Pingala Series

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Contents

***Abstract*—This manual provides a simple introduction to Transforms**

1 JEE 2019

Let α, β be the roots of the polynomial $x^2 - x - 1$

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1 \tag{1.1}$$

$$b_n = a_{n-1} + a_{n+1}, \quad n \geq 2, \quad b_1 = 1 \tag{1.2}$$

Verify the following using a python code.

1.1

$$\sum_{k=1}^{n} a_k = a_{n+2} - 1, \quad n \geq 1 \tag{1.3}$$

1.2

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{10}{89} \tag{1.4}$$

1.3

$$b_n = \alpha^n + \beta^n, \quad n \geq 1 \tag{1.5}$$

1.4

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{8}{89} \tag{1.6}$$

Solution: The following code verifies the above results:

```
wget https://raw.githubusercontent.com/Satwik-4/
    EE3900/master/pingala/codes/1.py
```

The first three results are true while the last one is false.

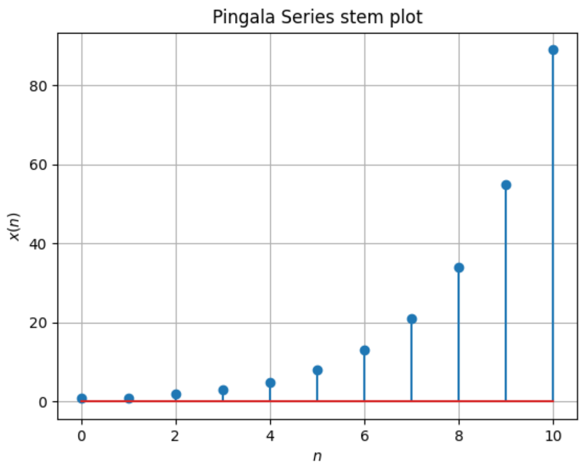

Fig. 2.2: x(n)

2 Pingala Series

2.1 The *one sided* Z-transform of $x(n)$ is defined as

$$X^+(z) = \sum_{n=0}^{\infty} x(n) z^{-n}, \quad z \in \mathbb{C} \tag{2.1}$$

2.2 The *Pingala* series is generated using the difference equation

$$x(n+2) = x(n+1) + x(n), \quad x(0) = x(1) = 1, n \geq 0 \tag{2.2}$$

Generate a stem plot for $x(n)$.
Solution: The following code plots 2.2

```
wget https://raw.githubusercontent.com/Satwik
    -4/EE3900/master/pingala/codes/2_2.py
```

2.3 Find $X^+(z)$.

Solution:

$$x(n+2) = x(n+1) + x(n) \tag{2.3}$$

$$\implies \mathcal{Z}^{+}\{x(n+2)\} = \mathcal{Z}^{+}\{x(n+1)\} + \mathcal{Z}^{+}\{x(n)\} \tag{2.4}$$

$$\implies z^2X^{+}(z) - z^2 - z = zX^{+}(z) - z + X^{+}(z) \tag{2.5}$$

$$\implies X^{+}(z)(z^2 - z - 1) = z^2 \tag{2.6}$$

$$\implies X^{+}(z) = \frac{1}{1 - z^{-1} - z^{-2}} \tag{2.7}$$

$$= \frac{1}{(1-\alpha z^{-1})(1-\beta z^{-1})} \tag{2.8}$$

ROC: $|z| > \frac{1+\sqrt{5}}{2}$

2.4 Find $x(n)$.

$$X^{+}(z) = \frac{1}{(1-\alpha z^{-1})(1-\beta z^{-1})} \tag{2.9}$$

$$= \left(\frac{1}{(\alpha-\beta)z^{-1}}\right)\left(\frac{1}{1-\alpha z^{-1}} - \frac{1}{1-\beta z^{-1}}\right) \tag{2.10}$$

$$= \frac{1}{z^{-1}(\alpha-\beta)}\left(\sum_{n=0}^{\infty}(\alpha z^{-1})^n - (\beta z^{-1})^n\right) \tag{2.11}$$

$$= \sum_{n=0}^{\infty}\frac{\alpha^n - \beta^n}{\alpha-\beta}z^{-n+1} \tag{2.12}$$

$$= \sum_{n=1}^{\infty}\frac{\alpha^n - \beta^n}{\alpha-\beta}z^{-n+1} \tag{2.13}$$

$$= \sum_{n=0}^{\infty}\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha-\beta}z^{-n} \tag{2.14}$$

$$\implies x(n) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha-\beta}u(n) \tag{2.15}$$

$$= a_{n+1}u(n) \tag{2.16}$$

2.5 Sketch

$$y(n) = x(n-1) + x(n+1), \quad n \geq 0 \tag{2.17}$$

Solution: The following code plots 2.5

```
wget https://raw.githubusercontent.com/Satwik
    -4/EE3900/master/pingala/codes/2_5.py
```

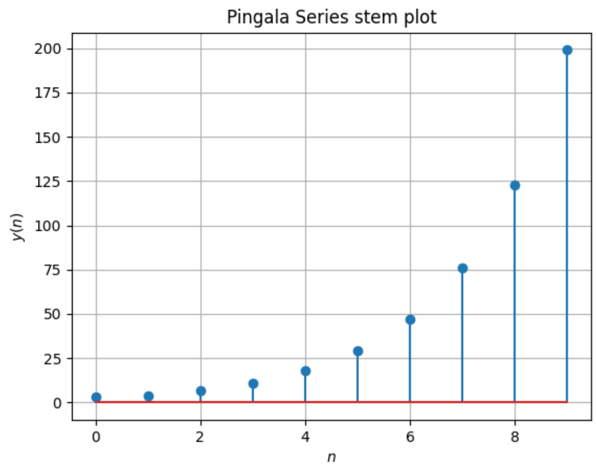

Fig. 2.5: y(n)

2.6 Find $Y^{+}(z)$.

Solution:

$$\mathcal{Z}^{+}\{y(n)\} = \mathcal{Z}^{+}\{x(n+1)\} + \mathcal{Z}^{+}\{x(n-1)\} \tag{2.18}$$

$$Y^{+}(z) = zX^{+}(z) - z + z^{-1}Z^{+}(z) \tag{2.19}$$

$$= (z + z^{-1})\frac{1}{1 - z^{-1} - z^{-2}} - z \tag{2.20}$$

$$= \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \tag{2.21}$$

ROC for the above z transform will again be $|z| > \frac{1+\sqrt{5}}{2}$

2.7 Find $y(n)$.

Solution:

$$
\begin{aligned}
Y^{+}(z) &= X^{+}(z) + \frac{2}{(z-\alpha)(z-\beta)} && (2.22)\\
&= X^{+}(z) + \frac{2}{\alpha-\beta}\left(\frac{1}{1-\alpha z^{-1}} + \frac{1}{1-\beta z^{-1}}\right) && (2.23)\\
&= X^{+}(z) + \frac{2}{\alpha-\beta}\left(\sum_{n=0}^{\infty}(\alpha^{n}+\beta^{n})z^{-n}\right) && (2.24)\\
\implies y(n) &= x(n) + 2u(n)\frac{\alpha^{n}+\beta^{n}}{\alpha-\beta} && (2.25)\\
&= \frac{\alpha^{n+1}+2\alpha^{n}-\beta^{n+1}+2\beta^{n}}{\alpha-\beta}u(n) && (2.26)\\
&= \frac{\alpha^{n}(\alpha+1)+\alpha^{n}-\beta^{n}(\beta+1)+\beta^{n}}{\alpha-\beta}u(n) && (2.27)\\
&= \frac{\alpha^{n+2}-\beta^{n+2}-\alpha\beta(\alpha^{n}+\beta^{n})}{\alpha-\beta}u(n) && (2.28)\\
&= \frac{(\alpha-\beta)(\alpha^{n+1}+\beta^{n+1})}{\alpha-\beta}u(n) && (2.29)\\
\implies y(n) &= \alpha^{n+1}+\beta^{n+1} && (2.30)
\end{aligned}
$$

3 Power of the Z transform

3.1 Show that

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(n) = x(n) * u(n-1) \qquad (3.1)$$

Solution: Using the definition of $x(n)$ from (2.16)

$$
\begin{aligned}
\sum_{k=1}^{n} a_k &= \sum_{k=0}^{n-1} a_{k+1} && (3.2)\\
&= \sum_{k=0}^{n-1} x(n) && (3.3)\\
&= \sum_{k=0}^{n-1} x(k) + x(n)\times 0 && (3.4)\\
&= \sum_{k=0}^{n-1} x(k)u(n-1-k) + x(n)\times u(-1) && (3.5)\\
&= \sum_{k=0}^{n} x(k)u(n-1-k) && (3.6)\\
&= x(n) * u(n-1) && (3.7)
\end{aligned}
$$

3.2 Show that

$$a_{n+2} - 1, \quad n \geq 1 \qquad (3.8)$$

can be expressed as

$$[x(n+1)-1]\,u(n) \qquad (3.9)$$

Solution: The above expression can be written for $n \geq 0$ as:

$$
\begin{aligned}
a_{n+1} - 1, &\quad n \geq 0 && (3.10)\\
= x(n) - 1, &\quad n \geq 0 && (3.11)\\
= (x(n)-1)u(n)& && (3.12)
\end{aligned}
$$

3.3 Show that

$$\sum_{k=1}^{\infty}\frac{a_k}{10^k} = \frac{1}{10}\sum_{k=0}^{\infty}\frac{x(k)}{10^k} = \frac{1}{10}X^{+}(10) \qquad (3.13)$$

Solution:

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \sum_{k=0}^{\infty} \frac{a_{k+1}}{10^{k+1}} \tag{3.14}$$
$$= \sum_{k=0}^{\infty} \frac{x(k)}{10^{k+1}} \tag{3.15}$$
$$= \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} \tag{3.16}$$
$$= \frac{1}{10} \sum_{k=0}^{\infty} x(k) 10^{-k} \tag{3.17}$$
$$= \frac{X^+(10)}{10} \tag{3.18}$$

3.4 Show that

$$\alpha^n + \beta^n, \quad n \geq 1 \tag{3.19}$$

can be expressed as

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right) u(n) \tag{3.20}$$

and find $W(z)$.
Solution: Replacing n with $n+1$:

$$\alpha^{n+1} + \beta^{n+1}, \quad n \geq 0 \tag{3.21}$$
$$= (\alpha^{n+1} + \beta^{n+1}) u(n) = w(n) \tag{3.22}$$

The z-transform can be computed as follows:

$$W(z) = \sum_{n=-\infty}^{\infty} w(n) z^{-n} \tag{3.23}$$
$$= \sum_{n=0}^{\infty} \alpha^{n+1} z^{-n} + \beta^{n+1} z^{-n} \tag{3.24}$$
$$= \alpha \sum_{n=0}^{\infty} \alpha^n z^{-n} + \beta \sum_{n=0}^{\infty} \beta^n z^{-n} \tag{3.25}$$
$$= \frac{\alpha}{1 - \alpha z^{-1}} + \frac{\beta}{1 - \beta z^{-1}} \tag{3.26}$$
$$= \frac{\alpha + \beta - 2\alpha\beta z^{-1}}{1 + \alpha\beta z^{-2} - (\alpha + \beta) z^{-1}} \tag{3.27}$$
$$= \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \tag{3.28}$$

ROC: $|z| > \frac{1+\sqrt{5}}{2}$

3.5 Show that

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} = \frac{1}{10} Y^+(10) \tag{3.29}$$

Solution:

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \sum_{k=0}^{\infty} \frac{b_{k+1}}{10^{k+1}} \tag{3.30}$$
$$= \sum_{k=0}^{\infty} \frac{x(k)}{10^{k+1}} \tag{3.31}$$
$$= \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} \tag{3.32}$$
$$= \frac{1}{10} \sum_{k=0}^{\infty} y(k) 10^{-k} \tag{3.33}$$
$$= \frac{Y^+(10)}{10} \tag{3.34}$$

3.6 Solve the JEE 2019 problem.
Solution:

a) From (3.7)

$$\sum_{k=1}^{n} a_k = x(n) * u(n-1) \tag{3.35}$$

Taking the positive z transform on the RHS:

$$\mathcal{Z}\{x(n) * u(n-1)\} = X^+(z) z^{-1} \frac{1}{1 - z^{-1}} \tag{3.36}$$
$$= \frac{z^{-1}}{(1 - z^{-1} - z^{-2})(1 - z^{-1})} \tag{3.37}$$
$$= z \left(\frac{1}{1 - z^{-1} - z^{-2}} - \frac{1}{1 - z^{-1}} \right) \tag{3.38}$$
$$= z \sum_{n=0}^{\infty} (x(n) - 1) z^{-n} \tag{3.39}$$
$$= \sum_{n=0}^{\infty} (x(n) - 1) z^{-n+1} \tag{3.40}$$
$$= \sum_{n=0}^{\infty} (x(n+1) - 1) z^{-n} \tag{3.41}$$

Taking the Inverse Z transform on (3.41):

$$x(n) * u(n-1) = (x(n+1) - 1) u(n) \tag{3.42}$$
$$\implies \sum_{k=1}^{n} a_k = a_{n+2} - 1, \quad n \geq 1 \tag{3.43}$$

b) From (3.18)

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{X^+(10)}{10} \tag{3.44}$$

$$= \frac{100}{100 - 10 - 1} \times \frac{1}{10} \tag{3.45}$$

$$= \frac{10}{89} \tag{3.46}$$

c) Using (1.2), (2.16), (2.30) we get:

$$b_n = a_{n+1} + a_{n-1} \tag{3.47}$$

$$= x(n) + x(n-2) \tag{3.48}$$

$$= y(n-1) \tag{3.49}$$

$$= \alpha^n + \beta^n \tag{3.50}$$

d) From (3.34)

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{Y^+(10)}{10} \tag{3.51}$$

$$= \frac{100 + 20}{100 - 10 - 1} \times \frac{1}{10} \tag{3.52}$$

$$= \frac{12}{89} \neq \frac{8}{89} \tag{3.53}$$

Thus, options (a),(b) and (c) are correct