EE3900 - Assignment 1

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Contents

***Abstract*—This manual provides a simple introduction to digital signal processing.**

1 Software Installation

Run the following commands

```
sudo apt−get update
sudo apt−get install libffi−dev libsndfile1 python3
    −scipy python3−numpy python3−matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file from

```
wget https://raw.githubusercontent.com/Satwik
    −4/EE3900/master/Assignment−1/codes/
    Sound_Noise.wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution:

```
import soundfile as sf
from scipy import signal

#read .wav file
input_signal,fs = sf.read('codes/
    Sound_Noise.wav')


#sampling frequency of Input signal
sampl_freq=fs

#order of the filter
order=4

#cutoff frquency 4kHz
cutoff_freq=4000.0

#digital frequency
Wn=2*cutoff_freq/sampl_freq

# b and a are numerator and denominator
    polynomials respectively
b, a = signal.butter(order,Wn, 'low')

#filter the input signal with butterworth filter
output_signal = signal.filtfilt(b, a,
    input_signal)
#output_signal = signal.lfilter(b, a,
    input_signal)

#write the output signal into .wav file

sf.write('codes/Sound_With_ReducedNoise.
    wav', output_signal, fs)
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_ReducedNoise.wav. Play

the file in the spectrogram in Problem 2.2. What do you observe?
Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

3 Difference Equation

3.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{3.1}$$

Sketch $x(n)$.
Solution: Fig. 3.2 contains the plot for $x(n)$

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$
$$y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$.
Solution: The following code yields Fig. 3.2.

```
wget https://raw.githubusercontent.com/Satwik
    -4/EE3900/master/Assignment-1/codes/3
    _2.py
```

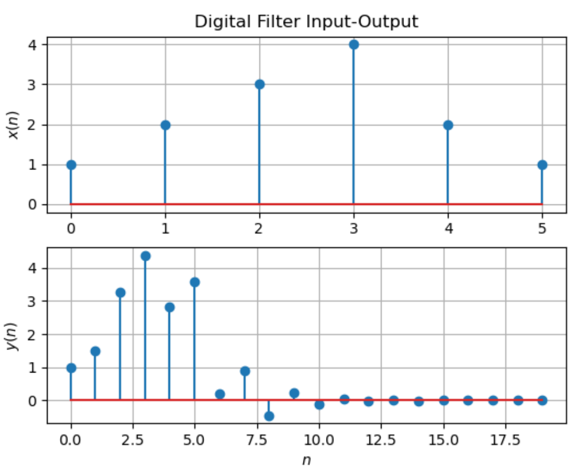

Fig. 3.2

3.3 Repeat the above exercise using a C code.
Solution:

```
#include <stdio.h>

int main(){
    float x[6] = {1.0,2.0,3.0,4.0,2.0,1.0};
    int k=20;
    float y[20] = {0};
    y[0]=x[0];
    y[1] = -0.5*y[0]+x[1];

    for(int n=2; n<k-1;n++){
        if(n<6){
            y[n] = -0.5*y[n-1]+x[n]+x[n
                -2];
        }
        else if(n>5 && n<8){
            y[n] = -0.5*y[n-1]+x[n-2];
        }
        else{
            y[n] = -0.5*y[n-1];
        }
    }
    FILE *fx,*fy;
    fx=fopen("2_x.txt","w");
    fy=fopen("2_y.txt","w");
      if(fx == NULL || fy == NULL)
  {
      printf("Error!");
      return 1;
  }

    for(int i=0;i<6;i++){
        fprintf(fx,"%f ",x[i]);
    }
    for(int i=0;i<20;i++){
        fprintf(fy,"%f ",y[i]);
    }
    fclose(fx);
    fclose(fy);
    return 0;
}
```

4 Z-transform

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution: From (4.1),

$$\mathcal{Z}\{x(n-1)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \tag{4.4}$$
$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} \tag{4.5}$$
$$= z^{-1}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.6}$$

resulting in (4.2). In the second case

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-k)z^{-n} \tag{4.7}$$
$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-k} \tag{4.8}$$
$$= z^{-k}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.9}$$

4.2 Obtain $X(z)$ for $x(n)$ defined in problem 3.1.
Solution:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.10}$$
$$= \sum_{n=0}^{5} x(n)z^{-n} \tag{4.11}$$
$$= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5} \tag{4.12}$$

Since $x[n]$ is of finite duration, the ROC will be the entire z-plane except $z = 0$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.13}$$

from (3.2) assuming that the Z-transform is a linear operation.
Solution: Applying (4.7) in (3.2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \tag{4.14}$$
$$\implies \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{4.15}$$

The pole of $H(z)$ is at $z = -1/2$ and its root is at $z = i$. ROC for $H(z)$ will be $|z| > 1/2$.

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.16}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.17}$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad z > 1 \tag{4.18}$$

Solution: It is easy to show that

$$\delta(n) \overset{\mathcal{Z}}{\rightleftharpoons} 1 \tag{4.19}$$

$$\mathcal{Z}\{\delta(n)\} = \sum_{n=-\infty}^{\infty} \delta(n)z^{-n} \tag{4.20}$$
$$= \delta(0)z^{0} \tag{4.21}$$
$$= 1 \tag{4.22}$$

and from (4.17),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{4.23}$$
$$= \frac{1}{1 - z^{-1}}, \quad z > 1 \tag{4.24}$$

using the formula for the sum of an infinite geometric progression.

4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad z > a \tag{4.25}$$

Solution:

$$\mathcal{Z}\{a^n u(n)\} = \sum_{n=-\infty}^{\infty} a^n u(n)z^{-n} \tag{4.26}$$
$$= \sum_{n=0}^{\infty} (az^{-1})^n \tag{4.27}$$
$$= \frac{1}{1 - az^{-1}} \tag{4.28}$$

4.6 Let

$$H\left(e^{\jmath\omega}\right) = H\left(z = e^{\jmath\omega}\right). \tag{4.29}$$

Plot $H\left(e^{\jmath\omega}\right)$. Is it periodic? If so, find the period. $H(e^{\jmath\omega})$ is known as the *Discrete Time Fourier Transform* (DTFT) of $h(n)$.

Solution:

$$H(e^{\jmath\omega}) = \frac{1+e^{-2\jmath\omega}}{1+\frac{1}{2}z^{-\jmath\omega}} \tag{4.30}$$

$$= \frac{1+cos2\omega-\jmath sin2\omega}{1+\frac{1}{2}(cos\omega-\jmath sin\omega)} \tag{4.31}$$

$$\implies |H(e^{\jmath\omega})|^2 = \frac{(1+cos2\omega)^2+(sin2\omega)^2}{(1+\frac{1}{2}cos\omega)^2+\frac{1}{4}sin^2\omega} \tag{4.32}$$

$$= \frac{2+2cos2\omega}{\frac{5}{4}+cos\omega} \tag{4.33}$$

$$= \frac{16cos^2\omega}{5+4cos\omega} \tag{4.34}$$

$$\implies |H(e^{\jmath\omega})| = \frac{4|cos\omega|}{\sqrt{5+4cos\omega}} \tag{4.35}$$

$\therefore$ Period $T = 2\pi$ The following code plots Fig. 4.6.

```
wget https://raw.githubusercontent.com/Satwik
    -4/EE3900/master/Assignment-1/codes/4
    _6.py
```

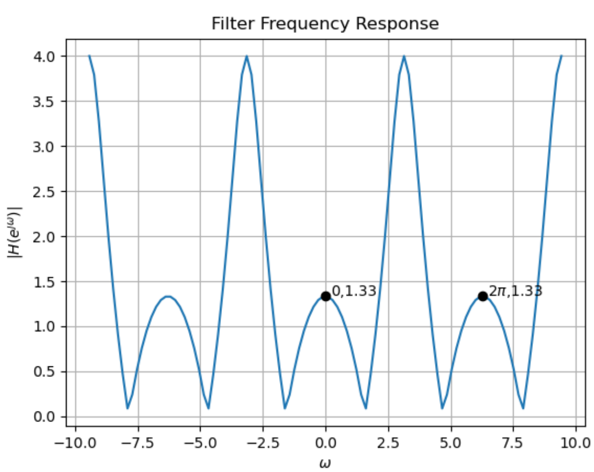

Fig. 4.6: $H(e^{\jmath\omega})$

4.7 Express $h(n)$ in terms of $H(e^{\jmath\omega})$.

Solution:

$$H(e^{\jmath\omega}) = \sum_{k=-\infty}^{\infty} h(k)e^{-k\jmath\omega} \tag{4.36}$$

$$\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{\jmath\omega n}d\omega = \int_{-\pi}^{\pi}\sum_{k=-\infty}^{\infty} h(k)e^{-k\jmath\omega}e^{\jmath\omega n}d\omega \tag{4.37}$$

$$= \sum_{k=-\infty}^{\infty} h(k)\int_{-\pi}^{\pi} e^{(n-k)\jmath\omega}d\omega \tag{4.38}$$

$$= (\pi+\pi)\sum_{k=-\infty}^{\infty} h(k)\delta(n-k) \tag{4.39}$$

$$= 2\pi h(n) \tag{4.40}$$

$$\implies h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{\jmath\omega n}d\omega \tag{4.41}$$

5 Impulse Response

5.1 Using long division, find

$$h(n), \quad n < 5 \tag{5.1}$$

for H(z) in (4.15).

Solution: Replacing z^-1 with x in (4.15):

$$H(x) = \frac{1+x^2}{1+\frac{1}{2}x} \tag{5.2}$$

Performing long division:

$$\begin{array}{r} 2x-4 \\ x+2)\overline{\;2x^2 \qquad +2} \\ -2x^2-4x \qquad \\ \hline -4x+2 \\ 4x+8 \\ \hline 10 \end{array}$$

$$\implies H(x) = 2x-4+\frac{5}{\frac{1}{2}x+1} \tag{5.3}$$

$$\implies H(z) = 2z^{-1}-4-\frac{5}{1+\frac{1}{2}z^{-1}} \tag{5.4}$$

Using (4.19) and (4.28), applying the inverse Z transform on both sides:

$$h(n) = 2\delta(n-1)-4\delta(n)-5\left(\frac{-1}{2}\right)^n u(n) \tag{5.5}$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.6}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).
Solution: From (4.15),

$$H(z) = \frac{1}{1+\frac{1}{2}z^{-1}} + \frac{z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.7}$$

$$\implies h(n) = \left(-\frac{1}{2}\right)^{n} u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.8}$$

using (4.25) and (4.7).

5.3 Sketch $h(n)$. Is it bounded? Convergent? Justify using the ratio test.
Solution: Using (5.8)

$$\lim_{n\to\infty}\left|\frac{h(n+1)}{h(n)}\right| = \left|\frac{\left(-\frac{1}{2}\right)^{n+1} u(n+1) + \left(-\frac{1}{2}\right)^{n-1} u(n-1)}{\left(-\frac{1}{2}\right)^{n} u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2)}\right| \tag{5.9}$$

$$\lim_{n\to\infty}\left|\frac{h(n+1)}{h(n)}\right| = \frac{\left(\frac{1}{2}\right)^{n+1} + \left(\frac{1}{2}\right)^{n-1}}{\left(\frac{1}{2}\right)^{n} + \left(\frac{1}{2}\right)^{n-2}} \tag{5.10}$$

$$= \frac{1+2^2}{2+2^3} < \infty \tag{5.11}$$

$\therefore h(n)$ is convergent.

$$|u(n)| \le 1, |u(n-2)| \le 1$$

$$\left|\left(-\frac{1}{2}\right)^{n}\right| \le 1, \left|\left(-\frac{1}{2}\right)^{n-2}\right| \le 1$$

$$\implies \left|\left(-\frac{1}{2}\right)^{n} u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2)\right| \le 2$$

$$\implies |h(n)| \le 2$$

$\therefore h(n)$ is bounded.
The following code plots Fig. 5.3.

```
wget https://raw.githubusercontent.com/Satwik-4/EE3900/master/Assignment-1/codes/5_3.py
```

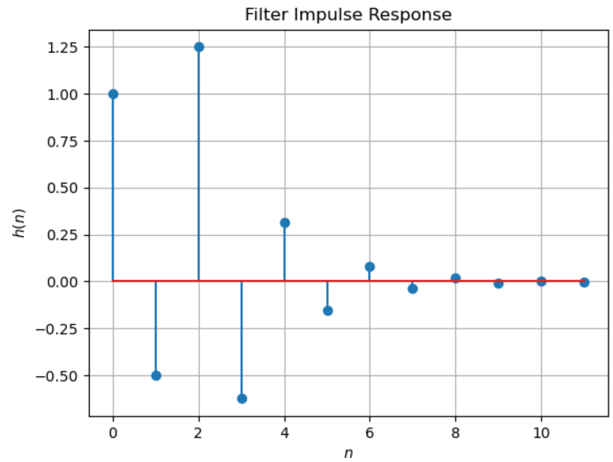

Fig. 5.3: $h(n)$ as the inverse of $H(z)$

5.4 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.12}$$

Is the system defined by (3.2) stable for the impulse response in (5.6)?
Solution: From (5.8):

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=-\infty}^{\infty} \left(-\frac{1}{2}\right)^{n} u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.13}$$

$$= \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^{n} + \sum_{n=2}^{\infty}\left(-\frac{1}{2}\right)^{n-2} \tag{5.14}$$

$$= 2\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^{n} \tag{5.15}$$

$$= 2 \times \frac{1}{1+1/2} = 1.33 < \infty \tag{5.16}$$

The system is stable.

5.5 Verify the above result using a python code.
Solution: The following python code can be used to verify the result:

```
wget https://raw.githubusercontent.com/Satwik-4/EE3900/master/Assignment-1/codes/4.py
```

5.6 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \tag{5.17}$$

This is the definition of $h(n)$.
Solution: The following code plots Fig. 5.6. Note that this is the same as Fig. 5.3.

```
wget https://raw.githubusercontent.com/Satwik-4/EE3900/master/Assignment-1/codes/5_6.py
```

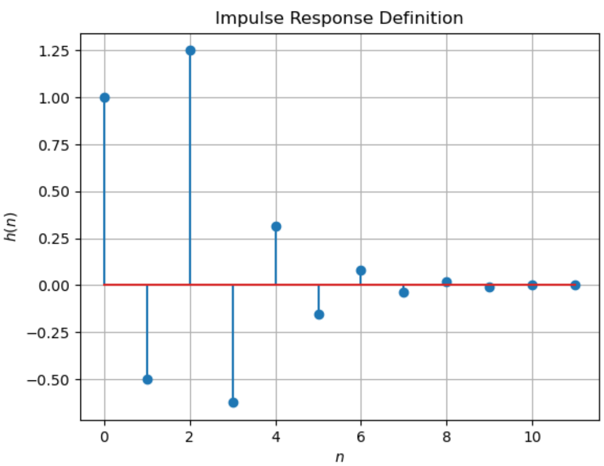

Fig. 5.6: $h(n)$ from the definition

5.7 Compute

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad (5.18)$$

Comment. The operation in (5.18) is known as *convolution*.

Solution: The following code plots Fig. 5.7. Note that this is the same as $y(n)$ in Fig. 3.2.

```
wget https://raw.githubusercontent.com/Satwik
    -4/EE3900/master/Assignment-1/codes/5
    _7.py
```

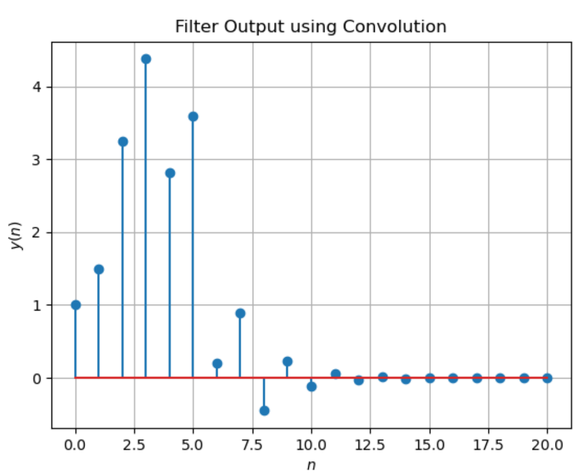

Fig. 5.7: $y(n)$ from the definition of convolution

5.8 Express the above convolution using a Toeplitz matrix. **Solution:** h(n) and x(n) can be represented as the following matrices:

$$X = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \quad H = \begin{pmatrix} 1 \\ -0.5 \\ 1.25 \\ -0.625 \\ 0.315 \\ 0.15625 \end{pmatrix} \tag{5.19}$$

A Toeplitz matrix can be constructed such that:

$$y = T(h)X \tag{5.20}$$

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -0.5 & 1 & 0 & 0 & 0 \\ 1.25 & -0.5 & 1 & 0 & 0 \\ -0.625 & 1.25 & -0.5 & 1 & 0 \\ 0.315 & -0.625 & 1.25 & -0.5 & 1 \\ 0.156 & 0.315 & -0.625 & 1.25 & -0.5 \\ 0 & 0.156 & 0.315 & -0.625 & 1.25 \\ 0 & 0 & 0.156 & 0.315 & -0.625 \\ 0 & 0 & 0 & 0.156 & 0.315 \\ 0 & 0 & 0 & 0 & 0.156 \end{pmatrix} \tag{5.21}$$

$$\implies y = \begin{pmatrix} 1 \\ 1.5 \\ 3.25 \\ 4.38 \\ 2.81 \\ 3.59 \\ 0.12 \\ 0.78 \\ -0.62 \\ 0 \\ -0.16 \end{pmatrix} \tag{5.22}$$

5.9 Show that

$$y(n) = \sum_{n=-\infty}^{\infty} x(n-k)h(k) \tag{5.23}$$

Solution: From the convolution operation in (5.18),

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.24}$$

Replacing k with $n-k$

$$= \sum_{n-k=-\infty}^{\infty} x(n-k)h(k) \tag{5.25}$$

$$= \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.26}$$

6 DFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-\jmath 2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{6.1}$$

and $H(k)$ using $h(n)$.

6.2 Compute

$$Y(k) = X(k)H(k) \tag{6.2}$$

6.3 Compute

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k)\cdot e^{\jmath 2\pi kn/N}, \quad n = 0, 1, \dots, N-1 \tag{6.3}$$

Solution: The following code computes $X(k), H(k), Y(k)$ and plots Fig. 5.7. Note that this is the same as $y(n)$ in Fig. 3.2.

```
wget https://raw.githubusercontent.com/Satwik
    -4/EE3900/master/filter/codes/6_3.py
```

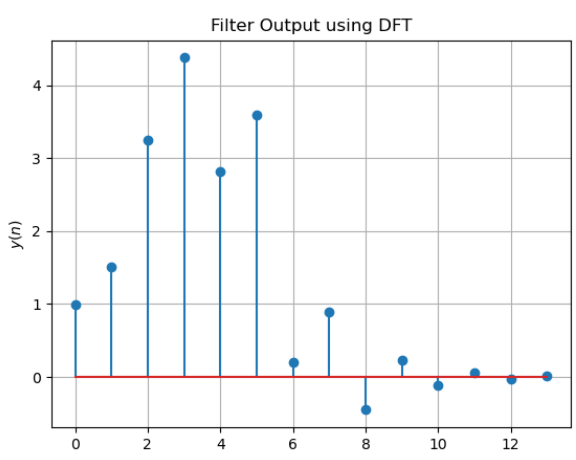

Fig. 6.3: $y(n)$ from the DFT

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.

Solution: The following code does the computations and plots Fig 6.4

```
wget https://raw.githubusercontent.com/Satwik
    -4/EE3900/master/Assignment-1/codes/6
    _4.py
```

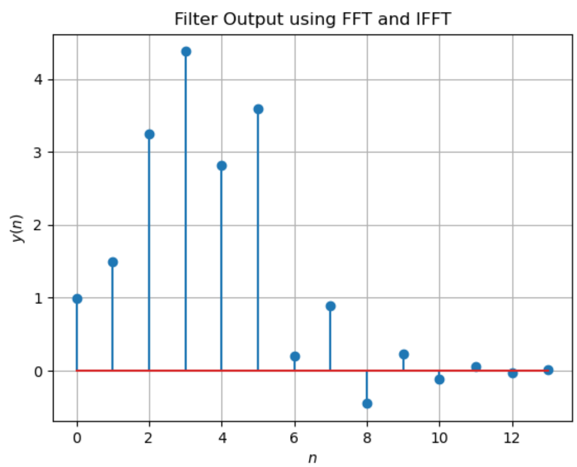

Fig. 6.4: $y(n)$ from FFT

The following code plots Figure 6.4 which contains the plots of $y(n)$ from the three definitions (difference equation, DFT, FFT)

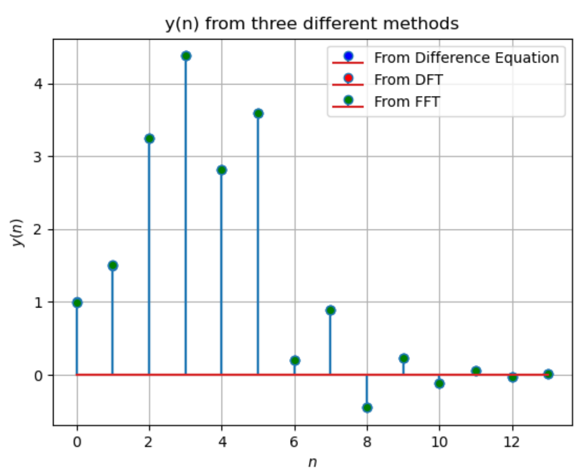

Fig. 6.4: y(n) from different definitions

7 FFT

1. The DFT of $x(n)$ is given by

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{7.1}$$

2. Let
$$W_N = e^{-j2\pi/N} \tag{7.2}$$
Then the N-point *DFT matrix* is defined as
$$\vec{F}_N = \left[W_N^{mn}\right], \quad 0 \le m, n \le N-1 \tag{7.3}$$
where W_N^{mn} are the elements of $\vec{F}_N$.
3. Let
$$\vec{I}_4 = \vec{e}_4^1\vec{e}_4^2 \qquad \vec{e}_4^3\vec{e}_4^4 \qquad (7.4)ref$$
be the 4×4 identity matrix. Then the 4 point *DFT permutation matrix* is defined as
$$\vec{P}_4 = \vec{e}_4^1\vec{e}_4^3 \qquad \vec{e}_4^2\vec{e}_4^4 \qquad (7.5)ref$$
4. The 4 point *DFT diagonal matrix* is defined as
$$\vec{D}_4 = \text{diag} W_8^0 W_8^1 \qquad W_8^2 W_8^3 \qquad ref(7.6)$$
5. Show that
$$W_N^2 = W_{N/2} \tag{7.7}$$

Solution:
$$W_N = e^{-\jmath 2\pi/N} \tag{7.8}$$
$$\implies W_N^2 = e^{-\jmath 4\pi/N} \tag{7.9}$$
$$= e^{-\jmath 2\pi/(N/2)} \tag{7.10}$$
$$= W_{N/2} \tag{7.11}$$
6. Show that
$$\vec{F}_4 = \begin{bmatrix} \vec{I}_2 & \vec{D}_2 \\ \vec{I}_2 & -\vec{D}_2 \end{bmatrix} \begin{bmatrix} \vec{F}_2 & 0 \\ 0 & \vec{F}_2 \end{bmatrix} \vec{P}_4 \tag{7.12}$$

Solution: Since P_4 is a permutation matrix, $P_4 P_4 = I$
$$\vec{D}_2\vec{F}_2 = \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & W_2 \end{bmatrix} \tag{7.13}$$
$$= \begin{bmatrix} 1 & 1 \\ W_4 & W_4 W_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ W_4 & W_4^3 \end{bmatrix} \tag{7.14}$$
$$\begin{bmatrix} \vec{I}_2 & \vec{D}_2 \\ \vec{I}_2 & -\vec{D}_2 \end{bmatrix} \begin{bmatrix} \vec{F}_2 & 0 \\ 0 & \vec{F}_2 \end{bmatrix} = \begin{bmatrix} \vec{F}_2 & \vec{D}_2\vec{F}_2 \\ \vec{F}_2 & -\vec{D}_2\vec{F}_2 \end{bmatrix} \tag{7.15}$$
$$= \begin{bmatrix} W_2^0 & W_2^0 & 1 & 1 \\ W_2^0 & W_2 & W_4 & W_4^3 \\ W_2^0 & W_2^0 & -1 & -1 \\ W_2^0 & W_2 & -W_4 & -W_4^3 \end{bmatrix} \tag{7.16}$$
$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^2 & W_4 & W_4^3 \\ 1 & 1 & W_4^2 & W_4^2 \\ 1 & W_4^2 & W_4^3 & W_4^5 \end{bmatrix} \tag{7.17}$$
we know that
$$\vec{P}_4 = [\vec{e_1}\vec{e_2}\vec{e_3}\vec{e_4}] \tag{7.18}$$
Which implies that a $\vec{P_4}$ would swap the second and third rows.
$$\begin{bmatrix} \vec{I}_2 & \vec{D}_2 \\ \vec{I}_2 & -\vec{D}_2 \end{bmatrix} \begin{bmatrix} \vec{F}_2 & 0 \\ 0 & \vec{F}_2 \end{bmatrix} \vec{P}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4 & W_4^2 & W_4^3 \\ 1 & W_4^2 & 1 & W_4^2 \\ 1 & W_4^3 & W_4^2 & W_4^5 \end{bmatrix} \tag{7.19}$$
$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \vec{F}_4 \tag{7.20}$$
7. Show that
$$\vec{F}_N = \begin{bmatrix} \vec{I}_{N/2} & \vec{D}_{N/2} \\ \vec{I}_{N/2} & -\vec{D}_{N/2} \end{bmatrix} \begin{bmatrix} \vec{F}_{N/2} & 0 \\ 0 & \vec{F}_{N/2} \end{bmatrix} \vec{P}_N \tag{7.21}$$

Solution: Let $\vec{F_N} = [\vec{f_N^1}\vec{f_N^2}\cdots\vec{f_N^N}]$,for even N:

$$\begin{bmatrix} I_{N/2}F_{N/2} \\ I_{N/2}F_{N/2} \end{bmatrix} = [\vec{f_N^1}\vec{f_N^3}\cdots\vec{f_N^{N-1}}] \tag{7.22}$$

$$\begin{bmatrix} D_{N/2}F_{N/2} \\ -D_{N/2}F_{N/2} \end{bmatrix} = [\vec{f_N^2}\vec{f_N^4}\cdots\vec{f_N^N}] \tag{7.23}$$

$$\implies \begin{bmatrix} I_{N/2}F_{N/2} & D_{N/2}F_{N/2} \\ I_{N/2}F_{N/2} & -D_{N/2}F_{N/2} \end{bmatrix} = [\vec{f_N^1}\vec{f_N^3}\cdots\vec{f_N^{N-1}}\vec{f_N^2}\vec{f_N^4}\cdots\vec{f_N^N}] \tag{7.24}$$

$$\implies \begin{bmatrix} \vec{I}_{N/2} & \vec{D}_{N/2} \\ \vec{I}_{N/2} & -\vec{D}_{N/2} \end{bmatrix}\begin{bmatrix} \vec{F}_{N/2} & 0 \\ 0 & \vec{F}_{N/2} \end{bmatrix}\vec{P}_N = [\vec{f_N^1}\vec{f_N^2}\cdots\vec{f_N^N}] \tag{7.25}$$

$$= \vec{F_N} \tag{7.26}$$

8. Find

$$\vec{P}_4\vec{x} \tag{7.27}$$

Solution:

$$\vec{x} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}, \vec{P_4} = \begin{pmatrix} \vec{e_1} & \vec{e_3} & \vec{e_2} & \vec{e_4} \end{pmatrix} \tag{7.28}$$

$$\vec{P_4}\vec{x} = x_0\vec{e_1} + x_2\vec{e_2} + x_1\vec{e_3} + x_3\vec{e_4} \tag{7.29}$$

$$= \begin{pmatrix} x_0 \\ x_2 \\ x_1 \\ x_3 \end{pmatrix} \tag{7.30}$$

9. Show that

$$\vec{X} = \vec{F}_N\vec{x} \tag{7.31}$$

where $\vec{x}, \vec{X}$ are the vector representations of $x(n), X(k)$ respectively.
Solution:

$$\vec{F_N}\vec{x}(k) = \sum_{i=0}^{N-1} F(ki)x(i) \tag{7.32}$$

$$= \sum_{i=0}^{N-1} W_N^{ki}x(i) \tag{7.33}$$

$$= \sum_{i=0}^{N-1} e^{\jmath 2\pi ki/N}x(i) = X(k) \tag{7.34}$$

10. Derive the following Step-by-step visualisation of 8-point FFTs into 4-point FFTs and so on

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix}\begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.35}$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix}\begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.36}$$

4-point FFTs into 2-point FFTs

$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix}\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.37}$$

$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix}\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.38}$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix}\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.39}$$

$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix}\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.40}$$

$$P_8\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \\ x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} \tag{7.41}$$

$$P_4\begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(4) \\ x(2) \\ x(6) \end{bmatrix} \tag{7.42}$$

$$P_4\begin{bmatrix} x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(1) \\ x(5) \\ x(3) \\ x(7) \end{bmatrix} \tag{7.43}$$

Therefore,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2\begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{7.44}$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2\begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{7.45}$$

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2\begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{7.46}$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.47}$$

Solution:

$$X(k) = \sum_{n=0}^{7} x(n) e^{\jmath 2\pi kn/8} \tag{7.48}$$

$$= \sum_{n=0}^{3} x(2n) e^{\jmath 4\pi kn/8} + \sum_{n=0}^{3} x(2n+1) e^{\jmath 2\pi k(2n+1)/8} \tag{7.49}$$

$$= \sum_{n=0}^{3} x(2n) e^{\jmath 4\pi kn/8} + e^{\jmath \pi k/4} x(2n+1) e^{\jmath 2\pi k(2n)/8} \tag{7.50}$$

$$= X_1(k) + e^{\jmath \pi k/4} X_2(k) \tag{7.51}$$

Where X_1, X_2 are the 4-point FFTs for the even and odd terms respectively. For the first four terms in X ie. $k = \{0, 1, 2, 3\}$:

$$\begin{pmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{pmatrix} = \begin{pmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{pmatrix} + \begin{pmatrix} e^{2\pi \jmath 0/8} X_2(0) \\ e^{2\pi \jmath 1/8} X_2(1) \\ e^{2\pi \jmath 2/8} X_2(2) \\ e^{2\pi \jmath 3/8} X_2(3) \end{pmatrix} \tag{7.52}$$

$$= \begin{pmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{pmatrix} + \begin{pmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{pmatrix} \begin{pmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{pmatrix} \tag{7.53}$$

Similarly for $k = \{4, 5, 6, 7\}$:

$$\begin{pmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{pmatrix} = \begin{pmatrix} X_1(4) \\ X_1(5) \\ X_1(6) \\ X_1(7) \end{pmatrix} + \begin{pmatrix} e^{2\pi \jmath 4/8} X_2(4) \\ e^{2\pi \jmath 5/8} X_2(5) \\ e^{2\pi \jmath 6/8} X_2(6) \\ e^{2\pi \jmath 7/8} X_2(7) \end{pmatrix} \tag{7.54}$$

Using $X_1(k) = X_1(k-4)$ and $e^{2\pi \jmath k/8} = -e^{2\pi \jmath (k-4)/8}$ for $k = \{4, 5, 6, 7\}$

$$\begin{pmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{pmatrix} = \begin{pmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{pmatrix} + \begin{pmatrix} -W_8^0 & 0 & 0 & 0 \\ 0 & -W_8 & 0 & 0 \\ 0 & 0 & -W_8^2 & 0 \\ 0 & 0 & 0 & -W_8^3 \end{pmatrix} \begin{pmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{pmatrix} \tag{7.55}$$

Similarly to convert the 4-point FFTs into 2-point FFTs, we can split them based on the even/odd indices. Let $x_1(n)$ and $x_2(n)$ represent $x(2n)$ and $x(2n+1)$ resp.

$$X_1(k) = \sum_{n=0}^{n=3} x_1(n) e^{\jmath 2\pi nk/4} \tag{7.56}$$

$$= \sum_{n=0}^{n=1} x_1(2n) e^{\jmath 2\pi 2nk/4} + x_1(2n+1) e^{\jmath 2\pi (2n+1)k/4} \tag{7.57}$$

$$= \sum_{n=0}^{n=1} x_1(2n) e^{\jmath 2\pi nk/2} + e^{\jmath 2\pi k/4} x_1(2n+1) e^{\jmath 2\pi nk/2} \tag{7.58}$$

$$= X_3(k) + e^{\jmath 2\pi k/4} X_4(k) \tag{7.59}$$

where X_3 and X_4 are now 2-point FFTs of the even and odd indexed terms in X_1. The above equation in matrix form can be written as follows for $k = \{0, 1\}$:

$$\begin{pmatrix} X_1(0) \\ X_1(1) \end{pmatrix} = \begin{pmatrix} X_3(0) \\ X_3(1) \end{pmatrix} + \begin{pmatrix} e^{\jmath 2\pi 0/4} X_4(0) \\ e^{\jmath 2\pi 1/4} X_4(1) \end{pmatrix} \tag{7.60}$$

$$= \begin{pmatrix} X_3(0) \\ X_3(1) \end{pmatrix} + \begin{pmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{pmatrix} \begin{pmatrix} X_4(0) \\ X_4(1) \end{pmatrix} \tag{7.61}$$

Similarly for $k = \{2, 3\}$ using the fact that $X_3(k) = X_3(k-2)$ and $X_4(k) = X_4(k-2)$

$$\begin{pmatrix} X_1(2) \\ X_1(3) \end{pmatrix} = \begin{pmatrix} X_3(0) \\ X_3(1) \end{pmatrix} + \begin{pmatrix} -W_4^0 & 0 \\ 0 & -W_4^1 \end{pmatrix} \begin{pmatrix} X_4(0) \\ X_4(1) \end{pmatrix} \tag{7.62}$$

Following the same procedure and introducing X_5, X_6 as 2-point FFTs for the even and odd indexed terms in X_2:

$$\begin{pmatrix} X_2(1) \\ X_2(0) \end{pmatrix} = \begin{pmatrix} X_5(0) \\ X_5(1) \end{pmatrix} + \begin{pmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{pmatrix} \begin{pmatrix} X_6(0) \\ X_6(1) \end{pmatrix} \tag{7.63}$$

$$\begin{pmatrix} X_2(2) \\ X_2(3) \end{pmatrix} = \begin{pmatrix} X_5(0) \\ X_5(1) \end{pmatrix} + \begin{pmatrix} -W_4^0 & 0 \\ 0 & -W_4^1 \end{pmatrix} \begin{pmatrix} X_6(0) \\ X_6(1) \end{pmatrix} \tag{7.64}$$

(7.61),(7.62),(7.63),(7.64) are the four equations that represent the 2-point FFTs of X_1 and X_2.

11. For

$$\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.65}$$

compute the DFT using (7.31)
Solution: The following code computes the DFT using matrix multiplication:

```
wget https://raw.githubusercontent.com/Satwik-4/EE3900/master/Assignment-1/codes/7_11.py
```

12. Repeat the above exercise using the FFT after zero padding $\vec{x}$.
Solution: The following code computes the DFT using matrix multiplication:

```
wget https://raw.githubusercontent.com/Satwik-4/EE3900/master/Assignment-1/codes/7_12.py
```

13. Write a C program to compute the 8-point FFT.
Solution: The following code computes the FFT using recursion:

```
wget https://raw.githubusercontent.com/Satwik-4/EE3900/master/Assignment-1/codes/7_13.c
```

8 Exercises

Answer the following questions by looking at the python code in Problem 2.3.

8.1 The command

```
output_signal = signal.lfilter(b, a, input_signal)
```

in Problem 2.3 is executed through the following difference equation

$$\sum_{m=0}^{M} a(m)\, y(n-m) = \sum_{k=0}^{N} b(k)\, x(n-k) \quad (8.1)$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.

8.2 Repeat all the exercises in the previous sections for the above a and b.

8.3 What is the sampling frequency of the input signal?
Solution: Sampling frequency(fs)=44.1kHZ.

8.4 What is type, order and cutoff-frequency of the above butterworth filter
Solution: The given butterworth filter is low pass with order=2 and cutoff-frequency=4kHz.

8.5 Modifying the code with different input parameters and to get the best possible output.