Chapter 11: Topics from Analytic Geometry

11.1 Exercises

Note: Let V, F, and l denote the vertex, focus, and directrix, respectively.

[1] $8y = x^2 \Rightarrow y = \frac{1}{8}x^2 \Rightarrow p = \frac{1}{4a} = \frac{1}{4(\frac{1}{8})} = \frac{1}{\frac{1}{2}} = 2$; $V(0, 0)$; $F(0, 2)$; l: $y = -2$

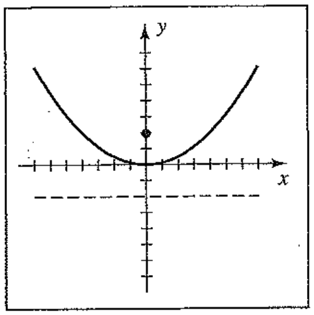

Figure 1

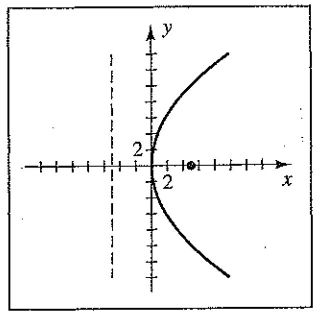

Figure 2

[2] $20x = y^2 \Rightarrow x = \frac{1}{20}y^2 \Rightarrow p = \frac{1}{4(\frac{1}{20})} = 5$; $V(0, 0)$; $F(5, 0)$; l: $x = -5$

[3] $2y^2 = -3x \Rightarrow (y - 0)^2 = -\frac{3}{2}(x - 0) \Rightarrow 4p = -\frac{3}{2} \Rightarrow p = -\frac{3}{8}$;

$V(0, 0)$; $F(-\frac{3}{8}, 0)$; l: $x = \frac{3}{8}$

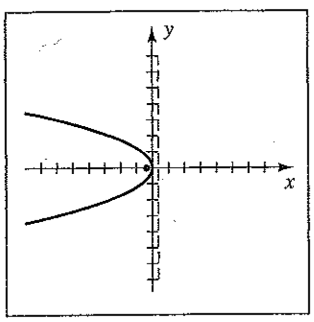

Figure 3

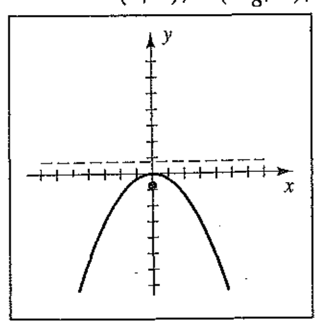

Figure 4

[4] $(x - 0)^2 = -3(y - 0) \Rightarrow 4p = -3 \Rightarrow p = -\frac{3}{4}$; $V(0, 0)$; $F(0, -\frac{3}{4})$; l: $y = \frac{3}{4}$

[5] $(x + 2)^2 = -8(y - 1) \Rightarrow 4p = -8 \Rightarrow p = -2$; $V(-2, 1)$; $F(-2, -1)$; l: $y = 3$

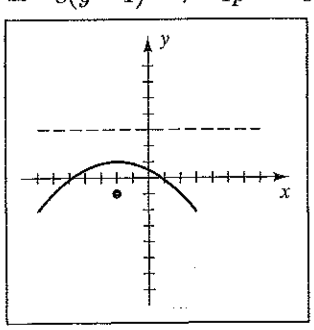

Figure 5

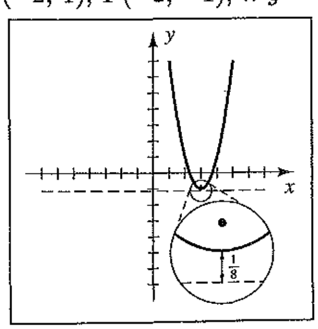

Figure 6

[6] $(x - 3)^2 = \frac{1}{2}(y + 1) \Rightarrow 4p = \frac{1}{2} \Rightarrow p = \frac{1}{8}$; $V(3, -1)$; $F(3, -\frac{7}{8})$; l: $y = -\frac{9}{8}$

$\boxed{7}$ $(y-2)^2 = \frac{1}{4}(x-3) \Rightarrow 4p = \frac{1}{4} \Rightarrow p = \frac{1}{16}$; $V(3, 2)$; $F(\frac{49}{16}, 2)$; l: $x = \frac{47}{16}$

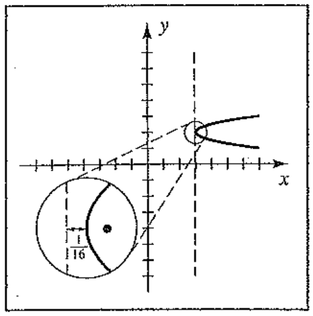

Figure 7

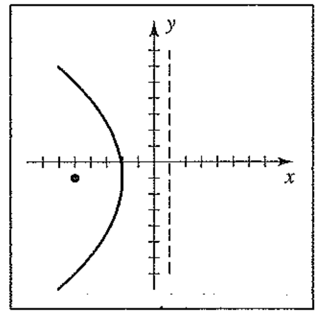

Figure 8

$\boxed{8}$ $(y+1)^2 = -12(x+2) \Rightarrow 4p = -12 \Rightarrow p = -3$; $V(-2, -1)$; $F(-5, -1)$; l: $x = 1$

$\boxed{9}$ $y = x^2 - 4x + 2 = (x^2 - 4x + \underline{4}) + 2 - \underline{4} \Rightarrow$

$(y+2) = 1(x-2)^2 \Rightarrow 4p = 1 \Rightarrow p = \frac{1}{4}$; $V(2, -2)$; $F(2, -\frac{7}{4})$; l: $y = -\frac{9}{4}$

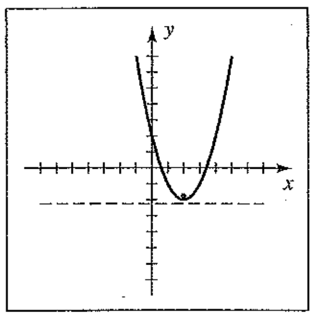

Figure 9

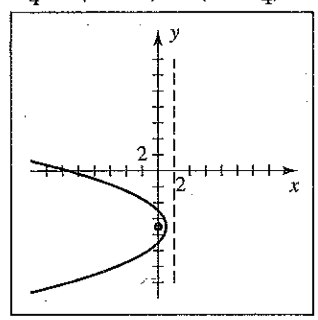

Figure 10

$\boxed{10}$ $y^2 + 14y + 4x + 45 = 0 \Rightarrow -4x = (y^2 + 14y + \underline{49}) + 45 - \underline{49} \Rightarrow$

$-4x + 4 = (y+7)^2 \Rightarrow (y+7)^2 = -4(x-1) \Rightarrow 4p = -4 \Rightarrow p = -1$;

$V(1, -7)$; $F(0, -7)$; l: $x = 2$

$\boxed{11}$ $x^2 + 20y = 10 \Rightarrow (x-0)^2 = -20(y - \frac{1}{2}) \Rightarrow 4p = -20 \Rightarrow p = -5$;

$V(0, \frac{1}{2})$; $F(0, -\frac{9}{2})$; l: $y = \frac{11}{2}$

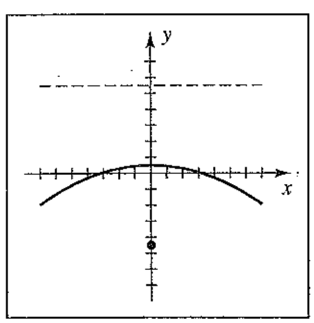

Figure 11

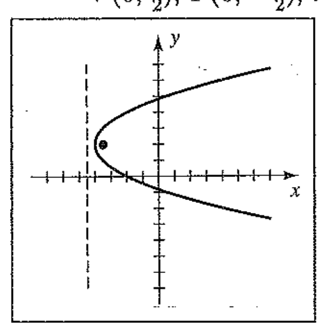

Figure 12

$\boxed{12}$ $y^2 - 4y - 2x - 4 = 0 \Rightarrow 2x = (y^2 - 4y + \underline{4}) - 4 - \underline{4} \Rightarrow$

$2(x+4) = (y-2)^2 \Rightarrow 4p = 2 \Rightarrow p = \frac{1}{2}$; $V(-4, 2)$; $F(-\frac{7}{2}, 2)$; l: $x = -\frac{9}{2}$

[13] $V(1,\ 0) \Rightarrow y^2 = 4p(x-1)$. $F(6,\ 0) \Rightarrow y^2 = 4(6-1)(x-1) \Rightarrow y^2 = 20(x-1)$.

[14] $V(0,\ -2) \Rightarrow x^2 = 4p(y+2)$.

$$F(0,\ 1) \Rightarrow x^2 = 4\big[1-(-2)\big](y+2) \Rightarrow x^2 = 12(y+2).$$

[15] $V(-2,\ 3) \Rightarrow (x+2)^2 = 4p(y-3)$.

$$x=2,\ y=2 \Rightarrow 16 = 4p(-1) \Rightarrow p=-4.\ (x+2)^2 = -16(y-3).$$

[16] $V(3,\ -2) \Rightarrow (y+2)^2 = 4p(x-3)$.

$$x=1,\ y=0 \Rightarrow 4 = 4p(-2) \Rightarrow p = -\tfrac{1}{2}.\ (y+2)^2 = -2(x-3).$$

[17] The distance from the focus $F(3,\ 2)$ to the directrix l: $y=-1$ is $2-(-1)=3$ units.

The vertex V is the point $(3,\ \frac{1}{2})$ {halfway between F and l}.

$V(3,\ \frac{1}{2}) \Rightarrow (x-3)^2 = 4p(y-\frac{1}{2})$.

$F(3,\ 2) \Rightarrow (x-3)^2 = 4\left[2-\frac{1}{2}\right](y+2) \Rightarrow x^2 = 6(y+2)$.

[18] The distance from the focus $F(-2,\ 1)$ to the directrix l: $x=3$ is $3-(-2)=5$ units.

The vertex V is the point $(\frac{1}{2},\ 1)$ {halfway between F and l}.

$V(\frac{1}{2},\ 1) \Rightarrow (y-1)^2 = 4p(x-\frac{1}{2})$.

$F(-2,\ 1) \Rightarrow (y-1)^2 = 4\left[-2-\frac{1}{2}\right](x-\frac{1}{2}) \Rightarrow (y-1)^2 = -10(x-\frac{1}{2})$.

[19] $F(2,\ 0)$ and l: $x=-2 \Rightarrow p=2$ and $V(0,\ 0)$. $(y-0)^2 = 4p(x-0) \Rightarrow y^2 = 8x$.

[20] $F(0,\ -4)$ and l: $y=4 \Rightarrow p=-4$ and $V(0,\ 0)$.

$$(x-0)^2 = 4p(y-0) \Rightarrow x^2 = -16y.$$

[21] $F(6,\ 4)$ and l: $y=-2 \Rightarrow p=3$ and $V(6,\ 1)$.

$$(x-6)^2 = 4p(y-1) \Rightarrow (x-6)^2 = 12(y-1).$$

[22] $F(-3,\ -2)$ and l: $y=1 \Rightarrow p=-\frac{3}{2}$ and $V(-3,\ -\frac{1}{2})$.

$$(x+3)^2 = 4p(y+\tfrac{1}{2}) \Rightarrow (x+3)^2 = -6(y+\tfrac{1}{2}).$$

[23] $V(3,\ -5)$ and l: $x=2 \Rightarrow p=1$. $(y+5)^2 = 4p(x-3) \Rightarrow (y+5)^2 = 4(x-3)$.

[24] $V(-2,\ 3)$ and l: $y=5 \Rightarrow p=-2$. $(x+2)^2 = 4p(y-3) \Rightarrow (x+2)^2 = -8(y-3)$.

[25] $V(-1,\ 0)$ and $F(-4,\ 0) \Rightarrow p=-3$. $(y-0)^2 = 4p(x+1) \Rightarrow y^2 = -12(x+1)$.

[26] $V(1,\ -2)$ and $F(1,\ 0) \Rightarrow p=2$. $(x-1)^2 = 4p(y+2) \Rightarrow (x-1)^2 = 8(y+2)$.

[27] The vertex at the origin and symmetric to the y-axis imply that the equation is of the form $y = ax^2$. Substituting $x=2$ and $y=-3$ into that equation yields

$$-3 = a\cdot 4 \Rightarrow a = -\tfrac{3}{4}.\ \text{Thus, an equation is } y = -\tfrac{3}{4}x^2, \text{ or } 3x^2 = -4y.$$

[28] $y = ax^2 \Rightarrow 3 = a(6)^2 \Rightarrow a = \frac{1}{12}$. $y = \frac{1}{12}x^2$, or $12y = x^2$.

[29] The vertex at $(-3,\ 5)$ and axis parallel to the x-axis imply that the equation is of the form $(y-5)^2 = 4p(x+3)$. Substituting $x=5$ and $y=9$ into that equation

yields $16 = 4p\cdot 8 \Rightarrow p = \frac{1}{2}$. Thus, an equation is $(y-5)^2 = 2(x+3)$.

$\boxed{30}$ $(y+2)^2 = 4p(x-3) \;\Rightarrow\; (1+2)^2 = 4p(0-3) \;\Rightarrow\; -\frac{9}{12} = p \;\Rightarrow$

$$(y+2)^2 = -3(x-3)$$

$\boxed{31}$ $P(0, 5)$ is the focus and l: $y = -3$ is the directrix.

The vertex V is halfway between them and is at $(0, 1)$. $p = d(V, F) = 5 - 1 = 4$.

$$(x-h)^2 = 4p(y-k) \;\Rightarrow\; (x-0)^2 = 4(4)(y-1) \;\Rightarrow\; x^2 = 16(y-1)$$

$\boxed{32}$ $P(7, 0)$ and l: $x = 1 \;\Rightarrow\; V(4, 0)$.

$$p = 7-4 = 3 \;\Rightarrow\; y^2 = 4p(x-4) \;\Rightarrow\; y^2 = 12(x-4).$$

$\boxed{33}$ $P(-6, 3)$ and l: $x = -2 \;\Rightarrow\; V(-4, 3)$.

$$p = -6-(-4) = -2 \;\Rightarrow\; (y-3)^2 = 4p(x+4) \;\Rightarrow\; (y-3)^2 = -8(x+4).$$

$\boxed{34}$ $P(5, -2)$ and l: $y = 4 \;\Rightarrow\; V(5, 1)$.

$$p = -2-1 = -3 \;\Rightarrow\; (x-5)^2 = 4p(y-1) \;\Rightarrow\; (x-5)^2 = -12(y-1).$$

Note: To find an equation for a lower or upper half, we need to solve for y (use $-$ or $+$ respectively). For the left or right half, solve for x (use $-$ or $+$ respectively).

$\boxed{35}$ $(y+1)^2 = x+3 \;\Rightarrow\; y+1 = \pm\sqrt{x+3} \;\Rightarrow\; y = -\sqrt{x+3} - 1$

$\boxed{36}$ $(y-2)^2 = x-4 \;\Rightarrow\; y-2 = \pm\sqrt{x-4} \;\Rightarrow\; y = \sqrt{x-4} + 2$

$\boxed{37}$ $(x+1)^2 = y-4 \;\Rightarrow\; x+1 = \pm\sqrt{y-4} \;\Rightarrow\; x = \sqrt{y-4} - 1$

$\boxed{38}$ $(x+3)^2 = y+2 \;\Rightarrow\; x+3 = \pm\sqrt{y+2} \;\Rightarrow\; x = -\sqrt{y+2} - 3$

$\boxed{39}$ The parabola has an equation of the form $y = ax^2 + bx + c$. Substituting the x and y values of $P(2, 5)$, $Q(-2, -3)$, and $R(1, 6)$ into this equation yields:

$$\left\{\begin{array}{rrrrll} 4a + & 2b + & c = & 5 & P & (\mathrm{E}_1) \\ 4a - & 2b + & c = & -3 & Q & (\mathrm{E}_2) \\ a + & b + & c = & 6 & R & (\mathrm{E}_3) \end{array}\right.$$

Solving E_3 for c $\{c = 6 - a - b\}$ and substituting into E_1 and E_2 yields:

$$\left\{\begin{array}{rrrl} 3a + & b = & -1 & (\mathrm{E}_4) \\ 3a - & 3b = & -9 & (\mathrm{E}_5) \end{array}\right.$$

$\mathrm{E}_4 - \mathrm{E}_5 \;\Rightarrow\; 4b = 8 \;\Rightarrow\; b = 2;\ a = -1;\ c = 5$. The equation is $y = -x^2 + 2x + 5$.

$\boxed{40}$ The parabola has an equation of the form $y = ax^2 + bx + c$. Substituting the x and y values of $P(3, -1)$, $Q(1, -7)$, and $R(-2, 14)$ into this equation yields:

$$\left\{\begin{array}{rrrrll} 9a + & 3b + & c = & -1 & P & (\mathrm{E}_1) \\ a + & b + & c = & -7 & Q & (\mathrm{E}_2) \\ 4a - & 2b + & c = & 14 & R & (\mathrm{E}_3) \end{array}\right.$$

Solving E_2 for c $\{c = -7 - a - b\}$ and substituting into E_1 and E_3 yields:

$$\left\{\begin{array}{rrrl} 8a + & 2b = & 6 & (\mathrm{E}_4) \\ 3a - & 3b = & 21 & (\mathrm{E}_5) \end{array}\right. \;\Rightarrow\; \left\{\begin{array}{rrrl} 4a + & b = & 3 & (\mathrm{E}_6) \\ a - & b = & 7 & (\mathrm{E}_7) \end{array}\right.$$

$\mathrm{E}_6 + \mathrm{E}_7 \;\Rightarrow\; 5a = 10 \;\Rightarrow\; a = 2;\ b = -5;\ c = -4$. The equation is $y = 2x^2 - 5x - 4$.

[41] The parabola has an equation of the form $x = ay^2 + by + c$. Substituting the x and y values of $P(-1, 1)$, $Q(11, -2)$, and $R(5, -1)$ into this equation yields:

$$\begin{cases} a + b + c = -1 & P\ (\mathrm{E}_1) \\ 4a - 2b + c = 11 & Q\ (\mathrm{E}_2) \\ a - b + c = 5 & R\ (\mathrm{E}_3) \end{cases}$$

Solving E_3 for c $\{c = 5 - a + b\}$ and substituting into E_1 and E_2 yields:

$$\begin{cases} 2b = -6 & (\mathrm{E}_4) \\ 3a - b = 6 & (\mathrm{E}_5) \end{cases}$$

$\mathrm{E}_4 \Rightarrow b = -3$; $a = 1$; $c = 1$. The equation is $x = y^2 - 3y + 1$.

[42] The parabola has an equation of the form $x = ay^2 + by + c$. Substituting the x and y values of $P(2, 1)$, $Q(6, 2)$, and $R(12, -1)$ into this equation yields:

$$\begin{cases} a + b + c = 2 & P\ (\mathrm{E}_1) \\ 4a + 2b + c = 6 & Q\ (\mathrm{E}_2) \\ a - b + c = 12 & R\ (\mathrm{E}_3) \end{cases}$$

Solving E_3 for c $\{c = 12 - a + b\}$ and substituting into E_1 and E_2 yields:

$$\begin{cases} 2b = -10 & (\mathrm{E}_4) \\ 3a + 3b = -6 & (\mathrm{E}_5) \end{cases}$$

$\mathrm{E}_4 \Rightarrow b = -5$; $a = 3$; $c = 4$. The equation is $x = 3y^2 - 5y + 4$.

[43] A cross section is a parabola with $V(0, 0)$ and passing through $P(4, 1)$. We need to find the focus F. $y = ax^2 \Rightarrow 1 = a(4)^2 \Rightarrow a = \frac{1}{16}$. $p = 1/(4a) = 1/(\frac{1}{4}) = 4$.

The light will collect 4 inches from the center of the mirror.

[44] $y = ax^2 \Rightarrow 3 = a(5)^2 \Rightarrow a = \frac{3}{25}$. $p = 1/(4a) = 1/(\frac{12}{25}) = \frac{25}{12}$ ft.

[45] $y = ax^2 \Rightarrow 1 = a(\frac{3}{2})^2 \Rightarrow a = \frac{4}{9}$. $p = 1/(4a) = 1/(\frac{16}{9}) = \frac{9}{16}$ ft.

[46] $y = ax^2 \Rightarrow \frac{3}{4} = a(2)^2 \Rightarrow a = \frac{3}{16}$. $p = 1/(4a) = 1/(\frac{3}{4}) = \frac{4}{3}$ in.

[47] $a = 1/(4p) = 1/(4 \cdot 5) = \frac{1}{20}$. $y = ax^2$ $\{y = 2\ \underline{\text{ft}}\}$ $\Rightarrow 24 = \frac{1}{20}x^2 \Rightarrow x = \sqrt{480}$.

The width is twice the value of x. Width $= 2\sqrt{480} \approx 43.82$ in.

[48] $a = 1/(4p) = 1/(4 \cdot 9) = \frac{1}{36}$. $24 = \frac{1}{36}x^2 \Rightarrow x = \sqrt{864}$. Width $= 2\sqrt{864} \approx 58.79$ in.

[49] (a) Let the parabola have the equation $x^2 = 4py$.

Since the point (r, h) is on the parabola, $r^2 = 4ph$ or $p = \dfrac{r^2}{4h}$.

(b) $p = 10$ and $h = 5 \Rightarrow r^2 = 4(10)(5) \Rightarrow r = 10\sqrt{2}$.

[50] Note that the value of p completely determines the parabola.

If (x_1, y_1) is on the parabola, then $y_1^2 = 4p(x_1 + p) \Rightarrow 4p^2 + 4x_1 p - y_1^2 = 0 \Rightarrow$

$$p = \frac{-4x_1 \pm \sqrt{16x_1^2 + 16y_1^2}}{8} = \frac{-x_1 \pm \sqrt{x_1^2 + y_1^2}}{2}.$$

If $y_1 \neq 0$, then there are exactly two values for p and hence, exactly two parabolas.

$\boxed{51}$ With $a = 125$ and $p = 50$, $S = \dfrac{8\pi p^2}{3}\left[\left(1 + \dfrac{a^2}{4p^2}\right)^{3/2} - 1\right] \approx 64{,}968\ \text{ft}^2$.

$\boxed{52}$ (a) From the figure we can see that the distance between Mars and the origin should be 58,000 miles. Thus, $p = 58{,}000$. $x = \dfrac{1}{4p}y^2 \Rightarrow x = \dfrac{1}{232{,}000}y^2$.

(b) Since $v = \sqrt{\frac{2k}{r}}$, the velocity of the satellite will be maximum when the distance r between the satellite and Mars is minimum. The minimum value of r is 58,000 miles, when the satellite is located at the vertex. First, convert r to meters. $58{,}000\ \text{mi} \times 1610\ \text{m/mi} \approx 9.34 \times 10^7$ m.

Thus, $v = \sqrt{\frac{2k}{r}} = \sqrt{\dfrac{2 \times 4.28 \times 10^{13}}{9.34 \times 10^7}} \approx 957$ m/sec.

(c) $y = 100{,}000 \Rightarrow x = \dfrac{100{,}000^2}{232{,}000} \approx 43{,}100$. If the satellite is located at the point (43,100, 100,000) and Mars is located at the point (58,000, 0), then $r = \sqrt{(58{,}000 - 43{,}100)^2 + (100{,}000 - 0)^2} \approx 101{,}100\ \text{mi} \approx 1.63 \times 10^8$ m.

Thus, $v = \sqrt{\frac{2k}{r}} = \sqrt{\dfrac{2 \times 4.28 \times 10^{13}}{1.63 \times 10^8}} \approx 725$ m/sec.

$\boxed{53}$ Depending on the type of calculator or software used, we may need to solve for y in terms of x. $x = -y^2 + 2y + 5 \Rightarrow y^2 - 2y + (x - 5) = 0$. This is a quadratic equation in y. Using the quadratic formula to solve for y yields

$$y = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(x-5)}}{2(1)} = 1 \pm \sqrt{6 - x}.$$

[−11, 10, 2] by [−7, 7]

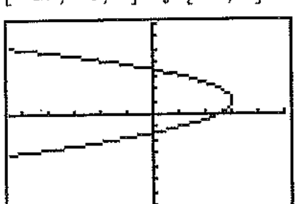

Figure 53

[−11, 10] by [−7, 7]

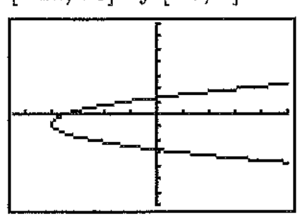

Figure 54

$\boxed{54}$ $x = 2y^2 + 3y - 7 \Rightarrow y = -\frac{3}{4} \pm \frac{1}{4}\sqrt{8x + 65}$

$\boxed{55}$ $x = y^2 + 1 \Rightarrow y = \pm\sqrt{x - 1}$. From the graph, we can see that there are 2 points of intersection. Their coordinates are approximately (2.08, −1.04) and (2.92, 1.38). See *Figure 55.*

[−2, 4] by [−3, 3]

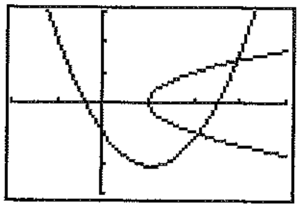

Figure 55

[−4, 4] by [−5, 3]

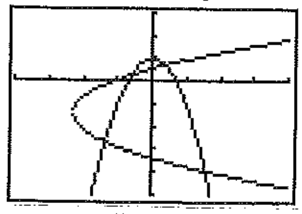

Figure 56

$\boxed{56}$ $x = 0.6y^2 + 1.7y - 1.1 \Rightarrow$

$$y = \frac{-1.7 \pm \sqrt{(1.7)^2 + 4(0.6)(x+1.1)}}{2(0.6)} = \frac{-1.7 \pm \sqrt{2.4x + 5.53}}{1.2}.$$

From the graph, we can see that there are 4 points of intersection. Their coordinates are approximately $(-1.34, -2.69)$, $(-0.65, 0.24)$, $(0.49, 0.74)$, and $(1.59, -3.97)$.

11.2 Exercises

Note: Let C, V, F, and M denote the center, the vertices, the foci, and the endpoints of the minor axis, respectively.

$\boxed{1}$ $\frac{x^2}{9} + \frac{y^2}{4} = 1$ • $c^2 = 9 - 4 \Rightarrow c = \pm\sqrt{5}$; $V(\pm 3, 0)$; $F(\pm\sqrt{5}, 0)$; $M(0, \pm 2)$

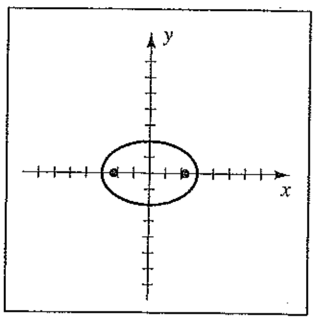

Figure 1

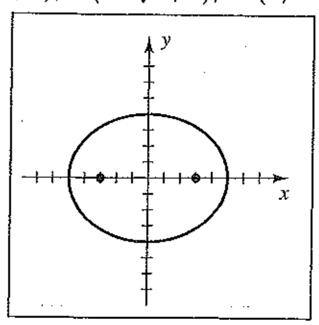

Figure 2

$\boxed{2}$ $\frac{x^2}{25} + \frac{y^2}{16} = 1$ • $c^2 = 25 - 16 \Rightarrow c = \pm 3$; $V(\pm 5, 0)$; $F(\pm 3, 0)$; $M(0, \pm 4)$

$\boxed{3}$ $\frac{x^2}{15} + \frac{y^2}{16} = 1$ • $c^2 = 16 - 15 \Rightarrow c = \pm 1$; $V(0, \pm 4)$; $F(0, \pm 1)$; $M(\pm\sqrt{15}, 0)$

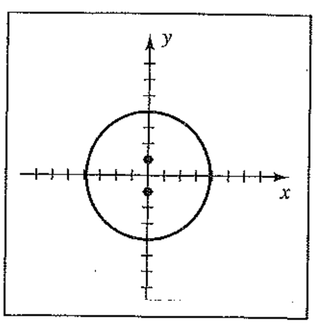

Figure 3

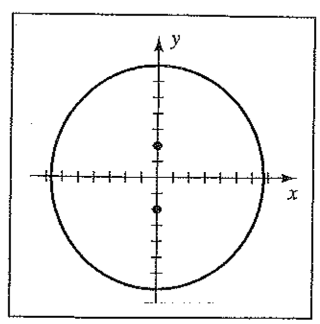

Figure 4

$\boxed{4}$ $\frac{x^2}{45} + \frac{y^2}{49} = 1$ • $c^2 = 49 - 45 \Rightarrow c = \pm 2$; $V(0, \pm 7)$; $F(0, \pm 2)$; $M(\pm\sqrt{45}, 0)$

$\boxed{5}$ $4x^2+y^2=16 \;\Rightarrow\; \frac{x^2}{4}+\frac{y^2}{16}=1;\; c^2=16-4 \;\Rightarrow\; c=\pm 2\sqrt{3};$

$V(0,\ \pm 4);\ F(0,\ \pm 2\sqrt{3});\ M(\pm 2,\ 0)$

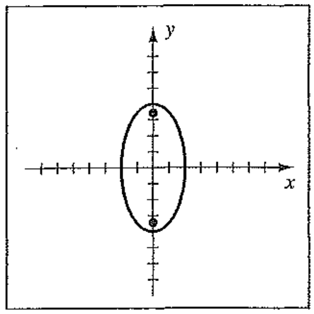

Figure 5

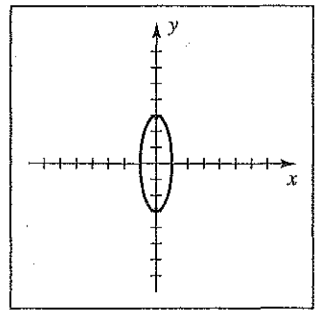

Figure 6

$\boxed{6}$ $y^2+9x^2=9 \;\Rightarrow\; \frac{x^2}{1}+\frac{y^2}{9}=1;\; c^2=9-1 \;\Rightarrow\; c=\pm 2\sqrt{2};$

$V(0,\ \pm 3);\ F(0,\ \pm 2\sqrt{2});\ M(\pm 1,\ 0)$

$\boxed{7}$ $4x^2+25y^2=1 \;\Rightarrow\; \frac{x^2}{\frac{1}{4}}+\frac{y^2}{\frac{1}{25}}=1;\; c^2=\frac{1}{4}-\frac{1}{25}=\frac{21}{100} \;\Rightarrow\; c=\pm\frac{1}{10}\sqrt{21};$

$V(\pm\frac{1}{2},\ 0);\ F(\pm\frac{1}{10}\sqrt{21},\ 0);\ M(0,\ \pm\frac{1}{5})$

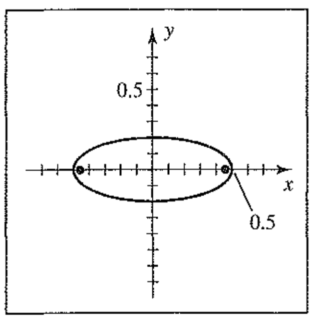

Figure 7

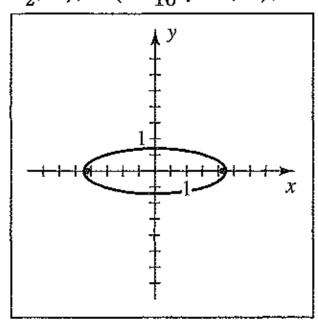

Figure 8

$\boxed{8}$ $10y^2+x^2=5 \;\Rightarrow\; \frac{x^2}{5}+\frac{y^2}{\frac{1}{2}}=1;\; c^2=5-\frac{1}{2}=\frac{9}{2} \;\Rightarrow\; c=\pm\frac{3}{2}\sqrt{2};$

$V(\pm\sqrt{5},\ 0);\ F(\pm\frac{3}{2}\sqrt{2},\ 0);\ M(0,\ \pm\frac{1}{2}\sqrt{2})$

$\boxed{9}$ $c^2=16-9 \;\Rightarrow\; c=\pm\sqrt{7};\ C(3,\ -4);\ V(3\pm 4,\ -4);\ F(3\pm\sqrt{7},\ -4);\ M(3,\ -4\pm 3)$

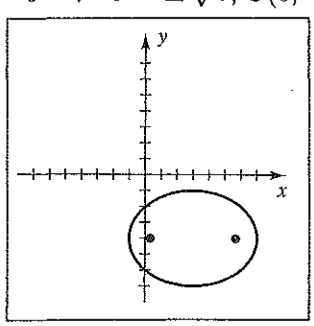

Figure 9

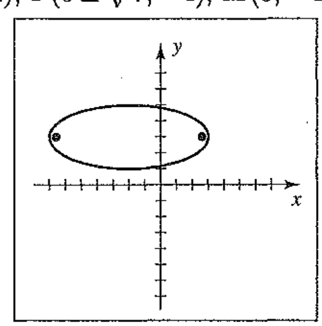

Figure 10

$\boxed{10}$ $c^2=25-4 \;\Rightarrow\; c=\pm\sqrt{21};\ C(-2,\ 3);\ V(-2\pm 5,\ 3);\ F(-2\pm\sqrt{21},\ 3);\ M(-2,\ 3\pm 2)$

$\boxed{11}$ $4x^2 + 9y^2 - 32x - 36y + 64 = 0 \;\Rightarrow$

$$4(x^2 - 8x + \underline{16}) + 9(y^2 - 4y + \underline{4}) = -64 + \underline{64} + \underline{36} \;\Rightarrow$$

$$4(x-4)^2 + 9(y-2)^2 = 36 \;\Rightarrow\; \frac{(x-4)^2}{9} + \frac{(y-2)^2}{4} = 1;$$

$$c^2 = 9 - 4 \;\Rightarrow\; c = \pm\sqrt{5};\; C(4,\,2);\; V(4 \pm 3,\,2);\; F(4 \pm \sqrt{5},\,2);\; M(4,\,2 \pm 2)$$

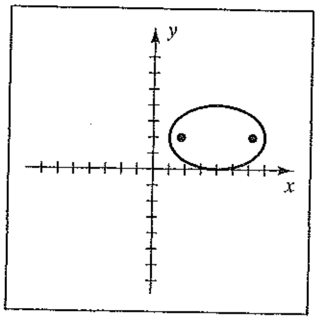

Figure 11

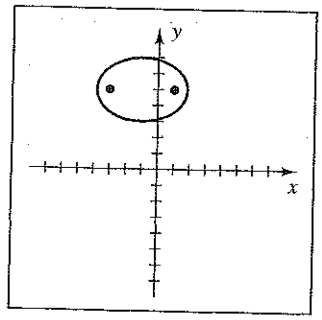

Figure 12

$\boxed{12}$ $x^2 + 2y^2 + 2x - 20y + 43 = 0 \;\Rightarrow$

$$(x^2 + 2x + \underline{1}) + 2(y^2 - 10y + \underline{25}) = -43 + \underline{1} + \underline{50} \;\Rightarrow$$

$$(x+1)^2 + 2(y-5)^2 = 8 \;\Rightarrow\; \frac{(x+1)^2}{8} + \frac{(y-5)^2}{4} = 1;\; c^2 = 8 - 4 \;\Rightarrow\; c = \pm 2;$$

$$C(-1,\,5);\; V(-1 \pm 2\sqrt{2},\,5);\; F(-1 \pm 2,\,5);\; M(-1,\,5 \pm 2)$$

$\boxed{13}$ $25x^2 + 4y^2 - 250x - 16y + 541 = 0 \;\Rightarrow$

$$25(x^2 - 10x + \underline{25}) + 4(y^2 - 4y + \underline{4}) = -541 + \underline{625} + \underline{16} \;\Rightarrow$$

$$25(x-5)^2 + 4(y-2)^2 = 100 \;\Rightarrow\; \frac{(x-5)^2}{4} + \frac{(y-2)^2}{25} = 1;\; c^2 = 25 - 4 \;\Rightarrow$$

$$c = \pm\sqrt{21};\; C(5,\,2);\; V(5,\,2 \pm 5);\; F(5,\,2 \pm \sqrt{21});\; M(5 \pm 2,\,2)$$

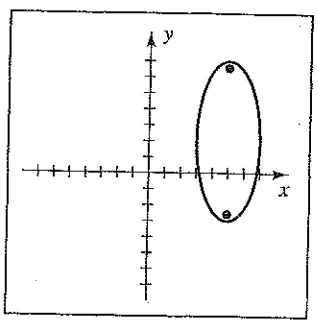

Figure 13

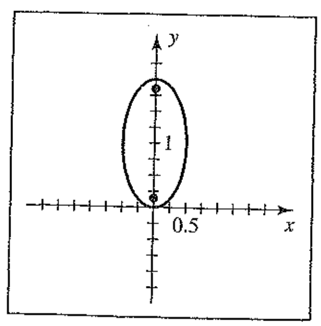

Figure 14

$\boxed{14}$ $4x^2 + y^2 = 2y \;\Rightarrow\; 4x^2 + y^2 - 2y + \underline{1} = \underline{1} \;\Rightarrow$

$$\frac{x^2}{\frac{1}{4}} + \frac{(y-1)^2}{1} = 1;\; c^2 = 1 - \tfrac{1}{4} \;\Rightarrow\; c = \pm\tfrac{1}{2}\sqrt{3};$$

$$C(0,\,1);\; V(0,\,1 \pm 1);\; F(0,\,1 \pm \tfrac{1}{2}\sqrt{3});\; M(0 \pm \tfrac{1}{2},\,1)$$

$\boxed{15}$ $a = 2$ and $b = 6$ $\Rightarrow$ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{x^2}{4} + \frac{y^2}{36} = 1$.

$\boxed{16}$ $a = 4$ and $b = 3$ $\Rightarrow$ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

$\boxed{17}$ The center of the ellipse is $(-2, 1)$. $a = 5$ and $b = 2$ give us $\frac{(x+2)^2}{25} + \frac{(y-1)^2}{4} = 1$.

$\boxed{18}$ The center of the ellipse is $(1, -2)$. $a = 2$ and $b = 4$ give us $\frac{(x-1)^2}{4} + \frac{(y+2)^2}{16} = 1$.

$\boxed{19}$ $b^2 = 8^2 - 5^2 = 39$. An equation is $\frac{x^2}{64} + \frac{y^2}{39} = 1$.

$\boxed{20}$ $b^2 = 7^2 - 2^2 = 45$. An equation is $\frac{x^2}{45} + \frac{y^2}{49} = 1$.

$\boxed{21}$ If the length of the minor axis is 3, then $b = \frac{3}{2}$. An equation is $\frac{4x^2}{9} + \frac{y^2}{25} = 1$.

$\boxed{22}$ If the length of the minor axis is 2, then $b = 1$. $a^2 = 3^2 + 1^2 = 10$.

An equation is $\frac{x^2}{10} + \frac{y^2}{1} = 1$.

$\boxed{23}$ With the vertices at $(0, \pm 6)$, an equation of the ellipse is $\frac{x^2}{b^2} + \frac{y^2}{36} = 1$. Substituting $x = 3$ and $y = 2$ and solving for b^2 yields $\frac{9}{b^2} + \frac{4}{36} = 1$ $\Rightarrow$ $\frac{9}{b^2} = \frac{8}{9}$ $\Rightarrow$ $b^2 = \frac{81}{8}$.

An equation is $\frac{8x^2}{81} + \frac{y^2}{36} = 1$.

$\boxed{24}$ Substituting the x and y values for $(2, 3)$ and $(6, 1)$ into $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ yields the equations $\frac{4}{a^2} + \frac{9}{b^2} = 1$ $\{E_1\}$ and $\frac{36}{a^2} + \frac{1}{b^2} = 1$ $\{E_2\}$, respectively. Solving,

$E_2 - 9E_1$ $\Rightarrow$ $-\frac{80}{b^2} = -8$ $\Rightarrow$ $b^2 = 10$ and $E_1 - 9E_2$ $\Rightarrow$ $-\frac{320}{a^2} = -8$ $\Rightarrow$ $a^2 = 40$.

An equation is $\frac{x^2}{40} + \frac{y^2}{10} = 1$.

$\boxed{25}$ With vertices $V(0, \pm 4)$, an equation of the ellipse is $\frac{x^2}{b^2} + \frac{y^2}{16} = 1$. $e = \frac{c}{a} = \frac{3}{4}$ and

$a = 4$ $\Rightarrow$ $c = 3$. Thus, $b^2 = 16 - 9 = 7$. An equation is $\frac{x^2}{7} + \frac{y^2}{16} = 1$.

$\boxed{26}$ An equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. $(1, 3)$ on the ellipse $\Rightarrow$ $\frac{1}{a^2} + \frac{9}{b^2} = 1$ $\Rightarrow$ $b^2 = \frac{9a^2}{a^2 - 1}$. $e = \frac{c}{a} = \frac{1}{2}$ $\Rightarrow$ $c = \frac{1}{2}a$. $b^2 = a^2 - c^2 = a^2 - \frac{1}{4}a^2 = \frac{3}{4}a^2$.

Thus, $\frac{9a^2}{a^2 - 1} = \frac{3}{4}a^2$ $\Rightarrow$ $a^2 = 13$ and $b^2 = \frac{39}{4}$. An equation is $\frac{x^2}{13} + \frac{4y^2}{39} = 1$.

$\boxed{27}$ $\frac{x^2}{2^2} + \frac{y^2}{(\frac{1}{3})^2} = 1$ $\Rightarrow$ $\frac{x^2}{4} + \frac{y^2}{\frac{1}{9}} = 1$ $\Rightarrow$ $\frac{x^2}{4} + 9y^2 = 1$

$\boxed{28}$ $\frac{x^2}{(\frac{1}{2})^2} + \frac{y^2}{4^2} = 1$ $\Rightarrow$ $\frac{x^2}{\frac{1}{4}} + \frac{y^2}{16} = 1$ $\Rightarrow$ $4x^2 + \frac{y^2}{16} = 1$

[29] $\dfrac{x^2}{(\frac{1}{2}\cdot 8)^2}+\dfrac{y^2}{(\frac{1}{2}\cdot 5)^2}=1 \;\Rightarrow\; \dfrac{x^2}{16}+\dfrac{4y^2}{25}=1$

[30] $\dfrac{x^2}{(\frac{1}{2}\cdot 6)^2}+\dfrac{y^2}{(\frac{1}{2}\cdot 7)^2}=1 \;\Rightarrow\; \dfrac{x^2}{9}+\dfrac{4y^2}{49}=1$

[31] Substituting $x=6-2y$ into $x^2+4y^2=20$ yields $(6-2y)^2+4y^2=20 \;\Rightarrow$
$8y^2-24y+16=0 \;\Rightarrow\; 8(y-1)(y-2)=0 \;\Rightarrow\; y=1,\ 2;\ x=4,\ 2.$

The two points of intersection are (2, 2) and (4, 1).

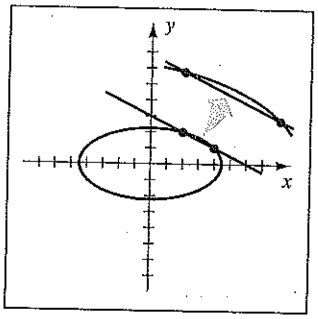

Figure 31

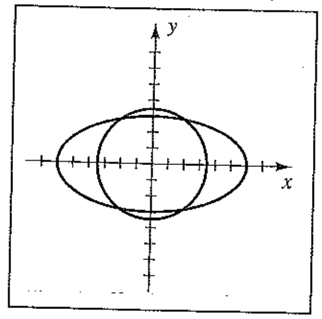

Figure 32

[32] $(x^2+4y^2=36)-(x^2+\ y^2\ =12) \;\Rightarrow\; 3y^2=24 \;\Rightarrow\; y^2=8;\ x^2=4.$

The four points of intersection are $(\pm 2,\ \pm 2\sqrt{2})$.

[33] $k=2a=10 \;\Rightarrow\; a=5.$ $F(3,\ 0)$ and $F'(-3,\ 0) \;\Rightarrow\; c=3.$

$b^2=a^2-c^2=25-9=16.$ An equation is $\dfrac{x^2}{25}+\dfrac{y^2}{16}=1.$

[34] $k=2a=26 \;\Rightarrow\; a=13.$ $F(12,\ 0)$ and $F'(-12,\ 0) \;\Rightarrow\; c=12.$

$b^2=a^2-c^2=169-144=25.$ An equation is $\dfrac{x^2}{169}+\dfrac{y^2}{25}=1.$

[35] $k=2a=34 \;\Rightarrow\; a=17.$ $F(0,\ 15)$ and $F'(0,\ -15) \;\Rightarrow\; c=15.$

$b^2=a^2-c^2=289-225=64.$ An equation is $\dfrac{x^2}{64}+\dfrac{y^2}{289}=1.$

[36] $k=2a=20 \;\Rightarrow\; a=10.$ $F(0,\ 8)$ and $F'(0,\ -8) \;\Rightarrow\; c=8.$

$b^2=a^2-c^2=100-64=36.$ An equation is $\dfrac{x^2}{36}+\dfrac{y^2}{100}=1.$

[37] $k=2a=7+3=10 \;\Rightarrow\; a=5.$ $F(4,\ 0)$ and $F'(-4,\ 0) \;\Rightarrow\; c=4.$

$b^2=a^2-c^2=25-16=9.$ An equation is $\dfrac{x^2}{25}+\dfrac{y^2}{9}=1.$

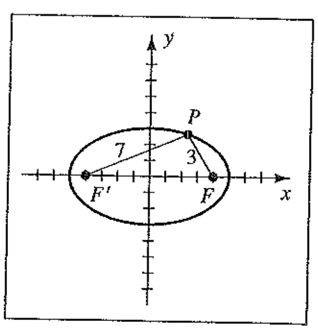

Figure 37

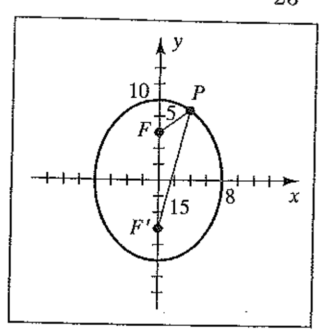

Figure 38

$\boxed{38}$ $k = 2a = 5 + 15 = 20 \Rightarrow a = 10$. $F(0, 6)$ and $F'(0, -6) \Rightarrow c = 6$.

$b^2 = a^2 - c^2 = 100 - 36 = 64$. An equation is $\frac{x^2}{64} + \frac{y^2}{100} = 1$. See *Figure 38.*

$\boxed{39}$ $y = 11\sqrt{1 - \frac{x^2}{49}} \Rightarrow \frac{y}{11} = \sqrt{1 - \frac{x^2}{49}} \Rightarrow \frac{x^2}{49} + \frac{y^2}{121} = 1$. Since y is nonnegative in the original equation, its graph is the upper half of the ellipse.

$\boxed{40}$ $y = -6\sqrt{1 - \frac{x^2}{25}} \Rightarrow \frac{y}{-6} = \sqrt{1 - \frac{x^2}{25}} \Rightarrow \frac{x^2}{25} + \frac{y^2}{36} = 1$; lower half

$\boxed{41}$ $x = -\frac{1}{3}\sqrt{9 - y^2} \Rightarrow -3x = \sqrt{9 - y^2} \Rightarrow 9x^2 = 9 - y^2 \Rightarrow x^2 + \frac{y^2}{9} = 1$; left half

$\boxed{42}$ $x = \frac{4}{5}\sqrt{25 - y^2} \Rightarrow \frac{5}{4}x = \sqrt{25 - y^2} \Rightarrow \frac{25}{16}x^2 = 25 - y^2 \Rightarrow \frac{x^2}{16} + \frac{y^2}{25} = 1$; right half

$\boxed{43}$ $x = 1 + 2\sqrt{1 - \frac{(y+2)^2}{9}} \Rightarrow \frac{x-1}{2} = \sqrt{1 - \frac{(y+2)^2}{9}} \Rightarrow \frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1$; right half

$\boxed{44}$ $x = -2 - 5\sqrt{1 - \frac{(y-1)^2}{16}} \Rightarrow \frac{x+2}{-5} = \sqrt{1 - \frac{(y-1)^2}{16}} \Rightarrow \frac{(x+2)^2}{25} + \frac{(y-1)^2}{16} = 1$; left half

$\boxed{45}$ $y = 2 - 7\sqrt{1 - \frac{(x+1)^2}{9}} \Rightarrow \frac{y-2}{-7} = \sqrt{1 - \frac{(x+1)^2}{9}} \Rightarrow \frac{(x+1)^2}{9} + \frac{(y-2)^2}{49} = 1$; lower half

$\boxed{46}$ $y = -1 + \sqrt{1 - \frac{(x-3)^2}{16}} \Rightarrow y + 1 = \sqrt{1 - \frac{(x-3)^2}{16}} \Rightarrow \frac{(x-3)^2}{16} + (y+1)^2 = 1$; upper half

$\boxed{47}$ Model this problem as an ellipse with $V(\pm 15, 0)$ and $M(0, \pm 10)$.

Substituting $x = 6$ into $\frac{x^2}{15^2} + \frac{y^2}{10^2} = 1$ yields $\frac{y^2}{100} = \frac{189}{225} \Rightarrow y^2 = 84$.

The desired height is $\sqrt{84} = 2\sqrt{21} \approx 9.165$ ft.

$\boxed{48}$ (a) This problem can be modeled as an ellipse with $V(\pm 100, 0)$ and passing through the point $(25, 30)$. Substituting $x = 25$ and $y = 30$ into $\frac{x^2}{100^2} + \frac{y^2}{b^2} = 1$ yields

$\frac{30^2}{b^2} = \frac{15}{16} \Rightarrow b^2 = 960$. An equation for the ellipse is $\frac{x^2}{10{,}000} + \frac{y^2}{960} = 1$.

An equation for the top half of the ellipse is $y = \sqrt{960\left(1 - \frac{x^2}{10{,}000}\right)}$.

(b) The height in the middle of the bridge is $\sqrt{960} = 8\sqrt{15} \approx 31$ ft.

$\boxed{49}$ $e = \frac{c}{a} = 0.017 \Rightarrow c = 0.017a = 0.017(93{,}000{,}000) = 1{,}581{,}000$. The maximum and minimum distances are $a + c = 94{,}581{,}000$ mi. and $a - c = 91{,}419{,}000$ mi.

$\boxed{50}$ $e = \frac{c}{a} \Rightarrow c = ae = (\frac{1}{2} \cdot 0.774)(0.206) = (0.387)(0.206) \approx 0.080$. As in Example 7, the maximum and minimum distances are $a + c \approx 0.387 + 0.080 = 0.467$ AU and $a - c \approx 0.387 - 0.080 = 0.307$ AU, respectively.

[51] (a) Let c denote the distance from the center of the hemi-ellipsoid to F. Hence,

$$(\tfrac{1}{2}k)^2 + c^2 = h^2 \Rightarrow c^2 = h^2 - \tfrac{1}{4}k^2 \Rightarrow c = \sqrt{h^2 - \tfrac{1}{4}k^2}.$$

$d = d(V, F) = h - c \Rightarrow d = h - \sqrt{h^2 - \frac{1}{4}k^2}$ and $d' = d(V, F') = h + c \Rightarrow$

$$d' = h + \sqrt{h^2 - \tfrac{1}{4}k^2}.$$

(b) From part (a), $d' = h + c \Rightarrow c = d' - h = 32 - 17 = 15$. $c = \sqrt{h^2 - \frac{1}{4}k^2} \Rightarrow$

$15 = \sqrt{17^2 - \frac{1}{4}k^2} \Rightarrow 225 = 289 - \frac{1}{4}k^2 \Rightarrow \frac{1}{4}k^2 = 64 \Rightarrow k^2 = 256 \Rightarrow$

$k = 16$ cm. $d = h - c = 17 - 15 = 2 \Rightarrow F$ should be located 2 cm from V.

[52] (a) From the previous exercise, $c = \sqrt{h^2 - \frac{1}{4}k^2} = \sqrt{15^2 - \frac{1}{4}(18)^2} = 12$.

The focus F should be located $d = h - c = 15 - 12 = 3$ cm from V.

(b) The kidney stone should be located $d' = h + c = 15 + 12 = 27$ cm from V.

[53] $c^2 = (\frac{1}{2} \cdot 50)^2 - 15^2 = 625 - 225 = 400 \Rightarrow c = 20$.

Their feet should be $25 - 20 = 5$ ft from the vertices.

[54] Since $a = p + c$, $b^2 = a^2 - c^2 = (p+c)^2 - c^2 = p^2 + 2pc = p(p + 2c)$.

Thus, the ellipse has the equation $\dfrac{[x - (p+c)]^2}{(p+c)^2} + \dfrac{y^2}{p(p+2c)} = 1 \Rightarrow$

$$y^2 = p(p+2c)\left[1 - \frac{(x-p-c)^2}{(p+c)^2}\right] = \frac{p(p+2c)(2xp + 2xc - x^2)}{(p+c)^2}.$$

Consider the expression to be a rational function of c with $4px$ as the coefficient of c^2 in the numerator and 1 as the coefficient of c^2 in the denominator.

Hence, as $c \to \infty$, $y^2 \to 4px$.

[55] First determine an equation of the ellipse for the orbit of Earth. $e = \frac{c}{a} \Rightarrow c = ae = 0.093 \times 149.6 = 13.9128$. $b^2 = a^2 - c^2 = 149.6^2 - 13.9128^2 \Rightarrow b \approx 148.95 \approx 149.0$.

An equation for the orbit of Earth is $\dfrac{x^2}{149.6^2} + \dfrac{y^2}{149.0^2} = 1$.

Graph $Y_1 = 149\sqrt{1 - (x^2/149.6^2)}$ and $Y_2 = -Y_1$.

The sun is at $(\pm 13.9128, 0)$. Plot the point $(13.9128, 0)$ for the sun.

[−300, 300, 100] by [−200, 200, 100]

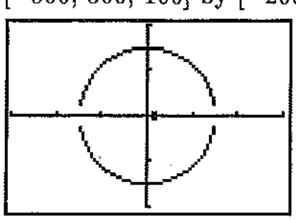

Figure 55

[−9000, 9000, 1000] by [−6000, 6000, 1000]

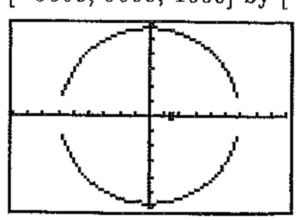

Figure 56

$\boxed{56}$ First determine an equation of the ellipse for the orbit of Pluto. $e = \frac{c}{a} \Rightarrow c = ae = 0.249 \times 5913 = 1472.337$. $b^2 = a^2 - c^2 = 5913^2 - 1472.337^2 \Rightarrow b \approx 5726.76 \approx 5727$.

An equation for the orbit of Pluto is $\dfrac{x^2}{5913^2} + \dfrac{y^2}{5727^2} = 1$.

Graph $Y_1 = 5727\sqrt{1 - (x^2/5913^2)}$ and $Y_2 = -Y_1$. See *Figure 56.*

The sun is at $(\pm 1472.337, 0)$. Plot the point $(1472.337, 0)$ for the sun.

$\boxed{57}$ $\dfrac{x^2}{2.9} + \dfrac{y^2}{2.1} = 1 \Rightarrow \dfrac{y^2}{2.1} = 1 - \dfrac{x^2}{2.9} \Rightarrow y^2 = 2.1\left(1 - \dfrac{x^2}{2.9}\right) \Rightarrow y = \pm\sqrt{2.1(1 - x^2/2.9)}$.

$\dfrac{x^2}{4.3} + \dfrac{(y-2.1)^2}{4.9} = 1 \Rightarrow \dfrac{(y-2.1)^2}{4.9} = 1 - \dfrac{x^2}{4.3} \Rightarrow (y-2.1)^2 = 4.9\left(1 - \dfrac{x^2}{4.3}\right) \Rightarrow$

$$y - 2.1 = \pm\sqrt{4.9(1 - x^2/4.3)} \Rightarrow y = 2.1 \pm \sqrt{4.9(1 - x^2/4.3)}.$$

From the graph, the points of intersection are approximately $(\pm 1.540, 0.618)$.

$[-6, 6]$ by $[-2, 6]$

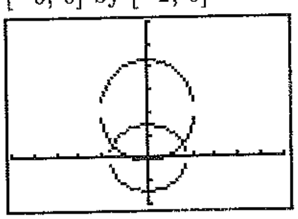

Figure 57

$[-6, 6]$ by $[-4, 4]$

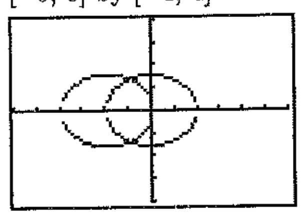

Figure 58

$\boxed{58}$ $\dfrac{x^2}{3.9} + \dfrac{y^2}{2.4} = 1 \Rightarrow \dfrac{y^2}{2.4} = 1 - \dfrac{x^2}{3.9} \Rightarrow y^2 = 2.4\left(1 - \dfrac{x^2}{3.9}\right) \Rightarrow y = \pm\sqrt{2.4(1 - x^2/3.9)}$.

$\dfrac{(x+1.9)^2}{4.1} + \dfrac{y^2}{2.5} = 1 \Rightarrow \dfrac{y^2}{2.5} = 1 - \dfrac{(x+1.9)^2}{4.1} \Rightarrow$

$$y^2 = 2.5[1 - (x+1.9)^2/4.1] \Rightarrow y = \pm\sqrt{2.5[1 - (x+1.9)^2/4.1]}.$$

From the graph, the points of intersection are approximately $(-0.905, \pm 1.377)$.

$\boxed{59}$ $\dfrac{(x+0.1)^2}{1.7} + \dfrac{y^2}{0.9} = 1 \Rightarrow y = \pm\sqrt{0.9[1 - (x+0.1)^2/1.7]}$.

$\dfrac{x^2}{0.9} + \dfrac{(y-0.25)^2}{1.8} = 1 \Rightarrow y = 0.25 \pm \sqrt{1.8(1 - x^2/0.9)}$.

From the graph, the points of intersection are approximately

$(-0.88, 0.76)$, $(-0.48, -0.91)$, $(0.58, -0.81)$, and $(0.92, 0.59)$.

$[-3, 3]$ by $[-2, 2]$

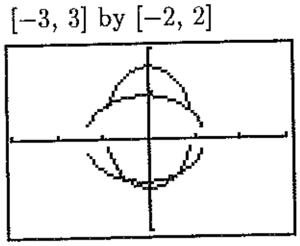

Figure 59

$[-4.5, 4.5]$ by $[-3, 3]$

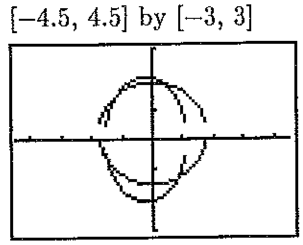

Figure 60

$\boxed{60}$ $\dfrac{x^2}{3.1}+\dfrac{(y-0.2)^2}{2.8}=1 \;\Rightarrow\; y=0.2\pm\sqrt{2.8(1-x^2/3.1)}$.

$\dfrac{(x+0.23)^2}{1.8}+\dfrac{y^2}{4.2}=1 \;\Rightarrow\; y=\pm\sqrt{4.2[1-(x+0.23)^2/1.8]}$.

From the graph, the points of intersection are approximately

$(-1.49, -0.68)$, $(-1.19, 1.44)$, $(0.36, 1.84)$, and $(0.82, -1.28)$. See *Figure 60.*

11.3 Exercises

Note: Let C, V, F, and W denote the center, the vertices, the foci, and the endpoints of the conjugate axis, respectively.

$\boxed{1}$ $\dfrac{x^2}{9}-\dfrac{y^2}{4}=1$ • $c^2=9+4 \;\Rightarrow\; c=\pm\sqrt{13}$;

$V(\pm 3, 0)$; $F(\pm\sqrt{13}, 0)$; $W(0, \pm 2)$; $y=\pm\frac{2}{3}x$

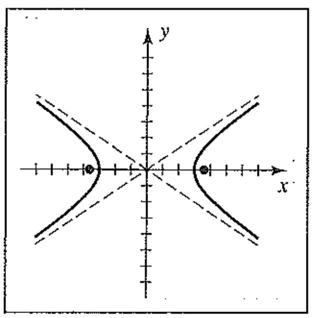

Figure 1

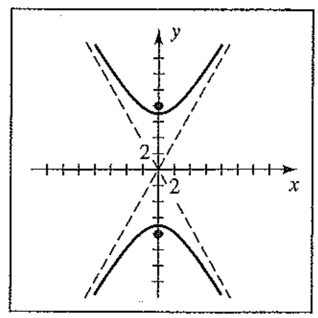

Figure 2

$\boxed{2}$ $\dfrac{y^2}{49}-\dfrac{x^2}{16}=1$ • $c^2=49+16 \;\Rightarrow\; c=\pm\sqrt{65}$;

$V(0, \pm 7)$; $F(0, \pm\sqrt{65})$; $W(\pm 4, 0)$; $y=\pm\frac{7}{4}x$

$\boxed{3}$ $\dfrac{y^2}{9}-\dfrac{x^2}{4}=1$ • $c^2=9+4 \;\Rightarrow\; c=\pm\sqrt{13}$;

$V(0, \pm 3)$; $F(0, \pm\sqrt{13})$; $W(\pm 2, 0)$; $y=\pm\frac{3}{2}x$

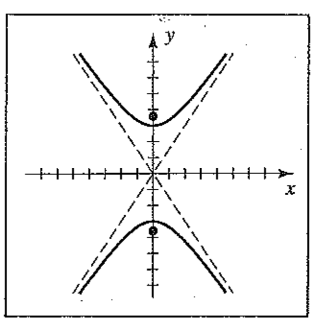

Figure 3

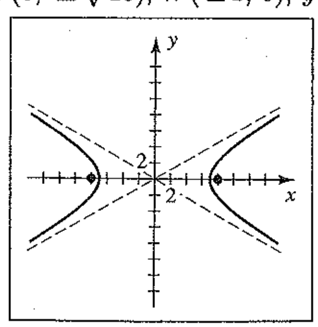

Figure 4

$\boxed{4}$ $\dfrac{x^2}{49}-\dfrac{y^2}{16}=1$ • $c^2=49+16 \;\Rightarrow\; c=\pm\sqrt{65}$;

$V(\pm 7, 0)$; $F(\pm\sqrt{65}, 0)$; $W(0, \pm 4)$; $y=\pm\frac{4}{7}x$

$\boxed{5}$ $x^2 - \frac{y^2}{24} = 1$ • $c^2 = 1 + 24 \Rightarrow c = \pm 5;$

$V(\pm 1, 0)$; $F(\pm 5, 0)$; $W(0, \pm\sqrt{24})$; $y = \pm\sqrt{24}x$

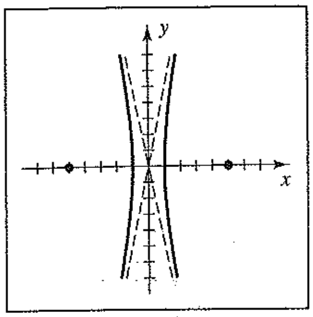

Figure 5

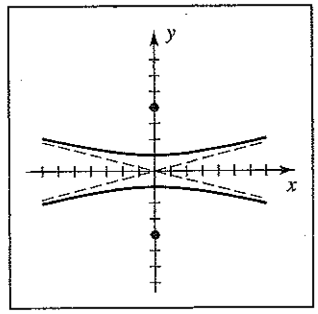

Figure 6

$\boxed{6}$ $y^2 - \frac{x^2}{15} = 1$ • $c^2 = 1 + 15 \Rightarrow c = \pm 4;$

$V(0, \pm 1)$; $F(0, \pm 4)$; $W(\pm\sqrt{15}, 0)$; $y = \pm(1/\sqrt{15})x$

$\boxed{7}$ $y^2 - 4x^2 = 16 \Rightarrow \frac{y^2}{16} - \frac{x^2}{4} = 1$; $c^2 = 16 + 4 \Rightarrow c = \pm 2\sqrt{5};$

$V(0, \pm 4)$; $F(0, \pm 2\sqrt{5})$; $W(\pm 2, 0)$; $y = \pm 2x$

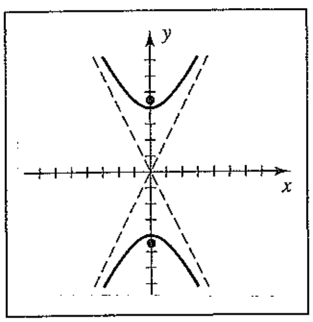

Figure 7

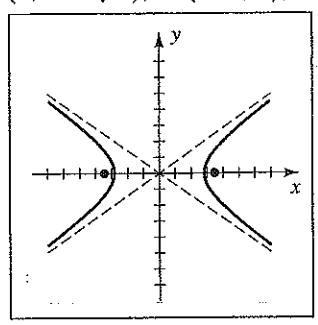

Figure 8

$\boxed{8}$ $x^2 - 2y^2 = 8 \Rightarrow \frac{x^2}{8} - \frac{y^2}{4} = 1$; $c^2 = 8 + 4 \Rightarrow c = \pm 2\sqrt{3};$

$V(\pm 2\sqrt{2}, 0)$; $F(\pm 2\sqrt{3}, 0)$; $W(0, \pm 2)$; $y = \pm\frac{1}{2}\sqrt{2}\,x$

$\boxed{9}$ $16x^2 - 36y^2 = 1 \Rightarrow \frac{x^2}{\frac{1}{16}} - \frac{y^2}{\frac{1}{36}} = 1$; $c^2 = \frac{1}{16} + \frac{1}{36} \Rightarrow c = \pm\frac{1}{12}\sqrt{13};$

$V(\pm\frac{1}{4}, 0)$; $F(\pm\frac{1}{12}\sqrt{13}, 0)$; $W(0, \pm\frac{1}{6})$; $y = \pm\frac{2}{3}x$

Note that the branches of the hyperbola almost coincide with the asymptotes.

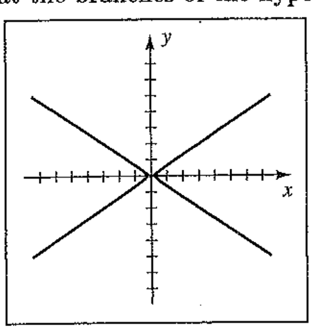

Figure 9

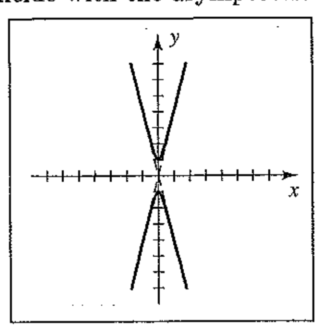

Figure 10

$\boxed{10}$ $y^2 - 16x^2 = 1 \;\Rightarrow\; \dfrac{y^2}{1} - \dfrac{x^2}{\frac{1}{16}} = 1$; $c^2 = 1 + \frac{1}{16} \;\Rightarrow\; c = \pm\frac{1}{4}\sqrt{17}$;

$V(0,\ \pm 1)$; $F(0,\ \pm\frac{1}{4}\sqrt{17})$; $W(\pm\frac{1}{4},\ 0)$; $y = \pm 4x$ (See *Figure 10.*)

$\boxed{11}$ $\dfrac{(y+2)^2}{9} - \dfrac{(x+2)^2}{4} = 1$; $c^2 = 9 + 4 \;\Rightarrow\; c = \pm\sqrt{13}$; $C(-2,\ -2)$;

$V(-2,\ -2 \pm 3)$; $F(-2,\ -2 \pm \sqrt{13})$; $W(-2 \pm 2,\ -2)$; $(y+2) = \pm\frac{3}{2}(x+2)$

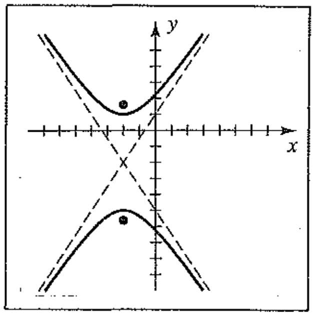

Figure 11

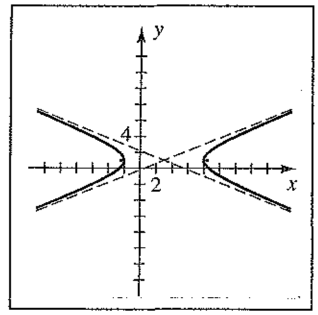

Figure 12

$\boxed{12}$ $\dfrac{(x-3)^2}{25} - \dfrac{(y-1)^2}{4} = 1$; $c^2 = 25 + 4 \;\Rightarrow\; c = \pm\sqrt{29}$;

$C(3,\ 1)$; $V(3 \pm 5,\ 1)$; $F(3 \pm \sqrt{29},\ 1)$; $W(3,\ 1 \pm 2)$; $(y-1) = \pm\frac{2}{5}(x-3)$

$\boxed{13}$ $144x^2 - 25y^2 + 864x - 100y - 2404 = 0 \;\Rightarrow$

$144(x^2 + 6x + \underline{\ 9\ }) - 25(y^2 + 4y + \underline{\ 4\ }) = 2404 + \underline{\ 1296\ } - \underline{\ 100\ } \;\Rightarrow$

$144(x+3)^2 - 25(y+2)^2 = 3600 \;\Rightarrow\; \dfrac{(x+3)^2}{25} - \dfrac{(y+2)^2}{144} = 1$;

$c^2 = 25 + 144 \;\Rightarrow\; c = \pm 13$;

$C(-3,\ -2)$; $V(-3 \pm 5,\ -2)$; $F(-3 \pm 13,\ -2)$; $W(-3,\ -2 \pm 12)$; $(y+2) = \pm\frac{12}{5}(x+3)$

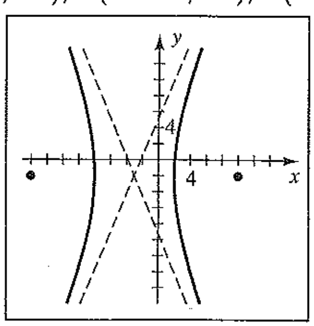

Figure 13

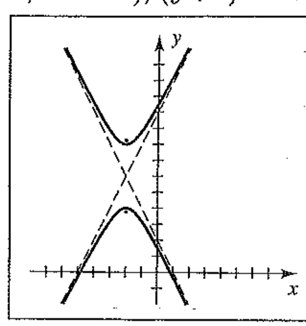

Figure 14

$\boxed{14}$ $y^2 - 4x^2 - 12y - 16x + 16 = 0 \;\Rightarrow$

$(y^2 - 12y + \underline{\ 36\ }) - 4(x^2 + 4x + \underline{\ 4\ }) = -16 + \underline{\ 36\ } - \underline{\ 16\ } \;\Rightarrow$

$(y-6)^2 - 4(x+2)^2 = 4 \;\Rightarrow\; \dfrac{(y-6)^2}{4} - \dfrac{(x+2)^2}{1} = 1$; $c^2 = 4 + 1 \;\Rightarrow\; c = \pm\sqrt{5}$;

$C(-2,\ 6)$; $V(-2,\ 6 \pm 2)$; $F(-2,\ 6 \pm \sqrt{5})$; $W(-2 \pm 1,\ 6)$; $(y-6) = \pm 2(x+2)$

$\boxed{15}$ $4y^2 - x^2 + 40y - 4x + 60 = 0 \Rightarrow$

$4(y^2 + 10y + \underline{25}) - 1(x^2 + 4x + \underline{4}) = -60 + \underline{100} - \underline{4} \Rightarrow$

$4(y+5)^2 - (x+2)^2 = 36 \Rightarrow \dfrac{(y+5)^2}{9} - \dfrac{(x+2)^2}{36} = 1;$

$c^2 = 9 + 36 \Rightarrow c = \pm 3\sqrt{5};$ $C(-2, -5);$

$V(-2, -5 \pm 3);\ F(-2, -5 \pm 3\sqrt{5});\ W(-2 \pm 6, -5);\ (y+5) = \pm\frac{1}{2}(x+2)$

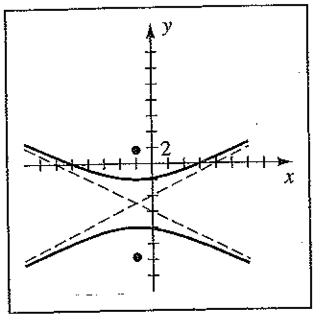

Figure 15

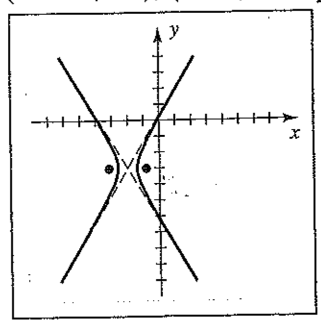

Figure 16

$\boxed{16}$ $25x^2 - 9y^2 + 100x - 54y + 10 = 0 \Rightarrow$

$25(x^2 + 4x + \underline{4}) - 9(y^2 + 6y + \underline{9}) = -10 + \underline{100} - \underline{81} \Rightarrow$

$25(x+2)^2 - 9(y+3)^2 = 9 \Rightarrow \dfrac{(x+2)^2}{\frac{9}{25}} - \dfrac{(y+3)^2}{1} = 1;$

$c^2 = \frac{9}{25} + 1 \Rightarrow c = \pm\frac{1}{5}\sqrt{34};$ $C(-2, -3);$

$V(-2 \pm \frac{3}{5}, -3);\ F(-2 \pm \frac{1}{5}\sqrt{34}, -3);\ W(-2, -3 \pm 1);\ (y+3) = \pm\frac{5}{3}(x+2)$

$\boxed{17}$ $a = 3$ and $c = 5 \Rightarrow b^2 = c^2 - a^2 = 16.$ $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is then $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1.$

$\boxed{18}$ $a = 4$ and $c = 6 \Rightarrow b^2 = c^2 - a^2 = 20.$ $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$ is then $\dfrac{y^2}{16} - \dfrac{x^2}{20} = 1.$

$\boxed{19}$ The center of the hyperbola is $(-2, -3)$.

$a = 1$ and $c = 2 \Rightarrow b^2 = 2^2 - 1^2 = 3$ and an equation is $(y+3)^2 - \dfrac{(x+2)^2}{3} = 1.$

$\boxed{20}$ The center of the hyperbola is $(1, 2)$.

$a = 1$ and $c = 3 \Rightarrow b^2 = 3^2 - 1^2 = 8$ and an equation is $(x-1)^2 - \dfrac{(y-2)^2}{8} = 1.$

$\boxed{21}$ $F(0, \pm 4)$ and $V(0, \pm 1) \Rightarrow W(\pm\sqrt{15}, 0)$. An equation is $\dfrac{y^2}{1} - \dfrac{x^2}{15} = 1.$

$\boxed{22}$ $F(\pm 8, 0)$ and $V(\pm 5, 0) \Rightarrow W(0, \pm\sqrt{39})$. An equation is $\dfrac{x^2}{25} - \dfrac{y^2}{39} = 1.$

$\boxed{23}$ $F(\pm 5, 0)$ and $V(\pm 3, 0) \Rightarrow W(0, \pm 4)$. An equation is $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1.$

$\boxed{24}$ $F(0, \pm 3)$ and $V(0, \pm 2) \Rightarrow W(\pm\sqrt{5}, 0)$. An equation is $\dfrac{y^2}{4} - \dfrac{x^2}{5} = 1.$

[25] Conjugate axis of length 4 and $F(0,\ \pm 5)$ $\Rightarrow$ $W(\pm 2,\ 0)$ and $V(0,\ \pm\sqrt{21})$.

An equation is $\frac{y^2}{21} - \frac{x^2}{4} = 1$.

[26] An equation of a hyperbola with vertices at $(\pm 4,\ 0)$ is $\frac{x^2}{16} - \frac{y^2}{b^2} = 1$. Substituting $x = 8$ and $y = 2$ yields $4 - \frac{4}{b^2} = 1$ $\Rightarrow$ $b^2 = \frac{4}{3}$. An equation is $\frac{x^2}{16} - \frac{3y^2}{4} = 1$.

[27] Asymptote equations of $y = \pm 2x$ and $V(\pm 3,\ 0)$ $\Rightarrow$ $W(0,\ \pm 6)$.

An equation is $\frac{x^2}{9} - \frac{y^2}{36} = 1$.

[28] Let the y value of V equal a and the x value of W equal b.

Now $a = \frac{1}{3}b$ {from the asymptote equation} and $a^2 + b^2 = 10^2$ {from the foci} $\Rightarrow$ $(\frac{1}{3}b)^2 + b^2 = 10^2$ $\Rightarrow$ $\frac{10}{9}b^2 = 100$ $\Rightarrow$ $b^2 = 90$ and $a^2 = 10$.

An equation is $\frac{y^2}{10} - \frac{x^2}{90} = 1$.

[29] $a = 5$, $b = 2(5) = 10$. $\frac{x^2}{5^2} - \frac{y^2}{10^2} = 1$ $\Rightarrow$ $\frac{x^2}{25} - \frac{y^2}{100} = 1$.

[30] $a = 2$, $2 = \frac{1}{4}(b)$ $\Rightarrow$ $b = 8$. $\frac{y^2}{2^2} - \frac{x^2}{8^2} = 1$ $\Rightarrow$ $\frac{y^2}{4} - \frac{x^2}{64} = 1$.

[31] $\frac{y^2}{(\frac{1}{2}\cdot 10)^2} - \frac{x^2}{(\frac{1}{2}\cdot 14)^2} = 1$ $\Rightarrow$ $\frac{y^2}{25} - \frac{x^2}{49} = 1$.

[32] $\frac{x^2}{(\frac{1}{2}\cdot 6)^2} - \frac{y^2}{(\frac{1}{2}\cdot 2)^2} = 1$ $\Rightarrow$ $\frac{x^2}{9} - \frac{y^2}{1} = 1$.

[33] $\frac{1}{3}(x + 2) = y^2$ • We have a first degree x-term and a second degree y-term, so this is a parabola with a horizontal axis.

[34] $y^2 = \frac{14}{3} - x^2$ $\Leftrightarrow$ $x^2 + y^2 = \frac{14}{3}$, a circle with center at the origin and radius $\sqrt{\frac{14}{3}}$.

[35] $x^2 + 6x - y^2 = 7$ • The coefficient of x^2 is positive and the coefficient of y^2 is negative, so this is a hyperbola.

[36] $x^2 + 4x + 4y^2 - 24y = -36$ • The coefficients of x^2 and y^2 are positive, so this is an ellipse.

[37] $-x^2 = y^2 - 25$ $\Leftrightarrow$ $x^2 + y^2 = 25$, a circle with center at the origin and radius 5.

[38] $x = 2x^2 - y + 4$ • We have a first degree y-term and a second degree x-term, so this is a parabola with a vertical axis.

[39] $4x^2 - 16x + 9y^2 + 36y = -16$ • The coefficients of x^2 and y^2 are positive, so this is an ellipse.

[40] $x + 4 = y^2 + y$ • We have a first degree x-term and a second degree y-term, so this is a parabola with a horizontal axis.

[41] $x^2 + 3x = 3y - 6$ • We have a first degree y-term and a second degree x-term, so this is a parabola with a vertical axis.

[42] $9x^2 - y^2 = 10 - 2y$ • The coefficient of x^2 is positive and the coefficient of y^2 is negative, so this is a hyperbola.

[43] Substituting $y = x + 4$ into $y^2 - 4x^2 = 16$ yields $(x+4)^2 - 4x^2 = 16 \Rightarrow$
$3x^2 - 8x = 0 \Rightarrow x(3x - 8) = 0 \Rightarrow x = 0, \frac{8}{3}$ and $y = 4, \frac{20}{3}$.

The two points of intersection are $(0, 4)$ and $(\frac{8}{3}, \frac{20}{3})$.

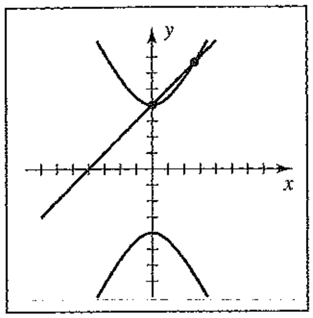

Figure 43

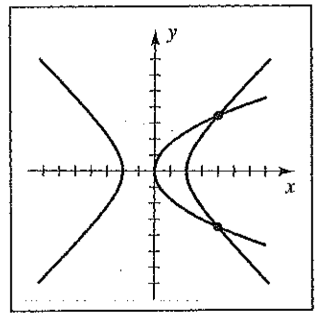

Figure 44

[44] Adding the two equations yields $x^2 - 3x = 4 \Rightarrow x^2 - 3x - 4 = 0 \Rightarrow$
$(x-4)(x+1) = 0 \Rightarrow x = 4, -1$. Substituting these values in the second equation, we find that for $x = 4$, $y = \pm 2\sqrt{3}$, and for $x = -1$, there are no real solutions for y.

The two points of intersection are $(4, \pm 2\sqrt{3})$.

[45] $k = 2a = 24 \Rightarrow a = 12$. $F(13, 0)$ and $F'(-13, 0) \Rightarrow c = 13$.
$b^2 = c^2 - a^2 = 169 - 144 = 25$. An equation is $\frac{x^2}{144} - \frac{y^2}{25} = 1$.

[46] $k = 2a = 8 \Rightarrow a = 4$. $F(5, 0)$ and $F'(-5, 0) \Rightarrow c = 5$.
$b^2 = c^2 - a^2 = 25 - 16 = 9$. An equation is $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

[47] $k = 2a = 16 \Rightarrow a = 8$. $F(0, 10)$ and $F'(0, -10) \Rightarrow c = 10$.
$b^2 = c^2 - a^2 = 100 - 64 = 36$. An equation is $\frac{y^2}{64} - \frac{x^2}{36} = 1$.

[48] $k = 2a = 30 \Rightarrow a = 15$. $F(0, 17)$ and $F'(0, -17) \Rightarrow c = 17$.
$b^2 = c^2 - a^2 = 289 - 225 = 64$. An equation is $\frac{y^2}{225} - \frac{x^2}{64} = 1$.

[49] $k = 2a = 11 - 3 = 8 \Rightarrow a = 4$. $F(0, 5)$ and $F'(0, -5) \Rightarrow c = 5$.
$b^2 = c^2 - a^2 = 25 - 16 = 9$. An equation is $\frac{y^2}{16} - \frac{x^2}{9} = 1$.

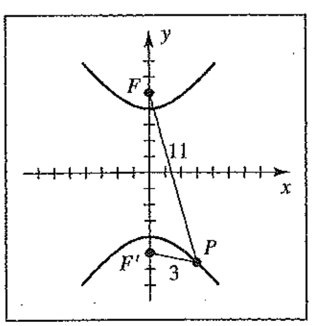

Figure 49

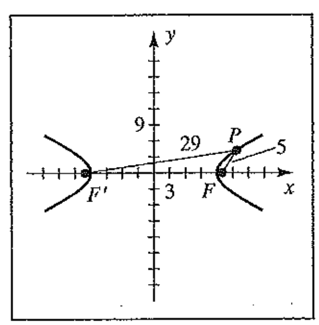

Figure 50

$\boxed{50}$ $k = 2a = 29 - 5 = 24 \Rightarrow a = 12$. $F(13, 0)$ and $F'(-13, 0) \Rightarrow c = 13$.

$b^2 = c^2 - a^2 = 169 - 144 = 25$. An equation is $\frac{x^2}{144} - \frac{y^2}{25} = 1$. See *Figure 50.*

$\boxed{51}$ $x = \frac{5}{4}\sqrt{y^2 + 16} \Rightarrow \frac{4}{5}x = \sqrt{y^2 + 16} \Rightarrow \frac{16}{25}x^2 = y^2 + 16 \Rightarrow \frac{16}{25}x^2 - y^2 = 16 \Rightarrow$

$\frac{x^2}{25} - \frac{y^2}{16} = 1$. Since x is positive in the original equation, its graph is the right branch of the hyperbola.

$\boxed{52}$ $x = -\frac{5}{4}\sqrt{y^2 + 16} \Rightarrow -\frac{4}{5}x = \sqrt{y^2 + 16} \Rightarrow \frac{16}{25}x^2 = y^2 + 16 \Rightarrow \frac{x^2}{25} - \frac{y^2}{16} = 1$;
left branch

$\boxed{53}$ $y = \frac{3}{7}\sqrt{x^2 + 49} \Rightarrow \frac{7}{3}y = \sqrt{x^2 + 49} \Rightarrow \frac{49}{9}y^2 = x^2 + 49 \Rightarrow \frac{y^2}{9} - \frac{x^2}{49} = 1$;
upper branch

$\boxed{54}$ $y = -\frac{3}{7}\sqrt{x^2 + 49} \Rightarrow -\frac{7}{3}y = \sqrt{x^2 + 49} \Rightarrow \frac{49}{9}y^2 = x^2 + 49 \Rightarrow \frac{y^2}{9} - \frac{x^2}{49} = 1$;
lower branch

$\boxed{55}$ $y = -\frac{9}{4}\sqrt{x^2 - 16} \Rightarrow -\frac{4}{9}y = \sqrt{x^2 - 16} \Rightarrow \frac{16}{81}y^2 = x^2 - 16 \Rightarrow \frac{x^2}{16} - \frac{y^2}{81} = 1$;
lower halves of the branches

$\boxed{56}$ $y = \frac{9}{4}\sqrt{x^2 - 16} \Rightarrow \frac{4}{9}y = \sqrt{x^2 - 16} \Rightarrow \frac{16}{81}y^2 = x^2 - 16 \Rightarrow \frac{x^2}{16} - \frac{y^2}{81} = 1$;
upper halves of the branches

$\boxed{57}$ $x = -\frac{2}{3}\sqrt{y^2 - 36} \Rightarrow -\frac{3}{2}x = \sqrt{y^2 - 36} \Rightarrow \frac{9}{4}x^2 = y^2 - 36 \Rightarrow \frac{y^2}{36} - \frac{x^2}{16} = 1$;
left halves of the branches

$\boxed{58}$ $x = \frac{2}{3}\sqrt{y^2 - 36} \Rightarrow \frac{3}{2}x = \sqrt{y^2 - 36} \Rightarrow \frac{9}{4}x^2 = y^2 - 36 \Rightarrow \frac{y^2}{36} - \frac{x^2}{16} = 1$;
right halves of the branches

$\boxed{59}$ Their equations are $\frac{x^2}{25} - \frac{y^2}{9} = 1$ and $\frac{x^2}{25} - \frac{y^2}{9} = -1$,

or, equivalently, $\frac{y^2}{9} - \frac{x^2}{25} = 1$.

Conjugate hyperbolas have the same asymptotes and exchange transverse and conjugate axes.

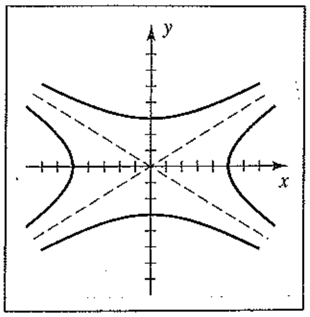

Figure 59

$\boxed{60}$ The center is $C(h, k)$ with $W(h, k \pm b)$. An equation is $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$.

[61] The path is a hyperbola with $V(\pm 3, 0)$ and $W(0, \pm\frac{3}{2})$.

An equation is $\dfrac{x^2}{(3)^2} - \dfrac{y^2}{(\frac{3}{2})^2} = 1$ or equivalently, $x^2 - 4y^2 = 9$.

If only the right branch is considered, then $x = \sqrt{9 + 4y^2}$ is an equation of the path.

[62] Let A and P be the points $(3, 0)$ and (x, y), respectively.

$S = [d(A, P)]^2 = (x-3)^2 + (y-0)^2 = x^2 - 6x + 9 + y^2 =$

$x^2 - 6x + 9 + (\frac{1}{2}x^2 + 4)$ {from $2y^2 - x^2 = 8$} $= \frac{3}{2}x^2 - 6x + 13$.

Since S is a quadratic function, its minimum occurs at $x = -\dfrac{b}{2a} = -\dfrac{-6}{2(\frac{3}{2})} = 2$.

If $x = 2$, $S = 7$, and the plane comes within $\sqrt{7}$ miles of A.

[63] Set up a coordinate system like Example 6. Then, $d_1 - d_2 = 2a = 160 \Rightarrow a = 80$.

$b^2 = c^2 - a^2 = 100^2 - 80^2 \Rightarrow b = 60$. The equation of the hyperbola with focus A, passing through the coordinates of the ship at $P(x, y)$ is $\dfrac{x^2}{80^2} - \dfrac{y^2}{60^2} = 1$. Now,

$y = 100 \Rightarrow x = \frac{80}{3}\sqrt{34}$. The ship's coordinates are $(\frac{80}{3}\sqrt{34}, 100) \approx (155.5, 100)$.

[64] By the reflective property of parabolic mirrors (see page 585) parallel rays striking the parabolic mirror will be reflected toward F_1. By the reflective property of hyperbolic mirrors (see page 612), these rays will be reflected toward the exterior focus of the hyperbolic mirror, which is located below the parabolic mirror in the figure.

[65] $\dfrac{(y-0.1)^2}{1.6} - \dfrac{(x+0.2)^2}{0.5} = 1 \Rightarrow 0.5(y-0.1)^2 - 1.6(x+0.2)^2 = 0.8 \Rightarrow$

$y = 0.1 \pm \sqrt{1.6 + 3.2(x+0.2)^2}$.

$\dfrac{(y-0.5)^2}{2.7} - \dfrac{(x-0.1)^2}{5.3} = 1 \Rightarrow 5.3(y-0.5)^2 - 2.7(x-0.1)^2 = 14.31 \Rightarrow$

$y = 0.5 \pm \sqrt{\frac{1}{5.3}[14.31 + 2.7(x-0.1)^2]}$.

From the graph,

the point of intersection in the first quadrant is approximately $(0.741, 2.206)$.

[−15, 15] by [−10, 10]

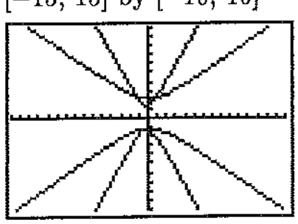

Figure 65

[−4.5, 4.5] by [−3, 3]

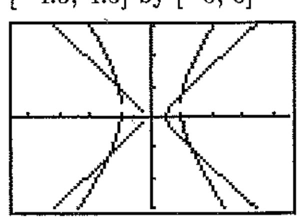

Figure 66

$\boxed{66}$ $\dfrac{(x-0.1)^2}{0.12}-\dfrac{y^2}{0.1}=1 \;\Rightarrow\; 0.1(x-0.1)^2-0.12y^2=0.012 \;\Rightarrow$

$$y=\pm\sqrt{\frac{1}{0.12}[0.1(x-0.1)^2-0.012]}.$$

$\dfrac{x^2}{0.9}-\dfrac{(y-0.3)^2}{2.1}=1 \;\Rightarrow\; 2.1x^2-0.9(y-0.3)^2=1.89 \;\Rightarrow$

$$y=0.3\pm\sqrt{\frac{1}{0.9}[2.1x^2-1.89]}.$$

From the graph of *Figure 66*,

the point of intersection in the first quadrant is approximately (0.994, 0.752).

$\boxed{67}$ $\dfrac{(x-0.3)^2}{1.3}-\dfrac{y^2}{2.7}=1 \;\Rightarrow\; y=\pm\sqrt{2.7[-1+(x-0.3)^2/1.3]}.$

$\dfrac{y^2}{2.8}-\dfrac{(x-0.2)^2}{1.2}=1 \;\Rightarrow\; y=\pm\sqrt{2.8[1+(x-0.2)^2/1.2]}.$

The two graphs nearly intersect in the second and fourth quadrants,

but there are no points of intersection.

[−15, 15, 2] by [−10, 10, 2]

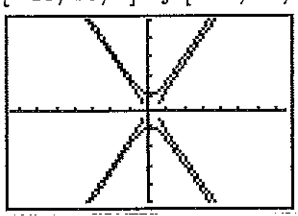

Figure 67

[−25, 25, 5] by [−25, 25, 5]

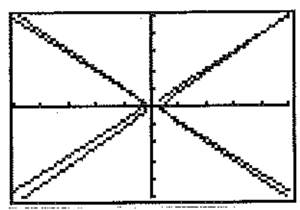

Figure 68

$\boxed{68}$ $\dfrac{(x+0.2)^2}{1.75}-\dfrac{(y-0.5)^2}{1.6}=1 \;\Rightarrow\; y=0.5\pm\sqrt{1.6[-1+(x+0.2)^2/1.75]}.$

$\dfrac{(x-0.6)^2}{2.2}-\dfrac{(y+0.4)^2}{2.35}=1 \;\Rightarrow\; y=-0.4\pm\sqrt{2.35[-1+(x-0.6)^2/2.2]}.$

From the graph, we can see that there are 2 points of intersection.

One is in the first quadrant, near the point (23, 23),

and the other is in the fourth quadrant, near the point (3, −2).

$\boxed{69}$ (a) The comet's path is hyperbolic with $a^2=26\times10^{14}$ and $b^2=18\times10^{14}$.

$c^2=a^2+b^2=26\times10^{14}+18\times10^{14}=44\times10^{14} \;\Rightarrow\; c\approx 6.63\times10^7$.

The coordinates of the sun are approximately $(6.63\times10^7,\ 0)$.

(b) The minimum distance between the comet and the sun will be

$$c-a=\sqrt{44\times10^{14}}-\sqrt{26\times10^{14}}=1.53\times10^7 \text{ mi}.$$

Since r must be in meters, 1.53×10^7 mi $\times$ 1610 m/mi $\approx 2.47\times10^{10}$ m. At this distance, v must be greater than $\sqrt{\dfrac{2k}{r}}\approx\sqrt{\dfrac{2(1.325\times10^{20})}{2.47\times10^{10}}}\approx 103{,}600$ m/sec.

11.4 Exercises

$\boxed{1}$ $x = t - 2 \;\Rightarrow\; t = x + 2.$ $y = 2t + 3 = 2(x + 2) + 3 = 2x + 7.$

As t varies from 0 to 5, (x, y) varies from $(-2, 3)$ to $(3, 13)$.

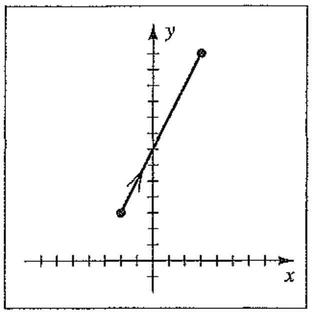

Figure 1

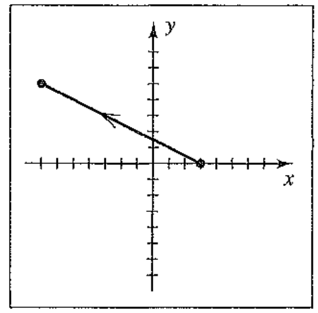

Figure 2

$\boxed{2}$ $y = 1 + t \;\Rightarrow\; t = y - 1.$ $x = 1 - 2t = 1 - 2(y - 1) = -2y + 3.$

As t varies from -1 to 4, (x, y) varies from $(3, 0)$ to $(-7, 5)$.

$\boxed{3}$ $x = t^2 + 1 \;\Rightarrow\; t^2 = x - 1.$ $y = t^2 - 1 = x - 2.$ As t varies from -2 to 2,

(x, y) varies from $(5, 3)$ to $(1, -1)$ $\{$when $t = 0\}$ and back to $(5, 3)$.

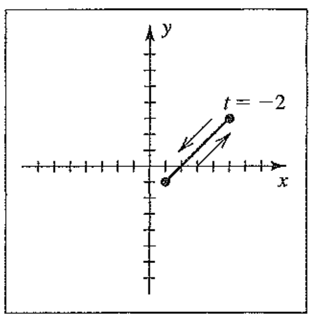

Figure 3

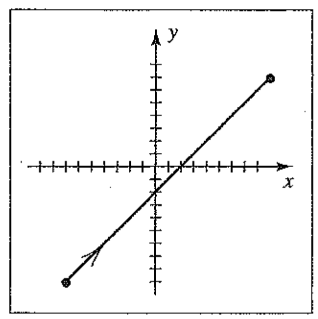

Figure 4

$\boxed{4}$ $x = t^3 + 1 \;\Rightarrow\; t^3 = x - 1.$ $y = t^3 - 1 = x - 2.$

As t varies from -2 to 2, (x, y) varies from $(-7, -9)$ to $(9, 7)$.

$\boxed{5}$ $y = 2t + 3 \;\Rightarrow\; t = \frac{1}{2}(y - 3).$

$x = 4\left[\frac{1}{2}(y - 3)\right]^2 - 5 \;\Rightarrow\; (y - 3)^2 = x + 5.$

This is a parabola with vertex at $(-5, 3)$. Since t takes on all real values, so does y, and the curve C is the entire parabola.

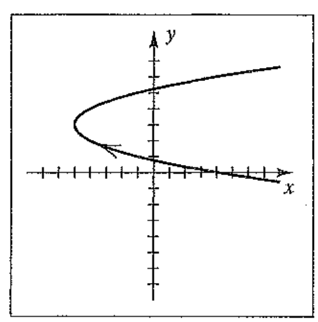

Figure 5

$\boxed{6}$ $x = t^3 \Rightarrow t = \sqrt[3]{x}$. $y = t^2 = x^{2/3}$.

x takes on all real values.

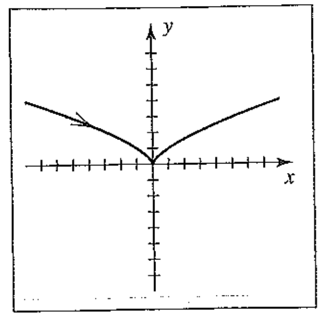

Figure 6

$\boxed{7}$ $y = e^{-2t} = (e^t)^{-2} = x^{-2} = 1/x^2$.

As t varies from $-\infty$ to ∞, x varies from 0 to ∞, excluding 0.

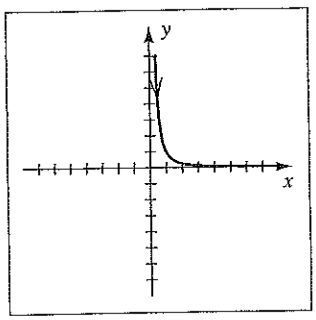

Figure 7

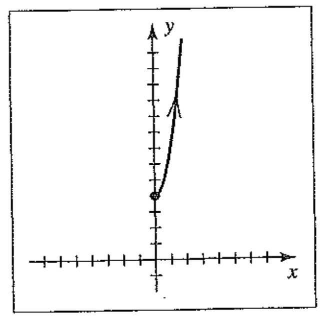

Figure 8

$\boxed{8}$ $x = \sqrt{t} \Rightarrow t = x^2$. $y = 3t + 4 = 3x^2 + 4$. As t varies from 0 to ∞,

x varies from 0 to ∞ and the graph is the right half of the parabola.

$\boxed{9}$ $x = 2\sin t$ and $y = 3\cos t \Rightarrow \frac{x}{2} = \sin t$ and $\frac{y}{3} = \cos t \Rightarrow$

$\frac{x^2}{4} + \frac{y^2}{9} = \sin^2 t + \cos^2 t = 1$. As t varies from 0 to 2π,

(x, y) traces the ellipse from $(0, 3)$ in a clockwise direction back to $(0, 3)$.

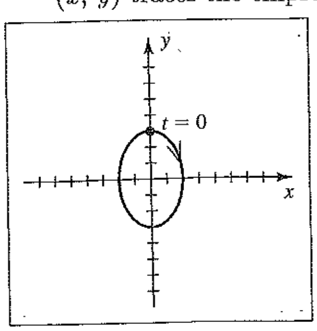

Figure 9

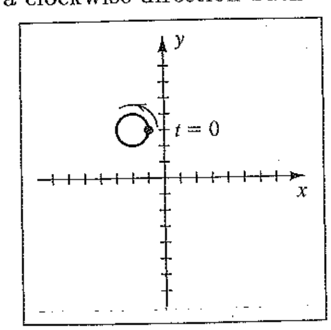

Figure 10

$\boxed{10}$ $x = \cos t - 2$ and $y = \sin t + 3 \Rightarrow x + 2 = \cos t$ and $y - 3 = \sin t \Rightarrow$

$(x+2)^2 + (y-3)^2 = \cos^2 t + \sin^2 t = 1$. As t varies from 0 to 2π,

(x, y) traces the circle from $(-1, 3)$ in a counterclockwise direction back to $(-1, 3)$.

$\boxed{11}$ $x = \sec t$ and $y = \tan t \;\Rightarrow\; x^2 - y^2 = \sec^2 t - \tan^2 t = 1$.

As t varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, $(x,\ y)$ traces the right branch of the hyperbola along the asymptote $y = -x$ to $(1,\ 0)$ and then along the asymptote $y = x$.

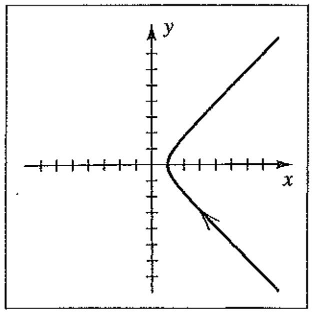

Figure 11

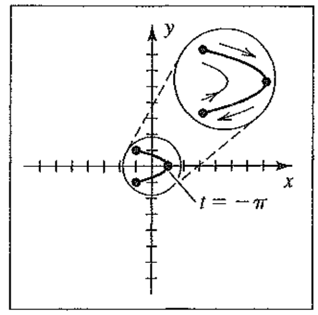

Figure 12

$\boxed{12}$ $x = \cos 2t = 1 - 2\sin^2 t = 1 - 2y^2$. As t varies from $-\pi$ to π, $(x,\ y)$ varies from $(1,\ 0)$ { the vertex } down to $(-1,\ -1)$ { when $t = -\frac{\pi}{2}$ }, back to the vertex when $t = 0$, up to $(-1,\ 1)$ { when $t = \frac{\pi}{2}$ }, and finally back to the vertex.

$\boxed{13}$ $y = 2\ln t = \ln t^2$ { since $t > 0$ } $= \ln x$.

As t varies from 0 to ∞, so does x, and y varies from $-\infty$ to ∞.

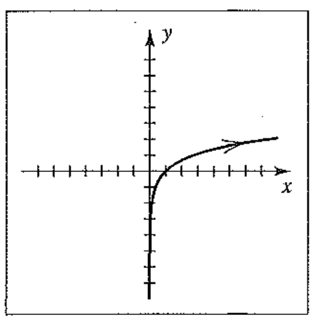

Figure 13

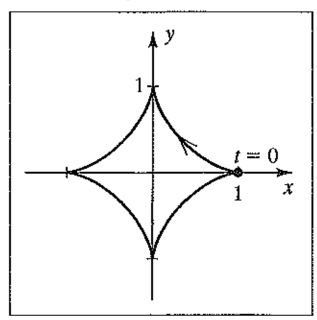

Figure 14

$\boxed{14}$ $x = \cos^3 t$ and $y = \sin^3 t \;\Rightarrow\; x^{2/3} = \cos^2 t$ and $y^{2/3} = \sin^2 t \;\Rightarrow\; x^{2/3} + y^{2/3} = 1$ or $y = \pm(1 - x^{2/3})^{3/2}$. As t varies from 0 to 2π, $(x,\ y)$ traces the astroid from $(1,\ 0)$ in a counterclockwise direction back to $(1,\ 0)$.

$\boxed{15}$ $y = \csc t = \dfrac{1}{\sin t} = \dfrac{1}{x}$.

As t varies from 0 to $\frac{\pi}{2}$,

$(x,\ y)$ varies asymptotically from the positive y-axis to $(1,\ 1)$.

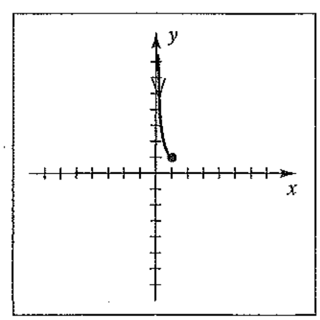

Figure 15

[16] $y = e^{-t} = (e^t)^{-1} = x^{-1} = \frac{1}{x}$.

As t varies from $-\infty$ to ∞,

(x, y) varies asymptotically from

the positive y-axis to the positive x-axis.

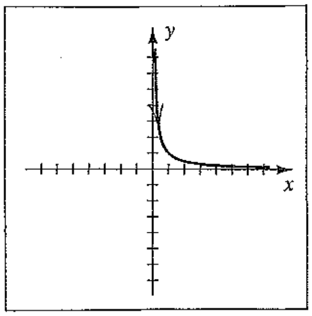

Figure 16

[17] $x = t$ and $y = \sqrt{t^2 - 1} \Rightarrow y = \sqrt{x^2 - 1} \Rightarrow x^2 - y^2 = 1$.

Since y is nonnegative, the graph is the top half of both branches of the hyperbola.

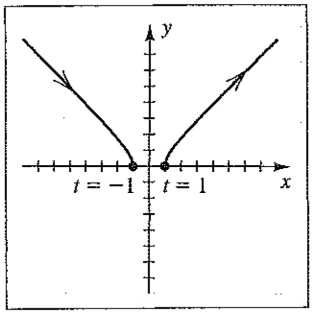

Figure 17

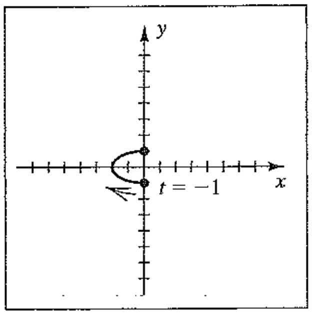

Figure 18

[18] $y = t$ and $x = -2\sqrt{1 - t^2} \Rightarrow x = -2\sqrt{1 - y^2} \Rightarrow x^2 = 4 - 4y^2 \Rightarrow x^2 + 4y^2 = 4$.

As t varies from -1 to 1, (x, y) traces the ellipse from $(0, -1)$ to $(0, 1)$.

[19] $x = t$ and $y = \sqrt{t^2 - 2t + 1} \Rightarrow y = \sqrt{x^2 - 2x + 1} = \sqrt{(x-1)^2} = |x - 1|$.

As t varies from 0 to 4, (x, y) traces $y = |x - 1|$ from $(0, 1)$ to $(4, 3)$.

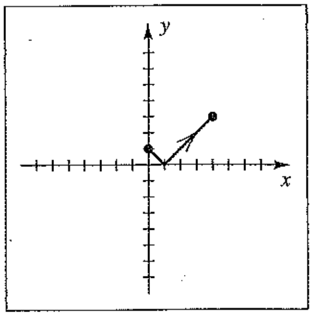

Figure 19

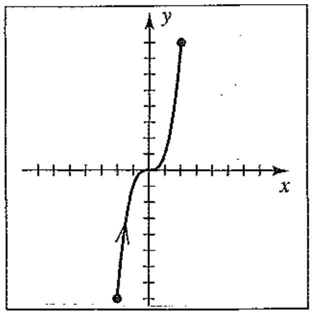

Figure 20

[20] $y = 8t^3 = (2t)^3 = x^3$. As t varies from -1 to 1, (x, y) varies from $(-2, -8)$ to $(2, 8)$.

$\boxed{21}$ $x = (t+1)^3 \Rightarrow t = x^{1/3} - 1$. $y = (t+2)^2 = (x^{1/3}+1)^2$.

As t varies from 0 to 2, (x, y) varies from $(1, 4)$ to $(27, 16)$.

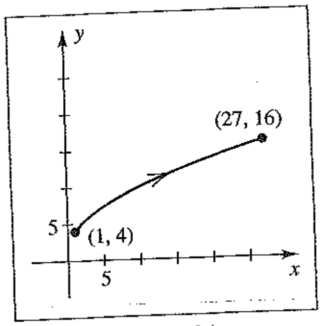

Figure 21

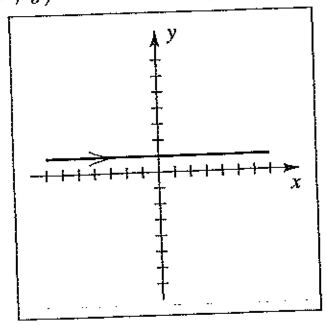

Figure 22

$\boxed{22}$ As t varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, x varies from $-\infty$ to ∞.

y is always 1 so we have the graph of $y = 1$.

$\boxed{23}$ (a) $x = 3 + 2\sin t$, $y = -2 + 2\cos t$; $0 \le t \le 2\pi$. We'll make use of the identity

$\sin^2 t + \cos^2 t = 1$. $x - 3 = 2\sin t$, $y + 2 = 2\cos t \Rightarrow$

$(x-3)^2 = 4\sin^2 t$, $(y+2)^2 = 4\cos^2 t \Rightarrow$

$(x-3)^2 + (y+2)^2 = 4\sin^2 t + 4\cos^2 t \Rightarrow$

$(x-3)^2 + (y+2)^2 = 4(\sin^2 t + \cos^2 t) \Rightarrow (x-3)^2 + (y+2)^2 = 4$. So the curve C is a circle with center $(3, -2)$ and radius $\sqrt{4} = 2$. As t varies from 0 to 2π,

(x, y) traces the circle from $(3, 0)$ in a clockwise direction back to $(3, 0)$.

(b) $x = 3 - 2\sin t$, $y = -2 + 2\cos t$; $0 \le t \le 2\pi$.

The orientation changes to counterclockwise.

(c) $x = 3 - 2\sin t$, $y = -2 - 2\cos t$; $0 \le t \le 2\pi$.

The starting and ending point changes to $(3, -4)$.

$\boxed{24}$ (a) $x = -2 + 3\sin t$, $y = 3 - 3\cos t$; $0 \le t \le 2\pi$. We'll make use of the identity

$\sin^2 t + \cos^2 t = 1$. $x + 2 = 3\sin t$, $y - 3 = -3\cos t \Rightarrow$

$(x+2)^2 = 9\sin^2 t$, $(y-3)^2 = 9\cos^2 t \Rightarrow$

$(x+2)^2 + (y-3)^2 = 9\sin^2 t + 9\cos^2 t \Rightarrow$

$(x+2)^2 + (y-3)^2 = 9(\sin^2 t + \cos^2 t) \Rightarrow (x+2)^2 + (y-3)^2 = 9$.

So the curve C is a circle with center $(-2, 3)$ and radius $\sqrt{9} = 3$.

As t varies from 0 to 2π, (x, y) traces the circle from $(-2, 0)$ in a

counterclockwise direction back to $(-2, 0)$.

(b) $x = -2 - 3\sin t$, $y = 3 + 3\cos t$; $0 \le t \le 2\pi$.

The starting and ending point changes to $(-2, 6)$.

(c) $x = -2 + 3\sin t$, $y = 3 + 3\cos t$; $0 \le t \le 2\pi$. The starting and ending point

changes to $(-2, 6)$ and the orientation changes to clockwise.

[25] All of the curves are a portion of the parabola $x = y^2$.

C_1: $x = t^2 = y^2$. y takes on all real values and we have the entire parabola.

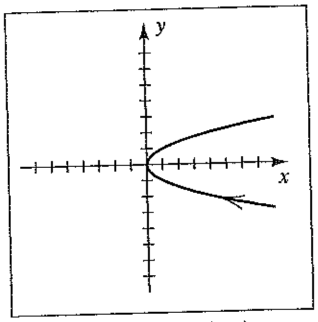

Figure 25 (C_1)

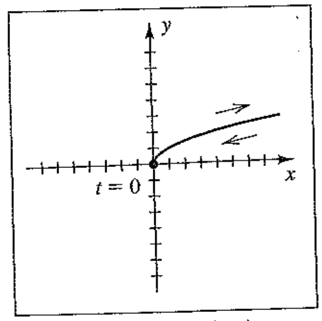

Figure 25 (C_2)

C_2: $x = t^4 = (t^2)^2 = y^2$. C_2 is only the top half since $y = t^2$ is nonnegative.

As t varies from $-\infty$ to ∞, the top portion is traced twice.

C_3: $x = \sin^2 t = (\sin t)^2 = y^2$. C_3 is the portion of the curve from (1, −1) to (1, 1).

The point (1, 1) is reached at $t = \frac{\pi}{2} + 2\pi n$ and the point (1, −1) when

$t = \frac{3\pi}{2} + 2\pi n$.

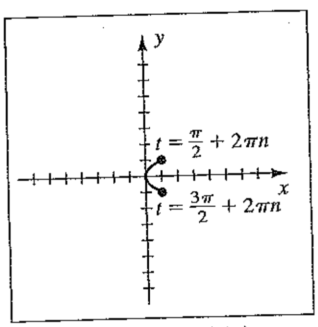

Figure 25 (C_3)

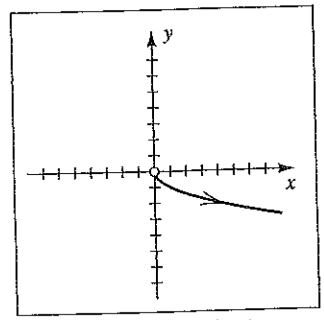

Figure 25 (C_4)

C_4: $x = e^{2t} = (e^t)^2 = (-e^t)^2 = y^2$. C_4 is the bottom half of the parabola since y

is negative. As t approaches $-\infty$, the parabola approaches the origin.

[26] All of the curves are a portion of the line $x + y = 1$.

C_1: $x + y = t + (1 - t) = 1$. C_1 is the entire line since x takes on all real values.

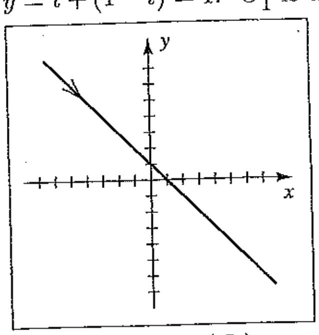

Figure 26 (C_1)

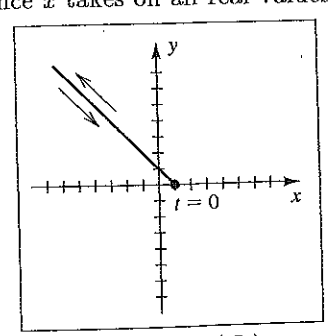

Figure 26 (C_2)

C_2: $x + y = (1 - t^2) + t^2 = 1$. C_2 is the portion of the line where $y \geq 0$ since $y = t^2$.

(continued)

C_3: $x + y = \cos^2 t + \sin^2 t = 1$. C_3 is only the portion from (0, 1) to (1, 0) since $\sin^2 t$ and $\cos^2 t$ are bounded by 0 and 1.

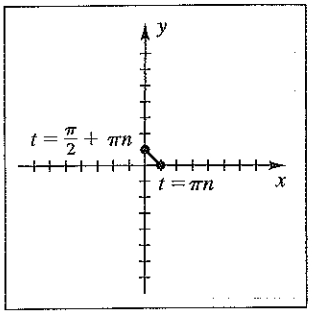

Figure 26 (C_3)

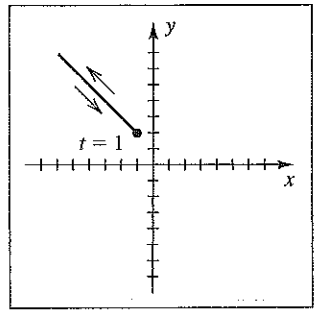

Figure 26 (C_4)

C_4: $x + y = (\ln t - t) + (1 + t - \ln t) = 1$. C_4 is defined when $t > 0$.

When $t = 1$, $(x, y) = (-1, 2)$. As t approaches 0 or ∞, x approaches $-\infty$.

$\boxed{27}$ In each part, the motion is on the unit circle since $x^2 + y^2 = 1$.

(a) $P(x, y)$ moves from (1, 0) counterclockwise to (−1, 0).

(b) $P(x, y)$ moves from (0, 1) clockwise to (0, −1).

(c) $P(x, y)$ moves from (−1, 0) clockwise to (1, 0).

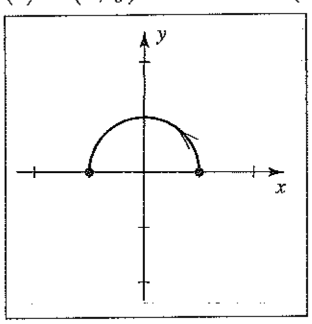

Figure 27(a)

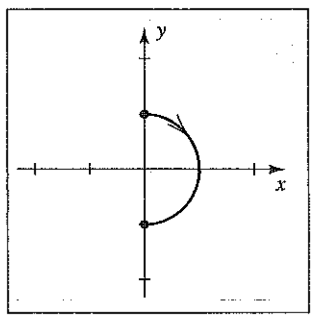

Figure 27(b)

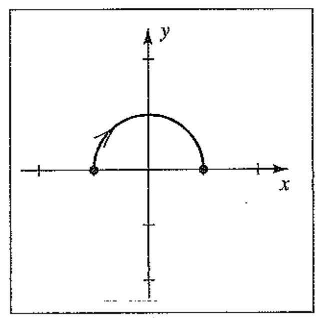

Figure 27(c)

$\boxed{28}$ In each part, the motion is on a line since $x + y = 1$.

(a) $P(x, y)$ moves from (0, 1) to (1, 0).

(b) $P(x, y)$ moves from (1, 0) to (0, 1).

(c) $P(x, y)$ moves from (1, 0) to (0, 1) twice in each direction.

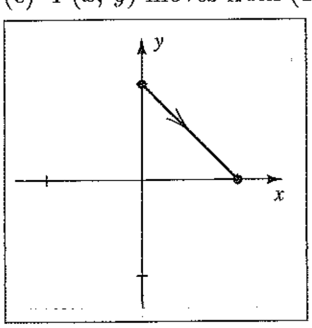

Figure 28(a)

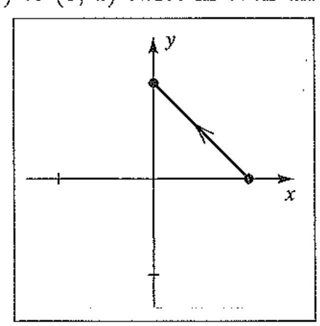

Figure 28(b)

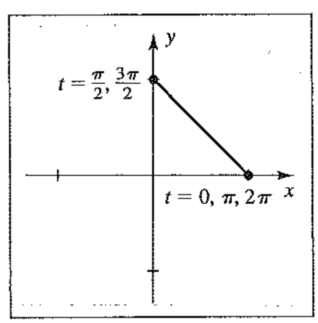

Figure 28(c)

[29] $x = a\cos t + h$ and $y = b\sin t + k \Rightarrow \dfrac{x-h}{a} = \cos t$ and $\dfrac{y-k}{b} = \sin t \Rightarrow$
$\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = \cos^2 t + \sin^2 t = 1$. This is the equation of an ellipse with center (h, k) and semiaxes of lengths a and b (axes of lengths $2a$ and $2b$).

[30] $x = a\sec t + h$, $y = b\tan t + k \Rightarrow \dfrac{x-h}{a} = \sec t$, $\dfrac{y-k}{b} = \tan t \Rightarrow$
$\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = \sec^2 t - \tan^2 t = 1$. This is an equation of a hyperbola with center (h, k), vertices $V(h \pm a, k)$ and $W(h, k \pm b)$. The transverse axis has length $2a$ and the conjugate axis has length $2b$. The right branch corresponds to $-\frac{\pi}{2} < t < \frac{\pi}{2}$ and the left branch corresponds to $\frac{\pi}{2} < t < \frac{3\pi}{2}$.

[31] Some choices for parts (a) and (b) are given—there are an infinite number of choices.

(a) (1) $x = t$, $y = t^2$; $t \in \mathbb{R}$
(2) $x = \tan t$, $y = \tan^2 t$; $-\frac{\pi}{2} < t < \frac{\pi}{2}$
(3) $x = t^3$, $y = t^6$; $t \in \mathbb{R}$

(b) (1) $x = e^t$, $y = e^{2t}$; $t \in \mathbb{R}$ (only gives $x > 0$)
(2) $x = \sin t$, $y = \sin^2 t$; $t \in \mathbb{R}$ (only gives $-1 \le x \le 1$)
(3) $x = \tan^{-1} t$, $y = (\tan^{-1} t)^2$; $t \in \mathbb{R}$ (only gives $-\frac{\pi}{2} < x < \frac{\pi}{2}$)

[32] (a) (1) $x = t$, $y = \ln t$; $t > 0$
(2) $x = e^t$, $y = t$; $t \in \mathbb{R}$
(3) $x = t^2$, $y = 2\ln t$; $t > 0$

(b) (1) $x = \sec t$, $y = \ln \sec t$; $-\frac{\pi}{2} < t < \frac{\pi}{2}$ (only gives $x \ge 1$)
(2) $x = \sin t$, $y = \ln \sin t$; $0 < t < \pi$ (only gives $0 < x \le 1$)
(3) $x = \tan^{-1} t$, $y = \ln \tan^{-1} t$; $t > 0$ (only gives $0 < x < \frac{\pi}{2}$)

[33] We use $x(t) = (s\cos\alpha)t$ and $y(t) = -\frac{1}{2}gt^2 + (s\sin\alpha)t + h$ with $s = 256\sqrt{3}$, $\alpha = 60^\circ$, and $h = 400$: $x = 256\sqrt{3}(\frac{1}{2})t$ and $y = -\frac{1}{2}(32)t^2 + 256\sqrt{3}(\sqrt{3}/2)t + 400 \Rightarrow$

$$x = 128\sqrt{3}t \quad \text{and} \quad y = -16t^2 + 384t + 400.$$

To find the range, we let $y = 0 \Rightarrow -16t^2 + 384t + 400 = 0 \Rightarrow$ $t^2 - 24t - 25 = 0$ {divide by -16} $\Rightarrow (t-25)(t+1) = 0 \Rightarrow t = 25$ seconds, and then substitute 25 for t in the equation for x. $x = 128\sqrt{3}(25) = 3200\sqrt{3} \approx 5542.56$ feet. To find the maximum altitude, we'll determine the values of t for which the height is 400, and then use the symmetry property of a parabola to find the vertex. $y = 400 \Rightarrow -16t^2 + 384t + 400 = 400 \Rightarrow -16t^2 + 384t = 0 \Rightarrow$ $-16t(t-24) = 0 \Rightarrow t = 0, 24$. Thus, the maximum altitude occurs when $t = \frac{1}{2}(24) = 12$. This value of h is $y = -16(12)^2 + 384(12) + 400 = 2704$ feet.

$\boxed{34}$ $x = 512\sqrt{2}(\sqrt{2}/2)t$ and $y = -\frac{1}{2}(32)t^2 + 512\sqrt{2}(\sqrt{2}/2)t + 1088 \Rightarrow$

$$x = 512t \quad \text{and} \quad y = -16t^2 + 512t + 1088.$$

$y = 0 \Rightarrow -16t^2 + 512t + 1088 = 0 \Rightarrow t^2 - 32t - 68 = 0 \Rightarrow$

$(t - 34)(t + 2) = 0 \Rightarrow t = 34$ seconds. $x = 512(34) = 17{,}408$ feet.

$y = 1088 \Rightarrow -16t^2 + 512t + 1088 = 1088 \Rightarrow -16t^2 + 512t = 0 \Rightarrow$

$-16t(t - 32) = 0 \Rightarrow t = 0, 32.$ The maximum altitude occurs when

$t = \frac{1}{2}(32) = 16$ and has value $y = -16(16)^2 + 512(16) + 1088 = 5184$ feet.

$\boxed{35}$ $x = 704(\sqrt{2}/2)t$ and $y = -\frac{1}{2}(32)t^2 + 704(\sqrt{2}/2)t + 0 \Rightarrow$

$$x = 352\sqrt{2}\,t \quad \text{and} \quad y = -16t^2 + 352\sqrt{2}\,t.$$

$y = 0 \Rightarrow -16t^2 + 352\sqrt{2}\,t = 0 \Rightarrow -16t(t - 22\sqrt{2}) = 0 \Rightarrow t = 22\sqrt{2}$ seconds.

$x = 352\sqrt{2}(22\sqrt{2}) = 15{,}488$ feet.

The maximum altitude occurs when $t = \frac{1}{2}(22\sqrt{2}) = 11\sqrt{2}$ and has value

$y = -16(11\sqrt{2})^2 + 352\sqrt{2}(11\sqrt{2}) = 3872$ feet.

$\boxed{36}$ $x = 2448(\sqrt{3}/2)t$ and $y = -\frac{1}{2}(32)t^2 + 2448(\frac{1}{2})t + 0 \Rightarrow$

$$x = 1224\sqrt{3}\,t \quad \text{and} \quad y = -16t^2 + 1224t.$$

$y = 0 \Rightarrow -16t^2 + 1224t = 0 \Rightarrow -8t(2t - 153) = 0 \Rightarrow t = 76.5$ seconds.

$x = 1224\sqrt{3}(76.5) = 93{,}636\sqrt{3} \approx 162{,}182.31$ feet.

The maximum altitude occurs when $t = \frac{1}{2}(76.5) = 38.25$ and has value

$y = -16(38.25)^2 + 1224(38.25) = 23{,}409$ feet.

$\boxed{37}$ (a) $x = a \sin \omega t$ and $y = b \cos \omega t \Rightarrow \frac{x}{a} = \sin \omega t$ and $\frac{y}{b} = \cos \omega t \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

The figure is an ellipse with center $(0, 0)$ and axes of lengths $2a$ and $2b$.

(b) $f(t + p) = a \sin[\omega_1(t + p)] = a \sin[\omega_1 t + \omega_1 p] = a \sin[\omega_1 t + 2\pi n] =$

$a \sin \omega_1 t = f(t).$

$g(t + p) = b \cos[\omega_2(t + p)] = b \cos\left[\omega_2 t + \frac{\omega_2}{\omega_1} 2\pi n\right] = b \cos[\omega_2 t + \frac{m}{n} 2\pi n] =$

$b \cos[\omega_2 t + 2\pi m] = b \cos \omega_2 t = g(t).$

Since f and g are periodic with period p,

the curve retraces itself every p units of time.

$\boxed{38}$ Since $x = 2 \sin 3t$ has period $\frac{2\pi}{3}$ and $y = 3 \sin(1.5t)$ has period $\frac{4\pi}{3}$,

the curve will repeat itself every $\frac{4\pi}{3}$ units of time.

$\boxed{39}$ (a) Let $x = 3 \sin(240\pi t)$ and $y = 4 \sin(240\pi t)$ for $0 \le t \le 0.01$. See *Figure 39.*

(b) From the graph, $y_{\text{int}} = 0$ and $y_{\text{max}} = 4$.

Thus, the phase difference is $\phi = \sin^{-1} \frac{y_{\text{int}}}{y_{\text{max}}} = \sin^{-1} \frac{0}{4} = 0°.$

$[-9,\ 9]$ by $[-6,\ 6]$

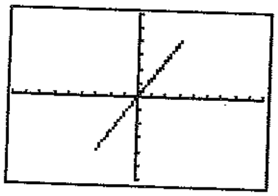

Figure 39

$[-9,\ 9]$ by $[-6,\ 6]$

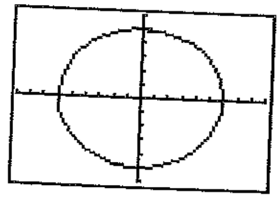

Figure 40

[40] (a) Let $x = 6\sin(120\pi t)$ and $y = 5\cos(120\pi t)$ for $0 \le t \le 0.02$.

(b) From the graph, $y_{\text{int}} = 5$ and $y_{\text{max}} = 5$.

Thus, the phase difference is $\phi = \sin^{-1}\dfrac{y_{\text{int}}}{y_{\text{max}}} = \sin^{-1}\dfrac{5}{5} = 90^\circ$.

Note: $5\cos(120\pi t) = 5\sin(120\pi t + \pi/2)$

[41] (a) Let $x = 80\sin(60\pi t)$ and $y = 70\cos(60\pi t - \pi/3)$ for $0 \le t \le 0.035$.

(b) From the graph, $y_{\text{int}} = 35$ and $y_{\text{max}} = 70$.

Thus, the phase difference is $\phi = \sin^{-1}\dfrac{y_{\text{int}}}{y_{\text{max}}} = \sin^{-1}\dfrac{35}{70} = 30^\circ$.

Note: Tstep must be sufficiently small to obtain the maximum of 70 on the graph. $70\cos(60\pi t - \pi/3) = 70\sin(60\pi t + \pi/6)$

$[-120,\ 120,\ 10]$ by $[-80,\ 80,\ 10]$

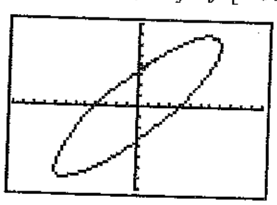

Figure 41

$[-300,\ 300,\ 100]$ by $[-200,\ 200,\ 100]$

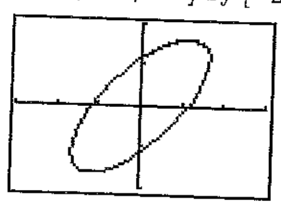

Figure 42

[42] (a) Let $x = 163\sin(120\pi t)$ and $y = 163\sin(120\pi t + \pi/4)$ for $0 \le t \le 0.02$.

(b) From the graph, $y_{\text{int}} \approx 115.26$ and $y_{\text{max}} = 163$.

Thus, the phase difference is $\phi = \sin^{-1}\dfrac{y_{\text{int}}}{y_{\text{max}}} = \sin^{-1}\dfrac{115}{163} \approx 45^\circ$.

[43] $x(t) = \sin(6\pi t)$, $y(t) = \cos(5\pi t)$ for $0 \le t \le 2$

$[-1,\ 1,\ 0.5]$ by $[-1,\ 1,\ 0.5]$

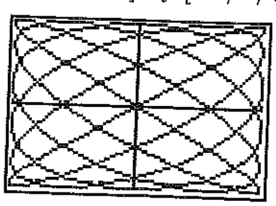

Figure 43

$[-1,\ 1,\ 0.5]$ by $[-1,\ 1,\ 0.5]$

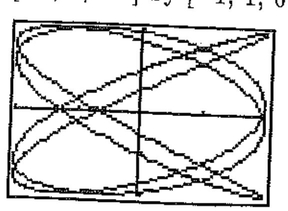

Figure 44

[44] $x(t) = \sin(4t)$, $y(t) = \sin(3t + \pi/6)$ for $0 \le t \le 6.5$

45 Let $\theta = \angle FDP$ and $\alpha = \angle GDP = \angle EDP$. Then $\angle ODG = (\frac{\pi}{2} - t)$ and $\alpha = \theta - (\frac{\pi}{2} - t) = \theta + t - \frac{\pi}{2}$. Arcs AF and PF are equal in length since each is the distance rolled. Thus, $at = b\theta$, or $\theta = (\frac{a}{b})t$ and $\alpha = \frac{a+b}{b}t - \frac{\pi}{2}$.

Note that $\cos\alpha = \sin\left(\frac{a+b}{b}t\right)$ and $\sin\alpha = -\cos\left(\frac{a+b}{b}t\right)$.

For the location of the points as illustrated, the coordinates of P are:

$$x = d(O,\,G) + d(G,\,B) = d(O,\,G) + d(E,\,P) = (a+b)\cos t + b\,\sin\alpha = (a+b)\cos t - b\,\cos\left(\frac{a+b}{b}t\right)$$

$$y = d(B,\,P) = d(G,\,D) - d(D,\,E) = (a+b)\sin t - b\,\cos\alpha = (a+b)\sin t - b\,\sin\left(\frac{a+b}{b}t\right)$$

It can be verified that these equations are valid for all locations of the points.

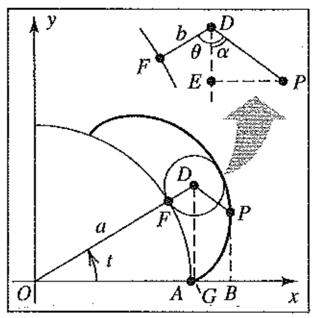

Figure 45

46 (a) See *Figures 46(a) & 46(b)*. The line from O to C, the center of the smaller circle, must also pass through B, the common point of tangency as the smaller circle rolls inside the larger. Note that:

(1) $t = \alpha$ by the properties of parallel lines

(2) $\overline{OB} = \overline{OA} = a$

(3) $\overline{CB} = \overline{CP} = b$

(4) $\overline{OC} = \overline{OB} - \overline{CB} = a - b$

(5) $\angle CPD = \beta$ by the properties of parallel lines

(6) $\angle BCP = \alpha + \beta = t + \beta$

(7) in $\triangle\, OCE$, $\overline{OE} = \overline{OC}\,\cos t$ and $\overline{EC} = \overline{OC}\,\sin t$

(8) in $\triangle\, DCP$, $\overline{DP} = \overline{CP}\,\cos\angle CPD$ and $\overline{DC} = \overline{CP}\,\sin\angle CPD$

(9) $\overline{DP} = \overline{EF}$

(10) in the smaller circle, $\widehat{BP} = b\,(\angle BCP)$

(11) in the larger circle, $\widehat{BA} = a\,(\angle BOA)$

(12) since $\widehat{BP} = \widehat{BA}$ (each is the distance rolled), $b\,(\angle BCP) = at \;\Rightarrow\; \angle BCP = \frac{a}{b}t$

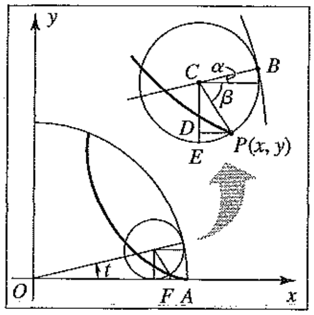

Figure 46(a)

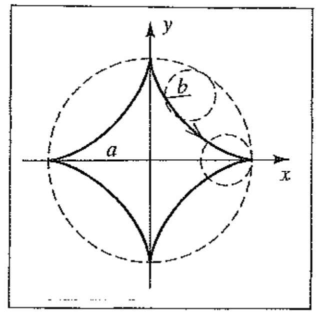

Figure 46(b)

$$
\begin{aligned}
\text{Now, } x &= \overline{OE} + \overline{EF} \\
&= \overline{OC}\cos t + \overline{CP}\cos\angle CPD \\
&= (a-b)\cos t + b\cos\beta \\
&= (a-b)\cos t + b\cos(\angle BCP - \alpha) \\
&= (a-b)\cos t + b\cos\left(\frac{a}{b}t - t\right) \\
&= (a-b)\cos t + b\cos\left(\frac{a-b}{b}t\right).
\end{aligned}
$$

$$
\begin{aligned}
\text{Similarly, } y &= \overline{EC} - \overline{DC} \\
&= \overline{OC}\sin t - \overline{CP}\sin\angle CPD \\
&= (a-b)\sin t - b\sin\beta \\
&= (a-b)\sin t - b\sin(\angle BCP - \alpha) \\
&= (a-b)\sin t - b\sin\left(\frac{a}{b}t - t\right) \\
&= (a-b)\sin t - b\sin\left(\frac{a-b}{b}\right).
\end{aligned}
$$

(b) If $b = \frac{1}{4}a$, then $x = (a - \frac{1}{4}a)\cos t + \frac{1}{4}a\cos\left(\dfrac{a-\frac{1}{4}a}{\frac{1}{4}a}t\right)$

$$= \tfrac{3}{4}a\cos t + \tfrac{1}{4}a\cos 3t = \tfrac{3}{4}a\cos t + \tfrac{1}{4}a(4\cos^3 t - 3\cos t) = a\cos^3 t.$$

Also, $y = (a - \frac{1}{4}a)\sin t - \frac{1}{4}a\sin\left(\dfrac{a-\frac{1}{4}a}{\frac{1}{4}a}t\right)$

$$= \tfrac{3}{4}a\sin t - \tfrac{1}{4}a\sin 3t = \tfrac{3}{4}a\sin t - \tfrac{1}{4}a(3\sin t - 4\sin^3 t) = a\sin^3 t.$$

The identities used for $\cos 3t$ and $\sin 3t$ can be derived by applying the addition, double angle, and fundamental identities.

$\boxed{47}$ $b = \frac{1}{3}a \Rightarrow a = 3b$. Substituting into the equations from Exercise 45 yields:

$$x = (3b + b)\cos t - b\cos\left(\frac{3b+b}{b}t\right) = 4b\cos t - b\cos 4t$$

$$y = (3b + b)\sin t - b\sin\left(\frac{3b+b}{b}t\right) = 4b\sin t - b\sin 4t$$

As an aid in graphing, to determine where the path of the smaller circle will intersect the path of the larger circle (for the original starting point of intersection at $A(a, 0)$), we can solve $x^2 + y^2 = a^2$ for t.

$$\begin{aligned} x^2 + y^2 &= 16b^2\cos^2 t - 8b^2\cos t\cos 4t + b^2\cos^2 4t + \\ &\qquad 16b^2\sin^2 t - 8b^2\sin t\sin 4t + b^2\sin^2 4t \\ &= 17b^2 - 8b^2(\cos t\cos 4t + \sin t\sin 4t) \\ &= 17b^2 - 8b^2[\cos(t - 4t)] = 17b^2 - 8b^2\cos 3t. \end{aligned}$$

Thus, $x^2 + y^2 = a^2 \Rightarrow 17b^2 - 8b^2\cos 3t = a^2 = 9b^2 \Rightarrow 8b^2 = 8b^2\cos 3t \Rightarrow$
$1 = \cos 3t \Rightarrow 3t = 2\pi n \Rightarrow t = \frac{2\pi}{3}n$.

It follows that the intersection points are at $t = \frac{2\pi}{3}, \frac{4\pi}{3}$, and 2π.

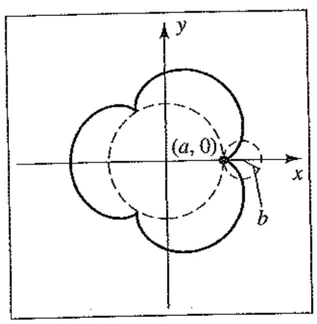

Figure 47

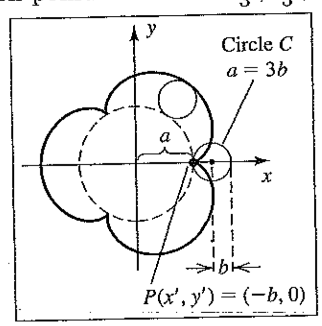

Figure 48

$\boxed{48}$ Let $C'(h, k)$ be the center of the circle C (in *Figure 48*) and $a = 3b$. Its coordinates are always $x = 4b\cos t$ and $y = 4b\sin t$ since it can be thought of as being on a circle of radius $4b$. Since the equations given in Exercise 47 are for a point P relative to the origin, we can see that the coordinates of P relative to C' are $P(x', y') =$ $P(-b\cos 4t, -b\sin 4t)$. At the starting point A, the coordinates of P relative to C' are $(-b, 0)$. Each time P has these relative coordinates, it will have made one revolution. Setting the coordinates equal to each other and solving for t we have:

$-b\cos 4t = -b \Rightarrow \cos 4t = 1 \Rightarrow 4t = 2\pi n \Rightarrow t = \frac{\pi}{2}n$ and also

$-b\sin 4t = 0 \Rightarrow \sin 4t = 0 \Rightarrow 4t = \pi n \Rightarrow t = \frac{\pi}{4}n$. These results

indicate that C will make one revolution every $\frac{\pi}{2}$ units, or 4 revolutions in 2π units.

49 Change to "Par" from "Func" under $\boxed{\text{MODE}}$. Make the assignments 3(sin T)^5 to X_{1T}, 3(cos T)^5 to Y_{1T}, 0 to Tmin, 2π to Tmax, and $\pi/30$ to Tstep. Algebraically, we have $x = 3\sin^5 t$ and $y = 3\cos^5 t \Rightarrow$

$$\frac{x}{3} = \sin^5 t \quad \text{and} \quad \frac{y}{3} = \cos^5 t \quad \Rightarrow \quad (\tfrac{x}{3})^{2/5} + (\tfrac{y}{3})^{2/5} = \sin^2 t + \cos^2 t = 1.$$

The graph traces an astroid.

[−6, 6] by [−4, 4]

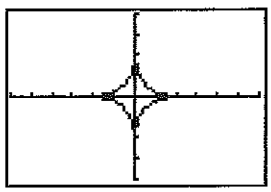

Figure 49

[−15, 15] by [−10, 10]

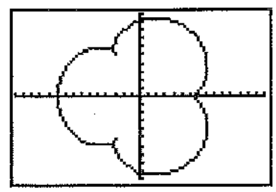

Figure 50

50 The graph traces an epicycloid. See Exercise 45 with $a = 6$ and $b = 2$.

51 The graph traces a curtate cycloid.

[−30, 30, 5] by [−20, 20, 5]

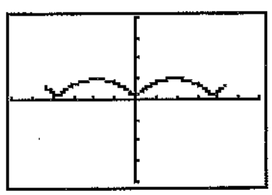

Figure 51

[−15, 15] by [−10, 10]

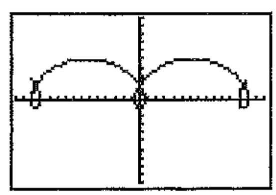

Figure 52

52 The graph traces a prolate cycloid.

53 The figure is a mask with a mouth, nose, and eyes. This graph may be obtained with a graphing utility that has the capability to graph 5 sets of parametric equations.

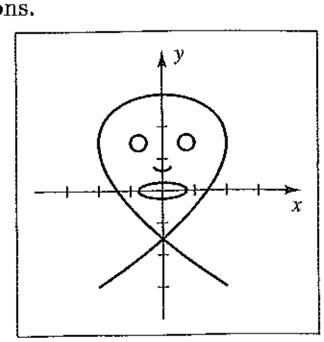

Figure 53

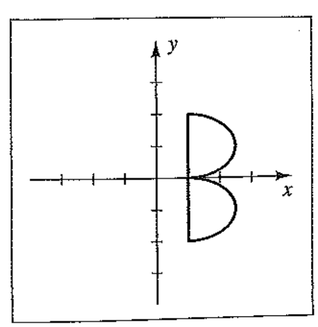

Figure 54

[54] C_1: $x = \frac{3}{2}\cos t + 1$ and $y = \sin t - 1$ $\Rightarrow$ $\frac{2}{3}(x-1) = \cos t$ and $(y+1) = \sin t$ $\Rightarrow$ $\dfrac{(x-1)^2}{\frac{9}{4}} + (y+1)^2 = 1$. This curve is part of an ellipse with center (1, −1) and endpoints (1, −2) and (1, 0). Similarly, C_2 is part of an ellipse with center (1, 1) and endpoints (1, 0) and (1, 2). C_3 is a vertical line, $x = 1$, with endpoints (1, −2) and (1, 2). The figure is the letter B. See *Figure 54.*

[55] C_1 is the line $y = 3x$ from (0, 0) to (1, 3). For C_2, $x - 1 = \tan t$ and $1 - \frac{1}{3}y = \tan t$ $\Rightarrow$ $x - 1 = 1 - \frac{1}{3}y$ $\Rightarrow$ $y = -3x + 6$. This line is sketched from (1, 3) to (2, 0). C_3 is the horizontal line $y = \frac{3}{2}$ from $(\frac{1}{2}, \frac{3}{2})$ to $(\frac{3}{2}, \frac{3}{2})$. The figure is the letter A.

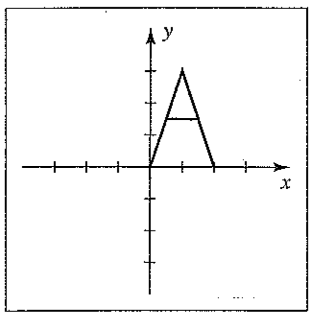

Figure 55

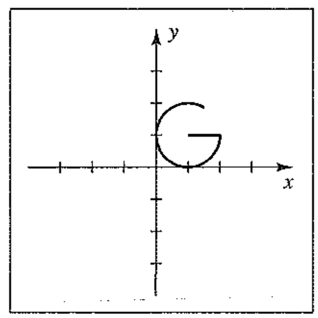

Figure 56

[56] C_1 is the circle $(x-1)^2 + (y-1)^2 = 1$ from $(\frac{3}{2},\ 1 + \frac{1}{2}\sqrt{3})$ to (2, 1). C_2 is the horizontal line $y = 1$ from (1, 1) to (2, 1). The figure is the letter G.

11.5 Exercises

Note: For the following exercises, the substitutions $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$, and $\tan\theta = \frac{y}{x}$ are used without mention. The numbers listed on each line of the r-θ chart correspond to the numbers labeled on the figures.

[1] Choices (a) $(3, 7\pi/3)$, (c) $(-3, 4\pi/3)$, and (e) $(-3, -2\pi/3)$ represent the same point as $(3, \pi/3)$.

[2] Choices (b) $(4, 7\pi/2)$, (d) $(4, -5\pi/2)$, (e) $(-4, -3\pi/2)$, and (f) $(-4, \pi/2)$ represent the same point as $(4, -\pi/2)$.

[3] (a) $x = r\cos\theta = 3\cos\frac{\pi}{4} = 3\left(\frac{\sqrt{2}}{2}\right) = \frac{3}{2}\sqrt{2}$. $y = r\sin\theta = 3\sin\frac{\pi}{4} = 3\left(\frac{\sqrt{2}}{2}\right) = \frac{3}{2}\sqrt{2}$.

(b) $x = -1\cos\frac{2\pi}{3} = -1(-\frac{1}{2}) = \frac{1}{2}$. $y = -1\sin\frac{2\pi}{3} = -1\left(\frac{\sqrt{3}}{2}\right) = -\frac{1}{2}\sqrt{3}$.

$\boxed{4}$ (a) $x = 5\cos\frac{5\pi}{6} = 5\left(-\frac{\sqrt{3}}{2}\right) = -\frac{5}{2}\sqrt{3}$. $y = 5\sin\frac{5\pi}{6} = 5(\frac{1}{2}) = \frac{5}{2}$.

(b) $x = -6\cos\frac{7\pi}{3} = -6(\frac{1}{2}) = -3$. $y = -6\sin\frac{7\pi}{3} = -6\left(\frac{\sqrt{3}}{2}\right) = -3\sqrt{3}$.

$\boxed{5}$ (a) $x = 8\cos(-\frac{2\pi}{3}) = 8(-\frac{1}{2}) = -4$. $y = 8\sin(-\frac{2\pi}{3}) = 8\left(-\frac{\sqrt{3}}{2}\right) = -4\sqrt{3}$.

(b) $x = -3\cos\frac{5\pi}{3} = -3(\frac{1}{2}) = -\frac{3}{2}$. $y = -3\sin\frac{5\pi}{3} = -3\left(-\frac{\sqrt{3}}{2}\right) = \frac{3}{2}\sqrt{3}$.

$\boxed{6}$ (a) $x = 4\cos(-\frac{\pi}{4}) = 4\left(\frac{\sqrt{2}}{2}\right) = 2\sqrt{2}$. $y = 4\sin(-\frac{\pi}{4}) = 4\left(-\frac{\sqrt{2}}{2}\right) = -2\sqrt{2}$.

(b) $x = -2\cos\frac{7\pi}{6} = -2\left(-\frac{\sqrt{3}}{2}\right) = \sqrt{3}$. $y = -2\sin\frac{7\pi}{6} = -2(-\frac{1}{2}) = 1$.

$\boxed{7}$ Let $\theta = \arctan\frac{3}{4}$. $x = 6\cos\theta = 6(\frac{4}{5}) = \frac{24}{5}$. $y = 6\sin\theta = 6(\frac{3}{5}) = \frac{18}{5}$.

$\boxed{8}$ Let $\theta = \arccos(-\frac{1}{3})$. $x = 10\cos\theta = 10(-\frac{1}{3}) = -\frac{10}{3}$.

$y = 10\sin\theta = 10\left(\frac{\sqrt{8}}{3}\right) = \frac{10}{3}\sqrt{8} = \frac{20}{3}\sqrt{2}$.

$\boxed{9}$ (a) $r^2 = x^2 + y^2 = (-1)^2 + (1)^2 = 2 \Rightarrow r = \sqrt{2}$. $\tan\theta = \frac{y}{x} = \frac{1}{-1} = -1 \Rightarrow$
$\theta = \frac{3\pi}{4}$ {θ in QII}.

(b) $r^2 = (-2\sqrt{3})^2 + (-2)^2 = 16 \Rightarrow r = 4$. $\tan\theta = \frac{-2}{-2\sqrt{3}} = \frac{1}{\sqrt{3}} \Rightarrow$
$\theta = \frac{7\pi}{6}$ {θ in QIII}.

$\boxed{10}$ (a) $r^2 = (3\sqrt{3})^2 + 3^2 = 36 \Rightarrow r = 6$. $\tan\theta = \frac{3}{3\sqrt{3}} = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$ {θ in QI}.

(b) $r^2 = 2^2 + (-2)^2 = 8 \Rightarrow r = 2\sqrt{2}$. $\tan\theta = \frac{-2}{2} = -1 \Rightarrow \theta = \frac{7\pi}{4}$ {θ in QIV}.

$\boxed{11}$ (a) $r^2 = 7^2 + (-7\sqrt{3})^2 = 196 \Rightarrow r = 14$.
$\tan\theta = \frac{-7\sqrt{3}}{7} = -\sqrt{3} \Rightarrow \theta = \frac{5\pi}{3}$ {θ in QIV}.

(b) $r^2 = 5^2 + 5^2 = 50 \Rightarrow r = 5\sqrt{2}$. $\tan\theta = \frac{5}{5} = 1 \Rightarrow \theta = \frac{\pi}{4}$ {θ in QI}.

$\boxed{12}$ (a) $r^2 = (-2\sqrt{2})^2 + (-2\sqrt{2})^2 = 16 \Rightarrow r = 4$.
$\tan\theta = \frac{-2\sqrt{2}}{-2\sqrt{2}} = 1 \Rightarrow \theta = \frac{5\pi}{4}$ {θ in QIII}.

(b) $r^2 = (-4)^2 + (4\sqrt{3})^2 = 64 \Rightarrow r = 8$. $\tan\theta = \frac{4\sqrt{3}}{-4} = -\sqrt{3} \Rightarrow$
$\theta = \frac{2\pi}{3}$ {θ in QII}.

$\boxed{13}$ $x = -3 \Rightarrow r\cos\theta = -3 \Rightarrow r = \frac{-3}{\cos\theta} \Rightarrow r = -3\sec\theta$

$\boxed{14}$ $y = 2 \Rightarrow r\sin\theta = 2 \Rightarrow r = \frac{2}{\sin\theta} \Rightarrow r = 2\csc\theta$

$\boxed{15}$ $x^2 + y^2 = 16 \Rightarrow r^2 = 16 \Rightarrow r = \pm 4$ {both are circles with radius 4}.

$\boxed{16}$ $x^2 + y^2 = 2 \Rightarrow r^2 = 2 \Rightarrow r = \pm\sqrt{2}$ {both are circles with radius $\sqrt{2}$}.

[17] $y^2 = 6x \;\Rightarrow\; r^2\sin^2\theta = 6r\cos\theta \;\Rightarrow\; r^2\sin^2\theta - 6r\cos\theta = 0 \;\Rightarrow$

$r(r\sin^2\theta - 6\cos\theta) = 0 \;\Rightarrow\; r\sin^2\theta - 6\cos\theta = 0 \;\Rightarrow\; r\sin^2\theta = 6\cos\theta \;\Rightarrow$

$r = \dfrac{6\cos\theta}{\sin^2\theta} = 6\cdot\dfrac{\cos\theta}{\sin\theta}\cdot\dfrac{1}{\sin\theta} = 6\cot\theta\,\csc\theta.$

Note that $r = 0$ { the pole } is included in the graph of $r = 6\cot\theta\,\csc\theta$.

[18] $x^2 = 8y \;\Rightarrow\; r^2\cos^2\theta = 8r\sin\theta \;\Rightarrow\; r^2\cos^2\theta - 8r\sin\theta = 0 \;\Rightarrow$

$r(r\cos^2\theta - 8\sin\theta) = 0 \;\Rightarrow\; r\cos^2\theta - 8\sin\theta = 0 \;\Rightarrow\; r\cos^2\theta = 8\sin\theta \;\Rightarrow$

$r = \dfrac{8\sin\theta}{\cos^2\theta} = 8\cdot\dfrac{\sin\theta}{\cos\theta}\cdot\dfrac{1}{\cos\theta} = 8\tan\theta\,\sec\theta.$

Note that $r = 0$ { the pole } is included in the graph of $r = 8\tan\theta\,\sec\theta$.

[19] $x + y = 3 \;\Rightarrow\; r\cos\theta + r\sin\theta = 3 \;\Rightarrow\; r(\cos\theta + \sin\theta) = 3 \;\Rightarrow\; r = \dfrac{3}{\cos\theta + \sin\theta}$

[20] $2y = -x + 4 \;\Rightarrow\; 2r\sin\theta = -r\cos\theta + 4 \;\Rightarrow\; 2r\sin\theta + r\cos\theta = 4 \;\Rightarrow$

$r(2\sin\theta + \cos\theta) = 4 \;\Rightarrow\; r = \dfrac{4}{2\sin\theta + \cos\theta}$

[21] $2y = -x \;\Rightarrow\; \frac{y}{x} = -\frac{1}{2} \;\Rightarrow\; \tan\theta = -\frac{1}{2} \;\Rightarrow\; \theta = \tan^{-1}(-\frac{1}{2})$

[22] $y = 6x \;\Rightarrow\; \frac{y}{x} = 6 \;\Rightarrow\; \tan\theta = 6 \;\Rightarrow\; \theta = \tan^{-1}6$ { ≈ 1.41 or 80.54° }

[23] $y^2 - x^2 = 4 \;\Rightarrow\; r^2\sin^2\theta - r^2\cos^2\theta = 4 \;\Rightarrow\; -r^2(\cos^2\theta - \sin^2\theta) = 4 \;\Rightarrow$

$-r^2\cos 2\theta = 4 \;\Rightarrow\; r^2 = \dfrac{-4}{\cos 2\theta} \;\Rightarrow\; r^2 = -4\sec 2\theta$

[24] $xy = 8 \;\Rightarrow\; (r\cos\theta)(r\sin\theta) = 8 \;\Rightarrow\; r^2(\frac{1}{2})(2\sin\theta\cos\theta) = 8 \;\Rightarrow$

$r^2\sin 2\theta = 16 \;\Rightarrow\; r^2 = \dfrac{16}{\sin 2\theta} \;\Rightarrow\; r^2 = 16\csc 2\theta$

[25] $(x-1)^2 + y^2 = 1 \;\Rightarrow\; x^2 - 2x + 1 + y^2 = 1 \;\Rightarrow\; x^2 + y^2 = 2x \;\Rightarrow$

$r^2 = 2r\cos\theta \;\Rightarrow\; r = 2\cos\theta$ { $r = 0$ is included }

[26] $(x+2)^2 + (y-3)^2 = 13 \;\Rightarrow\; x^2 + 4x + 4 + y^2 - 6y + 9 = 13 \;\Rightarrow$

$x^2 + y^2 = 6y - 4x \;\Rightarrow\; r^2 = 6r\sin\theta - 4r\cos\theta \;\Rightarrow\; r = 6\sin\theta - 4\cos\theta$

[27] $r\cos\theta = 5 \;\Rightarrow\; x = 5.$

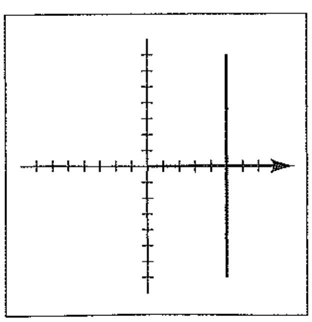

Figure 27

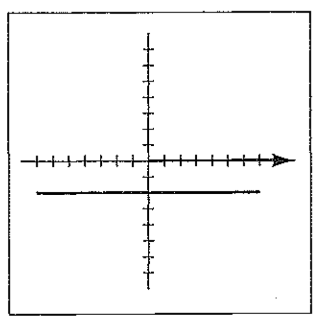

Figure 28

[28] $r\sin\theta = -2 \;\Rightarrow\; y = -2.$

29 $r - 6\sin\theta = 0 \;\Rightarrow\; r^2 = 6r\sin\theta \;\Rightarrow\; x^2 + y^2 = 6y \;\Rightarrow\; x^2 + y^2 - 6y + \underline{9} = \underline{9} \;\Rightarrow$
$x^2 + (y-3)^2 = 9.$

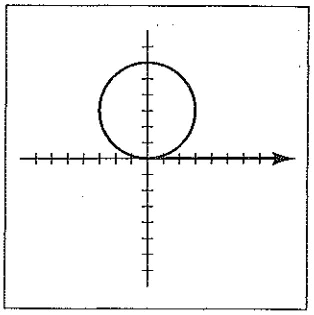

Figure 29

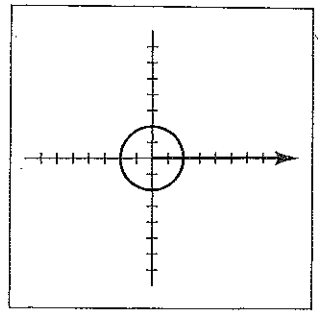

Figure 30

30 $r = 2 \;\Rightarrow\; r^2 = 4 \;\Rightarrow\; x^2 + y^2 = 4.$

31 $\theta = \frac{\pi}{4} \;\Rightarrow\; \tan\theta = \tan\frac{\pi}{4} \;\Rightarrow\; \frac{y}{x} = 1 \;\Rightarrow\; y = x.$

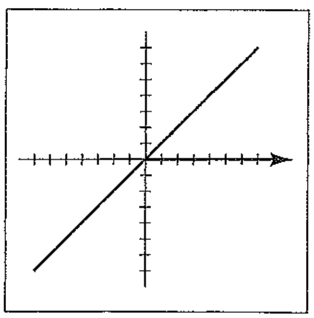

Figure 31

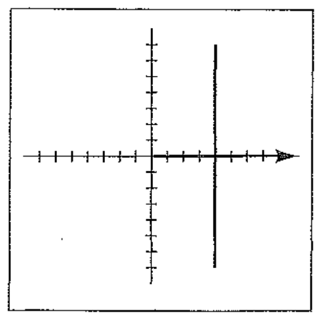

Figure 32

32 $r = 4\sec\theta \;\Rightarrow\; r\cos\theta = 4 \;\Rightarrow\; x = 4.$ θ is undefined at $\frac{\pi}{2} + \pi n.$

33 $r^2(4\sin^2\theta - 9\cos^2\theta) = 36 \;\Rightarrow\; 4r^2\sin^2\theta - 9r^2\cos^2\theta = 36 \;\Rightarrow$
$4y^2 - 9x^2 = 36 \;\Rightarrow\; \frac{y^2}{9} - \frac{x^2}{4} = 1.$

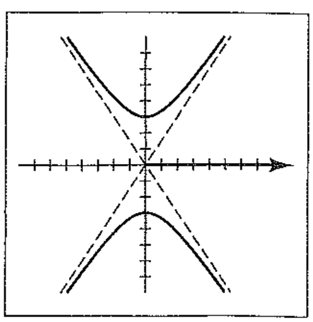

Figure 33

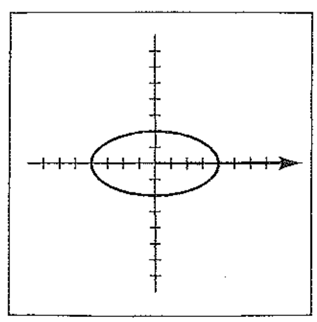

Figure 34

34 $r^2(\cos^2\theta + 4\sin^2\theta) = 16 \;\Rightarrow\; r^2\cos^2\theta + 4r^2\sin^2\theta = 16 \;\Rightarrow$
$x^2 + 4y^2 = 16 \;\Rightarrow\; \frac{x^2}{16} + \frac{y^2}{4} = 1.$

[35] $r^2 \cos 2\theta = 1 \Rightarrow r^2(\cos^2\theta - \sin^2\theta) = 1 \Rightarrow r^2 \cos^2\theta - r^2 \sin^2\theta = 1 \Rightarrow x^2 - y^2 = 1.$

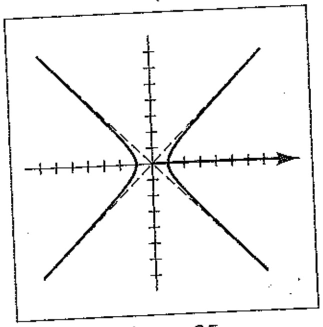

Figure 35

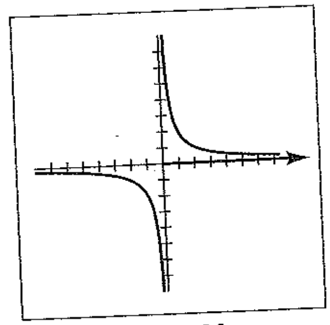

Figure 36

[36] $r^2 \sin 2\theta = 4 \Rightarrow r^2(2\sin\theta \cos\theta) = 4 \Rightarrow (r \sin\theta)(r \cos\theta) = 2 \Rightarrow xy = 2.$

[37] $r(\sin\theta - 2\cos\theta) = 6 \Rightarrow r \sin\theta - 2r \cos\theta = 6 \Rightarrow y - 2x = 6.$

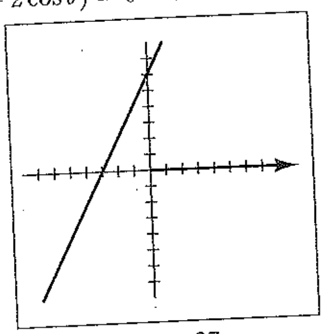

Figure 37

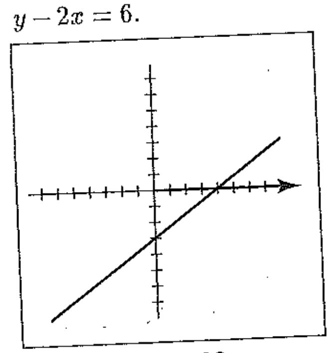

Figure 38

[38] $r(3\cos\theta - 4\sin\theta) = 12 \Rightarrow 3r \cos\theta - 4r \sin\theta = 12 \Rightarrow 3x - 4y = 12.$

[39] $r(\sin\theta + r \cos^2\theta) = 1 \Rightarrow r \sin\theta + r^2 \cos^2\theta = 1 \Rightarrow y + x^2 = 1$, or $y = -x^2 + 1.$

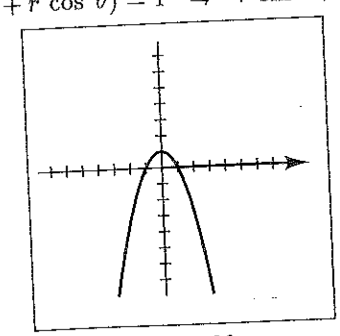

Figure 39

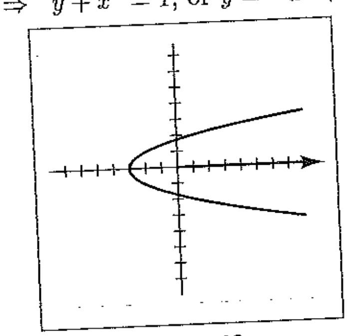

Figure 40

[40] $r(r \sin^2\theta - \cos\theta) = 3 \Rightarrow r^2 \sin^2\theta - r \cos\theta = 3 \Rightarrow y^2 - x = 3 \Rightarrow x = y^2 - 3.$

$\boxed{41}$ $r = 8\sin\theta - 2\cos\theta \;\Rightarrow\; r^2 = 8r\,\sin\theta - 2r\,\cos\theta \;\Rightarrow\; x^2 + y^2 = 8y - 2x \;\Rightarrow$

$$x^2 + 2x + \underline{1} + y^2 - 8y + \underline{16} = \underline{1} + \underline{16} \;\Rightarrow\; (x+1)^2 + (y-4)^2 = 17.$$

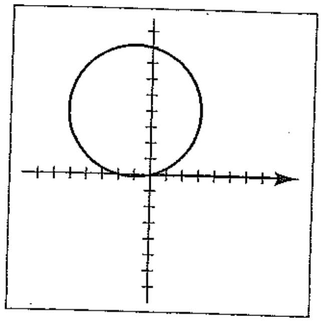

Figure 41

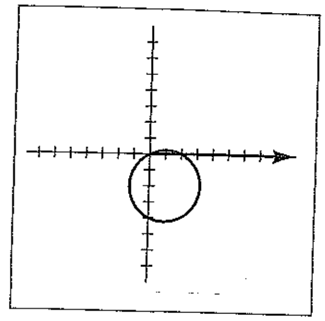

Figure 42

$\boxed{42}$ $r = 2\cos\theta - 4\sin\theta \;\Rightarrow\; r^2 = 2r\,\cos\theta - 4r\,\sin\theta \;\Rightarrow\; x^2 + y^2 = 2x - 4y \;\Rightarrow$

$$x^2 - 2x + \underline{1} + y^2 + 4y + \underline{4} = \underline{1} + \underline{4} \;\Rightarrow\; (x-1)^2 + (y+2)^2 = 5.$$

$\boxed{43}$ $r = \tan\theta \;\Rightarrow\; r^2 = \tan^2\theta \;\Rightarrow\; x^2 + y^2 = \dfrac{y^2}{x^2} \;\Rightarrow\; x^4 + x^2y^2 = y^2 \;\Rightarrow$

$$y^2 - x^2y^2 = x^4 \;\Rightarrow\; y^2(1-x^2) = x^4 \;\Rightarrow\; y^2 = \frac{x^4}{1-x^2}$$

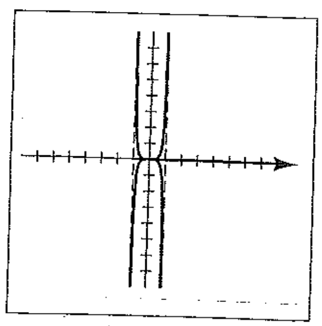

Figure 43

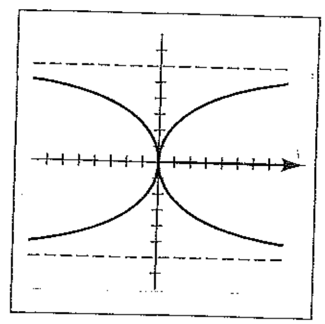

Figure 44

$\boxed{44}$ $r = 6\cot\theta \;\Rightarrow\; r^2 = 36\cot^2\theta \;\Rightarrow\; x^2 + y^2 = 36\left(\dfrac{x^2}{y^2}\right) \;\Rightarrow\; x^2y^2 + y^4 = 36x^2 \;\Rightarrow$

$$36x^2 - x^2y^2 = y^4 \;\Rightarrow\; x^2(36-y^2) = y^4 \;\Rightarrow\; x^2 = \frac{y^4}{36-y^2}$$

$\boxed{45}$ $r = 5 \;\Rightarrow\; r^2 = 25 \;\Rightarrow\; x^2 + y^2 = 25$, a circle centered at the origin with radius 5.

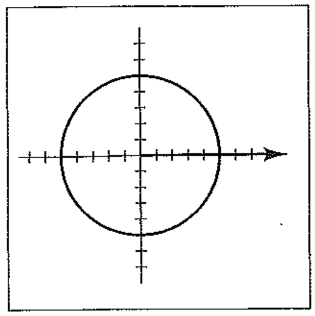

Figure 45

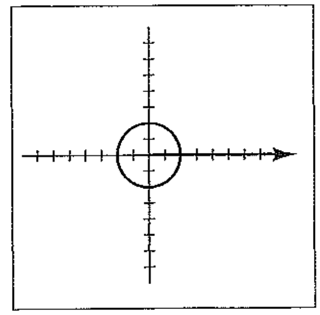

Figure 46

$\boxed{46}$ $r = -2 \;\Rightarrow\; r^2 = 4 \;\Rightarrow\; x^2 + y^2 = 4$, a circle centered at the origin with radius 2.

$\boxed{47}$ $\theta = -\frac{\pi}{6}$ and $r \in \mathbb{R}$. The line is $y = (\tan\theta)\,x$, or $y = -\frac{1}{3}\sqrt{3}\,x$.

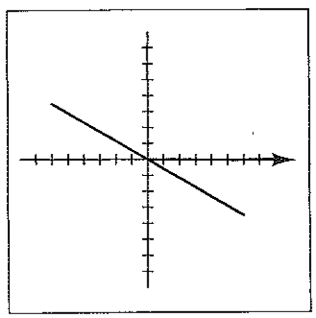

Figure 47

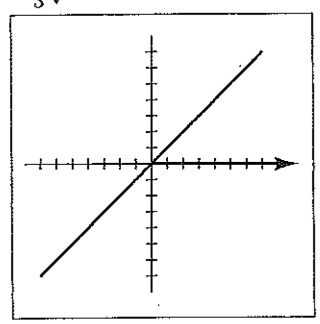

Figure 48

$\boxed{48}$ $\theta = \frac{\pi}{4}$ and $r \in \mathbb{R}$. The line is $y = (\tan\theta)\,x$, or $y = x$.

$\boxed{49}$ $r = 3\cos\theta \;\Rightarrow\; r^2 = 3r\cos\theta \;\Rightarrow\; x^2 + y^2 = 3x \;\Rightarrow$
$(x^2 - 3x + \underline{\tfrac{9}{4}}) + y^2 = \underline{\tfrac{9}{4}} \;\Rightarrow\; (x - \frac{3}{2})^2 + y^2 = \frac{9}{4}$.

Variation of θ	Variation of r
1) $0 \rightarrow \frac{\pi}{2}$	$3 \rightarrow 0$
2) $\frac{\pi}{2} \rightarrow \pi$	$0 \rightarrow -3$

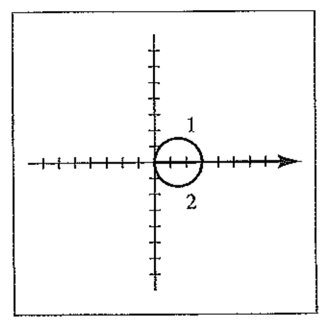

Figure 49

$\boxed{50}$ $r = -2\sin\theta \;\Rightarrow\; r^2 = -2r\sin\theta \;\Rightarrow$
$x^2 + y^2 = -2y \;\Rightarrow\; x^2 + y^2 + 2y + \underline{1} = \underline{1} \;\Rightarrow$
$x^2 + (y+1)^2 = 1$.

Variation of θ	Variation of r
1) $0 \rightarrow \frac{\pi}{2}$	$0 \rightarrow -2$
2) $\frac{\pi}{2} \rightarrow \pi$	$-2 \rightarrow 0$

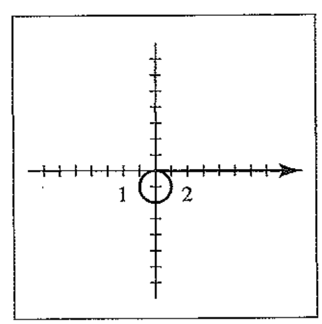

Figure 50

[51] $r = 4\cos\theta + 2\sin\theta \Rightarrow r^2 = 4r\cos\theta + 2r\sin\theta \Rightarrow$
$x^2 + y^2 = 4x + 2y \Rightarrow$
$x^2 - 4x + \underline{4} + y^2 - 2y + \underline{1} = \underline{4} + \underline{1} \Rightarrow$
$(x-2)^2 + (y-1)^2 = 5.$

Variation of θ				Variation of r	
1)	0	$\rightarrow$	$\frac{\pi}{2}$	$4 \rightarrow$	2
2)	$\frac{\pi}{2}$	$\rightarrow$	π	$2 \rightarrow$	-4

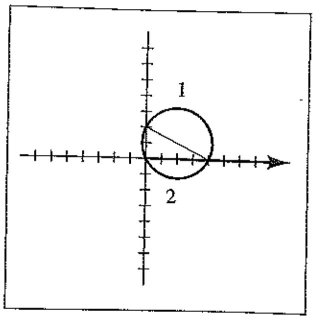

Figure 51

[52] $r = 6\cos\theta - 2\sin\theta \Rightarrow r^2 = 6r\cos\theta - 2r\sin\theta \Rightarrow$
$x^2 + y^2 = 6x - 2y \Rightarrow$
$x^2 - 6x + \underline{9} + y^2 + 2y + \underline{1} = \underline{9} + \underline{1} \Rightarrow$
$(x-3)^2 + (y+1)^2 = 10.$

Variation of θ				Variation of r	
1)	0	$\rightarrow$	$\frac{\pi}{2}$	$6 \rightarrow$	-2
2)	$\frac{\pi}{2}$	$\rightarrow$	π	$-2 \rightarrow$	-6

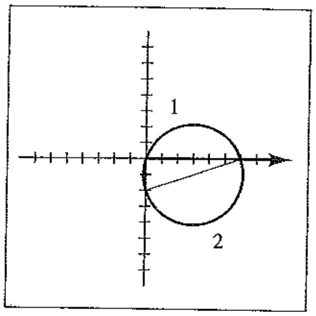

Figure 52

[53] $r = 4(1 - \sin\theta)$ is a cardioid since the coefficient of $\sin\theta$ has the same magnitude as the constant term. We next find the pole values by solving the equation $r = 0$.
$0 = 4(1 - \sin\theta) \Rightarrow \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2} + 2\pi n.$

Variation of θ				Variation of r	
1)	0	$\rightarrow$	$\frac{\pi}{2}$	$4 \rightarrow$	0
2)	$\frac{\pi}{2}$	$\rightarrow$	π	$0 \rightarrow$	4
3)	π	$\rightarrow$	$\frac{3\pi}{2}$	$4 \rightarrow$	8
4)	$\frac{3\pi}{2}$	$\rightarrow$	2π	$8 \rightarrow$	4

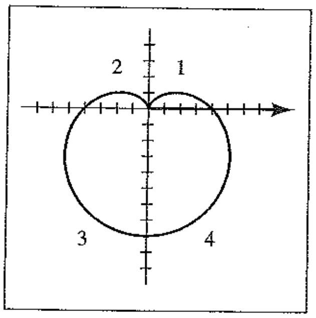

Figure 53

[54] $r = 3(1 + \cos\theta)$ is a cardioid.
$0 = 3(1 + \cos\theta) \Rightarrow \cos\theta = -1 \Rightarrow \theta = \pi + 2\pi n.$

Variation of θ				Variation of r	
1)	0	$\rightarrow$	$\frac{\pi}{2}$	$6 \rightarrow$	3
2)	$\frac{\pi}{2}$	$\rightarrow$	π	$3 \rightarrow$	0
3)	π	$\rightarrow$	$\frac{3\pi}{2}$	$0 \rightarrow$	3
4)	$\frac{3\pi}{2}$	$\rightarrow$	2π	$3 \rightarrow$	6

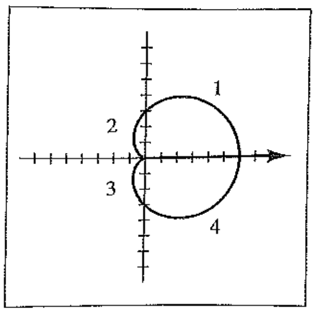

Figure 54

[55] $r = -6(1+\cos\theta)$ is a cardioid.

$0 = -6(1+\cos\theta) \Rightarrow \cos\theta = -1 \Rightarrow \theta = \pi + 2\pi n.$

	Variation of θ			Variation of r	
1)	0	$\to$	$\frac{\pi}{2}$	$-12 \to$	-6
2)	$\frac{\pi}{2}$	$\to$	π	$-6 \to$	0
3)	π	$\to$	$\frac{3\pi}{2}$	$0 \to$	-6
4)	$\frac{3\pi}{2}$	$\to$	2π	$-6 \to$	-12

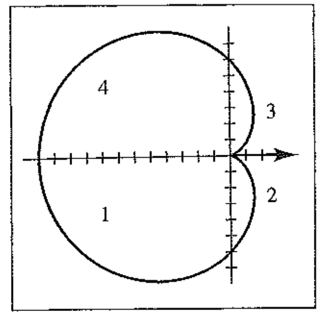

Figure 55

[56] $r = 2(1+\sin\theta)$ is a cardioid.

$0 = 2(1+\sin\theta) \Rightarrow \sin\theta = -1 \Rightarrow \theta = \frac{3\pi}{2} + 2\pi n.$

	Variation of θ			Variation of r	
1)	0	$\to$	$\frac{\pi}{2}$	$2 \to$	4
2)	$\frac{\pi}{2}$	$\to$	π	$4 \to$	2
3)	π	$\to$	$\frac{3\pi}{2}$	$2 \to$	0
4)	$\frac{3\pi}{2}$	$\to$	2π	$0 \to$	2

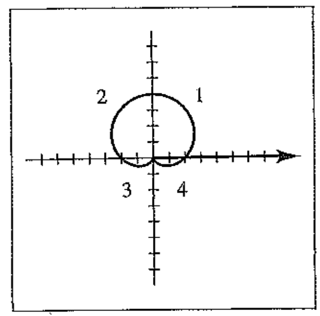

Figure 56

[57] $r = 2 + 4\sin\theta$ is a limaçon with a loop since the constant term has a smaller magnitude than the coefficient of $\sin\theta$.

$0 = 2 + 4\sin\theta \Rightarrow \sin\theta = -\frac{1}{2} \Rightarrow$

$\theta = \frac{7\pi}{6} + 2\pi n, \frac{11\pi}{6} + 2\pi n.$

We use the pole values as well as the quadrantal angles to set up the r-θ variation chart.

	Variation of θ			Variation of r	
1)	0	$\to$	$\frac{\pi}{2}$	$2 \to$	6
2)	$\frac{\pi}{2}$	$\to$	π	$6 \to$	2
3)	π	$\to$	$\frac{7\pi}{6}$	$2 \to$	0
4)	$\frac{7\pi}{6}$	$\to$	$\frac{3\pi}{2}$	$0 \to$	-2
5)	$\frac{3\pi}{2}$	$\to$	$\frac{11\pi}{6}$	$-2 \to$	0
6)	$\frac{11\pi}{6}$	$\to$	2π	$0 \to$	2

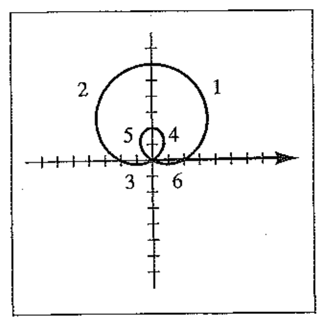

Figure 57

$\boxed{58}$ $r = 1 + 2\cos\theta$ is a limaçon with a loop.

$0 = 1 + 2\cos\theta \Rightarrow \cos\theta = -\frac{1}{2} \Rightarrow$

$\theta = \frac{2\pi}{3} + 2\pi n, \frac{4\pi}{3} + 2\pi n.$

	Variation of θ			Variation of r	
1)	0	$\to$	$\frac{\pi}{2}$	$3 \to$	1
2)	$\frac{\pi}{2}$	$\to$	$\frac{2\pi}{3}$	$1 \to$	0
3)	$\frac{2\pi}{3}$	$\to$	π	$0 \to$	-1
4)	π	$\to$	$\frac{4\pi}{3}$	$-1 \to$	0
5)	$\frac{4\pi}{3}$	$\to$	$\frac{3\pi}{2}$	$0 \to$	1
6)	$\frac{3\pi}{2}$	$\to$	2π	$1 \to$	3

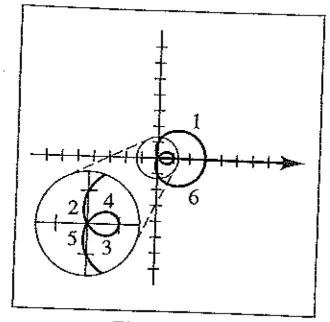

Figure 58

$\boxed{59}$ $r = \sqrt{3} - 2\sin\theta$ is a limaçon with a loop. $0 = \sqrt{3} - 2\sin\theta \Rightarrow \sin\theta = \sqrt{3}/2 \Rightarrow$
$\theta = \frac{\pi}{3} + 2\pi n, \frac{2\pi}{3} + 2\pi n.$ Let $a = \sqrt{3} - 2 \approx -0.27$ and $b = \sqrt{3} + 2 \approx 3.73$.

	Variation of θ			Variation of r	
1)	0	$\to$	$\frac{\pi}{3}$	$\sqrt{3} \to$	0
2)	$\frac{\pi}{3}$	$\to$	$\frac{\pi}{2}$	$0 \to$	a
3)	$\frac{\pi}{2}$	$\to$	$\frac{2\pi}{3}$	$a \to$	0
4)	$\frac{2\pi}{3}$	$\to$	π	$0 \to$	$\sqrt{3}$
5)	π	$\to$	$\frac{3\pi}{2}$	$\sqrt{3} \to$	b
6)	$\frac{3\pi}{2}$	$\to$	2π	$b \to$	$\sqrt{3}$

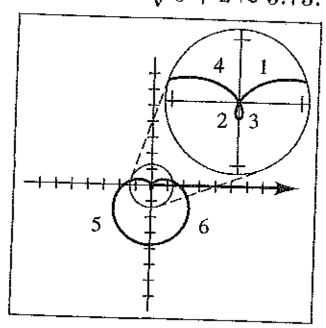

Figure 59

$\boxed{60}$ $r = 2\sqrt{3} - 4\cos\theta$ is a limaçon with a loop.

$0 = 2\sqrt{3} - 4\cos\theta \Rightarrow \cos\theta = \sqrt{3}/2 \Rightarrow \theta = \frac{\pi}{6} + 2\pi n, \frac{11\pi}{6} + 2\pi n.$

Let $a = 2\sqrt{3} - 4 \approx -0.54$ and $b = 2\sqrt{3} + 4 \approx 7.46$.

	Variation of θ			Variation of r	
1)	0	$\to$	$\frac{\pi}{6}$	$a \to$	0
2)	$\frac{\pi}{6}$	$\to$	$\frac{\pi}{2}$	$0 \to$	$2\sqrt{3}$
3)	$\frac{\pi}{2}$	$\to$	π	$2\sqrt{3} \to$	b
4)	π	$\to$	$\frac{3\pi}{2}$	$b \to$	$2\sqrt{3}$
5)	$\frac{3\pi}{2}$	$\to$	$\frac{11\pi}{6}$	$2\sqrt{3} \to$	0
6)	$\frac{11\pi}{6}$	$\to$	2π	$0 \to$	a

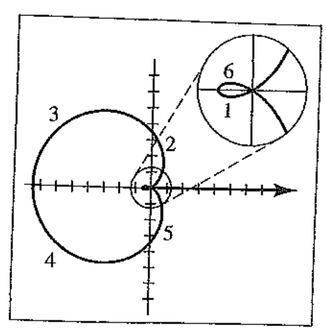

Figure 60

$\boxed{61}$ $r = 2 - \cos\theta$ •

$0 = 2 - \cos\theta \;\Rightarrow\; \cos\theta = 2 \;\Rightarrow\;$ no pole values.

	Variation of θ	Variation of r
1)	$0 \to \frac{\pi}{2}$	$1 \to 2$
2)	$\frac{\pi}{2} \to \pi$	$2 \to 3$
3)	$\pi \to \frac{3\pi}{2}$	$3 \to 2$
4)	$\frac{3\pi}{2} \to 2\pi$	$2 \to 1$

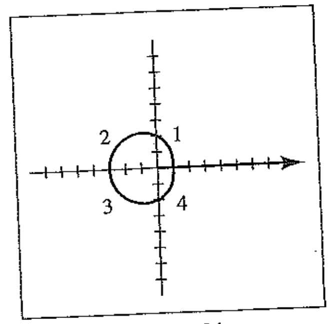

Figure 61

$\boxed{62}$ $r = 5 + 3\sin\theta$ •

$0 = 5 + 3\sin\theta \;\Rightarrow\; \sin\theta = -\frac{5}{3} \;\Rightarrow\;$ no pole values.

	Variation of θ	Variation of r
1)	$0 \to \frac{\pi}{2}$	$5 \to 8$
2)	$\frac{\pi}{2} \to \pi$	$8 \to 5$
3)	$\pi \to \frac{3\pi}{2}$	$5 \to 2$
4)	$\frac{3\pi}{2} \to 2\pi$	$2 \to 5$

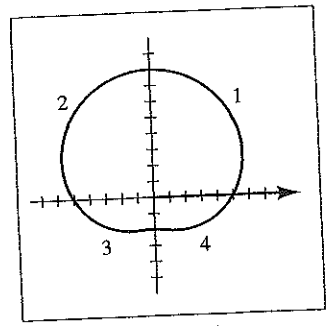

Figure 62

$\boxed{63}$ $r = 4\csc\theta \;\Rightarrow\; r\sin\theta = 4 \;\Rightarrow\; y = 4.$ r is undefined at $\theta = \pi n$.

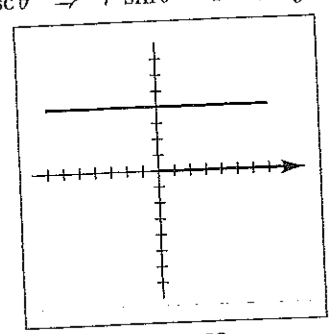
Figure 63

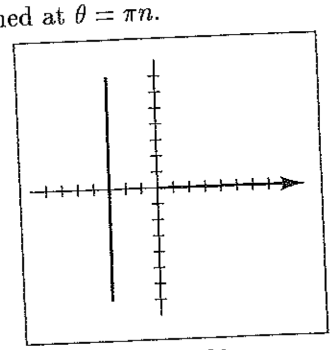
Figure 64

$\boxed{64}$ $r = -3\sec\theta \;\Rightarrow\; r\cos\theta = -3 \;\Rightarrow\; x = -3.$ r is undefined at $\theta = \frac{\pi}{2} + \pi n$.

$\boxed{65}$ $r = 8\cos 3\theta$ is a 3-leafed rose since 3 is odd.

$0 = 8\cos 3\theta \;\Rightarrow\; \cos 3\theta = 0 \;\Rightarrow\; 3\theta = \frac{\pi}{2} + \pi n \;\Rightarrow$

$\theta = \frac{\pi}{6} + \frac{\pi}{3}n.$

	Variation of θ	Variation of r
1)	$0 \to \frac{\pi}{6}$	$8 \to 0$
2)	$\frac{\pi}{6} \to \frac{\pi}{3}$	$0 \to -8$
3)	$\frac{\pi}{3} \to \frac{\pi}{2}$	$-8 \to 0$
4)	$\frac{\pi}{2} \to \frac{2\pi}{3}$	$0 \to 8$
5)	$\frac{2\pi}{3} \to \frac{5\pi}{6}$	$8 \to 0$
6)	$\frac{5\pi}{6} \to \pi$	$0 \to -8$

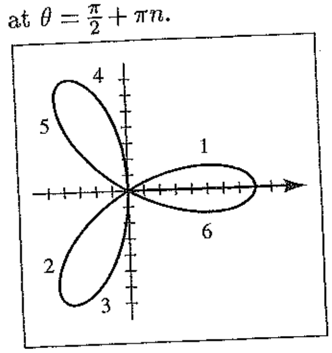

Figure 65

66 $r = 2\sin 4\theta$ is an 8-leafed rose since 4 is even. $0 = 2\sin 4\theta \Rightarrow \sin 4\theta = 0 \Rightarrow 4\theta = \pi n \Rightarrow \theta = \frac{\pi}{4}n$. Steps 9 through 16 follow a similar pattern to steps 1 through 8 and are labeled in the correct order.

	Variation of θ	Variation of r
1)	$0 \to \frac{\pi}{8}$	$0 \to 2$
2)	$\frac{\pi}{8} \to \frac{\pi}{4}$	$2 \to 0$
3)	$\frac{\pi}{4} \to \frac{3\pi}{8}$	$0 \to -2$
4)	$\frac{3\pi}{8} \to \frac{\pi}{2}$	$-2 \to 0$
5)	$\frac{\pi}{2} \to \frac{5\pi}{8}$	$0 \to 2$
6)	$\frac{5\pi}{8} \to \frac{3\pi}{4}$	$2 \to 0$
7)	$\frac{3\pi}{4} \to \frac{7\pi}{8}$	$0 \to -2$
8)	$\frac{7\pi}{8} \to \pi$	$-2 \to 0$

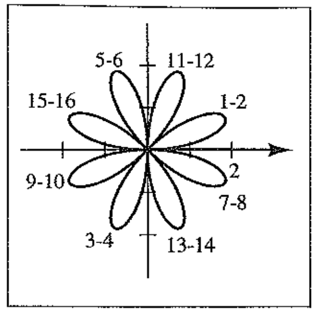

Figure 66

67 $r = 3\sin 2\theta$ is a 4-leafed rose. $0 = 3\sin 2\theta \Rightarrow \sin 2\theta = 0 \Rightarrow 2\theta = \pi n \Rightarrow \theta = \frac{\pi}{2}n$.

	Variation of θ	Variation of r
1)	$0 \to \frac{\pi}{4}$	$0 \to 3$
2)	$\frac{\pi}{4} \to \frac{\pi}{2}$	$3 \to 0$
3)	$\frac{\pi}{2} \to \frac{3\pi}{4}$	$0 \to -3$
4)	$\frac{3\pi}{4} \to \pi$	$-3 \to 0$
5)	$\pi \to \frac{5\pi}{4}$	$0 \to 3$
6)	$\frac{5\pi}{4} \to \frac{3\pi}{2}$	$3 \to 0$
7)	$\frac{3\pi}{2} \to \frac{7\pi}{4}$	$0 \to -3$
8)	$\frac{7\pi}{4} \to 2\pi$	$-3 \to 0$

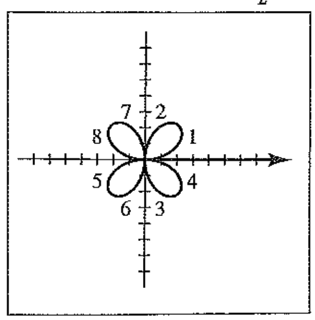

Figure 67

68 $r = 8\cos 5\theta$ is a 5-leafed rose.

$$0 = 8\cos 5\theta \Rightarrow \cos 5\theta = 0 \Rightarrow 5\theta = \frac{\pi}{2} + \pi n \Rightarrow \theta = \frac{\pi}{10} + \frac{\pi}{5}n.$$

	Variation of θ	Variation of r
1)	$0 \to \frac{\pi}{10}$	$8 \to 0$
2)	$\frac{\pi}{10} \to \frac{2\pi}{10}$	$0 \to -8$
3)	$\frac{2\pi}{10} \to \frac{3\pi}{10}$	$-8 \to 0$
4)	$\frac{3\pi}{10} \to \frac{4\pi}{10}$	$0 \to 8$
5)	$\frac{4\pi}{10} \to \frac{5\pi}{10}$	$8 \to 0$
6)	$\frac{5\pi}{10} \to \frac{6\pi}{10}$	$0 \to -8$
7)	$\frac{6\pi}{10} \to \frac{7\pi}{10}$	$-8 \to 0$
8)	$\frac{7\pi}{10} \to \frac{8\pi}{10}$	$0 \to 8$
9)	$\frac{8\pi}{10} \to \frac{9\pi}{10}$	$8 \to 0$
10)	$\frac{9\pi}{10} \to \pi$	$0 \to -8$

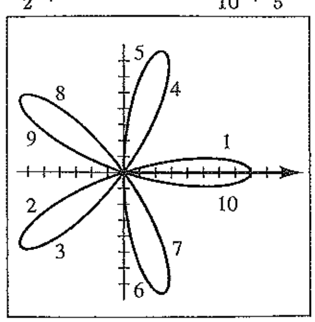

Figure 68

$\boxed{69}$ $r^2 = 4\cos 2\theta$ (lemniscate) •

$0 = 4\cos 2\theta \Rightarrow \cos 2\theta = 0 \Rightarrow 2\theta = \frac{\pi}{2} + \pi n \Rightarrow$
$\theta = \frac{\pi}{4} + \frac{\pi}{2}n.$

	Variation of θ	Variation of r
1)	$0 \to \frac{\pi}{4}$	$\pm 2 \to 0$
2)	$\frac{\pi}{4} \to \frac{\pi}{2}$	undefined
3)	$\frac{\pi}{2} \to \frac{3\pi}{4}$	undefined
4)	$\frac{3\pi}{4} \to \pi$	$0 \to \pm 2$

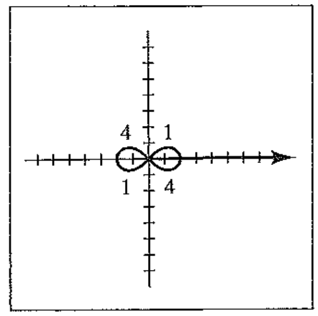

Figure 69

$\boxed{70}$ $r^2 = -16\sin 2\theta$ • $0 = -16\sin 2\theta \Rightarrow$
$\sin 2\theta = 0 \Rightarrow 2\theta = \pi n \Rightarrow \theta = \frac{\pi}{2}n.$

	Variation of θ	Variation of r
1)	$0 \to \frac{\pi}{4}$	undefined
2)	$\frac{\pi}{4} \to \frac{\pi}{2}$	undefined
3)	$\frac{\pi}{2} \to \frac{3\pi}{4}$	$0 \to \pm 4$
4)	$\frac{3\pi}{4} \to \pi$	$\pm 4 \to 0$

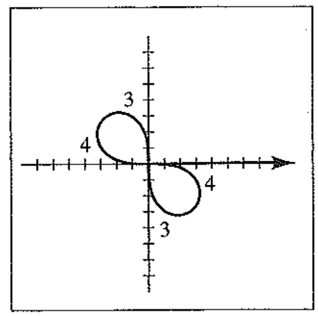

Figure 70

$\boxed{71}$ $r = 2^{\theta}$, $\theta \geq 0$ (spiral) •

	Variation of θ	Variation of r
1)	$0 \to \frac{\pi}{2}$	$1 \to 2.97$
2)	$\frac{\pi}{2} \to \pi$	$2.97 \to 8.82$
3)	$\pi \to \frac{3\pi}{2}$	$8.82 \to 26.22$
4)	$\frac{3\pi}{2} \to 2\pi$	$22.62 \to 77.88$

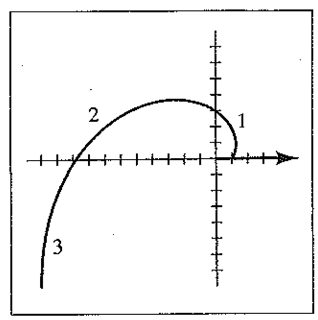

Figure 71

$\boxed{72}$ The values in the table are listed to emphasize the exponential growth. Note that $e^{2\theta} = (e^2)^{\theta}$.

	Variation of θ	Variation of r
1)	$0 \to \frac{\pi}{2}$	$1 \to 23.14$
2)	$\frac{\pi}{2} \to \pi$	$23.14 \to 535$
3)	$\pi \to \frac{3\pi}{2}$	$535 \to 12{,}392$
4)	$\frac{3\pi}{2} \to 2\pi$	$12{,}392 \to 286{,}751$

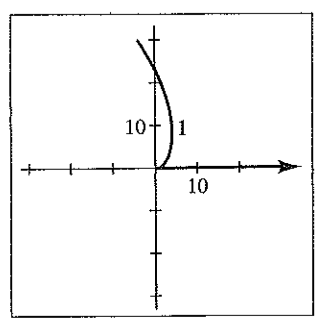

Figure 72

$\boxed{73}$ $r = 2\theta,\ \theta \geq 0$ •

Variation of θ			Variation of r		
1) 0	$\to$	$\frac{\pi}{2}$	0	$\to$	π
2) $\frac{\pi}{2}$	$\to$	π	π	$\to$	2π
3) π	$\to$	$\frac{3\pi}{2}$	2π	$\to$	3π

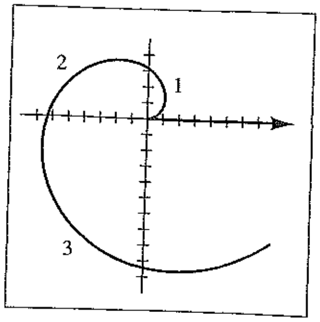

Figure 73

$\boxed{74}$ $r\theta = 1,\ \theta > 0$ (spiral) •

$r\theta = 1 \Rightarrow r = 1/\theta$. r is undefined at $\theta = 0$.

Variation of θ			Variation of r		
1) 0	$\to$	$\frac{\pi}{2}$	$+\infty$	$\to$	0.64
2) $\frac{\pi}{2}$	$\to$	π	0.64	$\to$	0.32
3) π	$\to$	$\frac{3\pi}{2}$	0.32	$\to$	0.21
4) $\frac{3\pi}{2}$	$\to$	2π	0.21	$\to$	0.16

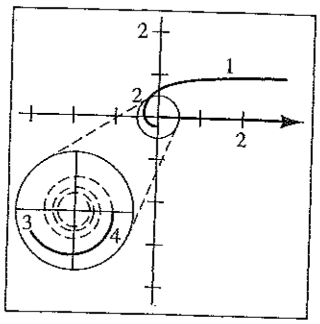

Figure 74

$\boxed{75}$ $r = 6\sin^2\left(\frac{\theta}{2}\right) = 6\left(\frac{1-\cos\theta}{2}\right) = 3(1-\cos\theta)$ is a cardioid.

$0 = 3(1-\cos\theta) \Rightarrow \cos\theta = 1 \Rightarrow \theta = 2\pi n$.

Variation of θ			Variation of r		
1) 0	$\to$	$\frac{\pi}{2}$	0	$\to$	3
2) $\frac{\pi}{2}$	$\to$	π	3	$\to$	6
3) π	$\to$	$\frac{3\pi}{2}$	6	$\to$	3
4) $\frac{3\pi}{2}$	$\to$	2π	3	$\to$	0

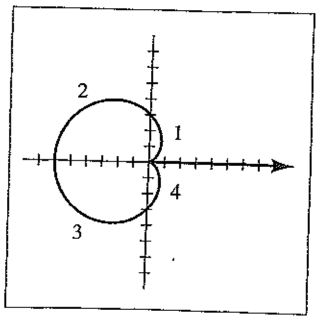

Figure 75

$\boxed{76}$ $r = -4\cos^2\left(\frac{\theta}{2}\right) = -4\left(\frac{1+\cos\theta}{2}\right) = -2(1+\cos\theta)$ is a cardioid. $0 = -2(1+\cos\theta) \Rightarrow \cos\theta = -1 \Rightarrow$ $\theta = \pi + 2\pi n$.

Variation of θ			Variation of r		
1) 0	$\to$	$\frac{\pi}{2}$	-4	$\to$	-2
2) $\frac{\pi}{2}$	$\to$	π	-2	$\to$	0
3) π	$\to$	$\frac{3\pi}{2}$	0	$\to$	-2
4) $\frac{3\pi}{2}$	$\to$	2π	-2	$\to$	-4

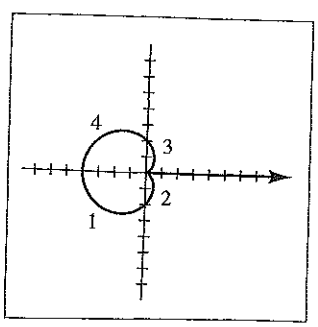

Figure 76

[77] Note that $r = 2\sec\theta$ is equivalent to $x = 2$. If $0 < \theta < \frac{\pi}{2}$ or $\frac{3\pi}{2} < \theta < 2\pi$, then $\sec\theta > 0$ and the graph of $r = 2 + 2\sec\theta$ is to the right of $x = 2$. If $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$, $\sec\theta < 0$ and $r = 2 + 2\sec\theta$ is to the left of $x = 2$. r is undefined at $\theta = \frac{\pi}{2} + \pi n$. $0 = 2 + 2\sec\theta \Rightarrow \sec\theta = -1 \Rightarrow \theta = \pi + 2\pi n$.

	Variation of θ	Variation of r
1)	$0 \rightarrow \frac{\pi}{2}$	$4 \rightarrow \infty$
2)	$\frac{\pi}{2} \rightarrow \pi$	$-\infty \rightarrow 0$
3)	$\pi \rightarrow \frac{3\pi}{2}$	$0 \rightarrow -\infty$
4)	$\frac{3\pi}{2} \rightarrow 2\pi$	$\infty \rightarrow 4$

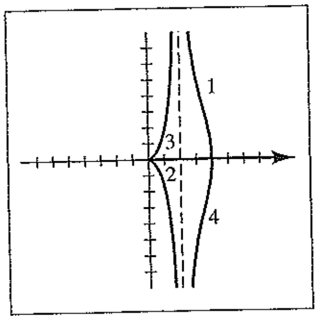

Figure 77

[78] Note that $r = -\csc\theta$ is equivalent to $y = -1$. If $0 < \theta < \pi$, then $\csc\theta > 0$ and the graph of $r = 1 - \csc\theta$ is above $y = -1$. If $\pi < \theta < 2\pi$, $\csc\theta < 0$ and $r = 1 - \csc\theta$ is below $y = -1$. r is undefined at $\theta = \pi n$. $0 = 1 - \csc\theta \Rightarrow \csc\theta = 1 \Rightarrow$ $\theta = \frac{\pi}{2} + 2\pi n$.

	Variation of θ	Variation of r
1)	$0 \rightarrow \frac{\pi}{2}$	$-\infty \rightarrow 0$
2)	$\frac{\pi}{2} \rightarrow \pi$	$0 \rightarrow -\infty$
3)	$\pi \rightarrow \frac{3\pi}{2}$	$\infty \rightarrow 2$
4)	$\frac{3\pi}{2} \rightarrow 2\pi$	$2 \rightarrow \infty$

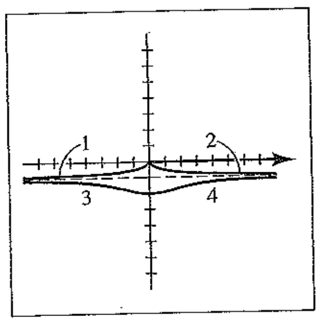

Figure 78

[79] Let $P_1(r_1, \theta_1)$ and $P_2(r_2, \theta_2)$ be points in an $r\theta$-plane. Let $a = r_1$, $b = r_2$, $c = d(P_1, P_2)$, and $\gamma = \theta_2 - \theta_1$. Substituting into the law of cosines, $c^2 = a^2 + b^2 - 2ab\cos\gamma$, gives us the formula: $[d(P_1, P_2)]^2 = r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)$

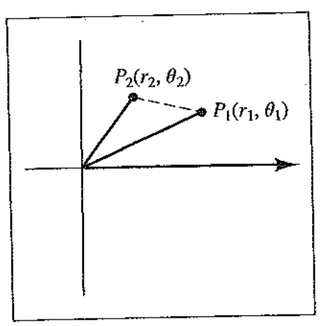

Figure 79

[80] (a) $r = a\sin\theta \Rightarrow r^2 = ar\sin\theta \Rightarrow x^2 + y^2 - ay = 0 \Rightarrow$
$x^2 + y^2 - ay + \frac{1}{4}a^2 = \frac{1}{4}a^2 \Rightarrow x^2 + (y - \frac{1}{2}a)^2 = \frac{1}{4}a^2$. $C(0, \frac{1}{2}a)$; $r = \frac{1}{2}\,|\,a\,|$

(b) $r = b\cos\theta \Rightarrow r^2 = br\cos\theta \Rightarrow x^2 + y^2 - bx = 0 \Rightarrow$
$x^2 + y^2 - bx + \frac{1}{4}b^2 = \frac{1}{4}b^2 \Rightarrow (x - \frac{1}{2}b)^2 + y^2 = \frac{1}{4}b^2$. $C(\frac{1}{2}b, 0)$; $r = \frac{1}{2}\,|\,b\,|$

(c) $r = a\sin\theta + b\cos\theta \Rightarrow r^2 = ar\sin\theta + br\cos\theta \Rightarrow x^2 - bx + y^2 - ay = 0 \Rightarrow$
$(x - \frac{1}{2}b)^2 + (y - \frac{1}{2}a)^2 = \frac{1}{4}b^2 + \frac{1}{4}a^2$. $C(\frac{1}{2}b, \frac{1}{2}a)$; $r = \frac{1}{2}\sqrt{b^2 + a^2}$

[81] (a) $I = \frac{1}{2}I_0[1 + \cos(\pi \sin \theta)] \Rightarrow r = 2.5[1 + \cos(\pi \sin \theta)]$ for $\theta \in [0, 2\pi]$.

(b) The signal is maximum in an east–west direction and minimum in a north–south direction.

$[-9, 9]$ by $[-6, 6]$

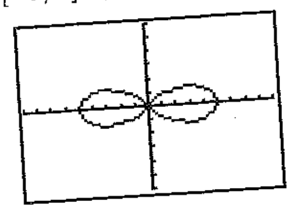

Figure 81

$[-9, 9]$ by $[-6, 6]$

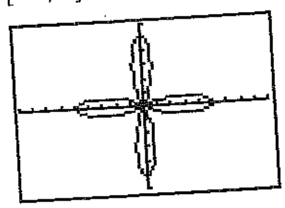

Figure 82

[82] (a) $I = \frac{1}{2}I_0[1 + \cos(\pi \sin 2\theta)] \Rightarrow r = 2.5[1 + \cos(\pi \sin 2\theta)]$ for $\theta \in [0, 2\pi]$.

(b) The signal is maximum in the east, west, north, and south directions. It is minimum in northwest, southeast, northeast, and southwest directions.

[83] Change to "Pol" mode under $\boxed{\text{MODE}}$, assign $2(\sin \theta)^2(\tan \theta)^2$ to r1 under $\boxed{\text{Y =}}$, and $-\pi/3$ to θmin, $\pi/3$ to θmax, and 0.04 to θstep under $\boxed{\text{WINDOW}}$. The graph is symmetric with respect to the polar axis.

$[-9, 9]$ by $[-6, 6]$

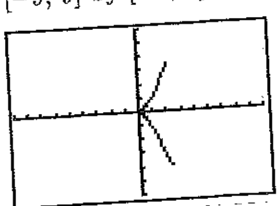

Figure 83

$[-9, 9]$ by $[-6, 6]$

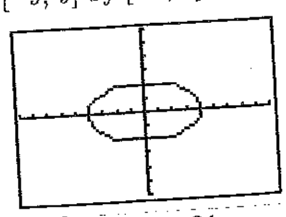

Figure 84

[84] Assign $4/(1 + (\sin \theta)^2)$ to r1, 0 to θmin, 2π to θmax, and $\pi/30$ to θstep. The graph is symmetric with respect to the polar axis, the line $\theta = \frac{\pi}{2}$, and the pole.

[85] Assign $8\cos(3\theta)$ to r1, $4 - 2.5\cos\theta$ to r2, 0 to θmin, 2π to θmax, and $\pi/30$ to θstep. From the graph, there are six points of intersection. The approximate polar coordinates are $(1.75, \pm 0.45)$, $(4.49, \pm 1.77)$, and $(5.76, \pm 2.35)$.

$[-12, 12]$ by $[-9, 9]$

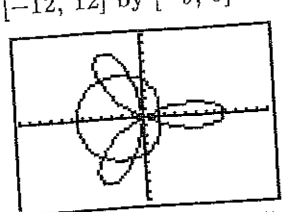

Figure 85

$[-3, 3]$ by $[-2, 2]$

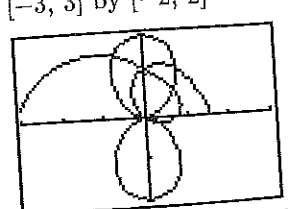

Figure 86

[86] See *Figure 86* on the preceding page. Assign $2(\sin\theta)^2$ to r1, $0.75(\theta + (\cos\theta)^2)$ to r2, $-\pi$ to θmin, π to θmax, and $\pi/30$ to θstep. Be sure to plot r2 for both positive and negative values of θ. From the graph, there are five points of intersection. The approximate polar coordinates are (0, 0), (0.32, −0.41), (0.96, 0.77), (1.39, 0.99), and (1.64, 2.01).

11.6 Exercises

Note: (1) For the ellipse, the major axis is vertical if the denominator contains $\sin\theta$, horizontal if the denominator contains $\cos\theta$.

(2) For the hyperbola, the transverse axis is vertical if the denominator contains $\sin\theta$, horizontal if the denominator contains $\cos\theta$. The focus at the pole is called F and V is the vertex associated with (or closest to) F. $d(V, F)$ denotes the distance from the vertex to the focus. The foci are not asked for in the directions, but are listed.

(3) For the parabola, the directrix is on the right, left, top, or bottom of the focus depending on the term "+cos", "−cos", "+sin", or "−sin", respectively, appearing in the denominator.

[1] Divide the numerator and denominator by the constant term in the denominator, i.e., 6. $r = \dfrac{12}{6 + 2\sin\theta} = \dfrac{2}{1 + \frac{1}{3}\sin\theta} \Rightarrow e = \frac{1}{3} < 1$, ellipse. From the preceding note, we see that the denominator has $\sin\theta$ and we have vertices at $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$.

$V(\frac{3}{2}, \frac{\pi}{2})$ and $V'(3, \frac{3\pi}{2})$. $d(V, F) = \frac{3}{2} \Rightarrow F' = (\frac{3}{2}, \frac{3\pi}{2})$.

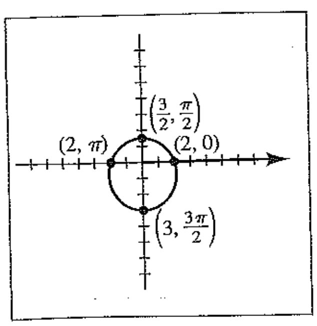

Figure 1

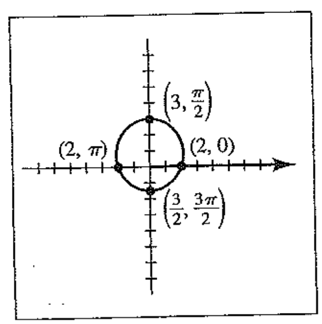

Figure 2

[2] $r = \dfrac{12}{6 - 2\sin\theta} = \dfrac{2}{1 - \frac{1}{3}\sin\theta} \Rightarrow e = \frac{1}{3} < 1$, ellipse.

$V(\frac{3}{2}, \frac{3\pi}{2})$ and $V'(3, \frac{\pi}{2})$. $d(V, F) = \frac{3}{2} \Rightarrow F' = (\frac{3}{2}, \frac{\pi}{2})$.

3 $r = \dfrac{12}{2 - 6\cos\theta} = \dfrac{6}{1 - 3\cos\theta} \;\Rightarrow\; e = 3 > 1$, hyperbola.

$V(\frac{3}{2}, \pi)$ and $V'(-3, 0)$. $d(V, F) = \frac{3}{2} \;\Rightarrow\; F' = (-\frac{9}{2}, 0)$.

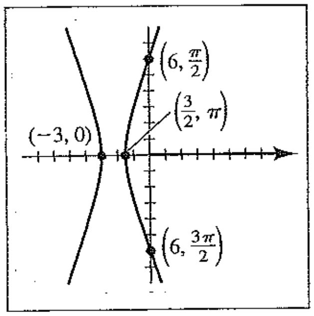

Figure 3

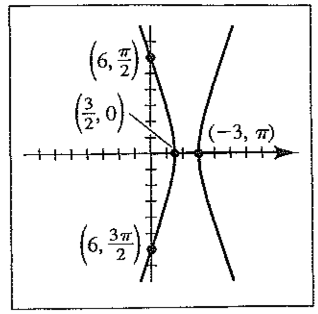

Figure 4

4 $r = \dfrac{12}{2 + 6\cos\theta} = \dfrac{6}{1 + 3\cos\theta} \;\Rightarrow\; e = 3 > 1$, hyperbola.

$V(\frac{3}{2}, 0)$ and $V'(-3, \pi)$. $d(V, F) = \frac{3}{2} \;\Rightarrow\; F' = (-\frac{9}{2}, \pi)$.

5 $r = \dfrac{3}{2 + 2\cos\theta} = \dfrac{\frac{3}{2}}{1 + 1\cos\theta} \;\Rightarrow\; e = 1$, parabola. Note that the expression is undefined in the $\theta = \pi$ direction. The vertex is in the $\theta = 0$ direction, $V(\frac{3}{4}, 0)$.

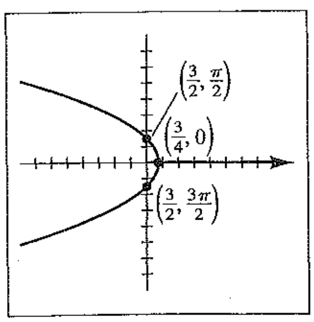

Figure 5

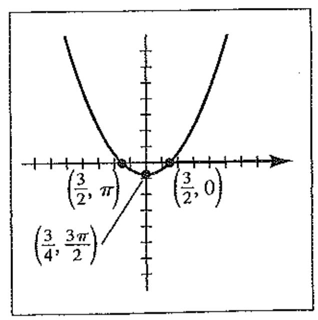

Figure 6

6 $r = \dfrac{3}{2 - 2\sin\theta} = \dfrac{\frac{3}{2}}{1 - 1\sin\theta} \;\Rightarrow\; e = 1$, parabola.

The vertex is in the $\theta = \frac{3\pi}{2}$ direction, $V(\frac{3}{4}, \frac{3\pi}{2})$.

7 $r = \dfrac{4}{\cos\theta - 2} = \dfrac{-2}{1 - \frac{1}{2}\cos\theta} \;\Rightarrow\; e = \frac{1}{2} < 1$, ellipse.

$V(-\frac{4}{3}, \pi)$ and $V'(-4, 0)$. $d(V, F) = \frac{4}{3} \;\Rightarrow\; F' = (-\frac{8}{3}, 0)$

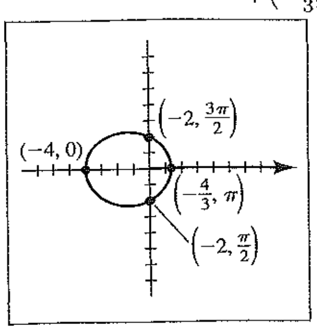

Figure 7

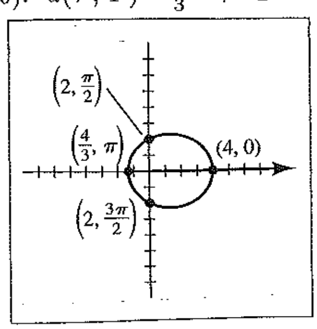

Figure 8

[8] $r = \dfrac{4\sec\theta}{2\sec\theta - 1} \cdot \dfrac{\cos\theta}{\cos\theta} = \dfrac{4}{2 - 1\cos\theta} = \dfrac{2}{1 - \frac{1}{2}\cos\theta} \Rightarrow e = \frac{1}{2} < 1$, ellipse.

$V(\frac{4}{3}, \pi)$ and $V'(4, 0)$. $d(V, F) = \frac{4}{3} \Rightarrow F' = (\frac{8}{3}, 0)$.

Since the original equation is undefined when $\sec\theta$ is undefined, the points

$(2, \frac{\pi}{2})$ and $(2, \frac{3\pi}{2})$ are excluded from the graph. See *Figure 8* on the preceding page.

[9] $r = \dfrac{6\csc\theta}{2\csc\theta + 3} \cdot \dfrac{\sin\theta}{\sin\theta} = \dfrac{6}{2 + 3\sin\theta} = \dfrac{3}{1 + \frac{3}{2}\sin\theta} \Rightarrow e = \frac{3}{2} > 1$, hyperbola.

$V(\frac{6}{5}, \frac{\pi}{2})$ and $V'(-6, \frac{3\pi}{2})$. $d(V, F) = \frac{6}{5} \Rightarrow F' = (-\frac{36}{5}, \frac{3\pi}{2})$.

Since the original equation is undefined when $\csc\theta$ is undefined,

the points $(3, 0)$ and $(3, \pi)$ are excluded from the graph.

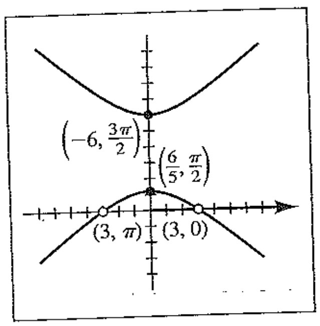

Figure 9

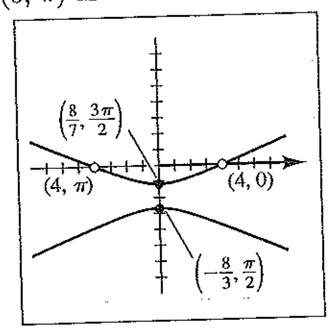

Figure 10

[10] $r = \dfrac{8\csc\theta}{2\csc\theta - 5} \cdot \dfrac{\sin\theta}{\sin\theta} = \dfrac{8}{2 - 5\sin\theta} = \dfrac{4}{1 - \frac{5}{2}\sin\theta} \Rightarrow e = \frac{5}{2} > 1$, hyperbola.

$V(\frac{8}{7}, \frac{3\pi}{2})$ and $V'(-\frac{8}{3}, \frac{\pi}{2})$. $d(V, F) = \frac{8}{7} \Rightarrow F' = (-\frac{80}{21}, \frac{\pi}{2})$.

Since the original equation is undefined when $\csc\theta$ is undefined,

the points $(4, 0)$ and $(4, \pi)$ are excluded from the graph.

[11] $r = \dfrac{4\csc\theta}{1 + \csc\theta} \cdot \dfrac{\sin\theta}{\sin\theta} = \dfrac{4}{1 + 1\sin\theta} \Rightarrow e = 1$, parabola. The vertex is in the $\theta = \frac{\pi}{2}$

direction, $V(2, \frac{\pi}{2})$. Since the original equation is undefined when $\csc\theta$ is undefined,

the points $(4, 0)$ and $(4, \pi)$ are excluded from the graph.

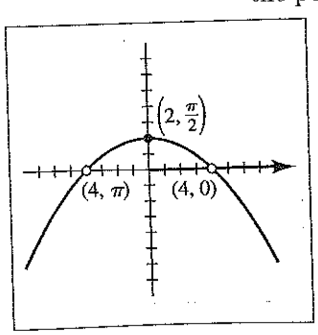

Figure 11

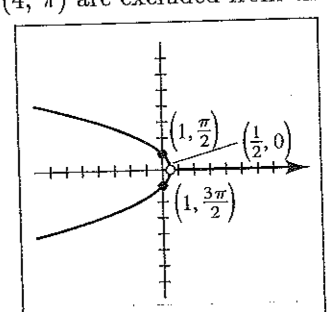

Figure 12

[12] $r = \csc\theta\,(\csc\theta - \cot\theta) = \dfrac{1}{\sin\theta}\left(\dfrac{1 - \cos\theta}{\sin\theta}\right) = \dfrac{1 - \cos\theta}{1 - \cos^2\theta} = \dfrac{1}{1 + 1\cos\theta} \Rightarrow e = 1$,

(continued)

parabola. The vertex is in the $\theta = 0$ direction, $V(\frac{1}{2}, 0)$. Since the original equation is undefined when $\csc\theta$ is undefined, the point $(\frac{1}{2}, 0)$ is excluded from the graph.

Note: For the following exercises, the substitutions $x = r\cos\theta$, $y = r\sin\theta$, and $r^2 = x^2 + y^2$ are made without mention.

$\boxed{13}$ $r = \dfrac{12}{6 + 2\sin\theta} \Rightarrow 6r + 2y = 12 \Rightarrow 3r = 6 - y \Rightarrow 9r^2 = 36 - 12y + y^2 \Rightarrow$
$9x^2 + 8y^2 + 12y - 36 = 0$

$\boxed{14}$ $r = \dfrac{12}{6 - 2\sin\theta} \Rightarrow 6r - 2y = 12 \Rightarrow 3r = y + 6 \Rightarrow 9r^2 = y^2 + 12y + 36 \Rightarrow$
$9x^2 + 8y^2 - 12y - 36 = 0$

$\boxed{15}$ $r = \dfrac{12}{2 - 6\cos\theta} \Rightarrow 2r - 6x = 12 \Rightarrow r = 3x + 6 \Rightarrow r^2 = 9x^2 + 36x + 36 \Rightarrow$
$8x^2 - y^2 + 36x + 36 = 0$

$\boxed{16}$ $r = \dfrac{12}{2 + 6\cos\theta} \Rightarrow 2r + 6x = 12 \Rightarrow r = 6 - 3x \Rightarrow r^2 = 36 - 36x + 9x^2 \Rightarrow$
$8x^2 - y^2 - 36x + 36 = 0$

$\boxed{17}$ $r = \dfrac{3}{2 + 2\cos\theta} \Rightarrow 2r + 2x = 3 \Rightarrow 2r = 3 - 2x \Rightarrow 4r^2 = 4x^2 - 12x + 9 \Rightarrow$
$4y^2 + 12x - 9 = 0$

$\boxed{18}$ $r = \dfrac{3}{2 - 2\sin\theta} \Rightarrow 2r - 2y = 3 \Rightarrow 2r = 2y + 3 \Rightarrow 4r^2 = 4y^2 + 12y + 9 \Rightarrow$
$4x^2 - 12y - 9 = 0$

$\boxed{19}$ $r = \dfrac{4}{\cos\theta - 2} \Rightarrow x - 2r = 4 \Rightarrow x - 4 = 2r \Rightarrow x^2 - 8x + 16 = 4r^2 \Rightarrow$
$3x^2 + 4y^2 + 8x - 16 = 0$

$\boxed{20}$ $r = \dfrac{4\sec\theta}{2\sec\theta - 1} \cdot \dfrac{\cos\theta}{\cos\theta} = \dfrac{4}{2 - 1\cos\theta} \Rightarrow 2r - x = 4 \Rightarrow 2r = x + 4 \Rightarrow$
$4r^2 = x^2 + 8x + 16 \Rightarrow 3x^2 + 4y^2 - 8x - 16 = 0.$
r is undefined when $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. For the rectangular equation, these points correspond to $x = 0$ (or $r\cos\theta = 0$). Substituting $x = 0$ into the above rectangular equation yields $4y^2 = 16$, or $y = \pm 2$. $\therefore$ exclude $(0, \pm 2)$

$\boxed{21}$ $r = \dfrac{6\csc\theta}{2\csc\theta + 3} \cdot \dfrac{\sin\theta}{\sin\theta} = \dfrac{6}{2 + 3\sin\theta} \Rightarrow 2r + 3y = 6 \Rightarrow 2r = 6 - 3y \Rightarrow$
$4r^2 = 36 - 36y + 9y^2 \Rightarrow 4x^2 - 5y^2 + 36y - 36 = 0.$
r is undefined when $\theta = 0$ or π. For the rectangular equation, these points correspond to $y = 0$ (or $r\sin\theta = 0$). Substituting $y = 0$ into the above rectangular equation yields $4x^2 = 36$, or $x = \pm 3$. $\therefore$ exclude $(\pm 3, 0)$

$\boxed{22}$ $r = \dfrac{8\csc\theta}{2\csc\theta - 5} \cdot \dfrac{\sin\theta}{\sin\theta} = \dfrac{8}{2 - 5\sin\theta} \Rightarrow 2r - 5y = 8 \Rightarrow 2r = 5y + 8 \Rightarrow$
$4r^2 = 25y^2 + 80y + 64 \Rightarrow 4x^2 - 21y^2 - 80y - 64 = 0.$
r is undefined when $\theta = 0$ or π. For the rectangular equation, these points correspond to $y = 0$ (or $r\sin\theta = 0$). Substituting $y = 0$ into the above rectangular equation yields $4x^2 = 64$, or $x = \pm 4$. $\therefore$ exclude $(\pm 4, 0)$

$\boxed{23}$ $r = \dfrac{4\csc\theta}{1+\csc\theta}\cdot\dfrac{\sin\theta}{\sin\theta} = \dfrac{4}{1+1\sin\theta} \Rightarrow r+y=4 \Rightarrow r=4-y \Rightarrow$

$r^2 = y^2 - 8y + 16 \Rightarrow x^2 + 8y - 16 = 0$. r is undefined when $\theta = 0$ or π.

For the rectangular equation, these points correspond to $y = 0$ (or $r\sin\theta = 0$).

Substituting $y = 0$ into the above rectangular equation yields $x^2 = 16$, or $x = \pm 4$.

$\therefore$ exclude $(\pm 4, 0)$

$\boxed{24}$ $r = \csc\theta(\csc\theta - \cot\theta) = \dfrac{1}{\sin\theta}\left(\dfrac{1-\cos\theta}{\sin\theta}\right) = \dfrac{1-\cos\theta}{1-\cos^2\theta} = \dfrac{1}{1+\cos\theta} \Rightarrow$

$r + x = 1 \Rightarrow r = 1 - x \Rightarrow r^2 = 1 - 2x + x^2 \Rightarrow y^2 + 2x - 1 = 0$.

r is undefined when $\theta = 0$ or π. For the rectangular equation, this point corresponds to $y = 0$ (or $r\sin\theta = 0$). Substituting $y = 0$ into the above rectangular equation yields $2x - 1 = 0$, or $x = \frac{1}{2}$. $\therefore$ exclude $(\frac{1}{2}, 0)$

$\boxed{25}$ $r = 2\sec\theta \Rightarrow r\cos\theta = 2 \Rightarrow x = 2$. Thus, $d = 2$ and since the directrix is on the right of the focus at the pole, we use "$+\cos\theta$". $r = \dfrac{2(\frac{1}{3})}{1+\frac{1}{3}\cos\theta}\cdot\dfrac{3}{3} = \dfrac{2}{3+\cos\theta}$.

$\boxed{26}$ $r\cos\theta = 5 \Rightarrow x = 5 \Rightarrow d = 5$ and use "$+\cos\theta$". $r = \dfrac{5(1)}{1+1\cos\theta} = \dfrac{5}{1+\cos\theta}$.

$\boxed{27}$ $r\cos\theta = -3 \Rightarrow x = -3$. Thus, $d = 3$ and since the directrix is on the left of the focus at the pole, we use "$-\cos\theta$". $r = \dfrac{3(\frac{4}{3})}{1-\frac{4}{3}\cos\theta}\cdot\dfrac{3}{3} = \dfrac{12}{3-4\cos\theta}$.

$\boxed{28}$ $r = -4\sec\theta \Rightarrow r\cos\theta = -4 \Rightarrow x = -4 \Rightarrow d = 4$ and use "$-\cos\theta$".

$$r = \frac{4(3)}{1-3\cos\theta} = \frac{12}{1-3\cos\theta}.$$

$\boxed{29}$ $r\sin\theta = -2 \Rightarrow y = -2$. Thus, $d = 2$ and since the directrix is under the focus at the pole, we use "$-\sin\theta$". $r = \dfrac{2(1)}{1-1\sin\theta} = \dfrac{2}{1-\sin\theta}$.

$\boxed{30}$ $r = -3\csc\theta \Rightarrow r\sin\theta = -3 \Rightarrow y = -3 \Rightarrow d = 3$ and use "$-\sin\theta$".

$$r = \frac{3(4)}{1-4\sin\theta} = \frac{12}{1-4\sin\theta}.$$

$\boxed{31}$ $r = 4\csc\theta \Rightarrow r\sin\theta = 4 \Rightarrow y = 4$. Thus, $d = 4$ and since the directrix is above the focus at the pole, we use "$+\sin\theta$". $r = \dfrac{4(\frac{2}{5})}{1+\frac{2}{5}\sin\theta}\cdot\dfrac{5}{5} = \dfrac{8}{5+2\sin\theta}$.

$\boxed{32}$ $r\sin\theta = 5 \Rightarrow y = 5 \Rightarrow d = 5$ and use "$+\sin\theta$". $r = \dfrac{5(\frac{3}{4})}{1+\frac{3}{4}\sin\theta}\cdot\dfrac{4}{4} = \dfrac{15}{4+3\sin\theta}$.

$\boxed{33}$ For a parabola, $e = 1$. The vertex is 4 units on top of the focus at the pole so $d = 2(4)$ and we should use "$+\sin\theta$" in the denominator. $r = \dfrac{8}{1+\sin\theta}$

$\boxed{34}$ For a parabola, $e = 1$. The vertex is 5 units to the right of the focus at the pole so $d = 2(5)$ and we should use "$+\cos\theta$" in the denominator. $r = \dfrac{10}{1+\cos\theta}$

$\boxed{35}$ (a) See *Figure 35.* $e = \frac{c}{a} = \dfrac{d(C,\,F)}{d(C,\,V)} = \dfrac{3}{4}$.

(b) Since the vertex is under the focus at the pole, use "$-\sin\theta$".

$$r = \frac{d(\frac{3}{4})}{1 - \frac{3}{4}\sin\theta} \text{ and } r = 1 \text{ when } \theta = \tfrac{3\pi}{2} \Rightarrow 1 = \frac{d(\frac{3}{4})}{1 - \frac{3}{4}(-1)} \Rightarrow 1 = \frac{\frac{3}{4}d}{\frac{7}{4}} \Rightarrow$$

$$d = \tfrac{7}{3}. \text{ Thus, } r = \frac{(\frac{7}{3})(\frac{3}{4})}{1 - \frac{3}{4}\sin\theta}\cdot\frac{4}{4} = \frac{7}{4 - 3\sin\theta}. \left\{\frac{x^2}{7} + \frac{(y-3)^2}{16} = 1\right\}$$

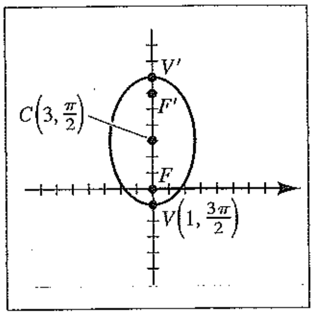

Figure 35

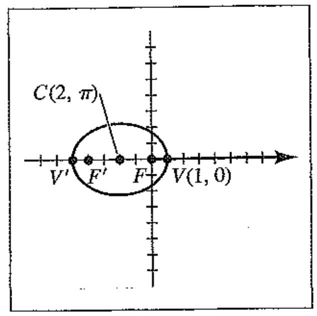

Figure 36

$\boxed{36}$ (a) See *Figure 36.* $e = \frac{c}{a} = \dfrac{d(C,\,F)}{d(C,\,V)} = \dfrac{2}{3}$.

(b) Since the vertex is to the right of the focus at the pole, use "$+\cos\theta$".

$$r = \frac{d(\frac{2}{3})}{1 + \frac{2}{3}\cos\theta} \text{ and } r = 1 \text{ when } \theta = 0 \Rightarrow 1 = \frac{d(\frac{2}{3})}{1 + \frac{2}{3}(1)} \Rightarrow 1 = \frac{\frac{2}{3}d}{\frac{5}{3}} \Rightarrow d = \tfrac{5}{2}.$$

$$\text{Thus, } r = \frac{(\frac{5}{2})(\frac{2}{3})}{1 + \frac{2}{3}\cos\theta}\cdot\frac{3}{3} = \frac{5}{3 + 2\cos\theta}. \left\{\frac{(x+2)^2}{9} + \frac{y^2}{5} = 1\right\}$$

$\boxed{37}$ (a) Let V and C denote the vertex closest to the sun and the center of the ellipse, respectively. Let s denote the distance from V to the directrix to the left of V.

$d(O,\,V) = d(C,\,V) - d(C,\,O) = a - c = a - ea = a(1-e)$.

Also, by the first theorem in §11.6, $\dfrac{d(O,\,V)}{s} = e \Rightarrow s = \dfrac{d(O,\,V)}{e} = \dfrac{a(1-e)}{e}$.

Now, $d = s + d(O,\,V) = \dfrac{a(1-e)}{e} + a(1-e) = \dfrac{a(1-e^2)}{e}$ and $de = a(1-e^2)$.

Thus, the equation of the orbit is $r = \dfrac{(1-e^2)a}{1 - e\cos\theta}$.

(b) The minimum distance occurs when $\theta = \pi$. $r_{\text{per}} = \dfrac{(1-e^2)a}{1-e(-1)} = a(1-e)$.

The maximum distance occurs when $\theta = 0$. $r_{\text{aph}} = \dfrac{(1-e^2)a}{1-e(1)} = a(1+e)$.

$\boxed{38}$ $r = \dfrac{(1-e^2)a}{1-e\cos\theta} = \dfrac{(1+e)[(1-e)a]}{1-e\cos\theta} = \dfrac{(1+0.249)(29.62)}{1-0.249\cos\theta}$ $\{\text{since } r_{\text{per}} = a(1-e)\}$

$\approx \dfrac{37.00}{1-0.249\cos\theta}$ is an equation of Pluto's orbit.

$$r_{\text{aph}} = a(1+e) = \left(\frac{r_{\text{per}}}{1-e}\right)(1+e) = \left(\frac{29.62}{1-0.249}\right)(1+0.249) \approx 49.26 \text{ AU}.$$

$\boxed{39}$ (a) Since $e = 0.9673 < 1$, the orbit of Halley's Comet is elliptical.

(b) The polar equation for the orbit of Saturn is $r = \dfrac{9.006(1+0.056)}{1-0.056\cos\theta}$.

The polar equation for Halley's comet is $r = \dfrac{0.5871(1+0.9673)}{1-0.9673\cos\theta}$.

[−36, 36, 3] by [−24, 24, 3]

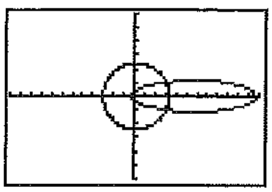

Figure 39

[−18, 18, 3] by [−12, 12, 3]

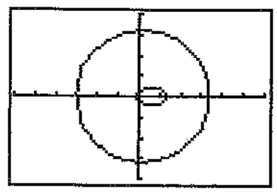

Figure 40

$\boxed{40}$ (a) Since $e = 0.8499 < 1$, the orbit of Encke's Comet is elliptical.

(b) The polar equation for Encke's comet is $r = \dfrac{0.3317(1+0.8499)}{1-0.8499\cos\theta}$.

$\boxed{41}$ (a) Since $e = 1.003 > 1$, the orbit of Comet 1959 III is hyperbolic.

(b) The polar equation for Comet 1959 III is $r = \dfrac{1.251(1+1.003)}{1-1.003\cos\theta}$.

[−18, 18, 3] by [−12, 12, 3]

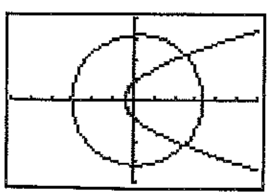

Figure 41

[−18, 18, 3] by [−12, 12, 3]

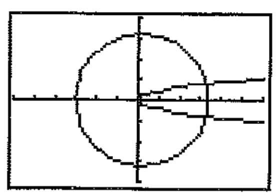

Figure 42

$\boxed{42}$ (a) Since $e = 1.000$, the orbit of Comet 1973.99 is parabolic.

(b) The polar equation for Comet 1973.99 is $r = \dfrac{0.142(1+1.000)}{1-1.000\cos\theta}$.

$\boxed{43}$ From Exercise 37, $r_{\text{aph}} = a(1+e) = a + ae$ and $r_{\text{per}} = a(1-e) = a - ae$.

Adding the two equations gives us $r_{\text{aph}} + r_{\text{per}} = 2a \Rightarrow a = \dfrac{r_{\text{aph}} + r_{\text{per}}}{2}$.

Subtracting the two equations gives us

$$r_{\text{aph}} - r_{\text{per}} = 2ae \Rightarrow e = \frac{r_{\text{aph}} - r_{\text{per}}}{2a} = \frac{r_{\text{aph}} - r_{\text{per}}}{2\cdot\dfrac{r_{\text{aph}} + r_{\text{per}}}{2}} = \frac{r_{\text{aph}} - r_{\text{per}}}{r_{\text{aph}} + r_{\text{per}}}.$$

Chapter 11 Review Exercises

Note: Let the notation be the same as in §11.1–11.3.

$\boxed{1}$ $y^2 = 64x \;\Rightarrow\; x = \frac{1}{64}y^2 \;\Rightarrow\; a = \frac{1}{64}.$ $p = \dfrac{1}{4(\frac{1}{64})} = 16.$ $V(0,\ 0);\ F(16,\ 0);\ l\colon x = -16$

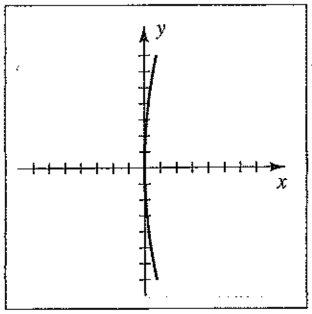

Figure 1

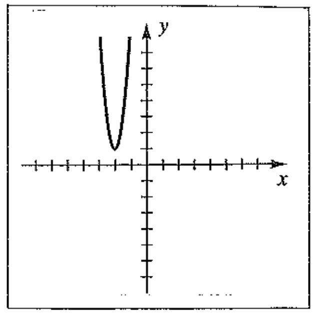

Figure 2

$\boxed{2}$ $y = 8x^2 + 32x + 33 \;\Rightarrow\; a = 8.$ $p = \dfrac{1}{4(8)} = \dfrac{1}{32}.$ $V(-2,\ 1);\ F(-2,\ \frac{33}{32});\ l\colon y = \frac{31}{32}$

$\boxed{3}$ $9y^2 = 144 - 16x^2 \;\Rightarrow\; \dfrac{x^2}{9} + \dfrac{y^2}{16} = 1;\ c^2 = 16 - 9 \;\Rightarrow\; c = \pm\sqrt{7};$

$V(0,\ \pm 4);\ F(0,\ \pm\sqrt{7});\ M(\pm 3,\ 0)$

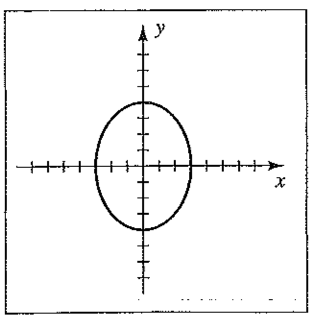

Figure 3

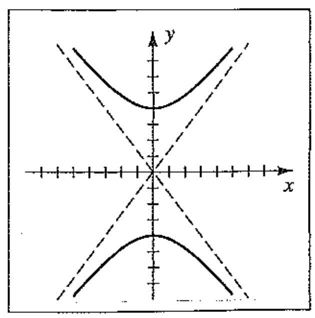

Figure 4

$\boxed{4}$ $9y^2 = 144 + 16x^2 \;\Rightarrow\; \dfrac{y^2}{16} - \dfrac{x^2}{9} = 1;\ c^2 = 16 + 9 \;\Rightarrow\; c = \pm 5;$

$V(0,\ \pm 4);\ F(0,\ \pm 5);\ W(\pm 3,\ 0);\ y = \pm\frac{4}{3}x$

$\boxed{5}$ $x^2 - y^2 - 4 = 0 \;\Rightarrow\; \dfrac{x^2}{4} - \dfrac{y^2}{4} = 1;\ c^2 = 4 + 4 \;\Rightarrow\; c = \pm 2\sqrt{2};$

$V(\pm 2,\ 0);\ F(\pm 2\sqrt{2},\ 0);\ W(0,\ \pm 2);\ y = \pm x$

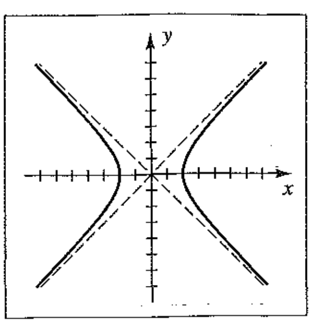

Figure 5

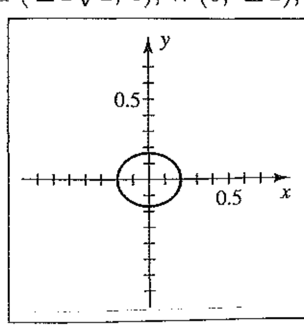

Figure 6

[6] $\dfrac{x^2}{\frac{1}{25}}+\dfrac{y^2}{\frac{1}{36}}=1$; $c^2=\frac{1}{25}-\frac{1}{36} \Rightarrow c=\pm\frac{1}{30}\sqrt{11}$; $V(\pm\frac{1}{5},\ 0)$; $F(\pm\frac{1}{30}\sqrt{11},\ 0)$; $M(0,\ \pm\frac{1}{6})$

See *Figure 6*.

[7] $25y=100-x^2 \Rightarrow y=4-\frac{1}{25}x^2 \Rightarrow a=-\frac{1}{25}$. $p=\dfrac{1}{4(-\frac{1}{25})}=-\dfrac{25}{4}$.

$V(0,\ 4)$; $F(0,\ -\frac{9}{4})$; $l\colon y=\frac{41}{4}$

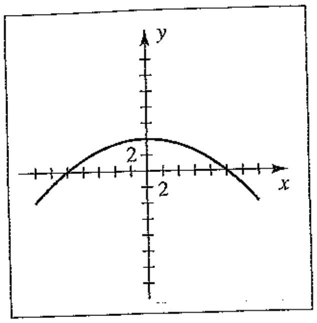

Figure 7

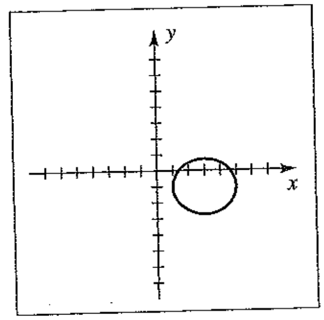

Figure 8

[8] $3x^2+4y^2-18x+8y+19=0 \Rightarrow$

$3(x^2-6x+\underline{\ 9\ })+4(y^2+2y+\underline{\ 1\ })=-19+\underline{\ 27\ }+\underline{\ 4\ } \Rightarrow$

$3(x-3)^2+4(y+1)^2=12 \Rightarrow \dfrac{(x-3)^2}{4}+\dfrac{(y+1)^2}{3}=1$;

$c^2=4-3 \Rightarrow c=\pm 1$; $C(3,\ -1)$; $V(3\pm 2,\ -1)$; $F(3\pm 1,\ -1)$; $M(3,\ -1\pm\sqrt{3})$

[9] $x^2-9y^2+8x+90y-210=0 \Rightarrow$

$(x^2+8x+\underline{\ 16\ })-9(y^2-10y+\underline{\ 25\ })=210+\underline{\ 16\ }-\underline{\ 225\ } \Rightarrow$

$(x+4)^2-9(y-5)^2=1 \Rightarrow \dfrac{(x+4)^2}{1}-\dfrac{(y-5)^2}{\frac{1}{9}}=1$; $c^2=1+\frac{1}{9} \Rightarrow c=\pm\frac{1}{3}\sqrt{10}$;

$C(-4,\ 5)$; $V(-4\pm 1,\ 5)$; $F(-4\pm\frac{1}{3}\sqrt{10},\ 5)$; $W(-4,\ 5\pm\frac{1}{3})$; $(y-5)=\pm\frac{1}{3}(x+4)$

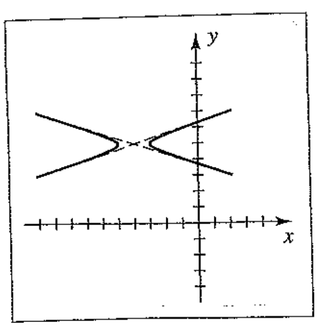

Figure 9

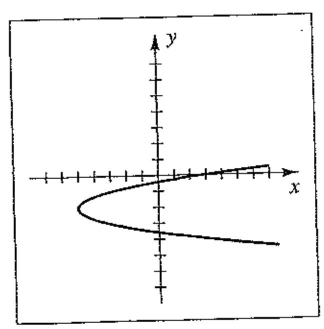

Figure 10

[10] $x=2y^2+8y+3 \Rightarrow a=2$. $p=\dfrac{1}{4(2)}=\dfrac{1}{8}$. $V(-5,\ -2)$; $F(-\frac{39}{8},\ -2)$; $l\colon x=-\frac{41}{8}$

$\boxed{11}$ $4x^2 + 9y^2 + 24x - 36y + 36 = 0 \;\Rightarrow$

$4(x^2 + 6x + \underline{\ 9\ }) + 9(y^2 - 4y + \underline{\ 4\ }) = -36 + \underline{\ 36\ } + \underline{\ 36\ } \;\Rightarrow$

$4(x+3)^2 + 9(y-2)^2 = 36 \;\Rightarrow\; \dfrac{(x+3)^2}{9} + \dfrac{(y-2)^2}{4} = 1;\; c^2 = 9 - 4 \;\Rightarrow\; c = \pm\sqrt{5};$

$C(-3, 2);\; V(-3 \pm 3, 2);\; F(-3 \pm \sqrt{5}, 2);\; M(-3, 2 \pm 2)$

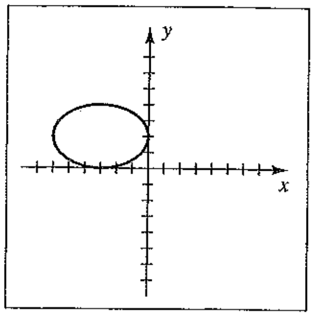

Figure 11

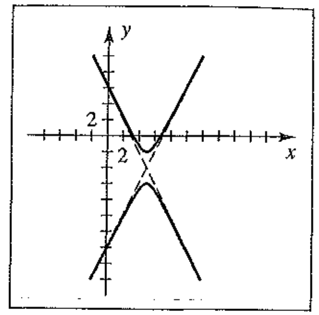

Figure 12

$\boxed{12}$ $4x^2 - y^2 - 40x - 8y + 88 = 0 \;\Rightarrow$

$4(x^2 - 10x + \underline{\ 25\ }) - (y^2 + 8y + \underline{\ 16\ }) = -88 + \underline{\ 100\ } - \underline{\ 16\ } \;\Rightarrow$

$4(x-5)^2 - (y+4)^2 = -4 \;\Rightarrow\; \dfrac{(y+4)^2}{4} - \dfrac{(x-5)^2}{1} = 1;\; c^2 = 4 + 1 \;\Rightarrow\; c = \pm\sqrt{5};$

$C(5, -4);\; V(5, -4 \pm 2);\; F(5, -4 \pm \sqrt{5});\; W(5 \pm 1, -4);\; (y+4) = \pm 2(x-5)$

$\boxed{13}$ $y^2 - 8x + 8y + 32 = 0 \;\Rightarrow\; x = \frac{1}{8}y^2 + y + 4 \;\Rightarrow\; a = \frac{1}{8}.\; p = \dfrac{1}{4(\frac{1}{8})} = 2.$

$V(2, -4);\; F(4, -4);\; l\!: x = 0$

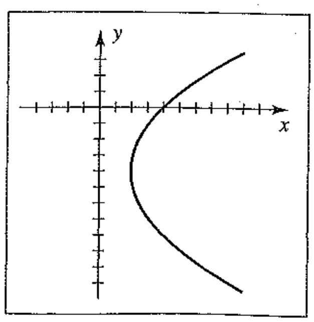

Figure 13

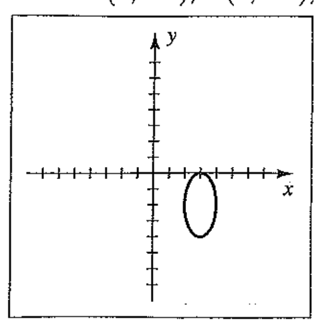

Figure 14

$\boxed{14}$ $4x^2 + y^2 - 24x + 4y + 36 = 0 \;\Rightarrow$

$4(x^2 - 6x + \underline{\ 9\ }) + (y^2 + 4y + \underline{\ 4\ }) = -36 + \underline{\ 36\ } + \underline{\ 4\ } \;\Rightarrow$

$4(x-3)^2 + (y+2)^2 = 4 \;\Rightarrow\; \dfrac{(x-3)^2}{1} + \dfrac{(y+2)^2}{4} = 1;\; c^2 = 4 - 1 \;\Rightarrow\; c = \pm\sqrt{3};$

$C(3, -2);\; V(3, -2 \pm 2);\; F(3, -2 \pm \sqrt{3});\; M(3 \pm 1, -2)$

$\boxed{15}$ $x^2 - 9y^2 + 8x + 7 = 0 \Rightarrow$

$(x^2 + 8x + \underline{\ 16\ }) - 9(y^2) = -7 + \underline{\ 16\ } \Rightarrow (x+4)^2 - 9(y^2) = 9 \Rightarrow$

$\dfrac{(x+4)^2}{9} - \dfrac{y^2}{1} = 1$; $c^2 = 9 + 1 \Rightarrow c = \pm\sqrt{10}$;

$C(-4,\ 0)$; $V(-4 \pm 3,\ 0)$; $F(-4 \pm \sqrt{10},\ 0)$; $W(-4,\ 0 \pm 1)$; $y = \pm\frac{1}{3}(x+4)$

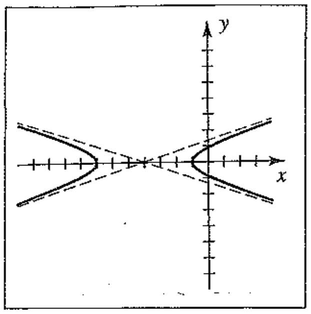

Figure 15

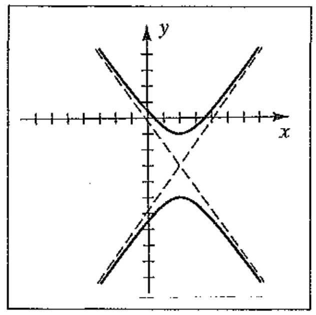

Figure 16

$\boxed{16}$ $y^2 - 2x^2 + 6y + 8x - 3 = 0 \Rightarrow$

$(y^2 + 6y + \underline{\ 9\ }) - 2(x^2 - 4x + \underline{\ 4\ }) = 3 + \underline{\ 9\ } - \underline{\ 8\ } \Rightarrow$

$(y+3)^2 - 2(x-2)^2 = 4 \Rightarrow \dfrac{(y+3)^2}{4} - \dfrac{(x-2)^2}{2} = 1$; $c^2 = 4 + 2 \Rightarrow c = \pm\sqrt{6}$;

$C(2,\ -3)$; $V(2,\ -3 \pm 2)$; $F(2,\ -3 \pm \sqrt{6})$; $W(2 \pm \sqrt{2},\ -3)$; $(y+3) = \pm\sqrt{2}\,(x-2)$

$\boxed{17}$ The vertex is $V(-7,\ k)$. $y = a(x+10)(x+4)$ and $x = 0$, $y = 80 \Rightarrow$

$80 = a(10)(4) \Rightarrow a = 2$. $x = -7 \Rightarrow y = 2(3)(-3) = -18$.

Hence, $y = 2(x+7)^2 - 18$.

$\boxed{18}$ The vertex is $V(-4,\ k)$. $y = a(x+11)(x-3)$ and $x = 2$, $y = 39 \Rightarrow$

$39 = a(13)(-1) \Rightarrow a = -3$. $x = -4 \Rightarrow y = -3(7)(-7) = 147$.

Hence, $y = -3(x+4)^2 + 147$.

$\boxed{19}$ An equation is $\dfrac{y^2}{7^2} - \dfrac{x^2}{3^2} = 1$ or $\dfrac{y^2}{49} - \dfrac{x^2}{9} = 1$.

$\boxed{20}$ $F(-4,\ 0)$ and $l\colon x = 4 \Rightarrow p = -4$ and $V(0,\ 0)$.

An equation is $(y-0)^2 = [4(-4)](x-0)$, or $y^2 = -16x$.

$\boxed{21}$ $F(0,\ -10)$ and $l\colon y = 10 \Rightarrow p = -10$ and $V(0,\ 0)$.

An equation is $(x-0)^2 = [4(-10)](y-0)$, or $x^2 = -40y$.

$\boxed{22}$ The general equation of a parabola that is symmetric to the x-axis and has its vertex at the origin is $x = ay^2$. Substituting $x = 5$ and $y = -1$ into that equation yields

$a = 5$. An equation is $x = 5y^2$.

$\boxed{23}$ $V(0,\ \pm 10)$ and $F(0,\ \pm 5) \Rightarrow b^2 = 10^2 - 5^2 = 75$.

An equation is $\dfrac{x^2}{75} + \dfrac{y^2}{10^2} = 1$ or $\dfrac{x^2}{75} + \dfrac{y^2}{100} = 1$.

$\boxed{24}$ $F(\pm 10,\ 0)$ and $V(\pm 5,\ 0)$ $\Rightarrow$ $b^2 = 10^2 - 5^2 = 75$.

An equation is $\dfrac{x^2}{5^2} - \dfrac{y^2}{75} = 1$ or $\dfrac{x^2}{25} - \dfrac{y^2}{75} = 1$.

$\boxed{25}$ Asymptote equations of $y = \pm 9x$ and $V(0,\ \pm 6)$ $\Rightarrow$ $b = \frac{6}{9} = \frac{2}{3}$.

An equation is $\dfrac{y^2}{6^2} - \dfrac{x^2}{(\frac{2}{3})^2} = 1$ or $\dfrac{y^2}{36} - \dfrac{x^2}{\frac{4}{9}} = 1$.

$\boxed{26}$ $F(\pm 2,\ 0)$ $\Rightarrow$ $c^2 = 4$. Now $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ can be written as $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2 - 4} = 1$ since $b^2 = a^2 - c^2$. Substituting $x = 2$ and $y = \sqrt{2}$ into that equation yields

$\dfrac{4}{a^2} + \dfrac{2}{a^2 - 4} = 1$ $\Rightarrow$ $4a^2 - 16 + 2a^2 = a^4 - 4a^2$ $\Rightarrow$ $a^4 - 10a^2 + 16 = 0$ $\Rightarrow$

$(a^2 - 2)(a^2 - 8) = 0$ $\Rightarrow$ $a^2 = 2, 8$. Since $a > c$, a^2 must be 8 and b^2 is equal to 4.

An equation is $\dfrac{x^2}{8} + \dfrac{y^2}{4} = 1$.

$\boxed{27}$ $M(\pm 5,\ 0)$ $\Rightarrow$ $b = 5$. $e = \frac{c}{a} = \dfrac{\sqrt{a^2 - b^2}}{a} = \dfrac{\sqrt{a^2 - 25}}{a} = \dfrac{2}{3}$ $\Rightarrow$ $\frac{2}{3}a = \sqrt{a^2 - 25}$ $\Rightarrow$

$\frac{4}{9}a^2 = a^2 - 25$ $\Rightarrow$ $\frac{5}{9}a^2 = 25$ $\Rightarrow$ $a^2 = 45$. An equation is $\dfrac{x^2}{25} + \dfrac{y^2}{45} = 1$.

$\boxed{28}$ $F(\pm 12,\ 0)$ $\Rightarrow$ $c = 12$. $e = \frac{c}{a} = \frac{12}{a} = \frac{3}{4}$ $\Rightarrow$ $a = 16$.

$b^2 = a^2 - c^2 = 16^2 - 12^2 = 112$. An equation is $\dfrac{x^2}{256} + \dfrac{y^2}{112} = 1$.

$\boxed{29}$ (a) Substituting $x = 2$ and $y = -3$ in $Ax^2 + 2y^2 = 4$ $\Rightarrow$ $A = -\frac{7}{2}$.

(b) The equation is $-\frac{7}{2}x^2 + 2y^2 = 4$ or $\dfrac{y^2}{2} - \dfrac{7x^2}{8} = 1$, a hyperbola.

$\boxed{30}$ The vertex of the square in the first quadrant has coordinates $(x,\ x)$.

Since it is on the ellipse, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ $\Rightarrow$ $b^2x^2 + a^2x^2 = a^2b^2$ $\Rightarrow$ $x^2 = \dfrac{a^2b^2}{a^2 + b^2}$.

x^2 is $\frac{1}{4}$ of the area of the square, hence $A = \dfrac{4a^2b^2}{a^2 + b^2}$.

$\boxed{31}$ The focus is a distance of $p = 1/(4a) = 1/(4 \cdot \frac{1}{8}) = 2$ units from the origin.

The equation of the circle is $x^2 + (y - 2)^2 = 2^2 = 4$.

$\boxed{32}$ $y = \frac{1}{64}\omega^2x^2 + k$ $\Rightarrow$ $x^2 = \dfrac{64}{\omega^2}(y - k)$ $\Rightarrow$ $4p = \dfrac{64}{\omega^2}$ $\Rightarrow$ $p = \dfrac{16}{\omega^2} = 2$ $\Rightarrow$

$\omega = 2\sqrt{2}$ rad/sec ≈ 0.45 rev/sec.

$\boxed{33}$ $y = t - 1 \Rightarrow t = y + 1$. $x = 3 + 4t = 3 + 4(y + 1) = 4y + 7$.

As t varies from -2 to 2, (x, y) varies from $(-5, -3)$ to $(11, 1)$.

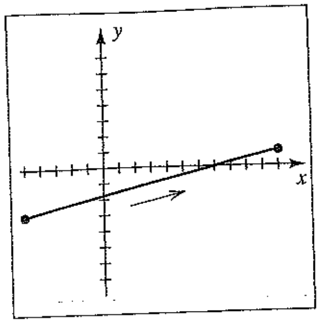

Figure 33

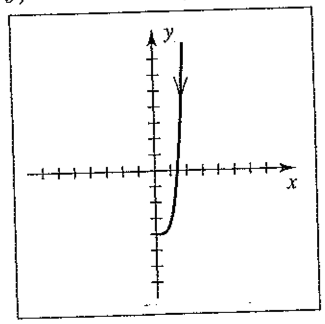

Figure 34

$\boxed{34}$ $x = \sqrt{-t} \Rightarrow t = -x^2$. $y = t^2 - 4 = x^4 - 4$. As t varies from $-\infty$ to 0, x varies from ∞ to 0 and the graph is the right half of the quartic.

$\boxed{35}$ $x = \cos^2 t - 2 \Rightarrow x + 2 = \cos^2 t$; $y = \sin t + 1 \Rightarrow (y - 1)^2 = \sin^2 t$.

$\sin^2 t + \cos^2 t = 1 = x + 2 + (y - 1)^2 \Rightarrow (y - 1)^2 = -(x + 1)$.

This is a parabola with vertex at $(-1, 1)$ and opening to the left. $t = 0$ corresponds to the vertex and as t varies from 0 to 2π, the point (x, y) moves to $(-2, 2)$ at $t = \frac{\pi}{2}$, back to the vertex at $t = \pi$, down to $(-2, 0)$ at $t = \frac{3\pi}{2}$, and finishes at the vertex at $t = 2\pi$.

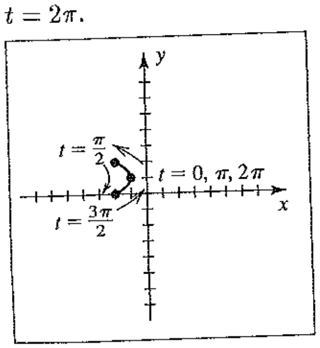

Figure 35

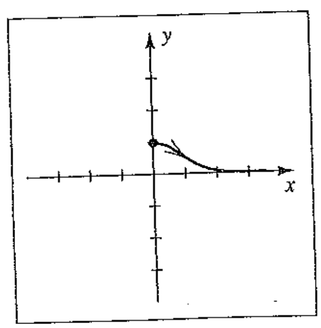

Figure 36

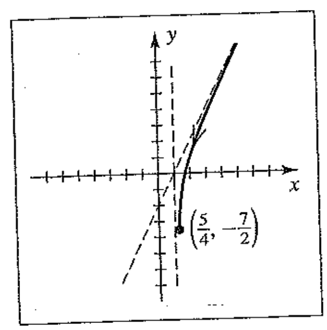

Figure 37

$\boxed{36}$ $x = \sqrt{t} \Rightarrow x^2 = t$ and $y = 2^{-x^2}$. The graph is a bell-shaped curve with a maximum point at $t = 0$, or $(0, 1)$. As t increases, x increases, and y gets close to 0.

$\boxed{37}$ $x = \frac{1}{t} + 1 \Rightarrow t = \frac{1}{x - 1}$ and $y = 2(x - 1) - \left(\frac{1}{x - 1}\right) = \frac{2(x^2 - 2x + 1) - 1}{x - 1} = \frac{2x^2 - 4x + 1}{x - 1}$. This is a rational function with a vertical asymptote at $x = 1$ and an oblique asymptote of $y = 2x - 2$. The graph has a minimum point at $(\frac{5}{4}, -\frac{7}{2})$ when $t = 4$ and then approaches the oblique asymptote as t approaches 0. See *Figure 37*.

$\boxed{38}$ All of the curves are a portion of the circle $x^2 + y^2 = 16$.

C_1: $y = \sqrt{16 - t^2} = \sqrt{16 - x^2}$.

Since $y = \sqrt{16 - t^2}$, y must be nonnegative and we have the top half of the circle.

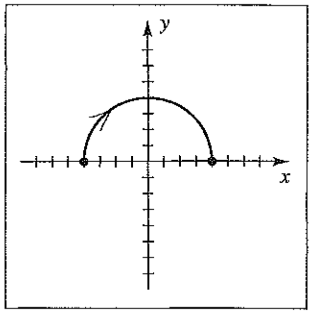

Figure 38 (C_1)

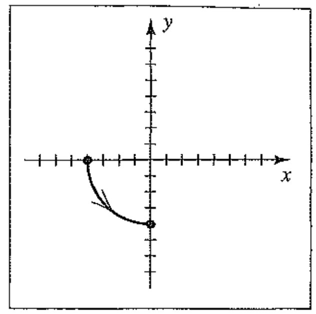

Figure 38 (C_2)

C_2: $x = -\sqrt{16 - t} = -\sqrt{16 - (-\sqrt{t})^2} = -\sqrt{16 - y^2}$.

This is the left half of the circle. Since $y = -\sqrt{t}$, y can only be nonpositive.

Hence we have only the third quadrant portion of the circle.

C_3: $x = 4\cos t$, $y = 4\sin t$ $\Rightarrow$ $\frac{x}{4} = \cos t$, $\frac{y}{4} = \sin t$ $\Rightarrow$ $\frac{x^2}{16} = \cos^2 t$, $\frac{y^2}{16} = \sin^2 t$ $\Rightarrow$

$\frac{x^2}{16} + \frac{y^2}{16} = \cos^2 t + \sin^2 t = 1$ $\Rightarrow$ $x^2 + y^2 = 16$. This is the entire circle.

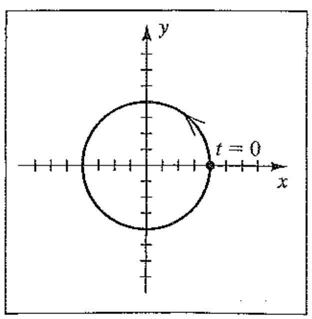

Figure 38 (C_3)

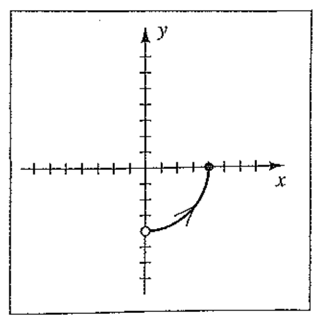

Figure 38 (C_4)

C_4: $y = -\sqrt{16 - e^{2t}} = -\sqrt{16 - (e^t)^2} = -\sqrt{16 - x^2}$.

This is the bottom half of the circle. Since e^t is positive, x takes on all positive

real values. Note that $(0, -4)$ is <u>not</u> included on the graph since $x \neq 0$.

$\boxed{39}$ $x = 1024(\sqrt{3}/2)t$ and $y = -\frac{1}{2}(32)t^2 + 1024(\frac{1}{2})t + 5120$ $\Rightarrow$

$$x = 512\sqrt{3}\,t \quad \text{and} \quad y = -16t^2 + 512t + 5120.$$

$y = 0$ $\Rightarrow$ $-16t^2 + 512t + 5120 = 0$ $\Rightarrow$ $t^2 - 32t - 320 = 0$ $\Rightarrow$

$(t - 40)(t + 8) = 0$ $\Rightarrow$ $t = 40$ seconds. $x = 512\sqrt{3}(40) = 20{,}480\sqrt{3} \approx 35{,}472.40$ feet.

$y = 5120$ $\Rightarrow$ $-16t^2 + 512t + 5120 = 5120$ $\Rightarrow$ $-16t^2 + 512t = 0$ $\Rightarrow$

$-16t(t - 32) = 0$ $\Rightarrow$ $t = 0, 32$. The maximum altitude occurs when

$t = \frac{1}{2}(32) = 16$ and has value $y = -16(16)^2 + 512(16) + 5120 = 9216$ feet.

[40] Two polar coordinate points that represent the same point as $(2, \pi/4)$ are $(-2, 5\pi/4)$ {negative r and opposite direction} and $(2, 9\pi/4)$ {same r with θ increased by 2π}.

[41] $x = r\cos\theta = 5\cos\frac{7\pi}{4} = 5\left(\frac{\sqrt{2}}{2}\right) = \frac{5}{2}\sqrt{2}$. $y = r\sin\theta = 5\sin\frac{7\pi}{4} = 5\left(-\frac{\sqrt{2}}{2}\right) = -\frac{5}{2}\sqrt{2}$.

[42] $r^2 = x^2 + y^2 = (2\sqrt{3})^2 + (-2)^2 = 16 \Rightarrow r = 4$.

$$\tan\theta = \frac{y}{x} = \frac{-2}{2\sqrt{3}} = -\frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{11\pi}{6} \ \{\theta \text{ in QIV}\}.$$

[43] $y^2 = 4x \Rightarrow r^2\sin^2\theta = 4r\cos\theta \Rightarrow$

$$r = \frac{4r\cos\theta}{r\sin^2\theta} = 4\cdot\frac{\cos\theta}{\sin\theta}\cdot\frac{1}{\sin\theta} \Rightarrow r = 4\cot\theta\csc\theta.$$

[44] $x^2 + y^2 - 3x + 4y = 0 \Rightarrow r^2 - 3r\cos\theta + 4r\sin\theta = 0 \Rightarrow$

$$r - 3\cos\theta + 4\sin\theta = 0 \Rightarrow r = 3\cos\theta - 4\sin\theta.$$

[45] $2x - 3y = 8 \Rightarrow 2r\cos\theta - 3r\sin\theta = 8 \Rightarrow r(2\cos\theta - 3\sin\theta) = 8$.

[46] $x^2 + y^2 = 2xy \Rightarrow r^2 = 2r^2\cos\theta\sin\theta \Rightarrow 1 = 2\sin\theta\cos\theta \Rightarrow \sin 2\theta = 1 \Rightarrow$

$2\theta = \frac{\pi}{2} + 2\pi\text{n} \Rightarrow \theta = \frac{\pi}{4}, \frac{5\pi}{4}$ on $[0, 2\pi)$, which are the same lines.

In rectangular coordinates: $x^2 + y^2 = 2xy \Rightarrow x^2 - 2xy + y^2 = 0 \Rightarrow$

$(x-y)^2 = 0 \Rightarrow x - y = 0$, or $y = x$.

[47] $r^2 = \tan\theta \Rightarrow x^2 + y^2 = \frac{y}{x} \Rightarrow x^3 + xy^2 = y$.

[48] $r = 2\cos\theta + 3\sin\theta \Rightarrow r^2 = 2r\cos\theta + 3r\sin\theta \Rightarrow x^2 + y^2 = 2x + 3y$.

[49] $r^2 = 4\sin 2\theta \Rightarrow r^2 = 4(2\sin\theta\cos\theta) \Rightarrow r^2 = 8\sin\theta\cos\theta \Rightarrow$

$$r^2\cdot r^2 = 8(r\sin\theta)(r\cos\theta) \Rightarrow (x^2 + y^2)^2 = 8xy.$$

[50] $\theta = \sqrt{3} \Rightarrow \tan^{-1}(\frac{y}{x}) = \sqrt{3} \Rightarrow \frac{y}{x} = \tan\sqrt{3} \Rightarrow y = (\tan\sqrt{3})\,x$.

Note that $\tan\sqrt{3} \approx -6.15$. This is a line through the origin making an angle of approximately 99.24° with the positive x-axis. The line is <u>not</u> $y = \frac{\pi}{3}x$.

[51] $r = 5\sec\theta + 3r\sec\theta \Rightarrow r\cos\theta = 5 + 3r \Rightarrow x - 5 = 3r \Rightarrow$

$$x^2 - 10x + 25 = 9r^2 \Rightarrow x^2 - 10x + 25 = 9x^2 + 9y^2 \Rightarrow 8x^2 + 9y^2 + 10x - 25 = 0$$

[52] $r^2\sin\theta = 6\csc\theta + r\cot\theta \Rightarrow$

$r^2\sin^2\theta = 6 + r\cos\theta$ {multiply by $\sin\theta$ to get $r^2\sin^2\theta$} $\Rightarrow y^2 = 6 + x$

[53] $r = -4\sin\theta \Rightarrow r^2 = -4r\sin\theta \Rightarrow$

$x^2 + y^2 = -4y \Rightarrow x^2 + y^2 + 4y + \underline{4} = \underline{4} \Rightarrow$

$x^2 + (y+2)^2 = 4$.

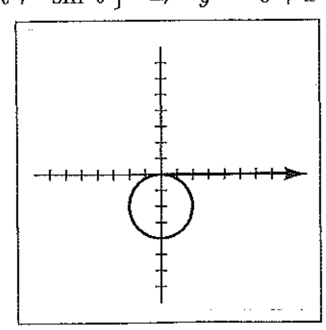

Figure 53

$\boxed{54}$ $r = 8\sec\theta \;\Rightarrow\; r\cos\theta = 8 \;\Rightarrow\; x = 8.$

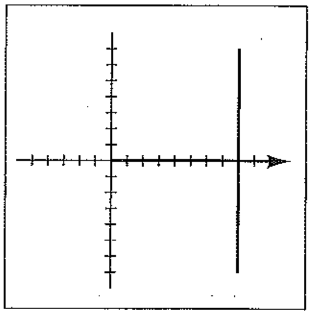

Figure 54

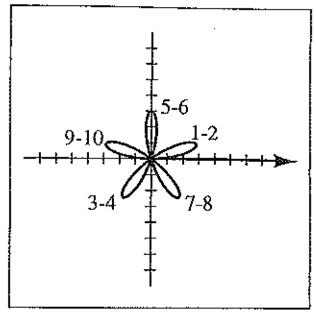

Figure 55

$\boxed{55}$ $r = 3\sin 5\theta$ is a 5-leafed rose. $0 = 3\sin 5\theta \;\Rightarrow\; \sin 5\theta = 0 \;\Rightarrow\; 5\theta = \pi n \;\Rightarrow\; \theta = \frac{\pi}{5}n.$

The numbers 1–10 correspond to θ ranging from 0 to π in $\frac{\pi}{10}$ increments.

$\boxed{56}$ $r = 6 - 3\cos\theta$ •

$0 = 6 - 3\cos\theta \;\Rightarrow\; \cos\theta = 2 \;\Rightarrow\;$ no pole values.

	Variation of θ	Variation of r
1)	$0 \to \frac{\pi}{2}$	$3 \to 6$
2)	$\frac{\pi}{2} \to \pi$	$6 \to 9$
3)	$\pi \to \frac{3\pi}{2}$	$9 \to 6$
4)	$\frac{3\pi}{2} \to 2\pi$	$6 \to 3$

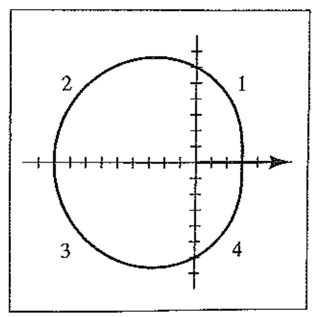

Figure 56

$\boxed{57}$ $r = 3 - 3\sin\theta$ is a cardioid since the coefficient of $\sin\theta$ has the same magnitude as the constant term.

$0 = 3 - 3\sin\theta \;\Rightarrow\; \sin\theta = 1 \;\Rightarrow\; \theta = \frac{\pi}{2} + 2\pi n.$

	Variation of θ	Variation of r
1)	$0 \to \frac{\pi}{2}$	$3 \to 0$
2)	$\frac{\pi}{2} \to \pi$	$0 \to 3$
3)	$\pi \to \frac{3\pi}{2}$	$3 \to 6$
4)	$\frac{3\pi}{2} \to 2\pi$	$6 \to 3$

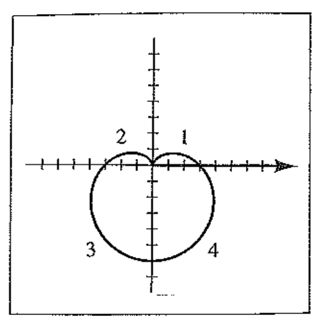

Figure 57

$\boxed{58}$ $r = 2 + 4\cos\theta$ is a limaçon with a loop.

$0 = 2 + 4\cos\theta \;\Rightarrow\; \cos\theta = -\frac{1}{2} \;\Rightarrow$

$\theta = \frac{2\pi}{3} + 2\pi n,\ \frac{4\pi}{3} + 2\pi n.$

	Variation of θ			Variation of r
1)	0	$\rightarrow$	$\frac{\pi}{2}$	$6 \rightarrow \ \ 2$
2)	$\frac{\pi}{2}$	$\rightarrow$	$\frac{2\pi}{3}$	$2 \rightarrow \ \ 0$
3)	$\frac{2\pi}{3}$	$\rightarrow$	π	$0 \rightarrow -2$
4)	π	$\rightarrow$	$\frac{4\pi}{3}$	$-2 \rightarrow \ \ 0$
5)	$\frac{4\pi}{3}$	$\rightarrow$	$\frac{3\pi}{2}$	$0 \rightarrow \ \ 2$
6)	$\frac{3\pi}{2}$	$\rightarrow$	2π	$2 \rightarrow \ \ 6$

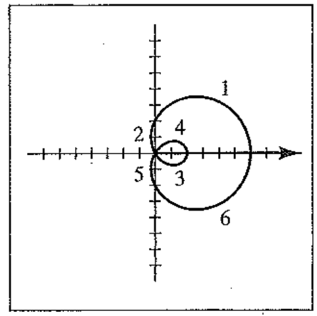

Figure 58

$\boxed{59}$ $r^2 = 9\sin 2\theta$

$0 = 9\sin 2\theta \;\Rightarrow\; \sin 2\theta = 0 \;\Rightarrow\; 2\theta = \pi n \;\Rightarrow\; \theta = \frac{\pi}{2}n.$

	Variation of θ			Variation of r
1)	0	$\rightarrow$	$\frac{\pi}{4}$	$0 \rightarrow \ \pm 3$
2)	$\frac{\pi}{4}$	$\rightarrow$	$\frac{\pi}{2}$	$\pm 3 \rightarrow \ \ 0$
3)	$\frac{\pi}{2}$	$\rightarrow$	$\frac{3\pi}{4}$	undefined
4)	$\frac{3\pi}{4}$	$\rightarrow$	π	undefined

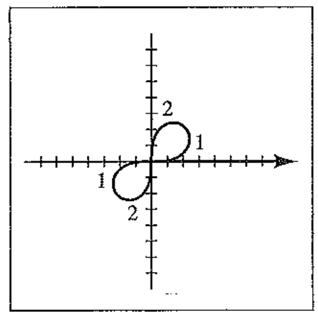

Figure 59

$\boxed{60}$ $2r = \theta \;\Rightarrow\; r = \frac{1}{2}\theta$. Positive values of θ yield the "counterclockwise spiral" while the "clockwise spiral" is obtained from the negative values of θ.

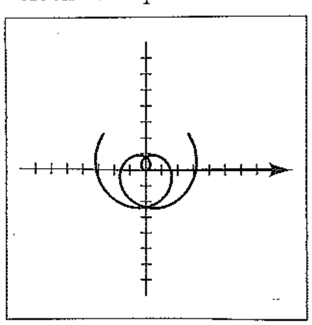
Figure 60

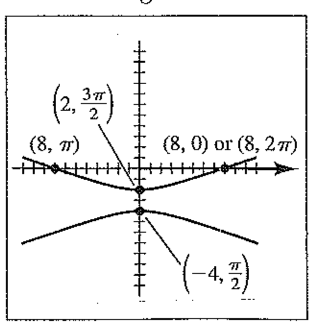

Figure 61

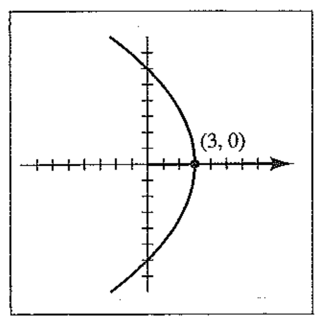

Figure 62

$\boxed{61}$ $r = \dfrac{8}{1 - 3\sin\theta} \;\Rightarrow\; e = 3 > 1$, hyperbola. See §11.6 for more details on this problem.

$V(2, \frac{3\pi}{2})$ and $V'(-4, \frac{\pi}{2})$. $d(V, F) = 2 \;\Rightarrow\; F'(-6, \frac{\pi}{2})$.

$\boxed{62}$ $r = 6 - r\cos\theta \;\Rightarrow\; r + r\cos\theta = 6 \;\Rightarrow\; r(1 + \cos\theta) = 6 \;\Rightarrow\; r = \dfrac{6}{1 + 1\cos\theta} \;\Rightarrow$

$e = 1$, parabola. The vertex is in the $\theta = 0$ direction, $V(3, 0)$.

[63] $r = \dfrac{6}{3+2\cos\theta} = \dfrac{2}{1+\frac{2}{3}\cos\theta} \Rightarrow e = \frac{2}{3} < 1$, ellipse.

$V(\frac{6}{5}, 0)$ and $V'(6, \pi)$. $d(V, F) = \frac{6}{5} \Rightarrow F' = (\frac{24}{5}, \pi)$.

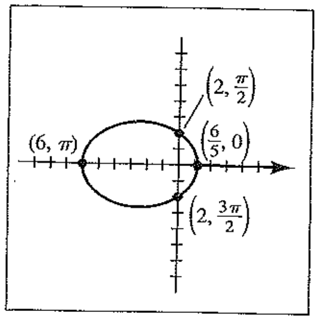

Figure 63

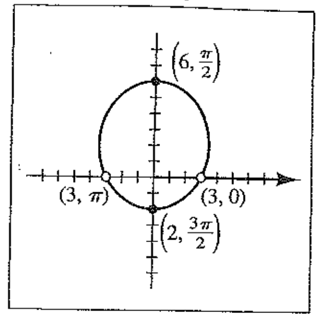

Figure 64

[64] $r = \dfrac{-6\csc\theta}{1-2\csc\theta} \cdot \dfrac{-\sin\theta}{-\sin\theta} = \dfrac{6}{2-\sin\theta} = \dfrac{3}{1-\frac{1}{2}\sin\theta} \Rightarrow e = \frac{1}{2} < 1$, ellipse.

$V(2, \frac{3\pi}{2})$ and $V'(6, \frac{\pi}{2})$. $d(V, F) = 2 \Rightarrow F' = (4, \frac{\pi}{2})$.

Since the original equation is undefined when $\csc\theta$ is undefined,

the points $(3, 0)$ and $(3, \pi)$ are excluded from the graph.

Chapter 11 Discussion Exercises

[1] For $y = ax^2$, the horizontal line through the focus is $y = p$. Since $a = 1/(4p)$, we have $p = (1/(4p))x^2 \Rightarrow x^2 = 4p^2 \Rightarrow x = 2|p|$. Doubling this value for the width gives us $w = 4|p|$.

[2] The circle goes through both foci and all four vertices of the auxiliary rectangle.

[3] Refer to Figure 2 and the derivation on text page 827.

[4] Refer to Figure 1 and the derivation on text page 840.

[5] $P(x, y)$ is a distance of $(2+d)$ from $(0, 0)$ and a distance of d from $(4, 0)$. The difference of these distances is $(2+d) - d = 2$, a positive constant. By the definition of a hyperbola, $P(x, y)$ lies on the right branch of the hyperbola with foci $(0, 0)$ and $(4, 0)$. The center of the hyperbola is halfway between the foci, that is, $(2, 0)$. The vertex is halfway from $(2, 0)$ to $(4, 0)$ since the distance from the circle to P equals the distance from P to $(4, 0)$.

Thus, the vertex is $(3, 0)$ and $a = 1$.

$b^2 = c^2 - a^2 = 2^2 - 1^2 = 3$ and

an equation of the right branch of the hyperbola is

$\dfrac{(x-2)^2}{1} - \dfrac{y^2}{3} = 1$, $x \geq 3$ or $x = 2 + \sqrt{1 + \dfrac{y^2}{3}}$.

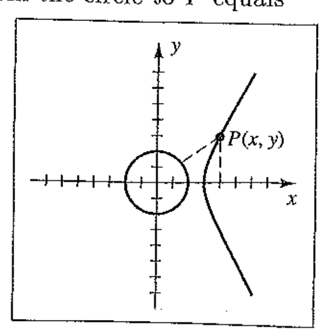

Figure 5

[6] $y = \frac{a}{b}\sqrt{x^2+b^2} \Rightarrow y^2 = \frac{a^2}{b^2}(x^2+b^2) \Rightarrow b^2y^2 = a^2x^2 + a^2b^2 \Rightarrow$

$b^2y^2 - a^2x^2 = a^2b^2 \Rightarrow \frac{b^2y^2}{a^2b^2} - \frac{a^2x^2}{a^2b^2} = 1 \Rightarrow \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$. This is an equation of a hyperbola with one focus at $c = \sqrt{a^2+b^2}$. The equation of the parabola is $y = dx^2$, so its focus is at $p = 1/(4d)$ and we have $d = \frac{1}{4p} = \frac{1}{4\sqrt{a^2+b^2}}$ {since $p = c$}.

[7] Solve $y = -16t^2 + 1024(\sin\alpha)t + 2304$ for t when $y = 0$. This gives $t_1 = 32\sin\alpha + 4\sqrt{64\sin^2\alpha + 9}$. The range r is given by $r = 1024(\cos\alpha)t_1$. Table r as a function of α to find that the maximum value of r occurs when $\alpha \approx 0.752528$ radians, or about 43.12°.

[8] (a) The projectile strikes the ground when $y = 0 \Rightarrow -\frac{1}{2}gt^2 + (s\sin\alpha)t = 0 \Rightarrow$

$t(-\frac{1}{2}gt + s\sin\alpha) = 0 \Rightarrow t = 0$ or $-\frac{1}{2}gt + s\sin\alpha = 0$, which gives us $t = \frac{2s\sin\alpha}{g}$.

(b) Use the answer from part (a) to get

$x = (s\cos\alpha)t = (s\cos\alpha)\frac{2s\sin\alpha}{g} = \frac{s^2(2\cos\alpha\sin\alpha)}{g}$, so $r = \frac{s^2\sin 2\alpha}{g}$.

(c) From part (b), $r = \frac{s^2\sin 2\alpha}{g}$ is a maximum when $\sin 2\alpha$ is a maximum; that is, when $\sin 2\alpha = 1 \Rightarrow 2\alpha = 90° \Rightarrow \alpha = 45°$.

(d) $x = (s\cos\alpha)t \Rightarrow t = \frac{x}{s\cos\alpha}$, so $y = -\frac{1}{2}g\left(\frac{x}{s\cos\alpha}\right)^2 + (s\sin\alpha)\left(\frac{x}{s\cos\alpha}\right)$, or

$$y = -\frac{g}{2s^2\cos^2\alpha}x^2 + (\tan\alpha)x.$$

(e) The time at which the maximum height is reached is one-half the time needed for the range; that is, one-half of the answer in part (a). So

$$t = \frac{1}{2}\left(\frac{2s\sin\alpha}{g}\right) = \frac{s\sin\alpha}{g}.$$

(f) Substitute the answer from part (e) for t in

$$\begin{aligned} y &= -\tfrac{1}{2}gt^2 + (s\sin\alpha)t \\ &= -\tfrac{1}{2}g\left(\frac{s\sin\alpha}{g}\right)^2 + (s\sin\alpha)\left(\frac{s\sin\alpha}{g}\right) \\ &= -\frac{1}{2}g\left(\frac{s^2\sin^2\alpha}{g^2}\right) + \frac{s^2\sin^2\alpha}{g} \\ &= -\frac{1}{2}\left(\frac{s^2\sin^2\alpha}{g}\right) + \frac{s^2\sin^2\alpha}{g} = \frac{s^2\sin^2\alpha}{2g}. \end{aligned}$$

$\boxed{9}$

$$
\begin{aligned}
x &= \sin 2t && \text{given} \\
x &= 2\sin t\,\cos t && \text{double-angle formula} \\
x^2 &= 4\sin^2 t\,\cos^2 t && \text{square both sides} \\
x^2 &= 4(1-\cos^2 t)\cos^2 t && \text{Pythagorean identity} \\
x^2 &= 4(1-y^2)y^2 && y = \cos t \\
4y^4 - 4y^2 + x^2 &= 0 && \text{equivalent equation} \\
y^2 &= \frac{4 \pm \sqrt{16-16x^2}}{8} && \text{use the quadratic formula to solve for } y^2 \\
&= \frac{1 \pm \sqrt{1-x^2}}{2} && \text{simplify} \\
y &= \pm\sqrt{\frac{1 \pm \sqrt{1-x^2}}{2}} && \text{take the square root}
\end{aligned}
$$

These complicated equations should indicate the advantage of expressing the curve in parametric form.

$\boxed{10}$ The graph of $r = f(\theta - \alpha)$ is the graph of $r = f(\theta)$ rotated counterclockwise through an angle α, whereas the graph $r = f(\theta + \alpha)$ is rotated clockwise.

$\boxed{11}$ ***n* even:** There are $2n$ leaves, each having a leaf angle of $(180/n)^\circ$. There is no open space between the leaves.

***n* odd:** There are n leaves, each having a leaf angle of $(180/n)^\circ$. There is 180° of open space—each space is $(180/n)^\circ$, equispaced between the leaves. If $n = 4k - 1$, where k is a natural number, there is a leaf centered on the $\theta = 3\pi/2$ axis; and if $n = 4k + 1$, there is a leaf centered on the $\theta = \pi/2$ axis.

For $r = \sin n\theta$, the pole values start at 0° and occur every $(180/n)^\circ$. For $r = \cos n\theta$, the pole values start at $(90/n)^\circ$ and occur every $(180/n)^\circ$.

$\boxed{12}$ $r = 4\sin\theta$ and $r = 4\cos\theta \Rightarrow \sin\theta = \cos\theta \Rightarrow \dfrac{\sin\theta}{\cos\theta} = \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4} + \pi n$.

So this gives us the solution $\theta = \frac{\pi}{4}$, but not the solution at the pole. Changing to rectangular equations, we have $r = 4\sin\theta \Rightarrow r^2 = 4r\sin\theta \Rightarrow x^2 + y^2 = 4y \Rightarrow x^2 + y^2 - 4y = 0 \Rightarrow x^2 + y^2 - 4y + 4 = 4 \Rightarrow x^2 + (y-2)^2 = 4$ and similarly, $r = 4\cos\theta \Leftrightarrow (x-2)^2 + y^2 = 4$. Solving for y we get $y = 2 \pm \sqrt{4 - x^2}$ and $y = \pm\sqrt{4 - (x-2)^2}$. Referring to Figure 17 of Section 11.5, we'll determine the values of x and y for which the lower semicircle {in red} intersects the upper semicircle {in blue}.

$2 - \sqrt{4 - x^2} = \sqrt{4 - (x-2)^2}$	lower red = upper blue
$4 - 4\sqrt{4 - x^2} + 4 - x^2 = 4 - (x-2)^2$	square both sides
$= 4 - (x^2 - 4x + 4)$	expand
$8 - 4x = 4\sqrt{4 - x^2}$	simplify
$2 - x = \sqrt{4 - x^2}$	divide by 4
$4 - 4x + x^2 = 4 - x^2$	square both sides
$2x^2 - 4x = 0$	set one side equal to zero
$2x(x-2) = 0$	factor
$2x = 0, \quad x - 2 = 0$	set each factor equal to 0
$x = 0, \quad x = 2$	solve for x

Substituting 0 and 2 for x in $y = 2 - \sqrt{4 - x^2}$ gives us $y = 0$ and $y = 2$; that is, the points $(0,\ 0)$ and $(2,\ 2)$, as expected. The point $(x,\ y) = (2,\ 2)$ corresponds to $(r,\ \theta) = (2\sqrt{2},\ \frac{\pi}{4})$ on both graphs. The point $(x,\ y) = (0,\ 0)$ corresponds to $(r,\ \theta) = (0,\ 0)$ on $r = 4\sin\theta$, but it corresponds to $(r,\ \theta) = (0,\ \frac{\pi}{2})$ on $r = 4\cos\theta$, and that's why the answer to the first system did not reveal the solution at the pole. Graph the polar equations in simultaneous mode to see the *collision point* (same values of r and θ) at $(2\sqrt{2},\ \frac{\pi}{4})$.