 10\sqrt{3}\rangle$.

The direction vector is $\langle 100\cos 30^\circ,\ 100\sin 30^\circ\rangle = \langle 50\sqrt{3},\ 50\rangle$. The work done is $\langle 10,\ 10\sqrt{3}\rangle \cdot \langle 50\sqrt{3},\ 50\rangle = 500\sqrt{3} + 500\sqrt{3} = 1000\sqrt{3} \approx 1732$ ft-lb. Note that the force in relation to the direction of movement is exactly the same as in Exercise 41.

[43] (a) The horizontal component has magnitude $93 \times 10^6$ and the vertical component has magnitude $0.432 \times 10^6$.

Thus, $\mathbf{v} = (93 \times 10^6)\mathbf{i} + (0.432 \times 10^6)\mathbf{j}$ and $\mathbf{w} = (93 \times 10^6)\mathbf{i} - (0.432 \times 10^6)\mathbf{j}$.

(b) $\cos\theta = \dfrac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} \approx 0.99995685 \;\Rightarrow\; \theta \approx 0.53^\circ$

[44] Let the vector $\mathbf{I}$ represent the magnitude and direction of the sun's rays and $\mathbf{h}$ a horizontal vector. Then, $\text{comp}_{\mathbf{h}}\mathbf{I} = \|\mathbf{I}\|\cos\phi = I\cos\phi$, so $\text{comp}_{\mathbf{h}}\mathbf{I} = 978\, e^{-0.136/\sin 30^\circ}\cos 30^\circ \approx 645$ watts/m$^2$. The total amount of radiation striking the wall is approximately $160 \times 645 = 103{,}200$ watts.

[45] $\mathbf{R} = 2(\mathbf{N}\cdot\mathbf{L})\mathbf{N} - \mathbf{L} = 2(\langle 0,\ 1\rangle \cdot \langle -\frac{4}{5},\ \frac{3}{5}\rangle)\langle 0,\ 1\rangle - \langle -\frac{4}{5},\ \frac{3}{5}\rangle = 2(\frac{3}{5})\langle 0,\ 1\rangle - \langle -\frac{4}{5},\ \frac{3}{5}\rangle =$

$\langle 0,\ \frac{6}{5}\rangle - \langle -\frac{4}{5},\ \frac{3}{5}\rangle = \langle \frac{4}{5},\ \frac{3}{5}\rangle$

[46] $\mathbf{R} = 2(\mathbf{N}\cdot\mathbf{L})\mathbf{N} - \mathbf{L} = 2(\langle \frac{1}{2}\sqrt{2},\ \frac{1}{2}\sqrt{2}\rangle \cdot \langle \frac{12}{13},\ -\frac{5}{13}\rangle)\langle \frac{1}{2}\sqrt{2},\ \frac{1}{2}\sqrt{2}\rangle - \langle \frac{12}{13},\ -\frac{5}{13}\rangle =$

$(\frac{7}{13}\sqrt{2})\langle \frac{1}{2}\sqrt{2},\ \frac{1}{2}\sqrt{2}\rangle - \langle \frac{12}{13},\ -\frac{5}{13}\rangle = \langle \frac{7}{13},\ \frac{7}{13}\rangle - \langle \frac{12}{13},\ -\frac{5}{13}\rangle = \langle -\frac{5}{13},\ \frac{12}{13}\rangle$

[47] Let horizontal ground be represented by $\mathbf{b} = \langle 1,\ 0\rangle$ (it could be any $\langle a,\ 0\rangle$).

$\text{comp}_{\mathbf{b}}\mathbf{a} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}\|} = \dfrac{\langle 2.6,\ 4.5\rangle \cdot \langle 1,\ 0\rangle}{\|\langle 1,\ 0\rangle\|} = \dfrac{2.6}{1} = 2.6 \;\Rightarrow\; |\,\text{comp}_{\mathbf{b}}\mathbf{a}\,| = 2.6$

[48] Let horizontal ground be represented by $\mathbf{b} = \langle 1,\ 0\rangle$.

$\text{comp}_{\mathbf{b}}\mathbf{a} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}\|} = \dfrac{\langle -3.1,\ 7.9\rangle \cdot \langle 1,\ 0\rangle}{\|\langle 1,\ 0\rangle\|} = \dfrac{-3.1}{1} = -3.1 \;\Rightarrow\; |\,\text{comp}_{\mathbf{b}}\mathbf{a}\,| = 3.1$

[49] Let the direction of the ground be represented by $\mathbf{b} = \langle \cos\theta,\ \sin\theta\rangle = \langle \cos 12^\circ,\ \sin 12^\circ\rangle$.

$\text{comp}_{\mathbf{b}}\mathbf{a} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}\|} = \dfrac{\langle 25.7,\ -3.9\rangle \cdot \langle \cos 12^\circ,\ \sin 12^\circ\rangle}{\|\langle \cos 12^\circ,\ \sin 12^\circ\rangle\|} \approx \dfrac{24.33}{1} = 24.33 \;\Rightarrow$

$|\,\text{comp}_{\mathbf{b}}\mathbf{a}\,| = 24.33$

[50] Let the direction of the ground be represented by

$\mathbf{b} = \langle \cos\theta,\ \sin\theta\rangle = \langle \cos(-17^\circ),\ \sin(-17^\circ)\rangle$.

$\text{comp}_{\mathbf{b}}\mathbf{a} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}\|} = \dfrac{\langle -13.8,\ 19.4\rangle \cdot \langle \cos(-17^\circ),\ \sin(-17^\circ)\rangle}{\|\langle \cos(-17^\circ),\ \sin(-17^\circ)\rangle\|} \approx \dfrac{-18.87}{1} = -18.87 \;\Rightarrow$

$|\,\text{comp}_{\mathbf{b}}\mathbf{a}\,| = 18.87$

[51] $P = \frac{1}{550}(\mathbf{F}\cdot\mathbf{v}) = \frac{1}{550}\|\mathbf{F}\|\|\mathbf{v}\|\cos\theta = \frac{1}{550}(2200)(8)\cos 30^\circ = 16\sqrt{3} \approx 27.7$ horsepower.

## *8.5 Exercises*

[1] $|3 - 4i| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$ [2] $|5 + 8i| = \sqrt{5^2 + 8^2} = \sqrt{89}$

$\boxed{3}$ $|-6-7i| = \sqrt{(-6)^2+(-7)^2} = \sqrt{85}$ $\boxed{4}$ $|1-i| = \sqrt{1^2+(-1)^2} = \sqrt{2}$

$\boxed{5}$ $|8i| = |0+8i| = \sqrt{0^2+8^2} = \sqrt{64} = 8$

$\boxed{6}$ $|i^7| = |i^4 \cdot i^2 \cdot i| = |(1)(-1)(i)| = |-i| = |0-i| = \sqrt{0^2+(-1)^2} = \sqrt{1} = 1$

*Note:* $|i^m| = 1$ for any integer $m$.

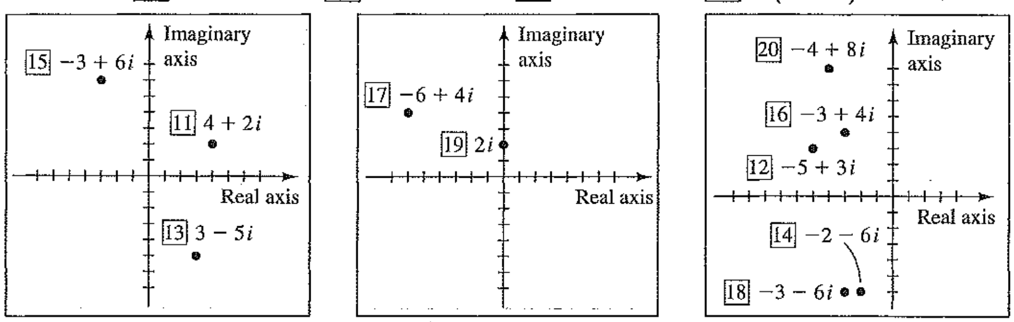
$\boxed{7}$ $|i^{500}| = |(i^4)^{125}| = |(1)^{125}| = |1| = |1+0i| = \sqrt{1^2+0^2} = \sqrt{1} = 1$

$\boxed{8}$ $|-15i| = |0-15i| = \sqrt{0^2+(-15)^2} = \sqrt{225} = 15$

$\boxed{9}$ $|0| = |0+0i| = \sqrt{0^2+0^2} = \sqrt{0} = 0$

$\boxed{10}$ $|-15| = |-15+0i| = \sqrt{(-15)^2+0^2} = \sqrt{225} = 15$

$\boxed{11}$ $4+2i$ $\boxed{12}$ $-5+3i$ $\boxed{13}$ $3-5i$ $\boxed{14}$ $-2-6i$ $\boxed{15}$ $-(3-6i) = -3+6i$




*Figures for Exercises 11–20*

$\boxed{16}$ $(1+2i)^2 = 1+2(1)(2i)+(2i)^2 = 1+4i-4 = -3+4i$

$\boxed{17}$ $2i(2+3i) = 4i+6i^2 = 4i-6 = -6+4i$

$\boxed{18}$ $(-3i)(2-i) = -6i+3i^2 = -6i-3 = -3-6i$

$\boxed{19}$ $(1+i)^2 = 1+2(1)(i)+i^2 = 1+2i-1 = 2i$

$\boxed{20}$ $4(-1+2i) = -4+8i$

$\boxed{21}$ $z = 1-i \Rightarrow r = \sqrt{1+(-1)^2} = \sqrt{2}$. $\tan\theta = \frac{-1}{1} = -1$ and $\theta$ in QIV $\Rightarrow \theta = \frac{7\pi}{4}$.

Thus, $z = 1-i = \sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$, or simply $\sqrt{2}\operatorname{cis}\frac{7\pi}{4}$.

$\boxed{22}$ $z = \sqrt{3}+i \Rightarrow r = \sqrt{(\sqrt{3})^2+1^2} = \sqrt{4} = 2$. $\tan\theta = \frac{1}{\sqrt{3}}$ and $\theta$ in QI $\Rightarrow \theta = \frac{\pi}{6}$.

$z = 2\operatorname{cis}\frac{\pi}{6}$.

$\boxed{23}$ $z = -4\sqrt{3}+4i \Rightarrow r = \sqrt{(-4\sqrt{3})^2+4^2} = \sqrt{64} = 8$.

$\tan\theta = \frac{4}{-4\sqrt{3}} = -\frac{1}{\sqrt{3}}$ and $\theta$ in QII $\Rightarrow \theta = \frac{5\pi}{6}$. $z = 8\operatorname{cis}\frac{5\pi}{6}$.

$\boxed{24}$ $z = -2-2i \Rightarrow r = \sqrt{(-2)^2+(-2)^2} = \sqrt{8} = 2\sqrt{2}$.

$\tan\theta = \frac{-2}{-2} = 1$ and $\theta$ in QIII $\Rightarrow \theta = \frac{5\pi}{4}$. $z = 2\sqrt{2}\operatorname{cis}\frac{5\pi}{4}$.

$\boxed{25}$ $z = 2\sqrt{3}+2i \Rightarrow r = \sqrt{(2\sqrt{3})^2+2^2} = \sqrt{16} = 4$.

$\tan\theta = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$ and $\theta$ in QI $\Rightarrow \theta = \frac{\pi}{6}$. $z = 4\operatorname{cis}\frac{\pi}{6}$.

$\boxed{26}$ $z = 3 - 3\sqrt{3}\,i \Rightarrow r = \sqrt{3^2 + (-3\sqrt{3})^2} = \sqrt{36} = 6.$

$\tan\theta = \frac{-3\sqrt{3}}{3} = -\sqrt{3}$ and $\theta$ in QIV $\Rightarrow \theta = \frac{5\pi}{3}$. $z = 6\operatorname{cis}\frac{5\pi}{3}$.

$\boxed{27}$ $z = -4 - 4i \Rightarrow r = \sqrt{(-4)^2 + (-4)^2} = \sqrt{32} = 4\sqrt{2}.$

$\tan\theta = \frac{-4}{-4} = 1$ and $\theta$ in QIII $\Rightarrow \theta = \frac{5\pi}{4}$. $z = 4\sqrt{2}\operatorname{cis}\frac{5\pi}{4}$.

$\boxed{28}$ $z = -10 + 10i \Rightarrow r = \sqrt{(-10)^2 + 10^2} = \sqrt{200} = 10\sqrt{2}.$

$\tan\theta = \frac{10}{-10} = -1$ and $\theta$ in QII $\Rightarrow \theta = \frac{3\pi}{4}$. $z = 10\sqrt{2}\operatorname{cis}\frac{3\pi}{4}$.

$\boxed{29}$ $z = -20i \Rightarrow r = 20$. $\theta$ on the negative $y$-axis $\Rightarrow \theta = \frac{3\pi}{2}$. $z = 20\operatorname{cis}\frac{3\pi}{2}$.

$\boxed{30}$ $z = -6i \Rightarrow r = 6$. $\theta$ on the negative $y$-axis $\Rightarrow \theta = \frac{3\pi}{2}$. $z = 6\operatorname{cis}\frac{3\pi}{2}$.

$\boxed{31}$ $z = 12 \Rightarrow r = 12$. $\theta$ on the positive $x$-axis $\Rightarrow \theta = 0$. $z = 12\operatorname{cis}0$.

$\boxed{32}$ $z = 15 \Rightarrow r = 15$. $\theta$ on the positive $x$-axis $\Rightarrow \theta = 0$. $z = 15\operatorname{cis}0$.

$\boxed{33}$ $z = -7 \Rightarrow r = 7$. $\theta$ on the negative $x$-axis $\Rightarrow \theta = \pi$. $z = 7\operatorname{cis}\pi$.

$\boxed{34}$ $z = -5 \Rightarrow r = 5$. $\theta$ on the negative $x$-axis $\Rightarrow \theta = \pi$. $z = 5\operatorname{cis}\pi$.

$\boxed{35}$ $z = 6i \Rightarrow r = 6$. $\theta$ on the positive $y$-axis $\Rightarrow \theta = \frac{\pi}{2}$. $z = 6\operatorname{cis}\frac{\pi}{2}$.

$\boxed{36}$ $z = 4i \Rightarrow r = 4$. $\theta$ on the positive $y$-axis $\Rightarrow \theta = \frac{\pi}{2}$. $z = 4\operatorname{cis}\frac{\pi}{2}$.

$\boxed{37}$ $z = -5 - 5\sqrt{3}\,i \Rightarrow r = \sqrt{(-5)^2 + (-5\sqrt{3})^2} = \sqrt{100} = 10.$

$\tan\theta = \frac{-5\sqrt{3}}{-5} = \sqrt{3}$ and $\theta$ in QIII $\Rightarrow \theta = \frac{4\pi}{3}$. $z = 10\operatorname{cis}\frac{4\pi}{3}$.

$\boxed{38}$ $z = \sqrt{3} - i \Rightarrow r = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2.$

$\tan\theta = \frac{-1}{\sqrt{3}}$ and $\theta$ in QIV $\Rightarrow \theta = \frac{11\pi}{6}$. $z = 2\operatorname{cis}\frac{11\pi}{6}$.

$\boxed{39}$ $z = 2 + i \Rightarrow r = \sqrt{2^2 + 1^2} = \sqrt{5}$. $\tan\theta = \frac{1}{2}$ and $\theta$ in QI $\Rightarrow \theta = \tan^{-1}\frac{1}{2}$.

$z = \sqrt{5}\operatorname{cis}(\tan^{-1}\frac{1}{2})$.

$\boxed{40}$ $z = 3 + 2i \Rightarrow r = \sqrt{3^2 + 2^2} = \sqrt{13}$. $\tan\theta = \frac{2}{3}$ and $\theta$ in QI $\Rightarrow \theta = \tan^{-1}\frac{2}{3}$.

$z = \sqrt{13}\operatorname{cis}(\tan^{-1}\frac{2}{3})$.

$\boxed{41}$ $z = -3 + i \Rightarrow r = \sqrt{(-3)^2 + 1^2} = \sqrt{10}.$

$\tan\theta = \frac{1}{-3}$ and $\theta$ in QII $\Rightarrow \theta = \tan^{-1}(-\frac{1}{3}) + \pi$. We must add $\pi$ to $\tan^{-1}(-\frac{1}{3})$ because $-\frac{\pi}{2} < \tan^{-1}(-\frac{1}{3}) < 0$ and we want $\theta$ to be in the interval $(\frac{\pi}{2}, \pi)$.

$z = \sqrt{10}\operatorname{cis}\left[\tan^{-1}(-\frac{1}{3}) + \pi\right]$.

$\boxed{42}$ $z = -4 + 2i \Rightarrow r = \sqrt{(-4)^2 + 2^2} = \sqrt{20} = 2\sqrt{5}.$

$\tan\theta = \frac{2}{-4} = -\frac{1}{2}$ and $\theta$ in QII $\Rightarrow \theta = \tan^{-1}(-\frac{1}{2}) + \pi$. $z = 2\sqrt{5}\operatorname{cis}\left[\tan^{-1}(-\frac{1}{2}) + \pi\right]$.

$\boxed{43}$ $z = -5 - 3i \Rightarrow r = \sqrt{(-5)^2 + (-3)^2} = \sqrt{34}$. $\tan\theta = \frac{-3}{-5} = \frac{3}{5}$ and $\theta$ in QIII $\Rightarrow$ $\theta = \tan^{-1}\frac{3}{5} + \pi$. We must add $\pi$ to $\tan^{-1}\frac{3}{5}$ because $0 < \tan^{-1}\frac{3}{5} < \frac{\pi}{2}$ and we want $\theta$ to be in the interval $(\pi, \frac{3\pi}{2})$. $z = \sqrt{34}\operatorname{cis}(\tan^{-1}\frac{3}{5} + \pi)$.

$\boxed{44}$ $z=-2-7i \Rightarrow r=\sqrt{(-2)^2+(-7)^2}=\sqrt{53}$.

$\tan\theta=\frac{-7}{-2}=\frac{7}{2}$ and $\theta$ in QIII $\Rightarrow$ $\theta=\tan^{-1}\frac{7}{2}+\pi$. $z=\sqrt{53}\operatorname{cis}(\tan^{-1}\frac{7}{2}+\pi)$.

$\boxed{45}$ $z=4-3i \Rightarrow r=\sqrt{4^2+(-3)^2}=\sqrt{25}=5$.

$\tan\theta=\frac{-3}{4}$ and $\theta$ in QIV $\Rightarrow$ $\theta=\tan^{-1}(-\frac{3}{4})+2\pi$. We must add $2\pi$ to $\tan^{-1}(-\frac{3}{4})$ because $-\frac{\pi}{2}<\tan^{-1}(-\frac{3}{4})<0$ and we want $\theta$ to be in the interval $(\frac{3\pi}{2}, 2\pi)$.

$$z=5\operatorname{cis}\left[\tan^{-1}(-\tfrac{3}{4})+2\pi\right].$$

$\boxed{46}$ $z=1-3i \Rightarrow r=\sqrt{1^2+(-3)^2}=\sqrt{10}$. $\tan\theta=\frac{-3}{1}=-3$ and $\theta$ in QIV $\Rightarrow$

$$\theta=\tan^{-1}(-3)+2\pi.\quad z=\sqrt{10}\operatorname{cis}\left[\tan^{-1}(-3)+2\pi\right].$$

$\boxed{47}$ $4(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})=4\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)=2\sqrt{2}+2\sqrt{2}\,i$

$\boxed{48}$ $8(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4})=8\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)=4\sqrt{2}-4\sqrt{2}\,i$

$\boxed{49}$ $6(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3})=6\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=-3+3\sqrt{3}\,i$

$\boxed{50}$ $12(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3})=12\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=-6-6\sqrt{3}\,i$

$\boxed{51}$ $5(\cos\pi+i\sin\pi)=5(-1+0i)=-5$

$\boxed{52}$ $3(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2})=3(0-1i)=-3i$

$\boxed{53}$ $\sqrt{34}\operatorname{cis}(\tan^{-1}\frac{3}{5})=\sqrt{34}\left[\cos(\tan^{-1}\frac{3}{5})+i\sin(\tan^{-1}\frac{3}{5})\right]=\sqrt{34}\left(\frac{5}{\sqrt{34}}+\frac{3}{\sqrt{34}}i\right)=5+3i$

$\boxed{54}$ $\sqrt{53}\operatorname{cis}\left[\tan^{-1}(-\frac{2}{7})\right]=\sqrt{53}\left\{\cos\left[\tan^{-1}(-\frac{2}{7})\right]+i\sin\left[\tan^{-1}(-\frac{2}{7})\right]\right\}=$

$$\sqrt{53}\left(\frac{7}{\sqrt{53}}-\frac{2}{\sqrt{53}}i\right)=7-2i$$

$\boxed{55}$ $\sqrt{5}\operatorname{cis}\left[\tan^{-1}(-\frac{1}{2})\right]=\sqrt{5}\left\{\cos\left[\tan^{-1}(-\frac{1}{2})\right]+i\sin\left[\tan^{-1}(-\frac{1}{2})\right]\right\}=$

$$\sqrt{5}\left(\frac{2}{\sqrt{5}}-\frac{1}{\sqrt{5}}i\right)=2-i$$

$\boxed{56}$ $\sqrt{10}\operatorname{cis}(\tan^{-1}3)=\sqrt{10}\left[\cos(\tan^{-1}3)+i\sin(\tan^{-1}3)\right]=$

$$\sqrt{10}\left(\frac{1}{\sqrt{10}}+\frac{3}{\sqrt{10}}i\right)=1+3i$$

$\boxed{57}$ $z_1=\sqrt{2}\operatorname{cis}\frac{3\pi}{4}$ and $z_2=\sqrt{2}\operatorname{cis}\frac{\pi}{4}$. $z_1z_2=\sqrt{2}\cdot\sqrt{2}\operatorname{cis}(\frac{3\pi}{4}+\frac{\pi}{4})=2\operatorname{cis}\pi=-2+0i$.

$$\frac{z_1}{z_2}=\frac{\sqrt{2}}{\sqrt{2}}\operatorname{cis}(\tfrac{3\pi}{4}-\tfrac{\pi}{4})=1\operatorname{cis}\tfrac{\pi}{2}=0+i$$

$\boxed{58}$ $z_1=2\operatorname{cis}\frac{11\pi}{6}$ and $z_2=2\operatorname{cis}\frac{7\pi}{6}$. $z_1z_2=2\cdot2\operatorname{cis}(\frac{11\pi}{6}+\frac{7\pi}{6})=4\operatorname{cis}3\pi=-4+0i$.

$$\frac{z_1}{z_2}=\tfrac{2}{2}\operatorname{cis}(\tfrac{11\pi}{6}-\tfrac{7\pi}{6})=1\operatorname{cis}\tfrac{2\pi}{3}=-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i.$$

$\boxed{59}$ $z_1=4\operatorname{cis}\frac{4\pi}{3}$ and $z_2=5\operatorname{cis}\frac{\pi}{2}$. $z_1z_2=4\cdot5\operatorname{cis}(\frac{4\pi}{3}+\frac{\pi}{2})=20\operatorname{cis}\frac{11\pi}{6}=10\sqrt{3}-10i$.

$$\frac{z_1}{z_2}=\tfrac{4}{5}\operatorname{cis}(\tfrac{4\pi}{3}-\tfrac{\pi}{2})=\tfrac{4}{5}\operatorname{cis}\tfrac{5\pi}{6}=-\tfrac{2}{5}\sqrt{3}+\tfrac{2}{5}i.$$

[60] $z_1 = 5\sqrt{2}\operatorname{cis}\frac{3\pi}{4}$ and $z_2 = 3\operatorname{cis}\frac{3\pi}{2}$. $z_1 z_2 = 5\sqrt{2}\cdot 3\operatorname{cis}\left(\frac{3\pi}{4}+\frac{3\pi}{2}\right) = 15\sqrt{2}\operatorname{cis}\frac{9\pi}{4} =$

$15 + 15i$. $\frac{z_1}{z_2} = \frac{5\sqrt{2}}{3}\operatorname{cis}\left(\frac{3\pi}{4}-\frac{3\pi}{2}\right) = \frac{5\sqrt{2}}{3}\operatorname{cis}\left(-\frac{3\pi}{4}\right) = -\frac{5}{3}-\frac{5}{3}i$.

[61] $z_1 = 10\operatorname{cis}\pi$ and $z_2 = 4\operatorname{cis}\pi$. $z_1 z_2 = 10\cdot 4\operatorname{cis}(\pi+\pi) = 40\operatorname{cis}2\pi = 40+0i$.

$\frac{z_1}{z_2} = \frac{10}{4}\operatorname{cis}(\pi-\pi) = \frac{5}{2}\operatorname{cis}0 = \frac{5}{2}+0i$.

[62] $z_1 = 2\operatorname{cis}\frac{\pi}{2}$ and $z_2 = 3\operatorname{cis}\frac{3\pi}{2}$. $z_1 z_2 = 2\cdot 3\operatorname{cis}\left(\frac{\pi}{2}+\frac{3\pi}{2}\right) = 6\operatorname{cis}2\pi = 6+0i$.

$\frac{z_1}{z_2} = \frac{2}{3}\operatorname{cis}\left(\frac{\pi}{2}-\frac{3\pi}{2}\right) = \frac{2}{3}\operatorname{cis}(-\pi) = -\frac{2}{3}+0i$.

[63] $z_1 = 4\operatorname{cis}0$ and $z_2 = \sqrt{5}\operatorname{cis}\left[\tan^{-1}\left(-\frac{1}{2}\right)\right]$. Let $\theta = \tan^{-1}\left(-\frac{1}{2}\right)$.

$$z_1 z_2 = 4\cdot\sqrt{5}\operatorname{cis}(0+\theta) = 4\sqrt{5}(\cos\theta + i\sin\theta) = 4\sqrt{5}\left(\frac{2}{\sqrt{5}}+\frac{-1}{\sqrt{5}}i\right) = 8-4i.$$

$$\frac{z_1}{z_2} = \frac{4}{\sqrt{5}}\operatorname{cis}(0-\theta) = \frac{4}{\sqrt{5}}[\cos(-\theta)+i\sin(-\theta)] = \frac{4}{\sqrt{5}}\left(\frac{2}{\sqrt{5}}+\frac{1}{\sqrt{5}}i\right) = \frac{8}{5}+\frac{4}{5}i.$$

[64] $z_1 = 3\operatorname{cis}\pi$ and $z_2 = \sqrt{29}\operatorname{cis}\left(\tan^{-1}\frac{2}{5}\right)$. Let $\theta = \tan^{-1}\frac{2}{5}$. $z_1 z_2 = 3\cdot\sqrt{29}\operatorname{cis}(\pi+\theta) =$

$$3\sqrt{29}[\cos(\pi+\theta)+i\sin(\pi+\theta)] = 3\sqrt{29}\left(\frac{-5}{\sqrt{29}}+\frac{-2}{\sqrt{29}}i\right) = -15-6i.$$

We simplify the above using the addition formulas for the sine and cosine as follows:

$$\cos(\pi+\theta) = \cos\pi\cos\theta - \sin\pi\sin\theta = -\cos\theta = \frac{-5}{\sqrt{29}}$$

and $$\sin(\pi+\theta) = \sin\pi\cos\theta + \cos\pi\sin\theta = -\sin\theta = \frac{-2}{\sqrt{29}}$$

$$\frac{z_1}{z_2} = \frac{3}{\sqrt{29}}\operatorname{cis}(\pi-\theta) = \frac{3}{\sqrt{29}}\left(\frac{-5}{\sqrt{29}}+\frac{2}{\sqrt{29}}i\right)$$

$\{$since $\cos(\pi-\theta) = -\cos\theta$ and $\sin(\pi-\theta) = \sin\theta\,\} = -\frac{15}{29}+\frac{6}{29}i$.

[65] Let $z_1 = r_1\operatorname{cis}\theta_1$ and $z_2 = r_2\operatorname{cis}\theta_2$.

$$\frac{z_1}{z_2} = \frac{r_1\operatorname{cis}\theta_1}{r_2\operatorname{cis}\theta_2} = \frac{r_1(\cos\theta_1+i\sin\theta_1)(\cos\theta_2-i\sin\theta_2)}{r_2(\cos\theta_2+i\sin\theta_2)(\cos\theta_2-i\sin\theta_2)}$$

$\{$multiplying by the conjugate of the denominator$\}$

$$= \frac{r_1\left[(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2)+i(\sin\theta_1\cos\theta_2-\sin\theta_2\cos\theta_1)\right]}{r_2\left[(\cos^2\theta_2+\sin^2\theta_2)+i(\sin\theta_2\cos\theta_2-\cos\theta_2\sin\theta_2)\right]}$$

$$= \frac{r_1[\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)]}{r_2(1+0i)} = \frac{r_1}{r_2}\operatorname{cis}(\theta_1-\theta_2).$$

[66] (a) Let $z_1 = r_1\operatorname{cis}\theta_1$, $z_2 = r_2\operatorname{cis}\theta_2$, and $z_3 = r_3\operatorname{cis}\theta_3$.

Then $z_1z_2z_3 = (z_1z_2)z_3 = [r_1r_2\operatorname{cis}(\theta_1+\theta_2)](r_3\operatorname{cis}\theta_3) =$

$[(r_1r_2)r_3]\operatorname{cis}[(\theta_1+\theta_2)+\theta_3] = (r_1r_2r_3)\operatorname{cis}(\theta_1+\theta_2+\theta_3)$.

(b) The generalization is $z_1z_2\cdots z_n = (r_1r_2\cdots r_n)\operatorname{cis}(\theta_1+\theta_2+\cdots+\theta_n)$.

$\boxed{67}$ The unknown quantity is $V$: $I = V/Z \Rightarrow$

$V = IZ = (10 \operatorname{cis} 35^\circ)(3 \operatorname{cis} 20^\circ) = (10 \times 3) \operatorname{cis}(35^\circ + 20^\circ) = 30 \operatorname{cis} 55^\circ \approx 17.21 + 24.57i.$

$\boxed{68}$ The unknown quantity is $V$: $I = V/Z \Rightarrow$

$$\begin{aligned} V = IZ &= (12 \operatorname{cis} 5^\circ)(100 \operatorname{cis} 90^\circ) \\ &= (12 \times 100) \operatorname{cis}(5^\circ + 90^\circ) = 1200 \operatorname{cis} 95^\circ \approx -104.59 + 1195.43i \end{aligned}$$

$\boxed{69}$ The unknown quantity is $Z$: $I = V/Z \Rightarrow$

$$Z = \frac{V}{I} = \frac{115 \operatorname{cis} 45^\circ}{8 \operatorname{cis} 5^\circ} = (115 \div 8) \operatorname{cis}(45^\circ - 5^\circ) = 14.375 \operatorname{cis} 40^\circ \approx 11.01 + 9.24i$$

$\boxed{70}$ The unknown quantity is $I$:

$$I = \frac{V}{Z} = \frac{163 \operatorname{cis} 17^\circ}{78 \operatorname{cis} 61^\circ} = (163 \div 78) \operatorname{cis}(17^\circ - 61^\circ) = \frac{163}{78} \operatorname{cis}(-44^\circ) \approx 1.50 - 1.45i$$

$\boxed{71}$ $Z = 14 - 13i \Rightarrow |Z| = \sqrt{14^2 + (-13)^2} = \sqrt{365} \approx 19.1$ ohms

$\boxed{72}$ $I = \frac{V}{Z} \Rightarrow Z = \frac{V}{I} = \frac{220 \operatorname{cis} 34^\circ}{5 \operatorname{cis} 90^\circ} = 44 \operatorname{cis}(-56^\circ) \approx 24.60 - 36.48i.$

The resistance is 24.60 ohms and the reactance is 36.48 ohms.

$\boxed{73}$ $I = \frac{V}{Z} \Rightarrow V = IZ = (4 \operatorname{cis} 90^\circ)[18 \operatorname{cis}(-78^\circ)] = 72 \operatorname{cis} 12^\circ \approx 70.43 + 14.97i.$

The actual voltage is 70.43 volts.

$\boxed{74}$ $I = \frac{V}{Z} = \frac{163 \operatorname{cis} 43^\circ}{100 \operatorname{cis} 17^\circ} = 1.63 \operatorname{cis} 26^\circ \approx 1.47 + 0.71i.$ The actual current is 1.47 amps.

## 8.6 Exercises

$\boxed{1}$ $(3+3i)^5 = (3\sqrt{2} \operatorname{cis} \frac{\pi}{4})^5 = (3\sqrt{2})^5 \operatorname{cis} \frac{5\pi}{4} = 972\sqrt{2}\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) = -972 - 972i$

$\boxed{2}$ $(1+i)^{12} = (\sqrt{2} \operatorname{cis} \frac{\pi}{4})^{12} = (\sqrt{2})^{12} \operatorname{cis} 3\pi = 64 \operatorname{cis} \pi = 64(-1 + 0i) = -64$

$\boxed{3}$ $(1-i)^{10} = (\sqrt{2} \operatorname{cis} \frac{7\pi}{4})^{10} = (\sqrt{2})^{10} \operatorname{cis} \frac{35\pi}{2} = 32 \operatorname{cis} \frac{3\pi}{2} = 32(0 - i) = -32i$

$\boxed{4}$ $(-1+i)^8 = (\sqrt{2} \operatorname{cis} \frac{3\pi}{4})^8 = (\sqrt{2})^8 \operatorname{cis} 6\pi = 16 \operatorname{cis} 0 = 16(1 + 0i) = 16$

$\boxed{5}$ $(1 - \sqrt{3}\,i)^3 = (2 \operatorname{cis} \frac{5\pi}{3})^3 = 2^3 \operatorname{cis} 5\pi = 8 \operatorname{cis} \pi = 8(-1 + 0i) = -8$

$\boxed{6}$ $(1 - \sqrt{3}\,i)^5 = (2 \operatorname{cis} \frac{5\pi}{3})^5 = 2^5 \operatorname{cis} \frac{25\pi}{3} = 32 \operatorname{cis} \frac{\pi}{3} = 32\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 16 + 16\sqrt{3}\,i$

$\boxed{7}$ $\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^{15} = (1 \operatorname{cis} \frac{3\pi}{4})^{15} = 1^{15} \operatorname{cis} \frac{45\pi}{4} = \operatorname{cis} \frac{5\pi}{4} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

$\boxed{8}$ $\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^{25} = (1 \operatorname{cis} \frac{\pi}{4})^{25} = 1^{25} \operatorname{cis} \frac{25\pi}{4} = \operatorname{cis} \frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

$\boxed{9}$ $\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)^{20} = (1 \operatorname{cis} \frac{7\pi}{6})^{20} = 1^{20} \operatorname{cis} \frac{70\pi}{3} = \operatorname{cis} \frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

[10] $\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)^{50} = (1\operatorname{cis}\frac{7\pi}{6})^{50} = 1^{50}\operatorname{cis}\frac{175\pi}{3} = \operatorname{cis}\frac{\pi}{3} = \frac{1}{2}+\frac{\sqrt{3}}{2}i$

[11] $(\sqrt{3}+i)^7 = (2\operatorname{cis}\frac{\pi}{6})^7 = 2^7\operatorname{cis}\frac{7\pi}{6} = 128\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right) = -64\sqrt{3}-64i$

[12] $(-2-2i)^{10} = (2\sqrt{2}\operatorname{cis}\frac{5\pi}{4})^{10} = (2\sqrt{2})^{10}\operatorname{cis}\frac{25\pi}{2} = 32{,}768\operatorname{cis}\frac{\pi}{2} = 32{,}768(0+i) = 32{,}768i$

[13] $1+\sqrt{3}\,i = 2\operatorname{cis}60^\circ$. $w_k = \sqrt{2}\operatorname{cis}\left(\frac{60^\circ+360^\circ k}{2}\right)$ for $k = 0,\ 1$.

$$w_0 = \sqrt{2}\operatorname{cis}30^\circ = \sqrt{2}\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right) = \frac{\sqrt{6}}{2}+\frac{\sqrt{2}}{2}i.$$

$$w_1 = \sqrt{2}\operatorname{cis}210^\circ = \sqrt{2}\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right) = -\frac{\sqrt{6}}{2}-\frac{\sqrt{2}}{2}i.$$

[14] $-9i = 9\operatorname{cis}270^\circ$. $w_k = \sqrt{9}\operatorname{cis}\left(\frac{270^\circ+360^\circ k}{2}\right)$ for $k = 0,\ 1$.

$$w_0 = 3\operatorname{cis}135^\circ = 3\left(-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right) = -\frac{3\sqrt{2}}{2}+\frac{3\sqrt{2}}{2}i.$$

$$w_1 = 3\operatorname{cis}315^\circ = 3\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right) = \frac{3\sqrt{2}}{2}-\frac{3\sqrt{2}}{2}i.$$

[15] $-1-\sqrt{3}\,i = 2\operatorname{cis}240^\circ$. $w_k = \sqrt[4]{2}\operatorname{cis}\left(\frac{240^\circ+360^\circ k}{4}\right)$ for $k = 0,\ 1,\ 2,\ 3$.

$$w_0 = \sqrt[4]{2}\operatorname{cis}60^\circ = \sqrt[4]{2}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = \frac{\sqrt[4]{2}}{2}+\frac{\sqrt[4]{18}}{2}i.$$

$\{\text{since } \sqrt[4]{2}\cdot\sqrt{3} = \sqrt[4]{2}\cdot\sqrt[4]{9} = \sqrt[4]{18}\}$

$$w_1 = \sqrt[4]{2}\operatorname{cis}150^\circ = \sqrt[4]{2}\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right) = -\frac{\sqrt[4]{18}}{2}+\frac{\sqrt[4]{2}}{2}i.$$

$$w_2 = \sqrt[4]{2}\operatorname{cis}240^\circ = \sqrt[4]{2}\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) = -\frac{\sqrt[4]{2}}{2}-\frac{\sqrt[4]{18}}{2}i.$$

$$w_3 = \sqrt[4]{2}\operatorname{cis}330^\circ = \sqrt[4]{2}\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right) = \frac{\sqrt[4]{18}}{2}-\frac{\sqrt[4]{2}}{2}i.$$

[16] $-8+8\sqrt{3}i = 16\operatorname{cis}120^\circ$. $w_k = \sqrt[4]{16}\operatorname{cis}\left(\frac{120^\circ+360^\circ k}{4}\right)$ for $k = 0,\ 1,\ 2,\ 3$.

$$w_0 = 2\operatorname{cis}30^\circ = 2\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right) = \sqrt{3}+i.$$

$$w_1 = 2\operatorname{cis}120^\circ = 2\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = -1+\sqrt{3}\,i.$$

$$w_2 = 2\operatorname{cis}210^\circ = 2\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right) = -\sqrt{3}-i.$$

$$w_3 = 2\operatorname{cis}300^\circ = 2\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) = 1-\sqrt{3}\,i.$$

[17] $-27i = 27\operatorname{cis}270°$. $w_k = \sqrt[3]{27}\operatorname{cis}\left(\frac{270° + 360°k}{3}\right)$ for $k = 0, 1, 2$.

$$w_0 = 3\operatorname{cis}90° = 3(0+i) = 3i.$$
$$w_1 = 3\operatorname{cis}210° = 3\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = -\frac{3\sqrt{3}}{2} - \frac{3}{2}i.$$
$$w_2 = 3\operatorname{cis}330° = 3\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = \frac{3\sqrt{3}}{2} - \frac{3}{2}i.$$

[18] $64i = 64\operatorname{cis}90°$. $w_k = \sqrt[3]{64}\operatorname{cis}\left(\frac{90° + 360°k}{3}\right)$ for $k = 0, 1, 2$.

$$w_0 = 4\operatorname{cis}30° = 4\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = 2\sqrt{3} + 2i.$$
$$w_1 = 4\operatorname{cis}150° = 4\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = -2\sqrt{3} + 2i.$$
$$w_2 = 4\operatorname{cis}270° = 4(0-i) = -4i.$$

[19] $1 = 1\operatorname{cis}0°$. $w_k = \sqrt[6]{1}\operatorname{cis}\left(\frac{0° + 360°k}{6}\right)$ for $k = 0, 1, 2, 3, 4, 5$.

$w_0 = 1\operatorname{cis}0° = 1 + 0i.$ $\qquad w_1 = 1\operatorname{cis}60° = \frac{1}{2} + \frac{\sqrt{3}}{2}i.$

$w_2 = 1\operatorname{cis}120° = -\frac{1}{2} + \frac{\sqrt{3}}{2}i.$ $\qquad w_3 = 1\operatorname{cis}180° = -1 + 0i.$

$w_4 = 1\operatorname{cis}240° = -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$ $\qquad w_5 = 1\operatorname{cis}300° = \frac{1}{2} - \frac{\sqrt{3}}{2}i.$

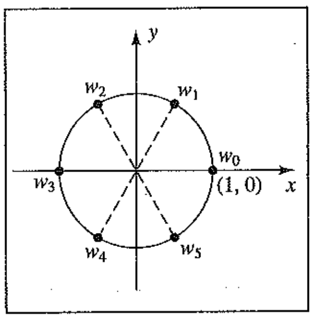


Figure 19

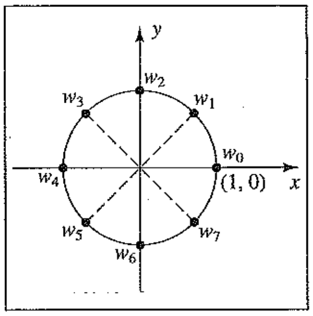


Figure 20

[20] $1 = 1\operatorname{cis}0°$. $w_k = \sqrt[8]{1}\operatorname{cis}\left(\frac{0° + 360°k}{8}\right)$ for $k = 0, 1, 2, 3, 4, 5, 6, 7$.

$w_0 = 1\operatorname{cis}0° = 1 + 0i.$ $\qquad w_1 = 1\operatorname{cis}45° = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i.$

$w_2 = 1\operatorname{cis}90° = 0 + i.$ $\qquad w_3 = 1\operatorname{cis}135° = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i.$

$w_4 = 1\operatorname{cis}180° = -1 + 0i.$ $\qquad w_5 = 1\operatorname{cis}225° = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i.$

$w_6 = 1\operatorname{cis}270° = 0 - i.$ $\qquad w_7 = 1\operatorname{cis}315° = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i.$

[21] $1+i=\sqrt{2}\operatorname{cis}45^\circ$. $w_k=\sqrt{\sqrt[5]{2}}\operatorname{cis}\left(\dfrac{45^\circ+360^\circ k}{5}\right)$ for $k=0, 1, 2, 3, 4$.

$w_k=\sqrt[10]{2}\operatorname{cis}\theta$ with $\theta=9^\circ, 81^\circ, 153^\circ, 225^\circ, 297^\circ$.

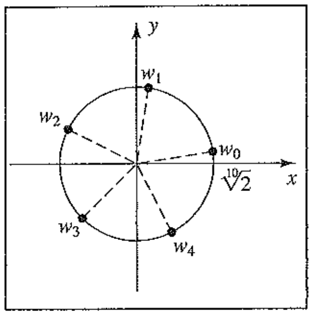



Figure 21

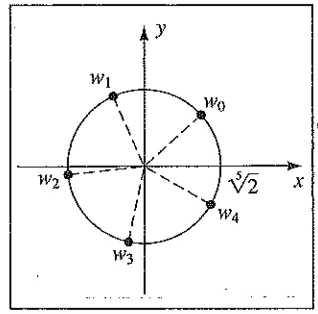



Figure 22

[22] $-\sqrt{3}-i=2\operatorname{cis}210^\circ$. $w_k=\sqrt[5]{2}\operatorname{cis}\left(\dfrac{210^\circ+360^\circ k}{5}\right)$ for $k=0, 1, 2, 3, 4$.

$w_k=\sqrt[5]{2}\operatorname{cis}\theta$ with $\theta=42^\circ, 114^\circ, 186^\circ, 258^\circ, 330^\circ$.

[23] $x^4-16=0 \Rightarrow x^4=16$. The problem is now to find the 4 fourth roots of 16.

$16=16+0i=16\operatorname{cis}0^\circ$. $w_k=\sqrt[4]{16}\operatorname{cis}\left(\dfrac{0^\circ+360^\circ k}{4}\right)$ for $k=0, 1, 2, 3$.

$w_0=2\operatorname{cis}0^\circ=2(1+0i)=2$. $\quad w_1=2\operatorname{cis}90^\circ=2(0+i)=2i$.

$w_2=2\operatorname{cis}180^\circ=2(-1+0i)=-2$. $\quad w_3=2\operatorname{cis}270^\circ=2(0-i)=-2i$.

[24] $x^6-64=0 \Rightarrow x^6=64$. The problem is now to find the 6 sixth roots of 64.

$64=64+0i=64\operatorname{cis}0^\circ$. $w_k=\sqrt[6]{64}\operatorname{cis}\left(\dfrac{0^\circ+360^\circ k}{6}\right)$ for $k=0, 1, 2, 3, 4, 5$.

$w_0=2\operatorname{cis}0^\circ=2(1+0i)=2$.

$w_1=2\operatorname{cis}60^\circ=2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=1+\sqrt{3}\,i$.

$w_2=2\operatorname{cis}120^\circ=2\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=-1+\sqrt{3}\,i$.

$w_3=2\operatorname{cis}180^\circ=2(-1+0i)=-2$.

$w_4=2\operatorname{cis}240^\circ=2\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=-1-\sqrt{3}\,i$.

$w_5=2\operatorname{cis}300^\circ=2\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=1-\sqrt{3}\,i$.

[25] $x^6+64=0 \Rightarrow x^6=-64$. The problem is now to find the 6 sixth roots of $-64$.

$-64=-64+0i=64\operatorname{cis}180^\circ$. $w_k=\sqrt[6]{64}\operatorname{cis}\left(\dfrac{180^\circ+360^\circ k}{6}\right)$ for $k=0, 1, \ldots, 5$.

$w_0=2\operatorname{cis}30^\circ=2\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)=\sqrt{3}+i$.

$w_1=2\operatorname{cis}90^\circ=2(0+i)=2i$.

$w_2=2\operatorname{cis}150^\circ=2\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)=-\sqrt{3}+i$. (continued)

$w_3 = 2\operatorname{cis}210^\circ = 2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = -\sqrt{3} - i.$
$w_4 = 2\operatorname{cis}270^\circ = 2(0 - i) = -2i.$
$w_5 = 2\operatorname{cis}330^\circ = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = \sqrt{3} - i.$

$\boxed{26}$ $x^5 + 1 = 0 \Rightarrow x^5 = -1$. The problem is now to find the 5 fifth roots of $-1$.

$-1 = -1 + 0i = 1\operatorname{cis}180^\circ$. $w_k = \sqrt[5]{1}\operatorname{cis}\left(\frac{180^\circ + 360^\circ k}{5}\right)$ for $k = 0, 1, 2, 3, 4$.

$w_k = 1\operatorname{cis}\theta$ with $\theta = 36^\circ, 108^\circ, 180^\circ, 252^\circ, 324^\circ$.

$\boxed{27}$ $x^3 + 8i = 0 \Rightarrow x^3 = -8i$. The problem is now to find the 3 cube roots of $-8i$.

$-8i = 0 - 8i = 8\operatorname{cis}270^\circ$. $w_k = \sqrt[3]{8}\operatorname{cis}\left(\frac{270^\circ + 360^\circ k}{3}\right)$ for $k = 0, 1, 2$.
$w_0 = 2\operatorname{cis}90^\circ = 2(0 + i) = 2i.$
$w_1 = 2\operatorname{cis}210^\circ = 2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = -\sqrt{3} - i.$
$w_2 = 2\operatorname{cis}330^\circ = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = \sqrt{3} - i.$

$\boxed{28}$ $x^3 - 64i = 0 \Rightarrow x^3 = 64i$. The problem is now to find the 3 cube roots of $64i$.

$64i = 0 + 64i = 64\operatorname{cis}90^\circ$. $w_k = \sqrt[3]{64}\operatorname{cis}\left(\frac{90^\circ + 360^\circ k}{3}\right)$ for $k = 0, 1, 2$.

$w_0 = 4\operatorname{cis}30^\circ = 4\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = 2\sqrt{3} + 2i.$

$w_1 = 4\operatorname{cis}150^\circ = 4\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = -2\sqrt{3} + 2i.$
$w_2 = 4\operatorname{cis}270^\circ = 4(0 - i) = -4i.$

$\boxed{29}$ $x^5 - 243 = 0 \Rightarrow x^5 = 243$. The problem is now to find the 5 fifth roots of 243.

$243 = 243 + 0i = 243\operatorname{cis}0^\circ$. $w_k = \sqrt[5]{243}\operatorname{cis}\left(\frac{0^\circ + 360^\circ k}{5}\right)$ for $k = 0, 1, 2, 3, 4$.

$w_k = 3\operatorname{cis}\theta$ with $\theta = 0^\circ, 72^\circ, 144^\circ, 216^\circ, 288^\circ$.

$\boxed{30}$ $x^4 + 81 = 0 \Rightarrow x^4 = -81$. The problem is now to find the 4 fourth roots of $-81$.

$-81 = -81 + 0i = 81\operatorname{cis}180^\circ$. $w_k = \sqrt[4]{81}\operatorname{cis}\left(\frac{180^\circ + 360^\circ k}{4}\right)$ for $k = 0, 1, 2, 3$.

$w_0 = 3\operatorname{cis}45^\circ = 3\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = \frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i.$

$w_1 = 3\operatorname{cis}135^\circ = 3\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = -\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i.$

$w_2 = 3\operatorname{cis}225^\circ = 3\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) = -\frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}i.$

$w_3 = 3\operatorname{cis}315^\circ = 3\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) = \frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}i.$

$\boxed{31}$ $[r(\cos\theta + i\sin\theta)]^n = \left[r(e^{i\theta})\right]^n = r^n(e^{i\theta})^n = r^n e^{i(n\theta)} = r^n(\cos n\theta + i\sin n\theta)$

## Chapter 8 Review Exercises

[1] $a = \sqrt{b^2 + c^2 - 2bc\cos\alpha} = \sqrt{6^2 + 7^2 - 2(6)(7)\cos 60^\circ} = \sqrt{43}.$

$$\beta = \cos^{-1}\left(\frac{a^2 + c^2 - b^2}{2ac}\right) = \cos^{-1}\left(\frac{43 + 49 - 36}{2\sqrt{43}\,(7)}\right) = \cos^{-1}\left(\frac{4}{\sqrt{43}}\right).$$

$$\gamma = \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2ab}\right) = \cos^{-1}\left(\frac{43 + 36 - 49}{2\sqrt{43}\,(6)}\right) = \cos^{-1}\left(\frac{5}{2\sqrt{43}}\right).$$

[2] $\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c} \Rightarrow \alpha = \sin^{-1}\left(\frac{a\sin\gamma}{c}\right) = \sin^{-1}\left(\frac{2\sqrt{3}\cdot\frac{1}{2}}{2}\right) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = 60^\circ$ or $120^\circ$.

There are two triangles possible since in either case $\alpha + \gamma < 180^\circ$.

$\beta = (180^\circ - \gamma) - \alpha = (180^\circ - 30^\circ) - (60^\circ \text{ or } 120^\circ) = 90^\circ$ or $30^\circ$.

$$\frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \Rightarrow b = \frac{c\sin\beta}{\sin\gamma} = \frac{2\sin(90^\circ \text{ or } 30^\circ)}{\sin 30^\circ} = 4 \text{ or } 2.$$

[3] $\gamma = 180^\circ - \alpha - \beta = 180^\circ - 60^\circ - 45^\circ = 75^\circ.$

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} \Rightarrow a = \frac{b\sin\alpha}{\sin\beta} = \frac{100\sin 60^\circ}{\sin 45^\circ} = \frac{100\cdot(\sqrt{3}/2)}{\sqrt{2}/2}\cdot\frac{\sqrt{2}}{\sqrt{2}} = 50\sqrt{6}.$$

$$\frac{c}{\sin\gamma} = \frac{b}{\sin\beta} \Rightarrow c = \frac{b\sin\gamma}{\sin\beta} = \frac{100\sin(45^\circ + 30^\circ)}{\sqrt{2}/2} =$$

$$100\sqrt{2}\,(\sin 45^\circ\cos 30^\circ + \cos 45^\circ\sin 30^\circ) = 100\sqrt{2}\left(\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2}\right) =$$

$$\tfrac{100}{4}\sqrt{2}\,(\sqrt{6} + \sqrt{2}) = 25(2\sqrt{3} + 2) = 50(1 + \sqrt{3}).$$

[4] $\alpha = \cos^{-1}\left(\frac{b^2 + c^2 - a^2}{2bc}\right) = \cos^{-1}\left(\frac{9 + 16 - 4}{2(3)(4)}\right) = \cos^{-1}\left(\frac{7}{8}\right).$

$$\beta = \cos^{-1}\left(\frac{a^2 + c^2 - b^2}{2ac}\right) = \cos^{-1}\left(\frac{4 + 16 - 9}{2(2)(4)}\right) = \cos^{-1}\left(\frac{11}{16}\right).$$

$$\gamma = \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2ab}\right) = \cos^{-1}\left(\frac{4 + 9 - 16}{2(2)(3)}\right) = \cos^{-1}\left(-\frac{1}{4}\right).$$

[5] $\alpha = 180^\circ - \beta - \gamma = 180^\circ - 67^\circ - 75^\circ = 38^\circ.$

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} \Rightarrow a = \frac{b\sin\alpha}{\sin\beta} = \frac{12\sin 38^\circ}{\sin 67^\circ} \approx 8.0.$$

$$\frac{c}{\sin\gamma} = \frac{b}{\sin\beta} \Rightarrow c = \frac{b\sin\gamma}{\sin\beta} = \frac{12\sin 75^\circ}{\sin 67^\circ} \approx 12.6, \text{ or } 13.$$

[6] $\frac{\sin\gamma}{c} = \frac{\sin\alpha}{a} \Rightarrow \gamma = \sin^{-1}\left(\frac{c\sin\alpha}{a}\right) = \sin^{-1}\left(\frac{125\sin 23^\circ 30'}{152}\right) \approx \sin^{-1}(0.3279) \approx$

$19^\circ 10'$ or $160^\circ 50'$ {rounded to the nearest 10 minutes}. Reject $160^\circ 50'$ because then $\alpha + \gamma \geq 180^\circ$. $\beta = 180^\circ - \alpha - \gamma \approx 180^\circ - 23^\circ 30' - 19^\circ 10' = 137^\circ 20'$.

$$\frac{b}{\sin\beta} = \frac{a}{\sin\alpha} \Rightarrow b = \frac{a\sin\beta}{\sin\alpha} = \frac{152\sin 137^\circ 20'}{\sin 23^\circ 30'} \approx 258.3, \text{ or } 258.$$

[7] $b = \sqrt{a^2 + c^2 - 2ac\cos\beta} \approx \sqrt{102.8} \approx 10.1.$

$\alpha = \cos^{-1}\left(\dfrac{b^2 + c^2 - a^2}{2bc}\right) \approx \cos^{-1}(0.9116) \approx 24^\circ.$

$\gamma = 180^\circ - \alpha - \beta \approx 180^\circ - 24^\circ - 115^\circ = 41^\circ.$

[8] $\alpha = \cos^{-1}\left(\dfrac{b^2 + c^2 - a^2}{2bc}\right) \approx \cos^{-1}(0.7410) \approx 42^\circ.$

$\beta = \cos^{-1}\left(\dfrac{a^2 + c^2 - b^2}{2ac}\right) \approx \cos^{-1}(0.0607) \approx 87^\circ.$

$\gamma = 180^\circ - \alpha - \beta \approx 180^\circ - 42^\circ - 87^\circ = 51^\circ.$

[9] $\mathcal{A} = \frac{1}{2}bc\sin\alpha = \frac{1}{2}(20)(30)\sin 75^\circ \approx 289.8$, or 290 square units.

[10] $s = \frac{1}{2}(a + b + c) = \frac{1}{2}(4 + 7 + 10) = 10.5.$

$\mathcal{A} = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{(10.5)(6.5)(3.5)(0.5)} \approx 10.9$ square units.

[11] (a) $\mathbf{a} = \langle -4, 5\rangle$ and $\mathbf{b} = \langle 2, -8\rangle \Rightarrow$

$\mathbf{a} + \mathbf{b} = \langle -4 + 2, 5 + (-8)\rangle = \langle -2, -3\rangle.$

(b) $\mathbf{a} - \mathbf{b} = \langle -4 - 2, 5 - (-8)\rangle = \langle -6, 13\rangle.$

(c) $2\mathbf{a} = 2\langle -4, 5\rangle = \langle -8, 10\rangle.$

(d) $-\frac{1}{2}\mathbf{b} = -\frac{1}{2}\langle 2, -8\rangle = \langle -1, 4\rangle.$

Terminal points are $(-2, -3)$, $(-6, 13)$, $(-8, 10)$, $(-1, 4)$.

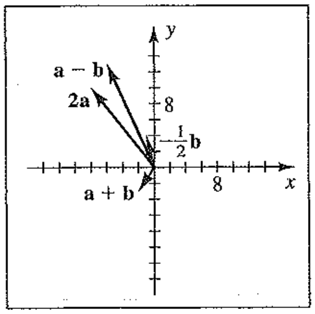



*Figure 11*

[12] (a) $4\mathbf{a} + \mathbf{b} = 4(2\mathbf{i} + 5\mathbf{j}) + (4\mathbf{i} - \mathbf{j}) = 8\mathbf{i} + 20\mathbf{j} + 4\mathbf{i} - \mathbf{j} = 12\mathbf{i} + 19\mathbf{j}.$

(b) $2\mathbf{a} - 3\mathbf{b} = 2(2\mathbf{i} + 5\mathbf{j}) - 3(4\mathbf{i} - \mathbf{j}) = 4\mathbf{i} + 10\mathbf{j} - 12\mathbf{i} + 3\mathbf{j} = -8\mathbf{i} + 13\mathbf{j}.$

(c) $\|\mathbf{a} - \mathbf{b}\| = \|(2\mathbf{i} + 5\mathbf{j}) - (4\mathbf{i} - \mathbf{j})\| = \|-2\mathbf{i} + 6\mathbf{j}\| = \sqrt{40} = 2\sqrt{10} \approx 6.32.$

(d) $\|\mathbf{a}\| - \|\mathbf{b}\| = \|2\mathbf{i} + 5\mathbf{j}\| - \|4\mathbf{i} - \mathbf{j}\| = \sqrt{29} - \sqrt{17} \approx 1.26.$

[13] S50°E is the same as 320°, or $-40^\circ$, on the $xy$-plane.

$\langle 14\cos(-40^\circ), 14\sin(-40^\circ)\rangle = \langle 14\cos 40^\circ, -14\sin 40^\circ\rangle \approx \langle 10.72, -9.00\rangle.$

[14] S60°E is equivalent to 330° and N74°E is equivalent to 16°.

$\langle 72\cos 330^\circ, 72\sin 330^\circ\rangle + \langle 46\cos 16^\circ, 46\sin 16^\circ\rangle = \mathbf{r} \approx \langle 106.57, -23.32\rangle.$

$\|\mathbf{r}\| \approx 109$ kg. $\tan\theta \approx \frac{-23.32}{106.57} \Rightarrow \theta \approx -12^\circ$, or equivalently, S78°E.

[15] $-2\mathbf{a} = -2(8\mathbf{i} - 6\mathbf{j}) = -16\mathbf{i} + 12\mathbf{j}$

[16] $\mathbf{a} = \langle -3, 7\rangle \Rightarrow \|\mathbf{a}\| = \sqrt{58}.$ $\mathbf{v} = 4\left(\dfrac{\mathbf{a}}{\|\mathbf{a}\|}\right) = \left\langle -\dfrac{12}{\sqrt{58}}, \dfrac{28}{\sqrt{58}}\right\rangle.$

[17] $\|\mathbf{r} - \mathbf{a}\| = c \Rightarrow \|\langle x - a_1, y - a_2\rangle\| = c \Rightarrow \sqrt{(x - a_1)^2 + (y - a_2)^2} = c \Rightarrow$

$(x - a_1)^2 + (y - a_2)^2 = c^2$. This is a circle with center $(a_1, a_2)$ and radius $c$.

[18] The vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{a} - \mathbf{b}$ form a triangle with the vector $\mathbf{a} - \mathbf{b}$ opposite angle $\theta$. The conclusion is a direct application of the law of cosines with sides $\|\mathbf{a}\|$, $\|\mathbf{b}\|$, and $\|\mathbf{a} - \mathbf{b}\|$.

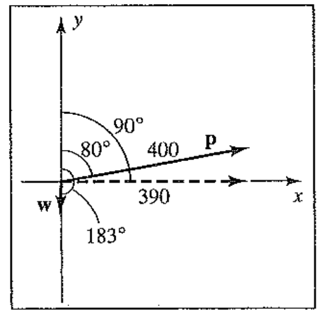


Figure 19

$\boxed{19}$ $\mathbf{p} = \langle 400\cos 10°,\ 400\sin 10°\rangle \approx \langle 393.92,\ 69.46\rangle$.

$\mathbf{r} = \langle 390\cos 0°,\ 390\sin 0°\rangle = \langle 390,\ 0\rangle$.

$\mathbf{w} = \mathbf{r} - \mathbf{p} \approx \langle -3.92,\ -69.46\rangle$ and

$\|\mathbf{w}\| \approx 69.57$, or 70 mi/hr.

$\tan\theta \approx \frac{-69.46}{-3.92} \Rightarrow \theta \approx 267°$, or in the direction of 183°.

$\boxed{20}$ (a) $\mathbf{a}\cdot\mathbf{b} = \langle 2,\ -3\rangle\cdot\langle -1,\ -4\rangle = (2)(-1)+(-3)(-4) = -2+12 = 10$

(b) The angle between $\mathbf{a}$ and $\mathbf{b}$ is $\theta = \cos^{-1}\left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}\right) = \cos^{-1}\left(\frac{10}{\sqrt{13}\sqrt{17}}\right) \approx 47°44'$.

(c) $\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{b}\cdot\mathbf{a}}{\|\mathbf{a}\|} = \frac{10}{\sqrt{13}} \approx 2.77$

$\boxed{21}$ (a) $(2\mathbf{a}-3\mathbf{b})\cdot\mathbf{a} = \left[2(6\mathbf{i}-2\mathbf{j}) - 3(\mathbf{i}+3\mathbf{j})\right]\cdot(6\mathbf{i}-2\mathbf{j})$

$= (9\mathbf{i}-13\mathbf{j})\cdot(6\mathbf{i}-2\mathbf{j}) = 54+26 = 80.$

(b) $\mathbf{c} = \mathbf{a}+\mathbf{b} = (6\mathbf{i}-2\mathbf{j})+(\mathbf{i}+3\mathbf{j}) = 7\mathbf{i}+\mathbf{j}$. The angle between $\mathbf{a}$ and $\mathbf{c}$ is

$$\theta = \cos^{-1}\left(\frac{\mathbf{a}\cdot\mathbf{c}}{\|\mathbf{a}\|\|\mathbf{c}\|}\right) = \cos^{-1}\left(\frac{(6\mathbf{i}-2\mathbf{j})\cdot(7\mathbf{i}+\mathbf{j})}{\|6\mathbf{i}-2\mathbf{j}\|\|7\mathbf{i}+\mathbf{j}\|}\right) = \cos^{-1}\left(\frac{40}{\sqrt{40}\sqrt{50}}\right) \approx 26°34'.$$

(c) $\text{comp}_{\mathbf{a}}(\mathbf{a}+\mathbf{b}) = \text{comp}_{\mathbf{a}}\mathbf{c} = \frac{\mathbf{c}\cdot\mathbf{a}}{\|\mathbf{a}\|} = \frac{40}{\sqrt{40}} = \sqrt{40} = 2\sqrt{10} \approx 6.32.$

$\boxed{22}$ $\mathbf{a}\cdot\overrightarrow{PQ} = \langle 7,\ 4\rangle\cdot\langle 3-(-5),\ 0-0\rangle = 56+0 = 56.$

$\boxed{23}$ $z = -10+10i \Rightarrow r = \sqrt{(-10)^2+10^2} = \sqrt{200} = 10\sqrt{2}.$

$\tan\theta = \frac{10}{-10} = -1$ and $\theta$ in QII $\Rightarrow \theta = \frac{3\pi}{4}$. $z = 10\sqrt{2}\,\text{cis}\,\frac{3\pi}{4}$.

$\boxed{24}$ $z = 2-2\sqrt{3}\,i \Rightarrow r = \sqrt{2^2+(-2\sqrt{3})^2} = \sqrt{16} = 4.$

$\tan\theta = \frac{-2\sqrt{3}}{2} = -\sqrt{3}$ and $\theta$ in QIV $\Rightarrow \theta = \frac{5\pi}{3}$. $z = 4\,\text{cis}\,\frac{5\pi}{3}$.

$\boxed{25}$ $z = -17 \Rightarrow r = 17$. $\theta$ on the negative $x$-axis $\Rightarrow \theta = \pi$. $z = 17\,\text{cis}\,\pi$.

$\boxed{26}$ $z = -12i \Rightarrow r = 12$. $\theta$ on the negative $y$-axis $\Rightarrow \theta = \frac{3\pi}{2}$. $z = 12\,\text{cis}\,\frac{3\pi}{2}$.

$\boxed{27}$ $z = -5\sqrt{3}-5i \Rightarrow r = \sqrt{(-5\sqrt{3})^2+(-5)^2} = \sqrt{100} = 10.$

$\tan\theta = \frac{-5}{-5\sqrt{3}} = \frac{1}{\sqrt{3}}$ and $\theta$ in QIII $\Rightarrow \theta = \frac{7\pi}{6}$. $z = 10\,\text{cis}\,\frac{7\pi}{6}$.

$\boxed{28}$ $z = 4+5i \Rightarrow r = \sqrt{4^2+5^2} = \sqrt{41}$. $\tan\theta = \frac{5}{4}$ and $\theta$ in QI $\Rightarrow \theta = \tan^{-1}\frac{5}{4}$.

$z = \sqrt{41}\,\text{cis}\,(\tan^{-1}\frac{5}{4})$.

$\boxed{29}$ $20\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right) = 20\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = 10\sqrt{3} - 10i$

$\boxed{30}$ $13\,\text{cis}\,(\tan^{-1}\frac{5}{12}) = 13\left[\cos(\tan^{-1}\frac{5}{12}) + i\sin(\tan^{-1}\frac{5}{12})\right] = 13\left(\frac{12}{13}+\frac{5}{13}i\right) = 12+5i$

$\boxed{31}$ $z_1 = -3\sqrt{3} - 3i = 6\operatorname{cis}\frac{7\pi}{6}$ and $z_2 = 2\sqrt{3} + 2i = 4\operatorname{cis}\frac{\pi}{6}$.

$$z_1 z_2 = 6 \cdot 4\operatorname{cis}\left(\tfrac{7\pi}{6} + \tfrac{\pi}{6}\right) = 24\operatorname{cis}\tfrac{4\pi}{3} = 24\left(-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right) = -12 - 12\sqrt{3}\,i.$$

$$\frac{z_1}{z_2} = \frac{6}{4}\operatorname{cis}\left(\tfrac{7\pi}{6} - \tfrac{\pi}{6}\right) = \frac{3}{2}\operatorname{cis}\pi = \frac{3}{2}(-1 + 0i) = -\frac{3}{2}.$$

$\boxed{32}$ $z_1 = 2\sqrt{2} + 2\sqrt{2}i = 4\operatorname{cis}\frac{\pi}{4}$ and $z_2 = -1 - i = \sqrt{2}\operatorname{cis}\frac{5\pi}{4}$.

$$z_1 z_2 = 4 \cdot \sqrt{2}\operatorname{cis}\left(\tfrac{\pi}{4} + \tfrac{5\pi}{4}\right) = 4\sqrt{2}\operatorname{cis}\tfrac{3\pi}{2} = 4\sqrt{2}\,(0 - i) = -4\sqrt{2}\,i.$$

$$\frac{z_1}{z_2} = \frac{4}{\sqrt{2}}\operatorname{cis}\left(\tfrac{\pi}{4} - \tfrac{5\pi}{4}\right) = 2\sqrt{2}\operatorname{cis}(-\pi) = 2\sqrt{2}\,(-1 + 0i) = -2\sqrt{2}.$$

$\boxed{33}$ $(-\sqrt{3} + i)^9 = (2\operatorname{cis}\frac{5\pi}{6})^9 = 2^9\operatorname{cis}\frac{15\pi}{2} = 512\operatorname{cis}\frac{3\pi}{2} = 512(0 - i) = -512i$

$\boxed{34}$ $\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)^{30} = (1\operatorname{cis}\frac{7\pi}{4})^{30} = 1^{30}\operatorname{cis}\frac{105\pi}{2} = \operatorname{cis}\frac{\pi}{2} = 0 + i$

$\boxed{35}$ $(3 - 3i)^5 = (3\sqrt{2}\operatorname{cis}\frac{7\pi}{4})^5 = (3\sqrt{2})^5\operatorname{cis}\frac{35\pi}{4} = 972\sqrt{2}\operatorname{cis}\frac{3\pi}{4} = 972\sqrt{2}\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) =$

$-972 + 972i$

$\boxed{36}$ $(2 + 2\sqrt{3}\,i)^{10} = (4\operatorname{cis}\frac{\pi}{3})^{10} = 4^{10}\operatorname{cis}\frac{10\pi}{3} = 2^{20}\operatorname{cis}\frac{4\pi}{3} = 2^{20}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) =$

$-2^{19} - 2^{19}\sqrt{3}\,i$

$\boxed{37}$ $-27 + 0i = 27\operatorname{cis}180^\circ$. $w_k = \sqrt[3]{27}\operatorname{cis}\left(\dfrac{180^\circ + 360^\circ k}{3}\right)$ for $k = 0,\ 1,\ 2$.

$w_0 = 3\operatorname{cis}60^\circ = 3\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = \frac{3}{2} + \frac{3\sqrt{3}}{2}i.$

$w_1 = 3\operatorname{cis}180^\circ = 3(-1 + 0i) = -3.$

$w_2 = 3\operatorname{cis}300^\circ = 3\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = \frac{3}{2} - \frac{3\sqrt{3}}{2}i.$

$\boxed{38}$ (a) $z^{24} = (1 - \sqrt{3}\,i)^{24} = (2\operatorname{cis}\frac{5\pi}{3})^{24} = 2^{24}\operatorname{cis}40\pi = 2^{24}(1 + 0i) = 2^{24}$

(b) $1 - \sqrt{3}\,i = 2\operatorname{cis}300^\circ$. $w_k = \sqrt[3]{2}\operatorname{cis}\left(\dfrac{300^\circ + 360^\circ k}{3}\right)$ for $k = 0,\ 1,\ 2$.

$w_k = \sqrt[3]{2}\operatorname{cis}\theta$ with $\theta = 100^\circ,\ 220^\circ,\ 340^\circ$.

$\boxed{39}$ $x^5 - 32 = 0 \;\Rightarrow\; x^5 = 32$. The problem is now to find the 5 fifth roots of 32.

$32 = 32 + 0i = 32\operatorname{cis}0^\circ$. $w_k = \sqrt[5]{32}\operatorname{cis}\left(\dfrac{0^\circ + 360^\circ k}{5}\right)$ for $k = 0,\ 1,\ 2,\ 3,\ 4$.

$w_k = 2\operatorname{cis}\theta$ with $\theta = 0^\circ,\ 72^\circ,\ 144^\circ,\ 216^\circ,\ 288^\circ$.

$\boxed{40}$ $\frac{\sin\gamma}{150} = \frac{\sin 27.4^\circ}{200} \Rightarrow$

$\gamma = \sin^{-1}\left(\frac{150\sin 27.4^\circ}{200}\right) \approx \sin^{-1}(0.3451) \approx 20.2^\circ$.

$\beta = 180^\circ - \alpha - \beta \approx 180^\circ - 27.4^\circ - 20.2^\circ = 132.4^\circ$.

The angle between the hill and the horizontal is then

$180^\circ - 132.4^\circ = 47.6^\circ$.

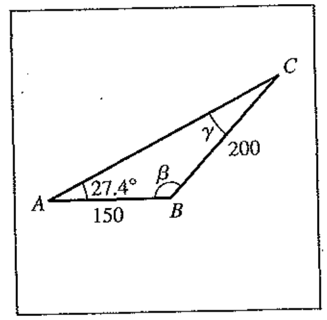



*Figure 40*

$\boxed{41}$ Let $a$ be the Earth–Venus distance, $b$ be the Earth–sun distance, and $c$ be the Venus–sun distance. Then, by the law of cosines (with $a$, $b$, and $c$ in millions), $a^2 = b^2 + c^2 - 2bc\cos\alpha = 93^2 + 67^2 - 2(93)(67)\cos 34^\circ \approx 2807 \Rightarrow$

$a \approx 53$—that is, 53,000,000 miles.

$\boxed{42}$ Let $P$ denote the point at the base of the shorter building, $S$ the point at the top of the shorter building, $T$ the point at the top of the skyscraper, $Q$ the point 50 feet up the side of the skyscraper, and $h$ the height of the skyscraper.

(a) $\angle SPT = 90^\circ - 62^\circ = 28^\circ$. $\angle PST = 90^\circ + 59^\circ = 149^\circ$.

Thus, $\angle STP = 180^\circ - 28^\circ - 149^\circ = 3^\circ$. $\frac{\overline{ST}}{\sin 28^\circ} = \frac{50}{\sin 3^\circ} \Rightarrow \overline{ST} \approx 448.52$, or 449 ft.

(b) $h = \overline{QT} + 50 = \overline{ST}\sin 59^\circ + 50 \approx 434.45$, or 434 ft.

$\boxed{43}$ (a) $\angle LAS = 180^\circ - 47.2^\circ - 66.4^\circ = 66.4^\circ$. $\frac{\overline{AL}}{\sin 47.2^\circ} = \frac{41}{\sin 66.4^\circ} \Rightarrow$

$\overline{AL} = \frac{41\sin 47.2^\circ}{\sin 66.4^\circ} \approx 32.83$, or 33 miles. $\overline{AS} = \overline{LS} = 41$ since $\triangle LAS$ is isosceles.

(b) Let $\overline{AP}$ be perpendicular to $\overline{LS}$. $\sin 47.2^\circ = \frac{\overline{AP}}{\overline{AS}} \Rightarrow \overline{AP} \approx 30.08$, or 30 miles.

$\boxed{44}$ Let $E$ denote the middle point. $\angle CDA = \angle BDC - \angle BDA = 125^\circ - 100^\circ = 25^\circ$.

In $\triangle CAD$, $\angle CAD = 180^\circ - \angle ACD - \angle CDA = 180^\circ - 115^\circ - 25^\circ = 40^\circ$.

$\frac{\overline{AD}}{\sin 115^\circ} = \frac{120}{\sin 40^\circ} \Rightarrow \overline{AD} \approx 169.20$. $\angle DCB = \angle ACD - \angle ACB = 115^\circ - 92^\circ = 23^\circ$.

In $\triangle DBC$, $\angle DBC = 180^\circ - \angle BDC - \angle DCB = 180^\circ - 125^\circ - 23^\circ = 32^\circ$.

$\frac{\overline{BD}}{\sin 23^\circ} = \frac{120}{\sin 32^\circ} \Rightarrow \overline{BD} \approx 88.48$.

In $\triangle ADB$, $\overline{AB}^2 = \overline{AD}^2 + \overline{BD}^2 - 2(\overline{AD})(\overline{BD})\cos\angle BDA \Rightarrow$

$\overline{AB} \approx \sqrt{(169.20)^2 + (88.48)^2 - 2(169.20)(88.48)\cos 100^\circ} \approx 204.1$, or 204 ft.

[45] If $d$ denotes the distance each girl walks before losing contact with each other, then $d = 5t$, where $t$ is in hours. Using the law of cosines,

$10^2 = d^2 + d^2 - 2(d)(d)\cos 105^\circ \Rightarrow 100 = 2d^2(1 - \cos 105^\circ) \Rightarrow d \approx 6.30 \Rightarrow$

$t = d/5 \approx 1.26$ hours, or 1 hour and 16 minutes.

[46] (a) Draw a vertical line $l$ through $C$ and label its $x$-intercept $D$. Since we have alternate interior angles, $\angle ACD = \theta_1$. $\angle DCP = 180^\circ - \theta_2$.

Thus $\angle ACP = \angle ACD + \angle DCP = \theta_1 + (180^\circ - \theta_2) = 180^\circ - (\theta_2 - \theta_1)$.

(b) Let $k = d(A, P)$. $k^2 = 17^2 + 17^2 - 2(17)(17)\cos[180^\circ - (\theta_2 - \theta_1)]$.

Since $\cos(180^\circ - \alpha) = \cos 180^\circ \cos\alpha + \sin 180^\circ \sin\alpha = -\cos\alpha$, we have $k^2 = 578 + 578\cos(\theta_2 - \theta_1) = 578[1 + \cos(\theta_2 - \theta_1)]$. Using the distance formula with the points $A(0, 26)$ and $P(x, y)$, we also have $k^2 = x^2 + (y - 26)^2$.

Hence, $578[1 + \cos(\theta_2 - \theta_1)] = x^2 + (y - 26)^2 \Rightarrow 1 + \cos(\theta_2 - \theta_1) = \dfrac{x^2 + (y - 26)^2}{578}$.

(c) If $x = 25$, $y = 4$, and $\theta_1 = 135^\circ$, then

$1 + \cos(\theta_2 - 135^\circ) = \dfrac{25^2 + (-22)^2}{578} = \dfrac{1109}{578} \Rightarrow \cos(\theta_2 - 135^\circ) = \frac{531}{578} \Rightarrow$

$\theta_2 - 135^\circ \approx 23.3^\circ \Rightarrow \theta_2 \approx 158.3^\circ$, or $158^\circ$.

[47] (a) Let $d$ denote the length of the rescue tunnel. Using the law of cosines,

$d^2 = 45^2 + 50^2 - 2(45)(50)\cos 78^\circ \Rightarrow d \approx 59.91$ ft. Now using the law of sines,

$\dfrac{\sin\theta}{45} = \dfrac{\sin 78^\circ}{d} \Rightarrow \theta = \sin^{-1}\left(\dfrac{45\sin 78^\circ}{d}\right) \approx 47.28^\circ$, or $47^\circ$.

(b) If $x$ denotes the number of hours needed, then

$d$ ft $= (3 \text{ ft/hr})(x \text{ hr}) \Rightarrow x = \frac{1}{3}d = \frac{1}{3}(59.91) \approx 20$ hr.

[48] (a) $\angle CBA = 180^\circ - 136^\circ = 44^\circ$ and

$d = \overline{AC} = \sqrt{22.9^2 + 17.2^2 - 2(22.9)(17.2)\cos 44^\circ} \approx 15.9$. Let $\alpha = \angle BAC$.

Using the law of sines, $\dfrac{\sin\alpha}{22.9} = \dfrac{\sin 44^\circ}{d} \Rightarrow \alpha = \sin^{-1}\left(\dfrac{22.9\sin 44^\circ}{d}\right) \approx 87.4^\circ$.

Let $\beta = \angle CAD$. Using the law of cosines, $5.7^2 = d^2 + 16^2 - 2(d)(16)\cos\beta \Rightarrow$

$\beta = \cos^{-1}\left(\dfrac{d^2 + 16^2 - 5.7^2}{2(d)(16)}\right) \approx 20.6^\circ$. $\phi \approx 180^\circ - 87.4^\circ - 20.6^\circ = 72^\circ$.

(b) The area of $ABCD$ is the sum of the areas of $\triangle CBA$ and $\triangle ADC$.

Area $= \frac{1}{2}(\text{base } \overline{BC})(\text{height to } A) + \frac{1}{2}(\text{base } \overline{AC})(\text{height to } D)$

$= \frac{1}{2}(\overline{BC})(\overline{BA})\sin\angle CBA + \frac{1}{2}(\overline{AC})(\overline{AD})\sin\angle CAD$

$= \frac{1}{2}(22.9)(17.2)\sin 44^\circ + \frac{1}{2}(15.9)(16)\sin 20.6^\circ \approx 136.8 + 44.8 = 181.6 \text{ ft}^2$.

(c) Let $h$ denote the perpendicular distance from $\overline{BA}$ to $C$.

$\sin 44^\circ = \dfrac{h}{22.9} \Rightarrow h \approx 15.9$. The wing span $\overline{CC'}$ is $2h + 5.8 \approx 37.6$ ft.

## Chapter 8 Discussion Exercises

[1] (a) $\dfrac{a}{\sin\alpha}=\dfrac{c}{\sin\gamma} \Rightarrow \dfrac{a}{c}=\dfrac{\sin\alpha}{\sin\gamma}$ and $\dfrac{b}{\sin\beta}=\dfrac{c}{\sin\gamma} \Rightarrow \dfrac{b}{c}=\dfrac{\sin\beta}{\sin\gamma}$.

Adding the equations yields $\frac{a}{c}+\frac{b}{c}=\dfrac{\sin\alpha}{\sin\gamma}+\dfrac{\sin\beta}{\sin\gamma} \Rightarrow \dfrac{a+b}{c}=\dfrac{\sin\alpha+\sin\beta}{\sin\gamma}$.

(b) $\dfrac{a+b}{c}=\dfrac{\sin\alpha+\sin\beta}{\sin\gamma} \Rightarrow \dfrac{a+b}{c}=\dfrac{[\mathrm{S1}]\ 2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)}{2\sin\frac{1}{2}\gamma\cos\frac{1}{2}\gamma}$.

Now $\gamma=180^\circ-(\alpha+\beta) \Rightarrow \frac{1}{2}\gamma=\left[90^\circ-\frac{1}{2}(\alpha+\beta)\right]$ and

$\sin\frac{1}{2}(\alpha+\beta)=\cos\left[90^\circ-\frac{1}{2}(\alpha+\beta)\right]=\cos\frac{1}{2}\gamma$. Thus, $\dfrac{a+b}{c}=\dfrac{\cos\frac{1}{2}(\alpha-\beta)}{\sin\frac{1}{2}\gamma}$.

*Note:* This is an interesting result and gives an answer to the question, "How can I check these triangle problems?" Some of my students have written programs for their graphing calculators to utilize this check.

[2] $z^{-n}=r^{-n}\left[\cos(-n\theta)+i\sin(-n\theta)\right]=\dfrac{\cos n\theta-i\sin n\theta}{r^n}$ {cos is even, sin is odd}

$$=\frac{(\cos n\theta-i\sin n\theta)(\cos n\theta+i\sin n\theta)}{r^n(\cos n\theta+i\sin n\theta)}=\frac{\cos^2 n\theta-i^2\sin^2 n\theta}{r^n(\cos n\theta+i\sin n\theta)}$$

$$=\frac{\cos^2 n\theta+\sin^2 n\theta}{r^n(\cos n\theta+i\sin n\theta)}=\frac{1}{r^n(\cos n\theta+i\sin n\theta)}=\frac{1}{z^n}$$

[3] Algebraic: $\sqrt[3]{a}$, $\sqrt[3]{a}\operatorname{cis}\frac{2\pi}{3}$, $\sqrt[3]{a}\operatorname{cis}\frac{4\pi}{3}$

Geometric: All roots lie on a circle of radius $\sqrt[3]{a}$, they are all $120^\circ$ apart, one is on the real axis, one is on $\theta=\frac{2\pi}{3}$, and one is on $\theta=\frac{4\pi}{3}$

[4] (a) The vector $\mathbf{v}-\mathbf{w}$ is the vector that would need to be added to $\mathbf{w}$ to equal $\mathbf{v}$. That is, if the initial point of $\mathbf{v}-\mathbf{w}$ (assuming $\mathbf{v}$ and $\mathbf{w}$ have the same initial point) is placed on the terminal point of $\mathbf{w}$, the terminal point of $\mathbf{v}-\mathbf{w}$ would coincide with the terminal point of $\mathbf{v}$ and "complete the triangle."

(b) Use the law of cosines to obtain

$$\|\mathbf{v}-\mathbf{w}\|=\sqrt{\|\mathbf{v}\|^2+\|\mathbf{w}\|^2-2\|\mathbf{v}\|\,\|\mathbf{w}\|\cos\theta}.$$

[5] (a) $\mathbf{c}=\mathbf{b}+\mathbf{a}=(\|\mathbf{b}\|\cos\alpha\,\mathbf{i}+\|\mathbf{b}\|\sin\alpha\,\mathbf{j})+(\|\mathbf{a}\|\cos(-\beta)\,\mathbf{i}+\|\mathbf{a}\|\sin(-\beta)\,\mathbf{j})$

$=\|\mathbf{b}\|\cos\alpha\,\mathbf{i}+\|\mathbf{b}\|\sin\alpha\,\mathbf{j}+\|\mathbf{a}\|\cos\beta\,\mathbf{i}-\|\mathbf{a}\|\sin\beta\,\mathbf{j}$

$=(\|\mathbf{b}\|\cos\alpha+\|\mathbf{a}\|\cos\beta)\mathbf{i}+(\|\mathbf{b}\|\sin\alpha-\|\mathbf{a}\|\sin\beta)\mathbf{j}$

(b) $\|\mathbf{c}\|^2 = (\|\mathbf{b}\|\cos\alpha + \|\mathbf{a}\|\cos\beta)^2 + (\|\mathbf{b}\|\sin\alpha - \|\mathbf{a}\|\sin\beta)^2$

$= \|\mathbf{b}\|^2\cos^2\alpha + 2\|\mathbf{a}\|\|\mathbf{b}\|\cos\alpha\,\cos\beta + \|\mathbf{a}\|^2\cos^2\beta +$

$\|\mathbf{b}\|^2\sin^2\alpha - 2\|\mathbf{a}\|\|\mathbf{b}\|\sin\alpha\,\sin\beta + \|\mathbf{a}\|^2\sin^2\beta$

$= (\|\mathbf{b}\|^2\cos^2\alpha + \|\mathbf{b}\|^2\sin^2\alpha) + (\|\mathbf{a}\|^2\cos^2\beta + \|\mathbf{a}\|^2\sin^2\beta) +$

$2\|\mathbf{a}\|\|\mathbf{b}\|\cos\alpha\,\cos\beta - 2\|\mathbf{a}\|\|\mathbf{b}\|\sin\alpha\,\sin\beta$

$= \|\mathbf{b}\|^2 + \|\mathbf{a}\|^2 + 2\|\mathbf{a}\|\|\mathbf{b}\|(\cos\alpha\,\cos\beta - \sin\alpha\,\sin\beta)$

$= \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + 2\|\mathbf{a}\|\|\mathbf{b}\|\cos(\alpha+\beta)$

$= \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + 2\|\mathbf{a}\|\|\mathbf{b}\|\cos(\pi-\gamma)\ \{\alpha+\beta+\gamma=\pi\}$

$= \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\|\|\mathbf{b}\|\cos\gamma\ \{\cos(\pi-\gamma) = -\cos\gamma\}$

(c) From part (a), we let $(\|\mathbf{b}\|\sin\alpha - \|\mathbf{a}\|\sin\beta) = 0$.

Thus, $\|\mathbf{b}\|\sin\alpha = \|\mathbf{a}\|\sin\beta$, and $\dfrac{\sin\alpha}{\|\mathbf{a}\|} = \dfrac{\sin\beta}{\|\mathbf{b}\|}$.

$\boxed{6}$ (a) $e^{2\pi i} = \cos 2\pi + i\sin 2\pi = 1 + i\cdot 0 = 1$

(b) $\text{LN}(-1) = \text{LN}(-1+0i) = \ln|-1+0i| + i(\pi + 2\pi\cdot 0) =$

$\ln\sqrt{(-1)^2+0^2} + i(\pi) = \ln 1 + \pi i = 0 + \pi i = \pi i$

$\text{LN}\, i = \text{LN}(0+i) = \ln|0+i| + i(\frac{\pi}{2} + 2\pi\cdot 0) = \ln\sqrt{0^2+1^2} + i(\frac{\pi}{2}) =$

$\ln 1 + \frac{\pi}{2}i = 0 + \frac{\pi}{2}i = \frac{\pi}{2}i$

(c) $\sqrt{i} = i^{1/2} = e^{(1/2)\,\text{LN}\,i} = e^{(1/2)\,(\ln 1 + i(\pi/2))} = e^{(1/2)\,(i(\pi/2))} = e^{(\pi/4)i} =$

$\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

$i^i = e^{i\,\text{LN}\,i} = e^{i\,(\ln 1 + i(\pi/2))} = e^{i(i(\pi/2))} = e^{-\pi/2} \approx 0.2079$ { a real number! }

$\boxed{7}$ If we check the statement

$$\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\,\tan\beta\,\tan\gamma\ (*)$$

for a couple sets of values of $\alpha$, $\beta$, and $\gamma$ such that $\alpha+\beta+\gamma=\pi$, we find that the statement is true, so we'll try to prove that $(*)$ is an identity.

$\text{LS} = \tan\alpha + \tan\beta + \tan\gamma$

$= \dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\sin\beta}{\cos\beta} + \dfrac{\sin\gamma}{\cos\gamma}$

$= \dfrac{\sin\alpha\,\cos\beta\,\cos\gamma + \sin\beta\,\cos\alpha\,\cos\gamma + \sin\gamma\,\cos\alpha\,\cos\beta}{\cos\alpha\,\cos\beta\,\cos\gamma}$

$= \dfrac{\cos\gamma\,(\sin\alpha\,\cos\beta + \sin\beta\,\cos\alpha) + \sin\gamma\,\cos\alpha\,\cos\beta}{\cos\alpha\,\cos\beta\,\cos\gamma}$

$= \dfrac{\cos\gamma\,\sin(\alpha+\beta) + \sin\gamma\,\cos\alpha\,\cos\beta}{\cos\alpha\,\cos\beta\,\cos\gamma}$

(continued)

Note that $\sin(\alpha+\beta) = \sin(\pi-\gamma) = \sin\pi\,\cos\gamma - \cos\pi\,\sin\gamma = 0-(-1)\sin\gamma = \sin\gamma$.

$$= \frac{\cos\gamma\,\sin\gamma + \sin\gamma\,\cos\alpha\,\cos\beta}{\cos\alpha\,\cos\beta\,\cos\gamma}$$

$$= \frac{\sin\gamma\,(\cos\gamma + \cos\alpha\,\cos\beta)}{\cos\alpha\,\cos\beta\,\cos\gamma}$$

Note that $\cos\gamma = \cos[\pi-(\alpha+\beta)] = \cos\pi\,\cos(\alpha+\beta) + \sin\pi\,\sin(\alpha+\beta)$

$= -\cos(\alpha+\beta) = -\cos\alpha\,\cos\beta + \sin\alpha\,\sin\beta$.

$$= \frac{\sin\gamma\,(-\cos\alpha\,\cos\beta + \sin\alpha\,\sin\beta + \cos\alpha\,\cos\beta)}{\cos\alpha\,\cos\beta\,\cos\gamma}$$

$$= \frac{\sin\gamma\,(\sin\alpha\,\sin\beta)}{\cos\alpha\,\cos\beta\,\cos\gamma} = \frac{\sin\alpha}{\cos\alpha}\cdot\frac{\sin\beta}{\cos\beta}\cdot\frac{\sin\gamma}{\cos\gamma} = \tan\alpha\,\tan\beta\,\tan\gamma = \text{RS}$$

$\boxed{8}$ If $\mathbf{W}$ represents the force of the ornament, then $\mathbf{T}_1 = \|\mathbf{T}_1\|\langle -\cos\alpha, \sin\alpha\rangle$ (*), $\mathbf{T}_2 = \|\mathbf{T}_2\|\langle \cos\beta, \sin\beta\rangle$, and $\mathbf{W} = \langle 0, -5\rangle$. The sum of the forces is $\mathbf{0} = \langle 0, 0\rangle$, so

$$-\|\mathbf{T}_1\|\cos\alpha + \|\mathbf{T}_2\|\cos\beta = 0 \ (\text{E}_1) \quad \text{and} \quad \|\mathbf{T}_1\|\sin\alpha + \|\mathbf{T}_2\|\sin\beta - 5 = 0 \ (\text{E}_2).$$

Solve $\text{E}_1$ for $\|\mathbf{T}_2\|$ and substitute into $\text{E}_2$.

$\|\mathbf{T}_2\|\cos\beta = \|\mathbf{T}_1\|\cos\alpha \;\Rightarrow\; \|\mathbf{T}_2\| = \dfrac{\|\mathbf{T}_1\|\cos\alpha}{\cos\beta}$, so $\text{E}_2$ becomes

$\|\mathbf{T}_1\|\sin\alpha + \dfrac{\|\mathbf{T}_1\|\cos\alpha}{\cos\beta}\cdot\sin\beta = 5 \;\Rightarrow$

$\|\mathbf{T}_1\|\sin\alpha\,\cos\beta + \|\mathbf{T}_1\|\cos\alpha\,\sin\beta = 5\cos\beta \;\Rightarrow$

$\|\mathbf{T}_1\|(\sin\alpha\,\cos\beta + \cos\alpha\,\sin\beta) = 5\cos\beta \;\Rightarrow\; \|\mathbf{T}_1\|\sin(\alpha+\beta) = 5\cos\beta \;\Rightarrow$

$\|\mathbf{T}_1\| = \dfrac{5\cos\beta}{\sin(\alpha+\beta)}$. Thus, $\|\mathbf{T}_2\| = \dfrac{\|\mathbf{T}_1\|\cos\alpha}{\cos\beta} = \dfrac{5\cos\beta\,\cos\alpha}{\sin(\alpha+\beta)\cos\beta} = \dfrac{5\cos\alpha}{\sin(\alpha+\beta)}$.

(*) $\cos(180^\circ - \alpha) = \cos 180^\circ\,\cos\alpha + \sin 180^\circ\,\sin\alpha$

$= -1\cdot\cos\alpha + 0\cdot\sin\alpha = -\cos\alpha$