Date

Relation and Function

* Cartesian Product

→ Suppose A & B are sets. and
$A = \{1, 2, 3\}$ and $B = \{1, 2\}$

Then $A \times B = \{(1,1)(1,2)(2,1)(2,2),(3,1)(3,2)\}$

This is called Cartesian product.

Now, $n(A) = 3$ and $n(B) = 2$
So $n(A \times B) = 3 \times 2 = 6$
Generally $\boxed{n(A \times B) = p \times q}$ where $p = n(A)$ and $q = n(B)$

Relation :-
→ A relation is a subset of the Cartesian product of two sets obtained by describing a particular condition

(Total no. of subsets of A×B.) Total no. of relations, of a Cartesian Product. $= 2^{p \times q}$

{* A relation R in the set A means, A relation in A×A.}

Two forms of representing a relation :-

(1) Roster form
eg $R = \{(1,1), (1,2), (1,3)\}$

(2) Set - builder form
eg $R = \{(a, b) : |a-b| \geq 0\}$

Page No. ____
Date

Types of Relation :-

- Empty Relation R, in A is given by,

$$R = \phi \subset A \times A.$$

- Universal Relation R, in A is given by,

$$R = A \times A \subseteq A \times A.$$

a.k.a trivial relations.

- Reflexive relation R in A is given by,

$$(a, a) \in R \quad \forall a \in A,$$

if $A = \{1, 2\}$
Eg $= \{(1,1)(2,2)(1,2)\}$
This is Reflexive but not identity relation

~~Symmetric relation~~

- Identity relation R in A is given by,

if & only if $(a, a) \in R \ \forall a \in A$

Eg. $= \{(1,1)(2,2)\}$
This is identity relation & also Reflexive.

- Symmetric Relation R in A is a relation satisfying $(a, b) \in R \Rightarrow (b, a) \in R$.

- Transitive relation R in A is a relation satisfying $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$.

→ if $(a, b) \in R$ but we cannot find $(b, c) \in R$ then also it is considered transitive.

Date

(*) Equivalence Relation R in A is a relation which is reflexive, symmetric and transitive.

□ Another method for representing a relation

Suppose $A = \{1, 2, 3, 4\}$ then.

→ $R = \{(a, b) : b = a + 1\}$

New Method

$a R b$ if and only if $b = a + 1$

if $(a, b) \in R$ we say a is related to b and we denote it as $a R b$.

Example: Let T be the set of all Δs in a plane with R a relation in T given by $R = \{(T_1, T_2) : T_1 \cong T_2\}$. Show that R is an equivalence relation.

Sol. T = set of all Δs in a plane (1)

$R = \{(T_1, T_2) : T_1 \cong T_2\}$ (2)

(3) Reflexive

R is reflexive since every triangle is congruent to itself i.e. $(T_1, T_2) \in R$ for all $T_1, T_2 \in T$

(4) Symmetric

$(T_1, T_2) \in R \Rightarrow T_1 \cong T_2 \Rightarrow T_2 \cong T_1 \Rightarrow (T_2, T_1) \in R$

Hence R is symmetric

(5) transitive

$(T_1, T_2) \in R \Rightarrow T_1 \cong T_2$

$(T_2, T_3) \in R \Rightarrow T_2 \cong T_3$

$\Rightarrow (T_1 \cong T_3) \Rightarrow (T_1, T_3) \in R$

Therefore R is an equivalence relation

Note :-

* If the conditions of Reflexive etc. are failing then solve by giving an example.
* If not then write theory of Short theory.
(Mathematical)

Equivalence Class (denoted by [a]) :-

Equivalence class [a] containing a ∈ A for an equivalence relation R in A is the subset of A containing all elements b related to a.

~~If R is an equivalence relation in A. Is~~

Detail

$Z =$ integers

$R = \{(a,b) : 2 \text{ divides } a-b\}$

→ Note that R is an equivalence relation.

- all even integers are related to zero. $E =$ all even int. $\subset Z$
- all odd integers are related to one. $O =$ all odd int. $\subset Z$.

Conditions

(i) All elements of E are related to each other
(ii) all elements of O are related to each other
(ii) & No element of E is related to any element of O & vice-versa
(iii) E and O are disjoint and $\boxed{Z = E \cup O}$ [As E and O are disjoint.]
also $E \cap O = \phi$.

(1) Therefore, E is called equivalence class containing zero denoted by, [0] and $[0] = 2r$ where $r \in Z$.

(2) and O is called equivalence class containing one. denoted by [1] and $[1] = 2r+1$ where $r \in Z$.

(Equivalence class)

→ ~~The result is true for n no. of equivalence relations in R, even if they are divided into n no. of equivalence classes.~~

~~Example Let R be the relation in set $A = \{1, 2, 3, 4\}$~~

P.T.O

Example

Equivalence CLASS

Let $A = \{a, b, c\}$

$R = \{(a,a), (b,b), (c,c), (b,c), (c,b)\}$

The Equivalence class of a under given relation is denoted by $[a]$, and it is defined as

$[a]$ = set of all elements related to a = $\{a\}$

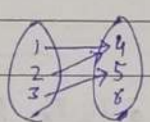
$[b]$ = set of all elements related to b = $\{b, c\}$

$[c]$ = set of all elements related to c

$= \{c, b\}$

Functions

→ function is a special type of relation, ~~which~~ in which every element of A (when $f : A \to B$) must have one and only one image in set B.



here $A = \{1, 2, 3\}$ is Domain
$B = \{4, 5, 6\}$ is Co-Domain
~~element~~ $R = \{4, 5\}$ is Range
(set of images)

- 1 is pre-image of 4.

TYPES OF FUNCTIONS :-

(*) **One - one** :- (one element → one image)

A function $f : X \to Y$ is one-one (or injective) if

$f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \quad \forall x_1, x_2 \in X$.

otherwise f is called many - one. (if $x_1 \neq x_2$ or $x_1 = \pm x_2$)
(Many element → one image)

[Methods to Prove a functions one - one] :-

①	②	③
$f(x_1) = f(x_2)$ $\Rightarrow x_1 = x_2 \ \forall x_1, x_2 \in X$.	$x_1 = x_2$ $\Rightarrow f(x_1) = f(x_2)$ $\forall x_1, x_2 \in X$	if $x_1 \neq x_2$ $\Rightarrow f(x_1) \neq f(x_2) \ \forall x_1, x_2 \in X$ if this fails not one-one.

Date

* ~~Onto (every element preimage of set X~~ :-

Onto ~~(every exist preimage of set x~~ (or surjective) if

A function $f: X \to Y$ is onto (or surjective) if given any $y \in Y$, $\exists$ (there exists) $x \in X$ such that $f(x) = y$.

(or)

Simple def: (~~A function f: X → Y is said to be onto if every~~ element of Y is the image of some element of X under ~~~~ f.)

~~(or)~~ {$f: X \to Y$ is onto if and only if Range of $f = Y$}

[Codomain = Range]

if Codomain ≠ Range ⇒ into function

METHOD to solve :-

(1) Write Normal theory ~~~~ :-

~~(2)~~ give example (if failing).

(2) Let $y = f(x)$; ~~and convert it in form of x.~~

Eg show that $f: N \to N$ given by $f(x) = 2x$ is not onto

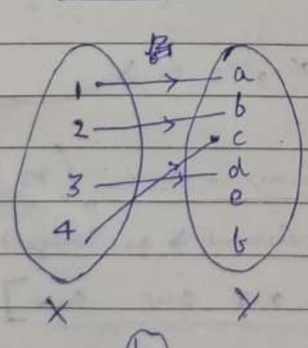
$y = f(x) \Rightarrow y = 2x \Rightarrow \frac{y}{2} = x \notin N$ for ~~every~~ some $y \in N$.

~~~~ $\frac{1}{2} = x \notin N$.

* if not failing then ~~~~ for all y. [proved]

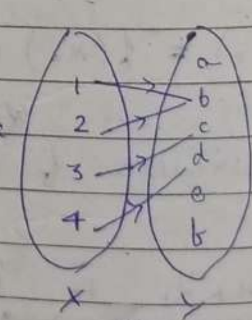
Pictorial Representation

- one-one



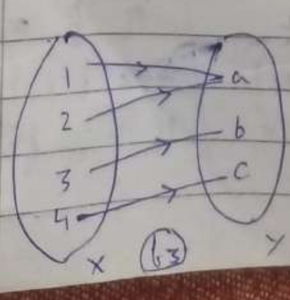
(f1)

- many-one



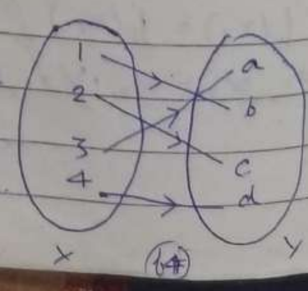
(f2)

- many-one
- Onto



(f3)

- one-one
- Onto
- Bijective



(f4)
~~~~

Page No.
Date

(*) Bijective function :-

⇒ A function $f : X \to Y$ is said to be one-one and onto (or bijective) if it is both one-one and onto.

(*) $\boxed{\text{A Bijective function is Invertible.}}$

→ its inverse exists.
i.e if $y = f(x)$ then
$f^{-1}(y) = x$ is possible.

PTO

~~MORE~~

Q. $f(x) = x + \frac{1}{x}$ is one-one or not ?

Sol. $f(x_1) = f(x_2) \Rightarrow x_1 + \frac{1}{x_1} = x_2 + \frac{1}{x_2} \Rightarrow (x_1 - x_2) + \left(\frac{1}{x_1} - \frac{1}{x_2}\right) = 0.$

$(x_1 - x_2) \times \left(1 - \frac{1}{x_1 x_2}\right) = 0 \Rightarrow \boxed{x_1 = x_2}$

$1 - \frac{1}{x_1 x_2} = 0 \Rightarrow 1 = \frac{1}{x_1 x_2} \Rightarrow \boxed{x_1 = \frac{1}{x_2}}$

$\boxed{x_1 \neq x_2}$

Not one-one, it is many-one

(check) Alternate Method :-

$\frac{dy}{dx} \geq 0 \ \forall x \in A$, $\frac{dy}{dx} \leq 0 \ \forall x \in A$ then

then one-one

Date

More Formulae :-

(I) if $n(A) = p$
$n(B) = q$

No. of Relation from A to B $= 2^{p \times q}$

(II) if $n(A) = p$
$n(B) = q$

No. of function from A to B $= q^{p}$

(III) $n(A) = p$
$n(B) = q$

No. of one - one
$n(A) \le n(B)$

No. of one - one from A to B
$$= \begin{cases} 0, & p > q \\ {}^{q}P_{p}, & p \le q \end{cases}$$

(iv) Onto $n(A) \ge n(B)$

$$\text{No. of Onto from A to B} = \begin{cases} 0, & p < q \\ \sum_{r=1}^{q} (-1)^{q-r}\, {}^{q}C_{r}\, r^{p}, & p \ge q \\ p!, & p = q \end{cases}$$

(V) Bijective $n(A) = n(B)$

$$\text{no. of Bijective} = \begin{cases} p! & p = q \\ 0 & p \ne 0 \end{cases}$$

If $A = \{a_1, a_2, a_3, \ldots n\}$ $\boxed{n(A) = n}$

$R = \{(a, b) : a, b \in A\}$

$R : A \to A$

* then no. of identity relations $= 2^n$

* And no. of Reflexive relations $= 2^{n^2 - n} = 2^{n(n-1)}$

* no. of Symmetric relations $= 2^{\frac{n(n+1)}{2}}$

* no. of Anti-Symmetric relations $= 2^n \times 3^{\frac{n^2 - n}{2}}$

$= 2^n \times 3^{\frac{n(n-1)}{2}}$

Ch - 3

MATRIX

Matrix :-

* A matrix is an ordered rectangular array of nos or functions. The no. or functions in the array are called elements/entries of matrix

- Rows of Matrix : The Horizontal line of elements.
- Columns of Matrix : The Vertical lines of elements.

~~General Format/Re :-~~

Representation of Matrix -

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \end{bmatrix}$$

Generalised Format -

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

(Or)

$$A = [a_{ij}]_{m \times n} \quad \text{where, } 1 \le i \le m, \; 1 \le j \le n$$

- ith row elements are $a_{i1}, a_{i2} \ldots a_{in}$
- jth Column elements are $a_{1j}, a_{2j} \ldots a_{mj}$

Order of a Matrix

* If a matrix has 'm' rows and 'n' columns, then order of the matrix is $m \times n$.
* The total no. of elements in the matrix of $m \times n$ will be mn.
* Eg. order of the below matrix is 2×3.

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 1 & 5 & 9 \end{bmatrix}$$

with total no. of elements = product of 3 and 2 = 6.

* Any point (x, y) in a plane can be represented in the matrix format as below:

$$P = [x, y] \quad \text{or} \quad \begin{bmatrix} x \\ y \end{bmatrix}$$

* ~~Suppose if the~~
* Suppose if the matrix has 8 elements then all possible order of the matrix can be found as below.
  - Find the factors of 8 → 1, 2, 4, 8.
  - ~~Find~~ Pair up them in all possible ways → ~~such that~~ such that their product is 8.

    (1, 8), (2, 4), (8, 1), (4, 2).

- Representation :-

(i) $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ or $\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ or $\left\|\begin{matrix} 1 & 2 \\ 4 & 3 \end{matrix}\right\|$

TYPES OF MATRIX :-

1.) **Column MATRIX**

- A matrix having only one column and any no. of rows.

Eg: ~~$A = [a_{ij}]_{1\times n}$~~ $A = [a_{ij}]_{m\times 1}$

- ~~A matrix having only one~~

2.) **ROW MATRIX**

- A matrix having only one row and any no. of columns.

$$A = [a_{ij}]_{1\times n}$$

3.) **Square Matrix**

- A matrix of order $m \times n$, such that $m = n$.

$A = [a_{ij}]_{m\times m}$, ~~order = m~~ is also called (Square matrix of order m.)

4.) **Diagonal matrix**

- A square matrix is said to be a diagonal matrix if all of its non-diagonal elements are zero.

$A = [b_{ij}]_{m\times m}$ and,

~~i.e.~~ $b_{ij} = 0$ when $i \neq j$.

Eg $[4]$ $\quad B = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ $\quad C = \begin{bmatrix} -1.1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

are Diagonal Matrix

- Horizontal Matrix: ~~n > m~~ $m < n$ Eg. $\begin{bmatrix} 1 & 2 & 4 \\ 4 & 2 & 1 \end{bmatrix}_{2\times 3}$

- Vertical Matrix: $m > n$ $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}_{3\times 2}$

Page No. ____
Date ____

5.) Scalar Matrix.

- A diagonal matrix is said to be a scalar matrix if its diagonal elements are equal. $A = [b_{ij}]_{m\times m}$ ~~and~~ where
- ~~Eg~~

$$b_{ij} = 0 \; ; \text{ when } i \neq j$$
$$\text{and } b_{ij} = k \; ; \text{ when } i = j \text{ for some constant } k.$$

~~Eg: [3]~~ Eg: $A = [3]$ $(k = 3)$ $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ $(k = -1)$

vi) Identity Matrix

It is a type of Scalar matrix in which $k = 1$.

- We denote Identity matrix of order n by ~~In~~ I_n or I.

where

$$I_n = [b_{ij}]_{m\times m} \; ~~and~~ \begin{Bmatrix} b_{ij} = 1 & \text{if } i = j \\ \& \; b_{ij} = 0 & \text{if } i \neq j \end{Bmatrix}$$

Eg:- $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(Square matrix) $\boxed{\text{order} = 2}$

vii) Zero matrix $\boxed{\text{denoted by } O.}$

All elements are zero. Eg. $[0]_{1\times 1}$ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$ $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$

~~order~~

viii) Equal Matrix

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if

(i) They are of the same order; and, if

(ii) $a_{ij} = b_{ij}$ {Every corresponding element is equal}

Triangular Matrix } Lower Δ Matrix $\begin{bmatrix} 1 & 0 & 0 \\ 5 & 2 & 0 \\ 6 & 7 & 4 \end{bmatrix}$

Upper Δ Matrix $\begin{bmatrix} 1 & 4 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$

Page No.
Date

Operation on Matrices

~~Addition of Mat~~

(*) ~~At~~ Addition of Matrices.

- Let A and B be two matrices each of order $m \times n$. Then, the sum of the matrices $A+B$ is defined only if matrices A and B are of the same order.

~~or $A = [a_{ij}]_{m\times n}$~~

- $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$

  Then, $A+B = [a_{ij} + b_{ij}]_{m\times n}$

- General Format

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ & $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$

Then, we define, $A+B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix}$.

- Example:-

-11+5

$$\begin{bmatrix} 5 & 2 \\ 4 & 9 \end{bmatrix} + \begin{bmatrix} -11 & 0 \\ 7 & 1 \end{bmatrix} = \begin{bmatrix} -6 & 2 \\ 11 & 10 \end{bmatrix}$$

Note When Adding two matrix A & B, if the order is not same then $A+B$ is not defined.

Properties of Addition of Matrices

- If $A = [a_{ij}]$, $B = [b_{ij}]$ & $C = [c_{ij}]$ are ~~the~~ three matrices of order $m \times n$, then.

⇒ Commutative Law

$$A + B = B + A$$

→ Associative Law

$$(A+B) + C = A + (B+C)$$

→ Existence of Additive Identity.

A Zero matrix (0) of order $m \times n$ ~~(same as~~ (same as of A), is additive identity, if

$$A + O = A = O + A.$$

→ Existence of Additive Inverse.

If A is a square matrix, then the matrix $(-A)$ is called additive inverse, if

$$A + (-A) = O = (-A) + A.$$

- <u>$-A$ is the additive inverse of A or negative of A.</u>

→ Cancellation Law

$$A + B = A + C \Rightarrow B = C \quad (\text{Left Cancellation Law})$$

$$B + A = C + A \Rightarrow B = C \quad (\text{Right Cancellation Law})$$

Subtraction of Matrices

- Let A and B be two matrices of the same order, then subtraction of matrices A − B, is defined as :-

$$A - B = [a_{ij} - b_{ij}]_{m\times n}, \text{ where } A = [a_{ij}]_{m\times n},$$
$$B = [b_{ij}]_{m\times n}$$

Eg:- If $A = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 3 & 6 \end{bmatrix}$ & $B = \begin{bmatrix} 0 & -4 & 3 \\ 9 & -4 & -3 \end{bmatrix}$

- Here order of A is 3×2 and order of B is 3×2, therefore subtraction is possible.

$$A - B = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 3 & 6 \end{bmatrix} - \begin{bmatrix} 0 & -4 & 3 \\ 9 & -4 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 6 & -3 \\ -9 & 7 & 9 \end{bmatrix}$$

Multiplication of a MATRIX By A SCALAR

- Let $A = [a_{ij}]_{m\times n}$ be a matrix
- Let k be any Scalar. than,

$$KA = \cancel{[k(a_{ij})} \; \cancel{[ka_{ij}} \; [k \times a_{ij}]_{m\times n}$$

Properties of Scalar Multiplication

(i) $K(A+B) = KA + KB$ (A & B are matrices) (K is scalar)

(ii) $K + l\,(A) = KA + lA$ (K & l are scalars) (A is matrix)

(iii) $K \times l \times A = K(l \times A) = l(K \times A)$

(iv) $(-K)A = -(KA) = K(-A)$

(negative of a matrix).

MULTIPLICATION OF MATRICES

* ~~Consider two matrices A & B, then for the multiplication the multip~~ for the ~~to multiplication to be~~

(*) The Product of two matrices (say A & B) is defined only if the no. of columns of A is equal to the no. of rows of B.

(*) If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$ then the product $C = [c_{ij}]$ will be a matrix of the order $m \times p$.

~~Then A*~~

Eg: $A = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 3 & -1 \end{bmatrix}$ of order $\cancel{3 \times 2}$ $\underset{m}{2} \times \underset{n}{3}$

$B = \begin{bmatrix} 0 & 3 \\ -2 & -1 \\ 0 & 4 \end{bmatrix}$ of order $\underset{n}{3} \times \underset{p}{2}$

Then $A \times B =$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 3 & -1 \end{bmatrix} \times \begin{bmatrix} 0 & 3 \\ -2 & -1 \\ 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1\times 0 + 0\times -2 + 0\times -2 & 1\times 3 + 0\times -1 + -2\times 4 \\ 0\times 0 + 3\times -2 + -1\times 0 & 0\times 3 + 3\times -1 + -1\times 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -5 \\ -6 & -7 \end{bmatrix}$$

~~Note~~

not that imp

Note :-

- ~~If AB is defined, then BA need not be defined.~~
- ~~If $A = [a_{ij}]$ of order $m \times n$ and $B = [b_{ij}]$ of order $n \times p$ then the product $C = [c_{ij}]$ will be defined.~~

just something not that imp

Let, $A = [a_{ij}]$ be an $m \times n$ matrix
Let, $B = [b_{jk}]$ be an $n \times p$ matrix.

~~there we know that~~, ~~$C =$~~

Let, then, $C = [c_{ik}]_{m \times p}$ is the product of A & B.

~~To~~ ~~To get the ik^{th} element~~

(To get the $(i, k)^{th}$ element c_{ik} of the matrix C, we take the i^{th} row of A and k^{th} column of B, multiply them elementwise & take the sum of all these products.

i^{th} row of A is $[a_{i1} \ a_{i2} \ \dots \ a_{in}]$

k^{th} column of B is $\begin{bmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{nk} \end{bmatrix}$

than ;

$$c_{ik} = a_{i1} \times b_{1k} + a_{i2} \times b_{2k} \dots a_{in} + b_{nk} = \sum_{j=1}^{n} a_{ij} b_{jk}$$

Note:-

- If AB is defined then, BA need not be defined
- If A, B are respectively $m \times n$, $k \times l$ matrices, then both AB & BA are defined if & only if $n = k$ and $l = m$.
- If both A & B are square matrices of the same order, then both AB & BA are defined.

Non-Commutative Property

- If AB & BA are both defined it is not necessary that $\boxed{AB = BA}$, ~~but it is possible~~

But

- if A and B are diagonal matrices of same order. then $\boxed{AB = BA}$;

Eg: $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$, $AB = BA = \begin{bmatrix} 3 & 0 \\ 0 & 8 \end{bmatrix}$

if $AB = BA$ then A & B are known as commute matrices.

- If the product of two matrices is a zero matrix, it is not necessary that one of the matrix is a zero matrix or both are zero matrices.

Eg. $A = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$ & $B = \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$ then $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Properties of Multiplication of Matrices

(1.) Associative Law (For any three matrices A, B and C.)

$(AB)C = A(BC)$, whenever both sides of equality are defined. (i.e. Product of matrices is defined)

(2.) The Distributive Law (For any three matrices A, B and C.)

(i.) $A(B+C) = AB + AC$

(ii) $(A+B)C = AC + BC$, whenever both sides of equality are defined

(3.) The existence of Multiplicative identity.

→ For every square matrix A, there exist an identity matrix of same order such that

$$AI = IA = A$$

Note :-

If $AB = -BA$, then A & B are known as Anti-Commute matrices.

if A & B are commute i.e. $AB = BA$ then,

$$\boxed{(A+B)^2 = A^2 + B^2 + AB + BA}$$

Transpose of a Matrix

- Let $A = [a_{ij}]_{m \times n}$, be a matrix of order $m \times n$. Then the $n \times m$ matrix obtained by interchanging the rows & columns of A is called the transpose of A and is denoted by A'. (or A^T).

- $\cancel{A'(A^T) = [a_{ij}]}$

- $\underset{(A^T)}{A'} = [a_{ji}]_{n \times m}$

Eg: $A \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, $A' = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

Properties of Transpose

- $(A')' = A$
- $(A \pm B)' = A' \pm B'$
- $(AB)' = B'A'$
- $(KA)' = KA'$
- $(ABC)' = C' \times B' \times A'$
- $(A^n)' = (A')^n$

Note :-

if for a square matrix A $AA' = I$, then A is said to be orthogonal Matrix.

Page No. ____
Date ____

Symmetric & Skew Symmetric Matrices.

- A Square matrix $A = [a_{ij}]$ is said to be symmetric if $A' = A$.

- Eg: $A = \begin{bmatrix} \sqrt{3} & 2 & 3 \\ 2 & -1.5 & -1 \\ 3 & -1 & 1 \end{bmatrix}$. Here $A' = A$

- A Square matrix $A = [a_{ij}]$ is said to be skew symmetric matrix if $A' = -A$.

- Eg: $A = \begin{bmatrix} 0 & e & f \\ -e & 0 & g \\ -f & -g & 0 \end{bmatrix}$, $A' = -A$

Note:-

* If A is symmetric then, $[a_{ij}] = [a_{ji}]$
* If A is skew symmetric then, $[a_{ij}] = -[a_{ji}]$
* All the diagonal elements of a skew symmetric matrix are $\boxed{\text{zero}}$

Square matrix $\Rightarrow \boxed{i=j} \Rightarrow \{ a_{ii} = -a_{ii} \Rightarrow \boxed{2a_{ii}=0}$

Theorem 1
For any Square matrix A with real number entries, $A + A'$ is a symmetric matrix & $A - A'$ is a skew symmetric matrix

~~Theorem 2 Any Square Matrix A with real no entries,~~

$A + A'$

Page No. ____

Date

Theorem 2

Any square matrix can be expressed as the sum of a symmetric & a skew symmetric matrix.

(Let A be a Matrix)

$$A = A$$

$$2A = A + A$$

$$2A = A + A' + A - A' \quad \text{(why?)}$$

$$\boxed{A = \frac{1}{2}(A+A') + \frac{1}{2}(A-A')}$$

Suppose, $B = A + A'$, B is symmetric

~~then,~~ $~~(KB)' = KB'~~$

& suppose $\boxed{C = KB}$

$C' = (KB)' = KB' = KB = C$

So C is Symmetric

$\boxed{\therefore (KB)' = KB'}$

So $\frac{1}{2} \times (A+A')$ is symmetric.

& $\frac{1}{2} \times (A-A')$ is skew ".

i)

ii.)

iii)

Conclusion :-

Multiply a symmetric or skew sym by a K, ($K \neq 0$) then it is still a symmetric or skew symmetric matrix.

Page No.
Date

3 Elementary Operation (Transformation) of a Matrix

→ There are six operations, three due to rows & ~~three do~~ other three due to columns. Known as Elementary operations.

(i) The Interchanging of any two rows or two columns.

$\boxed{\text{So } C \text{ is Symmetric}}$

$R_i \leftrightarrow R_j$ and $C_i \leftrightarrow C_j$

Rows Columns

- Interchange of i^{th} row and j^{th} row & i^{th} column and j^{th} column.

(ii.) The Multiplication of the elements of any row or column by a non-zero number.

$R_i \to KR_i$ and $C_i \to KC_j$

- Multiplication ~~Interchange~~ of i^{th} row & i^{th} column. ~~by constant K (K ≠ 0)~~. by any non-zero no. K.

iii) Addition to the elements of any row or column, the corresponding elements of any other row or column multiplied by a non-zero number.

$R_i \to R_i + KR_j$ and $C_i \to C_i + KC_j$

Eg:- $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ ~~applying R_2 $R_2 = 2R_1$~~

applying $R_2 \to R_2 + (-2)R_1$

than we get, $\begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix}$

Invertible (Inverse) Matrices.

~~If A is a sq~~

(*) If A & B are square matrices of same order, let be m such that;

$$\boxed{AB = BA = I}$$

then B is called the inverse matrix of A and is denoted by A^{-1}. Thus.

~~$A = B^{-1}$ and $B = A^{-1}$~~

$$\boxed{B = A^{-1}} \boxed{\text{and} \quad A = B^{-1}}$$

Note :-

- Rectangular matrices does not have inverse matrix.
- If $B = A^{-1}$ then, $A = B^{-1}$.

Theorem 3 Uniqueness of Inverse of a Square matrix.

~~If it exists it is Unique.~~

- If Inverse of a Square matrix exists it is Unique.

Let 'A' be a Square Matrix.
Let 'B' and 'C' be Inverses of A,
than we get,

$$AB = BA = I$$
$$\text{and} \quad AC = CA = I$$

~~Let~~

$$B = BI = B(AC) = BAC = IC = C$$

hence $\boxed{B = C}$

Theorem 4

If A and B are invertible matrices of the same order, then $(AB)^{-1} = B^{-1}A^{-1}$.

$$\boxed{(AB)^{-1} = B^{-1}A^{-1}}$$

(4) Inverse of a Matrix by Elementary Operations.

→ If A is a matrix such that A^{-1} exists, then to find A^{-1} using elementary row operations, write, ~~$A=$~~

$\boxed{A = IA}$ & apply a sequence of row operation on $A = IA$ till we get, $\boxed{I = BA}$

Example

Find inverse of $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ by elementary operations.

~~Note:- We can find A' using column operations by writing A = AI & finding I = AB~~

Sol. $A = IA$ or $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

then,

$$\begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A$$

$[R_2 \to R_2 - 2R_1]$

$$\rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix} A$$

$[R_2 \to -\frac{1}{5} R_2]$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix} A$$

$(R_1 \to R_1 - 2R_2)$

$$\rightarrow A' = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix}$$

Page No.
Date

Similarly, we can find A^{-1} using Column Operations, then, write $\boxed{A = AI}$ & ~~write~~ apply a sequence of column operation on $(A = AI)$, till we get, $\boxed{I = AB}$ (1)

Example

~~A~~ Find inverse of $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ by elementary operations (2)

Sol. $A = AI$

$$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(3.)

$$\begin{bmatrix} 1 & 0 \\ 2 & -5 \end{bmatrix} = A \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}, \text{ Applying } (C_2 \to C_2 - 2C_1)$$

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = A \begin{bmatrix} 1 & 2/5 \\ 0 & -1/5 \end{bmatrix}, \text{ Applying } \left(C_2 \to -\frac{1}{5}C_2\right)$$

(4.)

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & -1/5 \end{bmatrix}, \text{ Applying } (C_1 \to C_1 - 2C_2)$$

$$I = AB \Rightarrow A^{-1} = \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & -1/5 \end{bmatrix}$$ Eg.)

(*) By ~~using~~ applying one or more elementary operations, in case if we obtain all zeros in one or more rows of the Matrix A on L.H.S. then A^{-1} does not exist. Sol)

Some Special Types of Matrices :-

(1) Idempotent Matrix:

$$A^2 = A \Rightarrow \boxed{A^n = A}, n > 2 \text{ and } n \in N$$

(2) Nilpotent Matrix (of order m) :-

$\boxed{A^m = O}$, $m \in N$, and

$\{A^{m-1}$ is not $O\}$,

(3.) Periodic Matrix :-

$A^{k+1} = A$, for some positive integer k.

Period: least value of k which satisfies above equation

(Period of idempotent is 1 } ($A^{1+1} = A$)

(4.) Involutary Matrix :-

$A^2 = I$, [$A = A^{-1}$ for involutary matrix]

Eg.) A be involutary matrix such that $A^x = I$, for $x > 0$

find $\left\{ \sum_x \sin^x\theta + \cos^x\theta \right\}_{min} = ?$

Sol) $x = 2, 4, 6, 8 \ldots$

$S_\infty = \frac{a}{1-r}$

(when $r < 1$)

Now, $\sin^2\theta + \cos^2\theta + \sin^4\theta + \cos^4\theta + \ldots \infty$

$\{\sin^2\theta + \sin^4\theta + \ldots\} + \{\cos^2\theta + \cos^4\theta \ldots\}$

$= \frac{\sin^2\theta}{1-\sin^2\theta} + \frac{\cos^2\theta}{1-\cos^2\theta} = \left\{ \tan^2\theta + \cot^2\theta \right\}_{min} = 2$

$\because$ AM $\geq$ GM $x + \frac{1}{x} \geq 2$ when $x > 0$

PAGE No
DATE: / / 201

Properties of adjoint

- $adj(O) = O$ $\{ n = $ order of given Square matrix
- $adj(I_n) = I_n$
- $adj(adj(A)) = |A|^{n-2} A$
- $|adj(adj(A))| = |A|^{(n-1)^2}$

Example
$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \quad n = 2$$
$$adj(adj(A)) = |A|^{2-2} \cdot A = A$$

$$|\underbrace{adj(adj \ldots adj(A))}_{r \text{ times}}| = |A|^{(n-1)^r}$$

- $Adj(KA) = K^{n-1} Adj(A)$
(K is scalar)

- $|KA| = K^n |A|$
- $|A^{-1}| = \frac{1}{|A|}$

- $|A'| = |A|$

Chapter - 4

Determinant

- ~~Only Square matrix~~ determinant is calculated for Square Matrix only.
- The Answer (Value) of Determinant is Real Number.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \\ 3 & 4 & 2 \end{bmatrix}$$

$$|A| = \det A = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \\ 3 & 4 & 2 \end{vmatrix}$$

Determinant of A.

(*) Determinant of matrix of order :-

(1) 1×1 (1 by 1) :-

$$A = [5]$$

$$|A| = |5|_{1\times1} = 5 \text{ ans}$$

(2) ~~$2 \times 2 \rightarrow$~~

~~$A = \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix}_{2\times2}$~~

Date

(2) $2\times 2 \to (2 \text{ by } 2) \to$

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}_{2\times 2}$$

$$|A| = \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix}$$

$$= 2\times 4 - 1\times 3$$
$$= 8 - 3$$
$$= \boxed{5} \text{ ans}$$

(3.) 3×3 (3 by 3) $\to$

~~(3) 3×3 (3 by 3)~~

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

(*) We can find Determinant of 3×3 Matrix by many ways :-

- ~~By~~ By Expanding According to :-

(i) Row 1
(ii) Row 2
(iii) Row 3
(iv) Column 1
(v) Column 2
(vi) Column 3

By Row 1 $\{a_{11} \; a_{12} \; a_{13}\}$ -

$$|A| = (-1)^{1+1} \times 1 \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} + (-1)^{1+2} \times 2 \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} + (-1)^{1+3} \times 1 \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}$$

$$= 1\times 1 (3 - 1) - 1\times 2 (1 - 2) + 1\times 1 (1 - 6)$$

$$= 2 + 2 - 5$$

$$= \boxed{-1} \text{ ans}$$

~~Re~~ Note :-

(i) For Easier Calculation, Expand along more Zeroes (0_s).

(ii) instead of multiplying by $(-1)^{i+j}$ while Expanding, We can multiply by +1 or −1 according as $(i+j)$ is even or odd.

(iii) Let A $\begin{bmatrix} 2 & 2 \\ 4 & 0 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$. ~~Then, we can see~~

Then, we can see. $\boxed{A = 2B}$ Also

$|A| = 0 - 8 = -8$ $\boxed{\text{and}}$ $|B| = 0 - 2 = -2$

or $|A| = 2^n |B|$, where $n = 2$ is the order of Square matrices A & B.

(*) In General if $A = kB$ where A and B are ~~Square~~ square matrices of order n, ~~then $|A| = k^n |B|$~~ then, $|A| = k^n |B|$ where $n = 1, 2, 3$.

(order of Matrix)

Page No.
Date

Example

$$|A| = \begin{vmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 4 & 1 & 0 \end{vmatrix} \quad a_{13} = +$$

Along Column 3 :-

$$|A| = +4 \begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix}$$

$$= 4(-1-12)$$

$$= \boxed{-52} \text{ ans}$$

Example

$$\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix} \quad \begin{matrix} a_{21} & a_{22} & a_{23} \\ - & + & - \end{matrix}$$

Along Row 2 :-

$$= -6 \begin{vmatrix} -3 & 5 \\ 5 & -7 \end{vmatrix} + 0 - 4 \begin{vmatrix} 2 & -3 \\ 1 & 5 \end{vmatrix}$$

$$= \boxed{-28} \text{ ans}$$

Example

$$|A| = \begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix} \quad \boxed{a_{11} = +}, \boxed{a_{12} = -}, \boxed{a_{13} = +}$$

Alog Row 1 :-

$$|A| = +0 - 1 \begin{vmatrix} -1 & -3 \\ -2 & 0 \end{vmatrix} + 2 \begin{vmatrix} -1 & 0 \\ -2 & 3 \end{vmatrix}$$

$$= -1(0-6) + 2(-3)$$

$$= 6 - 6$$

$$= \boxed{0} \text{ ans}$$

Page No.
Date

Previous formula

2. Area of Triangle - $\left\{ \frac{1}{2} \left[x_1(y_2-y_3) + x_2(y_3-y_1) + x_3(y_1-y_2) \right] \right\}$

$(x_1, y_1) \quad (x_2, y_2) \quad (x_3, y_3)$

$$\text{Area of Triangle} = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Example

$(3, 8) \quad (-4, 2) \quad (5, 1)$

$x_1, y_1 \quad x_2, y_2 \quad x_3, y_3$

$$(\text{Area})\ \Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2} \begin{vmatrix} 3 & 8 & 1 \\ -4 & 2 & 1 \\ 5 & 1 & 1 \end{vmatrix}$$

$a_{13} = +$
$a_{23} = -$
$a_{33} = +$

Along Column 3 :-

$$= \frac{1}{2} \left[+1 \begin{vmatrix} -4 & 2 \\ 5 & 1 \end{vmatrix} - 1 \begin{vmatrix} 3 & 8 \\ 5 & 1 \end{vmatrix} + 1 \begin{vmatrix} 3 & 8 \\ -4 & 2 \end{vmatrix} \right]$$

$$= \frac{1}{2} \left[+1(-4-10) - 1(3-40) + 1(6+32) \right]$$

$$= \frac{1}{2} \left[-14 + 37 + 38 \right] \Rightarrow \cancel{\frac{1}{2} \times 61} \ \boxed{\frac{61}{2}} \text{ ans.}$$

Page No:
Date

Determinant.

Note :-

~~Remarks~~

(i) Area is Positive Quantity, So we take only Absolute value of Area.

(ii) If Area is given use both +ve & -ve values of the Determinant to find other things.

(iii) The Area of triangle formed by three ~~colene~~ collinear points is Zero.

Properties of Determinants

Property 1 :

- The value of the Determinant remains Unchanged ~~same~~ ~~if you find it by~~ if its rows & and columns are interchanged. ~~Transpose~~ (i.e. if it ~~Matrix~~ is Transposed).

$$\text{Let } \Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

~~then~~ ~~and them,~~ ~~then,~~

then,
$$\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(Transpose of Matrix Δ.)

So, $\Delta = \Delta_1$

Property 2 :

- If any two rows (or columns) of a Determinant are interchanged ~~to~~ then sign of Determinant changes.

Let $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 1 & 6 \\ 1 & 6 & 7 \end{vmatrix}$

$R_1 \leftrightarrow R_2$

than, we get

$\Delta = - \begin{vmatrix} 4 & 1 & 6 \\ 1 & 2 & 3 \\ 1 & 6 & 7 \end{vmatrix}$

Property 3 :

- If any two rows (or columns) of a Determinant are identical ~~(each corresponds~~ (all corresponding elements are same), then value of the Determinant is zero.

$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix}$

$R_1 = R_3$

$\Delta = 0$

Property 4 :

- If each element of a row (or a column) of a determinant is multiplied by a constant k, then its value get multiplied by k.

Let Δ (determinant) $= \begin{vmatrix} 1 & 3 & 1 \\ 1 & 2 & 4 \\ 1 & 2 & 3 \end{vmatrix}$

$R_1 \to 4R_1$

$\Delta_1 = \begin{vmatrix} 4 & 12 & 4 \\ 1 & 2 & 4 \\ 1 & 2 & 3 \end{vmatrix}$ So, $\Delta_1 = 4\Delta$

$C_2 \to 2C_2$

$\Delta_2 = \begin{vmatrix} 4 & 24 & 4 \\ 1 & 4 & 4 \\ 1 & 4 & 3 \end{vmatrix}$ So, $\Delta_2 = 2\Delta_1$

or

$\boxed{\Delta_2 = 4 \times 2\Delta}$

(*) **We can also take common :-**

let $\Delta = \begin{vmatrix} 6 & 1 & 1 \\ 4 & 3 & 5 \\ 8 & 2 & 1 \end{vmatrix}$

$\Delta = 2\begin{vmatrix} 3 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 2 & 1 \end{vmatrix}$

Page No.
Date

Property 5 :

Let $\Delta = \begin{vmatrix} a+2 & b+3 & c+1 \\ 6 & 8 & 1 \\ 3 & 2 & 1 \end{vmatrix}$

then,

$$\Delta = \begin{vmatrix} a & b & c \\ 6 & 8 & 1 \\ 3 & 2 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 3 & 1 \\ 6 & 8 & 1 \\ 3 & 2 & 1 \end{vmatrix}$$

- If some or all elements of a row or column of a determinant are expressed as sum of two (or more) terms, then the determinant can be expressed as sum of two (or more) Determinants.

Property 6 :

- The value of Determinant remain same if we apply the operation $R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$.

Let $\Delta = \begin{vmatrix} 1 & 3 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 4 \end{vmatrix}$

$R_2 \to R_2 + 3R_1$

$$\Delta = \begin{vmatrix} 1 & 3 & 2 \\ 2+3(1) & 3+3(3) & 1+3(2) \\ 1 & 2 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 3 & 2 \\ 5 & 12 & 7 \\ 1 & 2 & 4 \end{vmatrix}$$

Minors & Co-Factors :-

Minor

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

(determinant)

(*) $\boxed{\text{Minor of } a_{11}}$ $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22} \times a_{33} - a_{32} \times a_{23}$

(*) $\boxed{\text{Minor of } a_{22}}$ $M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = a_{11} \times a_{33} - a_{31} \times a_{13}$

Cofactor

$A_{11} = (-1)^{1+1} \times M_{11}$

$\boxed{\text{Cofactor of } a_{11}}$

$A_{22} = (-1)^{2+2} \cdot M_{22}$

$\boxed{\text{Cofactor of } a_{22}}$

Remark :-

(*) Minor of an element of a determinant such that Determinant of order n $(n \geq 2)$, is a determinant of order $(n-1)$.

Page No.
Date

Note.

New Definition

Δ = sum of the product of elements of any row (or column) with their corresponding co factors.

(Determinant)

Note

(*) If Element of a row (or column) are multiplied with cofactors of any other row (or column) then, their sum is zero.

Example: $\Delta = a_{11} A_{21} + a_{12} A_{22} + a_{13} A_{23} = 0$

or

$a_{11} A_{31} + a_{12} A_{32} + a_{13} A_{33} = 0$

Adjoint of a Matrix.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Then adj A = Transpose of $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

adjoint of A

$= \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

(*) where, A_{ij} is the co-factor of the element a_{ij}.

Date

Remark

- For a Square matrix of $\boxed{\text{order } 2}$, given by

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$ ~~the~~ the adjoint of A is $\Rightarrow \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

change sign   interchange

~~only~~ (*) We can use previous method also but this is a short cut.

(*) Theorem 1 :-

If A be any square matrix of order n, then

$$A(\operatorname{adj} A) = (\operatorname{adj} A)A = |A| I.$$

Definition 4

A Square matrix A is said to be singular if $\boxed{|A| = 0}$.

Definition 5

A Square matrix A is said to be non-singular if $\boxed{|A| \neq 0}$

Theorem 2

If A & B are non-singular matrices of the same order then AB & BA are ~~rates~~ also non-singular matrices of the same order.

(*) if $|A| \neq 0$ and $|B| \neq 0 \Rightarrow |AB| \neq 0$ and $|BA| \neq 0$

Page No.
Date

Theorem 3

$$|AB| = |A|\ |B|$$ A & B are square matrices of same order.

The Determinant of the product of matrices is equal to product of their respective Determinants.

~~Theorem 4~~ also Let A be a square matrix of order n. Then $|(\text{adj } A)| = |A|^{n-1}$ where n is order of square matrix A.

~~If A is a square matrix of order n.~~

Theorem 4.

A Square matrix A is invertible if and only if A is a non-singular matrix

~~True~~ Proof:- if A is invertible then ~~True~~

Conversely, Let A is a non-singular matrix.

Then $|A| \neq 0$ ~~and~~

Now,
$$A(\text{adj } A) = (\text{adj } A) A = |A| I \quad (\text{Theorem 1})$$

$$A\left(\frac{1}{|A|}\text{adj } A\right) = \left(\frac{1}{|A|}\text{adj } A\right) A = I$$

$$AB = BA = I$$

So $$\boxed{B = \frac{1}{|A|}\text{adj } A}$$

Thus A is invertible and $$\boxed{A^{-1} = \frac{1}{|A|}\text{adj } A}$$

Date

(*) Solving Linear Equations with Matrices & Determinants.

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

CASE 1 If $|A| \neq 0$ (Non-singular matrix)

- ~~Now~~ We can write,

$$AX = B$$

$$(A^{-1}A)X = A^{-1}B$$

$$\boxed{X = A^{-1}B}$$

CASE 2 if $|A| = 0$ (singular)

(*) Then if $(\text{adj } A) \cdot B \neq 0$

~~that~~ implies No Solution (Inconsistent)

(*) but if ~~Then~~ $(\text{adj } A) \cdot B = 0$ implies

Consistent $\boxed{\text{or}}$ Inconsistent

(many solutions) (No Solution)

Thee

Example

Solve the system of equn. $2x + 5y = 1$, $3x + 2y = 7$

Sol. equn can be written as $\boxed{AX = B}$

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

Now $|A| = -11 \neq 0$

A is Non singular.

So A has unique solution

$$A^{-1} = -\frac{1}{11}\begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix} \Rightarrow X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \Rightarrow \boxed{x = 3} \quad \boxed{y = -1}$$

Properties of Determinant (Advanced)

(*) $$\begin{vmatrix} 0 & 0 & 0 \\ a & b & c \\ d & e & f \end{vmatrix} = \begin{vmatrix} 0 & a & d \\ 0 & b & e \\ 0 & c & f \end{vmatrix} = 0$$

(*) $$\begin{vmatrix} R_1 + R_2 \\ R_2 + R_3 \\ R_3 + R_1 \end{vmatrix} = 2\begin{vmatrix} R_1 \\ R_2 \\ R_3 \end{vmatrix}$$ $$\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

(*) Some Standard Circular (Symmetric) Determinant :-

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ca & ab \end{vmatrix} = (a-b)(b-c)(c-a)$$

$$\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$$

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^4 & b^4 & c^4 \end{vmatrix} = (a-b)(b-c)(c-a)(a^2+b^2+c^2-ab-bc-ca)$$

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3+b^3+c^3-3abc)$$
$$= -\{(a+b+c)(a^2+b^2+c^2-ab-bc-ca)\}$$

PAGE No
DATE: / 201

Eg $\begin{vmatrix} x^n & x^{n+2} & x^{n+3} \\ y^n & y^{n+2} & y^{n+3} \\ z^n & z^{n+2} & z^{n+3} \end{vmatrix} = (x-y)(y-z)(z-x)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$, find '$n$'.

Sol) $(xyz)^n \begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix} = (x-y)(y-z)(z-x) \times xy+yz+zx$

$(xyz)^n = (xyz)^{-1} \Rightarrow \boxed{n=-1}$

PTO

(*) $d(uvw) = \frac{du}{dx} vw + \frac{dv}{dx} uw + \frac{dw}{dx} uv$

$$\Delta(x) = \begin{vmatrix} R_1 \\ R_2 \\ R_3 \end{vmatrix} \text{ or } \begin{vmatrix} C_1 & C_2 & C_3 \end{vmatrix}$$

$$\frac{d\Delta}{dx} = \begin{vmatrix} R_1' \\ R_2 \\ R_3 \end{vmatrix} + \begin{vmatrix} R_1 \\ R_2' \\ R_3 \end{vmatrix} + \begin{vmatrix} R_1 \\ R_2 \\ R_3' \end{vmatrix} \text{ as in Column.}$$

Eg: $\Delta(x) = \begin{vmatrix} (1+x)^{1947} & (1+x)^{1952} & (1+x)^{1967} \\ (1+x)^{1748} & (1+x)^{1942} & (1+x)^{2050} \\ (1+x)^{2020} & (1+x)^{2007} & (1+x)^{4096} \end{vmatrix}$

$$\Delta'(0) = \begin{vmatrix} 1947(1+0)^{1946} & 1952(1+0)^{1951} & 1967(1+0)^{1966} \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$+ \begin{vmatrix} 1 & 1 & 1 \\ - & - & - \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ - & - & - \end{vmatrix} = 0$$

System of Equations:

(i) For two Variables: $a_1x + b_1y = c_1$
$a_2x + b_2y = c_2$

has unique soln. $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$
has ∞ soln $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ } Consistent.

has no Soln. $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ } Inconsistent.

Homogeneous eqn

- If, $a_1x + b_1y = 0$ has non-zero solution
  $a_2x + b_2y = 0$ then,

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$$

Note:-
if $\Delta = 0$
∞ Solutions.
if $\Delta \neq 0$
Unique Solution

Homogeneous Equation

- if $a_1x + b_1y + c_1z = 0$
  $a_2x + b_2y + c_2z = 0$
  $a_3x + b_3y + c_3z = 0$ } has non-trivial/non-zero Solution then,

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Eg. $ax + y + z = 0$
$x + by + z = 0$
$x + y + cz = 0$
has non-zero Solutions
then $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = ?$

Sol. $\Delta = \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$ $\quad C_1 \to C_1 - C_2,\ C_2 \to C_2 - C_3$ $\begin{vmatrix} a-1 & 0 & 1 \\ 1-b & b-1 & 1 \\ 0 & 1-c & c \end{vmatrix} = 0$

$1-b-1+c$

$(a-1)\{c(b-1) - (1-c)\} + (1-b)(1-c) = 0$

$c(1-a)\times(1-b) + (1-a)(1-c) + (1-b)(1-c) =$

$\frac{c}{1-c} + \frac{1}{1-b} + \frac{1}{1-a} = 0 \Rightarrow$ Required =

So A has unique Solution $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \Rightarrow x = 3,\ y = -1$

Heterogeneous Equations.

Note:-
Non trivial, $\Delta = 0$
trivial $\Rightarrow \Delta \neq 0$

Cramer's Rule

$$\left.\begin{array}{l} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{array}\right\} \qquad x = \frac{\Delta_1}{\Delta} \qquad y = \frac{\Delta_2}{\Delta} \qquad z = \frac{\Delta_3}{\Delta}$$

~~find~~

find $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

→ $\Delta \neq 0$ [Unique Solution].

→ $\Delta = 0$, find $\Delta_1, \Delta_2, \Delta_3$

→ $\Delta_1 = \Delta_2 = \Delta_3 = 0$ (∞ Solutions)

→ at least one of $\Delta_1, \Delta_2, \Delta_3$ is non zero. (no solution).

where $\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_3 \\ d_3 & b_3 & c_3 \end{vmatrix}$

Similarly, $\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$

PAGE No
DATE: / / 201

Non-singular

Q) → The no. of 3×3 matrices with four entries as 1 & and all other as 0 is 6.

→ The no. of 3×3 symmetric matrices of above configuration will be 12

- If $\omega \neq 1$ is complex cube root of unity $\Rightarrow 1+\omega+\omega^2 = 0$

$\omega^3 = 1 \;;\; \omega^{3k} = 1$

Q.) If the trivial solution is the only solution of the system of equations

$x - ky + z = 0$
$kx + 3y - kz = 0$
$3x + y - z = 0$

then the set of all values of k is

Sol. trivial $\Rightarrow \Delta \neq 0$

$$\Delta = \begin{vmatrix} 1 & -k & 1 \\ k & 3 & -k \\ 3 & 1 & -1 \end{vmatrix} = 2(k+3)(k-2)$$

as $\Delta \neq 0 \Rightarrow \boxed{k \neq -3, 2}$

Ans $R - \{-3, 2\}$

Q) Let ω be a complex no. such that $2\omega + 1 = z$ where $z = \sqrt{-3}$. If

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2-1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix} = 3k, \text{ then } k = ?$$

Sol) $2\omega + 1 = \sqrt{3}i \Rightarrow \omega = \dfrac{-1+\sqrt{3}i}{2} \Rightarrow$ ω is complex cube root of unity

$1+\omega+\omega^2 = 0$ & $\omega^3 = 1$

$$3k = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix} = \begin{vmatrix} 3 & 1 & 1 \\ 0 & \omega & \omega^2 \\ 0 & \omega^2 & \omega \end{vmatrix}$$

$C_1 \to C_1 + C_2 + C_3$

$3(\omega^2 - \omega^4) = 3k \Rightarrow (\omega+\omega^2)(\omega-\omega^2) = 3k$

$(\omega^2 - \omega) = 3k$

$2k =$

Δ has unique solution $|x| = |3|$

PAGE No.
DATE: / 201

$$w^2 - w = k$$

$$k = \left(\frac{-1-\sqrt{3}i}{2}\right) - \left(\frac{-1+i\sqrt{3}}{2}\right)$$

$$= -\sqrt{3}i = -z = \bar{z}$$

2) If S is the set of distinct values of b for which the following system of linear equations

$x + y + z = 1$

$x + ay + z = 1$

$ax + by + z = 0.$

has no solution, then S is

Sol If we put $a = 1$ then first two equations are identical

Now, $x + y + z = 1$ } has no solution
~~at~~ & $x + by + z = 0$ }

i.e for $\boxed{b=1}$ we have $x + y + z = 0$. (not possible)

i.e. ~~the~~ the equations don't have any solutions for $\boxed{b=1}$.

~~x+y+z~~

Ans $S = \{1\}$

Chapter - 2

(~~Bijective~~ = ~~Invertible~~)

Real number $\rightarrow \boxed{y = \sin x}$ angle

~~then~~ than,

$\boxed{\sin^{-1} y = x}$ angle

real number

$x \in$ All real Angle

than Domain = R

Range $\rightarrow [-1, 1]$

Domain $\Rightarrow [-1, 1]$

&

Range $\Rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $\leftarrow$ Principle Range.

		Domain		Range (Principal Branch)
$\sin^{-1}$	:	$[-1, 1]$	$\rightarrow$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1}$	:	$[-1, 1]$	$\rightarrow$	$[0, \pi]$
$\operatorname{cosec}^{-1}$	:	$R - (-1, 1)$	$\rightarrow$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$\sec^{-1}$	:	$R - (-1, 1)$	$\rightarrow$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
$\tan^{-1}$	:	R	$\rightarrow$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1}$	:	R	$\rightarrow$	$(0, \pi)$

Date ______ Page ____

Q) (1) $y = \sin^{-1}\left(-\frac{1}{2}\right)$

$\sin 30° = \frac{1}{2}$

$\sin y = -\frac{1}{2}$

Range of $\sin^{-1} \Rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ that is the value of y lies between $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Then,

$\sin y = -\sin\left(\frac{\pi}{6}\right)$

$\sin y = \sin\left(-\frac{\pi}{6}\right)$

$\boxed{y = -\frac{\pi}{6}}$

(2) $y = \cos^{-1}\left(-\frac{1}{2}\right)$

$\cos y = -\frac{1}{2}$

Range of $\cos^{-1} \Rightarrow [0, \pi]$ i.e value of y lies b/w $[0, \pi]$

$\cos y = -\cos 60°$

$\cos y = \cos 120°$

$\boxed{y = 120°}$

Date ______ Page ____

Q) $y = \cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

$\cot y = \frac{-1}{\sqrt{3}}$

Range of $\cot^{-1} \Rightarrow (0, \pi)$ ~~$\Rightarrow y \in (0, \pi)$~~ i.e. $y \in (0, \pi)$

$\cot 60^\circ = \frac{+1}{\sqrt{3}}$

$-\cot 60^\circ = \frac{-1}{\sqrt{3}}$

$\cot(180^\circ - 60^\circ) = \frac{-1}{\sqrt{3}}$

$\cot(120^\circ) = \frac{-1}{\sqrt{3}} \Rightarrow y = 120^\circ$

$\sin y = \frac{x}{1}$

$\boxed{\text{New Concept}}$

$y = \sin^{-1}(x)$

$\sin y = \frac{x}{1} \begin{matrix} \leftarrow P \\ \leftarrow H \end{matrix}$ than Base $\Rightarrow \sqrt{1-x^2}$

$\cos y = \frac{\sqrt{1-x^2}}{1} \Rightarrow y = \cos^{-1}\left(\sqrt{1-x^2}\right)$

$\tan y = \frac{x}{\sqrt{1-x^2}} \Rightarrow y = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

$\cot y = \frac{\sqrt{1-x^2}}{x} \Rightarrow y = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

$\sec y = \frac{1}{\sqrt{1-x^2}} \Rightarrow y = \cot^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$

$\operatorname{cosec} = \frac{1}{x} \Rightarrow y = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$

$P = x$
$H = 1$
$B = \sqrt{1-x^2}$

Conclusion :-

$y = \sin^{-1}(x) = \cos^{-1}\left(\sqrt{1-x^2}\right)$

$= \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

$= \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$

Date ____ Page ____

Example

$y = \tan^{-1}\left(\frac{3}{4}\right)$ $\boxed{P=3}$ $\boxed{B=4}$ $\boxed{H=5}$

than,

$y = \tan^{-1}\left(\frac{3}{4}\right) = \sin^{-1}\left(\frac{3}{5}\right) = \sec^{-1}\left(\frac{5}{4}\right)$

More Formulas:

- $\sin^{-1}(-x) = -\sin^{-1}x$
- $\tan^{-1}(-x) = -\tan^{-1}x$
- ~~$\cosec^{-1}(-x$~~ $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}(x)$
- ☆ $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$
- ☆ $\sec^{-1}(-x) = \pi - \sec^{-1}(x)$
- ☆ $\cot^{-1}(-x) = \pi - \cot^{-1}(x)$

(I) $\sin^{-1}(x) + \cos^{-1}(x) = 90°$
(II) $\tan^{-1}(x) + \cot^{-1}(x) = 90°$
(III) $\operatorname{cosec}^{-1}(x) + \sec^{-1}(x) = 90°$

(☆) $\tan^{-1}(x) + \tan^{-1}(y) =$ ~~$\frac{\tan^{-1}(x+y)}{1-xy}$~~ $\tan^{-1}\left(\frac{x+y}{1-xy}\right)$

tan 1

- only when $xy < 1$

(☆) $\tan^{-1}(x) - \tan^{-1}(y) = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

tan 2

- only when $xy > -1$

Date ______ Page ______

Note:-

* $\operatorname{cosec}^{-1} x = 90° - \sec^{-1} x$

$\sin^{-1} x = 90° - \cos^{-1} x$

$\tan^{-1} x = 90° - \cot^{-1} x$

* similar for others.

* $\cancel{\frac{1 - \tan x}{1 + \tan x} = \tan(45° - x)}$

$2\tan^{-1}(x) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

if $\tan^{-1}(\theta) = \tan^{-1}(x)$ than $\boxed{\theta = x}$

* similar for others

$\tan^{-1}\left(\frac{1 - \tan x}{1 + \tan x}\right) = \tan^{-1}(1) - \tan^{-1}(\tan x)$

using identity tan 2

* $\tan\left(\tan^{-1} \frac{17}{6}\right) = \frac{17}{6}$

We know that $\tan^{-1}(\tan \theta) = \theta$ is true and $\tan(\tan^{-1}(x)) = \boxed{x}$ is also true

MORE CONCEPTS :→

* $\sin^{-1}(\sin \theta) = \theta$

similar formula for other ratios as well. ~~sin ---- to --- cosec.~~

* $\sin(\sin^{-1} x) = x$

similar formula for all other ratios as well.

* $\cos^{-1} x + \cos^{-1} y = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$

* $\cos^{-1} x - \cos^{-1} y = \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right)$

* $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \tan^{-1}\left(\frac{x + y + z - xyz}{1 - xy - yz - zx}\right)$

$p^2 = (1+y^2)^2 - (1-y^2)^2$

$= 4y^2$

$p = 2y$