CHAPTER 5
Logarithmic, Exponential, and Other Transcendental Functions

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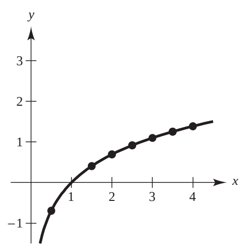
Section 5.1 The Natural Logarithmic Function: Differentiation

1. Simpson's Rule: $n = 10$

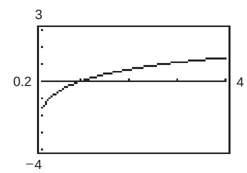
x	0.5	1.5	2	2.5	3	3.5	4
$\int_1^x \frac{1}{t}\,dt$	-0.6932	0.4055	0.6932	0.9163	1.0987	1.2529	1.3865

Note: $\int_1^{0.5} \frac{1}{t}\,dt = -\int_{0.5}^{1} \frac{1}{t}\,dt$

2. (a)



(b)



The graphs are identical.

3. (a) $\ln 45 \approx 3.8067$

(b) $\int_1^{45} \frac{1}{t}\,dt \approx 3.8067$

4. (a) $\ln 8.3 \approx 2.1163$

(b) $\int_1^{8.3} \frac{1}{t}\,dt \approx 2.1163$

5. (a) $\ln 0.8 \approx -0.2231$

(b) $\int_1^{0.8} \frac{1}{t}\,dt \approx -0.2231$

6. (a) $\ln 0.6 \approx -0.5108$

(b) $\int_1^{0.6} \frac{1}{t}\,dt \approx -0.5108$

7. $f(x) = \ln x + 2$

Vertical shift 2 units upward

Matches (b)

8. $f(x) = -\ln x$

Reflection in the x-axis

Matches (d)

9. $f(x) = \ln(x - 1)$

Horizontal shift 1 unit to the right

Matches (a)

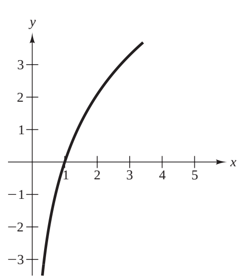
10. $f(x) = -\ln(-x)$

Reflection in the y-axis and the x-axis

Matches (c)

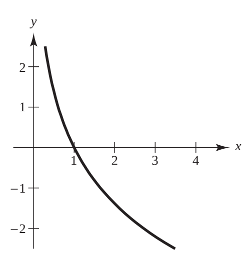
11. $f(x) = 3 \ln x$

Domain: $x > 0$



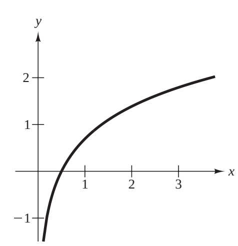
12. $f(x) = -2 \ln x$

Domain: $x > 0$



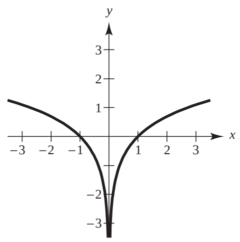
13. $f(x) = \ln 2x$

Domain: $x > 0$



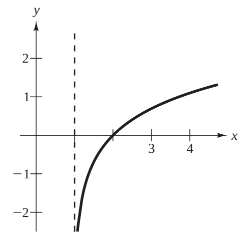
14. $f(x) = \ln|x|$

Domain: $x \neq 0$



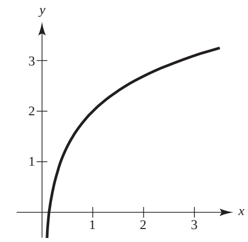
15. $f(x) = \ln(x - 1)$

Domain: $x > 1$



16. $g(x) = 2 + \ln x$

Domain: $x > 0$



17. (a) $\ln 6 = \ln 2 + \ln 3 \approx 1.7917$

(b) $\ln \frac{2}{3} = \ln 2 - \ln 3 \approx -0.4055$

(c) $\ln 81 = \ln 3^4 = 4 \ln 3 \approx 4.3944$

(d) $\ln \sqrt{3} = \ln 3^{1/2} = \frac{1}{2} \ln 3 \approx 0.5493$

18. (a) $\ln 0.25 = \ln \frac{1}{4} = \ln 1 - 2 \ln 2 \approx -1.3862$

(b) $\ln 24 = 3 \ln 2 + \ln 3 \approx 3.1779$

(c) $\ln \sqrt[3]{12} = \frac{1}{3}(2 \ln 2 + \ln 3) \approx 0.8283$

(d) $\ln \frac{1}{72} = \ln 1 - (3 \ln 2 + 2 \ln 3) \approx -4.2765$

19. $\ln \dfrac{2}{3} = \ln 2 - \ln 3$

20. $\ln \sqrt{2^3} = \ln 2^{3/2} = \dfrac{3}{2} \ln 2$

21. $\ln \dfrac{xy}{z} = \ln x + \ln y - \ln z$

22. $\ln(xyz) = \ln x + \ln y + \ln z$

23. $\ln \sqrt[3]{a^2 + 1} = \ln(a^2 + 1)^{1/3} = \frac{1}{3} \ln(a^2 + 1)$

24. $\ln \sqrt{a - 1} = \ln(a - 1)^{1/2} = \left(\dfrac{1}{2}\right) \ln(a - 1)$

25. $$\ln\left(\frac{x^2 - 1}{x^3}\right)^3 = 3[\ln(x^2 - 1) - \ln x^3]$$
$$= 3[\ln(x + 1) + \ln(x - 1) - 3 \ln x]$$

26. $\ln 3e^2 = \ln 3 + 2 \ln e = 2 + \ln 3$

27. $$\ln z(z - 1)^2 = \ln z + \ln(z - 1)^2$$
$$= \ln z + 2 \ln(z - 1)$$

28. $\ln \dfrac{1}{e} = \ln 1 - \ln e = -1$

29. $\ln(x - 2) - \ln(x + 2) = \ln \dfrac{x - 2}{x + 2}$

30. $$3 \ln x + 2 \ln y - 4 \ln z = \ln x^3 + \ln y^2 - \ln z^4$$
$$= \ln \frac{x^3 y^2}{z^4}$$

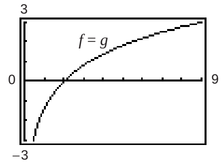
31. $$\frac{1}{3}[2 \ln(x + 3) + \ln x - \ln(x^2 - 1)] = \frac{1}{3} \ln \frac{x(x + 3)^2}{x^2 - 1}$$
$$= \ln \sqrt[3]{\frac{x(x + 3)^2}{x^2 - 1}}$$

32. $$2[\ln x - \ln(x + 1) - \ln(x - 1)] = 2 \ln \frac{x}{(x + 1)(x - 1)}$$
$$= \ln\left(\frac{x}{x^2 - 1}\right)^2$$

33. $2 \ln 3 - \dfrac{1}{2} \ln(x^2 + 1) = \ln 9 - \ln \sqrt{x^2 + 1} = \ln \dfrac{9}{\sqrt{x^2 + 1}}$

34. $\dfrac{3}{2}[\ln(x^2 + 1) - \ln(x + 1) - \ln(x - 1)] = \dfrac{3}{2} \ln \dfrac{x^2 + 1}{(x + 1)(x - 1)} = \ln \sqrt{\left(\dfrac{x^2 + 1}{x^2 - 1}\right)^3}$

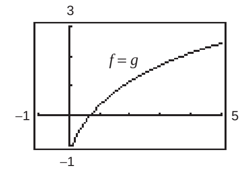
35. (a)



(b) $f(x) = \ln \frac{x^2}{4} = \ln x^2 - \ln 4 = 2 \ln x - \ln 4 = g(x)$

since $x > 0$.

36. (a)



(b) $f(x) = \ln \sqrt{x(x^2 + 1)} = \frac{1}{2} \ln[x(x^2 + 1)]$

$= \frac{1}{2}[\ln x + \ln(x^2 + 1)] = g(x)$

37. $\lim_{x \to 3^+} \ln(x - 3) = -\infty$

38. $\lim_{x \to 6^-} \ln(6 - x) = -\infty$

39. $\lim_{x \to 2^-} \ln[x^2(3 - x)] = \ln 4$

≈ 1.3863

40. $\lim_{x \to 5^+} \ln \frac{x}{\sqrt{x - 4}} = \ln 5 \approx 1.6094$

41. $y = \ln x^3 = 3 \ln x$

$y' = \frac{3}{x}$

Slope at $(1, 0)$ is $\frac{3}{1} = 3$.

$y - 0 = 3(x - 1)$

$y = 3x - 3$ Tangent line

42. $y = \ln x^{3/2} = \frac{3}{2} \ln x$

$y' = \frac{3}{2x}$

Slope at $(1, 0)$ is $\frac{3}{2}$.

$y - 0 = \frac{3}{2}(x - 1)$

$y = \frac{3}{2}x - \frac{3}{2}$ Tangent line

43. $y = \ln x^2 = 2 \ln x$

$y' = \frac{2}{x}$

Slope at $(1, 0)$ is 2.

$y - 0 = 2(x - 1)$

$y = 2x - 2$ Tangent line

44. $y = \ln x^{1/2} = \frac{1}{2} \ln x$

$y' = \frac{1}{2x}$

At $(1, 0)$, slope is $\frac{1}{2}$.

$y - 0 = \frac{1}{2}(x - 1)$

$y = \frac{1}{2}x - \frac{1}{2}$ Tangent line

45. $g(x) = \ln x^2 = 2 \ln x$

$g'(x) = \frac{2}{x}$

46. $h(x) = \ln(2x^2 + 1)$

$h'(x) = \frac{1}{2x^2 + 1}(4x) = \frac{4x}{2x^2 + 1}$

47. $y = (\ln x)^4$

$\frac{dy}{dx} = 4(\ln x)^3\left(\frac{1}{x}\right) = \frac{4(\ln x)^3}{x}$

48. $y = x \ln x$

$\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \ln x = 1 + \ln x$

49. $y = \ln\left[x\sqrt{x^2 - 1}\right] = \ln x + \frac{1}{2} \ln(x^2 - 1)$

$\frac{dy}{dx} = \frac{1}{x} + \frac{1}{2}\left(\frac{2x}{x^2 - 1}\right) = \frac{2x^2 - 1}{x(x^2 - 1)}$

50. $y = \ln \sqrt{x^2 - 4} = \frac{1}{2} \ln(x^2 - 4)$

$\frac{dy}{dx} = \frac{1}{2}\left(\frac{2x}{x^2 - 4}\right) = \frac{x}{x^2 - 4}$

51. $f(x) = \ln \frac{x}{x^2 + 1} = \ln x - \ln(x^2 + 1)$

$f'(x) = \frac{1}{x} - \frac{2x}{x^2 + 1} = \frac{1 - x^2}{x(x^2 + 1)}$

52. $f(x) = \ln\left(\frac{2x}{x + 3}\right) = \ln 2x - \ln(x + 3)$

$f'(x) = \frac{1}{x} - \frac{1}{x + 3} = \frac{3}{x(x + 3)}$

53. $g(t) = \dfrac{\ln t}{t^2}$

$$g'(t) = \frac{t^2(1/t) - 2t\ln t}{t^4} = \frac{1 - 2\ln t}{t^3}$$

54. $h(t) = \dfrac{\ln t}{t}$

$$h'(t) = \frac{t(1/t) - \ln t}{t^2} = \frac{1 - \ln t}{t^2}$$

55. $y = \ln(\ln x^2)$

$$\frac{dy}{dx} = \frac{1}{\ln x^2}\frac{d}{dx}(\ln x^2) = \frac{(2x/x^2)}{\ln x^2} = \frac{2}{x\ln x^2} = \frac{1}{x\ln x}$$

56. $y = \ln(\ln x)$

$$\frac{dy}{dx} = \frac{1/x}{\ln x} = \frac{1}{x\ln x}$$

57. $y = \ln\sqrt{\dfrac{x+1}{x-1}} = \dfrac{1}{2}[\ln(x+1) - \ln(x-1)]$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x+1} - \frac{1}{x-1}\right] = \frac{1}{1-x^2}$$

58. $y = \ln\sqrt[3]{\dfrac{x-1}{x+1}} = \dfrac{1}{3}[\ln(x-1) - \ln(x+1)]$

$$y' = \frac{1}{3}\left[\frac{1}{x-1} - \frac{1}{x+1}\right] = \frac{1}{3}\frac{2}{x^2-1} = \frac{2}{3(x^2-1)}$$

59. $f(x) = \ln\dfrac{\sqrt{4+x^2}}{x} = \dfrac{1}{2}\ln(4+x^2) - \ln x$

$$f'(x) = \frac{x}{4+x^2} - \frac{1}{x} = \frac{-4}{x(x^2+4)}$$

60. $f(x) = \ln\left(x + \sqrt{4+x^2}\right)$

$$f'(x) = \frac{1}{x+\sqrt{4+x^2}}\left(1 + \frac{x}{\sqrt{4+x^2}}\right) = \frac{1}{\sqrt{4+x^2}}$$

61. $y = \dfrac{-\sqrt{x^2+1}}{x} + \ln\left(x + \sqrt{x^2+1}\right)$

$$\frac{dy}{dx} = \frac{-x\left(x/\sqrt{x^2+1}\right) + \sqrt{x^2+1}}{x^2} + \left(\frac{1}{x+\sqrt{x^2+1}}\right)\left(1 + \frac{x}{\sqrt{x^2+1}}\right)$$

$$= \frac{1}{x^2\sqrt{x^2+1}} + \left(\frac{1}{x+\sqrt{x^2+1}}\right)\left(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right)$$

$$= \frac{1}{x^2\sqrt{x^2+1}} + \frac{1}{\sqrt{x^2+1}} = \frac{1+x^2}{x^2\sqrt{x^2+1}} = \frac{\sqrt{x^2+1}}{x^2}$$

62. $y = \dfrac{-\sqrt{x^2+4}}{2x^2} - \dfrac{1}{4}\ln\left(\dfrac{2+\sqrt{x^2+4}}{x}\right) = \dfrac{-\sqrt{x^2+4}}{2x^2} - \dfrac{1}{4}\ln\left(2+\sqrt{x^2+4}\right) + \dfrac{1}{4}\ln x$

$$\frac{dy}{dx} = \frac{-2x^2\left(x/\sqrt{x^2+4}\right) + 4x\sqrt{x^2+4}}{4x^4} - \frac{1}{4}\left(\frac{1}{2+\sqrt{x^2+4}}\right)\left(\frac{x}{\sqrt{x^2+4}}\right) + \frac{1}{4x}$$

Note that $\dfrac{1}{2+\sqrt{x^2+4}} = \dfrac{1}{2+\sqrt{x^2+4}} \cdot \dfrac{2-\sqrt{x^2+4}}{2-\sqrt{x^2+4}} = \dfrac{2-\sqrt{x^2+4}}{-x^2}$.

Hence, $\dfrac{dy}{dx} = \dfrac{-1}{2x\sqrt{x^2+4}} + \dfrac{\sqrt{x^2+4}}{x^3} - \dfrac{1}{4}\dfrac{\left(2-\sqrt{x^2+4}\right)}{-x^2}\left(\dfrac{x}{\sqrt{x^2+4}}\right) + \dfrac{1}{4x}$

$$= \frac{-1 + (1/2)\left(2-\sqrt{x^2+4}\right)}{2x\sqrt{x^2+4}} + \frac{\sqrt{x^2+4}}{x^3} + \frac{1}{4x}$$

$$= \frac{-\sqrt{x^2+4}}{4x\sqrt{x^2+4}} + \frac{\sqrt{x^2+4}}{x^3} + \frac{1}{4x} = \frac{\sqrt{x^2+4}}{x^3}.$$

63. $y = \ln|\sin x|$

$$\frac{dy}{dx} = \frac{\cos x}{\sin x} = \cot x$$

64. $y = \ln|\csc x|$

$$y' = \frac{-\csc x \cdot \cot x}{\csc x} = -\cot x$$

65. $y = \ln\left|\dfrac{\cos x}{\cos x - 1}\right|$

$$= \ln|\cos x| - \ln|\cos x - 1|$$

$$\frac{dy}{dx} = \frac{-\sin x}{\cos x} - \frac{-\sin x}{\cos x - 1}$$

$$= -\tan x + \frac{\sin x}{\cos x - 1}$$

66. $y = \ln|\sec x + \tan x|$

$$\frac{dy}{dx} = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$$

$$= \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} = \sec x$$

67. $y = \ln\left|\dfrac{-1 + \sin x}{2 + \sin x}\right|$

$$= \ln|-1 + \sin x| - \ln|2 + \sin x|$$

$$\frac{dy}{dx} = \frac{\cos x}{-1 + \sin x} - \frac{\cos x}{2 + \sin x}$$

$$= \frac{3 \cos x}{(\sin x - 1)(\sin x + 2)}$$

68. $y = \ln\sqrt{2 + \cos^2 x}$

$$= \frac{1}{2}\ln(2 + \cos^2 x)$$

$$y' = \frac{1}{2}\,\frac{-2\cos x \sin x}{2 + \cos^2 x}$$

$$= \frac{-\cos x \sin x}{2 + \cos^2 x}$$

69. $f(x) = \displaystyle\int_2^{\ln(2x)} (t + 1)\,dt$

$$f'(x) = [\ln(2x) + 1]\left(\frac{1}{x}\right) = \frac{\ln(2x) + 1}{x}$$

Second solution:

$$f(x) = \int_2^{\ln(2x)} (t + 1)\,dt$$

$$= \left[\frac{t^2}{2} + t\right]_2^{\ln 2x} = \left[\frac{[\ln(2x)]^2}{2} + \ln(2x)\right] - [2 + 2]$$

$$f'(x) = \frac{1}{2}\,2\ln(2x)\frac{1}{x} + \frac{1}{2x}(2) = \frac{\ln(2x) + 1}{x}$$

70. $g(x) = \displaystyle\int_1^{\ln x} (t^2 + 3)\,dt$

$$g'(x) = [(\ln x)^2 + 3]\frac{d}{dx}(\ln x) = \frac{(\ln x)^2 + 3}{x}$$

(Second Fundamental Theorem of Calculus)

71. (a) $y = 3x^2 - \ln x, \quad (1, 3)$

$$\frac{dy}{dx} = 6x - \frac{1}{x}$$

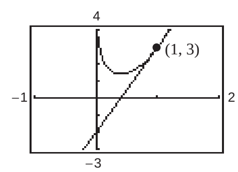
When $x = 1$, $\dfrac{dy}{dx} = 5$.

Tangent line: $y - 3 = 5(x - 1)$

$$y = 5x - 2$$

$$0 = 5x - y - 2$$

(b)



72. (a) $y = 4 - x^2 - \ln\left(\dfrac{1}{2}x + 1\right), \quad (0, 4)$

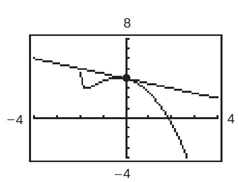
$$\frac{dy}{dx} = -2x - \frac{1}{(1/2)x + 1}\left(\frac{1}{2}\right) = -2x - \frac{1}{x + 2}$$

When $x = 0$, $\dfrac{dy}{dx} = -\dfrac{1}{2}$.

Tangent line: $y - 4 = -\dfrac{1}{2}(x - 0)$

$$y = -\frac{1}{2}x + 4$$

(b)



73. (a) $f(x) = \ln\sqrt{1 + \sin^2 x}$

$$= \frac{1}{2}\ln(1 + \sin^2 x), \quad \left(\frac{\pi}{4}, \ln\sqrt{\frac{3}{2}}\right)$$

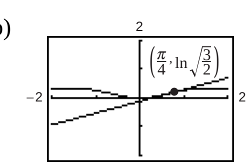
$$f'(x) = \frac{2\sin x \cos x}{2(1 + \sin^2 x)} = \frac{\sin x \cos x}{1 + \sin^2 x}$$

$$f'\left(\frac{\pi}{4}\right) = \frac{\left(\sqrt{2}/2\right)\left(\sqrt{2}/2\right)}{(3/2)} = \frac{1}{3}$$

Tangent line: $y - \ln\sqrt{\dfrac{3}{2}} = \dfrac{1}{3}\left(x - \dfrac{\pi}{4}\right)$

$$y = \frac{1}{3}x + \frac{1}{2}\ln\left(\frac{3}{2}\right) - \frac{\pi}{12}$$

(b)



74. (a) $f(x) = \sin(2x)\ln(x^2) = 2\sin(2x)\ln x, \quad (1, 0)$

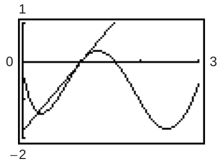
$$f'(x) = 4\cos(2x)\ln x + \frac{2\sin(2x)}{x}$$

$$f'(1) = 2\sin(2)$$

Tangent line: $y - 0 = 2\sin(2)(x - 1)$

$$y = 2\sin(2)x - 2\sin(2)$$

(b)



75. (a) $f(x) = x^3\ln x, \quad (1, 0)$

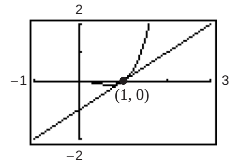
$$f'(x) = 3x^2\ln x + x^2$$

$$f'(1) = 1$$

Tangent line: $y - 0 = 1(x - 1)$

$$y = x - 1$$

(b)



76. (a) $f(x) = \frac{1}{2}x\ln(x^2), \quad (-1, 0)$

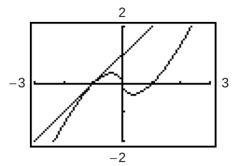
$$f'(x) = \frac{1}{2}\ln(x^2) + \frac{1}{2}x\left(\frac{2x}{x^2}\right) = \frac{1}{2}\ln(x^2) + 1$$

$$f'(-1) = 1$$

Tangent line: $y - 0 = 1(x + 1)$

$$y = x + 1$$

(b)



77. $x^2 - 3\ln y + y^2 = 10$

$$2x - \frac{3}{y}\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$2x = \frac{dy}{dx}\left(\frac{3}{y} - 2y\right)$$

$$\frac{dy}{dx} = \frac{2x}{(3/y) - 2y} = \frac{2xy}{3 - 2y^2}$$

78. $\ln(xy) + 5x = 30$

$$\ln x + \ln y + 5x = 30$$

$$\frac{1}{x} + \frac{1}{y}\frac{dy}{dx} + 5 = 0$$

$$\frac{1}{y}\frac{dy}{dx} = -\frac{1}{x} - 5$$

$$\frac{dy}{dx} = -\frac{y}{x} - 5y = -\left(\frac{y + 5xy}{x}\right)$$

79. $x + y - 1 = \ln(x^2 + y^2), \quad (1, 0)$

$$1 + y' = \frac{2x + 2yy'}{x^2 + y^2}$$

$$x^2 + y^2 + (x^2 + y^2)y' = 2x + 2yy'$$

At $(1, 0)$: $1 + y' = 2$

$$y' = 1$$

Tangent line: $y = x - 1$

80. $y^2 + \ln(xy) = 2, \quad (e, 1)$

$$2yy' + \frac{xy' + y}{xy} = 0$$

$$2xy^2y' + xy' + y = 0$$

At $(e, 1)$: $2ey' + ey' + 1 = 0$

$$y' = \frac{-1}{3e}$$

Tangent line: $y - 1 = \frac{-1}{3e}(x - e)$

$$y = \frac{-1}{3e}x + \frac{4}{3}$$

81. $y = 2(\ln x) + 3$

$y' = \dfrac{2}{x}$

$y'' = -\dfrac{2}{x^2}$

$xy'' + y' = x\left(-\dfrac{2}{x^2}\right) + \dfrac{2}{x} = 0$

82. $y = x(\ln x) - 4x$

$y' = x\left(\dfrac{1}{x}\right) + \ln x - 4 = -3 + \ln x$

$(x + y) - xy' = x + x \ln x - 4x - x(-3 + \ln x) = 0$

83. $y = \dfrac{x^2}{2} - \ln x$

Domain: $x > 0$

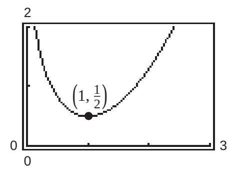
$y' = x - \dfrac{1}{x}$

$= \dfrac{(x + 1)(x - 1)}{x}$

$= 0$ when $x = 1$.

$y'' = 1 + \dfrac{1}{x^2} > 0$

Relative minimum: $\left(1, \dfrac{1}{2}\right)$



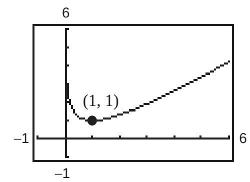
84. $y = x - \ln x$

Domain: $x > 0$

$y' = 1 - \dfrac{1}{x} = 0$ when $x = 1$.

$y'' = \dfrac{1}{x^2} > 0$

Relative minimum: $(1, 1)$



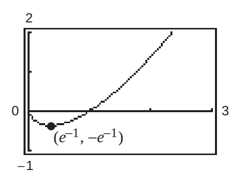
85. $y = x \ln x$

Domain: $x > 0$

$y' = x\left(\dfrac{1}{x}\right) + \ln x = 1 + \ln x = 0$ when $x = e^{-1}$.

$y'' = \dfrac{1}{x} > 0$

Relative minimum: $(e^{-1}, -e^{-1})$



86. $y = \dfrac{\ln x}{x}$

Domain: $x > 0$

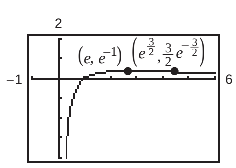
$y' = \dfrac{x(1/x) - \ln x}{x^2} = \dfrac{1 - \ln x}{x^2} = 0$ when $x = e$.

$y'' = \dfrac{x^2(-1/x) - (1 - \ln x)(2x)}{x^4}$

$= \dfrac{2(\ln x) - 3}{x^3} = 0$ when $x = e^{3/2}$.

Relative maximum: (e, e^{-1})

Point of inflection: $\left(e^{3/2}, \dfrac{3}{2}e^{-3/2}\right)$



87. $y = \dfrac{x}{\ln x}$

Domain: $0 < x < 1, x > 1$

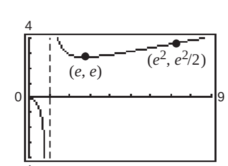
$y' = \dfrac{(\ln x)(1) - (x)(1/x)}{(\ln x)^2} = \dfrac{\ln x - 1}{(\ln x)^2} = 0$ when $x = e$.

$y'' = \dfrac{(\ln x)^2(1/x) - (\ln x - 1)(2/x) \ln x}{(\ln x)^4}$

$= \dfrac{2 - \ln x}{x(\ln x)^3} = 0$ when $x = e^2$.

Relative minimum: (e, e)

Point of inflection: $\left(e^2, \dfrac{e^2}{2}\right)$



88. $y = x^2 \ln \frac{x}{4}$, Domain: $x > 0$

$$y' = x^2\left(\frac{1}{x}\right) + 2x \ln \frac{x}{4} = x\left(1 + 2 \ln \frac{x}{4}\right) = 0 \text{ when:}$$

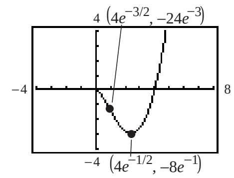
$$-1 = 2 \ln \frac{x}{4} \implies \ln \frac{x}{4} = -\frac{1}{2} \implies x = 4e^{-1/2}$$

$$y'' = 1 + 2 \ln \frac{x}{4} + 2x\left(\frac{1}{x}\right) = 3 + 2 \ln \frac{x}{4}$$

$y'' = 0$ when $x = 4e^{-3/2}$

Relative minimum: $(4e^{-1/2}, -8e^{-1})$

Point of inflection: $(4e^{-3/2}, -24e^{-3})$



89. $f(x) = \ln x$, $\quad f(1) = 0$

$f'(x) = \frac{1}{x}$, $\quad f'(1) = 1$

$f''(x) = -\frac{1}{x^2}$, $\quad f''(1) = -1$

$P_1(x) = f(1) + f'(1)(x - 1) = x - 1$, $\quad P_1(1) = 0$

$$P_2(x) = f(1) + f'(1)(x - 1) + \frac{1}{2}f''(1)(x - 1)^2$$

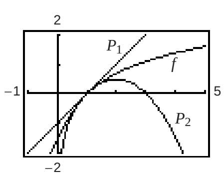
$= (x - 1) - \frac{1}{2}(x - 1)^2$, $\quad P_2(1) = 0$

$P_1'(x) = 1$, $\quad P_1'(1) = 1$

$P_2'(x) = 1 - (x - 1) = 2 - x$, $\quad P_2'(1) = 1$

$P_2''(x) = -1$, $\quad P_2''(1) = -1$

The values of f, P_1, P_2, and their first derivatives agree at $x = 1$. The values of the second derivatives of f and P_2 agree at $x = 1$.



90. $f(x) = x \ln x$, $\quad f(1) = 0$

$f'(x) = 1 + \ln x$, $\quad f'(1) = 1$

$f''(x) = \frac{1}{x}$, $\quad f''(1) = 1$

$P_1(x) = f(1) + f'(1)(x - 1) = x - 1$, $\quad P_1(1) = 0$

$$P_2(x) = f(1) + f'(1)(x - 1) + \frac{1}{2}f''(1)(x - 1)^2$$

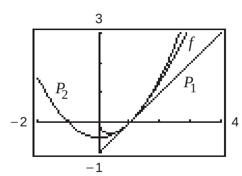
$= (x - 1) + \frac{1}{2}(x - 1)^2$, $\quad P_2(1) = 0$

$P_1'(x) = 1$, $\quad P_1'(1) = 1$

$P_2'(x) = 1 + (x - 1) = x$, $\quad P_2'(1) = 1$

$P_2''(x) = x$, $\quad P_2''(1) = 1$

The values of f, P_1, P_2, and their first derivatives agree at $x = 1$. The values of the second derivatives of f and P_2 agree at $x = 1$.



91. Find x such that $\ln x = -x$.

$f(x) = \ln x + x = 0$

$f'(x) = \frac{1}{x} + 1$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n\left[\frac{1 - \ln x_n}{1 + x_n}\right]$$

n	1	2	3
x_n	0.5	0.5644	0.5671
$f(x_n)$	−0.1931	−0.0076	−0.0001

Approximate root: $x = 0.567$

92. Find x such that $\ln x = 3 - x$.

$f(x) = x + (\ln x) - 3 = 0$

$f'(x) = 1 + \frac{1}{x}$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n\left[\frac{4 - \ln x_n}{1 + x_n}\right]$$

n	1	2	3
x_n	2	2.2046	2.2079
$f(x_n)$	−0.3069	−0.0049	0.0000

Approximate root: $x = 2.208$

93.
$$y = x\sqrt{x^2 - 1}$$
$$\ln y = \ln x + \frac{1}{2}\ln(x^2 - 1)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x} + \frac{x}{x^2 - 1}$$
$$\frac{dy}{dx} = y\left[\frac{2x^2 - 1}{x(x^2 - 1)}\right] = \frac{2x^2 - 1}{\sqrt{x^2 - 1}}$$

94.
$$y = \sqrt{(x - 1)(x - 2)(x - 3)}$$
$$\ln y = \frac{1}{2}[\ln(x - 1) + \ln(x - 2) + \ln(x - 3)]$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{2}\left[\frac{1}{x - 1} + \frac{1}{x - 2} + \frac{1}{x - 3}\right]$$
$$= \frac{1}{2}\left[\frac{3x^2 - 12x + 11}{(x - 1)(x - 2)(x - 3)}\right]$$
$$\frac{dy}{dx} = \frac{3x^2 - 12x + 11}{2y}$$
$$= \frac{3x^2 - 12x + 11}{2\sqrt{(x - 1)(x - 2)(x - 3)}}$$

95.
$$y = \frac{x^2\sqrt{3x - 2}}{(x - 1)^2}$$
$$\ln y = 2\ln x + \frac{1}{2}\ln(3x - 2) - 2\ln(x - 1)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{2}{x} + \frac{3}{2(3x - 2)} - \frac{2}{x - 1}$$
$$\frac{dy}{dx} = y\left[\frac{3x^2 - 15x + 8}{2x(3x - 2)(x - 1)}\right]$$
$$= \frac{3x^3 - 15x^2 + 8x}{2(x - 1)^3\sqrt{3x - 2}}$$

96.
$$y = \sqrt{\frac{x^2 - 1}{x^2 + 1}}$$
$$\ln y = \frac{1}{2}[\ln(x^2 - 1) - \ln(x^2 + 1)]$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{2}\left[\frac{2x}{x^2 - 1} - \frac{2x}{x^2 + 1}\right]$$
$$\frac{dy}{dx} = \sqrt{\frac{x^2 - 1}{x^2 + 1}}\left[\frac{2x}{x^4 - 1}\right]$$
$$= \frac{(x^2 - 1)^{1/2}2x}{(x^2 + 1)^{1/2}(x^2 - 1)(x^2 + 1)}$$
$$= \frac{2x}{(x^2 + 1)^{3/2}(x^2 - 1)^{1/2}}$$

97.
$$y = \frac{x(x - 1)^{3/2}}{\sqrt{x + 1}}$$
$$\ln y = \ln x + \frac{3}{2}\ln(x - 1) - \frac{1}{2}\ln(x + 1)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x} + \frac{3}{2}\left(\frac{1}{x - 1}\right) - \frac{1}{2}\left(\frac{1}{x + 1}\right)$$
$$\frac{dy}{dx} = \frac{y}{2}\left[\frac{2}{x} + \frac{3}{x - 1} - \frac{1}{x + 1}\right]$$
$$= \frac{y}{2}\left[\frac{4x^2 + 4x - 2}{x(x^2 - 1)}\right] = \frac{(2x^2 + 2x - 1)\sqrt{x - 1}}{(x + 1)^{3/2}}$$

98.
$$y = \frac{(x + 1)(x + 2)}{(x - 1)(x - 2)}$$
$$\ln y = \ln(x + 1) + \ln(x + 2) - \ln(x - 1) - \ln(x - 2)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x + 1} + \frac{1}{x + 2} - \frac{1}{x - 1} - \frac{1}{x - 2}$$
$$\frac{dy}{dx} = y\left[\frac{-2}{x^2 - 1} + \frac{-4}{x^2 - 4}\right] = y\left[\frac{-6x^2 + 12}{(x^2 - 1)(x^2 - 4)}\right]$$
$$= \frac{(x + 1)(x + 2)}{(x - 1)(x - 2)} \cdot \frac{-6(x^2 - 2)}{(x + 1)(x - 1)(x + 2)(x - 2)}$$
$$= -\frac{6(x^2 - 2)}{(x - 1)^2(x - 2)^2}$$

99. Answers will vary. See Theorems 5.1 and 5.2.

100. The base of the natural logarithmic function is e.

101. $g(x) = \ln f(x), \quad f(x) > 0$

$$g'(x) = \frac{f'(x)}{f(x)}$$

(a) Yes. If the graph of g is increasing, then $g'(x) > 0$. Since $f(x) > 0$, you know that $f'(x) = g'(x)f(x)$ and thus, $f'(x) > 0$. Therefore, the graph of f is increasing.

(b) No. Let $f(x) = x^2 + 1$ (positive and concave up). $g(x) = \ln(x^2 + 1)$ is not concave up.

102. (a) $f(1) \neq f(3)$

(b) $f'(x) = 1 - \dfrac{2}{x} = 0$ for $x = 2$.

103. False

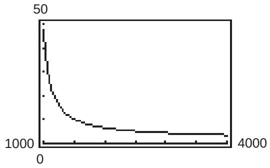
$$\ln x + \ln 25 = \ln(25x) \neq \ln(x + 25)$$

104. False; π is a constant.

$$\frac{d}{dx}[\ln \pi] = 0$$

105. $t = \dfrac{5.315}{-6.7968 + \ln x}, \quad 1000 < x$

(a)



(b) $t(1167.41) \approx 20$ years

$$T = (1167.41)(20)(12) = \$280,178.40$$

(c) $t(1068.45) \approx 30$ years

$$T = (1068.45)(30)(12) = \$384,642.00$$

(d) $$\frac{dt}{dx} = -5.315(-6.7968 + \ln x)^{-2}\left(\frac{1}{x}\right) = -\frac{5.315}{x(-6.7968 + \ln x)^2}$$

When $x = 1167.41$, $dt/dx \approx -0.0645$. When $x = 1068.45$, $dt/dx \approx -0.1585$.

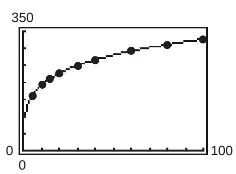
(e) There are two obvious benefits to paying a higher monthly payment:

1. The term is lower
2. The total amount paid is lower.

106. $$\beta = 10 \log_{10}\left(\frac{I}{10^{-16}}\right) = \frac{10}{\ln 10}\ln\left(\frac{I}{10^{-16}}\right) = \frac{10}{\ln 10}[\ln I + 16 \ln 10] = 160 + 10 \log_{10} I$$

$$\beta(10^{-10}) = \frac{10}{\ln 10}[\ln 10^{-10} + 16 \ln 10] = \frac{10}{\ln 10}[-10 \ln 10 + 16 \ln 10] = \frac{10}{\ln 10}[6 \ln 10] = 60 \text{ decibels}$$

107. (a)

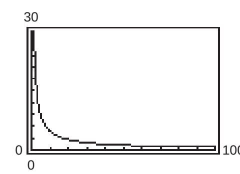


(b) $T'(p) = \dfrac{34.96}{p} + \dfrac{3.955}{\sqrt{p}}$

$T'(10) \approx 4.75$ deg/lb/in^2

$T'(70) \approx 0.97$ deg/lb/in^2

(c)



$$\lim_{p \to \infty} T'(p) = 0$$

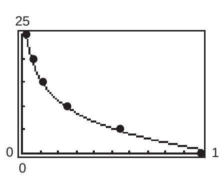
108. (a) You get an error message because $\ln h$ does not exist for $h = 0$.

(b) Reversing the data, you obtain

$h = 0.8627 - 6.4474 \ln p$.

[**Note:** Fit a line to the data $(x, y) = (\ln p, h)$.]

(c)



(d) If $p = 0.75$, $h \approx 2.72$ km.

(e) If $h = 13$ km, $p \approx 0.15$ atmosphere.

(f) $h = 0.8627 - 6.4474 \ln p$

$1 = -6.4474 \dfrac{1}{p}\dfrac{dp}{dh}$ (implicit differentiation)

$$\frac{dp}{dh} = \frac{p}{-6.4474}$$

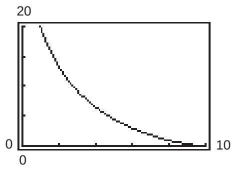
For $h = 5$, $p = 0.5264$ and $dp/dh = -0.0816$ atmos/km.

For $h = 20$, $p = 0.0514$ and $dp/dh = -0.0080$ atmos/km.

As the altitude increases, the rate of change of pressure decreases.

109. $y = 10\ln\left(\dfrac{10 + \sqrt{100 - x^2}}{x}\right) - \sqrt{100 - x^2} = 10\left[\ln\left(10 + \sqrt{100 - x^2}\right) - \ln x\right] - \sqrt{100 - x^2}$

(a)

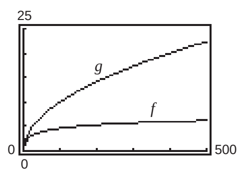


(b) $$\frac{dy}{dx} = 10\left[\frac{-x}{\sqrt{100 - x^2}\left(10 + \sqrt{100 - x^2}\right)} - \frac{1}{x}\right] + \frac{x}{\sqrt{100 - x^2}}$$
$$= \frac{x}{\sqrt{100 - x^2}}\left[\frac{-10}{10 + \sqrt{100 - x^2}}\right] - \frac{10}{x} + \frac{x}{\sqrt{100 - x^2}}$$
$$= \frac{x}{\sqrt{100 - x^2}}\left[\frac{-10}{10 + \sqrt{100 - x^2}} + 1\right] - \frac{10}{x}$$
$$= \frac{x}{\sqrt{100 - x^2}}\left[\frac{\sqrt{100 - x^2}}{10 + \sqrt{100 - x^2}}\right] - \frac{10}{x}$$
$$= \frac{x}{10 + \sqrt{100 - x^2}} - \frac{10}{x}$$
$$= \frac{x\left(10 - \sqrt{100 - x^2}\right)}{x^2} - \frac{10}{x} = -\frac{\sqrt{100 - x^2}}{x}$$

When $x = 5$, $dy/dx = -\sqrt{3}$. When $x = 9$, $dy/dx = -\sqrt{19}/9$.

(c) $\displaystyle\lim_{x\to 10^-} \frac{dy}{dx} = 0$

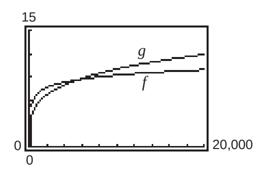
110. (a) $f(x) = \ln x$, $g(x) = \sqrt{x}$



$f'(x) = \dfrac{1}{x}$, $g'(x) = \dfrac{1}{2\sqrt{x}}$

For $x > 4$, $g'(x) > f'(x)$. g is increasing at a faster rate than f for "large" values of x.

(b) $f(x) = \ln x$, $g(x) = \sqrt[4]{x}$



$f'(x) = \dfrac{1}{x}$, $g'(x) = \dfrac{1}{4\sqrt[4]{x^3}}$

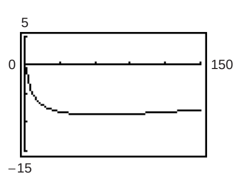
For $x > 256$, $g'(x) > f'(x)$. g is increasing at a faster rate than f for "large" values of x. $f(x) = \ln x$ increases very slowly for "large" values of x.

111. $y = \ln x$

$y' = \dfrac{1}{x} > 0$ for $x > 0$.

Since $\ln x$ is increasing on its entire domain $(0, \infty)$, it is a strictly monotonic function and therefore, is one-to-one.

112. (a)



(b) Using a graphing utility, there is a relative minimum at $(64, -8.6355)$.

(c) $f'(x) = \dfrac{1}{2\sqrt{x}} - \dfrac{4}{x} = 0$

$$2\sqrt{x} = \frac{x}{4}$$
$$8\sqrt{x} = x$$
$$64x = x^2$$
$$x = 64$$

By the First Derivative Test, $x = 64$ is a relative minimum.

Section 5.2 The Natural Logarithmic Function: Integration

1. $\int \frac{5}{x}\,dx = 5\int \frac{1}{x}\,dx = 5\ln|x| + C$

2. $\int \frac{10}{x}\,dx = 10\int \frac{1}{x}\,dx = 10\ln|x| + C$

3. $u = x + 1,\ du = dx$

$$\int \frac{1}{x+1}\,dx = \ln|x+1| + C$$

4. $u = x - 5,\ du = dx$

$$\int \frac{1}{x-5}\,dx = \ln|x-5| + C$$

5. $u = 3 - 2x,\ du = -2\,dx$

$$\int \frac{1}{3-2x}\,dx = -\frac{1}{2}\int \frac{1}{3-2x}(-2)\,dx = -\frac{1}{2}\ln|3-2x| + C$$

6. $\int \frac{1}{3x+2}\,dx = \frac{1}{3}\int \frac{1}{3x+2}(3)\,dx = \frac{1}{3}\ln|3x+2| + C$

7. $u = x^2 + 1,\ du = 2x\,dx$

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{1}{x^2+1}(2x)\,dx = \frac{1}{2}\ln(x^2+1) + C = \ln\sqrt{x^2+1} + C$$

8. $u = 3 - x^3,\ du = -3x^2\,dx$

$$\int \frac{x^2}{3-x^3}\,dx = -\frac{1}{3}\int \frac{1}{3-x^3}(-3x^2)\,dx = -\frac{1}{3}\ln|3-x^3| + C$$

9. $\int \frac{x^2-4}{x}\,dx = \int \left(x - \frac{4}{x}\right)dx = \frac{x^2}{2} - 4\ln|x| + C$

10. $u = 9 - x^2,\ du = -2x\,dx$

$$\int \frac{x}{\sqrt{9-x^2}}\,dx = -\frac{1}{2}\int (9-x^2)^{-1/2}(-2x)\,dx = -\sqrt{9-x^2} + C$$

11. $u = x^3 + 3x^2 + 9x,\ du = 3(x^2 + 2x + 3)\,dx$

$$\int \frac{x^2+2x+3}{x^3+3x^2+9x}\,dx = \frac{1}{3}\int \frac{3(x^2+2x+3)}{x^3+3x^2+9x}\,dx = \frac{1}{3}\ln|x^3+3x^2+9x| + C$$

12. $\int \frac{x(x+2)}{x^3+3x^2-4}\,dx = \frac{1}{3}\int \frac{3x^2+6x}{x^3+3x^2-4}\,dx \quad (u = x^3 + 3x^2 - 4)$

$$= \frac{1}{3}\ln|x^3+3x^2-4| + C$$

13. $\int \frac{x^2-3x+2}{x+1}\,dx = \int \left(x - 4 + \frac{6}{x+1}\right)dx = \frac{x^2}{2} - 4x + 6\ln|x+1| + C$

14. $\int \frac{2x^2+7x-3}{x-2}\,dx = \int \left(2x + 11 + \frac{19}{x-2}\right)dx = x^2 + 11x + 19\ln|x-2| + C$

15. $\int \frac{x^3-3x^2+5}{x-3}\,dx = \int \left(x^2 + \frac{5}{x-3}\right)dx = \frac{x^3}{3} + 5\ln|x-3| + C$

16. $\int \frac{x^3-6x-20}{x+5}\,dx = \int \left(x^2 - 5x + 19 - \frac{115}{x+5}\right)dx = \frac{x^3}{3} - \frac{5x^2}{2} + 19x - 115\ln|x+5| + C$

17. $\int \frac{x^4+x-4}{x^2+2}\,dx = \int \left(x^2 - 2 + \frac{x}{x^2+2}\right)dx = \frac{x^3}{3} - 2x + \frac{1}{2}\ln(x^2+2) + C$

18. $\int \frac{x^3-3x^2+4x-9}{x^2+3}\,dx = \int \left(-3 + x + \frac{x}{x^2+3}\right)dx = -3x + \frac{x^2}{2} + \frac{1}{2}\ln(x^2+3) + C$

19. $u = \ln x,\ du = \frac{1}{x}\,dx$

$$\int \frac{(\ln x)^2}{x}\,dx = \frac{1}{3}(\ln x)^3 + C$$

20. $\displaystyle\int \frac{1}{x\ln(x^3)}\,dx = \frac{1}{3}\int \frac{1}{\ln x}\cdot\frac{1}{x}\,dx$

$$= \frac{1}{3}\ln\left|\ln|x|\right| + C$$

21. $u = x + 1,\ du = dx$

$$\int \frac{1}{\sqrt{x+1}}\,dx = \int (x+1)^{-1/2}\,dx$$
$$= 2(x+1)^{1/2} + C$$
$$= 2\sqrt{x+1} + C$$

22. $u = 1 + x^{1/3},\ du = \frac{1}{3x^{2/3}}\,dx$

$$\int \frac{1}{x^{2/3}(1 + x^{1/3})}\,dx = 3\int \frac{1}{1 + x^{1/3}}\left(\frac{1}{3x^{2/3}}\right)dx$$
$$= 3\ln|1 + x^{1/3}| + C$$

23. $\displaystyle\int \frac{2x}{(x-1)^2}\,dx = \int \frac{2x - 2 + 2}{(x-1)^2}\,dx$

$$= \int \frac{2(x-1)}{(x-1)^2}\,dx + 2\int \frac{1}{(x-1)^2}\,dx$$
$$= 2\int \frac{1}{x-1}\,dx + 2\int \frac{1}{(x-1)^2}\,dx$$
$$= 2\ln|x-1| - \frac{2}{(x-1)} + C$$

24. $\displaystyle\int \frac{x(x-2)}{(x-1)^3}\,dx = \int \frac{x^2 - 2x + 1 - 1}{(x-1)^3}\,dx$

$$= \int \frac{(x-1)^2}{(x-1)^3}\,dx - \int \frac{1}{(x-1)^3}\,dx$$
$$= \int \frac{1}{x-1}\,dx - \int \frac{1}{(x-1)^3}\,dx$$
$$= \ln|x-1| + \frac{1}{2(x-1)^2} + C$$

25. $u = 1 + \sqrt{2x},\ du = \frac{1}{\sqrt{2x}}\,dx \implies (u-1)\,du = dx$

$$\int \frac{1}{1 + \sqrt{2x}}\,dx = \int \frac{(u-1)}{u}\,du = \int\left(1 - \frac{1}{u}\right)du$$
$$= u - \ln|u| + C_1$$
$$= \left(1 + \sqrt{2x}\right) - \ln\left|1 + \sqrt{2x}\right| + C_1$$
$$= \sqrt{2x} - \ln\left(1 + \sqrt{2x}\right) + C$$

where $C = C_1 + 1$.

26. $u = 1 + \sqrt{3x},\ du = \frac{3}{2\sqrt{3x}}\,dx \implies dx = \frac{2}{3}(u-1)\,du$

$$\int \frac{1}{1 + \sqrt{3x}}\,dx = \int \frac{1}{u}\,\frac{2}{3}(u-1)\,du$$
$$= \frac{2}{3}\int\left(1 - \frac{1}{u}\right)du$$
$$= \frac{2}{3}\left[u - \ln|u|\right] + C$$
$$= \frac{2}{3}\left[1 + \sqrt{3x} - \ln\left(1 + \sqrt{3x}\right)\right] + C$$
$$= \frac{2}{3}\sqrt{3x} - \frac{2}{3}\ln\left(1 + \sqrt{3x}\right) + C_1$$

27. $u = \sqrt{x} - 3,\ du = \frac{1}{2\sqrt{x}}\,dx \implies 2(u+3)\,du = dx$

$$\int \frac{\sqrt{x}}{\sqrt{x} - 3}\,dx = 2\int \frac{(u+3)^2}{u}\,du$$
$$= 2\int \frac{u^2 + 6u + 9}{u}\,du = 2\int\left(u + 6 + \frac{9}{u}\right)du$$
$$= 2\left[\frac{u^2}{2} + 6u + 9\ln|u|\right] + C_1$$
$$= u^2 + 12u + 18\ln|u| + C_1$$
$$= \left(\sqrt{x} - 3\right)^2 + 12\left(\sqrt{x} - 3\right) + 18\ln\left|\sqrt{x} - 3\right| + C_1$$
$$= x + 6\sqrt{x} + 18\ln\left|\sqrt{x} - 3\right| + C$$

where $C = C_1 - 27$.

28. $u = x^{1/3} - 1,\ du = \dfrac{1}{3x^{2/3}}\,dx \implies dx = 3(u+1)^2\,du$

$$\begin{aligned}\int \frac{\sqrt[3]{x}}{\sqrt[3]{x}-1}\,dx &= \int \frac{u+1}{u}3(u+1)^2\,du\\ &= 3\int \frac{u+1}{u}(u^2+2u+1)\,du\\ &= 3\int \left(u^2+3u+3+\frac{1}{u}\right)du\\ &= 3\left[\frac{u^3}{3}+\frac{3u^2}{2}+3u+\ln|u|\right]+C\\ &= 3\left[\frac{(x^{1/3}-1)^3}{3}+\frac{3(x^{1/3}-1)^2}{2}+3(x^{1/3}-1)+\ln|x^{1/3}-1|\right]+C\\ &= 3\ln|x^{1/3}-1|+\frac{3x^{2/3}}{2}+3x^{1/3}+x+C_1\end{aligned}$$

29. $\displaystyle\int \frac{\cos\theta}{\sin\theta}\,d\theta = \ln|\sin\theta| + C$

$(u = \sin\theta,\ du = \cos\theta\,d\theta)$

30. $$\begin{aligned}\int \tan 5\theta\,d\theta &= \frac{1}{5}\int\frac{5\sin 5\theta}{\cos 5\theta}\,d\theta\\ &= -\frac{1}{5}\ln|\cos 5\theta| + C\end{aligned}$$

31. $$\begin{aligned}\int \csc 2x\,dx &= \frac{1}{2}\int(\csc 2x)(2)\,dx\\ &= -\frac{1}{2}\ln|\csc 2x + \cot 2x| + C\end{aligned}$$

32. $$\begin{aligned}\int \sec\frac{x}{2}\,dx &= 2\int\sec\frac{x}{2}\left(\frac{1}{2}\right)dx\\ &= 2\ln\left|\sec\frac{x}{2}+\tan\frac{x}{2}\right| + C\end{aligned}$$

33. $\displaystyle\int \frac{\cos t}{1+\sin t}\,dt = \ln|1+\sin t| + C$

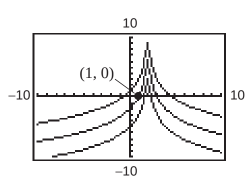
34. $u = \cot t,\ du = -\csc^2 t\,dt$

$$\int\frac{\csc^2 t}{\cot t}\,dt = -\ln|\cot t| + C$$

35. $\displaystyle\int \frac{\sec x\tan x}{\sec x - 1}\,dx = \ln|\sec x - 1| + C$

36. $$\begin{aligned}\int(\sec t + \tan t)\,dt &= \ln|\sec t + \tan t| - \ln|\cos t| + C\\ &= \ln\left|\frac{\sec t + \tan t}{\cos t}\right| + C\\ &= \ln|\sec t(\sec t + \tan t)| + C\end{aligned}$$

37. $$\begin{aligned}y &= \int\frac{3}{2-x}\,dx\\ &= -3\int\frac{1}{x-2}\,dx\\ &= -3\ln|x-2| + C\end{aligned}$$



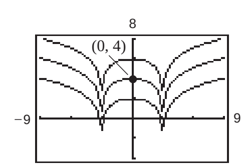
$(1, 0)$: $0 = -3\ln|1-2| + C \implies C = 0$

$y = -3\ln|x-2|$

38. $$\begin{aligned}y &= \int\frac{2x}{x^2-9}\,dx\\ &= \ln|x^2-9| + C\end{aligned}$$

$(0, 4)$: $4 = \ln|0-9| + C \implies C = 4 - \ln 9$

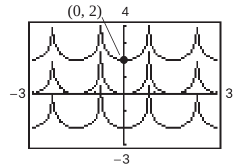
$y = \ln|x^2-9| + 4 - \ln 9$



39. $s = \int \tan(2\theta)\, d\theta$

$= \frac{1}{2}\int \tan(2\theta)(2\, d\theta)$

$= -\frac{1}{2}\ln|\cos 2\theta| + C$



$(0, 2)$: $2 = -\frac{1}{2}\ln|\cos(0)| + C \Rightarrow C = 2$

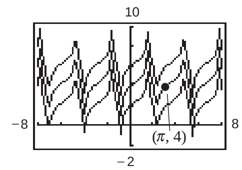
$s = -\frac{1}{2}\ln|\cos 2\theta| + 2$

40. $r = \int \frac{\sec^2 t}{\tan t + 1}\, dt$

$= \ln|\tan t + 1| + C$

$(\pi, 4)$: $4 = \ln|0 + 1| + C \Rightarrow C = 4$

$r = \ln|\tan t + 1| + 4$



41. $f''(x) = \frac{2}{x^2} = 2x^{-2}, \quad x > 0$

$f'(x) = \frac{-2}{x} + C$

$f'(1) = 1 = -2 + C \Rightarrow C = 3$

$f'(x) = \frac{-2}{x} + 3$

$f(x) = -2\ln x + 3x + C_1$

$f(1) = 1 = -2(0) + 3 + C_1 \Rightarrow C_1 = -2$

$f(x) = -2\ln x + 3x - 2$

42. $f''(x) = \frac{-4}{(x-1)^2} - 2 = -4(x-1)^{-2} - 2, \quad x > 1$

$f'(x) = \frac{4}{(x-1)} - 2x + C$

$f'(2) = 0 = 4 - 4 + C \Rightarrow C = 0$

$f'(x) = \frac{4}{x-1} - 2x$

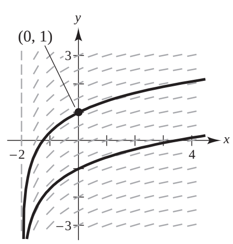
$f(x) = 4\ln(x-1) - x^2 + C_1$

$f(2) = 3 = 4(0) - 4 + C_1 \Rightarrow C_1 = 7$

$f(x) = 4\ln(x-1) - x^2 + 7$

43. $\frac{dy}{dx} = \frac{1}{x+2}$, $(0, 1)$

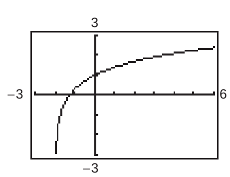
(a)



(b) $y = \int \frac{1}{x+2}\, dx = \ln|x+2| + C$

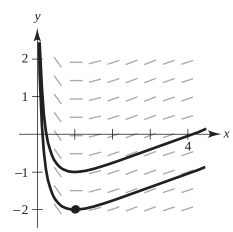
$y(0) = 1 \Rightarrow 1 = \ln 2 + C \Rightarrow C = 1 - \ln 2$

Hence, $y = \ln|x+2| + 1 - \ln 2 = \ln\left|\frac{x+2}{2}\right| + 1.$



44. $\frac{dy}{dx} = \frac{\ln x}{x}$, $(1, -2)$

(a)

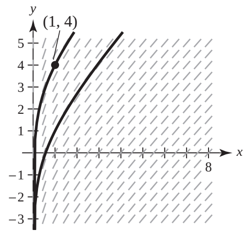


(b) $y = \int \frac{\ln x}{x}\, dx = \frac{(\ln x)^2}{2} + C$

$y(1) = -2 \Rightarrow -2 = \frac{(\ln 1)^2}{2} + C \Rightarrow C = -2$

Hence, $y = \frac{(\ln x)^2}{2} - 2.$

45. (a)

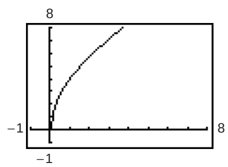


(b) $\dfrac{dy}{dx} = 1 + \dfrac{1}{x}, \quad (1, 4)$

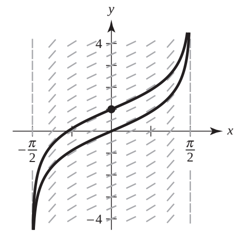
$y = x + \ln x + C$

$4 = 1 + 0 + C \implies C = 3$

$y = x + \ln x + 3$



46. (a)

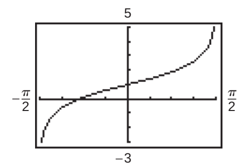


(b) $\dfrac{dy}{dx} = \sec x, \quad (0, 1)$

$y = \ln|\sec x + \tan x| + C$

$1 = \ln|1 + 0| + C \implies C = 1$

$y = \ln|\sec x + \tan x| + 1$



47. $\displaystyle\int_0^4 \frac{5}{3x+1}\,dx = \left[\frac{5}{3}\ln|3x+1|\right]_0^4$

$\displaystyle = \frac{5}{3}\ln 13 \approx 4.275$

48. $\displaystyle\int_{-1}^{1} \frac{1}{x+2}\,dx = \Big[\ln|x+2|\Big]_{-1}^{1}$

$= \ln 3 - \ln 1 = \ln 3$

49. $u = 1 + \ln x,\ du = \dfrac{1}{x}\,dx$

$\displaystyle\int_1^e \frac{(1+\ln x)^2}{x}\,dx = \left[\frac{1}{3}(1+\ln x)^3\right]_1^e$

$\displaystyle = \frac{7}{3}$

50. $u = \ln x,\ du = \dfrac{1}{x}\,dx$

$\displaystyle\int_e^{e^2} \frac{1}{x\ln x}\,dx = \int_e^{e^2}\left(\frac{1}{\ln x}\right)\frac{1}{x}\,dx = \Big[\ln|\ln|x||\Big]_e^{e^2} = \ln 2$

51. $\displaystyle\int_0^2 \frac{x^2-2}{x+1}\,dx = \int_0^2\left(x - 1 - \frac{1}{x+1}\right)dx$

$\displaystyle = \left[\frac{1}{2}x^2 - x - \ln|x+1|\right]_0^2 = -\ln 3$

52. $\displaystyle\int_0^1 \frac{x-1}{x+1}\,dx = \int_0^1 1\,dx + \int_0^1 \frac{-2}{x+1}\,dx$

$\displaystyle = \Big[x - 2\ln|x+1|\Big]_0^1 = 1 - 2\ln 2$

53. $\displaystyle\int_1^2 \frac{1-\cos\theta}{\theta - \sin\theta}\,d\theta = \Big[\ln|\theta - \sin\theta|\Big]_1^2$

$\displaystyle = \ln\left|\frac{2-\sin 2}{1-\sin 1}\right| \approx 1.929$

54. $\displaystyle\int_{0.1}^{0.2} (\csc 2\theta - \cot 2\theta)^2\,d\theta = \int_{0.1}^{0.2} (\csc^2 2\theta - 2\csc 2\theta\cot 2\theta + \cot^2 2\theta)\,d\theta$

$\displaystyle = \int_{0.1}^{0.2} (2\csc^2 2\theta - 2\csc 2\theta\cot 2\theta - 1)\,d\theta$

$\displaystyle = \Big[-\cot 2\theta + \csc 2\theta - \theta\Big]_{0.1}^{0.2} \approx 0.0024$

55. $\displaystyle\int \frac{1}{1+\sqrt{x}}\,dx = 2\left(1+\sqrt{x}\right) - 2\ln\left(1+\sqrt{x}\right) + C_1$

$= 2\left[\sqrt{x} - \ln\left(1+\sqrt{x}\right)\right] + C$ where $C = C_1 + 2$.

56. $\displaystyle\int \frac{1-\sqrt{x}}{1+\sqrt{x}}\,dx = -\left(1+\sqrt{x}\right)^2 + 6\left(1+\sqrt{x}\right) - 4\ln\left(1+\sqrt{x}\right) + C_1$

$\displaystyle = 4\sqrt{x} - x - 4\ln\left(1+\sqrt{x}\right) + C$ where $C = C_1 + 5$.

57. $\displaystyle\int \frac{\sqrt{x}}{x-1}\,dx = \ln\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right) + 2\sqrt{x} + C$

58. $\displaystyle\int \frac{x^2}{x-1}\,dx = \ln|x-1| + \frac{x^2}{2} + x + C$

59. $\displaystyle\int_{\pi/4}^{\pi/2} (\csc x - \sin x)\,dx = \Big[-\ln|\csc x + \cot x| + \cos x\Big]_{\pi/4}^{\pi/2}$

$\displaystyle = \ln\left(\sqrt{2}+1\right) - \frac{\sqrt{2}}{2} \approx 0.174$

60. $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{\sin^2 x - \cos^2 x}{\cos x}\,dx = \Big[\ln|\sec x + \tan x| - 2\sin x\Big]_{-\pi/4}^{\pi/4}$

$\displaystyle = \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) - 2\sqrt{2} \approx -1.066$

Note: In Exercises 61–64, you can use the Second Fundamental Theorem of Calculus or integrate the function.

61. $\displaystyle F(x) = \int_1^x \frac{1}{t}\,dt$

$\displaystyle F'(x) = \frac{1}{x}$

62. $\displaystyle F(x) = \int_0^x \tan t\,dt$

$F'(x) = \tan x$

63. $\displaystyle F(x) = \int_1^{3x} \frac{1}{t}\,dt$

$\displaystyle F'(x) = \frac{1}{3x}(3) = \frac{1}{x}$

(by Second Fundamental Theorem of Calculus)

Alternate Solution:

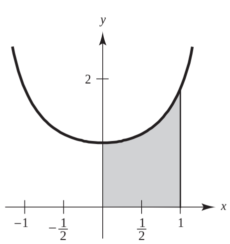
$\displaystyle F(x) = \int_1^{3x} \frac{1}{t}\,dt = \Big[\ln|t|\Big]_1^{3x} = \ln|3x|$

$\displaystyle F'(x) = \frac{1}{3x}(3) = \frac{1}{x}$

64. $\displaystyle F(x) = \int_1^{x^2} \frac{1}{t}\,dt$

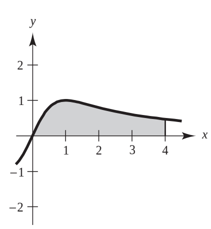
$\displaystyle F'(x) = \frac{2x}{x^2} = \frac{2}{x}$

65.



$A \approx 1.25$; Matches (d)

66.



$A \approx 3$; Matches (a)

67. $\displaystyle A = \int_1^3 \frac{4}{x}\,dx = 4\ln|x|\Big]_1^3 = 4\ln 3$

68. $A = \int_2^4 \frac{2}{x \ln x}\,dx = 2\int_2^4 \frac{1}{\ln x}\frac{1}{x}\,dx$

$= 2\ln|\ln x|\Big]_2^4$

$= 2[\ln(\ln 4) - \ln(\ln 2)]$

$= 2\ln\left(\frac{2\ln 2}{\ln 2}\right) = 2\ln 2$

69. $A = \int_0^{\pi/4} \tan x\,dx = -\ln|\cos x|\Big]_0^{\pi/4}$

$= -\ln\frac{\sqrt{2}}{2} + 0$

$= \ln\sqrt{2} = \frac{\ln 2}{2} \approx 0.3466$

70. $A = \int_{\pi/4}^{3\pi/4} \frac{\sin x}{1 + \cos x}\,dx$

$= -\ln|1 + \cos x|\Big]_{\pi/4}^{3\pi/4}$

$= -\ln\left(1 - \frac{\sqrt{2}}{2}\right) + \ln\left(1 + \frac{\sqrt{2}}{2}\right)$

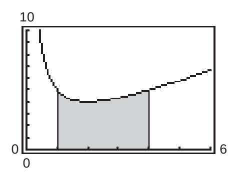
$= \ln\left(\frac{2 + \sqrt{2}}{2 - \sqrt{2}}\right)$

$= \ln\left(3 + 2\sqrt{2}\right)$

71. $A = \int_1^4 \frac{x^2 + 4}{x}\,dx = \int_1^4 \left(x + \frac{4}{x}\right)dx$

$= \left[\frac{x^2}{2} + 4\ln x\right]_1^4 = (8 + 4\ln 4) - \frac{1}{2}$

$= \frac{15}{2} + 8\ln 2 \approx 13.045$ square units

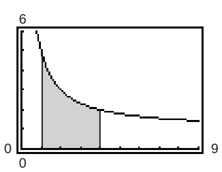


72. $A = \int_1^4 \frac{x + 4}{x}\,dx = \int_1^4 \left(1 + \frac{4}{x}\right)dx$

$= \Big[x + 4\ln x\Big]_1^4$

$= 4 + 4\ln 4 - 1$

$= 3 + 4\ln 4 \approx 8.5452$

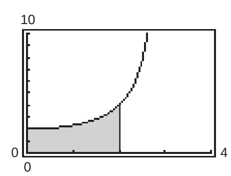


73. $\int_0^2 2\sec\frac{\pi x}{6}\,dx = \frac{12}{\pi}\int_0^2 \sec\left(\frac{\pi x}{6}\right)\frac{\pi}{6}\,dx$

$= \left[\frac{12}{\pi}\ln\left|\sec\frac{\pi x}{6} + \tan\frac{\pi x}{6}\right|\right]_0^2$

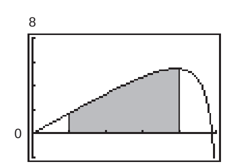
$= \frac{12}{\pi}\ln\left|\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right| - \frac{12}{\pi}\ln|1 + 0|$

$= \frac{12}{\pi}\ln\left(2 + \sqrt{3}\right) \approx 5.03041$



74. $\int_1^4 (2x - \tan(0.3x))\,dx = \left[x^2 + \frac{10}{3}\ln|\cos(0.3x)|\right]_1^4$

$= \left[16 + \frac{10}{3}\ln\cos(1.2)\right] - \left[1 + \frac{10}{3}\ln\cos(0.3)\right] \approx 11.7686$



75. $f(x) = \frac{12}{x},\ b - a = 5 - 1 = 4,\ n = 4$

Trapezoid: $\frac{4}{2(4)}[f(1) + 2f(2) + 2f(3) + 2f(4) + f(5)] = \frac{1}{2}[12 + 12 + 8 + 6 + 2.4] = 20.2$

Simpson: $\frac{4}{3(4)}[f(1) + 4f(2) + 2f(3) + 4f(4) + f(5)] = \frac{1}{3}[12 + 24 + 8 + 12 + 2.4] \approx 19.4667$

Calculator: $\int_1^5 \frac{12}{x}\,dx \approx 19.3133$

Exact: $12 \ln 5$

76. $f(x) = \frac{8x}{x^2 + 4},\ b - a = 4 - 0 = 4,\ n = 4$

Trapezoid: $\frac{4}{2(4)}[f(0) + 2f(1) + 2f(2) + 2f(3) + f(4)] = \frac{1}{2}[0 + 3.2 + 4 + 3.6923 + 1.6] \approx 6.2462$

Simpson: $\frac{4}{3(4)}[f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)] \approx 6.4615$

Calculator: $\int_0^4 \frac{8x}{x^2 + 4}\,dx \approx 6.438$

Exact: $4 \ln 5$

77. $f(x) = \ln x,\ b - a = 6 - 2 = 4,\ n = 4$

Trapezoid: $\frac{4}{2(4)}[f(2) + 2f(3) + 2f(4) + 2f(5) + f(6)] = \frac{1}{2}[0.6931 + 2.1972 + 2.7726 + 3.2189 + 1.7918] \approx 5.3368$

Simpson: $\frac{4}{3(4)}[f(2) + 4f(3) + 2f(4) + 4f(5) + f(6)] \approx 5.3632$

Calculator: $\int_2^6 \ln x\,dx \approx 5.3643$

78. $f(x) = \sec x, b - a = \frac{\pi}{3} - \left(-\frac{\pi}{3}\right) = \frac{2\pi}{3},\ n = 4$

Trapezoid: $\frac{2\pi/3}{2(4)}\left[f\left(-\frac{\pi}{3}\right) + 2f\left(-\frac{\pi}{6}\right) + 2f(0) + 2f\left(\frac{\pi}{6}\right) + f\left(\frac{\pi}{3}\right)\right] \approx \frac{\pi}{12}[2 + 2.3094 + 2 + 2.3094 + 2] \approx 2.780$

Simpson: $\frac{2\pi/3}{3(4)}\left[f\left(-\frac{\pi}{3}\right) + 4f\left(-\frac{\pi}{6}\right) + 2f(0) + 4f\left(\frac{\pi}{6}\right) + f\left(\frac{\pi}{3}\right)\right] \approx 2.6595$

Calculator: $\int_{-\pi/3}^{\pi/3} \sec x\,dx \approx 2.6339$

79. Power Rule

80. Substitution: $(u = x^2 + 4)$ and Power Rule

81. Substitution: $(u = x^2 + 4)$ and Log Rule

82. Substitution: $(u = \tan x)$ and Log Rule

83. $-\ln|\cos x| + C = \ln\left|\frac{1}{\cos x}\right| + C$
$= \ln|\sec x| + C$

84. $\ln|\sin x| + C = \ln\left|\frac{1}{\csc x}\right| + C$
$= -\ln|\csc x| + C$

85. $$\ln\left|\sec x + \tan x\right| + C = \ln\left|\frac{(\sec x + \tan x)(\sec x - \tan x)}{(\sec x - \tan x)}\right| + C = \ln\left|\frac{\sec^2 x - \tan^2 x}{\sec x - \tan x}\right| + C$$
$$= \ln\left|\frac{1}{\sec x - \tan x}\right| + C = -\ln\left|\sec x - \tan x\right| + C$$

86. $$-\ln\left|\csc x + \cot x\right| + C = -\ln\left|\frac{(\csc x + \cot x)(\csc x - \cot x)}{(\csc x - \cot x)}\right| + C = -\ln\left|\frac{\csc^2 x - \cot^2 x}{\csc x - \cot x}\right| + C$$
$$= -\ln\left|\frac{1}{\csc x - \cot x}\right| + C = \ln\left|\csc x - \cot x\right| + C$$

87. $$\text{Average value} = \frac{1}{4-2}\int_2^4 \frac{8}{x^2}\,dx$$
$$= 4\int_2^4 x^{-2}\,dx$$
$$= \left[-4\frac{1}{x}\right]_2^4$$
$$= -4\left(\frac{1}{4} - \frac{1}{2}\right) = 1$$

88. $$\text{Average value} = \frac{1}{4-2}\int_2^4 \frac{4(x+1)}{x^2}\,dx$$
$$= 2\int_2^4 \left(\frac{1}{x} + \frac{1}{x^2}\right) dx$$
$$= 2\left[\ln x - \frac{1}{x}\right]_2^4$$
$$= 2\left[\ln 4 - \frac{1}{4} - \ln 2 + \frac{1}{2}\right]$$
$$= 2\left[\ln 2 + \frac{1}{4}\right] = \ln 4 + \frac{1}{2} \approx 1.8863$$

89. $$\text{Average value} = \frac{1}{e-1}\int_1^e \frac{\ln x}{x}\,dx$$
$$= \frac{1}{e-1}\left[\frac{(\ln x)^2}{2}\right]_1^e$$
$$= \frac{1}{e-1}\left(\frac{1}{2}\right)$$
$$= \frac{1}{2e-2} \approx 0.291$$

90. $$\text{Average value} = \frac{1}{2-0}\int_0^2 \sec\frac{\pi x}{6}\,dx$$
$$= \left[\frac{1}{2}\left(\frac{6}{\pi}\right)\ln\left|\sec\frac{\pi x}{6} + \tan\frac{\pi x}{6}\right|\right]_0^2$$
$$= \frac{3}{\pi}\left[\ln\left(2 + \sqrt{3}\right) - \ln(1+0)\right]$$
$$= \frac{3}{\pi}\ln\left(2 + \sqrt{3}\right)$$

91. $$P(t) = \int \frac{3000}{1 + 0.25t}\,dt = (3000)(4)\int \frac{0.25}{1 + 0.25t}\,dt$$
$$= 12{,}000 \ln\left|1 + 0.25t\right| + C$$
$$P(0) = 12{,}000 \ln\left|1 + 0.25(0)\right| + C = 1000$$
$$C = 1000$$
$$P(t) = 12{,}000 \ln\left|1 + 0.25t\right| + 1000$$
$$= 1000\left[12 \ln\left|1 + 0.25t\right| + 1\right]$$
$$P(3) = 1000\left[12(\ln 1.75) + 1\right] \approx 7715$$

92. $$t = \frac{10}{\ln 2}\int_{250}^{300} \frac{1}{T - 100}\,dT$$
$$= \frac{10}{\ln 2}\left[\ln(T - 100)\right]_{250}^{300} = \frac{10}{\ln 2}\left[\ln 200 - \ln 150\right]$$
$$= \frac{10}{\ln 2}\left[\ln\left(\frac{4}{3}\right)\right] \approx 4.1504 \text{ units of time}$$

93. $$\frac{1}{50-40}\int_{40}^{50} \frac{90{,}000}{400 + 3x}\,dx = \left[3000 \ln\left|400 + 3x\right|\right]_{40}^{50}$$
$$\approx \$168.27$$

94. $\dfrac{dS}{dt} = \dfrac{k}{t}$

$$S(t) = \int \frac{k}{t}\,dt = k\ln|t| + C = k\ln t + C \text{ since } t > 1.$$

$S(2) = k\ln 2 + C = 200$

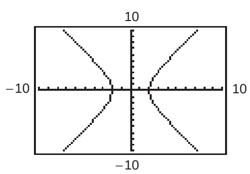
$S(4) = k\ln 4 + C = 300$

Solving this system yields $k = 100/\ln 2$ and $C = 100$. Thus,

$$S(t) = \frac{100\ln t}{\ln 2} + 100 = 100\left[\frac{\ln t}{\ln 2} + 1\right].$$

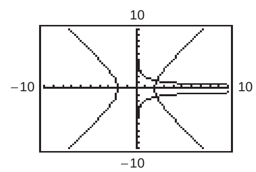
95. (a) $2x^2 - y^2 = 8$

$$y^2 = 2x^2 - 8$$
$$y_1 = \sqrt{2x^2 - 8}$$
$$y_2 = -\sqrt{2x^2 - 8}$$



(b) $y^2 = e^{-\int(1/x)\,dx} = e^{-\ln x + C} = e^{\ln(1/x)}(e^C) = \dfrac{1}{x}k$

Let $k = 4$ and graph $y^2 = \dfrac{4}{x}$. $\left(y_1 = \dfrac{2}{\sqrt{x}},\ y_2 = -\dfrac{2}{\sqrt{x}}\right)$



(c) In part (a): $2x^2 - y^2 = 8$ In part (b): $y^2 = \dfrac{4}{x} = 4x^{-1}$

$$4x - 2yy' = 0 \qquad 2yy' = \frac{-4}{x^2}$$

$$y' = \frac{2x}{y} \qquad y' = \frac{-2}{yx^2} = \frac{-2y}{y^2x^2} = \frac{-2y}{4x} = \frac{-y}{2x}$$

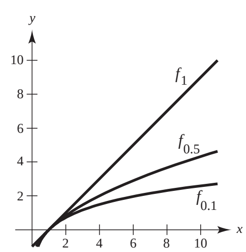
Using a graphing utility the graphs intersect at $(2.214, 1.344)$. The slopes are 3.295 and $-0.304 = (-1)/3.295$, respectively.

96. $k = 1$: $f_1(x) = x - 1$

$k = 0.5$: $f_{0.5}(x) = \dfrac{\sqrt{x} - 1}{0.5} = 2(\sqrt{x} - 1)$

$k = 0.1$: $f_{0.1}(x) = \dfrac{\sqrt[10]{x} - 1}{0.1} = 10(\sqrt[10]{x} - 1)$

$\lim\limits_{k\to 0^+} f_k(x) = \ln x$



97. False

$$\frac{1}{2}(\ln x) = \ln(x^{1/2})$$
$$\neq (\ln x)^{1/2}$$

98. False

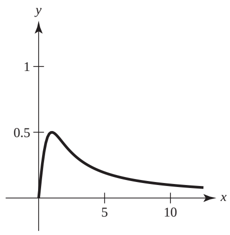
$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

99. True

$$\int \frac{1}{x}\,dx = \ln|x| + C_1$$
$$= \ln|x| + \ln|C|$$
$$= \ln|Cx|,\ C \neq 0$$

100. False; the integrand has a nonremovable discontinuity at $x = 0$.

101. $f(x) = \dfrac{x}{1 + x^2}$



(a) $y = \dfrac{1}{2}x$ intersects $f(x) = \dfrac{x}{1 + x^2}$:

$$\frac{1}{2}x = \frac{x}{1 + x^2}$$

$$1 + x^2 = 2$$

$$x = 1$$

$$A = \int_0^1 \left(\left[\frac{x}{1 + x^2}\right] - \frac{1}{2}x\right) dx$$

$$= \left[\frac{1}{2}\ln(x^2 + 1) - \frac{x^2}{4}\right]_0^1$$

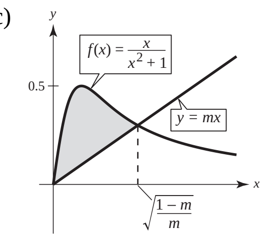
$$= \frac{1}{2}\ln 2 - \frac{1}{4}$$

(b) $f'(x) = \dfrac{(1 + x^2) - x(2x)}{(1 + x^2)^2} = \dfrac{1 - x^2}{(1 + x^2)^2}$

$f'(0) = 1$

Hence, for $0 < m < 1$, the graphs of f and $y = mx$ enclose a finite region.

(c)



$$\frac{x}{1 + x^2} = mx$$

$$1 = m + mx^2$$

$$x^2 = \frac{1 - m}{m}$$

$$x = \sqrt{\frac{1 - m}{m}}, \quad \text{intersection point}$$

$$A = \int_0^{\sqrt{(1-m)/m}} \left(\frac{x}{1 + x^2} - mx\right) dx, \quad 0 < m < 1$$

$$= \left[\frac{1}{2}\ln(1 + x^2) - \frac{mx^2}{2}\right]_0^{\sqrt{(1-m)/m}}$$

$$= \frac{1}{2}\ln\left(1 + \frac{1 - m}{m}\right) - \frac{1}{2}m\left(\frac{1 - m}{m}\right)$$

$$= \frac{1}{2}\ln\left(\frac{1}{m}\right) - \frac{1}{2}(1 - m)$$

$$= \frac{1}{2}[m - \ln(m) - 1]$$

102. $F(x) = \displaystyle\int_x^{2x} \frac{1}{t}\,dt, \quad x > 0$

$F'(x) = \dfrac{1}{2x}(2) - \dfrac{1}{x} = 0 \implies F$ is constant on $(0, \infty)$.

Alternate Solution:

$$F(x) = \ln t\Big]_x^{2x} = \ln(2x) - \ln x$$

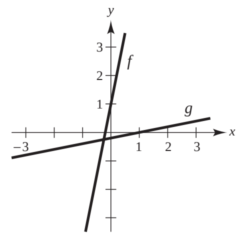
$$= \ln 2 + \ln x - \ln x$$

$$= \ln 2$$

Section 5.3 Inverse Functions

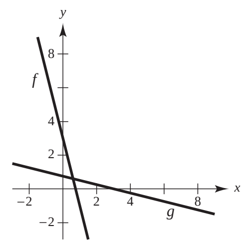
1. (a)
$$f(x) = 5x + 1$$
$$g(x) = \frac{x-1}{5}$$
$$f(g(x)) = f\left(\frac{x-1}{5}\right) = 5\left(\frac{x-1}{5}\right) + 1 = x$$
$$g(f(x)) = g(5x+1) = \frac{(5x-1)-1}{5} = x$$

(b)



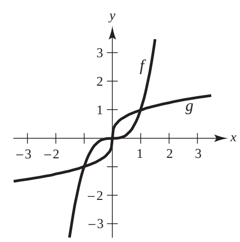
2. (a)
$$f(x) = 3 - 4x$$
$$g(x) = \frac{3-x}{4}$$
$$f(g(x)) = f\left(\frac{3-x}{4}\right) = 3 - 4\left(\frac{3-x}{4}\right) = x$$
$$g(f(x)) = g(3-4x) = \frac{3-(3-4x)}{4} = x$$

(b)



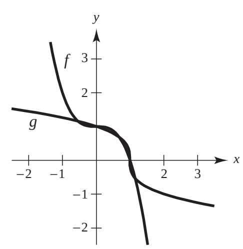
3. (a)
$$f(x) = x^3$$
$$g(x) = \sqrt[3]{x}$$
$$f(g(x)) = f\left(\sqrt[3]{x}\right) = \left(\sqrt[3]{x}\right)^3 = x$$
$$g(f(x)) = g(x^3) = \sqrt[3]{x^3} = x$$

(b)



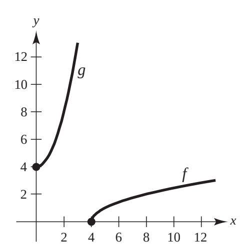
4. (a)
$$f(x) = 1 - x^3$$
$$g(x) = \sqrt[3]{1-x}$$
$$f(g(x)) = f\left(\sqrt[3]{1-x}\right) = 1 - \left(\sqrt[3]{1-x}\right)^3$$
$$= 1 - (1-x) = x$$
$$g(f(x)) = g(1-x^3)$$
$$= \sqrt[3]{1-(1-x^3)} = \sqrt[3]{x^3} = x$$

(b)



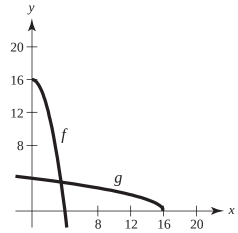
5. (a)
$$f(x) = \sqrt{x-4}$$
$$g(x) = x^2 + 4, \quad x \ge 0$$
$$f(g(x)) = f(x^2+4)$$
$$= \sqrt{(x^2+4)-4} = \sqrt{x^2} = x$$
$$g(f(x)) = g\left(\sqrt{x-4}\right)$$
$$= \left(\sqrt{x-4}\right)^2 + 4 = x - 4 + 4 = x$$

(b)



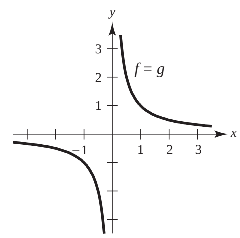
6. (a)
$$f(x) = 16 - x^2, \quad x \ge 0$$
$$g(x) = \sqrt{16 - x}$$
$$f(g(x)) = f\left(\sqrt{16 - x}\right) = 16 - \left(\sqrt{16 - x}\right)^2$$
$$= 16 - (16 - x) = x$$
$$g(f(x)) = g(16 - x^2) = \sqrt{16 - (16 - x^2)}$$
$$= \sqrt{x^2} = x$$

(b)



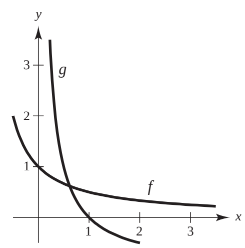
7. (a)
$$f(x) = \frac{1}{x}$$
$$g(x) = \frac{1}{x}$$
$$f(g(x)) = \frac{1}{1/x} = x$$
$$g(f(x)) = \frac{1}{1/x} = x$$

(b)



8. (a)
$$f(x) = \frac{1}{1 + x}, \quad x \ge 0$$
$$g(x) = \frac{1 - x}{x}, \quad 0 < x \le 1$$
$$f(g(x)) = f\left(\frac{1 - x}{x}\right) = \frac{1}{1 + \dfrac{1 - x}{x}} = \frac{1}{\dfrac{1}{x}} = x$$
$$g(f(x)) = g\left(\frac{1}{1 + x}\right) = \frac{1 - \dfrac{1}{1 + x}}{\dfrac{1}{1 + x}} = \frac{x}{1 + x} \cdot \frac{1 + x}{1} = x$$

(b)



9. Matches (c)

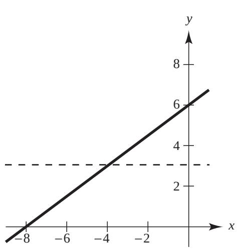
10. Matches (b)

11. Matches (a)

12. Matches (d)

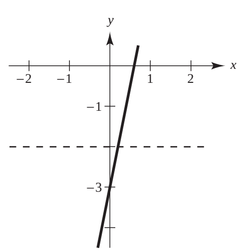
13. $f(x) = \frac{3}{4}x + 6$

One-to-one; has an inverse



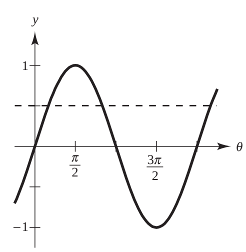
14. $f(x) = 5x - 3$

One-to-one; has an inverse



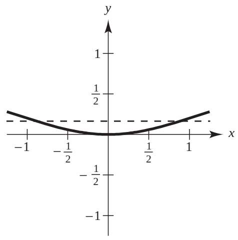
15. $f(\theta) = \sin\theta$

Not one-to-one; does not have an inverse



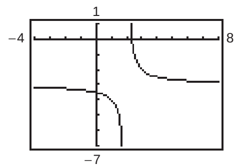
16. $F(x) = \dfrac{x^2}{x^2 + 4}$

Not one-to-one; does not have an inverse



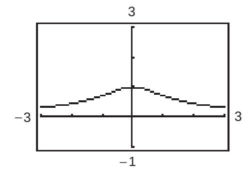
17. $h(s) = \dfrac{1}{s - 2} - 3$

One-to-one; has an inverse



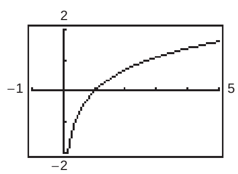
18. $g(t) = \dfrac{1}{\sqrt{t^2 + 1}}$

Not one-to-one; does not have an inverse



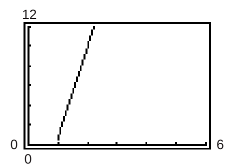
19. $f(x) = \ln x$

One-to-one; has an inverse



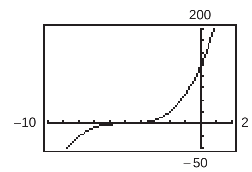
20. $f(x) = 5x\sqrt{x - 1}$

One-to-one; has an inverse



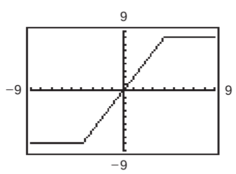
21. $g(x) = (x + 5)^3$

One-to-one; has an inverse



22. $h(x) = |x + 4| - |x - 4|$

Not one-to-one; does not have an inverse



23. $f(x) = \ln(x - 3), \quad x > 3$

$f'(x) = \dfrac{1}{x - 3} > 0$ for $x > 3$

f is increasing on $(3, \infty)$. Therefore, f is strictly monotonic and has an inverse.

24. $f(x) = \cos \dfrac{3x}{2}$

$f'(x) = -\dfrac{3}{2}\sin\dfrac{3x}{2} = 0$ when $x = 0, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \ldots$

f is not strictly monotonic on $(-\infty, \infty)$. Therefore, f does not have an inverse.

25. $f(x) = \dfrac{x^4}{4} - 2x^2$

$f'(x) = x^3 - 4x = 0$ when $x = 0, 2, -2$

f is not strictly monotonic on $(-\infty, \infty)$. Therefore, f does not have an inverse.

26. $f(x) = x^3 - 6x^2 + 12x$

$f'(x) = 3x^2 - 12x + 12 = 3(x - 2)^2 \geq 0$ for all x

f is increasing on $(-\infty, \infty)$. Therefore, f is strictly monotonic and has an inverse.

27. $f(x) = 2 - x - x^3$

$f'(x) = -1 - 3x^2 < 0$ for all x

f is decreasing on $(-\infty, \infty)$. Therefore, f is strictly monotonic and has an inverse.

28. $f(x) = (x + a)^3 + b$

$f'(x) = 3(x + a)^2 \geq 0$ for all x

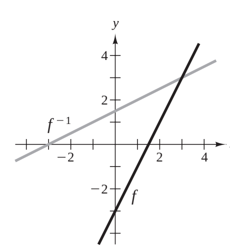
f is increasing on $(-\infty, \infty)$. Therefore, f is strictly monotonic and has an inverse.

29. $f(x) = 2x - 3 = y$

$$x = \frac{y + 3}{2}$$

$$y = \frac{x + 3}{2}$$

$$f^{-1}(x) = \frac{x + 3}{2}$$

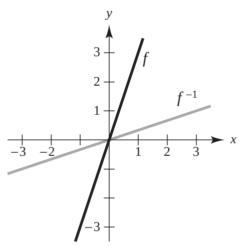


30. $f(x) = 3x = y$

$$x = \frac{y}{3}$$

$$y = \frac{x}{3}$$

$$f^{-1}(x) = \frac{x}{3}$$

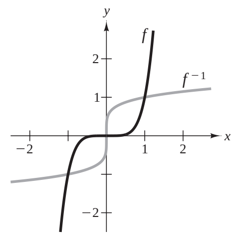


31. $f(x) = x^5 = y$

$$x = \sqrt[5]{y}$$

$$y = \sqrt[5]{x}$$

$$f^{-1}(x) = \sqrt[5]{x} = x^{1/5}$$

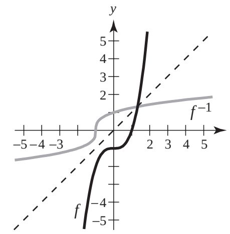


32. $f(x) = x^3 - 1 = y$

$$x = \sqrt[3]{y + 1}$$

$$y = \sqrt[3]{x + 1}$$

$$f^{-1}(x) = \sqrt[3]{x + 1}$$

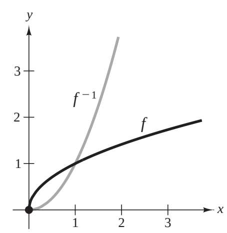


33. $f(x) = \sqrt{x} = y$

$$x = y^2$$

$$y = x^2$$

$$f^{-1}(x) = x^2, \quad x \geq 0$$

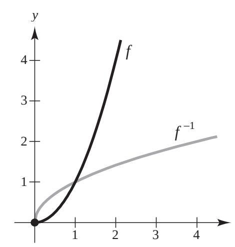


34. $f(x) = x^2 = y, \quad 0 \leq x$

$$x = \sqrt{y}$$

$$y = \sqrt{x}$$

$$f^{-1}(x) = \sqrt{x}$$

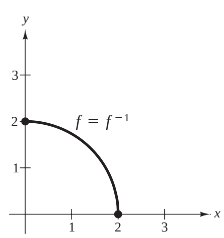


35. $f(x) = \sqrt{4 - x^2} = y, \quad 0 \leq x \leq 2$

$$x = \sqrt{4 - y^2}$$

$$y = \sqrt{4 - x^2}$$

$$f^{-1}(x) = \sqrt{4 - x^2}, \quad 0 \leq x \leq 2$$

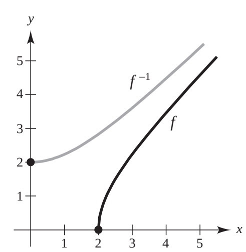


36. $f(x) = \sqrt{x^2 - 4} = y, \quad x \geq 2$

$$x = \sqrt{y^2 + 4}$$

$$y = \sqrt{x^2 + 4}$$

$$f^{-1}(x) = \sqrt{x^2 + 4}, \quad x \geq 0$$

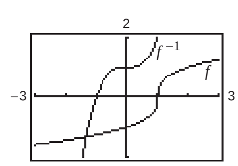


37. $f(x) = \sqrt[3]{x - 1} = y$

$$x = y^3 + 1$$

$$y = x^3 + 1$$

$$f^{-1}(x) = x^3 + 1$$



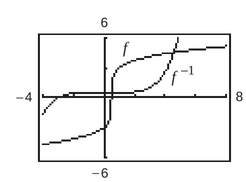
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

38. $f(x) = 3\sqrt[5]{2x - 1} = y$

$$x = \frac{y^5 + 243}{486}$$

$$y = \frac{x^5 + 243}{486}$$

$$f^{-1}(x) = \frac{x^5 + 243}{486}$$



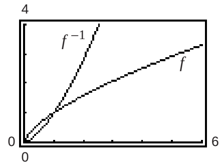
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

39. $f(x) = x^{2/3} = y, \quad x \geq 0$

$x = y^{3/2}$

$y = x^{3/2}$

$f^{-1}(x) = x^{3/2}, \quad x \geq 0$



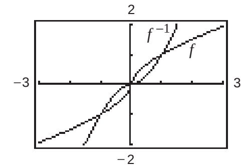
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

40. $f(x) = x^{3/5} = y$

$x = y^{5/3}$

$y = x^{5/3}$

$f^{-1}(x) = x^{5/3}$



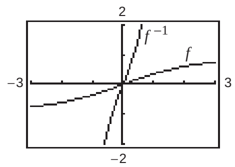
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

41. $f(x) = \dfrac{x}{\sqrt{x^2 + 7}} = y$

$x = \dfrac{\sqrt{7}y}{\sqrt{1 - y^2}}$

$y = \dfrac{\sqrt{7}x}{\sqrt{1 - x^2}}$

$f^{-1}(x) = \dfrac{\sqrt{7}x}{\sqrt{1 - x^2}}, \quad -1 < x < 1$



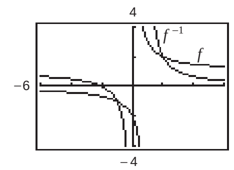
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

42. $f(x) = \dfrac{x + 2}{x} = y$

$x = \dfrac{2}{y - 1}$

$y = \dfrac{2}{x - 1}$

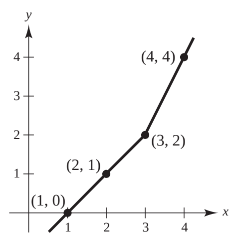
$f^{-1}(x) = \dfrac{2}{x - 1}$



The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

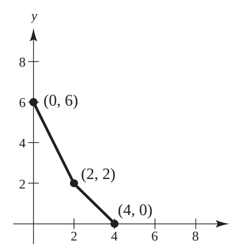
43.

x	1	2	3	4
$f^{-1}(x)$	0	1	2	4



44.

x	0	2	4
$f^{-1}(x)$	6	2	0



45. (a) Let x be the number of pounds of the commodity costing 1.25 per pound. Since there are 50 pounds total, the amount of the second commodity is $50 - x$. The total cost is

$$y = 1.25x + 1.60(50 - x)$$
$$= -0.35x + 80, \quad 0 \leq x \leq 50.$$

(b) We find the inverse of the original function:

$$y = -0.35x + 80$$
$$0.35x = 80 - y$$
$$x = \tfrac{100}{35}(80 - y)$$

Inverse: $y = \frac{100}{35}(80 - x) = \frac{20}{7}(80 - x)$

x represents cost and y represents pounds.

(c) Domain of inverse is $62.5 \leq x \leq 80$.

(d) If $x = 73$ in the inverse function, $y = \frac{100}{35}(80 - 73) = \frac{100}{5} = 20$ pounds.

46. $C = \frac{5}{9}(F - 32), \quad F \geq -459.6$

(a) $\frac{9}{5}C = F - 32$

$F = 32 + \frac{9}{5}C$

(b) The inverse function gives the temperature F corresponding to the Celsius temperature C.

(c) For $F \geq -459.6$, $C = \frac{5}{9}(F - 32) \geq -273.1\overline{1}$. Therefore, domain is $C \geq -273.\overline{1} = -273\frac{1}{9}$.

(d) If $C = 22°$, then $F = 32 + \frac{9}{5}(22) = 71.6°\text{F}$.

47. $f(x) = (x - 4)^2$ on $[4, \infty)$

$f'(x) = 2(x - 4) > 0$ on $(4, \infty)$

f is increasing on $[4, \infty)$. Therefore, f is strictly monotonic and has an inverse.

48. $f(x) = |x + 2|$ on $[-2, \infty)$

$f'(x) = \dfrac{|x + 2|}{x + 2}(1) = 1 > 0$ on $(-2, \infty)$

f is increasing on $[-2, \infty)$. Therefore, f is strictly monotonic and has an inverse.

49. $f(x) = \dfrac{4}{x^2}$ on $(0, \infty)$

$f'(x) = -\dfrac{8}{x^3} < 0$ on $(0, \infty)$

f is decreasing on $(0, \infty)$. Therefore, f is strictly monotonic and has an inverse.

50. $f(x) = \cot x$ on $(0, \pi)$

$f'(x) = -\csc^2 x < 0$ on $(0, \pi)$

f is decreasing on $(0, \pi)$. Therefore, f is strictly monotonic and has an inverse.

51. $f(x) = \cos x$ on $[0, \pi]$

$f'(x) = -\sin x < 0$ on $(0, \pi)$

f is decreasing on $[0, \pi]$. Therefore, f is strictly monotonic and has an inverse.

52. $f(x) = \sec x$ on $\left[0, \dfrac{\pi}{2}\right)$

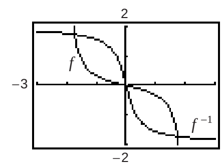
$f'(x) = \sec x \tan x > 0$ on $\left(0, \dfrac{\pi}{2}\right)$

f is increasing on $[0, \pi/2)$. Therefore, f is strictly monotonic and has an inverse.

53.
$$f(x) = \frac{x}{x^2 - 4} = y \text{ on } (-2, 2)$$
$$x^2y - 4y = x$$
$$x^2y - x - 4y = 0$$
$$a = y,\ b = -1,\ c = -4y$$
$$x = \frac{1 \pm \sqrt{1 - 4(y)(-4y)}}{2y} = \frac{1 \pm \sqrt{1 + 16y^2}}{2y}$$
$$y = f^{-1}(x) = \begin{cases} \left(1 - \sqrt{1 + 16x^2}\right)/2x, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

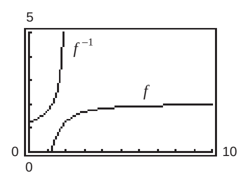
Domain of f^{-1}: all x

Range of f^{-1}: $-2 < y < 2$



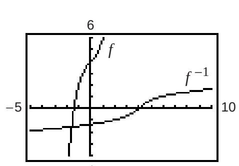
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

54.
$$f(x) = 2 - \frac{3}{x^2} = y \text{ on } (0, 10)$$
$$2x^2 - 3 = x^2y$$
$$x^2(2 - y) = 3$$
$$x = \pm\sqrt{\frac{3}{2 - y}}$$
$$y = \pm\sqrt{\frac{3}{2 - x}}$$
$$f^{-1}(x) = \sqrt{\frac{3}{2 - x}}, \quad x < 2$$



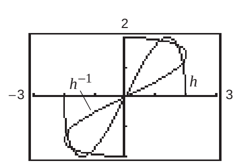
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

55. (a), (b)



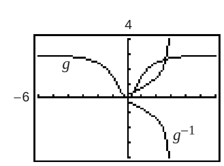
(c) Yes, f is one-to-one and has an inverse. The inverse relation is an inverse function.

56. (a), (b)



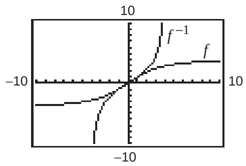
(c) h is not one-to-one and does not have an inverse. The inverse relation is not an inverse function.

57. (a), (b)



(c) g is not one-to-one and does not have an inverse. The inverse relation is not an inverse function.

58. (a), (b)



(c) Yes, f is one-to-one and has an inverse. The inverse relation is an inverse function.

59. $f(x) = \sqrt{x-2}$, Domain: $x \geq 2$

$f'(x) = \dfrac{1}{2\sqrt{x-2}} > 0$ for $x > 2$

f is one-to-one; has an inverse

$$\sqrt{x-2} = y$$
$$x - 2 = y^2$$
$$x = y^2 + 2$$
$$y = x^2 + 2$$
$$f^{-1}(x) = x^2 + 2, \quad x \geq 0$$

60. $f(x) = -3$

Not one-to-one; does not have an inverse

61. $f(x) = |x-2|, \quad x \leq 2$

$$= -(x-2)$$
$$= 2 - x$$

f is one-to-one; has an inverse

$$2 - x = y$$
$$2 - y = x$$
$$f^{-1}(x) = 2 - x, \quad x \geq 0$$

62. $f(x) = ax + b$

f is one-to-one; has an inverse

$$ax + b = y$$
$$x = \frac{y-b}{a}$$
$$y = \frac{x-b}{a}$$
$$f^{-1}(x) = \frac{x-b}{a}, \quad a \neq 0$$

63. $f(x) = (x-3)^2$ is one-to-one for $x \geq 3$.

$$(x-3)^2 = y$$
$$x - 3 = \sqrt{y}$$
$$x = \sqrt{y} + 3$$
$$y = \sqrt{x} + 3$$
$$f^{-1}(x) = \sqrt{x} + 3, \quad x \geq 0$$

(Answer is not unique.)

64. $f(x) = 16 - x^4$ is one-to-one for $x \geq 0$.

$$16 - x^4 = y$$
$$16 - y = x^4$$
$$\sqrt[4]{16-y} = x$$
$$\sqrt[4]{16-x} = y$$
$$f^{-1}(x) = \sqrt[4]{16-x}, \quad x \leq 16$$

65. $f(x) = |x+3|$ is one-to-one for $x \geq -3$.

$$x + 3 = y$$
$$x = y - 3$$
$$y = x - 3$$
$$f^{-1}(x) = x - 3, \quad x \geq 0$$

(Answer is not unique.)

66. $f(x) = |x-3|$ is one-to-one for $x \geq 3$.

$$x - 3 = y$$
$$x = y + 3$$
$$y = x + 3$$
$$f^{-1}(x) = x + 3, \quad x \geq 0$$

67. Yes, the volume is an increasing function, and hence one-to-one. The inverse function gives the time t corresponding to the volume V.

68. No, there could be two times $t_1 \neq t_2$ for which $h(t_1) = h(t_2)$.

69. No, $C(t)$ is not one-to-one because long distance costs are step functions. A call lasting 2.1 minutes costs the same as one lasting 2.2 minutes.

70. Yes, the area function is increasing and hence one-to-one. The inverse function gives the radius r corresponding to the area A.

71.
$$f(x) = x^3 + 2x - 1,\ f(1) = 2 = a$$
$$f'(x) = 3x^2 + 2$$
$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(1)} = \frac{1}{3(1)^2 + 2} = \frac{1}{5}$$

72.
$$f(x) = \frac{1}{27}(x^5 + 2x^3)$$
$$f(-3) = \frac{1}{27}(-243 - 54) = -11 = a$$
$$f'(x) = \frac{1}{27}(5x^4 + 6x^2)$$
$$(f^{-1})'(-11) = \frac{1}{f'(f^{-1}(-11))}$$
$$= \frac{1}{f'(-3)} = \frac{27}{5(-3)^4 + 6(-3)^2} = \frac{1}{17}$$

73.
$$f(x) = \sin x,\ f\left(\frac{\pi}{6}\right) = \frac{1}{2} = a$$
$$f'(x) = \cos x$$
$$(f^{-1})'\left(\frac{1}{2}\right) = \frac{1}{f'(f^{-1}(1/2))} = \frac{1}{f'(\pi/6)} = \frac{1}{\cos(\pi/6)}$$
$$= \frac{1}{\sqrt{3}/2} = \frac{2\sqrt{3}}{3}$$

74.
$$f(x) = \cos 2x,\ \ f(0) = 1 = a$$
$$f'(x) = -2\sin 2x$$
$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)}$$
$$= \frac{1}{-2\sin 0} = \frac{1}{0} \text{ which is undefined.}$$

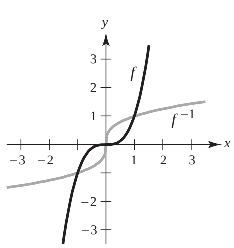
75.
$$f(x) = x^3 - \frac{4}{x},\ f(2) = 6 = a$$
$$f'(x) = 3x^2 + \frac{4}{x^2}$$
$$(f^{-1})'(6) = \frac{1}{f'(f^{-1}(6))} = \frac{1}{f'(2)} = \frac{1}{3(2)^2 + (4/2^2)} = \frac{1}{13}$$

76.
$$f(x) = \sqrt{x - 4},\ \ f(8) = 2 = a$$
$$f'(x) = \frac{1}{2\sqrt{x - 4}}$$
$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(8)}$$
$$= \frac{1}{1/\left(2\sqrt{8 - 4}\right)} = \frac{1}{1/4} = 4$$

77. (a) Domain $f =$ Domain $f^{-1} = (-\infty, \infty)$

(b) Range $f =$ Range $f^{-1} = (-\infty, \infty)$

(c)

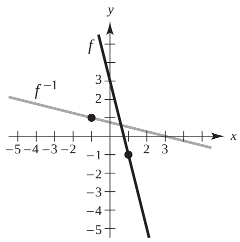


(d)
$$f(x) = x^3,\ \left(\frac{1}{2}, \frac{1}{8}\right)$$
$$f'(x) = 3x^2$$
$$f'\left(\frac{1}{2}\right) = \frac{3}{4}$$
$$f^{-1}(x) = \sqrt[3]{x},\ \left(\frac{1}{8}, \frac{1}{2}\right)$$
$$(f^{-1})'(x) = \frac{1}{3\sqrt[3]{x^2}}$$
$$(f^{-1})'\left(\frac{1}{8}\right) = \frac{4}{3}$$

78. (a) Domain $f =$ Domain $f^{-1} = (-\infty, \infty)$

(b) Range $f =$ Range $f^{-1} = (-\infty, \infty)$

(c)

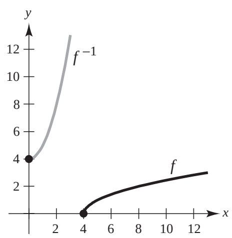


(d)
$$f(x) = 3 - 4x, \ (1, -1)$$
$$f'(x) = -4$$
$$f'(1) = -4$$
$$f^{-1}(x) = \frac{3-x}{4}, \ (-1, 1)$$
$$(f^{-1})'(x) = -\frac{1}{4}$$
$$(f^{-1})'(-1) = -\frac{1}{4}$$

79. (a) Domain $f = [4, \infty)$, Domain $f^{-1} = [0, \infty)$

(b) Range $f = [0, \infty)$, Range $f^{-1} = [4, \infty)$

(c)

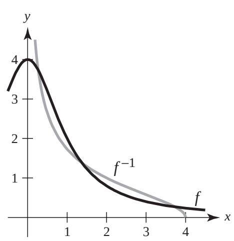


(d)
$$f(x) = \sqrt{x-4}, \ (5, 1)$$
$$f'(x) = \frac{1}{2\sqrt{x-4}}$$
$$f'(5) = \frac{1}{2}$$
$$f^{-1}(x) = x^2 + 4, \ (1, 5)$$
$$(f^{-1})'(x) = 2x$$
$$(f^{-1})'(1) = 2$$

80. (a) Domain $f = [0, \infty)$, Domain $f^{-1} = (0, 4]$

(b) Range $f = (0, 4]$, Range $f^{-1} = [0, \infty)$

(c)



(d)
$$f(x) = \frac{4}{1+x^2}$$
$$f'(x) = \frac{-8x}{(x^2+1)^2}, \ f'(1) = -2$$
$$f^{-1}(x) = \sqrt{\frac{4-x}{x}}$$
$$(f^{-1})'(x) = \frac{-2}{x^2\sqrt{(4-x)/x}}, \ (f^{-1})'(2) = -\frac{1}{2}$$

81. $x = y^3 - 7y^2 + 2$

$$1 = 3y^2\frac{dy}{dx} - 14y\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{1}{3y^2 - 14y}$$

At $(-4, 1)$, $\dfrac{dy}{dx} = \dfrac{1}{3 - 14} = \dfrac{-1}{11}$.

Alternate Solution:

Let $f(x) = x^3 - 7x^2 + 2$. Then $f'(x) = 3x^2 - 14x$ and $f'(1) = -11$. Hence,

$$\frac{dy}{dx} = \frac{1}{-11} = \frac{-1}{11}.$$

82. $x = 2\ln(y^2 - 3)$

$$1 = 2\frac{1}{y^2-3}2y\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{y^2-3}{4y}$$

At $(0, 4)$, $\dfrac{dy}{dx} = \dfrac{16-3}{16} = \dfrac{13}{16}$.

In Exercises 83–86, use the following.

$f(x) = \frac{1}{8}x - 3$ **and** $g(x) = x^3$

$f^{-1}(x) = 8(x + 3)$ **and** $g^{-1}(x) = \sqrt[3]{x}$

83. $(f^{-1} \circ g^{-1})(1) = f^{-1}(g^{-1}(1)) = f^{-1}(1) = 32$

84. $(g^{-1} \circ f^{-1})(-3) = g^{-1}(f^{-1}(-3) = g^{-1}(0) = 0$

85. $(f^{-1} \circ f^{-1})(6) = f^{-1}(f^{-1}(6)) = f^{-1}(72) = 600$

86. $(g^{-1} \circ g^{-1})(-4) = g^{-1}(g^{-1}(-4)) = g^{-1}\left(\sqrt[3]{-4}\right)$

$$= \sqrt[3]{\sqrt[3]{-4}} = -\sqrt[9]{4}$$

In Exercises 87–90, use the following.

$f(x) = x + 4$ **and** $g(x) = 2x - 5$

$f^{-1}(x) = x - 4$ **and** $g^{-1}(x) = \frac{x + 5}{2}$

87. $$\begin{aligned}(g^{-1} \circ f^{-1})(x) &= g^{-1}(f^{-1}(x)) \\ &= g^{-1}(x - 4) \\ &= \frac{(x - 4) + 5}{2} \\ &= \frac{x + 1}{2}\end{aligned}$$

88. $$\begin{aligned}(f^{-1} \circ g^{-1})(x) &= f^{-1}(g^{-1}(x)) \\ &= f^{-1}\left(\frac{x + 5}{2}\right) \\ &= \frac{x + 5}{2} - 4 \\ &= \frac{x - 3}{2}\end{aligned}$$

89. $$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(2x - 5) \\ &= (2x - 5) + 4 \\ &= 2x - 1\end{aligned}$$

Hence, $(f \circ g)^{-1}(x) = \frac{x + 1}{2}$.

Note: $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

90. $$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(x + 4) \\ &= 2(x + 4) - 5 \\ &= 2x + 3\end{aligned}$$

Hence, $(g \circ f)^{-1}(x) = \frac{x - 3}{2}$.

Note: $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

91. Answers will vary. See page 343 and Example 3.

92. The graphs of f and f^{-1} are mirror images with respect to the line $y = x$.

93. f is not one-to-one because many different x-values yield the same y-value.

Example: $f(0) = f(\pi) = 0$

Not continuous at $\frac{(2n - 1)\pi}{2}$, where n is an integer.

94. f is not one-to-one because different x-values yield the same y-value.

Example: $f(3) = f\left(-\frac{4}{3}\right) = \frac{3}{5}$

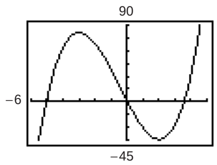
Not continuous at ± 2.

95. $f(x) = k(2 - x - x^3)$ is one-to-one. Since $f^{-1}(3) = -2$,

$f(-2) = 3 = k(2 - (-2) - (-2)^3) = 12k \implies k = \frac{1}{4}$.

96. (a) $f(x) = 2x^3 + 3x^2 - 36x$

f does not pass the horizontal line test.



(b) $f'(x) = 6x^2 + 6x - 36$

$= 6(x^2 + x - 6) = 6(x + 3)(x - 2)$

$f'(x) = 0$ at $x = 2, -3$

Hence, on the interval $[-2, 2]$, f is one-to-one.

97. Let f and g be one-to-one functions.

(a) Let $(f \circ g)(x_1) = (f \circ g)(x_2)$

$f(g(x_1)) = f(g(x_2))$

$g(x_1) = g(x_2)$ (Because f is one-to-one.)

$x_1 = x_2$ (Because g is one-to-one.)

Thus, $f \circ g$ is one-to-one.

(b) Let $(f \circ g)(x) = y$, then $x = (f \circ g)^{-1}(y)$. Also:

$(f \circ g)(x) = y$

$f(g(x)) = y$

$g(x) = f^{-1}(y)$

$x = g^{-1}(f^{-1}(y))$

$x = (g^{-1} \circ f^{-1})(y)$

Thus, $(f \circ g)^{-1}(y) = (g^{-1} \circ f^{-1})(y)$ and $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

98. If f has an inverse, then f and f^{-1} are both one-to-one. Let $(f^{-1})^{-1}(x) = y$ then $x = f^{-1}(y)$ and $f(x) = y$. Thus, $(f^{-1})^{-1} = f$.

99. Suppose $g(x)$ and $h(x)$ are both inverses of $f(x)$. Then the graph of $f(x)$ contains the point (a, b) if and only if the graphs of $g(x)$ and $h(x)$ contain the point (b, a). Since the graphs of $g(x)$ and $h(x)$ are the same, $g(x) = h(x)$. Therefore, the inverse of $f(x)$ is unique.

100. If f has an inverse and $f(x_1) = f(x_2)$, then $f^{-1}(f(x_1)) = f^{-1}(f(x_2)) \implies x_1 = x_2$. Therefore, f is one-to-one. If $f(x)$ is one-to-one, then for every value b in the range, there corresponds exactly one value a in the domain. Define $g(x)$ such that the domain of g equals the range of f and $g(b) = a$. By the reflexive property of inverses, $g = f^{-1}$.

101. False. Let $f(x) = x^2$.

102. True; if f has a y-intercept

103. True

104. False. Let $f(x) = x$ or $g(x) = 1/x$.

105. Not true.

Let $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 1 - x, & 1 < x \le 2 \end{cases}$.

f is one-to-one, but not strictly monotonic.

106. From Theorem 5.9, we have:

$$g'(x) = \frac{1}{f'(g(x))}$$

$$g''(x) = \frac{f'(g(x))(0) - f''(g(x))g'(x)}{[f'(g(x))]^2}$$

$$= -\frac{f''(g(x)) \cdot [1/(f'(g(x)))]}{[f'(g(x))]^2}$$

$$= -\frac{f''(g(x))}{[f'(g(x))]^3}$$

If f is increasing and concave down, then $f' > 0$ and $f'' < 0$ which implies that g is increasing and concave up.

107. $$f(x) = \int_2^x \frac{dt}{\sqrt{1 + t^4}}, \quad f(2) = 0$$

$$f'(x) = \frac{1}{\sqrt{1 + x^4}}$$

$$(f^{-1})'(0) = \frac{1}{f'(2)} = \frac{1}{1/\sqrt{17}} = \sqrt{17}$$

108. $f(x) = \int_2^x \sqrt{1+t^2}\,dt, \quad f(2) = 0$

$f'(x) = \sqrt{1+x^2}$

$f'(x) > 0$ for all $x \implies f$ increasing on $(-\infty, \infty) \implies f$ is one-to-one.

$(f^{-1})'(0) = \dfrac{1}{f'(2)} = \dfrac{1}{\sqrt{5}} = \dfrac{\sqrt{5}}{5}$

109. (a)

$$y = \frac{x-2}{x-1}$$
$$x = \frac{y-2}{y-1}$$
$$xy - x = y - 2$$
$$xy - y = x - 2$$
$$y = \frac{x-2}{x-1}$$

Hence, if $f(x) = \dfrac{x-2}{x-1}$, then $f^{-1}(x) = f(x)$.

(b) The graph of f is symmetric about the line $y = x$.

Section 5.4 Exponential Functions: Differentiation and Integration

1. $e^{\ln x} = 4$
$x = 4$

2. $e^{\ln 2x} = 12$
$2x = 12$
$x = 6$

3. $e^x = 12$
$x = \ln 12 \approx 2.485$

4. $4e^x = 83$
$e^x = \frac{83}{4}$
$x = \ln\left(\frac{83}{4}\right) \approx 3.033$

5. $9 - 2e^x = 7$
$2e^x = 2$
$e^x = 1$
$x = 0$

6. $-6 + 3e^x = 8$
$3e^x = 14$
$e^x = \frac{14}{3}$
$x = \ln\left(\frac{14}{3}\right)$
≈ 1.540

7. $50e^{-x} = 30$
$e^{-x} = \frac{3}{5}$
$-x = \ln\left(\frac{3}{5}\right)$
$x = \ln\left(\frac{5}{3}\right)$
≈ 0.511

8. $200e^{-4x} = 15$
$e^{-4x} = \frac{15}{200} = \frac{3}{40}$
$-4x = \ln\left(\frac{3}{40}\right)$
$x = \frac{1}{4}\ln\left(\frac{40}{3}\right)$
≈ 0.648

9. $\ln x = 2$
$x = e^2 \approx 7.389$

10. $\ln x^2 = 10$
$x^2 = e^{10}$
$x = \pm e^5 \approx \pm 148.4132$

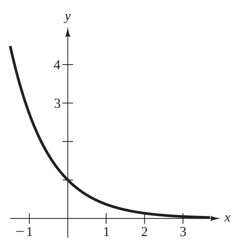
11. $\ln(x-3) = 2$
$x - 3 = e^2$
$x = 3 + e^2 \approx 10.389$

12. $\ln 4x = 1$
$4x = e' = e$
$x = \dfrac{e}{4} \approx 0.680$

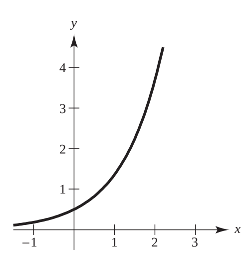
13. $\ln\sqrt{x+2} = 1$
$\sqrt{x+2} = e^1 = e$
$x + 2 = e^2$
$x = e^2 - 2 \approx 5.389$

14. $\ln(x-2)^2 = 12$
$(x-2)^2 = e^{12}$
$x - 2 = e^6$
$x = 2 + e^6 \approx 405.429$

15. $y = e^{-x}$



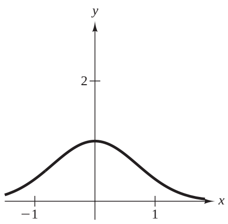
16. $y = \frac{1}{2}e^x$



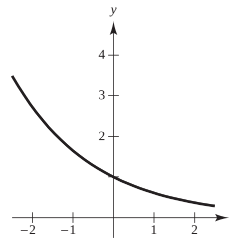
17. $y = e^{-x^2}$

Symmetric with respect to the y-axis

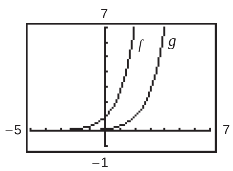
Horizontal asymptote: $y = 0$



18. $y = e^{-x/2}$

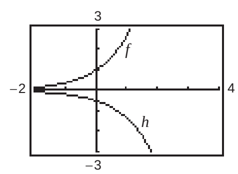


19. (a)



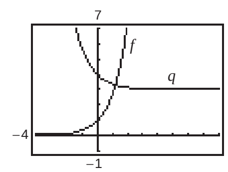
Horizontal shift 2 units to the right

(b)



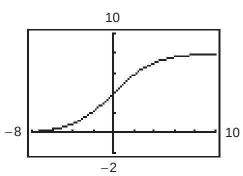
A reflection in the x-axis and a vertical shrink

(c)



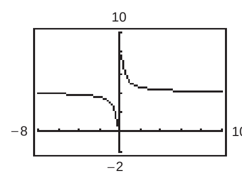
Vertical shift 3 units upward and a reflection in the y-axis

20. (a)



Horizontal asymptotes: $y = 0$ and $y = 8$

(b)



Horizontal asymptote: $y = 4$

21. $y = Ce^{ax}$

Horizontal asymptote: $y = 0$

Matches (c)

22. $y = Ce^{-ax}$

Horizontal asymptote: $y = 0$

Reflection in the y-axis

Matches (d)

23. $y = C(1 - e^{-ax})$

Vertical shift C units

Reflection in both the x- and y-axes

Matches (a)

24. $y = \dfrac{C}{1 + e^{-ax}}$

$$\lim_{x \to \infty} \frac{C}{1 + e^{-ax}} = C$$

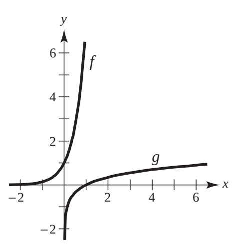
$$\lim_{x \to -\infty} \frac{C}{1 + e^{-ax}} = 0$$

Horizontal asymptotes:
$y = C$ and $y = 0$

Matches (b)

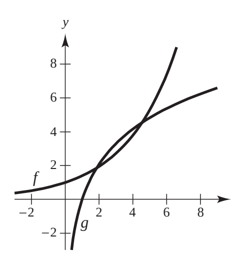
25. $f(x) = e^{2x}$

$g(x) = \ln\sqrt{x} = \dfrac{1}{2}\ln x$



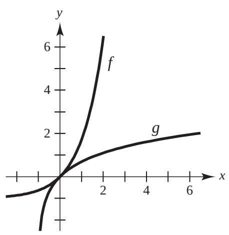
26. $f(x) = e^{x/3}$

$g(x) = \ln x^3 = 3\ln x$



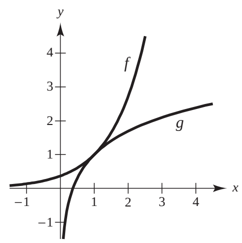
27. $f(x) = e^x - 1$

$g(x) = \ln(x + 1)$

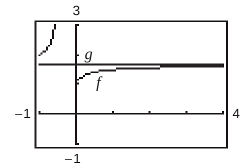


28. $f(x) = e^{x-1}$

$g(x) = 1 + \ln x$



29.



As $x \to \infty$, the graph of f approaches the graph of g.

$$\lim_{x\to\infty}\left(1 + \frac{0.5}{x}\right)^x = e^{0.5}$$

30. In the same way,

$$\lim_{x\to\infty}\left(1 + \frac{r}{x}\right)^x = e^r \text{ for } r > 0.$$

31. $\left(1 + \dfrac{1}{1{,}000{,}000}\right)^{1{,}000{,}000} \approx 2.718280469$

$e \approx 2.718281828$

$e > \left(1 + \dfrac{1}{1{,}000{,}000}\right)^{1{,}000{,}000}$

32. $1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} = 2.71\overline{825396}$

$e \approx 2.718281828$

$e > 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040}$

33. (a)
$$\begin{aligned} y &= e^{3x} \\ y' &= 3e^{3x} \\ y'(0) &= 3 \\ y - 1 &= 3(x - 0) \\ y &= 3x + 1 \quad \text{Tangent line} \end{aligned}$$

(b)
$$\begin{aligned} y &= e^{-3x} \\ y' &= -3e^{-3x} \\ y'(0) &= -3 \\ y - 1 &= -3(x - 0) \\ y &= -3x + 1 \quad \text{Tangent line} \end{aligned}$$

34. (a)
$$\begin{aligned} y &= e^{2x} \\ y' &= 2e^{2x} \\ y'(0) &= 2 \\ y - 1 &= 2(x - 0) \\ y &= 2x + 1 \end{aligned}$$

(b)
$$\begin{aligned} y &= e^{-2x} \\ y' &= -2e^{-2x} \\ y'(0) &= -2 \\ y - 1 &= -2(x - 0) \\ y &= -2x + 1 \end{aligned}$$

35. $f(x) = e^{2x}$

$f'(x) = 2e^{2x}$

36. $y = e^{-x^2}$

$\dfrac{dy}{dx} = -2xe^{-x^2}$

37. $y = e^{\sqrt{x}}$

$\dfrac{dy}{dx} = \dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$

38. $y = x^2e^{-x}$

$$\begin{aligned} \frac{dy}{dx} &= -x^2e^{-x} + 2xe^{-x} \\ &= xe^{-x}(2 - x) \end{aligned}$$

39. $g(t) = (e^{-t} + e^t)^3$

$g'(t) = 3(e^{-t} + e^t)^2(e^t - e^{-t})$

40. $g(t) = e^{-3/t^2}$

$g'(t) = e^{-3/t^2}(6t^{-3}) = \dfrac{6}{t^3e^{3/t^2}}$

41. $y = \ln(1 + e^{2x})$

$\dfrac{dy}{dx} = \dfrac{2e^{2x}}{1 + e^{2x}}$

42. $y = \ln\left(\dfrac{1 + e^x}{1 - e^x}\right) = \ln(1 + e^x) - \ln(1 - e^x)$

$\dfrac{dy}{dx} = \dfrac{e^x}{1 + e^x} + \dfrac{e^x}{1 - e^x} = \dfrac{2e^x}{1 - e^{2x}}$

43. $y = \dfrac{2}{e^x + e^{-x}} = 2(e^x + e^{-x})^{-1}$

$$\frac{dy}{dx} = -2(e^x + e^{-x})^{-2}(e^x - e^{-x}) = \frac{-2(e^x - e^{-x})}{(e^x + e^{-x})^2}$$

44. $y = \dfrac{e^x - e^{-x}}{2}$

$$\frac{dy}{dx} = \frac{e^x + e^{-x}}{2}$$

45. $y = e^x(\sin x + \cos x)$

$$\frac{dy}{dx} = e^x(\cos x - \sin x) + (\sin x + \cos x)(e^x)$$

$$= e^x(2\cos x) = 2e^x \cos x$$

46. $y = \ln e^x = x$

$$\frac{dy}{dx} = 1$$

47. $F(x) = \displaystyle\int_{\pi}^{\ln x} \cos e^t \, dt$

$$F'(x) = \cos(e^{\ln x}) \cdot \frac{1}{x} = \frac{\cos(x)}{x}$$

48. $F(x) = \displaystyle\int_0^{e^{2x}} \ln(t + 1)\, dt$

$$F'(x) = \ln(e^{2x} + 1)2e^{2x} = 2e^{2x}\ln(e^{2x} + 1)$$

49. $f(x) = e^{1-x}$, $(1, 1)$

$f'(x) = -e^{1-x}$, $f'(1) = -1$

$y - 1 = -1(x - 1)$

$y = -x + 2$ Tangent line

50. $y = e^{-2x+x^2}$, $(2, 1)$

$y' = (2x - 2)e^{-2x+x^2}$, $y'(2) = 2$

$y - 1 = 2(x - 2)$

$y = 2x - 3$ Tangent line

51. $y = \ln(e^{x^2}) = x^2$, $(-2, 4)$

$y' = 2x$, $y'(-2) = -4$

$y - 4 = -4(x + 2)$

$y = -4x - 4$ Tangent line

52. $y = \ln \dfrac{e^x + e^{-x}}{2}$, $(0, 0)$

$$y' = \frac{1}{[(e^x + e^{-x})/2]}[e^x - e^{-x}]$$

$y'(0) = 0$

$y = 0$ Tangent line

53. $y = x^2e^x - 2xe^x + 2e^x$, $(1, e)$

$y' = x^2e^x + 2xe^x - 2xe^x - 2e^x + 2e^x = x^2e^x$

$y'(1) = e$

$y - e = e(x - 1)$

$y = ex$ Tangent line

54. $y = xe^x - e^x$, $(1, 0)$

$y' = xe^x + e^x - e^x = xe^x$

$y'(1) = e$

$y - 0 = e(x - 1)$

$y = ex - e$ Tangent line

55. $f(x) = e^{-x}\ln x$, $(1, 0)$

$$f'(x) = e^{-x}\left(\frac{1}{x}\right) - e^{-x}\ln x = e^{-x}\left(\frac{1}{x} - \ln x\right)$$

$f'(1) = e^{-1}$

$y - 0 = e^{-1}(x - 1)$

$y = \dfrac{1}{e}x - \dfrac{1}{e}$ Tangent line

56. $f(x) = e^3 \ln x$, $(1, 0)$

$f'(x) = \dfrac{e^3}{x}$, $f'(1) = e^3$

$y - 0 = e^3(x - 1)$

$y = e^3(x - 1)$ Tangent line

57.
$$xe^y - 10x + 3y = 0$$
$$xe^y \frac{dy}{dx} + e^y - 10 + 3\frac{dy}{dx} = 0$$
$$\frac{dy}{dx}(xe^y + 3) = 10 - e^y$$
$$\frac{dy}{dx} = \frac{10 - e^y}{xe^y + 3}$$

58.
$$e^{xy} + x^2 - y^2 = 10$$
$$\left(x\frac{dy}{dx} + y\right)e^{xy} + 2x - 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx}(xe^{xy} - 2y) = -ye^{xy} - 2x$$
$$\frac{dy}{dx} = -\frac{ye^{xy} + 2x}{xe^{xy} - 2y}$$

59. $xe^y + ye^x = 1, \quad (0, 1)$

$xe^y y' + e^y + ye^x + y'e^x = 0$

At $(0, 1)$: $e + 1 + y' = 0$

$y' = -e - 1$

Tangent line: $y - 1 = (-e - 1)(x - 0)$

$y = (-e - 1)x + 1$

60. $1 + \ln(xy) = e^{x-y}, \quad (1, 1)$

$\frac{1}{xy}[xy' + y] = e^{x-y}[1 - y']$

At $(1, 1)$: $[y' + 1] = 1 - y'$

$y' = 0$

Tangent line: $y - 1 = 0(x - 1)$

$y = 1$

61. $f(x) = (3 + 2x)e^{-3x}$

$$f'(x) = (3 + 2x)(-3e^{-3x}) + 2e^{-3x} = (-7 - 6x)e^{-3x}$$
$$f''(x) = (-7 - 6x)(-3e^{-3x}) - 6e^{-3x} = 3(6x + 5)e^{-3x}$$

62. $g(x) = \sqrt{x} + e^x \ln x$

$$g'(x) = \frac{1}{2\sqrt{x}} + \frac{e^x}{x} + e^x \ln x$$
$$g''(x) = -\frac{1}{4x^{3/2}} + \frac{xe^x - e^x}{x^2} + \frac{e^x}{x} + e^x \ln x = -\frac{1}{4x\sqrt{x}} + \frac{e^x(2x - 1)}{x^2} + e^x \ln x$$

63.
$$y = e^x\left(\cos\sqrt{2}x + \sin\sqrt{2}x\right)$$
$$y' = e^x\left(-\sqrt{2}\sin\sqrt{2}x + \sqrt{2}\cos\sqrt{2}x\right) + e^x\left(\cos\sqrt{2}x + \sin\sqrt{2}x\right) = e^x\left[\left(1 + \sqrt{2}\right)\cos\sqrt{2}x + \left(1 - \sqrt{2}\right)\sin\sqrt{2}x\right]$$
$$y'' = e^x\left[-\left(\sqrt{2} + 2\right)\sin\sqrt{2}x + \left(\sqrt{2} - 2\right)\cos\sqrt{2}x\right] + e^x\left[\left(1 + \sqrt{2}\right)\cos\sqrt{2}x + \left(1 - \sqrt{2}\right)\sin\sqrt{2}x\right]$$
$$= e^x\left[\left(-1 - 2\sqrt{2}\right)\sin\sqrt{2}x + \left(-1 + 2\sqrt{2}\right)\cos\sqrt{2}x\right]$$
$$-2y' + 3y = -2e^x\left[\left(1 + \sqrt{2}\right)\cos\sqrt{2}x + \left(1 - \sqrt{2}\right)\sin\sqrt{2}x\right] + 3e^x\left[\cos\sqrt{2}x + \sin\sqrt{2}x\right]$$
$$= e^x\left[\left(1 - 2\sqrt{2}\right)\cos\sqrt{2}x + \left(1 + 2\sqrt{2}\right)\sin\sqrt{2}x\right] = -y''$$

Therefore, $-2y' + 3y = -y'' \Longrightarrow y'' - 2y' + 3y = 0$.

64.
$$y = e^x(3\cos 2x - 4\sin 2x)$$
$$y' = e^x(-6\sin 2x - 8\cos 2x) + e^x(3\cos 2x - 4\sin 2x) = e^x(-10\sin 2x - 5\cos 2x) = -5e^x(2\sin 2x + \cos 2x)$$
$$y'' = -5e^x(4\cos 2x - 2\sin 2x) - 5e^x(2\sin 2x + \cos 2x) = -5e^x(5\cos 2x) = -25e^x\cos 2x$$
$$y'' - 2y' = -25e^x\cos 2x - 2(-5e^x)(2\sin 2x + \cos 2x) = -5e^x(3\cos 2x - 4\sin 2x) = -5y$$

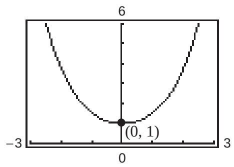
Therefore, $y'' - 2y' = -5y \Longrightarrow y'' - 2y' + 5y = 0$.

65. $f(x) = \dfrac{e^x + e^{-x}}{2}$

$f'(x) = \dfrac{e^x - e^{-x}}{2} = 0$ when $x = 0$.

$f''(x) = \dfrac{e^x + e^{-x}}{2} > 0$

Relative minimum: $(0, 1)$

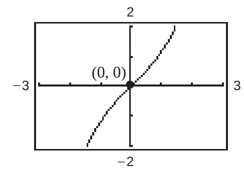


66. $f(x) = \dfrac{e^x - e^{-x}}{2}$

$f'(x) = \dfrac{e^x + e^{-x}}{2} > 0$

$f''(x) = \dfrac{e^x - e^{-x}}{2} = 0$ when $x = 0$.

Point of inflection: $(0, 0)$



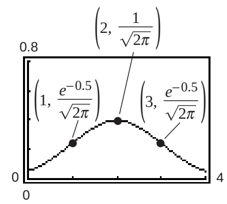
67. $g(x) = \dfrac{1}{\sqrt{2\pi}}e^{-(x-2)^2/2}$

$g'(x) = \dfrac{-1}{\sqrt{2\pi}}(x - 2)e^{-(x-2)^2/2}$

$g''(x) = \dfrac{1}{\sqrt{2\pi}}(x - 1)(x - 3)e^{-(x-2)^2/2}$

Relative maximum: $\left(2, \dfrac{1}{\sqrt{2\pi}}\right) \approx (2, 0.399)$

Points of inflection: $\left(1, \dfrac{1}{\sqrt{2\pi}}e^{-1/2}\right), \left(3, \dfrac{1}{\sqrt{2\pi}}e^{-1/2}\right) \approx (1, 0.242), (3, 0.242)$



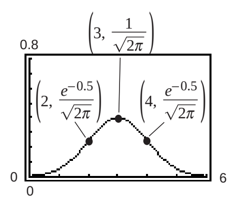
68. $g(x) = \dfrac{1}{\sqrt{2\pi}}e^{-(x-3)^2/2}$

$g'(x) = \dfrac{-1}{\sqrt{2\pi}}(x - 3)e^{-(x-3)^2/2}$

$g''(x) = \dfrac{1}{\sqrt{2\pi}}(x - 2)(x - 4)e^{-(x-3)^2/2}$

Relative maximum: $\left(3, \dfrac{1}{\sqrt{2\pi}}\right) \approx (3, 0.399)$

Points of inflection: $\left(2, \dfrac{1}{\sqrt{2\pi}}e^{-1/2}\right), \left(4, \dfrac{1}{\sqrt{2\pi}}e^{-1/2}\right) \approx (2, 0.242), (4, 0.242)$



69. $f(x) = x^2e^{-x}$

$f'(x) = -x^2e^{-x} + 2xe^{-x} = xe^{-x}(2 - x) = 0$ when $x = 0, 2$.

$f''(x) = -e^{-x}(2x - x^2) + e^{-x}(2 - 2x)$

$= e^{-x}(x^2 - 4x + 2) = 0$ when $x = 2 \pm \sqrt{2}$.

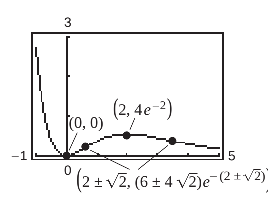
Relative minimum: $(0, 0)$

Relative maximum: $(2, 4e^{-2})$

$x = 2 \pm \sqrt{2}$

$y = \left(2 \pm \sqrt{2}\right)^2e^{-(2\pm\sqrt{2})}$

Points of inflection: $(3.414, 0.384), (0.586, 0.191)$



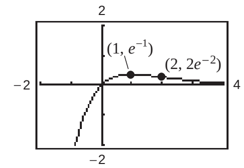
70. $f(x) = xe^{-x}$

$f'(x) = -xe^{-x} + e^{-x} = e^{-x}(1 - x) = 0$ when $x = 1$.

$f''(x) = -e^{-x} + (-e^{-x})(1 - x) = e^{-x}(x - 2) = 0$ when $x = 2$.

Relative maximum: $(1, e^{-1})$

Point of inflection: $(2, 2e^{-2})$



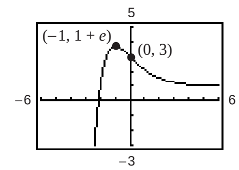
71. $g(t) = 1 + (2 + t)e^{-t}$

$g'(t) = -(1 + t)e^{-t}$

$g''(t) = te^{-t}$

Relative maximum: $(-1, 1 + e) \approx (-1, 3.718)$

Point of inflection: $(0, 3)$



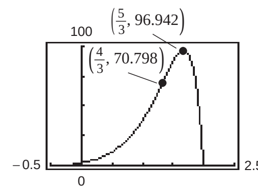
72. $f(x) = -2 + e^{3x}(4 - 2x)$

$f'(x) = e^{3x}(-2) + 3e^{3x}(4 - 2x) = e^{3x}(10 - 6x) = 0$ when $x = \frac{5}{3}$.

$f''(x) = e^{3x}(-6) + 3e^{3x}(10 - 6x) = e^{3x}(24 - 18x) = 0$ when $x = \frac{4}{3}$.

Relative maximum: $\left(\frac{5}{3}, 96.942\right)$

Point of inflection: $\left(\frac{4}{3}, 70.798\right)$

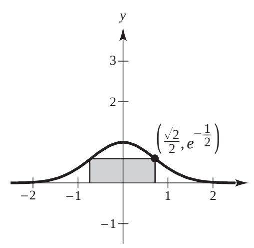


73. $A = \text{(base)(height)} = 2xe^{-x^2}$

$$\frac{dA}{dx} = -4x^2e^{-x^2} + 2e^{-x^2}$$

$$= 2e^{-x^2}(1 - 2x^2) = 0 \text{ when } x = \frac{\sqrt{2}}{2}.$$

$A = \sqrt{2}e^{-1/2}$



74. (a) $f(c) = f(c + x)$

$$10ce^{-c} = 10(c + x)e^{-(c+x)}$$

$$\frac{c}{e^c} = \frac{c + x}{e^{c+x}}$$

$$ce^{c+x} = (c + x)e^c$$

$$ce^x = c + x$$

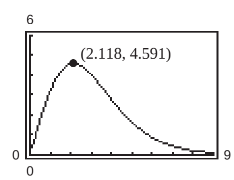
$$ce^x - c = x$$

$$c = \frac{x}{e^x - 1}$$

(b) $A(x) = xf(c) = x\left[10\left(\dfrac{x}{e^x - 1}\right)e^{-(x/(e^x - 1))}\right]$

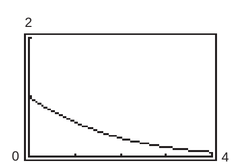
$$= \frac{10x^2}{e^x - 1}e^{x/(1 - e^x)}$$

(c) $A(x) = \dfrac{10x^2}{e^x - 1}e^{x/(1 - e^x)}$



The maximum area is 4.591 for $x = 2.118$ and $f(x) = 2.547$.

(d) $c = \dfrac{x}{e^x - 1}$



$$\lim_{x \to 0^+} c = 1$$

$$\lim_{x \to \infty} c = 0$$

75. $y = \dfrac{L}{1 + ae^{-x/b}}, \quad a > 0, b > 0, L > 0$

$$y' = \frac{-L\left(-\dfrac{a}{b}e^{-x/b}\right)}{(1 + ae^{-x/b})^2} = \frac{\dfrac{aL}{b}e^{-x/b}}{(1 + ae^{-x/b})^2}$$

$$y'' = \frac{(1 + ae^{-x/b})^2\left(\dfrac{-aL}{b^2}e^{-x/b}\right) - \left(\dfrac{aL}{b}e^{-x/b}\right)2(1 + ae^{-x/b})\left(\dfrac{-a}{b}e^{-x/b}\right)}{(1 + ae^{-x/b})^4}$$

$$= \frac{(1 + ae^{-x/b})\left(\dfrac{-aL}{b^2}e^{-x/b}\right) + 2\left(\dfrac{aL}{b}e^{-x/b}\right)\left(\dfrac{a}{b}e^{-x/b}\right)}{(1 + ae^{-x/b})^3}$$

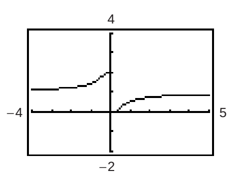
$$= \frac{Lae^{-x/b}[ae^{-x/b} - 1]}{(1 + ae^{-x/b})^3 b^2}$$

$$y'' = 0 \text{ if } ae^{-x/b} = 1 \Longrightarrow \frac{-x}{b} = \ln\left(\frac{1}{a}\right) \Longrightarrow x = b \ln a$$

$$y(b \ln a) = \frac{L}{1 + ae^{-(b \ln a)/b}} = \frac{L}{1 + a(1/a)} = \frac{L}{2}$$

Therefore, the y-coordinate of the inflection point is $L/2$.

76. (a)



(b) When x increases without bound, $1/x$ approaches zero, and $e^{1/x}$ approaches 1. Therefore, $f(x)$ approaches $2/(1 + 1) = 1$. Thus, $f(x)$ has a horizontal asymptote at $y = 1$. As x approaches zero from the right, $1/x$ approaches ∞, $e^{1/x}$ approaches ∞ and $f(x)$ approaches zero. As x approaches zero from the left, $1/x$ approaches $-\infty$, $e^{1/x}$ approaches zero, and $f(x)$ approaches 2. The limit does not exist since the left limit does not equal the right limit. Therefore, $x = 0$ is a nonremovable discontinuity.

77. $f(x) = e^{2x}$

$f'(x) = 2e^{2x}$

Let $(x, y) = (x, e^{2x})$ be the point on the graph where the tangent line passes through the origin. Equating slopes,

$$2e^{2x} = \frac{e^{2x} - 0}{x - 0}$$

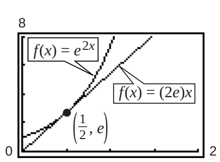
$$2 = \frac{1}{x}$$

$$x = \frac{1}{2}, y = e, y' = 2e.$$

Point: $\left(\dfrac{1}{2}, e\right)$

Tangent line: $y - e = 2e\left(x - \dfrac{1}{2}\right)$

$$y = 2ex$$



78. Let (x_0, y_0) be the desired point on $y = e^{-x}$.

$y = e^{-x}$

$y' = -e^{-x}$ (Slope of tangent line)

$-\dfrac{1}{y'} = e^x$ (Slope of normal line)

$y - e^{-x_0} = e^{x_0}(x - x_0)$

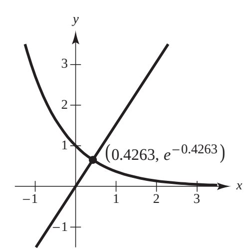
We want $(0, 0)$ to satisfy the equation:

$-e^{-x_0} = -x_0 e^{x_0}$

$1 = x_0 e^{2x_0}$

$x_0 e^{2x_0} - 1 = 0$

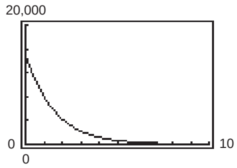
Solving by Newton's Method or using a computer, the solution is $x_0 \approx 0.4263$.



$(0.4263, e^{-0.4263})$

79. $V = 15{,}000e^{-0.6286t}, \quad 0 \le t \le 10$

(a)

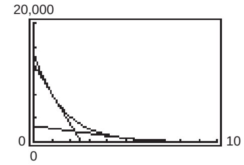


(b) $\dfrac{dV}{dt} = -9429e^{-0.6286t}$

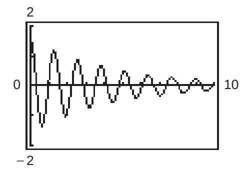
When $t = 1$, $\dfrac{dV}{dt} \approx -5028.84$.

When $t = 5$, $\dfrac{dV}{dt} \approx -406.89$.

(c)



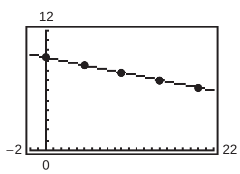
80. $1.56e^{-0.22t} \cos 4.9t \le 0.25$ (3 inches equals one-fourth foot.) Using a graphing utility or Newton's Method, we have $t \ge 7.79$ seconds.



81.

h	0	5	10	15	20
P	10,332	5583	2376	1240	517
$\ln P$	9.243	8.627	7.773	7.123	6.248

(a)



$y = -0.1499h + 9.3018$ is the regression line for data $(h, \ln P)$.

(b) $\ln P = ah + b$

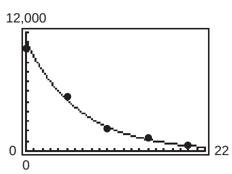
$P = e^{ah+b} = e^b e^{ah}$

$P = Ce^{ah}, C = e^b$

For our data, $a = -0.1499$ and $C = e^{9.3018} = 10{,}957.7$.

$P = 10{,}957.7e^{-0.1499h}$

(c)



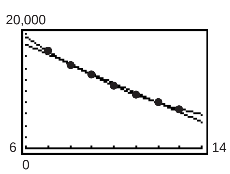
(d) $\dfrac{dP}{dh} = (10{,}957.71)(-0.1499)e^{-0.1499h}$

$= -1642.56e^{-0.1499h}$

For $h = 5$, $\dfrac{dP}{dh} = -776.3$. For $h = 18$, $\dfrac{dP}{dh} \approx -110.6$.

82. (a) Linear model: $V = -1686.8t + 28{,}242$

Quadratic model: $V = 109.52t^2 - 3877.3t + 38{,}756$



(b) The slope represents the average loss in value per year.

(c) Exponential model: $V = 49{,}591.06(0.8592)^t$

$= 49{,}591.06e^{-0.1518t}$

(d) As $t \to \infty$, $V \to 0$ for the exponential model. The value tends to zero.

(e) When $t = 8$, $V' \approx -2235$ dollars/year.

When $t = 12$, $V' \approx -1218$ dollars/year.

83. $f(x) = e^{x/2}, \qquad f(0) = 1$

$f'(x) = \frac{1}{2}e^{x/2}, \qquad f'(0) = \frac{1}{2}$

$f''(x) = \frac{1}{4}e^{x/2}, \qquad f''(0) = \frac{1}{4}$

$P_1(x) = 1 + \frac{1}{2}(x - 0) = \frac{x}{2} + 1, \qquad P_1(0) = 1$

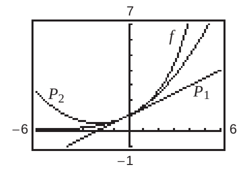
$P_1'(x) = \frac{1}{2}, \qquad P_1'(0) = \frac{1}{2}$

$P_2(x) = 1 + \frac{1}{2}(x - 0) + \frac{1}{8}(x - 0)^2 \qquad P_2(0) = 1$

$= \frac{x^2}{8} + \frac{x}{2} + 1$

$P_2'(x) = \frac{1}{4}x + \frac{1}{2}, \qquad P_2'(0) = \frac{1}{2}$

$P_2''(x) = \frac{1}{4}, \qquad P_2''(0) = \frac{1}{4}$



The values of f, P_1, P_2 and their first derivatives agree at $x = 0$. The values of the second derivatives of f and P_2 agree at $x = 0$.

84. $f(x) = e^{-x^2/2}, \qquad f(0) = 1$

$f'(x) = -xe^{-x^2/2}, \qquad f'(0) = 0$

$f''(x) = x^2e^{-x^2/2} - e^{-x^2/2} = e^{-x^2/2}(x^2 - 1), \qquad f''(0) = -1$

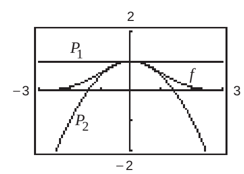
$P_1(x) = 1 + 0(x - 0) = 1, \qquad P_1(0) = 1$

$P_1'(x) = 0, \qquad P_1'(0) = 0$

$P_2(x) = 1 + 0(x - 0) - \frac{1}{2}(x - 0)^2 = 1 - \frac{x^2}{2}, \qquad P_2(0) = 1$

$P_2'(x) = -x, \qquad P_2'(0) = 0$

$P_2''(x) = -1, \qquad P_2''(0) = -1$



The values of f, P_1, P_2 and their first derivatives agree at $x = 0$. The values of the second derivatives of f and P_2 agree at $x = 0$.

85. Let $u = 5x$, $du = 5\,dx$.

$$\int e^{5x}(5)\,dx = e^{5x} + C$$

86. Let $u = -x^4$, $du = -4x^3\,dx$.

$$\int e^{-x^4}(-4x^3)\,dx = e^{-x^4} + C$$

87. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\right)dx = 2e^{\sqrt{x}} + C$

88. $\displaystyle\int \frac{e^{1/x^2}}{x^3}\,dx = -\frac{1}{2}\int e^{1/x^2}\left(\frac{-2}{x^3}\right)dx = -\frac{1}{2}e^{1/x^2} + C$

89. Let $u = 1 + e^{-x}$, $du = -e^{-x}\,dx$.

$$\int \frac{e^{-x}}{1 + e^{-x}}\,dx = -\int \frac{-e^{-x}}{1 + e^{-x}}\,dx = -\ln(1 + e^{-x}) + C = \ln\left(\frac{e^x}{e^x + 1}\right) + C = x - \ln(e^x + 1) + C$$

90. Let $u = 1 + e^{2x}$, $du = 2e^{2x}\,dx$.

$$\int \frac{e^{2x}}{1 + e^{2x}}\,dx = \frac{1}{2}\int \frac{2e^{2x}}{1 + e^{2x}}\,dx = \frac{1}{2}\ln(1 + e^{2x}) + C$$

91. Let $u = 1 - e^x$, $du = -e^x\,dx$.

$$\int e^x\sqrt{1 - e^x}\,dx = -\int (1 - e^x)^{1/2}(-e^x)\,dx$$
$$= -\frac{2}{3}(1 - e^x)^{3/2} + C$$

92. Let $u = e^x + e^{-x}$, $du = (e^x - e^{-x})\,dx$.

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \ln(e^x + e^{-x}) + C$$

93. Let $u = e^x - e^{-x}$, $du = (e^x + e^{-x})\,dx$.

$$\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx = \ln\left|e^x - e^{-x}\right| + C$$

94. Let $u = e^x + e^{-x}$, $du = (e^x - e^{-x})\,dx$.

$$\int \frac{2e^x - 2e^{-x}}{(e^x + e^{-x})^2}\,dx = 2\int (e^x + e^{-x})^{-2}(e^x - e^{-x})\,dx$$
$$= \frac{-2}{e^x + e^{-x}} + C$$

95. $$\int \frac{5 - e^x}{e^{2x}}\,dx = \int 5e^{-2x}\,dx - \int e^{-x}\,dx$$
$$= -\frac{5}{2}e^{-2x} + e^{-x} + C$$

96. $$\int \frac{e^{2x} + 2e^x + 1}{e^x}\,dx = \int (e^x + 2 + e^{-x})\,dx$$
$$= e^x + 2x - e^{-x} + C$$

97. $$\int e^{-x}\tan(e^{-x})\,dx = -\int [\tan(e^{-x})](-e^{-x})\,dx$$
$$= \ln\left|\cos(e^{-x})\right| + C$$

98. $$\int \ln(e^{2x-1})\,dx = \int (2x - 1)\,dx$$
$$= x^2 - x + C$$

99. Let $u = -2x$, $du = -2\,dx$.

$$\int_0^1 e^{-2x}\,dx = -\frac{1}{2}\int_0^1 e^{-2x}(-2)\,dx = \left[-\frac{1}{2}e^{-2x}\right]_0^1$$
$$= \frac{1}{2}(1 - e^{-2}) = \frac{e^2 - 1}{2e^2}$$

100. $$\int_3^4 e^{3-x}\,dx = \left[-e^{3-x}\right]_3^4 = -e^{-1} + 1 = 1 - \frac{1}{e}$$

101. $$\int_0^1 xe^{-x^2}\,dx = -\frac{1}{2}\int_0^1 e^{-x^2}(-2x)\,dx$$
$$= -\frac{1}{2}\left[e^{-x^2}\right]_0^1$$
$$= -\frac{1}{2}[e^{-1} - 1]$$
$$= \frac{1 - (1/e)}{2} = \frac{e - 1}{2e}$$

102. $$\int_{-2}^0 x^2e^{x^3/2}\,dx = \frac{2}{3}\int_{-2}^0 e^{x^3/2}\left(\frac{3}{2}x^2\right)dx$$
$$= \frac{2}{3}\left[e^{x^3/2}\right]_{-2}^0$$
$$= \frac{2}{3}[1 - e^{-4}]$$
$$= \frac{2}{3}\left[1 - \frac{1}{e^4}\right] = \frac{2(e^4 - 1)}{3e^4}$$

103. Let $u = \dfrac{3}{x}$, $du = -\dfrac{3}{x^2}\,dx$.

$$\int_1^3 \frac{e^{3/x}}{x^2}\,dx = -\frac{1}{3}\int_1^3 e^{3/x}\left(-\frac{3}{x^2}\right)dx$$
$$= \left[-\frac{1}{3}e^{3/x}\right]_1^3 = \frac{e}{3}(e^2 - 1)$$

104. Let $u = \dfrac{-x^2}{2}$, $du = -x\,dx$.

$$\int_0^{\sqrt{2}} xe^{-x^2/2}\,dx = -\int_0^{\sqrt{2}} e^{-x^2/2}(-x)\,dx$$
$$= \left[-e^{-x^2/2}\right]_0^{\sqrt{2}} = 1 - e^{-1} = \frac{e - 1}{e}$$

105. $\displaystyle\int_0^{\pi/2} e^{\sin \pi x}\cos \pi x\,dx = \frac{1}{\pi}\int_0^{\pi/2} e^{\sin \pi x}(\pi\cos \pi x)\,dx$

$$= \frac{1}{\pi}\Big[e^{\sin \pi x}\Big]_0^{\pi/2}$$

$$= \frac{1}{\pi}\left[e^{\sin(\pi^2/2)} - 1\right]$$

106. $\displaystyle\int_{\pi/3}^{\pi/2} e^{\sec 2x}\sec 2x\tan 2x\,dx = \frac{1}{2}\int_{\pi/3}^{\pi/2} e^{\sec 2x}(2\sec 2x\tan 2x)\,dx$

$$= \frac{1}{2}\Big[e^{\sec 2x}\Big]_{\pi/3}^{\pi/2}$$

$$= \frac{1}{2}\left[e^{-1} - e^{-2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{e} - \frac{1}{e^2}\right] = \frac{e-1}{2e^2}$$

107. Let $u = ax^2$, $du = 2ax\,dx$. (Assume $a \neq 0$.)

$$y = \int xe^{ax^2}\,dx$$

$$= \frac{1}{2a}\int e^{ax^2}(2ax)\,dx = \frac{1}{2a}e^{ax^2} + C$$

108. $\displaystyle y = \int (e^x - e^{-x})^2\,dx$

$$= \int (e^{2x} - 2 + e^{-2x})\,dx$$

$$= \frac{1}{2}e^{2x} - 2x - \frac{1}{2}e^{-2x} + C$$

109. $\displaystyle f'(x) = \int \frac{1}{2}(e^x + e^{-x})\,dx = \frac{1}{2}(e^x - e^{-x}) + C_1$

$$f'(0) = C_1 = 0$$

$$f(x) = \int \frac{1}{2}(e^x - e^{-x})\,dx = \frac{1}{2}(e^x + e^{-x}) + C_2$$

$$f(0) = 1 + C_2 = 1 \implies C_2 = 0$$

$$f(x) = \frac{1}{2}(e^x + e^{-x})$$

110. $\displaystyle f'(x) = \int (\sin x + e^{2x})\,dx = -\cos x + \frac{1}{2}e^{2x} + C_1$

$$f'(0) = -1 + \frac{1}{2} + C_1 = \frac{1}{2} \implies C_1 = 1$$

$$f'(x) = -\cos x + \frac{1}{2}e^{2x} + 1$$

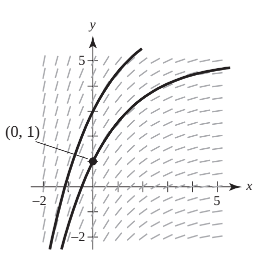
$$f(x) = \int\left(-\cos x + \frac{1}{2}e^{2x} + 1\right)dx$$

$$= -\sin x + \frac{1}{4}e^{2x} + x + C_2$$

$$f(0) = \frac{1}{4} + C_2 = \frac{1}{4} \implies C_2 = 0$$

$$f(x) = x - \sin x + \frac{1}{4}e^{2x}$$

111. (a)



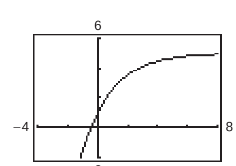
(b) $\displaystyle\frac{dy}{dx} = 2e^{-x/2}$, $(0, 1)$

$$y = \int 2e^{-x/2}\,dx = -4\int e^{-x/2}\left(-\frac{1}{2}\,dx\right)$$

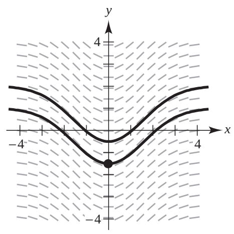
$$= -4e^{-x/2} + C$$

$(0, 1)$: $1 = -4e^0 + C = -4 + C \implies C = 5$

$$y = -4e^{-x/2} + 5$$



112. (a)



(b) $\dfrac{dy}{dx} = xe^{-0.2x^2}, \quad \left(0, -\dfrac{3}{2}\right)$

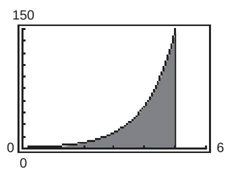
$$y = \int xe^{-0.2x^2}\,dx = \frac{1}{-0.4}\int e^{-0.2x^2}(-0.4x)\,dx$$

$$= -\frac{1}{0.4}e^{-0.2x^2} + C = -2.5e^{-0.2x^2} + C$$

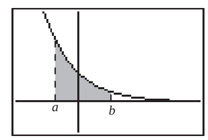
$$\left(0, -\frac{3}{2}\right): \; -\frac{3}{2} = -2.5e^0 + C = -2.5 + C \implies C = 1$$

$$y = -2.5e^{-0.2x^2} + 1$$

113. $\displaystyle\int_0^5 e^x\,dx = \Big[e^x\Big]_0^5 = e^5 - 1 \approx 147.413$

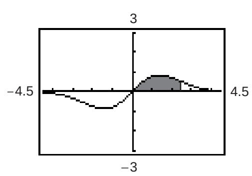


114. $\displaystyle\int_a^b e^{-x}\,dx = \Big[-e^{-x}\Big]_a^b = e^{-a} - e^{-b}$



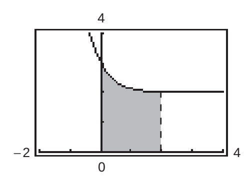
115. $\displaystyle\int_0^{\sqrt{6}} xe^{-x^2/4}\,dx = \Big[-2e^{-x^2/4}\Big]_0^{\sqrt{6}}$

$$= -2e^{-3/2} + 2 \approx 1.554$$



116. $\displaystyle\int_0^2 (e^{-2x} + 2)\,dx = \Big[-\frac{1}{2}e^{-2x} + 2x\Big]_0^2$

$$= -\frac{1}{2}e^{-4} + 4 + \frac{1}{2} \approx 4.491$$



117. $\displaystyle\int_0^4 \sqrt{x}\,e^x\,dx,\; n = 12$

Midpoint Rule: 92.1898

Trapezoidal Rule: 93.8371

Simpson's Rule: 92.7385

Graphing utility: 92.7437

118. $\displaystyle\int_0^2 2xe^{-x}\,dx,\; n = 12$

Midpoint Rule: 1.1906

Trapezoidal Rule: 1.1827

Simpson's Rule: 1.1880

Graphing utility: 1.18799

119. $\displaystyle 0.0665\int_{48}^{60} e^{-0.0139(t-48)^2}\,dt$

Graphing utility: $0.4772 = 47.72\%$

120. $\displaystyle\int_0^x 0.3^{-0.3t}\,dt = \frac{1}{2}$

$$\Big[-e^{-0.3t}\Big]_0^x = \frac{1}{2}$$

$$-e^{-0.3x} + 1 = \frac{1}{2}$$

$$e^{-0.3x} = \frac{1}{2}$$

$$-0.3x = \ln\frac{1}{2} = -\ln 2$$

$$x = \frac{\ln 2}{0.3} \approx 2.31 \text{ minutes}$$

121. $\int_0^x e^t\,dt \geq \int_0^x 1\,dt$

$\left[e^t\right]_0^x \geq \left[t\right]_0^x$

$e^x - 1 \geq x \implies e^x \geq 1 + x$ for $x \geq 0$

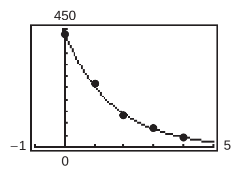
122.

t	0	1	2	3	4
R	425	240	118	71	36
$\ln R$	6.052	5.481	4.771	4.263	3.584

(a) $\ln R = -0.6155t + 6.0609$

$R = e^{-0.6155t + 6.0609} = 428.78e^{-0.6155t}$

(b)



(c) $\int_0^4 R(t)\,dt = \int_0^4 428.78e^{-0.6155t}\,dt$

≈ 637.2 liters

123. $f(x) = e^x$. Domain is $(-\infty, \infty)$ and range is $(0, \infty)$. f is continuous, increasing, one-to-one, and concave upwards on its entire domain.

$\lim_{x\to-\infty} e^x = 0$ and $\lim_{x\to\infty} e^x = \infty$.

124. The graphs of $f(x) = \ln x$ and $g(x) = e^x$ are mirror images across the line $y = x$.

125. Yes. $f(x) = Ce^x$, C a constant.

126. (a) Log Rule: $(u = e^x + 1)$

(b) Substitution: $(u = x^2)$

127. $e^{-x} = x \implies f(x) = x - e^{-x}$

$f'(x) = 1 + e^{-x}$

$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = x_n - \dfrac{x_n - e^{-x_n}}{1 + e^{-x_n}}$

$x_1 = 1$

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} \approx 0.5379$

$x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)} \approx 0.5670$

$x_4 = x_3 - \dfrac{f(x_3)}{f'(x_3)} \approx 0.5671$

We approximate the root of f to be $x = 0.567$.

128. Area $= \dfrac{8}{3} = \int_{-a}^{a} e^{-x}\,dx = -e^{-x}\Big]_{-a}^{a} = -e^{-a} + e^a$

Let $z = e^a$:

$\dfrac{8}{3} = \dfrac{-1}{z} + z$

$\dfrac{8}{3}z = -1 + z^2$

$3z^2 - 8z - 3 = 0$

$(3z + 1)(z - 3) = 0$

$z = 3 \implies e^a = 3 \implies a = \ln 3$

$\left(z = -\dfrac{1}{3} \implies e^a = -\dfrac{1}{3} \text{ impossible}\right)$

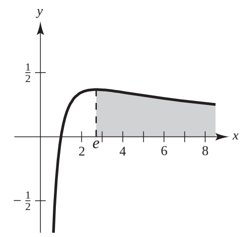
Answer: $a = \ln 3$

129. $\ln \dfrac{e^a}{e^b} = \ln e^a - \ln e^b = a - b$

$\ln e^{a-b} = a - b$

Therefore, $\ln \dfrac{e^a}{e^b} = \ln e^{a-b}$ and since $y = \ln x$ is one-to-one, we have $\dfrac{e^a}{e^b} = e^{a-b}$.

130. $f(x) = \dfrac{\ln x}{x}$



(a) $f'(x) = \dfrac{1 - \ln x}{x^2} = 0$ when $x = e$.

On $(0, e)$, $f'(x) > 0 \implies f$ is increasing.

On (e, ∞), $f'(x) < 0 \implies f$ is decreasing.

(b) For $e \le A < B$, we have

$$\frac{\ln A}{A} > \frac{\ln B}{B}$$

$$B \ln A > A \ln B$$

$$\ln A^B > \ln B^A$$

$$A^B > B^A.$$

(c) Since $e < \pi$, from part (b) we have $e^\pi > \pi^e$.

Section 5.5 Bases Other than *e* and Applications

1. $\log_2 \frac{1}{8} = \log_2 2^{-3} = -3$

2. $\log_{27} 9 = \log_{27} 27^{2/3} = \frac{2}{3}$

3. $\log_7 1 = 0$

4. $\log_a \dfrac{1}{a} = \log_a 1 - \log_a a = -1$

5. (a) $2^3 = 8$

$\log_2 8 = 3$

(b) $3^{-1} = \dfrac{1}{3}$

$\log_3 \dfrac{1}{3} = -1$

6. (a) $27^{2/3} = 9$

$\log_{27} 9 = \dfrac{2}{3}$

(b) $16^{3/4} = 8$

$\log_{16} 8 = \dfrac{3}{4}$

7. (a) $\log_{10} 0.01 = -2$

$10^{-2} = 0.01$

(b) $\log_{0.5} 8 = -3$

$0.5^{-3} = 8$

$\left(\frac{1}{2}\right)^{-3} = 8$

8. (a) $\log_3 \frac{1}{9} = -2$

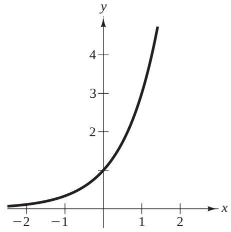
$3^{-2} = \frac{1}{9}$

(b) $49^{1/2} = 7$

$\log_{49} 7 = \frac{1}{2}$

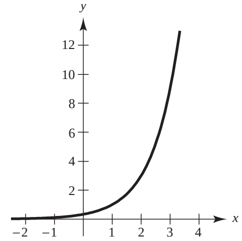
9. $y = 3^x$

x	-2	-1	0	1	2
y	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9



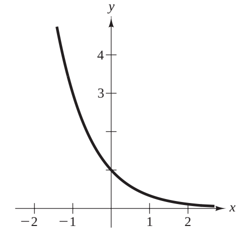
10. $y = 3^{x-1}$

x	-1	0	1	2	3
y	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9



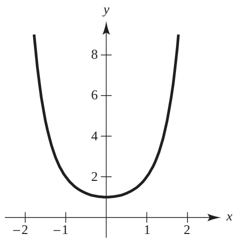
11. $y = \left(\frac{1}{3}\right)^x = 3^{-x}$

x	-2	-1	0	1	2
y	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$



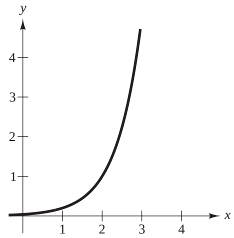
12. $y = 2^{x^2}$

x	-2	-1	0	1	2
y	16	2	1	2	16



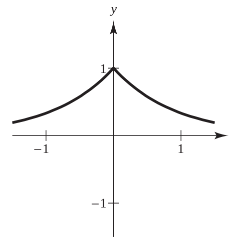
13. $h(x) = 5^{x-2}$

x	-1	0	1	2	3
y	$\frac{1}{125}$	$\frac{1}{25}$	$\frac{1}{5}$	1	5



14. $y = 3^{-|x|}$

x	0	± 1	± 2
y	1	$\frac{1}{3}$	$\frac{1}{9}$



15. (a) $\log_{10} 1000 = x$

$10^x = 1000$

$x = 3$

(b) $\log_{10} 0.1 = x$

$10^x = 0.1$

$x = -1$

16. (a) $\log_3 \frac{1}{81} = x$

$3^x = \frac{1}{81}$

$x = -4$

(b) $\log_6 36 = x$

$6^x = 36$

$x = 2$

17. (a) $\log_3 x = -1$

$3^{-1} = x$

$x = \frac{1}{3}$

(b) $\log_2 x = -4$

$2^{-4} = x$

$x = \frac{1}{16}$

18. (a) $\log_b 27 = 3$

$b^3 = 27$

$b = 3$

(b) $\log_b 125 = 3$

$b^3 = 125$

$b = 5$

19. (a)

$$\begin{aligned} x^2 - x &= \log_5 25 \\ x^2 - x &= \log_5 5^2 = 2 \\ x^2 - x - 2 &= 0 \\ (x+1)(x-2) &= 0 \\ x = -1 \quad &\text{OR} \quad x = 2 \end{aligned}$$

(b)

$$\begin{aligned} 3x + 5 &= \log_2 64 \\ 3x + 5 &= \log_2 2^6 = 6 \\ 3x &= 1 \\ x &= \tfrac{1}{3} \end{aligned}$$

20. (a)

$$\begin{aligned} \log_3 x + \log_3(x-2) &= 1 \\ \log_3[x(x-2)] &= 1 \\ x(x-2) &= 3^1 \\ x^2 - 2x - 3 &= 0 \\ (x+1)(x-3) &= 0 \\ x = -1 \quad &\text{OR} \quad x = 3 \end{aligned}$$

$x = 3$ is the only solution since the domain of the logarithmic function is the set of all *positive* real numbers.

(b)

$$\begin{aligned} \log_{10}(x+3) - \log_{10} x &= 1 \\ \log_{10} \frac{x+3}{x} &= 1 \\ \frac{x+3}{x} &= 10^1 \\ x + 3 &= 10x \\ 3 &= 9x \\ x &= \frac{1}{3} \end{aligned}$$

21.

$$\begin{aligned} 3^{2x} &= 75 \\ 2x \ln 3 &= \ln 75 \\ x &= \left(\frac{1}{2}\right)\frac{\ln 75}{\ln 3} \approx 1.965 \end{aligned}$$

22.

$$\begin{aligned} 5^{6x} &= 8320 \\ 6x \ln 5 &= \ln 8320 \\ x &= \frac{\ln 8320}{6 \ln 5} \approx 0.935 \end{aligned}$$

23.
$$\begin{aligned} 2^{3-z} &= 625 \\ (3-z)\ln 2 &= \ln 625 \\ 3-z &= \frac{\ln 625}{\ln 2} \\ z &= 3 - \frac{\ln 625}{\ln 2} \approx -6.288 \end{aligned}$$

24.
$$\begin{aligned} 3(5^{x-1}) &= 86 \\ 5^{x-1} &= \frac{86}{3} \\ (x-1)\ln 5 &= \ln\left(\frac{86}{3}\right) \\ x-1 &= \frac{\ln(86/3)}{\ln 5} \\ x &= 1 + \frac{\ln(86/3)}{\ln 5} \approx 3.085 \end{aligned}$$

25.
$$\begin{aligned} \left(1+\frac{0.09}{12}\right)^{12t} &= 3 \\ 12t\ln\left(1+\frac{0.09}{12}\right) &= \ln 3 \\ t &= \left(\frac{1}{12}\right)\frac{\ln 3}{\ln\left(1+\frac{0.09}{12}\right)} \approx 12.253 \end{aligned}$$

26.
$$\begin{aligned} \left(1+\frac{0.10}{365}\right)^{365t} &= 2 \\ 365t\ln\left(1+\frac{0.10}{365}\right) &= \ln 2 \\ t &= \left(\frac{1}{365}\right)\frac{\ln 2}{\ln\left(1+\frac{0.10}{365}\right)} \approx 6.932 \end{aligned}$$

27.
$$\begin{aligned} \log_2(x-1) &= 5 \\ x-1 &= 2^5 = 32 \\ x &= 33 \end{aligned}$$

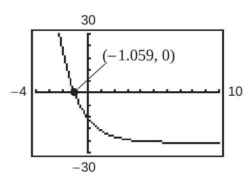
28.
$$\begin{aligned} \log_{10}(t-3) &= 2.6 \\ t-3 &= 10^{2.6} \\ t &= 3 + 10^{2.6} \approx 401.107 \end{aligned}$$

29.
$$\begin{aligned} \log_3 x^2 &= 4.5 \\ x^2 &= 3^{4.5} \\ x &= \pm\sqrt{3^{4.5}} \approx \pm 11.845 \end{aligned}$$

30.
$$\begin{aligned} \log_5 \sqrt{x-4} &= 3.2 \\ \sqrt{x-4} &= 5^{3.2} \\ x-4 &= (5^{3.2})^2 = 5^{6.4} \\ x &= 4 + 5^{6.4} \\ &\approx 29{,}748.593 \end{aligned}$$

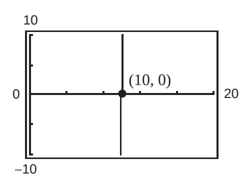
31. $g(x) = 6(2^{1-x}) - 25$

Zero: $x \approx -1.059$



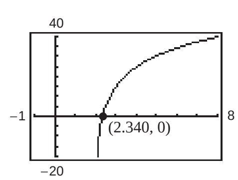
32. $f(t) = 300(1.0075^{12t}) - 735.41$

Zero: $t \approx 10$



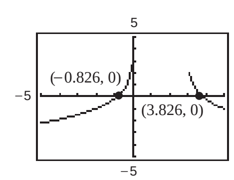
33. $h(s) = 32\log_{10}(s-2) + 15$

Zero: $s \approx 2.340$



34. $g(x) = 1 - 2\log_{10}[x(x-3)]$

Zeros: $x \approx -0.826, 3.826$

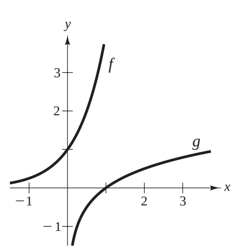


35. $f(x) = 4^x$

$g(x) = \log_4 x$

x	-2	-1	0	$\frac{1}{2}$	1
$f(x)$	$\frac{1}{16}$	$\frac{1}{4}$	1	2	4

x	$\frac{1}{16}$	$\frac{1}{4}$	1	2	4
$g(x)$	-2	-1	0	$\frac{1}{2}$	1

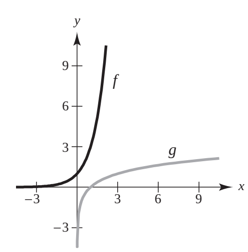


36. $f(x) = 3^x$

$g(x) = \log_3 x$

x	-2	-1	0	1	2
$f(x)$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
$g(x)$	-2	-1	0	1	2



37. $f(x) = 4^x$

$f'(x) = (\ln 4)4^x$

38. $y = x(6^{-2x})$

$$\frac{dy}{dx} = x[-2(\ln 6)6^{-2x}) + 6^{-2x}$$
$$= 6^{-2x}[-2x(\ln 6) + 1]$$
$$= 6^{-2x}(1 - 2x\ln 6)$$

39. $g(t) = t^2 2^t$

$$g'(t) = t^2(\ln 2)2^t + (2t)2^t$$
$$= t2^t(t\ln 2 + 2)$$
$$= 2^t t(2 + t\ln 2)$$

40. $f(t) = \dfrac{3^{2t}}{t}$

$$f'(t) = \frac{t(2\ln 3)3^{2t} - 3^{2t}}{t^2}$$
$$= \frac{3^{2t}(2t\ln 3 - 1)}{t^2}$$

41. $h(\theta) = 2^{-\theta}\cos \pi\theta$

$$h'(\theta) = 2^{-\theta}(-\pi\sin\pi\theta) - (\ln 2)2^{-\theta}\cos\pi\theta$$
$$= -2^{-\theta}[(\ln 2)\cos\pi\theta + \pi\sin\pi\theta]$$

42. $g(\alpha) = 5^{-\alpha/2}\sin 2\alpha$

$$g'(\alpha) = 5^{-\alpha/2}2\cos 2\alpha - \frac{1}{2}(\ln 5)5^{-\alpha/2}\sin 2\alpha$$

43. $f(x) = \log_2 \dfrac{x^2}{x-1}$

$$= 2\log_2 x - \log_2(x-1)$$
$$f'(x) = \frac{2}{x\ln 2} - \frac{1}{(x-1)\ln 2}$$
$$= \frac{x-2}{(\ln 2)x(x-1)}$$

44. $h(x) = \log_3 \dfrac{x\sqrt{x-1}}{2}$

$$= \log_3 x + \frac{1}{2}\log_3(x-1) - \log_3 2$$
$$h'(x) = \frac{1}{x\ln 3} + \frac{1}{2}\cdot\frac{1}{(x-1)\ln 3} - 0$$
$$= \frac{1}{\ln 3}\left[\frac{1}{x} + \frac{1}{2(x-1)}\right]$$
$$= \frac{1}{\ln 3}\left[\frac{3x-2}{2x(x-1)}\right]$$

45. $y = \log_5\sqrt{x^2-1} = \dfrac{1}{2}\log_5(x^2-1)$

$$\frac{dy}{dx} = \frac{1}{2}\cdot\frac{2x}{(x^2-1)\ln 5} = \frac{x}{(x^2-1)\ln 5}$$

46. $y = \log_{10}\dfrac{x^2-1}{x}$

$$= \log_{10}(x^2-1) - \log_{10} x$$
$$\frac{dy}{dx} = \frac{2x}{(x^2-1)\ln 10} - \frac{1}{x\ln 10}$$
$$= \frac{1}{\ln 10}\left[\frac{2x}{x^2-1} - \frac{1}{x}\right]$$
$$= \frac{1}{\ln 10}\left[\frac{x^2+1}{x(x^2-1)}\right]$$

47. $g(t) = \dfrac{10\log_4 t}{t} = \dfrac{10}{\ln 4}\left(\dfrac{\ln t}{t}\right)$

$$g'(t) = \frac{10}{\ln 4}\left[\frac{t(1/t) - \ln t}{t^2}\right]$$
$$= \frac{10}{t^2\ln 4}[1 - \ln t]$$
$$= \frac{5}{t^2\ln 2}(1 - \ln t)$$

48. $f(t) = t^{3/2}\log_2\sqrt{t+1} = t^{3/2}\dfrac{1}{2}\dfrac{\ln(t+1)}{\ln 2}$

$$f'(t) = \frac{1}{2\ln 2}\left[t^{3/2}\frac{1}{t+1} + \frac{3}{2}t^{1/2}\ln(t+1)\right]$$

49. $y = 2^{-x}, \quad (-1, 2)$

$y' = -2^{-x}\ln(2)$

At $(-1, 2)$, $y' = -2\ln(2)$.

Tangent line: $y - 2 = -2\ln(2)(x + 1)$

$$y = [-2\ln(2)]x + 2 - 2\ln(2)$$

50. $y = 5^{x-2}, \quad (2, 1)$

$y' = 5^{x-2}\ln 5$

At $(2, 1)$, $y' = \ln 5$.

Tangent line: $y - 1 = \ln(5)(x - 2)$

$$y = [\ln(5)]x + 1 - 2\ln(5)$$

51. $y = \log_3 x, \quad (27, 3)$

$y' = \dfrac{1}{x\ln 3}$

At $(27, 3)$, $y' = \dfrac{1}{27\ln 3}$.

Tangent line: $y - 3 = \dfrac{1}{27\ln 3}(x - 27)$

$$y = \frac{1}{27\ln 3}x + 3 - \frac{1}{\ln 3}$$

52. $y = \log_{10}(2x), \quad (5, 1)$

$y' = \dfrac{1}{x\ln 10}$

At $(5, 1)$, $y' = \dfrac{1}{5\ln 10}$.

Tangent line: $y - 1 = \dfrac{1}{5\ln 10}(x - 5)$

$$y = \frac{1}{5\ln 10}x + 1 - \frac{1}{\ln 10}$$

53.
$$\begin{aligned}
y &= x^{2/x}\\
\ln y &= \frac{2}{x}\ln x\\
\frac{1}{y}\left(\frac{dy}{dx}\right) &= \frac{2}{x}\left(\frac{1}{x}\right) + \ln x\left(-\frac{2}{x^2}\right) = \frac{2}{x^2}(1 - \ln x)\\
\frac{dy}{dx} &= \frac{2y}{x^2}(1 - \ln x) = 2x^{(2/x)-2}(1 - \ln x)
\end{aligned}$$

54.
$$\begin{aligned}
y &= x^{x-1}\\
\ln y &= (x - 1)(\ln x)\\
\frac{1}{y}\left(\frac{dy}{dx}\right) &= (x - 1)\left(\frac{1}{x}\right) + \ln x\\
\frac{dy}{dx} &= y\left[\frac{x - 1}{x} + \ln x\right]\\
&= x^{x-2}(x - 1 + x\ln x)
\end{aligned}$$

55.
$$\begin{aligned}
y &= (x - 2)^{x+1}\\
\ln y &= (x + 1)\ln(x - 2)\\
\frac{1}{y}\left(\frac{dy}{dx}\right) &= (x + 1)\left(\frac{1}{x - 2}\right) + \ln(x - 2)\\
\frac{dy}{dx} &= y\left[\frac{x + 1}{x - 2} + \ln(x - 2)\right]\\
&= (x - 2)^{x+1}\left[\frac{x + 1}{x - 2} + \ln(x - 2)\right]
\end{aligned}$$

56.
$$\begin{aligned}
y &= (1 + x)^{1/x}\\
\ln y &= \frac{1}{x}\ln(1 + x)\\
\frac{1}{y}\left(\frac{dy}{dx}\right) &= \frac{1}{x}\left(\frac{1}{1 + x}\right) + \ln(1 + x)\left(-\frac{1}{x^2}\right)\\
\frac{dy}{dx} &= \frac{y}{x}\left[\frac{1}{x + 1} - \frac{\ln(x + 1)}{x}\right]\\
&= \frac{(1 + x)^{1/x}}{x}\left[\frac{1}{x + 1} - \frac{\ln(x + 1)}{x}\right]
\end{aligned}$$

57. $y = x^{\sin x}, \quad \left(\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

$\ln y = \sin x \ln x$

$\dfrac{y'}{y} = \dfrac{\sin x}{x} + \cos x \ln x$

At $\left(\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$: $\dfrac{y'}{(\pi/2)} = \dfrac{1}{(\pi/2)} + 0$

$$y' = 1$$

Tangent line: $y - \dfrac{\pi}{2} = 1\left(x - \dfrac{\pi}{2}\right)$

$$y = x$$

58. $y = (\sin x)^{2x}, \quad \left(\dfrac{\pi}{2}, 1\right)$

$\ln y = 2x\ln(\sin x)$

$\dfrac{y'}{y} = \dfrac{2x}{\sin x}\cos x + 2\ln(\sin x)$

At $\left(\dfrac{\pi}{2}, 1\right)$, $y' = 0$.

Tangent line: $y = 1$

59. $y = (\ln x)^{\cos x}, \quad (e, 1)$

$\ln y = \cos x \cdot \ln(\ln x)$

$\dfrac{y'}{y} = \cos x \cdot \dfrac{1}{x \ln x} - \sin x \cdot \ln(\ln x)$

At $(e, 1)$, $y' = \cos(e)\dfrac{1}{e} - 0$.

Tangent line: $y - 1 = \dfrac{\cos(e)}{e}(x - e)$

$$y = \frac{\cos(e)}{e}x + 1 - \cos(e)$$

60. $y = x^{1/x}, \quad (1, 1)$

$\ln y = \dfrac{1}{x}\ln x$

$\dfrac{y'}{y} = \dfrac{1}{x^2} - \dfrac{\ln x}{x^2}$

At $(1, 1)$, $y' = 1 - 0 = 1$.

Tangent line: $y - 1 = 1(x - 1)$

$$y = x$$

61. $\displaystyle\int 3^x\,dx = \frac{3^x}{\ln 3} + C$

62. $\displaystyle\int 5^{-x}\,dx = \frac{-5^{-x}}{\ln 5} + C$

63.
$$\int x\left(5^{-x^2}\right)dx = -\frac{1}{2}\int 5^{-x^2}(-2x)\,dx$$
$$= -\left(\frac{1}{2}\right)\frac{5^{-x^2}}{\ln 5} + C$$
$$= \frac{-1}{2\ln 5}\left(5^{-x^2}\right) + C$$

64.
$$\int (3 - x)\,7^{(3-x)^2}\,dx = -\frac{1}{2}\int -2(3 - x)\,7^{(3-x)^2}\,dx$$
$$= -\frac{1}{2\ln 7}\left[7^{(3-x)^2}\right] + C$$

65. $\displaystyle\int \frac{3^{2x}}{1 + 3^{2x}}\,dx$, $u = 1 + 3^{2x}$, $du = 2(\ln 3)3^{2x}\,dx$

$$\frac{1}{2\ln 3}\int \frac{(2\ln 3)3^{2x}}{1 + 3^{2x}}\,dx = \frac{1}{2\ln 3}\ln(1 + 3^{2x}) + C$$

66. $\displaystyle\int 2^{\sin x}\cos x\,dx$, $u = \sin x$, $du = \cos x\,dx$

$$\frac{1}{\ln 2}2^{\sin x} + C$$

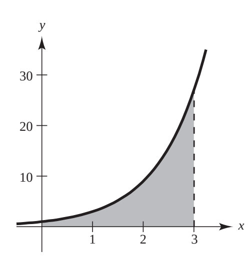
67.
$$\int_{-1}^{2} 2^x\,dx = \left[\frac{2^x}{\ln 2}\right]_{-1}^{2}$$
$$= \frac{1}{\ln 2}\left[4 - \frac{1}{2}\right] = \frac{7}{2\ln 2} = \frac{7}{\ln 4}$$

68.
$$\int_{-2}^{2} 4^{x/2}\,dx = 2\int_{-2}^{2} 4^{x/2}\left(\frac{1}{2}\,dx\right)$$
$$= \left[2\frac{1}{\ln 4}4^{x/2}\right]_{-2}^{2}$$
$$= \left[\frac{1}{\ln 2}4^{x/2}\right]_{-2}^{2}$$
$$= \frac{1}{\ln 2}\left[4 - \frac{1}{4}\right] = \frac{15}{4\ln 2}$$

69.
$$\int_0^1 (5^x - 3^x)\,dx = \left[\frac{5^x}{\ln 5} - \frac{3^x}{\ln 3}\right]_0^1$$
$$= \left(\frac{5}{\ln 5} - \frac{3}{\ln 3}\right) - \left(\frac{1}{\ln 5} - \frac{1}{\ln 3}\right)$$
$$= \frac{4}{\ln 5} - \frac{2}{\ln 3}$$

70.
$$\int_1^e (6^x - 2^x)\,dx = \left[\frac{6^x}{\ln 6} - \frac{2^x}{\ln 2}\right]_1^e$$
$$= \left(\frac{6^e}{\ln 6} - \frac{2^e}{\ln 2}\right) - \left(\frac{6}{\ln 6} - \frac{2}{\ln 2}\right)$$

71.
$$\text{Area} = \int_0^3 3^x\,dx$$
$$= \left[\frac{1}{\ln 3}3^x\right]_0^3$$
$$= \frac{1}{\ln 3}(27 - 1) = \frac{26}{\ln 3}$$



72. Area $= \int_0^{\pi} 3^{\cos x} \sin x \, dx$

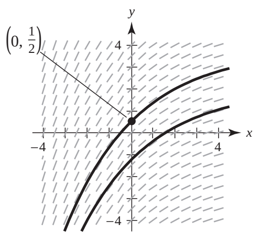
$$= \left[\frac{-3^{\cos x}}{\ln 3}\right]_0^{\pi}$$

$$= \frac{-1}{\ln 3}[3^{-1} - 3]$$

$$= \frac{8}{3 \ln 3} \approx 2.4273$$

73. $\frac{dy}{dx} = 0.4^{x/3}, \quad \left(0, \frac{1}{2}\right)$

(a)



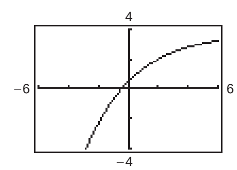
(b) $y = \int 0.4^{x/3} \, dx = 3\int 0.4^{x/3}\left(\frac{1}{3} \, dx\right)$

$$= \frac{3}{\ln 0.4} 0.4^{x/3} + C = 3(\ln 2.5)(0.4)^{x/3} + C$$

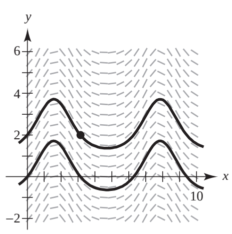
$$\frac{1}{2} = 3(\ln 2.5) + C \implies C = \frac{1}{2} - 3 \ln 2.5$$

$$y = 3(\ln 2.5)(0.4)^{x/3} + \frac{1}{2} - 3 \ln 2.5$$

$$= \frac{3(1 - 0.4^{x/3})}{\ln 2.5} + \frac{1}{2}$$



74. (a)



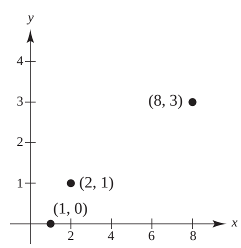
(b) $\frac{dy}{dx} = e^{\sin x} \cos x, \quad (\pi, 2)$

$$y = \int e^{\sin x} \cos x \, dx = e^{\sin x} + C$$

$$(\pi, 2): \; 2 = e^{\sin \pi} + C = 1 + C \implies C = 1$$

$$y = e^{\sin x} + 1$$

75.



x	1	2	8
y	0	1	3

(a) y is an exponential function of x: False

(b) y is a logarithmic function of x: True; $y = \log_2 x$

(c) x is an exponential function of y: True, $2^y = x$

(d) y is a linear function of x: False

76. $f(x) = \log_{10} x$

(a) Domain: $x > 0$

(b) $y = \log_{10} x$

$10^y = x$

$f^{-1}(x) = 10^x$

(c) $\log_{10} 1000 = \log_{10} 10^3 = 3$

$\log_{10} 10{,}000 = \log_{10} 10^4 = 4$

If $1000 \le x \le 10{,}000$, then $3 \le f(x) \le 4$.

(d) If $f(x) < 0$, then $0 < x < 1$.

(e) $f(x) + 1 = \log_{10} x + \log_{10} 10$

$= \log_{10}(10x)$

x must have been increased by a factor of 10.

(f) $\log_{10}\left(\frac{x_1}{x_2}\right) = \log_{10} x_1 - \log_{10} x_2$

$= 3n - n = 2n$

Thus, $x_1/x_2 = 10^{2n} = 100^n$.

77. $f(x) = \log_2 x \implies f'(x) = \dfrac{1}{x \ln 2}$

$g(x) = x^x \implies g'(x) = x^x(1 + \ln x)$

Note: Let $y = g(x)$. Then:

$\ln y = \ln x^x = x \ln x$

$\dfrac{1}{y}y' = x \cdot \dfrac{1}{x} + \ln x$

$y' = y(1 + \ln x)$

$y' = x^x(1 + \ln x) = g'(x)$

$h(x) = x^2 \implies h'(x) = 2x$

$k(x) = 2^x \implies k'(x) = (\ln 2)2^x$

From greatest to smallest rate of growth:

$g(x), k(x), h(x), f(x)$

78. (a) $y = x^a$

$y' = ax^{a-1}$

(b) $y = a^x$

$y' = (\ln a)a^x$

(c) $y = x^x$

$\ln y = x \ln x$

$\dfrac{1}{y}y' = x \cdot \dfrac{1}{x} + (1) \ln x$

$y' = y(1 + \ln x)$

$y' = x^x(1 + \ln x)$

(d) $y = a^a$

$y' = 0$

79. $C(t) = P(1.05)^t$

(a) $C(10) = 24.95(1.05)^{10}$

$\approx \$40.64$

(b) $\dfrac{dC}{dt} = P(\ln 1.05)(1.05)^t$

When $t = 1$, $\dfrac{dC}{dt} \approx 0.051P$.

When $t = 8$, $\dfrac{dC}{dt} \approx 0.072P$.

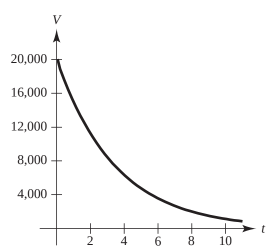
(c) $\dfrac{dC}{dt} = (\ln 1.05)[P(1.05)^t]$

$= (\ln 1.05)C(t)$

The constant of proportionality is ln 1.05.

80. $V(t) = 20{,}000\left(\dfrac{3}{4}\right)^t$

(a)



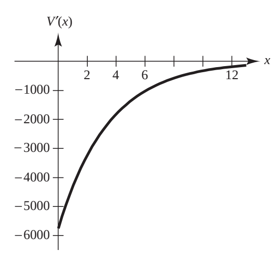
$V(2) = 20{,}000\left(\dfrac{3}{4}\right)^2 = \$11{,}250$

(b) $\dfrac{dV}{dt} = 20{,}000\left(\ln \dfrac{3}{4}\right)\left(\dfrac{3}{4}\right)^t$

When $t = 1$, $\dfrac{dV}{dt} \approx -4315.23$.

When $t = 4$, $\dfrac{dV}{dt} \approx -1820.49$.

(c)



Horizontal asymptote: $V' = 0$

As the car ages, it is worth less each year and depreciates less each year, but the value of the car will never reach \$0.

81. $P = \$1000,\ r = 3\tfrac{1}{2}\% = 0.035,\ t = 10$

$A = 1000\left(1 + \dfrac{0.035}{n}\right)^{10n}$

$A = 1000e^{(0.035)(10)} = 1419.07$

n	1	2	4	12	365	Continuous
A	1410.60	1414.78	1416.91	1418.34	1419.04	1419.07

82. $P = \$2500,\ r = 6\% = 0.06,\ t = 20$

$A = 2500\left(1 + \dfrac{0.06}{n}\right)^{20n}$

$A = 2500e^{(0.06)(20)} = 8300.29$

n	1	2	4	12	365	Continuous
A	8017.84	8155.09	8226.66	8275.51	8299.47	8300.29

83. $P = \$1000,\ r = 5\% = 0.05,\ t = 30$

$A = 1000\left(1 + \frac{0.05}{n}\right)^{30n}$

$A = 1000e^{(0.05)30} = 4481.69$

n	1	2	4	12	365	Continuous
A	4321.94	4399.79	4440.21	4467.74	4481.23	4481.69

84. $P = \$5000, r = 7\% = 0.07, t = 25$

$A = 5000\left(1 + \frac{0.07}{n}\right)^{25n}$

$A = 5000e^{0.07(25)}$

n	1	2	4	12	365	Continuous
A	27,137.16	27,924.63	28,340.78	28,627.09	28,768.19	28,773.01

85. $100{,}000 = Pe^{0.05t} \implies P = 100{,}000e^{-0.05t}$

t	1	10	20	30	40	50
P	95,122.94	60,653.07	36,787.94	22,313.02	13,533.53	8208.50

86. $100{,}000 = Pe^{0.06t} \implies P = 100{,}000e^{-0.06t}$

t	1	10	20	30	40	50
P	94,176.45	54,881.16	30,119.42	16,529.89	9071.80	4978.71

87. $100{,}000 = P\left(1 + \frac{0.05}{12}\right)^{12t} \implies P = 100{,}000\left(1 + \frac{0.05}{12}\right)^{-12t}$

t	1	10	20	30	40	50
P	95,132.82	60,716.10	36,864.45	22,382.66	13,589.88	8251.24

88. $100{,}000 = P\left(1 + \frac{0.07}{365}\right)^{365t} \implies P = 100{,}000\left(1 + \frac{0.07}{365}\right)^{-365t}$

t	1	10	20	30	40	50
P	93,240.01	49,661.86	24,663.01	12,248.11	6082.64	3020.75

89. (a) $A = 20{,}000\left(1 + \frac{0.06}{365}\right)^{(365)(8)} \approx \$32{,}320.21$

(b) $A = \$30{,}000$

(c) $A = 8000\left(1 + \frac{0.06}{365}\right)^{(365)(8)} + 20{,}000\left(1 + \frac{0.06}{365}\right)^{(365)(4)}$

$\approx \$12{,}928.09 + 25{,}424.48 = \$38{,}352.57$

(d) $A = 9000\left[\left(1 + \frac{0.06}{365}\right)^{(365)(8)} + \left(1 + \frac{0.06}{365}\right)^{(365)(4)} + 1\right]$

$\approx \$34{,}985.11$

Take option (c).

90. Let $P = \$100,\ 0 \le t \le 20.$

(a) $A = 100e^{0.03t}$

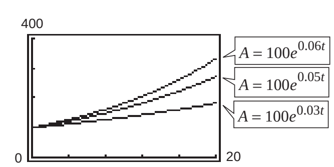
$A(20) \approx 182.21$

(b) $A = 100e^{0.05t}$

$A(20) \approx 271.83$

(c) $A = 100e^{0.06t}$

$A(20) \approx 332.01$



91. (a) $\lim_{t \to \infty} 6.7e^{(-48.1)/t} = 6.7e^0 = 6.7$ million ft^3

(b) $V' = \dfrac{322.27}{t^2}e^{-(48.1)/t}$

$V'(20) \approx 0.073$ million ft^3/yr

$V'(60) \approx 0.040$ million ft^3/yr

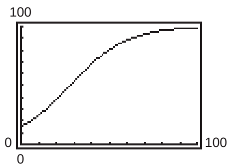
92. (a) $\lim_{n \to \infty} \dfrac{0.86}{1 + e^{-0.25n}} = 0.86$ or 86%

(b) $P' = \dfrac{-0.86(-0.25)(e^{-0.25n})}{(1 + e^{-0.25n})} = \dfrac{0.215e^{-0.25n}}{(1 + e^{-0.25n})}$

$P'(3) \approx 0.069$

$P'(10) \approx 0.016$

93. $y = \dfrac{300}{3 + 17e^{-0.0625x}}$

(a) 

(b) If $x = 2$ (2000 egg masses), $y \approx 16.67 \approx 16.7\%$.

(c) If $y = 66.67\%$, then $x \approx 38.8$ or 38,800 egg masses.

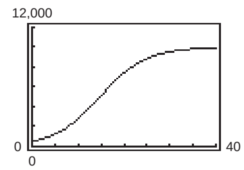
(d) $y = 300(3 + 17e^{-0.0625x})^{-1}$

$y' = \dfrac{318.75e^{-0.0625x}}{(3 + 17e^{-0.0625x})^2}$

$y'' = \dfrac{19.921875e^{-0.0625x}(17e^{-0.0625x} - 3)}{(3 + 17e^{-0.0625x})^3}$

$17e^{-0.0625x} - 3 = 0 \implies$

$x \approx 27.8$ or 27,800 egg masses.

94. (a) 

(b) Limiting size: 10,000 fish

(c) $p(t) = \dfrac{10{,}000}{1 + 19e^{-t/5}}$

$p'(t) = \dfrac{e^{-t/5}}{(1 + 19e^{t/5})^2}\left(\dfrac{19}{5}\right)(10{,}000) = \dfrac{38{,}000e^{-t/5}}{(1 + 19e^{-t/5})^2}$

$p'(1) \approx 113.5$ fish/month

$p'(10) \approx 403.2$ fish/month

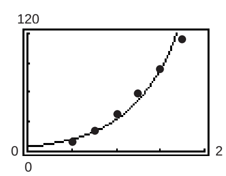
(d) $p''(t) = -\dfrac{38{,}000}{5}(e^{-t/5})\left[\dfrac{1 - 19e^{-t/5}}{(1 + 19e^{-t/5})^3}\right] = 0$

$19e^{-t/5} = 1$

$\dfrac{t}{5} = \ln 19$

$t = 5 \ln 19 \approx 14.72$

95. (a) $B = 4.7539(6.7744)^d = 4.7539e^{1.9132d}$

(b) 

(c) $B'(d) = 9.0952e^{1.9132d}$

$B'(0.8) \approx 42.03$ tons/inch

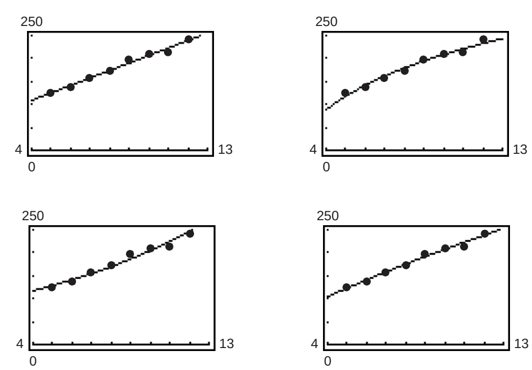
$B'(1.5) \approx 160.38$ tons/inch

96. (a) $y_1 = 16.32t + 43.4$, linear

$y_2 = -93.58 + 131.22 \ln x$

$y_3 = (80.99)1.097^x$

$y_4 = (36.55)x^{0.754}$

(b) 

(c) The amount given increases 16.32 billion on average per year.

(d) $y_1'(6) = 16.32$

$y_2'(6) = \dfrac{131.22}{6} = 21.87$

$y_3'(6) = (80.99)1.097^6(\ln 1.097) \approx 13.07$

$y_4'(6) = 36.55(0.754)6^{0.754-1} \approx 17.74$

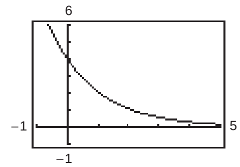
y_2 is increasing at the greatest rate.

97. (a) $\int_0^4 f(t)\,dt \approx 5.67$

$\int_0^4 g(t)\,dt \approx 5.67$

$\int_0^4 h(t)\,dt \approx 5.67$

(b)



(c) The functions appear to be equal: $f(t) = g(t) = h(t)$. Analytically,

$$f(t) = 4\left(\frac{3}{8}\right)^{2t/3} = 4\left[\left(\frac{3}{8}\right)^{2/3}\right]^t = 4\left(\frac{9^{1/3}}{4}\right)^t = g(t)$$

$$h(t) = 4e^{-0.653886t} = 4\left[e^{-0.653886}\right]^t \approx 4(0.52002)^t$$

$$g(t) = 4\left(\frac{9^{1/3}}{4}\right)^t \approx 4(0.52002)^t$$

No. The definite integrals over a given interval may be equal when the functions are not equal.

98.

x	1	10^{-1}	10^{-2}	10^{-4}	10^{-6}
$(1+x)^{1/x}$	2	2.594	2.705	2.718	2.718

99.

t	0	1	2	3	4
y	1200	720	432	259.20	155.52

$y = C(k^t)$

When $t = 0, y = 1200 \implies C = 1200.$

$y = 1200(k^t)$

$$\frac{720}{1200} = 0.6, \frac{432}{720} = 0.6, \frac{259.20}{432} = 0.6, \frac{155.52}{259.20} = 0.6$$

Let $k = 0.6.$

$y = 1200(0.6)^t$

100.

t	0	1	2	3	4
y	600	630	661.50	694.58	729.30

$y = C(k^t)$

When $t = 0,\ y = 600 \implies C = 600.$

$y = 600(k^t)$

$$\frac{630}{600} = 1.05, \frac{661.50}{630} = 1.05, \frac{694.58}{661.50} \approx 1.05,$$

$$\frac{729.30}{694.58} \approx 1.05$$

Let $k = 1.05.$

$y = 600(1.05)^t$

101. False. e is an irrational number.

102. True

$$\begin{aligned} f(e^{n+1}) - f(e^n) &= \ln e^{n+1} - \ln e^n \\ &= n + 1 - n \\ &= 1 \end{aligned}$$

103. True

$$\begin{aligned} f(g(x)) &= 2 + e^{\ln(x-2)} \\ &= 2 + x - 2 = x \\ g(f(x)) &= \ln(2 + e^x - 2) \\ &= \ln e^x = x \end{aligned}$$

104. True

$$\frac{d^n y}{dx^n} = Ce^x$$

$$= y \text{ for } n = 1, 2, 3, \ldots$$

105. True

$\frac{d}{dx}[e^x] = e^x$ and $\frac{d}{dx}[e^{-x}] = -e^{-x}$

$e^x = e^{-x}$ when $x = 0.$

$(e^0)(-e^{-0}) = -1$

106. True

$f(x) = g(x)e^x = 0 \implies$

$g(x) = 0$ since $e^x > 0$ for all x.

107. $$\frac{dy}{dt} = \frac{8}{25}y\left(\frac{5}{4} - y\right),\ y(0) = 1$$

$$\frac{dy}{y[(5/4) - y]} = \frac{8}{25}\,dt \implies \frac{4}{5}\int\left(\frac{1}{y} + \frac{1}{(5/4) - y}\right)dy = \int \frac{8}{25}\,dt \implies$$

$$\ln y - \ln\left(\frac{5}{4} - y\right) = \frac{2}{5}t + C$$

$$\ln\left(\frac{y}{(5/4) - y}\right) = \frac{2}{5}t + C$$

$$\frac{y}{(5/4) - y} = e^{(2/5)t + C} = C_1 e^{(2/5)t}$$

$$y(0) = 1 \implies C_1 = 4 \implies 4e^{(2/5)t} = \frac{y}{(5/4) - y}$$

$$\implies 4e^{(2/5)t}\left(\frac{5}{4} - y\right) = y \implies 5e^{(2/5)t} = 4e^{(2/5)t}y + y = (4e^{(2/5)t} + 1)y$$

$$\implies y = \frac{5e^{(2/5)t}}{4e^{(2/5)t} + 1} = \frac{5}{4 + e^{-0.4t}} = \frac{1.25}{1 + 0.25e^{-0.4t}}$$

108. $f(x) = a^x$

(a) $f(u + v) = a^{u+v} = a^u a^v = f(u)f(v)$

(b) $f(2x) = a^{2x} = (a^x)^2 = [f(x)]^2$

109. (a)
$$y^x = x^y$$
$$x \ln y = y \ln x$$
$$x\frac{y'}{y} + \ln y = \frac{y}{x} + y' \ln x$$
$$y'\left[\frac{x}{y} - \ln x\right] = \frac{y}{x} - \ln y$$
$$y' = \frac{(y/x) - \ln y}{(x/y) - \ln x}$$
$$y' = \frac{y^2 - xy \ln y}{x^2 - xy \ln x}$$

(b) (i) At (c, c): $y' = \dfrac{c^2 - c^2 \ln c}{c^2 - c^2 \ln c} = 1,\ (c \neq 0, e)$

(ii) At $(2, 4)$: $y' = \dfrac{16 - 8 \ln 4}{4 - 8 \ln 2} = \dfrac{4 - 4 \ln 2}{1 - 2 \ln 2} \approx -3.1774$

(iii) At $(4, 2)$: $y' = \dfrac{4 - 8 \ln 2}{16 - 8 \ln 4} = \dfrac{1 - 2 \ln 2}{4 - 4 \ln 2} \approx -0.3147$

(c) y' is undefined for

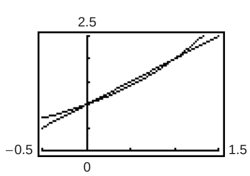
$x^2 = xy \ln x \implies x = y \ln x = \ln x^y \implies e^x = x^y$.

At (e, e), y' is undefined.

110. $f(x) = 1 + x,\ g(x) = b^x$

(a) $b = 2$

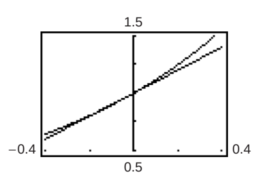
$g(x) = 2^x$



Intersection points: $(0, 1), (1, 2)$

(b) $b = 3$

$g(x) = 3^x$



Intersection point: $(0, 1)$

(c) $g(x) = e^x \geq 1 + x$ for all x.

$g'(0) = 1 = f'(0)$

Hence, $g(x) \geq f(x)$ for all $b \geq e$.

111. Let $f(x) = \dfrac{\ln x}{x}, \quad x > 0.$

$f'(x) = \dfrac{1 - \ln x}{x^2} < 0$ for $x > e \implies f$ is decreasing for $x \geq e$. Hence, for $e \leq x < y$:

$$f(x) > f(y)$$

$$\frac{\ln x}{x} > \frac{\ln y}{y}$$

$$(xy)\frac{\ln x}{x} > (xy)\frac{\ln y}{y}$$

$$\ln x^y > \ln y^x$$

$$x^y > y^x$$

For $n \geq 8$, $e < \sqrt{n} < \sqrt{n+1}$, $\left(\sqrt{8} \approx 2.828\right)$ and so letting $x = \sqrt{n}$, $y = \sqrt{n+1}$, we have

$$\left(\sqrt{n}\right)^{\sqrt{n+1}} > \left(\sqrt{n+1}\right)^{\sqrt{n}}.$$

Note: $\sqrt{8}^{\sqrt{9}} \approx 22.6$ and $\sqrt{9}^{\sqrt{8}} \approx 22.4$.

Note: This same argument shows $e^{\pi} > \pi^{e}$.

112. $$\log_e\left(1 + \frac{1}{x}\right) = \ln\left(1 + \frac{1}{x}\right)$$

$$= \int_x^{1+x} \frac{dt}{t}$$

$$> \int_x^{1+x} \frac{dt}{1+x} \qquad \begin{pmatrix} \text{because } 1 + x \geq t \\ \text{on } x \leq t \leq 1 + x \end{pmatrix}$$

$$= \left[\frac{t}{1+x}\right]_x^{1+x}$$

$$= \frac{1+x}{1+x} - \frac{x}{1+x}$$

$$= \frac{1}{1+x}$$

Note: You can confirm this result by graphing

$y_1 = \ln\left(1 + \dfrac{1}{x}\right)$ and

$y_2 = \dfrac{1}{1+x}$.

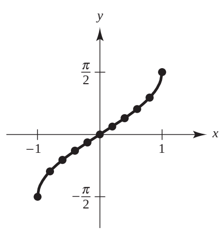
Section 5.6 Inverse Trigonometric Functions: Differentiation

1. $y = \arcsin x$

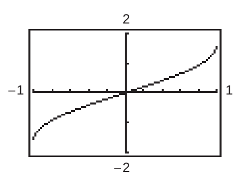
(a)

x	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
y	-1.571	-0.927	-0.644	-0.412	-0.201	0	0.201	0.412	0.644	0.927	1.571

(b)



(c)



(d) Symmetric about origin:

$\arcsin(-x) = -\arcsin x$

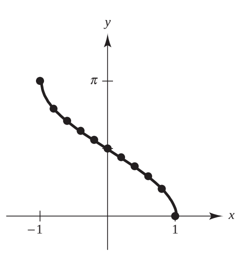
Intercept: $(0, 0)$

2. $y = \arccos x$

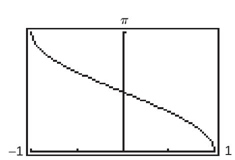
(a)

x	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
y	3.142	2.499	2.214	1.982	1.772	1.571	1.369	1.159	0.927	0.644	0

(b)



(c)



(d) Intercepts: $\left(0, \dfrac{\pi}{2}\right)$ and $(1, 0)$

No symmetry

3. $y = \arccos x$

$\left(-\frac{\sqrt{2}}{2}, \frac{3\pi}{4}\right)$ because $\cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$\left(\frac{1}{2}, \frac{\pi}{3}\right)$ because $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$\left(\frac{\sqrt{3}}{2}, \frac{\pi}{6}\right)$ because $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

4. $\left(\_\_, \frac{\pi}{4}\right) = \left(1, \frac{\pi}{4}\right)$

$\left(\_\_, -\frac{\pi}{6}\right) = \left(-\frac{\sqrt{3}}{3}, -\frac{\pi}{6}\right)$

$\left(-\sqrt{3}, \_\_\right) = \left(-\sqrt{3}, -\frac{\pi}{3}\right)$

5. $\arcsin \frac{1}{2} = \frac{\pi}{6}$

6. $\arcsin 0 = 0$

7. $\arccos \frac{1}{2} = \frac{\pi}{3}$

8. $\arccos 0 = \frac{\pi}{2}$

9. $\arctan \frac{\sqrt{3}}{3} = \frac{\pi}{6}$

10. $\text{arccot}(-\sqrt{3}) = \frac{5\pi}{6}$

11. $\text{arccsc}(-\sqrt{2}) = -\frac{\pi}{4}$

12. $\arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$

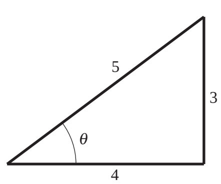
13. $\arccos(-0.8) \approx 2.50$

14. $\arcsin(-0.39) \approx -0.40$

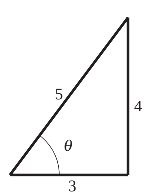
15. $\text{arcsec}(1.269) = \arccos\left(\frac{1}{1.269}\right) \approx 0.66$

16. $\arctan(-3) \approx -1.25$

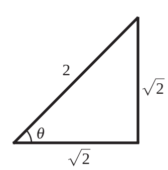
17. (a) $\sin\left(\arctan \frac{3}{4}\right) = \frac{3}{5}$



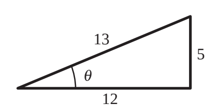
(b) $\sec\left(\arcsin \frac{4}{5}\right) = \frac{5}{3}$



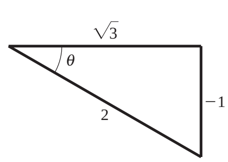
18. (a) $\tan\left(\arccos \frac{\sqrt{2}}{2}\right) = \tan\left(\frac{\pi}{4}\right) = 1$



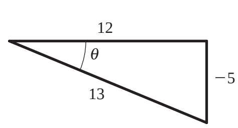
(b) $\cos\left(\arcsin \frac{5}{13}\right) = \frac{12}{13}$



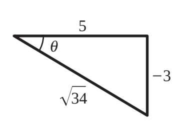
19. (a) $\cot\left[\arcsin\left(-\frac{1}{2}\right)\right] = \cot\left(-\frac{\pi}{6}\right) = -\sqrt{3}$



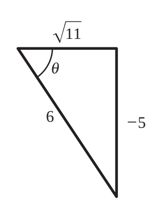
(b) $\csc\left[\arctan\left(-\frac{5}{12}\right)\right] = -\frac{13}{5}$



20. (a) $\sec\left[\arctan\left(-\frac{3}{5}\right)\right] = \frac{\sqrt{34}}{5}$



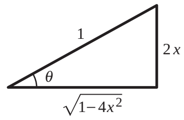
(b) $\tan\left[\arcsin\left(-\frac{5}{6}\right)\right] = -\frac{5\sqrt{11}}{11}$



21. $y = \cos(\arcsin 2x)$

$\theta = \arcsin 2x$

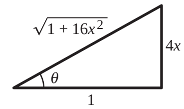
$y = \cos\theta = \sqrt{1 - 4x^2}$



22. $y = \sec(\arctan 4x)$

$\theta = \arctan 4x$

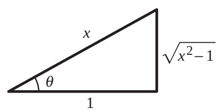
$y = \sec\theta = \sqrt{1 + 16x^2}$



23. $y = \sin(\operatorname{arcsec} x)$

$\theta = \operatorname{arcsec} x,\ 0 \le \theta \le \pi,\ \theta \ne \dfrac{\pi}{2}$

$y = \sin\theta = \dfrac{\sqrt{x^2 - 1}}{|x|}$

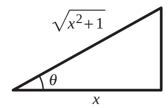


The absolute value bars on x are necessary because of the restriction $0 \le \theta \le \pi$, $\theta \ne \pi/2$, and $\sin\theta$ for this domain must always be nonnegative.

24. $y = \cos(\operatorname{arccot} x)$

$\theta = \operatorname{arccot} x$

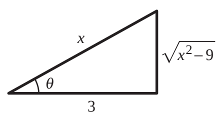
$y = \cos\theta = \dfrac{x}{\sqrt{x^2 + 1}}$



25. $y = \tan\left(\operatorname{arcsec}\dfrac{x}{3}\right)$

$\theta = \operatorname{arcsec}\dfrac{x}{3}$

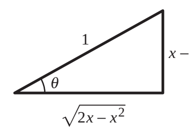
$y = \tan\theta = \dfrac{\sqrt{x^2 - 9}}{3}$



26. $y = \sec[\arcsin(x - 1)]$

$\theta = \arcsin(x - 1)$

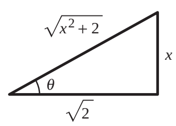
$y = \sec\theta = \dfrac{1}{\sqrt{2x - x^2}}$



27. $y = \csc\left(\arctan\dfrac{x}{\sqrt{2}}\right)$

$\theta = \arctan\dfrac{x}{\sqrt{2}}$

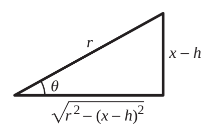
$y = \csc\theta = \dfrac{\sqrt{x^2 + 2}}{x}$



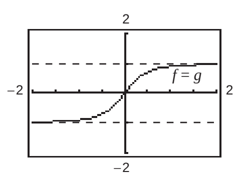
28. $y = \cos\left(\arcsin\dfrac{x - h}{r}\right)$

$\theta = \arcsin\dfrac{x - h}{r}$

$y = \cos\theta = \dfrac{\sqrt{r^2 - (x - h)^2}}{r}$



29. (a)

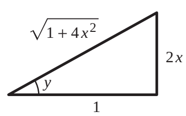


(b) Let $y = \arctan(2x)$

$\tan y = 2x$

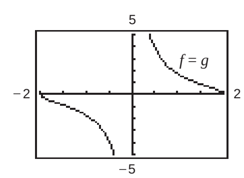
$\sin y = \sin(\arctan(2x))$

$= \dfrac{2x}{\sqrt{1 + 4x^2}}.$



(c) Asymptotes: $y = \pm 1$

30. (a)

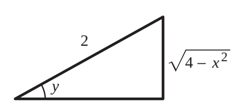


(b) Let $y = \arccos\dfrac{x}{2}$

$\cos y = \dfrac{x}{2}$

$\tan y = \tan\left(\arccos\dfrac{x}{2}\right)$

$= \dfrac{\sqrt{4 - x^2}}{x}.$



(c) No horizontal asymptotes; domain is $-2 \le x \le 0$, $0 < x \le 2$.

(Vertical asymptote: $x = 0$)

31. $\arcsin(3x - \pi) = \frac{1}{2}$

$$3x - \pi = \sin\left(\tfrac{1}{2}\right)$$

$$x = \tfrac{1}{3}\left[\sin\left(\tfrac{1}{2}\right) + \pi\right] \approx 1.207$$

32. $\arctan(2x - 5) = -1$

$$2x - 5 = \tan(-1)$$

$$x = \tfrac{1}{2}(\tan(-1) + 5) \approx 1.721$$

33. $\arcsin\sqrt{2x} = \arccos\sqrt{x}$

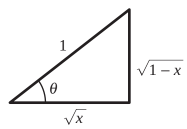
$$\sqrt{2x} = \sin\left(\arccos\sqrt{x}\right)$$

$$\sqrt{2x} = \sqrt{1 - x}, \quad 0 \le x \le 1$$

$$2x = 1 - x$$

$$3x = 1$$

$$x = \frac{1}{3}$$



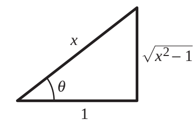
34. $\arccos x = \operatorname{arcsec} x$

$$x = \cos(\operatorname{arcsec} x)$$

$$x = \frac{1}{x}$$

$$x^2 = 1$$

$$x = \pm 1$$



35. (a) $\operatorname{arccsc} x = \arcsin\dfrac{1}{x}, \quad |x| \ge 1$

Let $y = \operatorname{arccsc} x$. Then for

$$-\frac{\pi}{2} \le y < 0 \text{ and } 0 < y \le \frac{\pi}{2},$$

$\csc y = x \implies \sin y = 1/x$. Thus, $y = \arcsin(1/x)$. Therefore, $\operatorname{arccsc} x = \arcsin(1/x)$.

(b) $\arctan x + \arctan\dfrac{1}{x} = \dfrac{\pi}{2}, \quad x > 0$

Let $y = \arctan x + \arctan(1/x)$. Then,

$$\tan y = \frac{\tan(\arctan x) + \tan[\arctan(1/x)]}{1 - \tan(\arctan x)\tan[\arctan(1/x)]}$$

$$= \frac{x + (1/x)}{1 - x(1/x)}$$

$$= \frac{x + (1/x)}{0} \text{ (which is undefined).}$$

Thus, $y = \pi/2$. Therefore, $\arctan x + \arctan(1/x) = \pi/2$.

36. (a) $\arcsin(-x) = -\arcsin x, \quad |x| \le 1$

Let $y = \arcsin(-x)$. Then,

$$-x = \sin y \implies x = -\sin y \implies x = \sin(-y).$$

Thus, $-y = \arcsin x \implies y = -\arcsin x$. Therefore, $\arcsin(-x) = -\arcsin x$.

(b) $\arccos(-x) = \pi - \arccos x, \quad |x| \le 1$

Let $y = \arccos(-x)$. Then,

$$-x = \cos y \implies x = -\cos y \implies x = \cos(\pi - y).$$

Thus, $\pi - y = \arccos x \implies y = \pi - \arccos x$. Therefore, $\arccos(-x) = \pi - \arccos x$.

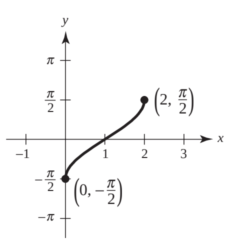
37. $f(x) = \arcsin(x - 1)$

$$x - 1 = \sin y$$

$$x = 1 + \sin y$$

Domain: $[0, 2]$

Range: $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$



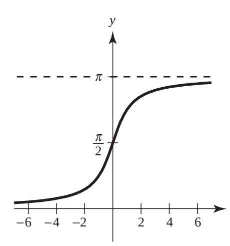
$f(x)$ is the graph of $\arcsin x$ shifted 1 unit to the right.

38. $f(x) = \arctan x + \dfrac{\pi}{2}$

$$x = \tan\left(y - \frac{\pi}{2}\right)$$

Domain: $(-\infty, \infty)$

Range: $(0, \pi)$



$f(x)$ is the graph of $\arctan x$ shifted $\pi/2$ units upward.

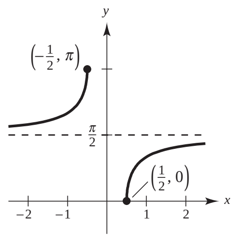
39. $f(x) = \operatorname{arcsec} 2x$

$2x = \sec y$

$x = \dfrac{1}{2}\sec y$

Domain: $\left(-\infty, -\dfrac{1}{2}\right], \left[\dfrac{1}{2}, \infty\right)$

Range: $\left[0, \dfrac{\pi}{2}\right), \left(\dfrac{\pi}{2}, \pi\right]$



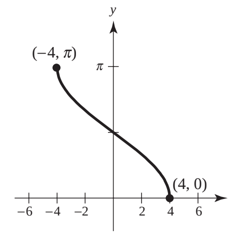
40. $f(x) = \arccos\left(\dfrac{x}{4}\right)$

$\dfrac{x}{4} = \cos y$

$x = 4\cos y$

Domain: $[-4, 4]$

Range: $[0, \pi]$



41. $f(x) = 2\arcsin(x-1)$

$$f'(x) = \frac{2}{\sqrt{1-(x-1)^2}} = \frac{2}{\sqrt{2x-x^2}}$$

42. $f(t) = \arcsin t^2$

$$f'(t) = \frac{2t}{\sqrt{1-t^4}}$$

43. $g(x) = 3\arccos\dfrac{x}{2}$

$$g'(x) = \frac{-3(1/2)}{\sqrt{1-(x^2/4)}} = \frac{-3}{\sqrt{4-x^2}}$$

44. $f(x) = \operatorname{arcsec} 2x$

$$f'(x) = \frac{2}{|2x|\sqrt{4x^2-1}} = \frac{1}{|x|\sqrt{4x^2-1}}$$

45. $f(x) = \arctan\dfrac{x}{a}$

$$f'(x) = \frac{1/a}{1+(x^2/a^2)} = \frac{a}{a^2+x^2}$$

46. $f(x) = \arctan\sqrt{x}$

$$f'(x) = \left(\frac{1}{1+x}\right)\left(\frac{1}{2\sqrt{x}}\right) = \frac{1}{2\sqrt{x}(1+x)}$$

47. $g(x) = \dfrac{\arcsin 3x}{x}$

$$g'(x) = \frac{x\left(3/\sqrt{1-9x^2}\right) - \arcsin 3x}{x^2} = \frac{3x - \sqrt{1-9x^2}\arcsin 3x}{x^2\sqrt{1-9x^2}}$$

48. $h(x) = x^2\arctan x$

$$h'(x) = 2x\arctan x + \frac{x^2}{1+x^2}$$

49. $h(t) = \sin(\arccos t) = \sqrt{1-t^2}$

$$h'(t) = \frac{1}{2}(1-t^2)^{-1/2}(-2t) = \frac{-t}{\sqrt{1-t^2}}$$

50. $f(x) = \arcsin x + \arccos x = \dfrac{\pi}{2}$

$$f'(x) = 0$$

51. $y = x\arccos x - \sqrt{1-x^2}$

$$y' = \arccos x - \frac{x}{\sqrt{1-x^2}} - \frac{1}{2}(1-x^2)^{-1/2}(-2x) = \arccos x$$

52. $y = \ln(t^2+4) - \dfrac{1}{2}\arctan\dfrac{t}{2}$

$$y' = \frac{2t}{t^2+4} - \frac{1}{2}\cdot\frac{1}{1+(t/2)^2}\left(\frac{1}{2}\right) = \frac{2t}{t^2+4} - \frac{1}{t^2+4} = \frac{2t-1}{t^2+4}$$

53. $y = \dfrac{1}{2}\left(\dfrac{1}{2}\ln\dfrac{x+1}{x-1} + \arctan x\right)$

$$= \frac{1}{4}[\ln(x+1) - \ln(x-1)] + \frac{1}{2}\arctan x$$

$$\frac{dy}{dx} = \frac{1}{4}\left(\frac{1}{x+1} - \frac{1}{x-1}\right) + \frac{1/2}{1+x^2} = \frac{1}{1-x^4}$$

54. $y = \frac{1}{2}\left[x\sqrt{4 - x^2} + 4\arcsin\left(\frac{x}{2}\right)\right]$

$$y' = \frac{1}{2}\left[x\frac{1}{2}(4 - x^2)^{-1/2}(-2x) + \sqrt{4 - x^2} + 2\frac{1}{\sqrt{1 - (x/2)^2}}\right]$$
$$= \frac{1}{2}\left[\frac{-x^2}{\sqrt{4 - x^2}} + \sqrt{4 - x^2} + \frac{4}{\sqrt{4 - x^2}}\right]$$
$$= \sqrt{4 - x^2}$$

55. $y = x\arcsin x + \sqrt{1 - x^2}$

$$\frac{dy}{dx} = x\left(\frac{1}{\sqrt{1 - x^2}}\right) + \arcsin x - \frac{x}{\sqrt{1 - x^2}} = \arcsin x$$

56. $y = x\arctan 2x - \frac{1}{4}\ln(1 + 4x^2)$

$$\frac{dy}{dx} = \frac{2x}{1 + 4x^2} + \arctan(2x) - \frac{1}{4}\left(\frac{8x}{1 + 4x^2}\right) = \arctan(2x)$$

57. $y = 8\arcsin\frac{x}{4} - \frac{x\sqrt{16 - x^2}}{2}$

$$y' = 2\frac{1}{\sqrt{1 - (x/4)^2}} - \frac{\sqrt{16 - x^2}}{2} - \frac{x}{4}(16 - x^2)^{-1/2}(-2x)$$
$$= \frac{8}{\sqrt{16 - x^2}} - \frac{\sqrt{16 - x^2}}{2} + \frac{x^2}{2\sqrt{16 - x^2}}$$
$$= \frac{16 - (16 - x^2) + x^2}{2\sqrt{16 - x^2}} = \frac{x^2}{\sqrt{16 - x^2}}$$

58. $y = 25\arcsin\frac{x}{5} - x\sqrt{25 - x^2}$

$$y' = 5\frac{1}{\sqrt{1 - (x/2)^2}} - \sqrt{25 - x^2} - x\frac{1}{2}(25 - x^2)^{-1/2}(-2x)$$
$$= \frac{25}{\sqrt{25 - x^2}} - \frac{(25 - x^2)}{\sqrt{25 - x^2}} + \frac{x^2}{\sqrt{25 - x^2}}$$
$$= \frac{2x^2}{\sqrt{25 - x^2}}$$

59. $y = \arctan x + \frac{x}{1 + x^2}$

$$y' = \frac{1}{1 + x^2} + \frac{(1 + x^2) - x(2x)}{(1 + x^2)^2}$$
$$= \frac{(1 + x^2) + (1 - x^2)}{(1 + x^2)^2}$$
$$= \frac{2}{(1 + x^2)^2}$$

60. $y = \arctan\frac{x}{2} - \frac{1}{2(x^2 + 4)}$

$$y' = \frac{1}{2}\frac{1}{1 + (x/2)^2} + \frac{1}{2}(x^2 + 4)^{-2}(2x)$$
$$= \frac{2}{x^2 + 4} + \frac{x}{(x^2 + 4)^2}$$
$$= \frac{2x^2 + 8 + x}{(x^2 + 4)^2}$$

61. $y = 2\arcsin x, \quad \left(\frac{1}{2}, \frac{\pi}{3}\right)$

$$y' = \frac{2}{\sqrt{1 - x^2}}$$

At $\left(\frac{1}{2}, \frac{\pi}{3}\right)$, $y' = \frac{2}{\sqrt{1 - (1/4)}} = \frac{4}{\sqrt{3}}$.

Tangent line: $y - \frac{\pi}{3} = \frac{4}{\sqrt{3}}\left(x - \frac{1}{2}\right)$

$$y = \frac{4}{\sqrt{3}}x + \frac{\pi}{3} - \frac{2}{\sqrt{3}}$$
$$y = \frac{4\sqrt{3}}{3}x + \frac{\pi}{3} - \frac{2\sqrt{3}}{3}$$

62. $y = \frac{1}{2}\arccos x, \quad \left(-\frac{\sqrt{2}}{2}, \frac{3\pi}{8}\right)$

$$y' = \frac{-1}{2\sqrt{1 - x^2}}$$

At $\left(-\frac{\sqrt{2}}{2}, \frac{3\pi}{8}\right)$, $y' = \frac{-1}{2\sqrt{1/2}} = -\frac{\sqrt{2}}{2}$.

Tangent line: $y - \frac{3\pi}{8} = -\frac{\sqrt{2}}{2}\left(x + \frac{\sqrt{2}}{2}\right)$

$$y = -\frac{\sqrt{2}}{2}x + \frac{3\pi}{8} - \frac{1}{2}$$

63. $y = \arctan\left(\frac{x}{2}\right), \quad \left(2, \frac{\pi}{4}\right)$

$$y' = \frac{1}{1 + (x^2/4)}\left(\frac{1}{2}\right) = \frac{2}{4 + x^2}$$

At $\left(2, \frac{\pi}{4}\right)$, $y' = \frac{2}{4+4} = \frac{1}{4}$.

Tangent line: $y - \frac{\pi}{4} = \frac{1}{4}(x - 2)$

$$y = \frac{1}{4}x + \frac{\pi}{4} - \frac{1}{2}$$

64. $y = \operatorname{arcsec}(4x), \quad \left(\frac{\sqrt{2}}{4}, \frac{\pi}{4}\right)$

$$y' = \frac{4}{|4x|\sqrt{16x^2 - 1}} = \frac{1}{x\sqrt{16x^2 - 1}} \text{ for } x > 0$$

At $\left(\frac{\sqrt{2}}{4}, \frac{\pi}{4}\right)$, $y' = \frac{1}{\left(\sqrt{2}/4\right)\sqrt{2 - 1}} = 2\sqrt{2}$.

Tangent line: $y - \frac{\pi}{4} = 2\sqrt{2}\left(x - \frac{\sqrt{2}}{4}\right)$

$$y = 2\sqrt{2}x + \frac{\pi}{4} - 1$$

65. $y = 4x \arccos(x - 1), \quad (1, 2\pi)$

$$y' = 4x\frac{-1}{\sqrt{1 - (x-1)^2}} + 4\arccos(x - 1)$$

At $(1, 2\pi)$, $y' = -4 + 2\pi$.

Tangent line: $y - 2\pi = (2\pi - 4)(x - 1)$

$$y = (2\pi - 4)x + 4$$

66. $y = 3x \arcsin x, \quad \left(\frac{1}{2}, \frac{\pi}{4}\right)$

$$y' = 3x\frac{1}{\sqrt{1 - x^2}} + 3\arcsin x$$

At $\left(\frac{1}{2}, \frac{\pi}{4}\right)$, $y' = \frac{3}{2}\frac{1}{\sqrt{3/4}} + 3\left(\frac{\pi}{6}\right) = \sqrt{3} + \frac{\pi}{2}$.

Tangent line: $y - \frac{\pi}{4} = \left(\sqrt{3} + \frac{\pi}{2}\right)\left(x - \frac{1}{2}\right)$

$$y = \left(\sqrt{3} + \frac{\pi}{2}\right)x - \frac{\sqrt{3}}{2}$$

67. $f(x) = \arctan x, \quad a = 0$

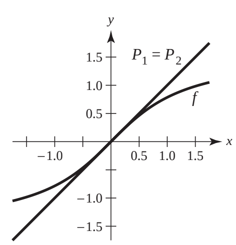
$f(0) = 0$

$f'(x) = \frac{1}{1 + x^2}, \qquad f'(0) = 1$

$f''(x) = \frac{-2x}{(1 + x^2)^2}, \quad f''(0) = 0$

$P_1(x) = f(0) + f'(0)x = x$

$P_2(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 = x$



68. $f(x) = \arccos x, \quad a = 0$

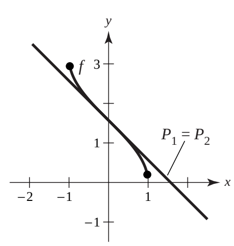
$f(0) = \frac{\pi}{2}$

$f'(x) = \frac{-1}{\sqrt{1 - x^2}}, \quad f'(0) = -1$

$f''(x) = \frac{-x}{(1 - x^2)^{3/2}}, \quad f''(0) = 0$

$P_1(x) = f(0) + f'(0)x = \frac{\pi}{2} - x$

$P_2(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 = \frac{\pi}{2} - x$



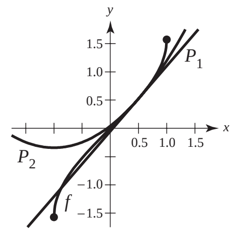
69. $f(x) = \arcsin x, \quad a = \dfrac{1}{2}$

$$f'(x) = \frac{1}{\sqrt{1 - x^2}}$$

$$f''(x) = \frac{x}{(1 - x^2)^{3/2}}$$

$$P_1(x) = f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) = \frac{\pi}{6} + \frac{2\sqrt{3}}{3}\left(x - \frac{1}{2}\right)$$

$$P_2(x) = f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) + \frac{1}{2}f''\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right)^2 = \frac{\pi}{6} + \frac{2\sqrt{3}}{3}\left(x - \frac{1}{2}\right) + \frac{2\sqrt{3}}{9}\left(x - \frac{1}{2}\right)^2$$



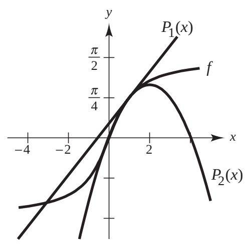
70. $f(x) = \arctan x, \quad a = 1$

$$f'(x) = \frac{1}{1 + x^2}$$

$$f''(x) = \frac{-2x}{(1 + x^2)^2}$$

$$P_1(x) = f(1) + f'(1)(x - 1) = \frac{\pi}{4} + \frac{1}{2}(x - 1)$$

$$P_2(x) = f(1) + f'(1)(x - 1) + \frac{1}{2}f''(1)(x - 1)^2 = \frac{\pi}{4} + \frac{1}{2}(x - 1) - \frac{1}{4}(x - 1)^2$$



71. $f(x) = \operatorname{arcsec} x - x$

$$f'(x) = \frac{1}{|x|\sqrt{x^2 - 1}} - 1$$

$$= 0 \text{ when } |x|\sqrt{x^2 - 1} = 1$$

$$x^2(x^2 - 1) = 1$$

$$x^4 - x^2 - 1 = 0 \text{ when } x^2 = \frac{1 + \sqrt{5}}{2} \text{ or}$$

$$x = \pm\sqrt{\frac{1 + \sqrt{5}}{2}} = \pm 1.272$$

Relative maximum: $(1.272, -0.606)$

Relative minimum: $(-1.272, 3.747)$

72. $f(x) = \arcsin x - 2x$

$$f'(x) = \frac{1}{\sqrt{1 - x^2}} - 2$$

$$= 0 \text{ when } \sqrt{1 - x^2} = \frac{1}{2} \text{ or } x = \pm\frac{\sqrt{3}}{2}$$

$$f''(x) = \frac{x}{(1 - x^2)^{3/2}}$$

$$f''\left(\frac{\sqrt{3}}{2}\right) > 0$$

Relative minimum: $\left(\dfrac{\sqrt{3}}{2}, -0.68\right)$

$$f''\left(-\frac{\sqrt{3}}{2}\right) < 0$$

Relative maximum: $\left(-\dfrac{\sqrt{3}}{2}, 0.68\right)$

73. $f(x) = \arctan x - \arctan(x - 4)$

$$f'(x) = \frac{1}{1 + x^2} - \frac{1}{1 + (x - 4)^2} = 0$$

$$1 + x^2 = 1 + (x - 4)^2$$

$$0 = -8x + 16$$

$$x = 2$$

By the First Derivative Test, $(2, 2.214)$ is a relative maximum.

74. $f(x) = \arcsin x - 2\arctan x$

$$f'(x) = \frac{1}{\sqrt{1 - x^2}} - \frac{2}{1 + x^2} = 0$$

$$1 + x^2 = 2\sqrt{1 - x^2}$$

$$1 + 2x^2 + x^4 = 4(1 - x^2)$$

$$x^4 + 6x^2 - 3 = 0$$

$$x = \pm 0.681$$

By the First Derivative Test, $(-0.681, 0.447)$ is a relative maximum and $(0.681, -0.447)$ is a relative minimum.

75. $x^2 + x \arctan y = y - 1, \quad \left(-\frac{\pi}{4}, 1\right)$

$$2x + \arctan y + \frac{x}{1+y^2}y' = y'$$

$$\left(1 - \frac{x}{1+y^2}\right)y' = 2x + \arctan y$$

$$y' = \frac{2x + \arctan y}{1 - \dfrac{x}{1+y^2}}$$

At $\left(-\frac{\pi}{4}, 1\right)$, $y' = \dfrac{-\dfrac{\pi}{2} + \dfrac{\pi}{4}}{1 - \dfrac{-\pi/4}{2}} = \dfrac{-\dfrac{\pi}{2}}{2 + \dfrac{\pi}{4}} = \dfrac{-2\pi}{8+\pi}.$

Tangent line: $y - 1 = \dfrac{-2\pi}{8+\pi}\left(x + \dfrac{\pi}{4}\right)$

$$y = \frac{-2\pi}{8+\pi}x + 1 - \frac{\pi^2}{16 + 2\pi}$$

76. $\arctan(xy) = \arcsin(x+y), \quad (0, 0)$

$$\frac{1}{1+(xy)^2}[y + xy'] = \frac{1}{\sqrt{1-(x+y)^2}}[1 + y']$$

At $(0, 0)$: $0 = 1 + y' \implies y' = -1$

Tangent line: $y = -x$

77. $\arcsin x + \arcsin y = \frac{\pi}{2}, \quad \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$$\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-y^2}}y' = 0$$

$$\frac{1}{\sqrt{1-y^2}}y' = \frac{-1}{\sqrt{1-x^2}}$$

At $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $\quad y' = -1$

Tangent line: $y - \dfrac{\sqrt{2}}{2} = -1\left(x - \dfrac{\sqrt{2}}{2}\right)$

$$y = -x + \sqrt{2}$$

78. $\arctan(x+y) = y^2 + \frac{\pi}{4}, \quad (1, 0)$

$$\frac{1}{1+(x+y)^2}[1 + y'] = 2yy'$$

At $(1, 0)$: $\frac{1}{2}[1 + y'] = 0 \implies y' = -1$

Tangent line: $y - 0 = -1(x - 1)$

$$y = -x + 1$$

79. The trigonometric functions are not one-to-one on $(-\infty, \infty)$, so their domains must be restricted to intervals on which they are one-to-one.

80. $\arctan 0 = 0$. π is not in the range of $y = \arctan x$.

81. $y = \operatorname{arccot} x, \quad 0 < y < \pi$

$x = \cot y$

$\tan y = \dfrac{1}{x}$

So, graph the function $y = \arctan(1/x)$ for $x > 0$ and $y = \arctan(1/x) + \pi$ for $x < 0$.

82. The derivatives are algebraic. See Theorem 5.18.

83. False

$\arccos \frac{1}{2} = \frac{\pi}{3}$

since the range is $[0, \pi]$.

84. False

$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, so

$\arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}.$

85. True

$\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2} > 0$ for all x.

86. False

The range of $y = \arcsin x$ is $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

87. True

$$\frac{d}{dx}[\arctan(\tan x)] = \frac{\sec^2 x}{1 + \tan^2 x} = \frac{\sec^2 x}{\sec^2 x} = 1$$

88. False

$$\arcsin^2 0 + \arccos^2 0 = 0 + \left(\frac{\pi}{2}\right)^2 \neq 1$$

89. (a) $\cot \theta = \dfrac{x}{5}$

$$\theta = \operatorname{arccot}\left(\frac{x}{5}\right)$$

(b) $\dfrac{d\theta}{dt} = \dfrac{-1/5}{1 + (x/5)^2}\dfrac{dx}{dt} = \dfrac{-5}{x^2 + 25}\dfrac{dx}{dt}$

If $\dfrac{dx}{dt} = -400$ and $x = 10$, $\dfrac{d\theta}{dt} = 16$ rad/hr.

If $\dfrac{dx}{dt} = -400$ and $x = 3$, $\dfrac{d\theta}{dt} \approx 58.824$ rad/hr.

90. (a) $\cot \theta = \dfrac{x}{3}$

$$\theta = \operatorname{arccot}\left(\frac{x}{3}\right)$$

(b) $\dfrac{d\theta}{dt} = \dfrac{-3}{x^2 + 9}\dfrac{dx}{dt}$

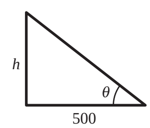
If $x = 10$, $\dfrac{d\theta}{dt} \approx 11.001$ rad/hr.

If $x = 3$, $\dfrac{d\theta}{dt} \approx 66.667$ rad/hr.

A lower altitude results in a greater rate of change of θ.

91. (a) $h(t) = -16t^2 + 256$

$-16t^2 + 256 = 0$ when $t = 4$ sec



(b) $\tan \theta = \dfrac{h}{500} = \dfrac{-16t^2 + 256}{500}$

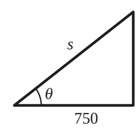
$$\theta = \arctan\left[\frac{16}{500}(-t^2 + 16)\right]$$

$$\frac{d\theta}{dt} = \frac{-8t/125}{1 + [(4/125)(-t^2 + 16)]^2} = \frac{-1000t}{15{,}625 + 16(16 - t^2)^2}$$

When $t = 1$, $d\theta/dt \approx -0.0520$ rad/sec.

When $t = 2$, $d\theta/dt \approx -0.1116$ rad/sec.

92. $\cos \theta = \dfrac{750}{s}$



$$\theta = \arccos\left(\frac{750}{s}\right)$$

$$\frac{d\theta}{dt} = \frac{d\theta}{ds}\cdot\frac{ds}{dt} = \frac{-1}{\sqrt{1 - (750/s)^2}}\left(\frac{-750}{s^2}\right)\frac{ds}{dt} = \frac{750}{s\sqrt{s^2 - 750^2}}\frac{ds}{dt}$$

93. (a) $$\tan(\arctan x + \arctan y) = \frac{\tan(\arctan x) + \tan(\arctan y)}{1 - \tan(\arctan x)\tan(\arctan y)} = \frac{x + y}{1 - xy}, \quad xy \neq 1$$

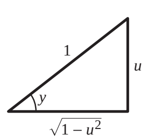
Therefore,

$\arctan x + \arctan y = \arctan\left(\dfrac{x + y}{1 - xy}\right)$, $xy \neq 1$.

(b) Let $x = \frac{1}{2}$ and $y = \frac{1}{3}$.

$$\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \arctan\frac{(1/2) + (1/3)}{1 - [(1/2)\cdot(1/3)]} = \arctan\frac{5/6}{1 - (1/6)} = \arctan\frac{5/6}{5/6} = \arctan 1 = \frac{\pi}{4}$$

94. (a) Let $y = \arcsin u$. Then



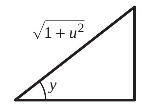
$$\sin y = u$$

$$\cos y \cdot y' = u'$$

$$\frac{dy}{dx} = \frac{u'}{\cos y} = \frac{u'}{\sqrt{1 - u^2}}.$$

—CONTINUED—

(b) Let $y = \arctan u$. Then



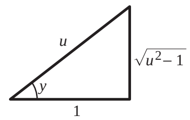
$$\tan y = u$$

$$\sec^2 y\frac{dy}{dx} = u'$$

$$\frac{dy}{dx} = \frac{u'}{\sec^2 y} = \frac{u'}{1 + u^2}.$$

94. —CONTINUED—

(c) Let $y = \operatorname{arcsec} u$. Then



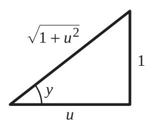
$$\sec y = u$$

$$\sec y \tan y \frac{dy}{dx} = u'$$

$$\frac{dy}{dx} = \frac{u'}{\sec y \tan y} = \frac{u'}{|u|\sqrt{u^2-1}}.$$

Note: The absolute value sign in the formula for the derivative of arcsec u is necessary because the inverse secant function has a positive slope at every value in its domain.

(e) Let $y = \operatorname{arccot} u$. Then

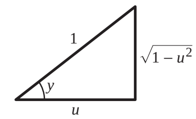


$$\cot y = u$$

$$-\csc^2 y \frac{dy}{dx} = u'$$

$$\frac{dy}{dx} = \frac{u'}{-\csc^2 y} = -\frac{u'}{1+u^2}.$$

(d) Let $y = \arccos u$. Then

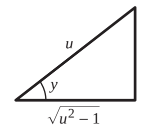


$$\cos y = u$$

$$-\sin y \frac{dy}{dx} = u'$$

$$\frac{dy}{dx} = -\frac{u'}{\sin y} = -\frac{u'}{\sqrt{1-u^2}}.$$

(f) Let $y = \operatorname{arccsc} u$. Then



$$\csc y = u$$

$$-\csc y \cot y \frac{dy}{dx} = u'$$

$$\frac{dy}{dx} = \frac{u'}{-\csc y \cot y} = -\frac{u'}{|u|\sqrt{u^2-1}}$$

Note: The absolute value sign in the formula for the derivative of arccsc u is necessary because the inverse cosecant function has a negative slope at every value in its domain.

95. $f(x) = kx + \sin x$

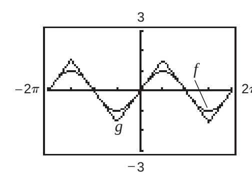
$f'(x) = k + \cos x \geq 0$ for $k \geq 1$

$f'(x) = k + \cos x \leq 0$ for $k \leq -1$

Therefore, $f(x) = kx + \sin x$ is strictly monotonic and has an inverse for $k \leq -1$ or $k \geq 1$.

96. $f(x) = \sin x$

$g(x) = \arcsin(\sin x)$

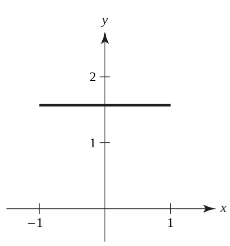


(a) The range of $y = \arcsin x$ is $-\pi/2 \leq y \leq \pi/2$.

(b) Maximum: $\pi/2$

Minimum: $-\pi/2$

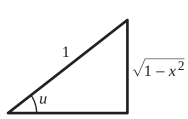
97. (a) $f(x) = \arccos x + \arcsin x$

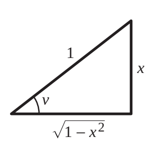


(b) The graph of f is the constant function $y = \pi/2$.

(c) Let $u = \arccos x$ and $v = \arcsin x$

$\cos u = x$ and $\sin v = x$.





$$\begin{aligned}\sin(u+v) &= \sin u \cos v + \sin v \cos u\\ &= \sqrt{1-x^2}\sqrt{1-x^2} + x \cdot x\\ &= 1 - x^2 + x^2 = 1\end{aligned}$$

Hence, $u + v = \pi/2$. Thus, $\arccos x + \arcsin x = \pi/2$.

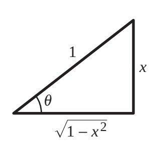
98. Let $\theta = \arctan\left(\dfrac{x}{\sqrt{1-x^2}}\right), \quad -1 < x < 1$

$$\tan \theta = \frac{x}{\sqrt{1-x^2}}$$

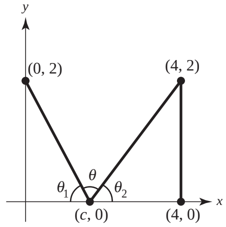
$$\sin \theta = \frac{x}{1} = x$$

$$\arcsin x = \theta.$$

Thus, $\arcsin x = \arctan\left(\dfrac{x}{\sqrt{1-x^2}}\right)$ for $-1 < x < 1$.



99.



$\tan\theta_1 = \dfrac{2}{c},\ \tan\theta_2 = \dfrac{2}{4-c},\ 0 < c < 4$

To maximize θ, we minimize $f(c) = \theta_1 + \theta_2$.

$$f(c) = \arctan\left(\frac{2}{c}\right) + \arctan\left(\frac{2}{4-c}\right)$$

$$f'(c) = \frac{-2}{c^2+4} + \frac{2}{(4-c)^2+4} = 0$$

$$\frac{1}{c^2+4} = \frac{1}{(4-c)^2+4}$$

$$c^2 + 4 = c^2 - 8c + 16 + 4$$

$$8c = 16$$

$$c = 2$$

By the First Derivative Test, $c = 2$ is a minimum. Hence, $(c, f(c)) = (2, \pi/2)$ is a relative maximum for the angle θ. Checking the endpoints:

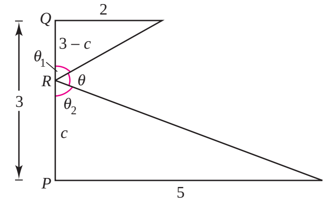
$c = 0$: $\tan\theta = \dfrac{4}{2} = 2 \implies \theta \approx 1.107$

$c = 4$: $\tan\theta = \dfrac{4}{2} = 2 \implies \theta \approx 1.107$

$c = 2$: $\theta = \pi - \theta_1 - \theta_2 = \dfrac{\pi}{2} \approx 1.5708$

Thus, $(2, \pi/2)$ is the absolute maximum.

100.



$\tan\theta_1 = \dfrac{2}{3-c},\ \tan\theta_2 = \dfrac{5}{c},\ 0 < c < 3$

To maximize θ, minimize $f(c) = \theta_1 + \theta_2$.

$$f(c) = \arctan\left(\frac{2}{3-c}\right) + \arctan\left(\frac{5}{c}\right)$$

$$f'(c) = \frac{2}{(3-c)^2+4} + \frac{-5}{c^2+25} = 0$$

$$2(c^2+25) = 5(c^2 - 6c + 9 + 4)$$

$$3c^2 - 30c + 15 = 0$$

$$c^2 - 10c + 5 = 0$$

$$c = 5 - 2\sqrt{5} \approx 0.5279 \text{ (since } c \in [0, 3])$$

$$\theta_1 + \theta_2 \approx 2.1458 \text{ and}$$

$$\theta \approx \pi - (\theta_1 + \theta_2) \approx 0.9958$$

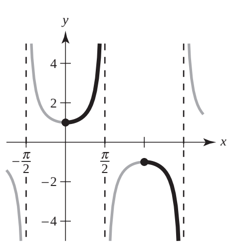
Checking the endpoints:

$c = 3 \implies \tan\theta = \dfrac{3}{5} \implies \theta \approx 0.5404$

$c = 0$: $\tan\theta = \dfrac{3}{2} \implies \theta \approx 0.9828$

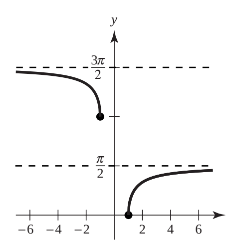
Thus, $c = 5 - 2\sqrt{5}$ yields the absolute maximum.

101. $f(x) = \sec x,\quad 0 \le x < \dfrac{\pi}{2},\ \pi \le x < \dfrac{3\pi}{2}$



(a) $y = \operatorname{arcsec} x,\quad x \le -1 \quad \text{or} \quad x \ge 1$

$$0 \le y < \frac{\pi}{2} \quad \text{or} \quad \pi \le y < \frac{3\pi}{2}$$



(b) $y = \operatorname{arcsec} x$

$x = \sec y$

$1 = \sec y \tan y \cdot y'$

$$y' = \frac{1}{\sec y \tan y}$$

$$= \frac{1}{x\sqrt{x^2-1}}$$

$\tan^2 y + 1 = \sec^2 y$

$\tan y = \pm\sqrt{\sec^2 y - 1}$

On $0 \le y < \pi/2$ and $\pi \le y < 3\pi/2$, $\tan y \ge 0$.

Section 5.7 Inverse Trigonometric Functions: Integration

1. $\displaystyle\int \frac{5}{\sqrt{9-x^2}}\,dx = 5\arcsin\left(\frac{x}{3}\right) + C$

2. $\displaystyle\int \frac{3}{\sqrt{1-4x^2}}\,dx = \frac{3}{2}\int \frac{2}{\sqrt{1-4x^2}}\,dx = \frac{3}{2}\arcsin(2x) + C$

3. $\displaystyle\int \frac{7}{16+x^2}\,dx = \frac{7}{4}\arctan\left(\frac{x}{4}\right) + C$

4. $\displaystyle\int \frac{4}{1+9x^2}\,dx = \frac{4}{3}\int \frac{3}{1+9x^2}\,dx = \frac{4}{3}\arctan(3x) + C$

5. $\displaystyle\int \frac{1}{x\sqrt{4x^2-1}}\,dx = \int \frac{2}{2x\sqrt{(2x)^2-1}}\,dx = \operatorname{arcsec}|2x| + C$

6. $\displaystyle\int \frac{1}{4+(x-1)^2}\,dx = \frac{1}{2}\arctan\left(\frac{x-1}{2}\right) + C$

7. $\displaystyle\int \frac{x^3}{x^2+1}\,dx = \int\left[x - \frac{x}{x^2+1}\right]dx = \int x\,dx - \frac{1}{2}\int \frac{2x}{x^2+1}\,dx = \frac{1}{2}x^2 - \frac{1}{2}\ln(x^2+1) + C$ (Use long division.)

8. $\displaystyle\int \frac{x^4-1}{x^2+1}\,dx = \int (x^2-1)\,dx = \frac{1}{3}x^3 - x + C$

9. $\displaystyle\int \frac{1}{\sqrt{1-(x+1)^2}}\,dx = \arcsin(x+1) + C$

10. Let $u = t^2$, $du = 2t\,dt$.

$$\int \frac{t}{t^4+16}\,dt = \frac{1}{2}\int \frac{1}{(4)^2+(t^2)^2}(2t)\,dt = \frac{1}{8}\arctan\frac{t^2}{4} + C$$

11. Let $u = t^2$, $du = 2t\,dt$.

$$\int \frac{t}{\sqrt{1-t^4}}\,dt = \frac{1}{2}\int \frac{1}{\sqrt{1-(t^2)^2}}(2t)\,dt = \frac{1}{2}\arcsin(t^2) + C$$

12. Let $u = x^2$, $du = 2x\,dx$.

$$\int \frac{1}{x\sqrt{x^4-4}}\,dx = \frac{1}{2}\int \frac{1}{x^2\sqrt{(x^2)^2-2^2}}(2x)\,dx = \frac{1}{4}\operatorname{arcsec}\frac{x^2}{2} + C$$

13. Let $u = e^{2x}$, $du = 2e^{2x}\,dx$.

$$\int \frac{e^{2x}}{4+e^{4x}}\,dx = \frac{1}{2}\int \frac{2e^{2x}}{4+(e^{2x})^2}\,dx = \frac{1}{4}\arctan\frac{e^{2x}}{2} + C$$

14. $\displaystyle\int \frac{1}{3+(x-2)^2}\,dx = \int \frac{1}{\left(\sqrt{3}\right)^2+(x-2)^2}\,dx = \frac{1}{\sqrt{3}}\arctan\left(\frac{x-2}{\sqrt{3}}\right) + C$

15. $\displaystyle\int \frac{1}{\sqrt{x}\sqrt{1-x}}\,dx$, $u = \sqrt{x}$, $x = u^2$, $dx = 2u\,du$

$$\int \frac{1}{u\sqrt{1-u^2}}(2u\,du) = 2\int \frac{du}{\sqrt{1-u^2}} = 2\arcsin u + C = 2\arcsin\sqrt{x} + C$$

16. $\displaystyle\int \frac{3}{2\sqrt{x}(1+x)}\,dx$, $u = \sqrt{x}$, $du = \dfrac{1}{2\sqrt{x}}\,dx$, $dx = 2u\,du$

$$\frac{3}{2}\int \frac{2u\,du}{u(1+u^2)} = 3\int \frac{du}{1+u^2} = 3\arctan u + C = 3\arctan\sqrt{x} + C$$

17. $\displaystyle\int \frac{x-3}{x^2+1}\,dx = \frac{1}{2}\int \frac{2x}{x^2+1}\,dx - 3\int \frac{1}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1) - 3\arctan x + C$

18. $\displaystyle\int \frac{4x+3}{\sqrt{1-x^2}}\,dx = (-2)\int \frac{-2x}{\sqrt{1-x^2}}\,dx + 3\int \frac{1}{\sqrt{1-x^2}}\,dx = -4\sqrt{1-x^2} + 3\arcsin x + C$

19. $\displaystyle\int \frac{x+5}{\sqrt{9-(x-3)^2}}\,dx = \int \frac{(x-3)}{\sqrt{9-(x-3)^2}}\,dx + \int \frac{8}{\sqrt{9-(x-3)^2}}\,dx$

$$= -\sqrt{9-(x-3)^2} - 8\arcsin\left(\frac{x-3}{3}\right) + C = -\sqrt{6x-x^2} + 8\arcsin\left(\frac{x}{3}-1\right) + C$$

20. $\displaystyle\int \frac{x-2}{(x+1)^2+4}\,dx = \frac{1}{2}\int \frac{2x+2}{(x+1)^2+4}\,dx - \int \frac{3}{(x+1)^2+4}\,dx$

$\displaystyle= \frac{1}{2}\ln(x^2+2x+5) - \frac{3}{2}\arctan\left(\frac{x+1}{2}\right) + C$

21. Let $u = 3x$, $du = 3\,dx$.

$\displaystyle\int_0^{1/6} \frac{1}{\sqrt{1-9x^2}}\,dx = \frac{1}{3}\int_0^{1/6} \frac{1}{\sqrt{1-(3x)^2}}(3)\,dx$

$\displaystyle= \left[\frac{1}{3}\arcsin(3x)\right]_0^{1/6} = \frac{\pi}{18}$

22. $\displaystyle\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx = \left[\arcsin\frac{x}{2}\right]_0^1 = \frac{\pi}{6}$

23. Let $u = 2x$, $du = 2\,dx$.

$\displaystyle\int_0^{\sqrt{3}/2} \frac{1}{1+4x^2}\,dx = \frac{1}{2}\int_0^{\sqrt{3}/2} \frac{2}{1+(2x)^2}\,dx$

$\displaystyle= \left[\frac{1}{2}\arctan(2x)\right]_0^{\sqrt{3}/2} = \frac{\pi}{6}$

24. $\displaystyle\int_{\sqrt{3}}^3 \frac{1}{9+x^2}\,dx = \left[\frac{1}{3}\arctan\frac{x}{3}\right]_{\sqrt{3}}^3 = \frac{\pi}{36}$

25. Let $u = \arcsin x$, $du = \dfrac{1}{\sqrt{1-x^2}}\,dx$.

$\displaystyle\int_0^{1/\sqrt{2}} \frac{\arcsin x}{\sqrt{1-x^2}}\,dx = \left[\frac{1}{2}\arcsin^2 x\right]_0^{1/\sqrt{2}} = \frac{\pi^2}{32} \approx 0.308$

26. Let $u = \arccos x$, $du = -\dfrac{1}{\sqrt{1-x^2}}\,dx$.

$\displaystyle\int_0^{1/\sqrt{2}} \frac{\arccos x}{\sqrt{1-x^2}}\,dx = -\int_0^{1/\sqrt{2}} \frac{-\arccos x}{\sqrt{1-x^2}}\,dx$

$\displaystyle= \left[-\frac{1}{2}\arccos^2 x\right]_0^{1/\sqrt{2}} = \frac{3\pi^2}{32} \approx 0.925$

27. Let $u = 1-x^2$, $du = -2x\,dx$.

$\displaystyle\int_{-1/2}^0 \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int_{-1/2}^0 (1-x^2)^{-1/2}(-2x)\,dx$

$\displaystyle= \left[-\sqrt{1-x^2}\right]_{-1/2}^0 = \frac{\sqrt{3}-2}{2}$

≈ -0.134

28. Let $u = 1+x^2$, $du = 2x\,dx$.

$\displaystyle\int_{-\sqrt{3}}^0 \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_{-\sqrt{3}}^0 \frac{1}{1+x^2}(2x)\,dx$

$\displaystyle= \left[\frac{1}{2}\ln(1+x^2)\right]_{-\sqrt{3}}^0 = -\ln 2$

29. Let $u = \cos x$, $du = -\sin x\,dx$.

$\displaystyle\int_{\pi/2}^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx = -\int_{\pi/2}^{\pi} \frac{-\sin x}{1+\cos^2 x}\,dx$

$\displaystyle= \left[-\arctan(\cos x)\right]_{\pi/2}^{\pi} = \frac{\pi}{4}$

30. $\displaystyle\int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x}\,dx = \arctan(\sin x)\Big]_0^{\pi/2} = \frac{\pi}{4}$

31. $\displaystyle\int_0^2 \frac{dx}{x^2-2x+2} = \int_0^2 \frac{1}{1+(x-1)^2}\,dx$

$\displaystyle= \left[\arctan(x-1)\right]_0^2 = \frac{\pi}{2}$

32. $\displaystyle\int_{-2}^2 \frac{dx}{x^2+4x+13} = \int_{-2}^2 \frac{dx}{(x+2)^2+9}$

$\displaystyle= \left[\frac{1}{3}\arctan\left(\frac{x+2}{3}\right)\right]_{-2}^2$

$\displaystyle= \frac{1}{3}\arctan\left(\frac{4}{3}\right)$

33. $$\int \frac{2x}{x^2+6x+13}\,dx = \int \frac{2x+6}{x^2+6x+13}\,dx - 6\int \frac{1}{x^2+6x+13}\,dx = \int \frac{2x+6}{x^2+6x+13}\,dx - 6\int \frac{1}{4+(x+3)^2}\,dx$$
$$= \ln\left|x^2+6x+13\right| - 3\arctan\left(\frac{x+3}{2}\right) + C$$

34. $$\int \frac{2x-5}{x^2+2x+2}\,dx = \int \frac{2x+2}{x^2+2x+2}\,dx - 7\int \frac{1}{1+(x+1)^2}\,dx = \ln\left|x^2+2x+2\right| - 7\arctan(x+1) + C$$

35. $$\int \frac{1}{\sqrt{-x^2-4x}}\,dx = \int \frac{1}{\sqrt{4-(x+2)^2}}\,dx = \arcsin\left(\frac{x+2}{2}\right) + C$$

36. $$\int \frac{2}{\sqrt{-x^2+4x}}\,dx = \int \frac{2}{\sqrt{4-(x^2-4x+4)}}\,dx = \int \frac{2}{\sqrt{4-(x-2)^2}}\,dx = 2\arcsin\left(\frac{x-2}{2}\right) + C$$

37. Let $u = -x^2 - 4x$, $du = (-2x-4)\,dx$.
$$\int \frac{x+2}{\sqrt{-x^2-4x}}\,dx = -\frac{1}{2}\int (-x^2-4x)^{-1/2}(-2x-4)\,dx = -\sqrt{-x^2-4x} + C$$

38. Let $u = x^2 - 2x$, $du = (2x-2)\,dx$.
$$\int \frac{x-1}{\sqrt{x^2-2x}}\,dx = \frac{1}{2}\int (x^2-2x)^{-1/2}(2x-2)\,dx = \sqrt{x^2-2x} + C$$

39. $$\int_2^3 \frac{2x-3}{\sqrt{4x-x^2}}\,dx = \int_2^3 \frac{2x-4}{\sqrt{4x-x^2}}\,dx + \int_2^3 \frac{1}{\sqrt{4x-x^2}}\,dx = -\int_2^3 (4x-x^2)^{-1/2}(4-2x)\,dx + \int_2^3 \frac{1}{\sqrt{4-(x-2)^2}}\,dx$$
$$= \left[-2\sqrt{4x-x^2} + \arcsin\left(\frac{x-2}{2}\right)\right]_2^3 = 4 - 2\sqrt{3} + \frac{\pi}{6} \approx 1.059$$

40. $$\int \frac{1}{(x-1)\sqrt{x^2-2x}}\,dx = \int \frac{1}{(x-1)\sqrt{(x-1)^2-1}}\,dx = \operatorname{arcsec}\left|x-1\right| + C$$

41. Let $u = x^2 + 1$, $du = 2x\,dx$.
$$\int \frac{x}{x^4+2x^2+2}\,dx = \frac{1}{2}\int \frac{2x}{(x^2+1)^2+1}\,dx = \frac{1}{2}\arctan(x^2+1) + C$$

42. Let $u = x^2 - 4$, $du = 2x\,dx$.
$$\int \frac{x}{\sqrt{9+8x^2-x^4}}\,dx = \frac{1}{2}\int \frac{2x}{\sqrt{25-(x^2-4)^2}}\,dx = \frac{1}{2}\arcsin\left(\frac{x^2-4}{5}\right) + C$$

43. Let $u = \sqrt{e^t-3}$. Then $u^2 + 3 = e^t$, $2u\,du = e^t\,dt$, and $\dfrac{2u\,du}{u^2+3} = dt$.
$$\int \sqrt{e^t-3}\,dt = \int \frac{2u^2}{u^2+3}\,du = \int 2\,du - \int 6\,\frac{1}{u^2+3}\,du$$
$$= 2u - 2\sqrt{3}\arctan\frac{u}{\sqrt{3}} + C = 2\sqrt{e^t-3} - 2\sqrt{3}\arctan\sqrt{\frac{e^t-3}{3}} + C$$

44. Let $u = \sqrt{x-2}$, $u^2 + 2 = x$, $2u\,du = dx$.
$$\int \frac{\sqrt{x-2}}{x+1}\,dx = \int \frac{2u^2}{u^2+3}\,du = \int \frac{2u^2+6-6}{u^2+3}\,du = 2\int du - 6\int \frac{1}{u^2+3}\,du$$
$$= 2u - \frac{6}{\sqrt{3}}\arctan\frac{u}{\sqrt{3}} + C = 2\sqrt{x-2} - 2\sqrt{3}\arctan\sqrt{\frac{x-2}{3}} + C$$

45. $\displaystyle\int_1^3 \frac{dx}{\sqrt{x}(1+x)}$

Let $u = \sqrt{x}$, $u^2 = x$, $2u\,du = dx$, $1 + x = 1 + u^2$.

$$\int_1^{\sqrt{3}} \frac{2u\,du}{u(1+u^2)} = \int_1^{\sqrt{3}} \frac{2}{1+u^2}\,du$$
$$= 2\arctan(u)\Big]_1^{\sqrt{3}}$$
$$= 2\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\pi}{6}$$

46. $\displaystyle\int_0^1 \frac{dx}{2\sqrt{3-x}\sqrt{x+1}}$

Let $u = \sqrt{x+1}$, $u^2 = x + 1$, $2u\,du = dx$, $\sqrt{3-x} = \sqrt{4-u^2}$.

$$\int_1^{\sqrt{2}} \frac{2u\,du}{2\sqrt{4-u^2}\,u} = \int_1^{\sqrt{2}} \frac{du}{\sqrt{4-u^2}}$$
$$= \arcsin\left(\frac{u}{2}\right)\Big]_1^{\sqrt{2}}$$
$$= \arcsin\left(\frac{\sqrt{2}}{2}\right) - \arcsin\left(\frac{1}{2}\right)$$
$$= \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$$

47. (a) $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C, \quad u = x$

(b) $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx = -\sqrt{1-x^2} + C, \quad u = 1 - x^2$

(c) $\displaystyle\int \frac{1}{x\sqrt{1-x^2}}\,dx$ cannot be evaluated using the basic integration rules.

48. (a) $\displaystyle\int e^{x^2}\,dx$ cannot be evaluated using the basic integration rules.

(b) $\displaystyle\int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + C, \quad u = x^2$

(c) $\displaystyle\int \frac{1}{x^2}e^{1/x}\,dx = -e^{1/x} + C, \quad u = \frac{1}{x}$

49. (a) $\displaystyle\int \sqrt{x-1}\,dx = \frac{2}{3}(x-1)^{3/2} + C, \quad u = x - 1$

(b) Let $u = \sqrt{x-1}$. Then $x = u^2 + 1$ and $dx = 2u\,du$.

$$\int x\sqrt{x-1}\,dx = \int (u^2+1)(u)(2u)\,du = 2\int (u^4 + u^2)\,du = 2\left(\frac{u^5}{5} + \frac{u^3}{3}\right) + C$$
$$= \frac{2}{15}u^3(3u^2 + 5) + C = \frac{2}{15}(x-1)^{3/2}[3(x-1) + 5] + C = \frac{2}{15}(x-1)^{3/2}(3x+2) + C$$

(c) Let $u = \sqrt{x-1}$. Then $x = u^2 + 1$ and $dx = 2u\,du$.

$$\int \frac{x}{\sqrt{x-1}}\,dx = \int \frac{u^2+1}{u}(2u)\,du = 2\int (u^2+1)\,du = 2\left(\frac{u^3}{3} + u\right) + C = \frac{2}{3}u(u^2+3) + C = \frac{2}{3}\sqrt{x-1}(x+2) + C$$

Note: In (b) and (c), substitution was necessary *before* the basic integration rules could be used.

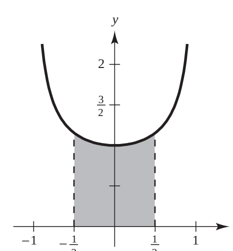
50. (a) $\displaystyle\int \frac{1}{1+x^4}\,dx$ cannot be evaluated using the basic integration rules.

(b) $\displaystyle\int \frac{x}{1+x^4}\,dx = \frac{1}{2}\int \frac{2x}{1+(x^2)^2}\,dx$
$$= \frac{1}{2}\arctan(x^2) + C, \quad u = x^2$$

(c) $\displaystyle\int \frac{x^3}{1+x^4}\,dx = \frac{1}{4}\int \frac{4x^3}{1+x^4}\,dx$
$$= \frac{1}{4}\ln(1+x^4) + C, \quad u = 1 + x^4$$

51. Area $\approx (1)(1) = 1$

Matches (c)



52. No. This integral does not correspond to any of the basic differentiation rules.

53. $y' = \dfrac{1}{\sqrt{4 - x^2}}, \quad (0, \pi)$

$$y = \int \frac{1}{\sqrt{4 - x^2}}\,dx = \arcsin\left(\frac{x}{2}\right) + C$$

$$y(0) = \pi = C$$

$$y = \arcsin\left(\frac{x}{2}\right) + \pi$$

54. $y' = \dfrac{1}{4 + x^2}, \quad (2, \pi)$

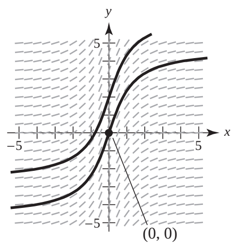
$$y = \int \frac{1}{4 + x^2}\,dx = \frac{1}{2}\arctan\frac{x}{2} + C$$

$$\pi = \frac{1}{2}\arctan\left(\frac{2}{2}\right) + C$$

$$= \frac{\pi}{8} + C \implies C = \frac{7\pi}{8}$$

$$y = \frac{1}{2}\arctan\left(\frac{x}{2}\right) + \frac{7\pi}{8}$$

55. (a)

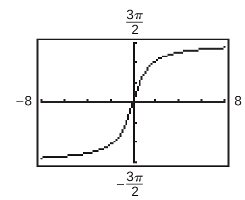


(b) $\dfrac{dy}{dx} = \dfrac{3}{1 + x^2}, \quad (0, 0)$

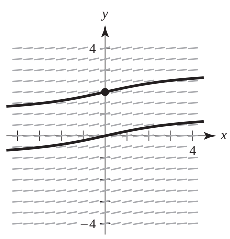
$$y = 3\int \frac{dx}{1 + x^2} = 3\arctan x + C$$

$$(0, 0)\text{: } 0 = 3\arctan(0) + C \implies C = 0$$

$$y = 3\arctan x$$



56. (a)

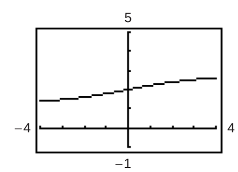


(b) $y' = \dfrac{2}{9 + x^2}, \ (0, 2)$

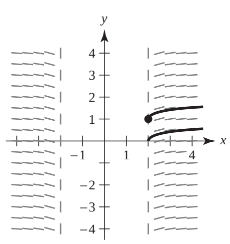
$$y = \int \frac{2}{9 + x^2}\,dx = \frac{2}{3}\arctan\left(\frac{x}{3}\right) + C$$

$$2 = C$$

$$y = \frac{2}{3}\arctan\left(\frac{x}{3}\right) + 2$$



57. (a)

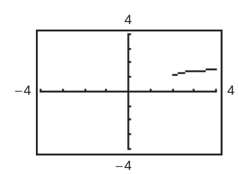


(b) $y' = \dfrac{1}{x\sqrt{x^2 - 4}}, \quad (2, 1)$

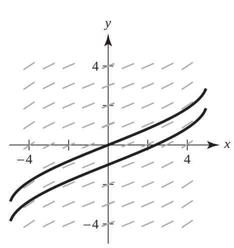
$$y = \int \frac{1}{x\sqrt{x^2 - 4}}\,dx = \frac{1}{2}\operatorname{arcsec}\frac{|x|}{2} + C$$

$$1 = \frac{1}{2}\operatorname{arcsec}(1) + C = C$$

$$y = \frac{1}{2}\operatorname{arcsec}\frac{x}{2} + 1, \quad x \geq 2$$



58. (a)

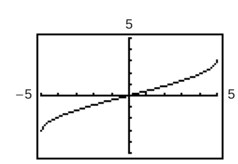


(b) $y' = \dfrac{2}{\sqrt{25 - x^2}}, \quad (5, \pi)$

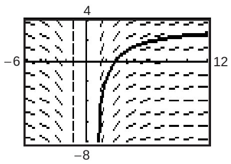
$$y = \int \frac{2}{\sqrt{25 - x^2}}\,dx = 2\arcsin\left(\frac{x}{5}\right) + C$$

$$\pi = 2\arcsin(1) + C \implies C = 0$$

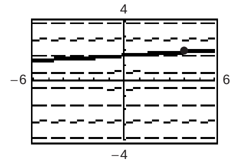
$$y = 2\arcsin\left(\frac{x}{5}\right)$$



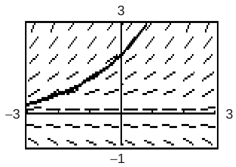
59. $\dfrac{dy}{dx} = \dfrac{10}{x\sqrt{x^2-1}}, \quad y(3) = 0$



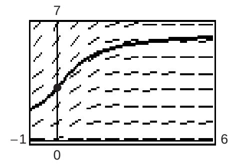
60. $\dfrac{dy}{dx} = \dfrac{1}{12+x^2}, \quad y(4) = 2$



61. $\dfrac{dy}{dx} = \dfrac{2y}{\sqrt{16-x^2}}, \quad y(0) = 2$



62. $\dfrac{dy}{dx} = \dfrac{\sqrt{y}}{1+x^2}, \quad y(0) = 4$



63. $A = \displaystyle\int_1^3 \frac{1}{x^2-2x+5}\,dx = \int_1^3 \frac{1}{(x-1)^2+4}\,dx$

$$= \frac{1}{2}\arctan\left(\frac{x-1}{2}\right)\Big]_1^3$$

$$= \frac{1}{2}\arctan(1) - \frac{1}{2}\arctan(0)$$

$$= \frac{\pi}{8}$$

64. $\text{Area} = \displaystyle\int_{-2}^0 \frac{2}{x^2+4x+8}\,dx = \int_{-2}^0 \frac{2}{(x+2)^2+4}\,dx$

$$= \arctan\left(\frac{x+2}{2}\right)\Big]_{-2}^0$$

$$= \arctan(1) - \arctan(0)$$

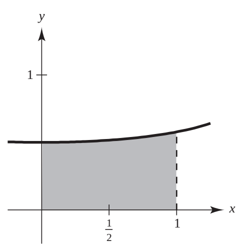
$$= \frac{\pi}{4}$$

65. $\text{Area} = \displaystyle\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx$

$$= \arcsin\left(\frac{x}{2}\right)\Big]_0^1$$

$$= \arcsin\left(\frac{1}{2}\right) - \arcsin(0)$$

$$= \frac{\pi}{6}$$



66. $\text{Area} = \displaystyle\int_{2/\sqrt{2}}^2 \frac{1}{x\sqrt{x^2-1}}\,dx$

$$= \operatorname{arcsec} x\Big]_{2/\sqrt{2}}^2$$

$$= \operatorname{arcsec}(2) - \operatorname{arcsec}\left(\frac{2}{\sqrt{2}}\right)$$

$$= \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

67. $\text{Area} = \displaystyle\int_{-\pi/2}^{\pi/2} \frac{3\cos x}{1+\sin^2 x}\,dx = 3\int_{-\pi/2}^{\pi/2} \frac{1}{1+\sin^2 x}(\cos x\,dx)$

$$= 3\arctan(\sin x)\Big]_{-\pi/2}^{\pi/2}$$

$$= 3\arctan(1) - 3\arctan(-1)$$

$$= \frac{3\pi}{4} + \frac{3\pi}{4} = \frac{3\pi}{2}$$

68. $A = \displaystyle\int_0^{\ln(\sqrt{3})} \frac{e^x}{1+e^{2x}}\,dx, \quad (u = e^x)$

$$= \arctan(e^x)\Big]_0^{\ln\sqrt{3}}$$

$$= \arctan\left(\sqrt{3}\right) - \arctan(1)$$

$$= \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

69. (a) $$\frac{d}{dx}\left[\ln x - \frac{1}{2}\ln(1+x^2) - \frac{\arctan x}{x} + C\right] = \frac{1}{x} - \frac{x}{1+x^2} - \left(\frac{x[1/(1+x^2)] - \arctan x}{x^2}\right)$$
$$= \frac{1+x^2-x^2}{x(1+x^2)} - \frac{1}{x(1+x^2)} + \frac{\arctan x}{x^2} = \frac{\arctan x}{x^2}$$

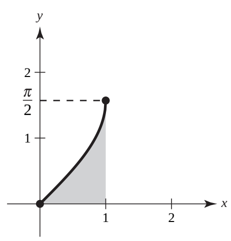
Thus, $\displaystyle\int \frac{\arctan x}{x^2}\,dx = \ln x - \frac{1}{2}\ln(1+x^2) - \frac{\arctan x}{x} + C.$

(b) $$A = \int_1^{\sqrt{3}} \frac{\arctan x}{x^2}\,dx$$
$$= \left[\ln x - \frac{1}{2}\ln(1+x^2) - \frac{\arctan x}{x}\right]_1^{\sqrt{3}}$$
$$= \left(\ln\sqrt{3} - \frac{1}{2}\ln(4) - \frac{\arctan\sqrt{3}}{\sqrt{3}}\right) - \left(\frac{-1}{2}\ln 2 - \arctan(1)\right)$$
$$= \frac{1}{2}\ln 3 - \frac{1}{2}\ln 2 - \frac{\pi\sqrt{3}}{9} + \frac{\pi}{4} \approx 0.3835$$

70. (a) $$\frac{d}{dx}[x(\arcsin x)^2 - 2x + 2\sqrt{1-x^2}\arcsin x + C]$$
$$= (\arcsin x)^2 + 2x(\arcsin x)\frac{1}{\sqrt{1-x^2}} - 2 - \frac{2x}{\sqrt{1-x^2}}\arcsin x + 2\sqrt{1-x^2}\frac{1}{\sqrt{1-x^2}} = (\arcsin x)^2$$

(b) $$A = \int_0^1 (\arcsin x)^2\,dx$$
$$= \left[x(\arcsin x)^2 - 2x + 2\sqrt{1-x^2}\arcsin x\right]_0^1$$
$$= \left(\left(\frac{\pi}{2}\right)^2 - 2\right) - (0)$$
$$= \frac{\pi^4}{4} - 2 \approx 0.4674$$

71. (a)



Shaded area is given by $\displaystyle\int_0^1 \arcsin x\,dx$.

(b) $\displaystyle\int_0^1 \arcsin x\,dx \approx 0.5708$

(c) Divide the rectangle into two regions.

$$\text{Area rectangle} = (\text{base})(\text{height}) = 1\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$$
$$\text{Area rectangle} = \int_0^1 \arcsin x\,dx + \int_0^{\pi/2} \sin y\,dy$$
$$\frac{\pi}{2} = \int_0^1 \arcsin x\,dx + (-\cos y)\Big]_0^{\pi/2}$$
$$= \int_0^1 \arcsin x\,dx + 1$$

Hence, $\displaystyle\int_0^1 \arcsin x\,dx = \frac{\pi}{2} - 1,\quad (\approx 0.5708).$

72. (a) $$\int_0^1 \frac{4}{1+x^2}\,dx = \left[4\arctan x\right]_0^1 = 4\arctan 1 - 4\arctan 0 = 4\left(\frac{\pi}{4}\right) - 4(0) = \pi$$

(b) Let $n = 6$.

$$4\int_0^1 \frac{1}{1+x^2}\,dx \approx 4\left(\frac{1}{18}\right)\left[1 + \frac{4}{1+(1/36)} + \frac{2}{1+(1/9)} + \frac{4}{1+(1/4)} + \frac{2}{1+(4/9)} + \frac{4}{1+(25/36)} + \frac{1}{2}\right] \approx 3.1415918$$

(c) 3.1415927

73. $F(x) = \dfrac{1}{2}\displaystyle\int_x^{x+2} \frac{2}{t^2+1}\,dt$

(a) $F(x)$ represents the average value of $f(x)$ over the interval $[x, x+2]$. Maximum at $x = -1$, since the graph is greatest on $[-1, 1]$.

(b) $F(x) = \left[\arctan t\right]_x^{x+2} = \arctan(x+2) - \arctan x$

$$F'(x) = \frac{1}{1+(x+2)^2} - \frac{1}{1+x^2} = \frac{(1+x^2)-(x^2+4x+5)}{(x^2+1)(x^2+4x+5)} = \frac{-4(x+1)}{(x^2+1)(x^2+4x+5)} = 0 \text{ when } x = -1.$$

74. $\displaystyle\int \frac{1}{\sqrt{6x-x^2}}\,dx$

(a) $6x - x^2 = 9 - (x^2 - 6x + 9) = 9 - (x-3)^2$

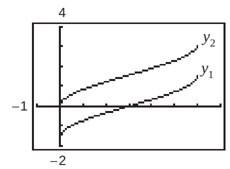
$$\int \frac{1}{\sqrt{6x-x^2}}\,dx = \int \frac{dx}{\sqrt{9-(x-3)^2}} = \arcsin\left(\frac{x-3}{3}\right) + C$$

(b) $u = \sqrt{x}$, $u^2 = x$, $2u\,du = dx$

$$\int \frac{1}{\sqrt{6u^2-u^4}}(2u\,du) = \int \frac{2}{\sqrt{6-u^2}}\,du$$

$$= 2\arcsin\left(\frac{u}{\sqrt{6}}\right) + C = 2\arcsin\left(\frac{\sqrt{x}}{\sqrt{6}}\right) + C$$

(c)



The antiderivatives differ by a constant, $\pi/2$.

Domain: $[0, 6]$

75. False, $\displaystyle\int \frac{dx}{3x\sqrt{9x^2-16}} = \frac{1}{12}\,\mathrm{arcsec}\,\frac{|3x|}{4} + C$

76. False, $\displaystyle\int \frac{dx}{25+x^2}\,dx = \frac{1}{5}\arctan\frac{x}{5} + C$

77. True

$$\frac{d}{dx}\left[-\arccos\frac{x}{2} + C\right] = \frac{1/2}{\sqrt{1-(x/2)^2}} = \frac{1}{\sqrt{4-x^2}}$$

78. False. Use substitution: $u = 9 - e^{2x}$, $du = -2e^{2x}\,dx$

79. $\dfrac{d}{dx}\left[\arcsin\left(\dfrac{u}{a}\right) + C\right] = \dfrac{1}{\sqrt{1-(u^2/a^2)}}\left(\dfrac{u'}{a}\right) = \dfrac{u'}{\sqrt{a^2-u^2}}$

Thus, $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right) + C.$

80. $\dfrac{d}{dx}\left[\dfrac{1}{a}\arctan\dfrac{u}{a} + C\right] = \dfrac{1}{a}\left[\dfrac{u'/a}{1+(u/a)^2}\right]$

$$= \frac{1}{a^2}\left[\frac{u'}{(a^2+u^2)/a^2}\right] = \frac{u'}{a^2+u^2}$$

Thus, $\displaystyle\int \frac{du}{a^2+u^2} = \int \frac{u'}{a^2+u^2}\,dx = \frac{1}{a}\arctan\frac{u}{a} + C.$

81. Assume $u > 0$.

$$\frac{d}{dx}\left[\frac{1}{a}\,\mathrm{arcsec}\,\frac{u}{a} + C\right] = \frac{1}{a}\left[\frac{u'/a}{(u/a)\sqrt{(u/a)^2-1}}\right] = \frac{1}{a}\left[\frac{u'}{u\sqrt{(u^2-a^2)/a^2}}\right] = \frac{u'}{u\sqrt{u^2-a^2}}.$$

The case $u < 0$ is handled in a similar manner. Thus,

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \int \frac{u'}{u\sqrt{u^2-a^2}}\,dx = \frac{1}{a}\,\mathrm{arcsec}\,\frac{|u|}{a} + C.$$

82. (a) $A = \displaystyle\int_0^1 \frac{1}{1+x^2}\,dx$

(b) Trapezoidal Rule: $n = 8$, $b - a = 1 - 0 = 1$

$A \approx 0.7847$

—CONTINUED—

82. —CONTINUED—

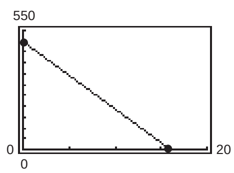
(c) Because

$$\int_0^1 \frac{1}{1+x^2}\,dx = \arctan x\Big]_0^1 = \frac{\pi}{4},$$

you can use the Trapezoidal Rule to approximate $\pi/4$, and hence, π. For example, using $n = 200$, you obtain

$\pi \approx 4(0.785397) = 3.141588.$

83. (a) $v(t) = -32t + 500$



(b) $s(t) = \int v(t)\,dt = \int (-32t + 500)\,dt$

$$= -16t^2 + 500t + C$$

$$s(0) = -16(0) + 500(0) + C = 0 \implies C = 0$$

$$s(t) = -16t^2 + 500t$$

When the object reaches its maximum height, $v(t) = 0$.

$$v(t) = -32t + 500 = 0$$

$$-32t = -500$$

$$t = 15.625$$

$$s(15.625) = -16(15.625)^2 + 500(15.625)$$

$$= 3906.25 \text{ ft (Maximum height)}$$

(c)
$$\int \frac{1}{32 + kv^2}\,dv = -\int dt$$

$$\frac{1}{\sqrt{32k}}\arctan\left(\sqrt{\frac{k}{32}}v\right) = -t + C_1$$

$$\arctan\left(\sqrt{\frac{k}{32}}v\right) = -\sqrt{32k}\,t + C$$

$$\sqrt{\frac{k}{32}}v = \tan\left(C - \sqrt{32k}\,t\right)$$

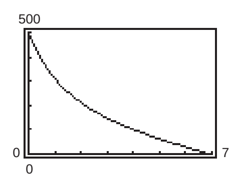
$$v = \sqrt{\frac{32}{k}}\tan\left(C - \sqrt{32k}\,t\right)$$

When $t = 0$, $v = 500$, $C = \arctan\left(500\sqrt{k/32}\right)$, and we have

$$v(t) = \sqrt{\frac{32}{k}}\tan\left[\arctan\left(500\sqrt{\frac{k}{32}}\right) - \sqrt{32k}\,t\right].$$

(d) When $k = 0.001$:

$$v(t) = \sqrt{32{,}000}\tan\left[\arctan\left(500\sqrt{0.00003125}\right) - \sqrt{0.032}\,t\right]$$



$v(t) = 0$ when $t_0 \approx 6.86$ sec.

(e) $h = \int_0^{6.86} \sqrt{32{,}000}\tan\left[\arctan\left(500\sqrt{0.00003125}\right) - \sqrt{0.032}\,t\right]dt$

Simpson's Rule: $n = 10$; $h \approx 1088$ feet

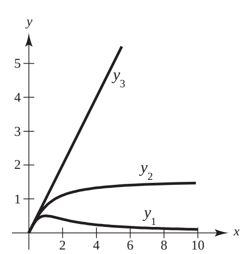
(f) Air resistance lowers the maximum height.

84. Let $f(x) = \arctan x - \dfrac{x}{1+x^2}$

$$f'(x) = \frac{1}{1+x^2} - \frac{1-x^2}{(1+x^2)^2} = \frac{2x^2}{(1+x^2)} > 0 \text{ for } x > 0.$$

Since $f(0) = 0$ and f is increasing for $x > 0$,

$\arctan x - \dfrac{x}{1+x^2} > 0$ for $x > 0$. Thus, $\arctan x > \dfrac{x}{1+x^2}$.



Let $g(x) = x - \arctan x$

$$g'(x) = 1 - \frac{1}{1+x^2} = \frac{x^2}{1+x^2} > 0 \text{ for } x > 0.$$

Since $g(0) = 0$ and g is increasing for $x > 0$, $x - \arctan x > 0$ for $x > 0$. Thus, $x > \arctan x$. Therefore,

$\dfrac{x}{1+x^2} < \arctan x < x.$

Section 5.8 Hyperbolic Functions

1. (a) $\sinh 3 = \dfrac{e^3 - e^{-3}}{2} \approx 10.018$

(b) $\tanh(-2) = \dfrac{\sinh(-2)}{\cosh(-2)} = \dfrac{e^{-2} - e^2}{e^{-2} + e^2} \approx -0.964$

2. (a) $\cosh 0 = \dfrac{e^0 + e^0}{2} = 1$

(b) $\operatorname{sech} 1 = \dfrac{2}{e + e^{-1}} \approx 0.648$

3. (a) $\operatorname{csch}(\ln 2) = \dfrac{2}{e^{\ln 2} - e^{-\ln 2}} = \dfrac{2}{2 - (1/2)} = \dfrac{4}{3}$

(b) $\coth(\ln 5) = \dfrac{\cosh(\ln 5)}{\sinh(\ln 5)} = \dfrac{e^{\ln 5} + e^{-\ln 5}}{e^{\ln 5} - e^{-\ln 5}}$

$= \dfrac{5 + (1/5)}{5 - (1/5)} = \dfrac{13}{12}$

4. (a) $\sinh^{-1} 0 = 0$

(b) $\tanh^{-1} 0 = 0$

5. (a) $\cosh^{-1} 2 = \ln\left(2 + \sqrt{3}\right) \approx 1.317$

(b) $\operatorname{sech}^{-1} \dfrac{2}{3} = \ln\left(\dfrac{1 + \sqrt{1 - (4/9)}}{2/3}\right) \approx 0.962$

6. (a) $\operatorname{csch}^{-1} 2 = \ln\left(\dfrac{1 + \sqrt{5}}{2}\right) \approx 0.481$

(b) $\coth^{-1} 3 = \dfrac{1}{2} \ln\left(\dfrac{4}{2}\right) \approx 0.347$

7. $\tanh^2 x + \operatorname{sech}^2 x = \left(\dfrac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2 + \left(\dfrac{2}{e^x + e^{-x}}\right)^2 = \dfrac{e^{2x} - 2 + e^{-2x} + 4}{(e^x + e^{-x})^2} = \dfrac{e^{2x} + 2 + e^{-2x}}{e^{2x} + 2 + e^{-2x}} = 1$

8. $\dfrac{1 + \cosh 2x}{2} = \dfrac{1 + (e^{2x} + e^{-2x})/2}{2} = \dfrac{e^{2x} + 2 + e^{-2x}}{4} = \left(\dfrac{e^x + e^{-x}}{2}\right)^2 = \cosh^2 x$

9. $\sinh x \cosh y + \cosh x \sinh y = \left(\dfrac{e^x - e^{-x}}{2}\right)\left(\dfrac{e^y + e^{-y}}{2}\right) + \left(\dfrac{e^x + e^{-x}}{2}\right)\left(\dfrac{e^y - e^{-y}}{2}\right)$

$= \dfrac{1}{4}[e^{x+y} - e^{-x+y} + e^{x-y} - e^{-(x+y)} + e^{x+y} + e^{-x+y} - e^{x-y} - e^{-(x+y)}]$

$= \dfrac{1}{4}[2(e^{x+y} - e^{-(x+y)})] = \dfrac{e^{(x+y)} - e^{-(x+y)}}{2} = \sinh(x + y)$

10. $2 \sinh x \cosh x = 2\left(\dfrac{e^x - e^{-x}}{2}\right)\left(\dfrac{e^x + e^{-x}}{2}\right) = \dfrac{e^{2x} - e^{-2x}}{2} = \sinh 2x$

11. $3 \sinh x + 4 \sinh^3 x = \sinh x(3 + 4 \sinh^2 x) = \left(\dfrac{e^x - e^{-x}}{2}\right)\left[3 + 4\left(\dfrac{e^x - e^{-x}}{2}\right)^2\right]$

$= \left(\dfrac{e^x - e^{-x}}{2}\right)[3 + e^{2x} - 2 + e^{-2x}] = \dfrac{1}{2}(e^x - e^{-x})(e^{2x} + e^{-2x} + 1)$

$= \dfrac{1}{2}[e^{3x} + e^{-x} + e^x - e^x - e^{-3x} - e^{-x}] = \dfrac{e^{3x} - e^{-3x}}{2} = \sinh(3x)$

12. $2 \cosh \dfrac{x + y}{2} \cosh \dfrac{x - y}{2} = 2\left[\dfrac{e^{(x+y)/2} + e^{-(x+y)/2}}{2}\right]\left[\dfrac{e^{(x-y)/2} + e^{-(x-y)/2}}{2}\right]$

$= 2\left[\dfrac{e^x + e^y + e^{-y} + e^{-x}}{4}\right] = \dfrac{e^x + e^{-x}}{2} + \dfrac{e^y + e^{-y}}{2}$

$= \cosh x + \cosh y$

13. $\sinh x = \dfrac{3}{2}$

$$\cosh^2 x - \left(\frac{3}{2}\right)^2 = 1 \implies \cosh^2 x = \frac{13}{4} \implies \cosh x = \frac{\sqrt{13}}{2}$$

$$\tanh x = \frac{3/2}{\sqrt{13}/2} = \frac{3\sqrt{13}}{13}$$

$$\operatorname{csch} x = \frac{1}{3/2} = \frac{2}{3}$$

$$\operatorname{sech} x = \frac{1}{\sqrt{13}/2} = \frac{2\sqrt{13}}{13}$$

$$\coth x = \frac{1}{3/\sqrt{13}} = \frac{\sqrt{13}}{3}$$

14. $\tanh x = \dfrac{1}{2}$

$$\left(\frac{1}{2}\right)^2 + \operatorname{sech}^2 x = 1 \implies \operatorname{sech}^2 x = \frac{3}{4} \implies \operatorname{sech} x = \frac{\sqrt{3}}{2}$$

$$\cosh x = \frac{1}{\sqrt{3}/2} = \frac{2\sqrt{3}}{3}$$

$$\coth x = \frac{1}{1/2} = 2$$

$$\sinh x = \tanh x \cosh x = \left(\frac{1}{2}\right)\left(\frac{2\sqrt{3}}{3}\right) = \frac{\sqrt{3}}{3}$$

$$\operatorname{csch} x = \frac{1}{\sqrt{3}/3} = \sqrt{3}$$

Putting these in order:

$$\sinh x = \frac{\sqrt{3}}{3} \qquad \operatorname{csch} x = \sqrt{3}$$

$$\cosh x = \frac{2\sqrt{3}}{3} \qquad \operatorname{sech} x = \frac{\sqrt{3}}{2}$$

$$\tanh x = \frac{1}{2} \qquad \coth x = 2$$

15. $y = \operatorname{sech}(x + 1)$

$y' = -\operatorname{sech}(x + 1)\tanh(x + 1)$

16. $y = \coth 3x$

$y' = -3\operatorname{csch}^2(3x)$

17. $f(x) = \ln(\sinh x)$

$$f'(x) = \frac{1}{\sinh x}(\cosh x) = \coth x$$

18. $g(x) = \ln(\cosh x)$

$$g'(x) = \frac{1}{\cosh x}(\sinh x) = \tanh x$$

19. $y = \ln\left(\tanh \dfrac{x}{2}\right)$

$$y' = \frac{1/2}{\tanh(x/2)}\operatorname{sech}^2\left(\frac{x}{2}\right) = \frac{1}{2\sinh(x/2)\cosh(x/2)} = \frac{1}{\sinh x} = \operatorname{csch} x$$

20. $y = x\cosh x - \sinh x$

$$y' = x\sinh x + \cosh x - \cosh x = x\sinh x$$

21. $h(x) = \dfrac{1}{4}\sinh 2x - \dfrac{x}{2}$

$$h'(x) = \frac{1}{2}\cosh(2x) - \frac{1}{2} = \frac{\cosh(2x) - 1}{2} = \sinh^2 x$$

22. $h(t) = t - \coth t$

$h'(t) = 1 + \operatorname{csch}^2 t = \coth^2 t$

23. $f(t) = \arctan(\sinh t)$

$$f'(t) = \frac{1}{1 + \sinh^2 t}(\cosh t) = \frac{\cosh t}{\cosh^2 t} = \operatorname{sech} t$$

24. $g(x) = \operatorname{sech}^2 3x$

$$g'(x) = -2\operatorname{sech}(3x)\operatorname{sech}(3x)\tanh(3x)(3) = -6\operatorname{sech}^2 3x \tanh 3x$$

25. $y = \sinh(1 - x^2), \quad (1, 0)$

$y' = \cosh(1 - x^2)(-2x)$

$y'(1) = -2$

Tangent line: $y - 0 = -2(x - 1)$

$y = -2x + 2$

26. $y = x^{\cosh x}, \quad (1, 1)$

$\ln y = \cosh x \ln x$

$$\frac{y'}{y} = \frac{\cosh x}{x} + \sinh x \ln x$$

At $(1, 1)$, $y' = \cosh(1)$.

Tangent line: $y - 1 = \cosh(1)(x - 1)$

$y = \cosh(1)x - \cosh(1) + 1$

Note: $\cosh(1) \approx 1.5431$

27. $y = (\cosh x - \sinh x)^2, \quad (0, 1)$

$y' = 2(\cosh x - \sinh x)(\sinh x - \cosh x)$

At $(0, 1)$, $y' = 2(1)(-1) = -2$.

Tangent line: $y - 1 = -2(x - 0)$

$y = -2x + 1$

28. $y = e^{\sinh x}, \quad (0, 1)$

$y' = e^{\sinh x} \cosh x$

$y'(0) = e^0(1) = 1$

Tangent line: $y - 1 = 1(x - 0)$

$y = x + 1$

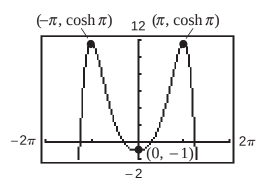
29. $f(x) = \sin x \sinh x - \cos x \cosh x, \quad -4 \le x \le 4$

$f'(x) = \sin x \cosh x + \cos x \sinh x - \cos x \sinh x + \sin x \cosh x$

$= 2 \sin x \cosh x = 0$ when $x = 0, \pm\pi$.

Relative maxima: $(\pm\pi, \cosh \pi)$

Relative minimum: $(0, -1)$

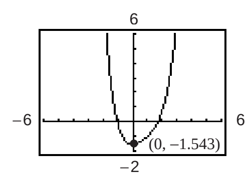


30. $f(x) = x \sinh(x - 1) - \cosh(x - 1)$

$f'(x) = x \cosh(x - 1) + \sinh(x - 1) - \sinh(x - 1) = x \cosh(x - 1)$

$f'(x) = 0$ for $x = 0$.

By the First Derivative Test, $(0, -\cosh(-1)) \approx (0, -1.543)$ is a relative minimum.



31. $g(x) = x \operatorname{sech} x$

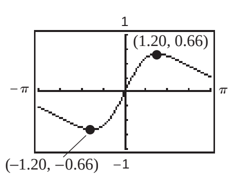
$g'(x) = \operatorname{sech} x - x \operatorname{sech} x \tanh x$

$= \operatorname{sech} x(1 - x \tanh x) = 0$

$x \tanh x = 1$

Using a graphing utility, $x \approx \pm 1.1997$.

By the First Derivative Test, $(1.1997, 0.6627)$ is a relative maximum and $(-1.1997, -0.6627)$ is a relative minimum.



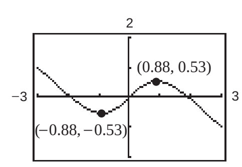
32. $h(x) = 2 \tanh x - x$

$h'(x) = 2 \operatorname{sech}^2 x - 1 = 0$

$\operatorname{sech}^2 x = \frac{1}{2}$

Using a graphing utility, $x \approx 0.8814$.

From the First Derivative Test, $(0.8814, 0.5328)$ is a relative maximum and $(-0.8814, -0.5328)$ is a relative minimum.



33. $y = a \sinh x$

$y' = a \cosh x$

$y'' = a \sinh x$

$y''' = a \cosh x$

Therefore, $y''' - y' = 0$.

34. $y = a \cosh x$

$y' = a \sinh x$

$y'' = a \cosh x$

Therefore, $y'' - y = 0$.

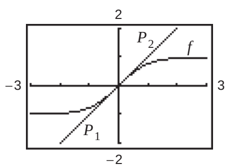
35. $f(x) = \tanh x, \qquad f(0) = 0$

$f'(x) = \operatorname{sech}^2 x, \qquad f'(0) = 1$

$f''(x) = -2 \operatorname{sech}^2 x \tanh x, \qquad f''(0) = 0$

$P_1(x) = f(0) + f'(0)(x - 0) = x$

$P_2(x) = f(0) + f'(0)(x - 0) + \frac{1}{2}f''(0)(x - 0)^2 = x$



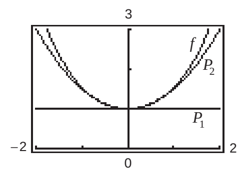
36. $f(x) = \cosh x \qquad f(1) = \cosh(0) \approx 1$

$f'(x) = \sinh x \qquad f'(1) = \sinh(0) \approx 0$

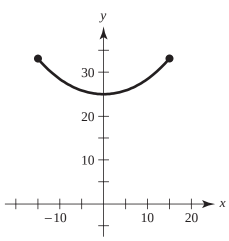
$f''(x) = \cosh x \qquad f''(1) = \cosh(0) \approx 1$

$P_1(x) = f(0) + f'(0)(x - 0) = 1$

$P_2(x) = 1 + \frac{1}{2}x^2$



37. (a) $y = 10 + 15 \cosh \dfrac{x}{15}, \quad -15 \le x \le 15$

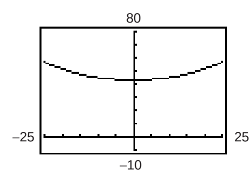


(b) At $x = \pm 15$, $y = 10 + 15 \cosh(1) \approx 33.146$.

At $x = 0$, $y = 10 + 15 \cosh(0) = 25$.

(c) $y' = \sinh \dfrac{x}{15}$. At $x = 15$, $y' = \sinh(1) \approx 1.175$.

38. (a) $y = 18 + 25 \cosh \dfrac{x}{25}, \quad -25 \le x \le 25$



(b) At $x = \pm 25$, $y = 18 + 25 \cosh(1) \approx 56.577$.

At $x = 0$, $y = 18 + 25 = 43$.

(c) $y' = \sinh \dfrac{x}{25}$. At $x = 25$, $y' = \sinh(1) \approx 1.175$.

39. Let $u = 1 - 2x$, $du = -2\,dx$.

$$\int \sinh(1 - 2x)\,dx = -\frac{1}{2}\int \sinh(1 - 2x)(-2)\,dx$$

$$= -\frac{1}{2}\cosh(1 - 2x) + C$$

40. Let $u = \sqrt{x}$, $du = \dfrac{1}{2\sqrt{x}}\,dx$.

$$\int \frac{\cosh \sqrt{x}}{\sqrt{x}}\,dx = 2\int \cosh \sqrt{x}\left(\frac{1}{2\sqrt{x}}\right)dx = 2 \sinh \sqrt{x} + C$$

41. Let $u = \cosh(x - 1)$, $du = \sinh(x - 1)\,dx$.

$$\int \cosh^2(x - 1) \sinh(x - 1)\,dx = \frac{1}{3}\cosh^3(x - 1) + C$$

42. Let $u = \cosh x$, $du = \sinh x\,dx$.

$$\int \frac{\sinh}{1 + \sinh^2 x}\,dx = \int \frac{\sinh x}{\cosh^2 x}\,dx = \frac{-1}{\cosh x} + C$$

$$= -\operatorname{sech} x + C$$

43. Let $u = \sinh x$, $du = \cosh x\, dx$.

$$\int \frac{\cosh x}{\sinh x}\, dx = \ln|\sinh x| + C$$

44. Let $u = 2x - 1$, $du = 2\, dx$.

$$\int \operatorname{sech}^2(2x - 1)\, dx = \frac{1}{2}\int \operatorname{sech}^2(2x - 1)(2)\, dx$$
$$= \frac{1}{2}\tanh(2x - 1) + C$$

45. Let $u = \dfrac{x^2}{2}$, $du = x\, dx$.

$$\int x \operatorname{csch}^2 \frac{x^2}{2}\, dx = \int \left(\operatorname{csch}^2 \frac{x^2}{2}\right) x\, dx = -\coth \frac{x^2}{2} + C$$

46. Let $u = \operatorname{sech} x$, $du = -\operatorname{sech} x \tanh x\, dx$.

$$\int \operatorname{sech}^3 x \tanh x\, dx = -\int \operatorname{sech}^2 x(-\operatorname{sech} x \tanh x)\, dx$$
$$= -\frac{1}{3}\operatorname{sech}^3 x + C$$

47. Let $u = \dfrac{1}{x}$, $du = -\dfrac{1}{x^2}\, dx$.

$$\int \frac{\operatorname{csch}(1/x)\coth(1/x)}{x^2}\, dx = -\int \operatorname{csch}\frac{1}{x}\coth\frac{1}{x}\left(-\frac{1}{x^2}\right) dx$$
$$= \operatorname{csch}\frac{1}{x} + C$$

48. Let $u = \sinh x$, $du = \cosh x\, dx$.

$$\int \frac{\cosh x}{\sqrt{9 - \sinh^2 x}}\, dx = \arcsin\left(\frac{\sinh x}{3}\right) + C$$
$$= \arcsin\left(\frac{e^x - e^{-x}}{6}\right) + C$$

49. Let $u = x^2$, $du = 2x\, dx$.

$$\int \frac{x}{x^4 + 1}\, dx = \frac{1}{2}\int \frac{2x}{(x^2)^2 + 1}\, dx = \frac{1}{2}\arctan(x^2) + C$$

50. $$\int \frac{2}{x\sqrt{1 + 4x^2}}\, dx = 2\int \frac{1}{(2x)\sqrt{1 + (2x)^2}}(2)\, dx$$
$$= 2\ln\left(\frac{-1 + \sqrt{1 + 4x^2}}{|2x|}\right) + C$$

51. $$\int_0^{\ln 2} \tanh x\, dx = \int_0^{\ln 2} \frac{\sinh x}{\cosh x}\, dx, \quad (u = \cosh x)$$
$$= \ln(\cosh x)\Big]_0^{\ln 2}$$
$$= \ln(\cosh(\ln 2) - \ln(\cosh(0))$$
$$= \ln\left(\frac{5}{4}\right) - 0 = \ln\left(\frac{5}{4}\right)$$

Note: $\cosh(\ln 2) = \dfrac{e^{\ln 2} + e^{-\ln 2}}{2} = \dfrac{2 + (1/2)}{2} = \dfrac{5}{4}$

52. $$\int_0^1 \cosh^2 x\, dx = \int_0^1 \frac{1 + \cosh(2x)}{2}\, dx$$
$$= \frac{1}{2}\left[x + \frac{1}{2}\sinh(2x)\right]_0^1$$
$$= \frac{1}{2}\left[1 + \frac{1}{2}\sinh(2)\right]$$
$$= \frac{1}{2} + \frac{1}{2}\sinh(1)\cosh(1)$$

53. $$\int_0^4 \frac{1}{25 - x^2}\, dx = \frac{1}{10}\int \frac{1}{5 - x}\, dx + \frac{1}{10}\int \frac{1}{5 + x}\, dx$$
$$= \left[\frac{1}{10}\ln\left|\frac{5 + x}{5 - x}\right|\right]_0^4 = \frac{1}{10}\ln 9 = \frac{1}{5}\ln 3$$

54. $$\int_0^4 \frac{1}{\sqrt{25 - x^2}}\, dx = \left[\arcsin\frac{x}{5}\right]_0^4 = \arcsin\frac{4}{5}$$

55. Let $u = 2x$, $du = 2\, dx$.

$$\int_0^{\sqrt{2}/4} \frac{2}{\sqrt{1 - 4x^2}}\, dx = \int_0^{\sqrt{2}/4} \frac{1}{\sqrt{1 - (2x)^2}}(2)\, dx$$
$$= \left[\arcsin(2x)\right]_0^{\sqrt{2}/4} = \frac{\pi}{4}$$

56. $2e^{-x}\cosh x = 2e^{-x}\left[\dfrac{e^x + e^{-x}}{2}\right] = 1 + e^{-2x}$

$$\int_0^{\ln 2} 2e^{-x}\cosh x\,dx = \int_0^{\ln 2}(1 + e^{-2x})\,dx$$
$$= \left[x - \frac{1}{2}e^{-2x}\right]_0^{\ln 2}$$
$$= \left[\ln 2 - \frac{1}{2}\left(\frac{1}{4}\right)\right] - \left[0 - \frac{1}{2}\right]$$
$$= \frac{3}{8} + \ln 2$$

57. $y = \cosh^{-1}(3x)$

$$y' = \frac{3}{\sqrt{9x^2 - 1}}$$

58. $y = \tanh^{-1}\dfrac{x}{2}$

$$y' = \frac{1}{1 - (x/2)^2}\left(\frac{1}{2}\right) = \frac{2}{4 - x^2}$$

59. $y = \sinh^{-1}(\tan x)$

$$y' = \frac{1}{\sqrt{\tan^2 x + 1}}(\sec^2 x) = |\sec x|$$

60. $y = \operatorname{sech}^{-1}(\cos 2x),\quad 0 < x < \dfrac{\pi}{4}$

$$y' = \frac{-1}{\cos 2x\sqrt{1 - \cos^2 2x}}(-2\sin 2x) = \frac{2\sin 2x}{\cos 2x|\sin 2x|} = \frac{2}{\cos 2x} = 2\sec 2x,$$

since $\sin 2x \geq 0$ for $0 < x < \pi/4$.

61. $y = \tanh^{-1}(\sin 2x)$

$$y' = \frac{1}{1 - \sin^2 2x}(2\cos 2x) = 2\sec 2x$$

62. $y = (\operatorname{csch}^{-1} x)^2$

$$y' = 2\operatorname{csch}^{-1} x\left(\frac{-1}{|x|\sqrt{1 + x^2}}\right) = \frac{-2\operatorname{csch}^{-1} x}{|x|\sqrt{1 + x^2}}$$

63. $y = 2x\sinh^{-1}(2x) - \sqrt{1 + 4x^2}$

$$y' = 2x\left(\frac{2}{\sqrt{1 + 4x^2}}\right) + 2\sinh^{-1}(2x) - \frac{4x}{\sqrt{1 + 4x^2}}$$
$$= 2\sinh^{-1}(2x)$$

64. $y = x\tanh^{-1} x + \ln\sqrt{1 - x^2} = x\tanh^{-1} x + \dfrac{1}{2}\ln(1 - x^2)$

$$y' = x\left(\frac{1}{1 - x^2}\right) + \tanh^{-1} x + \frac{-x}{1 - x^2} = \tanh^{-1} x$$

65. Answers will vary.

66. See the definitions and graphs in the textbook.

67. $\displaystyle\lim_{x\to\infty} \sinh x = \infty$

68. $\displaystyle\lim_{x\to\infty} \tanh x = 1$

69. $\displaystyle\lim_{x\to\infty} \operatorname{sech} x = 0$

70. $\displaystyle\lim_{x\to-\infty} \operatorname{csch} x = 0$

71. $\displaystyle\lim_{x\to 0}\frac{\sinh x}{x} = \lim_{x\to 0}\frac{e^x - e^{-x}}{2x} = 1$

72. $\displaystyle\lim_{x\to 0}\coth x$ does not exist.

($\coth x \to \infty$ for $x \to 0^+$, $\coth x \to -\infty$ for $x \to 0^-$)

73. $\displaystyle\int \frac{1}{\sqrt{1 + e^{2x}}}\,dx = \int \frac{e^x}{e^x\sqrt{1 + (e^x)^2}}\,dx = -\operatorname{csch}^{-1}(e^x) + C$

$$= -\ln\left(\frac{1 + \sqrt{1 + e^{2x}}}{e^x}\right) + C$$

74. $\displaystyle\int \frac{x}{9 - x^4}\,dx = -\frac{1}{2}\int \frac{-2x}{9 - (x^2)^2}\,dx$

$$= -\frac{1}{2}\left(\frac{1}{6}\right)\ln\left|\frac{3 - x^2}{3 + x^2}\right| + C$$
$$= -\frac{1}{12}\ln\left|\frac{3 - x^2}{3 + x^2}\right| + C$$

75. Let $u = \sqrt{x}$, $du = \dfrac{1}{2\sqrt{x}}\,dx$.

$$\int \frac{1}{\sqrt{x}\sqrt{1+x}}\,dx = 2\int \frac{1}{\sqrt{1+(\sqrt{x})^2}}\left(\frac{1}{2\sqrt{x}}\right)dx = 2\sinh^{-1}\sqrt{x} + C = 2\ln\left(\sqrt{x} + \sqrt{1+x}\right) + C$$

76. Let $u = x^{3/2}$, $du = \dfrac{3}{2}\sqrt{x}\,dx$.

$$\int \frac{\sqrt{x}}{\sqrt{1+x^3}}\,dx = \frac{2}{3}\int \frac{1}{\sqrt{1+(x^{3/2})^2}}\left(\frac{3}{2}\sqrt{x}\right)dx = \frac{2}{3}\sinh^{-1}(x^{3/2}) + C = \frac{2}{3}\ln\left(x^{3/2} + \sqrt{1+x^3}\right) + C$$

77. $$\int \frac{-1}{4x-x^2}\,dx = \int \frac{1}{(x-2)^2-4}\,dx = \frac{1}{4}\ln\left|\frac{(x-2)-2}{(x-2)+2}\right| = \frac{1}{4}\ln\left|\frac{x-4}{x}\right| + C$$

78. $$\int \frac{dx}{(x+2)\sqrt{x^2+4x+8}} = \int \frac{dx}{(x+2)\sqrt{(x+2)^2+4}} = -\frac{1}{2}\ln\left(\frac{2+\sqrt{(x+2)^2+4}}{|x+2|}\right) + C$$

79. $$\int \frac{1}{1-4x-2x^2}\,dx = \int \frac{1}{3-2(x+1)^2}\,dx$$
$$= \frac{1}{\sqrt{2}}\int \frac{\sqrt{2}}{\left(\sqrt{3}\right)^2 - \left[\sqrt{2}(x+1)\right]^2}\,dx$$
$$= \frac{1}{\sqrt{2}}\cdot\frac{1}{2\sqrt{3}}\ln\left|\frac{\sqrt{3}+\sqrt{2}(x+1)}{\sqrt{3}-\sqrt{2}(x+1)}\right| + C = \frac{1}{2\sqrt{6}}\ln\left|\frac{\sqrt{2}(x+1)+\sqrt{3}}{\sqrt{2}(x+1)-\sqrt{3}}\right| + C$$

80. $$\int \frac{1}{(x+1)\sqrt{2x^2+4x+8}}\,dx = \int \frac{1}{(x+1)\sqrt{2(x+1)^2+6}}\,dx$$
$$= \frac{1}{\sqrt{2}}\int \frac{1}{(x+1)\sqrt{(x+1)^2+(\sqrt{3})^2}}\,dx = -\frac{1}{\sqrt{6}}\ln\left(\frac{\sqrt{3}+\sqrt{(x+1)^2+3}}{x+1}\right) + C$$

81. Let $u = 4x - 1$, $du = 4\,dx$.

$$y = \int \frac{1}{\sqrt{80+8x-16x^2}}\,dx$$
$$= \frac{1}{4}\int \frac{4}{\sqrt{81-(4x-1)^2}}\,dx = \frac{1}{4}\arcsin\left(\frac{4x-1}{9}\right) + C$$

82. Let $u = 2(x-1)$, $du = 2\,dx$.

$$y = \int \frac{1}{(x-1)\sqrt{-4x^2+8x-1}}\,dx$$
$$= \int \frac{2}{2(x-1)\sqrt{(\sqrt{3})^2-[2(x-1)]^2}}\,dx$$
$$= -\frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}+\sqrt{-4x^2+8x-1}}{2(x-1)}\right| + C$$

83. $$y = \int \frac{x^3-21x}{5+4x-x^2}\,dx = \int\left(-x-4+\frac{20}{5+4x-x^2}\right)dx$$
$$= \int(-x-4)\,dx + 20\int \frac{1}{3^2-(x-2)^2}\,dx$$
$$= -\frac{x^2}{2} - 4x + \frac{20}{6}\ln\left|\frac{3+(x-2)}{3-(x-2)}\right| + C$$
$$= -\frac{x^2}{2} - 4x + \frac{10}{3}\ln\left|\frac{1+x}{5-x}\right| + C$$
$$= \frac{-x^2}{2} - 4x - \frac{10}{3}\ln\left|\frac{5-x}{x+1}\right| + C$$

84. $$y = \int \frac{1-2x}{4x-x^2}\,dx = \int \frac{4-2x}{4x-x^2}\,dx + 3\int \frac{1}{(x-2)^2-4}\,dx$$
$$= \ln|4x-x^2| + \frac{3}{4}\ln\left|\frac{(x-2)-2}{(x-2)+2}\right| + C$$
$$= \ln|4x-x^2| + \frac{3}{4}\ln\left|\frac{x-4}{x}\right| + C$$

85. $A = 2\int_0^4 \operatorname{sech}\frac{x}{2}\,dx$

$= 2\int_0^4 \frac{2}{e^{x/2} + e^{-x/2}}\,dx$

$= 4\int_0^4 \frac{e^{x/2}}{(e^{x/2})^2 + 1}\,dx$

$= \Big[8\arctan(e^{x/2})\Big]_0^4$

$= 8\arctan(e^2) - 2\pi \approx 5.207$

86. $A = \int_0^2 \tanh 2x\,dx = \int_0^2 \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\,dx$

$= \frac{1}{2}\int_0^2 \frac{1}{e^{2x} + e^{-2x}}(2)(e^{2x} - e^{-2x})\,dx$

$= \left[\frac{1}{2}\ln(e^{2x} + e^{-2x})\right]_0^2 = \frac{1}{2}\ln(e^4 + e^{-4}) - \frac{1}{2}\ln 2$

$= \ln\sqrt{\frac{e^4 + e^{-4}}{2}} \approx 1.654$

87. $A = \int_0^2 \frac{5x}{\sqrt{x^4 + 1}}\,dx$

$= \frac{5}{2}\int_0^2 \frac{2x}{\sqrt{(x^2)^2 + 1}}\,dx$

$= \left[\frac{5}{2}\ln\left(x^2 + \sqrt{x^4 + 1}\right)\right]_0^2$

$= \frac{5}{2}\ln\left(4 + \sqrt{17}\right) \approx 5.237$

88. $A = \int_3^5 \frac{6}{\sqrt{x^2 - 4}}\,dx$

$= \left[6\ln\left(x + \sqrt{x^2 - 4}\right)\right]_3^5$

$= 6\ln\left(5 + \sqrt{21}\right) - 6\ln\left(3 + \sqrt{5}\right)$

$= 6\ln\left(\frac{5 + \sqrt{21}}{3 + \sqrt{5}}\right) \approx 3.626$

89. (a) $\int_0^{\sqrt{3}} \frac{dx}{\sqrt{x^2 + 1}} = \ln\left(x + \sqrt{x^2 + 1}\right)\Big]_0^{\sqrt{3}}$

$= \ln\left(\sqrt{3} + 2\right) \approx 1.317$

(b) $\int_0^{\sqrt{3}} \frac{dx}{\sqrt{x^2 + 1}} = \sinh^{-1} x\Big]_0^{\sqrt{3}} = \sinh^{-1}\left(\sqrt{3}\right) \approx 1.317$

90. (a) $\int_{-1/2}^{1/2} \frac{dx}{1 - x^2} = \int_{-1/2}^{1/2}\left[\frac{1/2}{1 + x} + \frac{1/2}{1 - x}\right]dx$

$= \left[\frac{1}{2}\ln|1 + x| - \frac{1}{2}\ln|1 - x|\right]_{-1/2}^{1/2}$

$= \frac{1}{2}\left[\ln\frac{3}{2} - \ln\frac{1}{2} - \ln\frac{1}{2} + \ln\frac{3}{2}\right]$

$= \ln\frac{3}{2} - \ln\frac{1}{2} = \ln 3$

(b) $\int_{-1/2}^{1/2} \frac{dx}{1 - x^2} = \Big[\tanh^{-1} x\Big]_{-1/2}^{1/2}$

$= \frac{\ln 3}{2} - \left(-\frac{\ln 3}{2}\right) = \ln 3$

91. $\int \frac{3k}{16}\,dt = \int \frac{1}{x^2 - 12x + 32}\,dx$

$\frac{3kt}{16} = \int \frac{1}{(x - 6)^2 - 4}\,dx = \frac{1}{2(2)}\ln\left|\frac{(x - 6) - 2}{(x - 6) + 2}\right| + C = \frac{1}{4}\ln\left|\frac{x - 8}{x - 4}\right| + C$

When $x = 0$:

$t = 0$

$C = -\frac{1}{4}\ln(2)$

When $x = 1$:

$t = 10$

$\frac{30k}{16} = \frac{1}{4}\ln\left|\frac{-7}{-3}\right| - \frac{1}{4}\ln(2) = \frac{1}{4}\ln\left(\frac{7}{6}\right)$

$k = \frac{2}{15}\ln\left(\frac{7}{6}\right)$

When $t = 20$:

$\left(\frac{3}{16}\right)\left(\frac{2}{15}\right)\ln\left(\frac{7}{6}\right)(20) = \frac{1}{4}\ln\frac{x - 8}{2x - 8}$

$\ln\left(\frac{7}{6}\right)^2 = \ln\frac{x - 8}{2x - 8}$

$\frac{49}{36} = \frac{x - 8}{2x - 8}$

$62x = 104$

$x = \frac{104}{62} = \frac{52}{31} \approx 1.677$ kg

92. (a) $v(t) = -32t$

(b) $s(t) = \int v(t)\,dt = \int (-32t)\,dt = -16t^2 + C$

$s(0) = -16(0)^2 + C = 400 \implies C = 400$

$s(t) = -16t^2 + 400$

(c) $$\frac{dv}{dt} = -32 + kv^2$$

$$\int \frac{dv}{kv^2 - 32} = \int dt$$

$$\int \frac{dv}{32 - kv^2} = -\int dt$$

Let $u = \sqrt{k}\,v$, then $du = \sqrt{k}\,dv$.

$$\frac{1}{\sqrt{k}} \cdot \frac{1}{2\sqrt{32}} \ln\left|\frac{\sqrt{32} + \sqrt{k}\,v}{\sqrt{32} - \sqrt{k}\,v}\right| = -t + C$$

Since $v(0) = 0$, $C = 0$.

$$\ln\left|\frac{\sqrt{32} + \sqrt{k}\,v}{\sqrt{32} - \sqrt{k}\,v}\right| = -2\sqrt{32k}\,t$$

$$\frac{\sqrt{32} + \sqrt{k}\,v}{\sqrt{32} - \sqrt{k}\,v} = e^{-2\sqrt{32k}\,t}$$

$$\sqrt{32} + \sqrt{k}\,v = e^{-2\sqrt{32k}\,t}\left(\sqrt{32} - \sqrt{k}\,v\right)$$

$$v\left(\sqrt{k} + \sqrt{k}\,e^{-2\sqrt{32k}\,t}\right) = \sqrt{32}\left(e^{-2\sqrt{32k}\,t} - 1\right)$$

$$v = \frac{\sqrt{32}\left(e^{-2\sqrt{32k}\,t} - 1\right)}{\sqrt{k}\left(e^{-2\sqrt{32k}\,t} + 1\right)} \cdot \frac{e^{\sqrt{32k}\,t}}{e^{\sqrt{32k}\,t}}$$

$$= \frac{\sqrt{32}}{\sqrt{k}}\left[\frac{-\left(e^{\sqrt{32k}\,t} - e^{-\sqrt{32k}\,t}\right)}{e^{\sqrt{32k}\,t} + e^{-\sqrt{32k}\,t}}\right]$$

$$= -\frac{\sqrt{32}}{\sqrt{k}} \tanh\left(\sqrt{32k}\,t\right)$$

(d) $$\lim_{t\to\infty}\left[-\frac{\sqrt{32}}{\sqrt{k}} \tanh\left(\sqrt{32k}\,t\right)\right] = -\frac{\sqrt{32}}{\sqrt{k}}$$

The velocity is bounded by $-\sqrt{32}/\sqrt{k}$.

(e) Since $\int \tanh(ct)\,dt = (1/c)\ln\cosh(ct)$ (which can be verified by differentiation), then

$$s(t) = \int -\frac{\sqrt{32}}{\sqrt{k}} \tanh\left(\sqrt{32k}\,t\right) dt$$

$$= -\frac{\sqrt{32}}{\sqrt{k}} \frac{1}{\sqrt{32k}} \ln\left[\cosh\left(\sqrt{32k}\,t\right)\right] + C$$

$$= -\frac{1}{k} \ln\left[\cosh\left(\sqrt{32k}\,t\right)\right] + C.$$

When $t = 0$,

$s(0) = C = 400 \implies 400 - (1/k)\ln\left[\cosh\left(\sqrt{32k}\,t\right)\right]$.

When $k = 0.01$:

$s_2(t) = 400 - 100\ln\left(\cosh\sqrt{0.32}\,t\right)$

$s_1(t) = -16t^2 + 400$

$s_1(t) = 0$ when $t = 5$ seconds

$s_2(t) = 0$ when $t \approx 8.3$ seconds

When air resistance is not neglected, it takes approximately 3.3 more seconds to reach the ground.

(f) As k increases, the time required for the object to reach the ground increases.

93. $y = a\,\operatorname{sech}^{-1}\dfrac{x}{a} - \sqrt{a^2 - x^2}, \quad a > 0$

$$\frac{dy}{dx} = \frac{-1}{(x/a)\sqrt{1 - (x^2/a^2)}} + \frac{x}{\sqrt{a^2 - x^2}} = \frac{-a^2}{x\sqrt{a^2 - x^2}} + \frac{x}{\sqrt{a^2 - x^2}} = \frac{x^2 - a^2}{x\sqrt{a^2 - x^2}} = \frac{-\sqrt{a^2 - x^2}}{x}$$

94. Equation of tangent line through $P = (x_0, y_0)$:

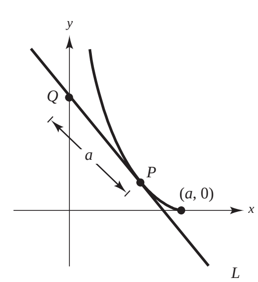
$$y - a\,\operatorname{sech}^{-1}\frac{x_0}{a} + \sqrt{a^2 - x_0^2} = -\frac{\sqrt{a^2 - x_0^2}}{x_0}(x - x_0)$$

When $x = 0$,

$$y = a\,\operatorname{sech}^{-1}\frac{x_0}{a} - \sqrt{a^2 - x_0^2} + \sqrt{a^2 - x_0^2} = a\,\operatorname{sech}^{-1}\frac{x_0}{a}.$$

Hence, Q is the point $\left[0, a\,\operatorname{sech}^{-1}(x_0/a)\right]$.

Distance from P to Q: $d = \sqrt{x_0^2 + \left(-\sqrt{a^2 - x_0^2}\right)^2} = a$



95. Let $u = \tanh^{-1} x, \quad -1 < x < 1$

$$\tanh u = x.$$

$$\frac{\sinh u}{\cosh u} = \frac{e^u - e^{-u}}{e^u + e^{-u}} = x$$

$$e^u - e^{-u} = xe^u + xe^{-u}$$

$$e^{2u} - 1 = xe^{2u} + x$$

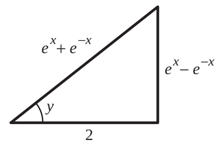
$$e^{2u}(1 - x) = 1 + x$$

$$e^{2u} = \frac{1 + x}{1 - x}$$

$$2u = \ln\left(\frac{1 + x}{1 - x}\right)$$

$$u = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right), \quad -1 < x < 1$$

96. Let $y = \arcsin(\tanh x)$. Then,



$$\sin y = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \text{ and}$$

$$\tan y = \frac{e^x - e^{-x}}{2} = \sinh x.$$

Thus, $y = \arctan(\sinh x)$. Therefore,

$\arctan(\sinh x) = \arcsin(\tanh x)$.

97.
$$\int_{-b}^{b} e^{xt}\,dt = \frac{e^{xt}}{x}\Big]_{-b}^{b}$$

$$= \frac{e^{xb}}{x} - \frac{e^{-xb}}{x}$$

$$= \frac{2}{x}\left[\frac{e^{xb} - e^{-xb}}{2}\right]$$

$$= \frac{2}{x}\sinh(xb)$$

98. $y = \cosh x = \dfrac{e^x + e^{-x}}{2}$

$$y' = \frac{e^x - e^{-x}}{2} = \sinh x$$

99.
$$y = \operatorname{sech}^{-1} x$$

$$\operatorname{sech} y = x$$

$$-(\operatorname{sech} y)(\tanh y)y' = 1$$

$$y' = \frac{-1}{(\operatorname{sech} y)(\tanh y)}$$

$$= \frac{-1}{(\operatorname{sech} y)\sqrt{1 - \operatorname{sech}^2 y}} = \frac{-1}{x\sqrt{1 - x^2}}$$

100.
$$y = \cosh^{-1} x$$

$$\cosh y = x$$

$$(\sinh y)(y') = 1$$

$$y' = \frac{1}{\sinh y} = \frac{1}{\sqrt{\cosh^2 y - 1}} = \frac{1}{\sqrt{x^2 - 1}}$$

101.
$$y = \sinh^{-1} x$$

$$\sinh y = x$$

$$(\cosh y)y' = 1$$

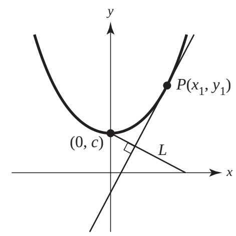
$$y' = \frac{1}{\cosh y} = \frac{1}{\sqrt{\sinh^2 y + 1}} = \frac{1}{\sqrt{x^2 + 1}}$$

102. $y = \operatorname{sech} x = \dfrac{2}{e^x + e^{-x}}$

$$y' = -2(e^x + e^{-x})^{-2}(e^x - e^{-x})$$

$$= \left(\frac{-2}{e^x + e^{-x}}\right)\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) = -\operatorname{sech} x \tanh x$$

103. $y = c \cosh \dfrac{x}{c}$



Let $P(x_1, y_1)$ be a point on the catenary.

$y' = \sinh \dfrac{x}{c}$

The slope at P is $\sinh(x_1/c)$. The equation of line L is

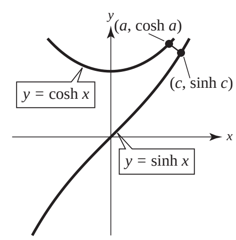
$$y - c = \frac{-1}{\sinh(x_1/c)}(x - 0).$$

When $y = 0$, $c = \dfrac{x}{\sinh(x_1/c)} \implies x = c \sinh\left(\dfrac{x_1}{c}\right)$. The length of L is

$$\sqrt{c^2 \sinh^2\left(\frac{x_1}{c}\right) + c^2} = c \cdot \cosh \frac{x_1}{c} = y_1,$$

the ordinate y_1 of the point P.

104. There is no such common normal. To see this, assume there is a common normal.
$y = \cosh x \implies y' = \sinh x$.



Normal line at $(a, \cosh a)$ is $y - \cosh a = \dfrac{-1}{\sinh a}(x - a)$.

Similarly, $y - \sinh c = \dfrac{-1}{\cosh c}(x - c)$ is normal at $(c, \sinh c)$. Also,

$$\frac{-1}{\sinh a} = \frac{-1}{\cosh c} \implies \cosh c = \sinh a.$$

The slope between the points is $\dfrac{\sinh c - \cosh a}{c - a}$. Therefore, $-\dfrac{a - c}{\cosh a - \sinh c} = \cosh c = \sinh a$.

$\cosh c > 0 \implies a > 0$

$\sinh x < \cosh x$ for all $x \implies \sinh c < \cosh c = \sinh a < \cosh a$. Hence, $c < a$. But,

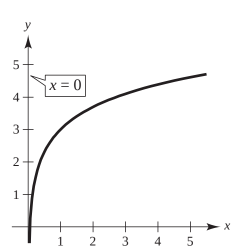
$$-\frac{a - c}{\cosh a - \sinh c} < 0, \text{ a contradiction.}$$

Review Exercises for Chapter 5

1. $f(x) = \ln x + 3$

Vertical shift 3 units upward

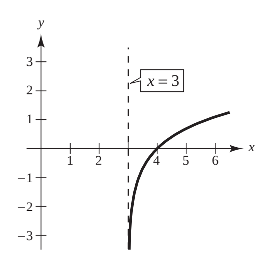
Vertical asymptote: $x = 0$



2. $f(x) = \ln(x - 3)$

Horizontal shift 3 units to the right

Vertical asymptote: $x = 3$



3. $\ln \sqrt[5]{\dfrac{4x^2 - 1}{4x^2 + 1}} = \dfrac{1}{5} \ln \dfrac{(2x - 1)(2x + 1)}{4x^2 + 1} = \dfrac{1}{5}[\ln(2x - 1) + \ln(2x + 1) - \ln(4x^2 + 1)]$

4. $\ln[(x^2 + 1)(x - 1)] = \ln(x^2 + 1) + \ln(x - 1)$

5. $\ln 3 + \dfrac{1}{3}\ln(4 - x^2) - \ln x = \ln 3 + \ln \sqrt[3]{4 - x^2} - \ln x$

$$= \ln\left(\frac{3\sqrt[3]{4 - x^2}}{x}\right)$$

6. $3[\ln x - 2\ln(x^2+1)] + 2\ln 5 = 3\ln x - 6\ln(x^2+1) + \ln 5^2$

$$= \ln x^3 - \ln(x^2+1)^6 + \ln 25 = \ln\left[\frac{25x^3}{(x^2+1)^6}\right]$$

7. $\ln\sqrt{x+1} = 2$

$$\sqrt{x+1} = e^2$$
$$x+1 = e^4$$
$$x = e^4 - 1 \approx 53.598$$

8. $\ln x + \ln(x-3) = 0$

$$\ln x(x-3) = 0$$
$$x(x-3) = e^0$$
$$x^2 - 3x - 1 = 0$$
$$x = \frac{3 \pm \sqrt{13}}{2}$$
$$x = \frac{3+\sqrt{13}}{2} \text{ only since } \frac{3-\sqrt{13}}{2} < 0.$$

9. $g(x) = \ln\sqrt{x} = \frac{1}{2}\ln x$

$$g'(x) = \frac{1}{2x}$$

10. $h(x) = \ln\frac{x(x-1)}{x-2} = \ln x + \ln(x-1) - \ln(x-2)$

$$h'(x) = \frac{1}{x} + \frac{1}{x-1} - \frac{1}{x-2} = \frac{x^2-4x+2}{x^3-3x^2+2x}$$

11. $f(x) = x\sqrt{\ln x}$

$$f'(x) = \left(\frac{x}{2}\right)(\ln x)^{-1/2}\left(\frac{1}{x}\right) + \sqrt{\ln x}$$
$$= \frac{1}{2\sqrt{\ln x}} + \sqrt{\ln x} = \frac{1+2\ln x}{2\sqrt{\ln x}}$$

12. $f(x) = \ln[x(x^2-2)^{2/3}] = \ln x + \frac{2}{3}\ln(x^2-2)$

$$f'(x) = \frac{1}{x} + \frac{2}{3}\left(\frac{2x}{x^2-2}\right) = \frac{7x^2-6}{3x^3-6x}$$

13. $y = \frac{1}{b^2}[a + bx - a\ln(a+bx)]$

$$\frac{dy}{dx} = \frac{1}{b^2}\left(b - \frac{ab}{a+bx}\right) = \frac{x}{a+bx}$$

14. $y = -\frac{1}{ax} + \frac{b}{a^2}\ln\frac{a+bx}{x}$

$$= -\frac{1}{ax} + \frac{b}{a^2}[\ln(a+bx) - \ln x]$$
$$\frac{dy}{dx} = -\frac{1}{a}\left(-\frac{1}{x^2}\right) + \frac{b}{a^2}\left[\frac{b}{a+bx} - \frac{1}{x}\right]$$
$$= \frac{1}{ax^2} + \frac{b}{a^2}\left[\frac{-a}{x(a+bx)}\right] = \frac{1}{ax^2} - \frac{b}{ax(a+bx)}$$
$$= \frac{(a+bx)-bx}{ax^2(a+bx)} = \frac{1}{x^2(a+bx)}$$

15. $y = \ln(2+x) + \frac{2}{2+x}, \quad (-1, 2)$

$$y' = \frac{1}{2+x} - \frac{2}{(2+x)^2}$$
$$y'(-1) = 1 - 2 = -1$$

Tangent line: $y - 2 = -1(x+1)$

$$y = -x + 1$$

16. $y = \ln\frac{1+x}{x} = \ln(1+x) - \ln x, \quad (1, \ln 2)$

$$y' = \frac{1}{1+x} - \frac{1}{x}$$
$$y'(1) = \frac{1}{2} - 1 = -\frac{1}{2}$$

Tangent line: $y - \ln 2 = -\frac{1}{2}(x-1)$

$$y = -\frac{1}{2}x + \ln 2 + \frac{1}{2}$$

17. $u = 7x - 2,\ du = 7\,dx$

$$\int \frac{1}{7x-2}\,dx = \frac{1}{7}\int \frac{1}{7x-2}(7)\,dx = \frac{1}{7}\ln|7x-2| + C$$

18. $u = x^2 - 1,\ du = 2x\,dx$

$$\int \frac{x}{x^2-1}\,dx = \frac{1}{2}\int \frac{2x}{x^2-1}\,dx = \frac{1}{2}\ln|x^2-1| + C$$

19. $$\int \frac{\sin x}{1+\cos x}\,dx = -\int \frac{-\sin x}{1+\cos x}\,dx$$
$$= -\ln|1+\cos x| + C$$

20. $u = \ln x,\ du = \dfrac{1}{x}\,dx$

$$\int \frac{\ln\sqrt{x}}{x}\,dx = \frac{1}{2}\int (\ln x)\left(\frac{1}{x}\right)dx = \frac{1}{4}(\ln x)^2 + C$$

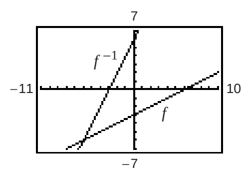
21. $$\int_1^4 \frac{x+1}{x}\,dx = \int_1^4 \left(1 + \frac{1}{x}\right)dx = \Big[x + \ln|x|\Big]_1^4$$
$$= 3 + \ln 4$$

22. $$\int_1^e \frac{\ln x}{x}\,dx = \int_1^e (\ln x)^1\left(\frac{1}{x}\right)dx = \left[\frac{1}{2}(\ln x)^2\right]_1^e = \frac{1}{2}$$

23. $$\int_0^{\pi/3} \sec\theta\,d\theta = \Big[\ln|\sec\theta + \tan\theta|\Big]_0^{\pi/3} = \ln\left(2 + \sqrt{3}\right)$$

24. $$\int_0^{\pi/4} \tan\left(\frac{\pi}{4} - x\right)dx = \left[\ln\left|\cos\left(\frac{\pi}{4} - x\right)\right|\right]_0^{\pi/4}$$
$$= 0 - \ln\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}\ln 2$$

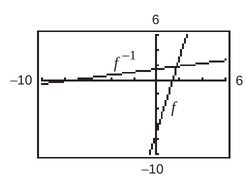
25. (a)
$$f(x) = \tfrac{1}{2}x - 3$$
$$y = \tfrac{1}{2}x - 3$$
$$2(y+3) = x$$
$$2(x+3) = y$$
$$f^{-1}(x) = 2x + 6$$

(b) 

(c) $f^{-1}(f(x)) = f^{-1}\left(\tfrac{1}{2}x - 3\right) = 2\left(\tfrac{1}{2}x - 3\right) + 6 = x$

$f(f^{-1}(x)) = f(2x+6) = \tfrac{1}{2}(2x+6) - 3 = x$

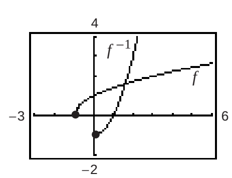
26. (a)
$$f(x) = 5x - 7$$
$$y = 5x - 7$$
$$\frac{y+7}{5} = x$$
$$\frac{x+7}{5} = y$$
$$f^{-1}(x) = \frac{x+7}{5}$$

(b) 

(c) $f^{-1}(f(x)) = f^{-1}(5x-7) = \dfrac{(5x-7)+7}{5} = x$

$f(f^{-1}(x)) = f\left(\dfrac{x+7}{5}\right) = 5\left(\dfrac{x+7}{5}\right) - 7 = x$

27. (a)
$$f(x) = \sqrt{x+1}$$
$$y = \sqrt{x+1}$$
$$y^2 - 1 = x$$
$$x^2 - 1 = y$$
$$f^{-1}(x) = x^2 - 1,\quad x \ge 0$$

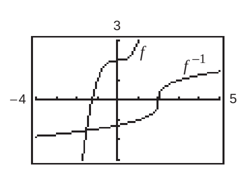
(b) 

(c) $f^{-1}(f(x)) = f^{-1}\left(\sqrt{x+1}\right) = \sqrt{(x^2-1)^2} - 1 = x$

$f(f^{-1}(x)) = f(x^2-1) = \sqrt{(x^2-1)+1}$

$= \sqrt{x^2} = x$ for $x \ge 0$.

28. (a)
$$f(x) = x^3 + 2$$
$$y = x^3 + 2$$
$$\sqrt[3]{y-2} = x$$
$$\sqrt[3]{x-2} = y$$
$$f^{-1}(x) = \sqrt[3]{x-2}$$

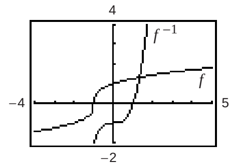
(b) 

(c) $f^{-1}(f(x)) = f^{-1}(x^3+2) = \sqrt[3]{(x^3+2)-2} = x$

$f(f^{-1}(x)) = f\left(\sqrt[3]{x-2}\right) = \left(\sqrt[3]{x-2}\right)^3 + 2 = x$

29. (a)
$$\begin{aligned} f(x) &= \sqrt[3]{x+1} \\ y &= \sqrt[3]{x+1} \\ y^3 - 1 &= x \\ x^3 - 1 &= y \\ f^{-1}(x) &= x^3 - 1 \end{aligned}$$

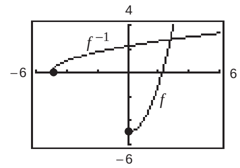
(b)



(c)
$$\begin{aligned} f^{-1}(f(x)) &= f^{-1}\left(\sqrt[3]{x+1}\right) \\ &= \left(\sqrt[3]{x+1}\right)^3 - 1 = x \\ f(f^{-1}(x)) &= f(x^3-1) = \sqrt[3]{(x^3-1)+1} = x \end{aligned}$$

30. (a)
$$\begin{aligned} f(x) &= x^2 - 5, \quad x \ge 0 \\ y &= x^2 - 5 \\ \sqrt{y+5} &= x \\ \sqrt{x+5} &= y \\ f^{-1}(x) &= \sqrt{x+5} \end{aligned}$$

(b)



(c)
$$\begin{aligned} f^{-1}(f(x)) &= f^{-1}(x^2 - 5) \\ &= \sqrt{(x^2-5)+5} = x \text{ for } x \ge 0. \\ f(f^{-1}(x)) &= f\left(\sqrt{x+5}\right) = \left(\sqrt{x+5}\right)^2 - 5 = x \end{aligned}$$

31.
$$\begin{aligned} f(x) &= x^3 + 2 \\ f^{-1}(x) &= (x-2)^{1/3} \\ (f^{-1})'(x) &= \frac{1}{3}(x-2)^{-2/3} \\ (f^{-1})'(-1) &= \frac{1}{3}(-1-2)^{-2/3} = \frac{1}{3(-3)^{2/3}} \\ &= \frac{1}{3^{5/3}} \approx 0.160 \end{aligned}$$

or

$$\begin{aligned} f(-3^{1/3}) &= -1 \\ f'(x) &= 3x^2 \\ f'(-3^{1/3}) &= 3(-3^{1/3})^2 = 3^{5/3} \\ (f^{-1})'(-1) &= \frac{1}{f'(-3^{1/3})} = \frac{1}{3^{5/3}} \end{aligned}$$

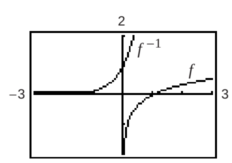
32.
$$\begin{aligned} f(x) &= x\sqrt{x-3} \\ f(4) &= 4 \\ f'(x) &= \sqrt{x-3} + \frac{1}{2}x(x-3)^{-1/2} \\ f'(4) &= 1 + 2 = 3 \\ (f^{-1})'(4) &= \frac{1}{f'(4)} = \frac{1}{3} \end{aligned}$$

33.
$$\begin{aligned} f(x) &= \tan x \\ f\left(\frac{\pi}{6}\right) &= \frac{\sqrt{3}}{3} \\ f'(x) &= \sec^2 x \\ f'\left(\frac{\pi}{6}\right) &= \frac{4}{3} \\ (f^{-1})'\left(\frac{\sqrt{3}}{3}\right) &= \frac{1}{f'(\pi/6)} = \frac{3}{4} \end{aligned}$$

34.
$$\begin{aligned} f(x) &= \ln x \\ f^{-1}(x) &= e^x \\ (f^{-1})'(x) &= e^x \\ (f^{-1})'(0) &= e^0 = 1 \end{aligned}$$

35. (a)
$$\begin{aligned} f(x) &= \ln\sqrt{x} \\ y &= \ln\sqrt{x} \\ e^y &= \sqrt{x} \\ e^{2y} &= x \\ e^{2x} &= y \\ f^{-1}(x) &= e^{2x} \end{aligned}$$

(b)

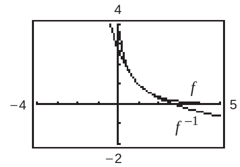


(c)
$$\begin{aligned} f^{-1}(f(x)) &= f^{-1}\left(\ln\sqrt{x}\right) = e^{2\ln\sqrt{x}} = e^{\ln x} = x \\ f(f^{-1}(x)) &= f(e^{2x}) = \ln\sqrt{e^{2x}} = \ln e^x = x \end{aligned}$$

36. (a) $f(x) = e^{1-x}$

$$y = e^{1-x}$$
$$\ln y = 1 - x$$
$$x = 1 - \ln y$$
$$y = 1 - \ln x$$
$$f^{-1}(x) = 1 - \ln x$$

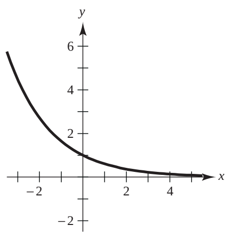
(b)



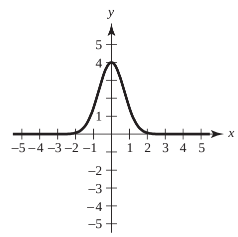
(c) $f^{-1}(f(x)) = f^{-1}(e^{1-x}) = 1 - \ln(e^{1-x})$

$$= 1 - (1 - x) = x$$
$$f(f^{-1}(x)) = f(1 - \ln x) = e^{1-(1-\ln x)} = e^{\ln x} = x$$

37. $y = e^{-x/2}$



38. $y = 4e^{-x^2}$



39. $g(t) = t^2e^t$

$$g'(t) = t^2e^t + 2te^t = te^t(t + 2)$$

40. $g(x) = \ln\left(\dfrac{e^x}{1 + e^x}\right)$

$$= \ln e^x - \ln(1 + e^x) = x - \ln(1 + e^x)$$
$$g'(x) = 1 - \frac{e^x}{1 + e^x} = \frac{1}{1 + e^x}$$

41. $y = \sqrt{e^{2x} + e^{-2x}}$

$$y' = \frac{1}{2}(e^{2x} + e^{-2x})^{-1/2}(2e^{2x} - 2e^{-2x}) = \frac{e^{2x} - e^{-2x}}{\sqrt{e^{2x} + e^{-2x}}}$$

42. $h(z) = e^{-z^2/2}$

$$h'(z) = -ze^{-z^2/2}$$

43. $g(x) = \dfrac{x^2}{e^x}$

$$g'(x) = \frac{e^x(2x) - x^2e^x}{e^{2x}} = \frac{x(2 - x)}{e^x}$$

44. $y = 3e^{-3/t}$

$$y' = 3e^{-3/t}(3t^{-2}) = \frac{9e^{-3/t}}{t^2}$$

45. $f(x) = \ln(e^{-x^2}) = -x^2, \quad (2, -4)$

$$f'(x) = -2x$$
$$f'(2) = -4$$

Tangent line: $y + 4 = -4(x - 2)$

$$y = -4x + 4$$

46. $f(\theta) = \frac{1}{2}e^{\sin 2\theta}, \quad \left(0, \frac{1}{2}\right)$

$$f'(\theta) = \tfrac{1}{2}e^{\sin 2\theta} \cdot 2\cos 2\theta$$
$$f'(0) = 1$$

Tangent line: $y - \frac{1}{2} = 1(x - 0)$

$$y = x + \tfrac{1}{2}$$

47. $y(\ln x) + y^2 = 0$

$$y\left(\frac{1}{x}\right) + (\ln x)\left(\frac{dy}{dx}\right) + 2y\left(\frac{dy}{dx}\right) = 0$$
$$(2y + \ln x)\frac{dy}{dx} = \frac{-y}{x}$$
$$\frac{dy}{dx} = \frac{-y}{x(2y + \ln x)}$$

48. $\cos x^2 = xe^y$

$$-2x \sin x^2 = xe^y\frac{dy}{dx} + e^y$$
$$\frac{dy}{dx} = -\frac{2x \sin x^2 + e^y}{xe^y}$$

49. $\displaystyle\int_0^1 xe^{-3x^2}\,dx = -\frac{1}{6}\int_0^1 e^{-3x^2}(-6x\,dx)$

$$= -\frac{1}{6}e^{-3x^2}\Big]_0^1$$

$$= -\frac{1}{6}[e^{-3} - 1]$$

$$= \frac{1}{6}\left(1 - \frac{1}{e^3}\right)$$

50. $\displaystyle\int_{1/2}^2 \frac{e^{1/x}}{x^2}dx$

Let $u = \dfrac{1}{x}$, $du = \dfrac{-1}{x^2}\,dx$.

$x = \dfrac{1}{2} \Rightarrow u = 2, \quad x = 2 \Rightarrow u = \dfrac{1}{2}$

$$-\int_2^{1/2} e^u\,du = -e^u\Big]_2^{1/2} = -e^{1/2} + e^2 = e^2 - \sqrt{e}$$

51. $\displaystyle\int \frac{e^{4x} - e^{2x} + 1}{e^x}\,dx = \int (e^{3x} - e^x + e^{-x})\,dx$

$$= \frac{1}{3}e^{3x} - e^x - e^{-x} + C$$

$$= \frac{e^{4x} - 3e^{2x} - 3}{3e^x} + C$$

52. Let $u = e^{2x} + e^{-2x}$, $du = (2e^{2x} - e^{-2x})\,dx$.

$$\int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\,dx = \frac{1}{2}\int \frac{2e^{2x} - 2e^{-2x}}{e^{2x} + e^{-2x}}\,dx$$

$$= \frac{1}{2}\ln(e^{2x} + e^{-2x}) + C$$

53. $\displaystyle\int xe^{1-x^2}\,dx = -\frac{1}{2}\int e^{1-x^2}(-2x)\,dx$

$$= -\frac{1}{2}e^{1-x^2} + C$$

54. Let $u = x^3 + 1$, $du = 3x^2\,dx$.

$$\int x^2 e^{x^3+1}\,dx = \frac{1}{3}\int e^{x^3+1}(3x^2)\,dx = \frac{1}{3}e^{x^3+1} + C$$

55. $\displaystyle\int_1^3 \frac{e^x}{e^x - 1}\,dx$

Let $u = e^x - 1$, $du = e^x\,dx$.

$$\int_1^3 \frac{e^x}{e^x - 1}\,dx = \ln|e^x - 1|\Big]_1^3$$

$$= \ln(e^3 - 1) - \ln(e - 1)$$

$$= \ln\left(\frac{e^3 - 1}{e - 1}\right)$$

$$= \ln(e^2 + e + 1)$$

56. $\displaystyle\int_0^2 \frac{e^{2x}}{e^{2x} + 1}\,dx$

Let $u = e^{2x} + 1$, $du = 2e^{2x}\,dx$.

$$\frac{1}{2}\int_0^2 \frac{2e^{2x}}{e^{2x} + 1}\,dx = \frac{1}{2}\ln(e^{2x} + 1)\Big]_0^2$$

$$= \frac{1}{2}\ln(e^4 + 1) - \frac{1}{2}\ln 2$$

$$= \frac{1}{2}\ln\left(\frac{e^4 + 1}{2}\right)$$

57.

$$y = e^x(a\cos 3x + b\sin 3x)$$

$$y' = e^x(-3a\sin 3x + 3b\cos 3x) + e^x(a\cos 3x + b\sin 3x)$$

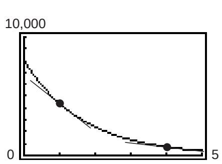
$$= e^x[(-3a + b)\sin 3x + (a + 3b)\cos 3x]$$

$$y'' = e^x[3(-3a + b)\cos 3x - 3(a + 3b)\sin 3x] + e^x[(-3a + b)\sin 3x + (a + 3b)\cos 3x]$$

$$= e^x[(-6a - 8b)\sin 3x + (-8a + 6b)\cos 3x]$$

$$y'' - 2y' + 10y = e^x\{[(-6a - 8b) - 2(-3a + b) + 10b]\sin 3x + [(-8a + 6b) - 2(a + 3b) + 10a]\cos 3x\} = 0$$

58. (a), (c)



(b) $V = 8000e^{-0.6t}, \quad 0 \le t \le 5$

$V'(t) = -4800e^{-0.6t}$

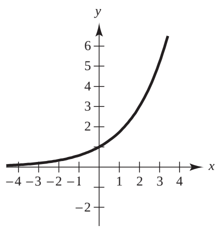
$V'(1) = -2634.3$ dollars/year

$V'(4) = -435.4$ dollars/year

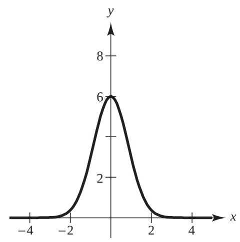
59. Area $= \displaystyle\int_0^4 xe^{-x^2}\,dx = \left[-\frac{1}{2}e^{-x^2}\right]_0^4 = -\frac{1}{2}(e^{-16} - 1) \approx 0.500$

60. Area $= \displaystyle\int_0^2 2e^{-x}\,dx = \left[-2e^{-x}\right]_0^2 = -2e^{-2} + 2 = 2 - \frac{2}{e^2} \approx 1.729$

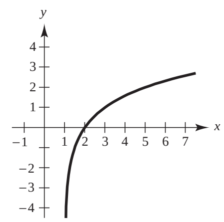
61. $y = 3^{x/2}$



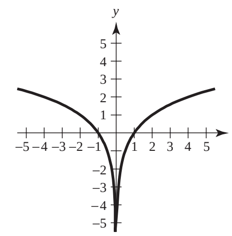
62. $g(x) = 6(2^{-x^2})$



63. $y = \log_2(x - 1)$



64. $y = \log_4 x^2$



65. $f(x) = 3^{x-1}$

$f'(x) = 3^{x-1}\ln 3$

66. $f(x) = 4^x e^x$

$f'(x) = 4^x e^x + (\ln 4)4^x e^x$

$= 4^x e^x(1 + \ln 4)$

67. $y = x^{2x+1}$

$\ln y = (2x + 1)\ln x$

$\dfrac{y'}{y} = \dfrac{2x + 1}{x} + 2\ln x$

$y' = y\left(\dfrac{2x + 1}{x} + 2\ln x\right) = x^{2x+1}\left(\dfrac{2x + 1}{x} + 2\ln x\right)$

68. $y = x(4^{-x})$

$y' = 4^{-x} - x \cdot 4^{-x}\ln 4$

69. $g(x) = \log_3\sqrt{1 - x} = \dfrac{1}{2}\log_3(1 - x)$

$g'(x) = \left(\dfrac{1}{2}\right)\dfrac{-1}{(1 - x)\ln 3} = \dfrac{1}{2(x - 1)\ln 3}$

70. $h(x) = \log_5 \dfrac{x}{x - 1} = \log_5 x - \log_5(x - 1)$

$h'(x) = \dfrac{1}{\ln 5}\left[\dfrac{1}{x} - \dfrac{1}{x - 1}\right] = \dfrac{1}{\ln 5}\left[\dfrac{-1}{x(x - 1)}\right]$

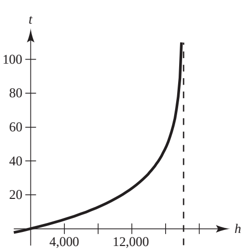
71. $\displaystyle\int (x + 1)5^{(x+1)^2}\,dx = \left(\frac{1}{2}\right)\frac{1}{\ln 5}5^{(x+1)^2} + C$

72. $\displaystyle\int \frac{2^{-1/t}}{t^2}\,dt = \frac{1}{\ln 2}2^{-1/t} + C$

73. $t = 50\log_{10}\left(\dfrac{18{,}000}{18{,}000 - h}\right)$

(a) Domain: $0 \le h < 18{,}000$

(b)



Vertical asymptote: $h = 18{,}000$

(c) $t = 50\log_{10}\left(\dfrac{18{,}000}{18{,}000 - h}\right)$

$10^{t/50} = \dfrac{18{,}000}{18{,}000 - h}$

$18{,}000 - h = 18{,}000(10^{-t/50})$

$h = 18{,}000(1 - 10^{-t/50})$

$\dfrac{dh}{dt} = 360\ln 10\left(\dfrac{1}{10}\right)^{t/50}$ is greatest when $t = 0$.

74. (a) $10{,}000 = Pe^{(0.07)(15)}$

$$P = \frac{10{,}000}{e^{1.05}} \approx \$3499.38$$

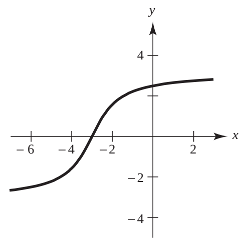
(b) $2P = Pe^{10r}$

$$2 = e^{10r}$$

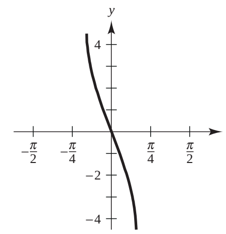
$$\ln 2 = 10r$$

$$r = \frac{\ln 2}{10} \approx 6.93\%$$

75. $f(x) = 2 \arctan(x + 3)$



76. $h(x) = -3 \arcsin(2x)$

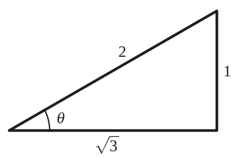


77. (a)

$$\text{Let } \theta = \arcsin \frac{1}{2}$$

$$\sin \theta = \frac{1}{2}$$

$$\sin\left(\arcsin \frac{1}{2}\right) = \sin \theta = \frac{1}{2}.$$



(b)

$$\text{Let } \theta = \arcsin \frac{1}{2}$$

$$\sin \theta = \frac{1}{2}$$

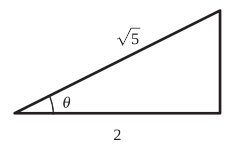
$$\cos\left(\arcsin \frac{1}{2}\right) = \cos \theta = \frac{\sqrt{3}}{2}.$$

78. (a)

$$\text{Let } \theta = \operatorname{arccot} 2$$

$$\cot \theta = 2$$

$$\tan(\operatorname{arccot} 2) = \tan \theta = \frac{1}{2}.$$

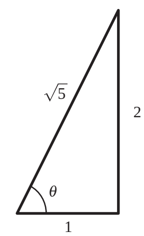


(b)

$$\text{Let } \theta = \operatorname{arcsec} \sqrt{5}$$

$$\sec \theta = \sqrt{5}$$

$$\cos\left(\operatorname{arcsec} \sqrt{5}\right) = \cos \theta = \frac{1}{\sqrt{5}}.$$



79. $y = \tan(\arcsin x) = \dfrac{x}{\sqrt{1 - x^2}}$

$$y' = \frac{(1 - x^2)^{1/2} + x^2(1 - x^2)^{-1/2}}{1 - x^2} = (1 - x^2)^{-3/2}$$

80. $y = \arctan(x^2 - 1)$

$$y' = \frac{2x}{1 + (x^2 - 1)^2} = \frac{2x}{x^4 - 2x^2 + 2}$$

81. $y = x \operatorname{arcsec} x$

$$y' = \frac{x}{|x|\sqrt{x^2 - 1}} + \operatorname{arcsec} x$$

82. $y = \dfrac{1}{2} \arctan e^{2x}$

$$y' = \frac{1}{2}\left(\frac{1}{1 + e^{4x}}\right)(2e^{2x}) = \frac{e^{2x}}{1 + e^{4x}}$$

83. $y = x(\arcsin x)^2 - 2x + 2\sqrt{1 - x^2} \arcsin x$

$$y' = \frac{2x \arcsin x}{\sqrt{1 - x^2}} + (\arcsin x)^2 - 2 + \frac{2\sqrt{1 - x^2}}{\sqrt{1 - x^2}} - \frac{2x}{\sqrt{1 - x^2}} \arcsin x = (\arcsin x)^2$$

84. $y = \sqrt{x^2 - 4} - 2 \operatorname{arcsec} \dfrac{x}{2}, \quad 2 < x < 4$

$$y' = \frac{x}{\sqrt{x^2 - 4}} - \frac{1}{\left(|x|/2\right)\sqrt{(x/2)^2 - 1}} = \frac{x}{\sqrt{x^2 - 4}} - \frac{4}{|x|\sqrt{x^2 - 4}} = \frac{x^2 - 4}{|x|\sqrt{x^2 - 4}} = \frac{\sqrt{x^2 - 4}}{x}$$

85. Let $u = e^{2x}$, $du = 2e^{2x}\,dx$.

$$\int \frac{1}{e^{2x} + e^{-2x}}\,dx = \int \frac{e^{2x}}{1 + e^{4x}}\,dx = \frac{1}{2}\int \frac{1}{1 + (e^{2x})^2}(2e^{2x})\,dx = \frac{1}{2}\arctan(e^{2x}) + C$$

86. Let $u = 5x$, $du = 5\,dx$.

$$\int \frac{1}{3 + 25x^2}\,dx = \frac{1}{5}\int \frac{1}{(\sqrt{3})^2 + (5x)^2}(5)\,dx = \frac{1}{5\sqrt{3}}\arctan\frac{5x}{\sqrt{3}} + C$$

87. Let $u = x^2$, $du = 2x\,dx$.

$$\int \frac{x}{\sqrt{1 - x^4}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{1 - (x^2)^2}}(2x)\,dx = \frac{1}{2}\arcsin x^2 + C$$

88. $\displaystyle\int \frac{1}{16 + x^2}\,dx = \frac{1}{4}\arctan\frac{x}{4} + C$

89. Let $u = \arctan\left(\frac{x}{2}\right)$, $du = \frac{2}{4 + x^2}\,dx$.

$$\int \frac{\arctan(x/2)}{4 + x^2}\,dx = \frac{1}{2}\int \left(\arctan\frac{x}{2}\right)\left(\frac{2}{4 + x^2}\right)dx$$

$$= \frac{1}{4}\left(\arctan\frac{x}{2}\right)^2 + C$$

90. Let $u = \arcsin x$, $du = \frac{1}{\sqrt{1 - x^2}}\,dx$.

$$\int \frac{\arcsin x}{\sqrt{1 - x^2}}\,dx = \frac{1}{2}(\arcsin x)^2 + C$$

91. $A = \displaystyle\int_0^1 \frac{4 - x}{\sqrt{4 - x^2}}\,dx$

$$= 4\int_0^1 \frac{1}{\sqrt{4 - x^2}}\,dx + \frac{1}{2}\int_0^1 (4 - x^2)^{-1/2}(-2x)\,dx$$

$$= \left[4\arcsin\left(\frac{x}{2}\right) + \sqrt{4 - x^2}\right]_0^1$$

$$= \left(4\arcsin\left(\frac{1}{2}\right) + \sqrt{3}\right) - 2$$

$$= \frac{2\pi}{3} + \sqrt{3} - 2 \approx 1.8264$$

92. $\displaystyle\int_0^4 \frac{x}{16 + x^2}\,dx = \frac{1}{2}\ln(16 + x^2)\Big]_0^4$

$$= \frac{1}{2}\ln 32 - \frac{1}{2}\ln 16$$

$$= \frac{1}{2}\ln 2$$

93. $\displaystyle\int \frac{dy}{\sqrt{A^2 - y^2}} = \int \sqrt{\frac{k}{m}}\,dt$

$$\arcsin\left(\frac{y}{A}\right) = \sqrt{\frac{k}{m}}\,t + C$$

Since $y = 0$ when $t = 0$, you have $C = 0$. Thus,

$$\sin\left(\sqrt{\frac{k}{m}}\,t\right) = \frac{y}{A}$$

$$y = A\sin\left(\sqrt{\frac{k}{m}}\,t\right).$$

94. $y = 2x - \cosh\sqrt{x}$

$$y' = 2 - \frac{1}{2\sqrt{x}}\left(\sinh\sqrt{x}\right) = 2 - \frac{\sinh\sqrt{x}}{2\sqrt{x}}$$

95. $y = x\tanh^{-1} 2x$

$$y' = x\left(\frac{2}{1 - 4x^2}\right) + \tanh^{-1} 2x = \frac{2x}{1 - 4x^2} + \tanh^{-1} 2x$$

96. Let $u = x^2$, $du = 2x\,dx$.

$$\int \frac{x}{\sqrt{x^4 - 1}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{(x^2)^2 - 1}}(2x)\,dx = \frac{1}{2}\ln\left(x^2 + \sqrt{x^4 - 1}\right) + C$$

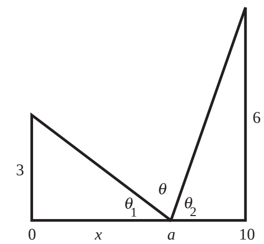
97. Let $u = x^3$, $du = 3x^2\,dx$.

$$\int x^2(\operatorname{sech} x^3)^2\,dx = \frac{1}{3}\int (\operatorname{sech} x^3)^2(3x^2)\,dx = \frac{1}{3}\tanh x^3 + C$$

Problem Solving for Chapter 5

1. $\tan\theta_1 = \dfrac{3}{x}$

$\tan\theta_2 = \dfrac{6}{10-x}$



Minimize $\theta_1 + \theta_2$:

$$f(x) = \theta_1 + \theta_2 = \arctan\left(\frac{3}{x}\right) + \arctan\left(\frac{6}{10-x}\right)$$

$$f'(x) = \frac{1}{1+\dfrac{9}{x^2}}\left(\frac{-3}{x^2}\right) + \frac{1}{1+\dfrac{36}{(10-x)^2}}\left(\frac{6}{(10-x)^2}\right) = 0$$

$$\frac{3}{x^2+9} = \frac{6}{(10-x)^2+36}$$

$$(10-x)^2 + 36 = 2(x^2+9)$$

$$100 - 20x + x^2 + 36 = 2x^2 + 18$$

$$x^2 + 20x - 118 = 0$$

$$x = \frac{-20 \pm \sqrt{20^2 - 4(-118)}}{2} = -10 \pm \sqrt{218}$$

$a = -10 + \sqrt{218} \approx 4.7648$ $\quad f(a) \approx 1.4153$

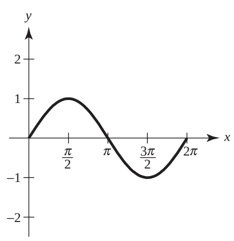
$\theta = \pi - (\theta_1 + \theta_2) \approx 1.7263$ or $98.9°$

Endpoints: $a = 0$: $\theta \approx 1.0304$

$a = 10$: $\theta \approx 1.2793$

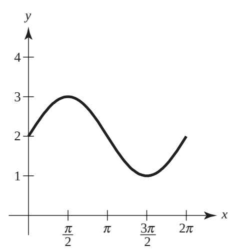
Maximum is 1.7263 at $a = -10 + \sqrt{218} \approx 4.7648$.

2. (a)



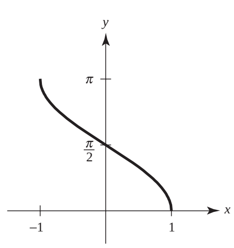
$$\int_0^{\pi} \sin x\,dx = -\int_{\pi}^{2\pi} \sin x\,dx \implies \int_0^{2\pi} \sin x\,dx = 0$$

(b)



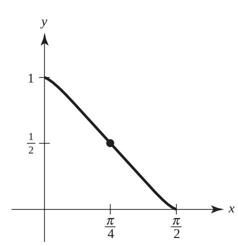
$$\int_0^{2\pi} (\sin x + 2)\,dx = 2(2\pi) = 4\pi$$

(c)



$$\int_{-1}^{1} \arccos x\,dx = 2\left(\frac{\pi}{2}\right) = \pi$$

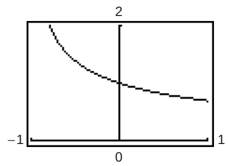
(d)



$y = \dfrac{1}{1+(\tan x)^{\sqrt{2}}}$ symmetric with respect to point $\left(\dfrac{\pi}{4}, \dfrac{1}{2}\right)$.

$$\int_0^{\pi/2} \frac{1}{1+(\tan x)^{\sqrt{2}}}\,dx = \frac{\pi}{2}\left(\frac{1}{2}\right) = \frac{\pi}{4}$$

3. (a) $f(x) = \dfrac{\ln(x+1)}{x}, \quad -1 \le x \le 1$



(b) $\lim_{x\to 0} f(x) = 1$

(c) Let $g(x) = \ln x$, $g'(x) = 1/x$, and $g'(1) = 1$. From the definition of derivative

$$g'(1) = \lim_{x\to 0} \frac{g(1+x) - g(1)}{x} = \lim_{x\to 0} \frac{\ln(1+x)}{x}.$$

Thus, $\lim_{x\to 0} f(x) = 1$.

4. $f(x) = \sin(\ln x)$

(a) Domain: $x > 0$ or $(0, \infty)$

(b) $f(x) = 1 = \sin(\ln x) \implies \ln x = \dfrac{\pi}{2} + 2k\pi$

Two values are $x = e^{\pi/2}, e^{(\pi/2)+2\pi}$.

(c) $f(x) = -1 = \sin(\ln x) \implies \ln x = \dfrac{3\pi}{2} + 2k\pi$

Two values are $x = e^{-\pi/2}, e^{3\pi/2}$.

(d) Since the range of the sine function is $[-1, 1]$, parts (b) and (c) show that the range of f is $[-1, 1]$.

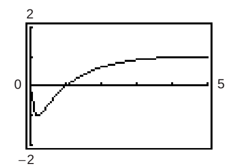
(e) $f'(x) = \dfrac{1}{x}\cos(\ln x)$

$f'(x) = 0 \implies \cos(\ln x) = 0 \implies \ln x = \dfrac{\pi}{2} + k\pi \implies$

$x = e^{\pi/2}$ on $[1, 10]$

$$\left.\begin{aligned} f(e^{\pi/2}) &= 1 \\ f(1) &= 0 \\ f(10) &\approx 0.7440 \end{aligned}\right\} \text{Maximum is 1 at } x = e^{\pi/2} \approx 4.8105$$

(f)



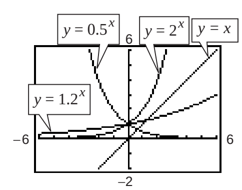
$\lim_{x\to 0^+} f(x)$ seems to be $-\frac{1}{2}$. (This is incorrect.)

(g) For the points $x = e^{\pi/2}, e^{-3\pi/2}, e^{-7\pi/2}, \ldots$ we have $f(x) = 1$.

For the points $x = e^{-\pi/2}, e^{-5\pi/2}, e^{-9\pi/2}, \ldots$ we have $f(x) = -1$.

That is, as $x \to 0^+$, there is an infinite number of points where $f(x) = 1$, and an infinite number where $f(x) = -1$. Thus, $\lim_{x\to 0+} \sin(\ln x)$ does not exist. You can verify this by graphing $f(x)$ on small intervals close to the origin.

5. $y = 0.5^x$ and $y = 1.2^x$ intersect $y = x$. $y = 2^x$ does not intersect $y = x$. Suppose $y = x$ is tangent to $y = a^x$ at (x, y).



$a^x = x \implies a = x^{1/x}$.

$y' = a^x \ln a = 1 \implies x \ln x^{1/x} = 1 \implies \ln x = 1 \implies x = e, a = e^{1/e}$

For $0 < a \le e^{1/e} \approx 1.445$, the curve $y = a^x$ intersects $y = x$.

6. (a) $\dfrac{\text{Area sector}}{\text{Area circle}} = \dfrac{t}{2\pi} \implies \text{Area sector} = \dfrac{t}{2\pi}(\pi) = \dfrac{t}{2}$

(b) Area $AOP = \dfrac{1}{2}(\text{base})(\text{height}) - \displaystyle\int_1^{\cosh t} \sqrt{x^2 - 1}\,dx$

$$A(t) = \frac{1}{2}\cosh t \cdot \sinh t - \int_1^{\cosh t} \sqrt{x^2-1}\,dx$$

$$A'(t) = \frac{1}{2}[\cosh^2 t + \sinh^2 t] - \sqrt{\cosh^2 t - 1}\,\sinh t = \frac{1}{2}[\cosh^2 t + \sinh^2 \mathrm{t}] - \sinh^2 t = \frac{1}{2}[\cosh^2 t - \sinh^2 \mathrm{t}] = \frac{1}{2}$$

$$A(t) = \frac{1}{2}t + C$$

But, $A(0) = C = 0 \implies C = 0$ Thus, $A(t) = \frac{1}{2}t$ or $t = 2A(t)$.

7. (a) $y = f(x) = \arcsin x$

$\sin y = x$

$$\text{Area } A = \int_{\pi/6}^{\pi/4} \sin y \cdot dy = -\cos y\Big]_{\pi/6}^{\pi/4} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} = \frac{\sqrt{3}-\sqrt{2}}{2} \approx 0.1589$$

$$\text{Area } B = \left(\frac{1}{2}\right)\left(\frac{\pi}{6}\right) = \frac{\pi}{12} \approx 0.2618$$

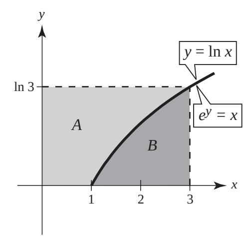
(b) $$\int_{1/2}^{\sqrt{2}/2} \arcsin x\, dx = \text{Area}(C) = \left(\frac{\pi}{4}\right)\left(\frac{\sqrt{2}}{2}\right) - A - B$$

$$= \frac{\pi\sqrt{2}}{8} - \frac{\sqrt{3}-\sqrt{2}}{2} - \frac{\pi}{12} = \pi\left(\frac{\sqrt{2}}{8} - \frac{1}{12}\right) + \frac{\sqrt{2}-\sqrt{3}}{2} \approx 0.1346$$

(c) $$\text{Area } A = \int_0^{\ln 3} e^y\, dy$$

$$= e^y\Big]_0^{\ln 3} = 3 - 1 = 2$$

$$\text{Area } B = \int_1^3 \ln x\, dx = 3(\ln 3) - A = 3\ln 3 - 2 = \ln 27 - 2 \approx 1.2958$$



(d) $\tan y = x$

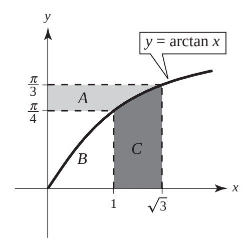
$$\text{Area } A = \int_{\pi/4}^{\pi/3} \tan y\, dy$$

$$= -\ln|\cos y|\Big]_{\pi/4}^{\pi/3}$$

$$= -\ln\frac{1}{2} + \ln\frac{\sqrt{2}}{2} = \ln\sqrt{2} = \frac{1}{2}\ln 2$$

$$\text{Area } C = \int_1^{\sqrt{3}} \arctan x\, dx = \left(\frac{\pi}{3}\right)\left(\sqrt{3}\right) - \frac{1}{2}\ln 2 - \left(\frac{\pi}{4}\right)(1)$$

$$= \frac{\pi}{12}\left(4\sqrt{3} - 3\right) - \frac{1}{2}\ln 2 \approx 0.6818$$



8. $y = \ln x$

$$y' = \frac{1}{x}$$

$$y - b = \frac{1}{a}(x - a)$$

$$y = \frac{1}{a}x + b - 1 \quad \text{Tangent line}$$

If $x = 0$, $c = b - 1$. Thus, $b - c = b - (b - 1) = 1$.

9. $y = e^x$

$$y' = e^x$$

$$y - b = e^a(x - a)$$

$$y = e^a x - ae^a + b \quad \text{Tangent line}$$

If $y = 0$: $e^a x = ae^a - b$

$$bx = ab - b \quad (b = e^a)$$

$$x = a - 1$$

$$c = a - 1$$

Thus, $a - c = a - (a - 1) = 1$.

10. Let $u = 1 + \sqrt{x}$, $\sqrt{x} = u - 1$, $x = u^2 - 2u + 1$, $dx = (2u - 2)\,du$.

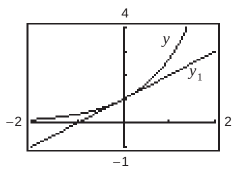
$$\begin{aligned}\text{Area} &= \int_1^4 \frac{1}{\sqrt{x} + x}\,dx = \int_2^3 \frac{2u - 2}{(u - 1) + (u^2 - 2u + 1)}\,du\\ &= \int_2^3 \frac{2(u - 1)}{u^2 - u}\,du\\ &= \int_2^3 \frac{2}{u}\,du = \Big[2\ln|u|\Big]_2^3\\ &= 2\ln 3 - 2\ln 2 = 2\ln\left(\frac{3}{2}\right)\\ &\approx 0.8109\end{aligned}$$

11. Let $u = \tan x$, $du = \sec^2 x\,dx$.

$$\begin{aligned}\text{Area} &= \int_0^{\pi/4} \frac{1}{\sin^2 x + 4\cos^2 x}\,dx = \int_0^{\pi/4} \frac{\sec^2 x}{\tan^2 x + 4}\,dx\\ &= \int_0^1 \frac{du}{u^2 + 4}\\ &= \left[\frac{1}{2}\arctan\left(\frac{u}{2}\right)\right]_0^1\\ &= \frac{1}{2}\arctan\left(\frac{1}{2}\right)\end{aligned}$$

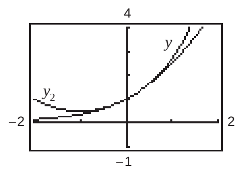
12. (a) (i) $y = e^x$

$y_1 = 1 + x$



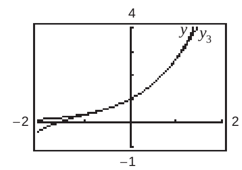
(ii) $y = e^x$

$y_2 = 1 + x + \left(\dfrac{x^2}{2}\right)$



(iii) $y = e^x$

$y_3 = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6}$

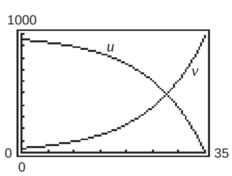


(b) n^{th} term is $x^n/n!$ in polynomial: $y_4 = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!}$

(c) Conjecture: $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$

13. (a) $u = 985.93 - \left(985.93 - \dfrac{(120{,}000)(0.095)}{12}\right)\left(1 + \dfrac{0.095}{12}\right)^{12t}$

$v = \left(985.93 - \dfrac{(120{,}000)(0.095)}{12}\right)\left(1 + \dfrac{0.095}{12}\right)^{12t}$



(b) The larger part goes for interest. The curves intersect when $t \approx 27.7$ years.

(c) The slopes are negatives of each other. Analytically,

$$u = 985.93 - v \implies \frac{du}{dt} = -\frac{dv}{dt}$$

$u'(15) = -v'(15) = -14.06.$

(d) $t = 12.7$ years

Again, the larger part goes for interest.