9.3.12B

EE24BTECH11027 - satwikagv

Question:
Plot the solution of the differential equation:

$$y'' + xy' + xy = x. \tag{0.1}$$

Solution:
RK4 method:
Let assume

$$y' = z \tag{0.2}$$

substituting the eq (0.2) in eq (0.1) we get,

$$z' + xz + xy = x \tag{0.3}$$

$$z' = x(1 - y - z) \tag{0.4}$$

Let

$$z' = f(x, y, z) \tag{0.5}$$

$$f(x, y, z) = x(1 - y - z) \tag{0.6}$$

To update y and z first we need to find the intermediate slopes of y and z those are given by

$$k_{1,y} = h(z_n) \tag{0.7}$$

$$k_{2,y} = h\left(z_n + \frac{k_{1,z}}{2}\right) \tag{0.8}$$

$$k_{3,y} = h\left(z_n + \frac{k_{2,z}}{2}\right) \tag{0.9}$$

$$k_{4,y} = h(z_n + k_{3,z}) \tag{0.10}$$

$$k_{1,z} = hf(x_n, y_n, z_n) \tag{0.11}$$

$$k_{2,z} = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_{1,y}}{2}, z_n + \frac{k_{1,z}}{2}\right) \tag{0.12}$$

$$k_{3,z} = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_{2,y}}{2}, z_n + \frac{k_{2,z}}{2}\right) \tag{0.13}$$

$$k_{4,z} = hf\left(x_n + h, y_n + k_{3,y}, z_n + k_{3,z}\right) \tag{0.14}$$

The next values of x, y and z are given by

$$x_{n+1} = x_n + h \tag{0.15}$$

$$y_{n+1} = y_n + \frac{1}{6}\left(k_{1,y} + 2k_{2,y} + 2k_{3,y} + k_{4,y}\right) \tag{0.16}$$

$$z_{n+1} = z_n + \frac{1}{6}\left(k_{1,z} + 2k_{2,z} + 2k_{3,z} + k_{4,z}\right) \tag{0.17}$$

On iterationg the eqs from (0.7) to (0.17) over 100 times we get some values of y. Here we assume the initial conditions as

$$x_0 = 0 \tag{0.18}$$

$$y_0 = 0 \tag{0.19}$$

$$z_0 = 1 \tag{0.20}$$

$$h = 0.1 \tag{0.21}$$

By plotting the values of y we get in eq (0.16) we get the graph of the solution of the given differential equation

Method of Differences

To plot the curve of the given differential equation (0.1) we can do it using the method of finite differences which is a numerical technique for solving complex differential equations by approximating derivatives with differences.

The approximated forward derivative of $y(x)$ is given as:

$$y_n' \approx \frac{y_{n+1} - y_n}{h} \tag{0.22}$$

On rearranging we get,

$$y_{n+1} = y_n + y_n'(h) \tag{0.23}$$

And also

$$x_{n+1} = x_n + h \tag{0.24}$$

The approximated forward derivative of second order of $y(x)$ is given as:

$$y_n'' \approx \frac{y_{n+1}' - y_n'}{h} \tag{0.25}$$

Substitute eq (0.22) in eq (0.25) we get,

$$y_n'' \approx \frac{y_{n+2} - 2y_{n+1} + y_n}{h^2} \tag{0.26}$$

Substitute eq (0.22) and eq (0.26) in eq (0.1) and on rearanging we get,

$$y_{n+2} = y_{n+1}(2 - hx_n) + y_n\left(-1 + hx_n - h^2x_n\right) + h^2x_n \tag{0.27}$$

We need to assume two initial conditions as it is a second order differential equation.

So here we assume the initial conditions as

$$x_0 = 0 \tag{0.28}$$
$$y_0 = 0 \tag{0.29}$$
$$y'_0 = 1 \tag{0.30}$$
$$h = 0.1 \tag{0.31}$$

substitute eq (0.28), eq (0.29) and eq (0.30) in eq (0.1)
we get

$$y''(0) = 0 \tag{0.32}$$

Substitute eq (0.29) and eq (0.30) in eq (0.23)

$$y_1 = y_0 + y'_0(0.1) \tag{0.33}$$
$$y_1 = 0.1 \tag{0.34}$$

For the rest of the points use eq (0.27) we get the other points.
The graph sim1 represents the graph obtained by the method of finite differences
The graph sim2 represents the graph obtained by the RK4 method