Date ____ Page ____

Ch-4 Limits

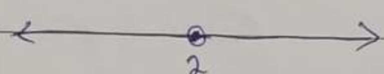
* There are two types of checking the limit of a function at a particular value.

Type (I) Eg: $\lim_{x \to 1} \frac{x-1}{x^2-1}$; This is not define form

So, $\lim_{x \to 1} \frac{1}{x+1} = \frac{1}{2}$

Function is defined for all Real value except -1.

Type (II) Eg: $f(x) = \begin{cases} 2x+3 & \text{if } x \le 2 \\ 2x-3 & \text{if } x > 2 \end{cases}$



CASE I

~~At $x = 2$,~~

~~if $x \le 2$ if $x = 2$~~

~~$f(x) = 2(2)+3$~~

$= 5$

At $x = 2$

$f(2) = 2(2)+3$

$= 5.$

L.H.L.

$\lim_{x \to 2^-} f(x) =$

$= \lim_{h \to 0} f(2-h)$

$= \lim_{h \to 0} 2(2-h)+3$

$= 7$

R.H.L.

$\lim_{x \to 2^+} f(x)$

$= \lim_{h \to 0} f(2+h)$

$= \lim_{h \to 0} 2(2+h)-3$

$= 1$

Date ______ Page ____

L.H.L $\neq$ R.H.L.

So $f(x)$ is not continous at $x = 2$.

CASE II

Let $x = c \in (-\infty, 2)$

$f(x) = 2x + 3$

$f(c) = 2c + 3$

$\lim_{x \to c} f(x) = \lim_{x \to c} 2x + 3 = 2c + 3$

So it is continous for all $c \in (-\infty, 0)$

CASE III

Let $x = c \in (20, \infty)$

$f(x) = 2x - 3$

$f(c) = 2c - 3$

$\lim_{x \to c} f(x) = \lim_{x \to c} 2x - 3 = 2c - 3$

So it is continous for all $c \in (0, \infty)$

Definition 1

Eg ~~Example~~

Suppose f is a real function ~~on a subset of the real numbers~~ and let c be a point in the domain of f, then f is continous at c if $\boxed{\lim_{x \to c} f(x) = f(c)}$

~~$f(x) = 2x + 3$~~

~~At $x = 1$,~~

~~$\lim_{x \to 1} f(x) = \lim_{x \to 1} 2x + 3 = 2(1) + 3 = 3$~~

~~$f(0) = -3$~~

~~hence f is continous at $x = 0$~~

Note : if LHL = RHL = Value of function(v) $\Rightarrow$ limit exists & continous
if only LHL = RHL $\neq$ V $\Rightarrow$ limit exist but not continous

Date ______ Page ____

Example 1 Check the continuity of the function f given by $f(x) = 2x+3$ at $x = 1$.

Sol. $f(x) = 2x+3$

~~At $x=1$~~, At $x = 1$,

~~At $x=1$~~, $\lim_{x\to 1} f(x) = \lim_{x\to 1} 2x+3 = 2(1)+3 = 5$

$f(1) = 2(1)+3 = 5$

hence f is continous at $x = 1$.

Definition 2

A real function f is said to be continous if it is continous at every point in the domain of f.

~~Ag~~ Algebra of Continous functions :→

(1) if $\lim_{x\to c} f(x) = f(c)$ and ~~$\lim_{x\to c} f(x)$~~ $\lim_{x\to c} g(x) = g(c)$

than; we get ~~f+g~~ $\lim_{x\to c} (f+g)(x) = (f+g)(c)$

or $\boxed{\text{if function continous at f and g than it is also continous at f + g}}$

Similar for $(-)$, $(\times)$ and $(\div)$.
such that g(x) ≠ 0

Read Special Cases.

Continuity of $\sin x$

$f(x) = \sin x$

It is defined for all real no.s.

Let c be a real no. (~~angle~~ real angle).

$f(c) = \sin c$

~~lim As~~ Put, ~~$x = c + h$~~

~~As $x \to c$, $h \to 0$.~~

$\lim_{x \to c^+} f(x) = \lim_{h \to 0} f(c+h) = \lim_{h \to 0} \sin[c+h]$

$= \lim_{h \to 0} [\sin c \cos h] + \lim_{h \to 0} [\cos c \times \sin h]$

$= \sin c + 0$

$\therefore \lim_{x \to c} f(x) = f(c)$

So The function is continuous for all real so it is a continuous function.

COMPOSITE FUNCTION

Eg:- Let $f(x) = x^2$ and $g(x) = \cos x$.

Their composite function is defined as.

$(f \circ g)(x) = f(g(x))$ In this case $(f \circ g)(x) = (\cos x)^2$

Comp. funct

This is defined whenever the range of ~~f~~ g (i.e. in this case $[-1, 1]$) is a subset of domain of f (i.e. in this case all Real).

THEOREM 2 ~~f an~~

Suppose f and g are real valued functions such that, $(f \circ g)$ is defined at c.

If $\lim_{x \to c} g(x) = g(c)$ & $\lim_{x \to g(c)} f(x) = f(g(c))$

than $\lim_{x \to c} (f \circ g)(x) = (f \circ g)(c)$ } $\lim_{x \to c} f(g(x)) = f(g(c))$

Or we can say ~~continuous at~~

1. ~~g~~ if g continuous at c.
2. and if f continuous at $g(c)$.
3. than $(f \circ g)$ continuous at c.

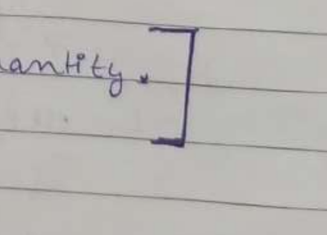
Continuity in a Closed interval {[ ]}

(*) A function f is said to be continuous in a closed interval $[a, b]$ if :

(i) f is cont. in open interval (a, b) &

(ii) f is right continuous at 'a' ~~i.e.~~

i.e. $\left[\lim_{x \to a^+} f(x) = f(a) = \text{a finite quantity}\right]$ ~~cont~~



(iii) f is left continuous at 'b'

i.e. $\left[\lim_{x \to b^-} f(x) = f(b) = \text{a finite quantity.}\right]$

* $|x|$, $|x-k|$ are continuous functions.

Date ______ Page ______

(*) $[x]$ is discontinous at all integral point.

Eg: $[4x-3]$; no. of points of discontinuity. when $1 < x < 3$

Sol) $1 < x < 3 \Rightarrow 4 < 4x < 12 \Rightarrow \boxed{1 < 4x-3 < 9}$

$1 < f(x) < 9$

$4x-3$ has integral value 2, 3, 4, 5, 6, 7, 8 where it is discontinous.

Eg: $f(x) = \frac{1}{1-x}$ is not cont. at $x=1$, ~~becau~~ why ?

Sol.) Bcoz it is not defined at $x=1$.

Advanced

Q) if $\lim_{x\to 0} [1 + x \ln(1+b^2)]^{1/x} = 2b\sin^2\theta$, $b>0$ & $\theta \in (-\pi, \pi]$, then $\theta = ?$

Sol)

1^∞ form : $\lim_{x\to a} (f(x))^{g(x)} = e^{\lim_{x\to a} g(x)(f(x)-1)}$

So, $\lim_{x\to 0} \left\{1 + x \ln(1+b^2)\right\}^{1/x} = e^{\lim_{x\to 0} \frac{1}{x} \times x \ln(1+b^2)}$

$= e^{\ln(1+b^2)} = 1+b^2$

Now, $1+b^2 = 2b\sin^2\theta$ (given)

For L.H.S = R.H.S,

$\sin^2\theta = 1 \Rightarrow \boxed{\theta = \pm \frac{\pi}{2}}$

Differentiability

$$\text{R.H.D} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
(Right hand derivative)

$$\text{L.H.D} = \lim_{h \to 0} \frac{f(x-h) - f(x)}{-h}$$
(Left hand derivative)

if, $\boxed{\text{L.H.D} = \text{R.H.D}}$

$f(x)$ will be differentiable.

~~derivat~~

DERIVATIVE FORMULAS :-

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = \cancel{\sec x} - \operatorname{cosec}^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \times \tan x$$

$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \times \cot x$$

$$\frac{d}{dx}(\log x) = \frac{1}{x}$$

$$\frac{d}{dx}(a^x) = a^x \log a.$$

$$\frac{d}{dx}(k) = 0$$

RULES

(a) $\frac{d}{dx}(cy) = c \times \frac{dy}{dx}$ {c is constant}

(b) $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$

(c) $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

(d) $= \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

(e) $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$

Date ______ Page ______

$4-2h+3$ $\quad$ $4-2h-3$

8) $f(x) = \begin{cases} 2x+3 & \text{if } x \le 2 \\ 2x-3 & x > 2 \end{cases}$

$f(2-h) = 2(2-h)+3 = 7-2h$

$f(2+h) = 2(2+h)-3 = 1+2h$

$f(2) = 2(2)+3 = 7$

$$L.H.D. = \lim_{h \to 0} \frac{f(2-h)-f(2)}{-h}$$

$$= \lim_{h \to 0} \frac{7-2h-7}{-h} = \lim_{h \to 0} 2$$

$$= 2$$

$$R.H.D. = \lim_{h \to 0} \frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h \to 0} \frac{1+2h-7}{h}$$

$$= \lim_{h \to 0} \frac{2h-6}{h}$$

$$= \cancel{6}\ \infty \text{ (undefined)}$$

$$\boxed{L.H.D \neq R.H.D.}$$

$y = f(x) = \cancel{\sin(x^2)^2}\ \sin(x^2)$

$$\frac{dy}{dx} = \frac{d}{dx}\sin(x^2) = \cos(x^2)\frac{d}{dx}(x^2) = \cos(x^2)\,2x$$

$$= 2x\cos(x^2)$$

Actual procedure -

$t = x^2$

$\frac{dt}{dx} = 2x$

$2x\,dx = \frac{dt}{2x}$

$y = \sin t$

$\frac{dy}{dt} = \cos t$

$\frac{dy}{dx} = \cos(x^2) \times 2x$

$y = f(x)$ $\quad x + y = x^2$

Date ______ Page ______

~~Implicit Function~~ $f(x)$ $y = f(x)$ $\frac{dy}{dx}$ $f(x+y) = x^2$

(*) Theorem 3

If a function f is differentiable at a point c, then it is also continuous at that point

(*) Corollary 1

Every differentiable function is continuous.

~~$y = f(x) = y$~~

IMPLICIT FUNCTION

Q. $x + y = x^2$ then find $\frac{dy}{dx} = ?$ & $\frac{dx}{dy} = ?$

dependent, independent; dependent, independent

$$\frac{d(x)}{dx} + \frac{d(y)}{dx} = \frac{d(x^2)}{dx}$$

$$\Rightarrow 1 \times \frac{dx}{dx} + \frac{dy}{dx} = 2x\frac{dx}{dx}$$

$$\boxed{\frac{dy}{dx} = 2x - 1}$$

~~$x + y = x^2$~~

~~$\frac{dx}{dy} + \frac{d}{dy} 1 \times \frac{dy}{dy} = \frac{d(x^2)}{dy}$~~

~~$1 = \frac{dx^2}{dy} - \frac{dx}{dy}$~~

~~$= \frac{dx}{dy}(x-1)$~~

Note:- $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

$$x + y = x^2$$

$$\frac{d(x)}{dy} + \frac{d(y)}{dy} = \frac{d(x^2)}{dy}$$

$$1 \times \frac{dx}{dy} + 1 \times \frac{dy}{dy} = 2x\frac{dx}{dy}$$

$$1 = 2x\frac{dx}{dy} - \frac{dx}{dy}$$

$$1 = \frac{dx}{dy}(2x - 1)$$

$$\boxed{\frac{1}{(2x-1)} = \frac{dx}{dy}}$$

Date ______ Page ____

Example.

$x^2 + y^2 = 8$, find $\frac{dy}{dx} = ?$

$\frac{d(x^2)}{dx} + \frac{d(y^2)}{dx} = \frac{d(8)}{dx}$

$2x\frac{dx}{dx} + 2y\frac{dy}{dx} = \cancel{8}\,0\,\frac{dx}{dx}$

$\cancel{2x} + 2y\frac{dy}{dx} = \cancel{0} - \frac{2x}{2y}$

$\boxed{\frac{dy}{dx} = \frac{-x}{y}}$

$\frac{dx}{dy} = ?$

$\frac{d(x^2)}{dy} + \frac{d(y^2)}{dy} = \frac{d(8)}{dy}$

$2x\frac{dx}{dy} + 2y\frac{dy}{dy} = 0$

$2x\frac{dx}{dy} = -2y$

$\boxed{\frac{dx}{dy} = -\frac{y}{x}}$

	Domain of f^{-1}
$\frac{d}{dx}\sin^{-1}x = +\frac{1}{\sqrt{1-x^2}}$	$(-1,1)$
$\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$	$(-1,1)$
$\frac{d}{dx}\tan^{-1}x = +\frac{1}{1+x^2}$	R
$\frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x^2}$	R
$\frac{d}{dx}\sec^{-1}x = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$ $\{\lvert x\rvert > 1\}$	$(-\infty,-1) \cup (1,\infty)$
$\frac{d}{dx}\operatorname{cosec}^{-1}x = -\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$R-[-1,1]$ $(-\infty,-1) \cup (1,\infty)$

(?)

derivative of $\boxed{\frac{dy}{dx} = ?}$

$y = \sin^{-1}(x)$

$\sin y = x$

$\cos y\,\frac{dy}{dy} = 1 \cdot \frac{dx}{dy}$

$\boxed{\frac{dy}{dx} = \frac{1}{\cos y}}$

$\cos y = \sqrt{1-\sin^2 y}$

$\cos y = \sqrt{1-x^2}$

$\boxed{\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}}$

~~EXPONENTIAL FUNCTION :->~~

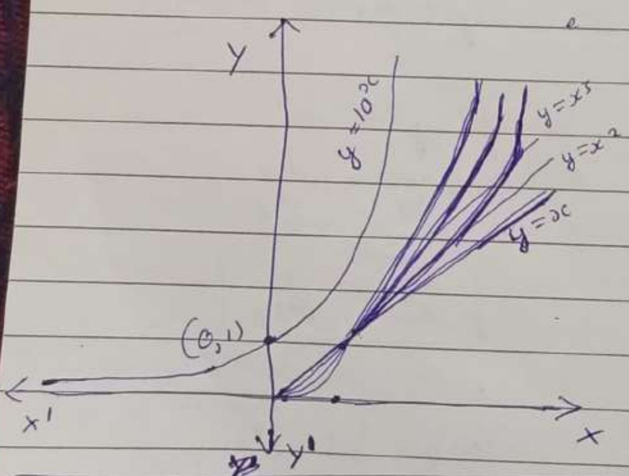
~~The function with the gives the highest value of y is the function.~~

$$\boxed{\cancel{y = 10^x}}$$

- ~~This is a kind of Exponential function~~
- ~~It~~

EXPONETIAL FUNCTION :-

$$\boxed{f(x) = y = b^x}$$



- $b > 1$ (b = Positive base)
- Domain all Real ($x \in \mathbb{R}$).
- Range all the Real no.s.
- Every Exponential function passes through $(0, 1)$
- Graph rises from ~~Left~~ left to Right
- For -ve values it comes closer to ~~0~~ 0 (zero).

$y = 10^x$ $\quad \boxed{y = 10^x > y = x^n}$

hence this function grows faster than any other function

This is called Common Exponential function.

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots = e \text{ (a no. b/w 2 \& 3)}$$

$$\boxed{y = e^x}$$

This is called Natural Exponential function

LOG FUNCTION :->

Example:-

~~$Log_2 8 = 3$~~ if, $2^3 = 8$ than $Log_2 8 = 3$

$Log_b a = x$ if $b^x = a$

* Log of a to the base b is x if $b^x = a$.

General form

$Log_b : R^+ \to R$.

* In Reality Log_b is a function from R^+ to R, such that

$\boxed{Log_b x = y}$ this implies, $b^y = x$.

where, $x \in R^+$ (Domain = +ve Real)

$y \in R$ (all Real)

$b > 1$

$Log_e x = y$ $\to$ o denoted by $\ln x$ or simply $Log\, x$.

* This is called Natural Log.

$Log_{10} x = y$

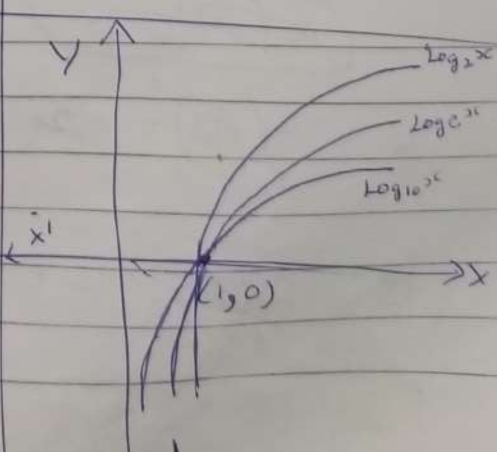
* This is called Common Log.

* o Log function ~~....~~ is ever increasing from Left to Right

o every log function passes through (1,0)

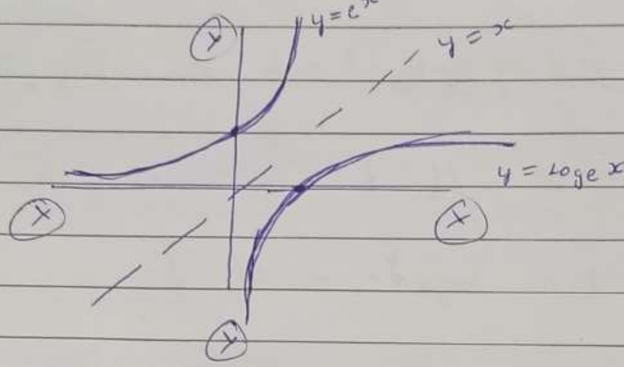
o ~~for x = -ve, the graph can be made going parallel to y-axis downwards~~

Graph :-



Note

(*) Log & Exponential functions are mirror images of each other reflected in the line $y = x$.



~~IA~~ IMP

FORMULAS

Imp Note:-

(*) $\boxed{\log_b 1 = 0}$ any base.

$\log_{10} 10 = 1$

$\log_e e = 1$

$\log_x x = 1$

Exponential Differentiation

constant

(*) $\frac{d}{dx}(a^x) = a^x \log_e a$

(*) $\frac{d}{dx}(e^x) = e^x (\log_e e) = 1$

$= e^x$.

$\cancel{\frac{d}{dx}(10^x) = 10^x}$

(*) $\frac{d}{dx}(e^{2x}) = e^{2x} \times 2$

(*) $\frac{d}{dx}(e^{-2x}) = e^{-2x} \times -2$

$\cancel{\frac{d}{dx}(e^{2x}) = e^{2x} \log_e e}$ and

$\cancel{= e^{2x}}$

$\frac{d}{dx}(a^{2x}) = a^{2x} \log_e a \times 2$

$\frac{d}{dx}(a^{-2x}) = a^{-2x} \log_e a \times -2$

$\frac{d}{dx}(e^x) =$

Properties of Log

$\left\{ \cancel{\frac{d(\log x)}{dx} = \frac{1}{x}} \right\}$

Change of base Property

$$\log_a b = \frac{\log_e b}{\log_e a}$$

$$\frac{d(\log_e x)}{dx} = \frac{1}{x}$$

$$\frac{d}{dx}(\log_7 x) = ?$$

$$\frac{d}{dx}\left(\frac{\log_e x}{\log_e 7}\right)$$

$$= \frac{1}{\log_e 7} \times \frac{1}{x} \text{ ans}$$

Log of product of no./function :→

(1) $\log_e(mn) = \log_e m + \log_e n$

(2) $\log_e \frac{m}{n} = \log_e m - \log_e n$

(3) $\log_e m^n = n \times \log_e m$

$$\log_e 4$$
$$= \log_e (2)^2$$
$$= 2\log_e 2 \text{ ans}$$

&

(4) $\log_e e = 1$

$\log_{10} 10 = 1$

$\log_a a = 1$

$e^{\log 5}$

$e^{\log 5} \times e$

Base

$e^{\log_e 5} = 5$

$e^{\log_e x^2} = x^2$

$a^{\log_a x^3} = x^3$

$e^{2\log_e x} = \cancel{x} \cdot e^{\log_e x^2} = x^2.$

Only valid for the values of x. $\boxed{(R+)}$. Domain

Date ____ Page ____

8)

$$y = x^x$$

Derivative Using Log

$$\log y = \log x^x$$

$$\cancel{\log y \; \frac{1}{y} \times \frac{dy}{dx}} = x \times \log x$$

$$\frac{1}{y} \times \frac{dy}{dx} = \frac{x}{x} + \log x$$

$$\frac{dy}{dx} = x^x (1 + \log x)$$

---

9)

$$\cancel{y = x^x + x^{\sin x}}$$

~~Let u = x^x and, v = x^{sin x}~~

Q) $y = x^x + x^{\sin x}$

Ex)

Let $u = x^x$ and $v = x^{\sin x}$

$$y = u + v$$

then, $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ — (1)

Now,

$$u = x^x$$

$$\log u = x \times \log x$$

$$\frac{1}{u} \times \frac{du}{dx} = \frac{x}{x} + \log x$$

$$\boxed{\frac{du}{dx} = x^x (1 + \log x)}$$

And,

$$v = x^{\sin x}$$

$$\log v = \log x^{\sin x}$$

$$\frac{1}{v} \times \frac{dv}{dx} = \sin x \times \log x$$

$$\frac{1}{v} \times \frac{dv}{dx} = \frac{\sin x}{x} + \log x \cos x$$

$$\frac{dv}{dx} = x^{\sin x} \left(\frac{\sin x}{x} + \log x \cos x \right)$$

$$\boxed{\frac{dy}{dx} = x^x (1 + \log x) + x^{\sin x} \left(\frac{\sin x}{x} + \cos x \log x \right)}$$

Date ____ Page ____

(a) $y = \sin x^{\cos x}$

$\log y = \cos x \times \log(\sin x)$ ← chain Rule

$\frac{1}{y} \times \frac{dy}{dx} = \frac{\cos^2 x}{\sin x} + \log(\sin x) \times -\sin x$

$\frac{dy}{dx} = \sin x^{\cos x}\left[\frac{\cos^2 x}{\sin x} + \log(\sin x)\times -\sin x\right]$

~~$\cos x \; y^6$~~

(b) $y = x^{\log x}$

$\log y = \log x^{\log x}$

$\frac{1}{y} \times \frac{dy}{dx} = \frac{d}{dx}(\log x \times \log x)$

$\frac{1}{y} \times \frac{dy}{dx} = 2\frac{\log x}{x}$

$\frac{dy}{dx} = x^{\log x} \times \left[\frac{2\log x}{x}\right]$

More Log Formulas.

(*) $\log_a b = \frac{1}{\log_b a}$

$\log_a \frac{1}{b} = -\log_a b$

$\log_{\frac{1}{a}} b = -\log_a b$

$\log_a b \log_b c = \log_a c$

$\log_{a^m} a^n = \frac{n}{m}, \; m \neq 0$

$\cancel{\frac{dy}{dx} = \sin x}$ ~~cos x~~

Date ______ Page ______

PARAMETRIC FORM :→

(Q) Find $\frac{dy}{dx} = ?$

$x = \cos t$ ← Parametric number.

$\frac{dx}{dt} = -\sin t$ — (i)

$y = \sin 2t$

$\frac{dy}{dt} = \cos 2t \times \frac{d}{dt}(2t)$

$\frac{dy}{dt} = \cos 2t \times 2$ — (ii)

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = 2\cos 2t \div -\sin t$ ⇒

$$\boxed{\frac{dy}{dx} = -\frac{2\cos 2t}{\sin t}}$$

DOUBLE DERIVATIVE :→

$y = x^3 + x^2 + x$

$\frac{dy}{dx} = 3x^2 + 2x + 1$

$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = 6x + 2$

Date ______ Page ____

~~Attached~~

~~Ex~~ Examples

Q Differentiate $\sin^2 x$ w.r.t. $e^{\cos x}$

(1)

Let $v = e^{\cos x}$ and $u = \sin^2 x$, so we have to find $\boxed{\frac{du}{dv} = ?}$

$u = \sin^2 x$

$\frac{du}{dx} = 2(\sin x) \times \cos x$

$v = e^{\cos x}$

$\frac{dv}{dx} = e^{\cos x} \times -\sin x$

Q

$\cancel{\frac{du}{dx}} \times \frac{du}{dx} \times \frac{dx}{dv} = \frac{2\sin x \cos x}{-\sin x \times e^{\cos x}}$

$\boxed{\frac{du}{dv} = -\frac{2\cos x}{e^{\cos x}}}$

~~7~~ ~~8~~ ~~11~~ ~~12~~ ~~13~~ ~~14~~ ~~15~~

(2) $\boxed{x = a(\cos t + t\sin t)}$ $\boxed{y = a(\sin t - t\cos t)}$

Q) $\frac{d^2y}{dx^2} = ?$

$\frac{dx}{dt} = a\left[-\sin t + t\cos t + \sin t\right]$

$\boxed{\frac{dx}{dt} = a\,t\cos t}$

$\frac{dy}{dt} = a\left[\cos t - (-t\sin t + \cos t)\right]$

$\boxed{\frac{dy}{dt} = +a\,t\sin t}$

$\Rightarrow \boxed{\frac{dy}{dx} = \tan t}$

$\frac{d^2y}{dx^2} = \sec^2 t \times \frac{dt}{dx}$

$= \frac{1}{\cos^2 t} \times \frac{1}{a\,t\cos t} \Rightarrow \frac{1}{\cos^3 t\, a t}$ ans

Mean Value Theorem :-

Theorem 6 (Rolle's Theorem)

Let $f : [a, b] \to R$ ← $\boxed{\text{Co-Domain}}$

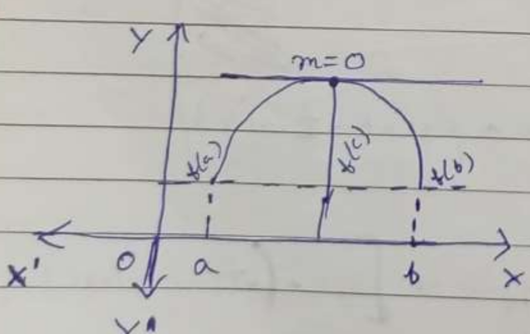
↑ $\boxed{\text{Domain}}$

a function from $[a, b]$ to R.

- ~~Let~~ if f ~~be~~ is continuous on $[a, b]$
- ~~Let~~ if f ~~be~~ is differentiable on (a, b)
- ~~and~~ ~~if, f(a) =~~ and if $\boxed{f(a) = f(b)}$

Then there exist some $c \in (a, b)$

& such that $f'(c) = 0$.



So $f'(c) = 0$

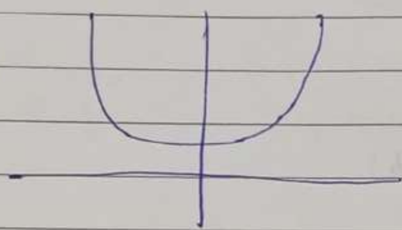
$\boxed{\text{Slope of tangent} = \text{Derivative of } f(x)}$

Example

$y = x^2 + 2$ ~~interv~~

Verify Rolle's Theorem for interval $[-2, 2]$

⇒ Sol.



- f is continuous ~~[~~ $[-2, 2]$
- f is differentiable $(-2, 2)$ as $f'x = 2x$ which defined for $(-2,2)$.
- $f(-2) = f(2)$

Rolle theorem: So there exist some $c \in (-2, 2)$

such that $\boxed{f'(c) = 0}$

f ~~Let~~ ~~$y = x^2+2$~~ ~~$f(x) = x^2+2$~~

~~$f'(x) = 2x$~~

~~Let $f'(x) = 0$, then,~~

~~$0 = 2x$~~

~~$\boxed{x = 0}$ At $x=0$ $f'(x) = 0$~~

At ~~$x = 0$; $f'(x) = 0$.~~

$f(x) = x^2+2$

$f'(x) = 2x$

Let ~~us~~ ~~sup~~ $f'(x) = 0$

$0 = 2x$

$\boxed{x = 0}$

At $x = 0$ $f'(x) = 0$.

~~So~~ ~~$\boxed{x = c = 0}$~~

&

~~and~~

~~$\in$~~ $x \in (-2, 2)$

$\boxed{\text{hence Verified}}$ ✓

Mean Value Theorem :- (Extension of Rolle's Theorem).

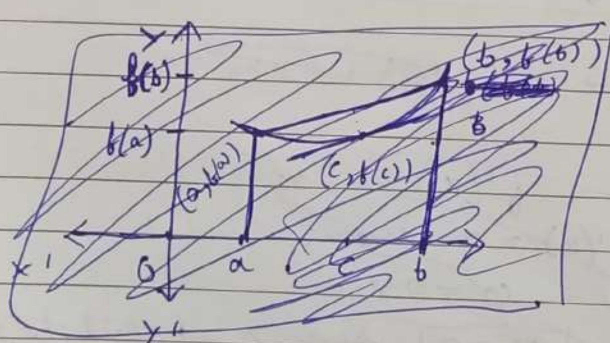
Let $f : [a, b] \rightarrow R$

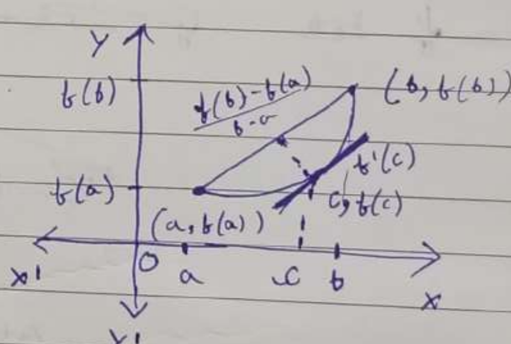
f ↑ a function. $[a, b]$ ↑ Domain R ↑ Co-Domain

- if f is continous on $[a, b]$
- if f is Differentiable on (a, b).
- if $f(a) \neq f(b)$

(*) then there exist ~~for~~ some $c \in (a, b)$

such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$





Date ______ Page ____

Example

$f(x) = x^2$ ~~fare interval~~

Verify MVT for interval $[2,4]$

Sol. ✓ f is continuous on $[2,4]$
✓ f is Differentiable on $(2,4)$.

$$f(a) = 4 \quad , f(b) = 16.$$

$$f(a) \neq f(b)$$

M.V.T ~~for there~~
So there exists some such $c \in (2,4)$.

such that $$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$\cancel{f(c) = \frac{16-4}{2}}$$

$$f'(c) = \frac{f(4) - f(2)}{4-2}$$

$$f'(c) = \frac{16-4}{2} = \frac{12}{2}$$

$$\boxed{f'(c) = 6}$$

$f(x) = x^2$
$f'(x) = 2x$

$2x = 6$
$\boxed{x = 3}$
At $x = 3$ $\boxed{f'c = \frac{f(b) - f(a)}{b-a}}$ ✓

Date ______ Page ____

Advanced ($\frac{\infty}{\infty}$ form)

a) If $\lim_{x\to\infty}\left(\frac{x^2+x+1}{x+1}-ax-b\right)=4$ then $a, b = ?$

Sol) $\lim_{x\to\infty}\frac{(x^2+x+1)-(ax+b)(x+1)}{(x+1)}=4$

$\Rightarrow \lim_{x\to\infty}\frac{(x^2+x+1)-(ax^2+ax+bx+b)}{x+1}=4$

$\Rightarrow \lim_{x\to\infty}\frac{(1-a)x^2+(1-a-b)x+(1-b)}{x+1}=4$

$\lim_{x\to\infty}\frac{(1-a)x^{\to 0}+(1-a-b)+\cancel{\frac{(1-b)}{x}}^{\to 0}}{1+\frac{1}{x}\to 0}=4$

Possible only if $(1-a)=0 \Rightarrow \boxed{a=1}$

$(1-a-b)=4 \Rightarrow \boxed{b=-4}$

Advanced

L Hospital's Rule (derivative $\Rightarrow$ Limit)

(1) If $\lim_{x\to c}f(x)=0$ & $\lim_{x\to c}g(x)=0$ & $\lim_{x\to c}\frac{f'(x)}{g'(x)}=L$

Then $\lim_{x\to c}\frac{f(x)}{g(x)}=L$

(2) If $\lim_{x\to c}f(x)=\pm\infty$ & $\lim_{x\to c}g(x)=\pm\infty$ & $\lim_{x\to c}\frac{f'(x)}{g'(x)}=L$

Then $\lim_{x\to c}\frac{f(x)}{g(x)}=L$

Example :-

$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\cos x}{1}=1$

Date ______ Page ____

- $\left(\begin{array}{c}\text{function is } \cancel{\text{derivable}} \\ \text{Differentiable}\end{array}\right) \cancel{\Rightarrow} \cancel{\text{derivate}} \Rightarrow \left(\begin{array}{c}\text{Derivative of function is} \\ \text{Continuous}\end{array}\right)$

Q.) $f(x) = \begin{cases} \sin x & \text{if } x < \pi \\ mx + n & \text{if } x \geq \pi \end{cases}$

m & n are constants. Determine m & n such that f is derivable on set of real nos.

Sol) f derivable ⇒ f' continuous

L.H.L = R.H.L: $\sin \pi = m\pi + n$

$m\pi + n = 0$ — (i)

$-\pi + n = 0$

$\boxed{n = \pi}$

$f'(x) = \begin{cases} \cos x & \text{if } x < \pi \\ m & \text{if } x \geq \pi \end{cases}$

$\cos \pi = m$

$\boxed{m = -1}$

Q.) The function $f(x) = (x^2-1)|x^2-3x+2| + \cos|x|$ is not differentiable at

Sol) $f(x) = (x+1)(x-1)|(x-1)(x-2)| + \cos(x)$ $[\because \cos(-a) = \cos a]$

(*) as Every polynomial, Inverse, Log, expone, ~~sin~~ simply Trigo. are continuous & Differentiable, in its Domain

$|x|$ is not diff at $x=0$ but $x|x|$ is differen everywhere.

$f(x) = (x+1)(x-1)|(x-1)(x-2)| + \cos x$ is not differentiable Only at $\boxed{x = 2}$

Q) The LHD of $f(x) = [x]\sin(\pi x)$ at $x = k$, k an integer is

Sol) LHD. $= \lim_{h\to 0} \frac{f(k-h) - f(k)}{-h} = \lim_{h\to 0} \frac{[k-h]\sin(\pi(k-h)) - [k]\sin(\pi k)}{-h}$

$= \lim_{h\to 0} \frac{(k-1)\sin(k\pi - \pi h) - k\,\cancel{\sin k\pi}^{\,0}}{-h} = \lim_{h\to 0} \frac{(k-1)(-1)^{k+1}\sin(\pi h)}{-h}$

$= \lim_{h\to 0} \frac{(k-1)(-1)^{k}\sin(\pi h)\,\pi}{\pi h} = \pi(k-1)(-1)^{k}$

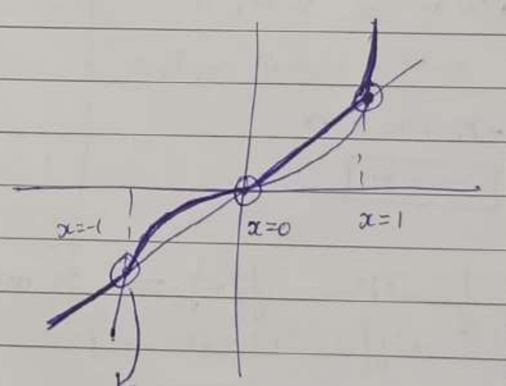
$[\sin(n\pi - a) = (-1)^{n+1}\sin a]$

Q.) Let $f: R \to R$, $f(x) = \max(x, x^3)$. The set of all points where $f(x)$ is not differentiable is.

Sol.) $f(x) = \max(x, x^3)$

from fig.

$f(x)$ is not differentiable at $x = -1, 0, \& \ 1$



Note:-
Curve breaks
⇓
discontinuous.

~~Function~~ function nature changes & ~~curve breaks~~
(Not. Differentiable).

Q) ~~...~~ If $y = f(x) = x^3 + x^5$ & $g(x) = f^{-1}(x)$ then $g'(2) = ?$

Sol)

$f^{-1}(x) = g(x)$

$g(f(x)) = x$

$g'(f(x)) \cdot f'(x) = 1$

$g'(f(x)) = \frac{1}{f'(x)}$

$f(x) = 2$ when $x = 1$

$g'(f(1)) = \frac{1}{3x^2 + 5x^4}\Big]_{x=1}$

$g'(2) = \frac{1}{8}$

$\boxed{g'(2) = \frac{1}{8}}$

Date ____ Page ____

Note :-

$f(x) = \begin{vmatrix} R_1 \\ R_2 \\ R_3 \end{vmatrix}$, $f'(x) = \begin{vmatrix} R_1' \\ R_2 \\ R_3 \end{vmatrix} + \begin{vmatrix} R_1 \\ R_2' \\ R_3 \end{vmatrix} + \begin{vmatrix} R_1 \\ R_2 \\ R_3' \end{vmatrix}$.

Q) Let $f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$

$f'''(0) = ?$

Sol) $f'(x) = \begin{vmatrix} 3x^2 & \cos x & -\sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix} + 0 + 0$

$f''(x) = \begin{vmatrix} 6x & -\sin x & -\cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix} + 0 + 0$

$f'''(x) = \begin{vmatrix} 6 & -\cos x & \sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$

$f'''(0) = \begin{vmatrix} 6 & -1 & 0 \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix} = 0$ ans

Ch-7

Integrals

(*) Integration is the reverse process of Differentiation and so it is also called anti-~~Derivative~~ Differentiation.

if $\frac{d}{dx}(\sin x) = \cos x$

than,

$$\int \cos x \, dx = \sin x$$

Note :-

$$\frac{d}{dx}(\sin x) + \frac{d}{dx}(5) = \cos x + 0$$

$$= \cos x$$

So $\boxed{\int \cos x \, dx = \sin x + C}$

Constant comes back

~~The above is called Indefinite integral of cos x (f) with respect to x~~ • C is known as arbitrary constant

~~The f~~

- This ~~above~~ type of integration is Indefinite Integration.

~~Generally~~ $\cancel{\int f'(x)\,dx = f(x) + C}$ ~~if~~ $\cancel{\frac{d}{dx}(f(x)) = f'(x)}$

~~Generally,~~

~~$\int f'(x)\,dx = f(x) + C$ if $\frac{d}{dx} f(x) = f'(x)$~~

~~Formulas for Integration :~~

~~$f'(x)$ $\int f'(x)\,dx = f(x)$~~

Generally ;

if $\frac{dy}{dx} = f(x)$ when $y = F(x)$

than, $\boxed{\int f(x)\,dx = F(x) + C}$

Note :-

$\boxed{\int dx = x + c}$

Formulas :-

$f(x)$	$F(x)$	$f(x)$	$F(x)$
$\sin x$	$-\cos x$	$x^n, n \neq -1$	$\frac{x^{n+1}}{n+1}$
$\cos x$	$\sin x$	$\frac{1}{x}$	$\log \lvert x \rvert$
$\sec^2 x$	$\tan x$		
$\operatorname{cosec}^2 x$	$-\cot x$		
$\sec x \tan x$	$\sec x$		
$\operatorname{cosec} x \cot x$	$-\operatorname{cosec} x$		

More Formulas

$f(x)$	$F(x)$	$f(x)$	$F(x)$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1}x$	e^x	e^x
$\frac{1}{\sqrt{1-x^2}}$	$-\cos^{-1}x$	a^x	$\frac{a^x}{\log a}$
$\frac{1}{1+x^2}$	$\tan^{-1}x$	$\cancel{\frac{1}{x^2+a^2}}$	$\cancel{\frac{1}{a}\tan^{-1}\frac{x}{a}}$
$\frac{1}{1+x^2}$	$-\cot^{-1}x$	$\cancel{\frac{1}{\sqrt{a^2-x^2}}}$	$\cancel{\sin^{-1}\frac{x}{a}}$
$\frac{1}{x\sqrt{x^2-1}}$	$\sec^{-1}x$		
$\frac{1}{x\sqrt{x^2-1}}$	$-\operatorname{cosec}^{-1}x$		

Note:-

$\int \cos 2x = \frac{\sin 2x}{2}$

$\int e^{3x} = \frac{e^{3x}}{3}$

- Similar for others.

- Example

- Write anti-derivative of $\cos 2x$ using method of inspection

Sol $\frac{d}{dx}(\sin 2x) = 2\cos 2x$

or $\cos 2x = \frac{1}{2}\frac{d}{dx}(\sin 2x) = \frac{d}{dx}\left(\frac{1}{2}\sin 2x\right)$

Therefore an anti-derivative of $\cos 2x$ is $\frac{1}{2}\sin 2x$.

- Do not write constant (+c) in inspection method.

Q) Find Integrals.

i) $\int (2x^2 + e^x)\, dx$

$\int 2x^2 dx + \int e^x dx$

$= \boxed{2\frac{x^3}{3} + e^x + C}$

ii) $\int \frac{x^3 + 5x^2 - 4}{x^2} dx$

$\int (x + 5 - 4x^{-2}) dx$

$\int x dx + \int 5 dx - \int 4x^{-2} dx$

$\boxed{\frac{x^2}{2} + 5x + 4 \times \frac{1}{x} + C}$

iii) $\int \frac{\sec^2 x}{\operatorname{cosec}^2 x} dx$

$\int \sec^2 x \times (\sin^2 x)\, dx$

$\int \sec^2 x \times (1 - \cos^2 x)\, dx$

$\int \sec^2 x - 1 \quad dx$

$\int \sec^2 x\, dx - \int dx$

$\boxed{\tan x - x + C}$

Methods of Integration.

1.) Integration by Substitution.
2.) Integration using Partial Fractions.
3.) Integration by Parts.

Integration by Substitution -

Example

$$\int\left(\frac{\sin(\tan^{-1}x)}{1+x^2}\right)dx$$

$t = \tan^{-1}x$

$\frac{dt}{dx} = \frac{-1}{1+x^2}$

$dt \times (1+x^2) = dx$

$\int \frac{\sin t}{\cancel{1+x^2}} dt \times \cancel{(1+x^2)}$

$\int \sin t \; dt$

$-\cos t + c$

$-\cos(\tan^{-1}x) + c$

Example

$$\int\left(\frac{1}{x+x\log x}\right)dx$$

$t = \int\left(\frac{1}{x(1+\log x)}\right)dx$

$t = 1+\log x$

$\frac{dt}{dx} = \frac{1}{x}$

$dt \times x = dx$

$\int \frac{1}{\cancel{x}(1+\log x)} dt \times \cancel{x}$

$\int \frac{1}{t} dt$

$\log|t| + c$

$\log|1+\log x| + c$

12.)

#

$\frac{7}{3}$

12.) $\int (x^3-1)^{1/3} x^5$

$t = x^3 - 1$

$\frac{dt}{dx} = 3x^2$

$dx = \frac{dt}{3x^2}$

$\int t^{1/3} x^5 \frac{dt}{3x^2}$

$\frac{1}{3}\int t^{1/3} x^3 \, dt$

$\frac{1}{3}\int t^{1/3} (t+1)\, dt$

$\frac{1}{3}\int (t^{4/3} + t^{1/3})\, dt$

$\frac{1}{3}\left[\int t^{4/3} dt + \int t^{1/3} dt\right]$

$\frac{1}{3}\left[t^{7/3} \times \frac{3}{7} + t^{4/3} \times \frac{3}{4}\right] + C$

$$\boxed{\frac{1}{7}(x^3-1)^{7/3} + \frac{1}{4}(x^3-1)^{4/3} + C}$$

Double Substitution

$\int \left(\frac{\tan^4 \sqrt{x} \; \sec^2 \sqrt{x}}{\sqrt{x}}\right) dx$

Let $t = \sqrt{x}$

$\frac{dt}{dx} = \frac{1}{2\sqrt{x}}$

$dt \times 2\sqrt{x} = dx$

$\int \left(\frac{\tan^4 t \times \sec^2 t}{\sqrt{x}}\right) \times dt \times 2\sqrt{x}$

$2 \times \int (\tan^4 t \times \sec^2 t)\, dt$

Let $u = \tan t$

$\frac{du}{dt} = \sec^2 t$

$\frac{du}{\sec^2 t} = dt$

$2\int (u^4 \times \cancel{\sec^2 t}) \times \frac{du}{\cancel{\sec^2 t}} = \frac{2}{5}u^5 + C$

$= \frac{2}{5}\tan^5(\sqrt{x}) + C$

More Formulas

(i) $\int \tan x\, dx = \log|\sec x| + c$

(ii) $\int \cot x\, dx = \log|\sin x| + c$

(iii) $\int \sec x\, dx = \log|\sec x + \tan x| + c$

(iv) $\int \operatorname{cosec} x\, dx = \log|\operatorname{cosec} x - \cot x| + c$
$= -\log|\operatorname{cosec} x + \cot x| + c$

Integral using Trignometric Functions :-

→ When the integrand involves some trigonometric functions, we use some known identities to find the integral.

Example

(i) $\int \cos^2 x\, dx$

$\cos 2x = 2\cos^2 x - 1 \Rightarrow \frac{\cos 2x + 1}{2} = \cos^2 x$

$\int (\cos 2x + 1)/2\, dx$

$\frac{1}{2}\int \cos 2x\, dx + \int dx$

Let $t = 2x$

$\frac{dt}{dx} = 2$

$dx = \frac{dt}{2}$

$\frac{1}{2}\int \cos t \frac{dt}{2} + \int dx$

$\frac{1}{2}\left[\frac{1}{2}\int \cos t\, dt + \int dx\right]$

$\frac{1}{2}\left[\frac{\sin t}{2} + x\right] + c$

$\frac{\sin 2x}{4} + \frac{x}{2} + c$

ii) $\int \sin 2x \cos 3x \, dx$

$\frac{1}{2} \int (2 \sin 2x \cos 3x) \, dx$

$\frac{1}{2} \int \left[\sin(2x+3x) + \sin(-x) \right] dx$

$\frac{1}{2} \int (\sin 5x - \sin x) \, dx$

$\frac{1}{2} \left[\int \sin 5x \, dx - \int \sin x \, dx \right]$

Let $t = 5x$

$\frac{dt}{dx} = 5 \Rightarrow dx = \frac{dt}{5}$

$\frac{1}{2} \left[\int \sin t \frac{dt}{5} - \int \sin x \, dx \right]$

$\frac{1}{2} \left[\frac{1}{5} \int \sin t \, dt - \int \sin x \, dx \right]$

$\frac{1}{2} \left[\frac{1}{5} \times - \cos t + \cos x \right] + C$

$\frac{-\cos t}{10} + \frac{\cos x}{2} + C$

$-\frac{\cos(5x)}{10} + \frac{\cos x}{2} + C$

iii) $\int \sin^4 x \, dx$

$\int (\sin^2 x)^2 \, dx$

$\int \left(\frac{1 - \cos 2x}{2} \right)^2 dx$ $\quad \therefore \cos 2x = 1 - 2\sin^2 x$

$\Rightarrow \sin^2 x = \frac{1 - \cos 2x}{2}$

$\int \left(\frac{1 + \cos^2 2x - 2\cos 2x}{4} \right) dx$

$\frac{1}{4} \int \left[1 + \cos^2 2x - 2\cos 2x \right] dx$

$\frac{1}{4} \left[\int dx + \int \cos^2 2x \, dx - 2 \int \cos 2x \, dx \right]$

$$\frac{1}{4}\left[\int dx + \frac{1}{2}\int \cos^2 t - \int \cos t\, dt\right]$$

$$\frac{1}{4}\left[x + \frac{1}{2}\left[\frac{\sin 2x}{4} + \frac{x}{2} + c_1\right] - \sin t\right] + c_2$$

$$\frac{x}{4} + \frac{\sin 2x}{8} + \frac{x}{4} + \frac{c_1}{2} - \frac{\sin t}{4} + c_2$$

$$\frac{x}{2} + \frac{\sin 2x}{8} - \frac{\sin 2x}{4} + \frac{c_1}{2} + c_2$$

$$\frac{x}{2} + \frac{\sin 2x}{4}\ \frac{x}{2} - \frac{\sin 2x}{8} + C$$

$$\frac{1}{4}\left[\int dx + \int \cos^2 2x\, dx - 2\int \cos 2x\, dx\right]$$

$$I = \int \cos^2 2x$$

$$= \int \left(\frac{\cos 4x + 1}{2}\right) = \frac{1}{2}\left[\int \cos 4x\, dx + \int dx\right]$$

$$= \frac{1}{2}\left[\frac{\sin 4x}{4} + x\right] = \frac{\sin 4x}{8} + \frac{x}{2}$$

$$N = \int \cos 2x\, dx = \frac{\sin 2x}{2}$$

$$\frac{1}{4}\left\{\left[x + \frac{\sin 4x}{8} + \frac{x}{2}\right] - 2\,\frac{\sin 2x}{2}\right\} + C$$

$$\frac{1}{4}\left[\frac{3x}{2} + \frac{\sin 4x}{8} - \sin 2x\right] + C$$

$$\frac{3x}{8} + \frac{\sin 4x}{32} - \frac{\sin 2x}{4} + C$$

Integral of some Particular Functions :-

(1) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + C$

(2) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + C$

(3) $\int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$

(4) $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x+\sqrt{x^2-a^2}\right| + C$

(5) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

(6) $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right| + C$

Example

if these are of the form ax^2+bx+c we take as denominator Do the completing the square after that same with (8)

$\int \frac{1}{x^2+(4)^2} = \frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right) + C$

(7) $\int \frac{1}{x^2+6x+10} = \frac{1}{x^2+6x+10+(3)^2-(3)^2} = \frac{1}{(x+3)^2+(1)^2}$

Let $t = (x+3)$

$\frac{dt}{dx} = 1$

$dt = dx$

$\int \frac{1}{t^2+(1)^2} = \frac{1}{1} \times \tan^{-1}\left(\frac{t}{1}\right) + C$

$= \tan^{-1}(x+3) + C$

(8) $\int \frac{1}{\sqrt{x^2+6x+10}}$ is evaluated in similar way as (7).

(9.) $\int \frac{x+3}{x^2+6x+10}$

Let $x+3 = A\frac{d}{dx}(x^2+6x+10)+B$

$x+3 = A(2x+6)+B$

$x+3 = 2Ax+6A+B$

$x = 2Ax$

$\boxed{A=\frac{1}{2}}$

$6A+B=3$

$\cancel{6}3\times\frac{1}{\cancel{2}}+B=3$

$\boxed{B=0}$

Putting the values we get -

$\int \frac{\frac{1}{2}(2x+6)\,dx}{x^2+6x+10}$ Let $t=x^2+6x+10 \Rightarrow \frac{dt}{dx}=2x+6$

$=\frac{1}{2}\int\frac{dt}{t} = \frac{1}{2}\log|t| = \boxed{\frac{1}{2}\log|x^2+6x+10|}$ ans

(10) $\int \frac{x+3}{\sqrt{x^2+6x+10}}$ is also evaluated in similar way as (9).

$\Rightarrow x+3 = A\frac{d}{dx}(x^2+6x+10)+B$

like this.

Integration by Partial Fractions :-

Rational function $= \dfrac{P(x)}{q(x)}$ where $P(x)$ & $q(x)$ are Polynomial functions and $q(x) \neq 0$

- If degree of Numerator < degree of Denominator, we call it a proper rational function otherwise improper.
- An improper function can be converted into a proper function as following :-

Eg $\dfrac{x^2+1}{x^2-5x+6}$,

$$x^2-5x+6 \,\overline{)\, x^2+1 \,}\,(1$$
$$\underline{x^2-5x+6}$$
$$5x-5$$

So we can write,

$$\frac{x^2+1}{x^2-5x+6} = 1 + \frac{5x-5}{x^2-5x+6} \Rightarrow 1 + \frac{5x-5}{(x-2)(x-3)}$$

Formulas :-

(S.No.)	(Form of rational function)	(Form of Partial fraction)
1.)	$\dfrac{px+q}{(x-a)(x-b)}$, $a \neq b$	$\dfrac{A}{(x-a)} + \dfrac{B}{(x-b)}$
2.)	$\dfrac{px+q}{(x-a)^2}$	$\dfrac{A}{(x-a)} + \dfrac{B}{(x-a)^2}$
3.)	$\dfrac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\dfrac{A}{(x-a)} + \dfrac{B}{(x-b)} + \dfrac{C}{(x-c)}$
4.)	$\dfrac{px^2+qx+r}{(x-a)^2(x-b)}$	$\dfrac{A}{(x-a)} + \dfrac{B}{(x-a)^2} + \dfrac{C}{(x-b)}$

5.) $\dfrac{px^2+qx+r}{(x-a)(x^2+bx+c)}$ | $\dfrac{A}{(x-a)}+\dfrac{Bx+C}{x^2+bx+c}$

→ where x^2+bx+c cannot be facterized further

Example

Q) $\int \dfrac{1}{(x+1)(x+2)}$ ~~A~~ Degree of num < degree of deno. hence it is proper.

Let $\dfrac{1}{(x+1)(x+2)} = \dfrac{A}{(x+1)} + \dfrac{B}{(x+2)}$ — (i)

$1 = A(x+2) + B(x+1)$

$1 = Ax + 2A + Bx + B$

$1 = Ax + Bx + 2A + B$

~~A+B~~

$\boxed{A+B=0}$ $\boxed{2A+B=1}$

~~A - 2B = 0~~ ~~2A = -B~~

~~-A = 0~~ ~~B = 2A~~

$A = -B$

$2(-B) + B = 1$

$-2B + B = 1$

$\boxed{B = -1}$

$\boxed{A = 1}$

Putting in (i)

$\dfrac{1}{(x+1)(x+2)} = \dfrac{1}{(x+1)} + -\dfrac{1}{(x+2)}$

$\int \dfrac{1}{(x+1)(x+2)} = \int \dfrac{1}{x+1} - \int \dfrac{1}{x+2}$

$= \log|x+1| - \log|x+2| + C$

$= \log\left|\dfrac{x+1}{x+2}\right| + C$

Example

Find $\int \frac{3x-2}{(x+1)^2(x+3)}\,dx$ ⊛ it is a Proper rational function

Let $\frac{3x-2}{(x+1)^2(x+3)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+3)}$ —(i)

$$3x-2 = A(x+1)(x+3) + B(x+3) + C(x+1)^2$$
$$3x-2 = A(x^2+4x+3) + Bx + 3B + C(x^2+1+2x)$$
$$3x-2 = Ax^2 + 4Ax + 3A + Bx + 3B + Cx^2 + C + 2Cx$$
$$3x-2 = Ax^2 + Cx^2 + 4Ax + Bx + 2Cx + 3A + 3B + C$$

$\boxed{A+C=0}$ $\quad$ $\boxed{4A+B+2C=3}$ $\quad$ $\boxed{3A+3B+C=-2}$

$A=-C$ $\quad$ $4(-C)+B+2C=3$ $\quad$ $3(-C)+3B+C=-2$

$-4C+2C+B=3$ $\quad$ $-3C+3B+C=-2$

$-2C+B=3$ —(ii) $\quad$ $-2C+3B=-2$

Solving we get,

$\boxed{B=-\frac{5}{2}}$

$-2C = \frac{11}{4} \Rightarrow \boxed{C=-\frac{11}{4}}$

$\boxed{A=\frac{11}{4}}$

Putting in (i)

$$\int \frac{3x-2}{(x+1)^2(x+3)} = \frac{11}{4}\int\frac{1}{(x+1)} - \frac{5}{2}\int\frac{1}{(x+1)^2} - \frac{11}{4}\int\frac{1}{x+3}$$

$$= \frac{11}{4}\log|x+1| + \frac{5}{2}\times\frac{1}{(x+1)} - \frac{11}{4}\log|x+3| + C$$

$$= \frac{11}{4}\log\left|\frac{x+1}{x+3}\right| + \frac{5}{2(x+1)} + C$$

Integration by Parts :-

$$\int f(x)\,g(x) = f(x)\int g(x)\,dx - \int\left[f'(x)\int g(x)\,dx\right]dx$$

→ The integral of the product of two functions
= first function × integral of second function
− Integral of [differential coefficient of first function × integral of second function]

$$\boxed{(uv)^{I} = uv^{I} - [u' v^{I}]^{I}}$$

Some Rules :-

(I) v usko let karte hain jiska integral pata he.

Example

~~$\int 2\log x\, dx$~~

$$\int \underset{v}{1} \cdot \underset{u}{\log x}\, dx = \log x \times x - \int\left(\frac{1}{\cancel{x}} \times \cancel{x}\right)dx$$

$$x\log x - x$$

$$x(\log x - 1) + C$$

(II) ~~IRAFE~~

Priority for u :-

ILATE

I — Inverse function
L — Logarithmic function
A — Algebraic function
T — Trigonometric function
E — Exponential function

Example

$$\int x^2 e^x$$

x^2 is u, e^x is v

o We know integral for both → Rule 1 ✓
o u is Algebraic v is exponent.

$$\int x^2 e^x = x^2 \int e^x - \int \left[2x \times \int e^x\right] dx$$
$$= x^2 e^x - \int (2x \times e^x)\, dx \quad -(i)$$

Let $I = \int (2x \times e^x)$, where $2x$ is u and e^x is v

$$I = 2x e^x - \int (2 \times e^x)\, dx$$
$$= 2xe^x - 2e^x$$

~~Putting in (i)~~

Putting in (i)

$$\int x^2 e^x = x^2 e^x - \left[2xe^x - 2e^x\right]$$
$$= x^2e^x - 2xe^x + 2e^x + C$$
$$= e^x (x^2 - 2x + 2) + C$$

Example

$$\int x^2 \log x$$

x^2 is v, $\log x$ is u

$$\int x^2 \log x = \log x \times \frac{x^3}{3} - \int \left[\frac{1}{x} \times \frac{x^{\cancel{3}\,2}}{3}\right] dx$$
$$= \frac{\log x \times x^3}{3} - \frac{1}{3} \times \frac{x^3}{3} + C$$
$$= \frac{\log x \times x^3}{3} - \frac{x^3}{9} + C$$

~~Note :- Add constant C~~
~~Integrate everything in one step & apply constant C~~
~~in 1 step for convenience in one step only for convenience.~~

Example

$$\int x \times \sec^2 x$$

x is u, $\sec^2 x$ is v

$$\int x \times \sec^2 x = x \int \sec^2 x - \int [1 \times \tan x]\, dx$$
$$x \tan x - \log|\sec x| + C$$
$$= x \tan x + \log|\cos x| + C. \qquad \therefore \log|x| = -\log\left|\frac{1}{x}\right|$$

~~Note :- Apply constant after integration.~~

Example

$\int x \sin^{-1}x$

Sol) Let $y = \sin^{-1}x$ or $\sin y = x$ ~~$\Rightarrow \frac{dy}{dx}\cos y = 1$~~

$\cos y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$ $\boxed{dx = \cos y\, dy}$

$\int x \sin^{-1}x\, dx = \int \sin y\, y \cos y\, dy = \frac{1}{2}\int \sin 2y\, y\, dy$

$\frac{1}{2}\int \sin 2y\; y\, dy = \frac{1}{2}\left[y \times -\frac{\cos 2y}{2} + \int\left[1 \times \frac{\cos 2y}{2}\right] dy\right]$

$= \frac{1}{2}\left[-\frac{\cos 2y\, y}{2} + \frac{\sin 2y}{4}\right] + C$

$= \frac{-\cos 2y\, y}{4} + \frac{\sin 2y}{8} + C$

$= \frac{-(1-2\sin^2 y)y}{4} + \frac{2\sin y \cos y}{8} + C$

$= \frac{(2x^2-1)\sin^{-1}x}{4} + \frac{x\sqrt{1-x^2}}{4} + C.$

Integration by Parts:-

→ It applicable only for those function which are differentiable. (or whose derivate can be found out)

Tip → Do not add constant after each integration. Only apply one C at the end.

→ ILATE is not a rule, just a trick for fast calculation

(*) $$\int e^x \left[f(x) + f'(x)\right] dx = e^x f(x) + c$$

Integral of some more Particular functions.

$$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + c$$

$$\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + c$$

Proof :-

For $\int\sqrt{a^2-x^2}$:-

Let $\int\sqrt{a^2-x^2} = I$

L.H.S.

$$\int 1\times\sqrt{a^2-x^2} = x\sqrt{a^2-x^2} - \int \frac{-2x^2}{2\sqrt{a^2-x^2}}\,dx$$

$\therefore$ By Parts

$$I = x\sqrt{a^2-x^2} - \int \frac{a^2-x^2-a^2}{\sqrt{a^2-x^2}}\,dx$$

$$= x\sqrt{a^2-x^2} - \int\left(\sqrt{a^2-x^2} - \frac{a^2}{\sqrt{a^2-x^2}}\right)dx$$

$$= x\sqrt{a^2-x^2} - \int\sqrt{a^2-x^2}\,dx + a^2\int\frac{1}{\sqrt{a^2-x^2}}\,dx$$

$$2I = x\sqrt{a^2-x^2} + a^2\sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\boxed{I = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C}$$

→ Similarly you can prove others.

Example

Q)

Sol

Example

Q)

Sol

Example

Sol

Example

Q) $\int \sqrt{4 - x^2}\, dx$

Sol $= \int \sqrt{(2)^2 - x^2}\, dx$ $a = 2$ $x \Rightarrow x$

$= \frac{x}{2} \sqrt{4 - x^2} + \frac{4}{2} \sin^{-1}\left(\frac{x}{2}\right) + c$

Example

Q) $\int \sqrt{x^2 + 4x + 1}\, dx$

Sol $\int \sqrt{(x+2)^2 - (\sqrt{3})^2}\, dx$ $x \Rightarrow x+2$ $a = \sqrt{3}$

$= \frac{x+2}{2} \sqrt{x^2 + 4x + 1} - \frac{3}{2} \log \left| x + 2 + \sqrt{x^2 + 4x + 1} \right| + C.$

Example $\int \sqrt{x^2 + 4x + 6}\, dx$

Sol $\int \sqrt{(x+2)^2 + (\sqrt{2})^2}$ $x \Rightarrow x+2$ $a = \sqrt{2}$

$= \frac{x+2}{2} \sqrt{x^2 + 4x + 6} - \frac{2}{2} \log \left| x + 2 + \sqrt{x^2 + 4x + 6} \right| + c$

(*) Definite Integral

~~As limit of sum~~

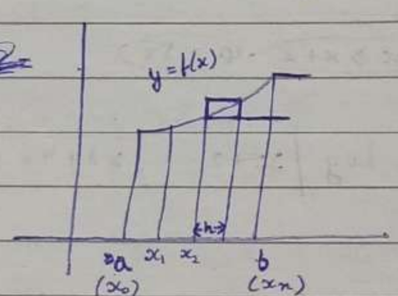
Upper Limit → $\int_a^b f(x)\,dx \implies$ Ways to Integrate this.

Lower Limit

(1) As a limit of a sum

(2) Direct Method (Using 2nd F.T.C.)

$S_n =$



- $h = \dfrac{b-a}{n}$

if $h \to 0, n \to \infty$.

(Stripe become Rectangle)

$◺ \approx \square$

$S_n = hf(a) + hf(a+h) \ldots\ldots$

$S_n = \lim\limits_{h\to 0} h\left[f(a) + f(a+h) + \ldots + f(a+(n-1)h)\right]$

$$\boxed{S_n = b-a \lim_{n\to\infty} \frac{1}{n} \times \left[f(a) + f(a+h) + \ldots + f(a+(n-1)h)\right]}$$

Ex. $f(x) = x^2$

$\int_2^4 f(x)\,dx$

$= \int_2^4 x^2\,dx = \left[\frac{x^3}{3}\right]_2^4$

$= \frac{(4)^3}{3} - \frac{(2)^3}{3}$ ans

$$\boxed{\begin{aligned} \int_a^b f(x)\,dx &= \left[F(x)\right]_a^b \\ \left[F(x)\right]_a^b &= F(b) - F(a) \end{aligned}}$$

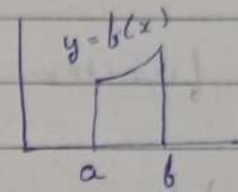
Concept of Dummy Variable

$$\int_a^b f(x)\,dx = \int_a^b f(u)\,du$$

↳ We can choose any Variable for Integration

Example

Q.) $\int_a^b x \, dx \quad \{f(x) = x\}$



Sol. $h = \dfrac{b-a}{n}$

$$S_n = (b-a) \lim_{n\to\infty} \frac{1}{n} \left[f(a) + f(a+h) \ldots\ldots f(a+(n-1)h) \right]$$

$$= (b-a) \lim_{n\to\infty} \frac{1}{n} \left[a + a+h + a+2h + \ldots a + (n-1)h \right]$$

$$= (b-a) \lim_{n\to\infty} \frac{1}{n} \left[a + \left(a + \frac{b-a}{n}\right) + \left(a + 2\frac{(b-a)}{n}\right) + \ldots \left(a + \frac{(n-1)(b-a)}{n}\right) \right]$$

$$= (b-a) \lim_{n\to\infty} \frac{1}{n} \left[an + \frac{b-a}{n} (1 + 2 + \ldots (n-1)) \right]$$

$$= (b-a) \lim_{n\to\infty} \frac{1}{n} \left[an + \frac{b-a}{\cancel{n}} \times \frac{(n-1)\cancel{n}}{2} \right]$$

$$= (b-a) \lim_{n\to\infty} \frac{1}{n} \left[\frac{\cancel{a}\, 2an + (b-a) \times (n-1)}{2} \right]$$

$$= \frac{b-a}{2} \lim_{n\to\infty} \left[2a + (b-a) \frac{(n-1)}{n} \right]$$

$$= \frac{b-a}{2} \lim_{n\to\infty} \left[2a + (b-a)\left(1 - \frac{1}{n}\right) \right]$$

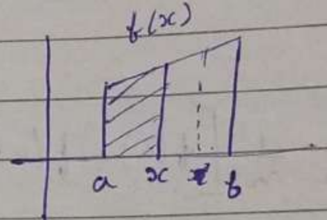
$$= \frac{b-a}{2} \times (a+b) \qquad \because \text{applying limit}$$

$$= \frac{b^2 - a^2}{2}$$

Fundamental Theorem of Calculus

⇒ Area function

$\int_a^x f(x)\,dx$



$$\boxed{A(x) = \int_a^x f(x)\,dx}$$

- x is arbitary.
- So Area changes as value of x changes.

{1st F.T.C}

Let f be a continous function in domain [a,b] and let A(x) be area function in this domain.

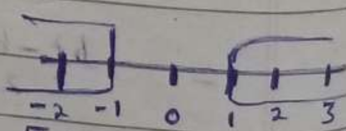
then, $\boxed{A'(x) = f(x)}$

{2nd F.T.C} Let f be a function which is continous in interval [a, b].

then, $$\boxed{\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)}$$

Note :-

$$\int_{-2}^{3} x\sqrt{(x^2-1)}\,dx \quad \left\{ \begin{array}{l} x^2-1 \geq 0 \\ x^2 \geq 1 \\ x \geq 1 \text{ or } x \leq -1 \end{array} \right.$$



[not defined in given interval]

→ This function is not continous or defined hence we cannot integrate this function.

Evaluation of Definite Integral by Substitution

Example

Q) $\int_0^1 \frac{\tan^{-1} x}{1+x^2}\, dx$

~~Sol.~~ $\cancel{\int \frac{\tan^{-1} x}{1+x^2}\, dx}$

~~Sol.~~ **Method 1**

Sol. $\int \frac{\tan^{-1} x}{1+x^2}\, dx$

Put, $t = \tan^{-1} x \Rightarrow \frac{dt}{dx} = \frac{1}{1+x^2}$

$dt \times (1+x^2) = dx$

$\int \frac{t}{\cancel{1+x^2}} \times dt\,\cancel{(1+x^2)} = \int t\, dt$

$= \frac{t^2}{2} = \frac{(\tan^{-1} x)^2}{2}$

Putting limits.

$\left[\frac{(\tan^{-1} x)^2}{2}\right]_0^1$ $\cancel{= \frac{(\tan^{-1}(1))^2}{2}}$

$= \frac{(\tan^{-1}(1))^2}{2} - \frac{(\tan^{-1}(0))^2}{2}$

$= \frac{(45^\circ)^2}{2} - 0$

$= \frac{(180^\circ)^2}{32}$

Method

$\int_0^1 \frac{\tan^{-1} x}{1+x^2}\, dx$

$t = \tan^{-1} x \Rightarrow \frac{dt}{dx} = \frac{1}{1+x^2}$

$dx = dt\,(1+x^2)$

$t = \tan^{-1} x$

when $\begin{cases} x = 1, & t = 45^\circ \\ x = 0, & t = 0 \end{cases}$

$\int_0^1 \frac{\tan^{-1} x}{1+x^2}\, dx = \int_0^{45^\circ} \frac{t}{\cancel{1+x^2}}\, dt\,\cancel{(1+x^2)}$

$= \left[\frac{t^2}{2}\right]_0^{45^\circ} = \frac{1}{2}\left[t^2\right]_{0^\circ}^{45^\circ}$

$= \frac{1}{2}\left[(45^\circ)^2\right] \cancel{- 0} = \frac{(45^\circ)^2}{2}$

$= \frac{(180^\circ)^2}{32}$

(✱) $\int \frac{1}{x^4+1}\,dx$

$\frac{1}{2}\int \frac{2}{x^4+1}\,dx = \frac{1}{2}\int \frac{x^2+1-x^2+1}{x^4+1}\,dx$

$\frac{1}{2}\left[\int \frac{x^2+1}{x^4+1} - \int \frac{x^2-1}{x^4+1}\right]dx$ ~~[divide by x^2]~~

(✱) $\int \frac{x^2+1}{x^4+1} = \int \frac{1+x^{-2}}{x^2+x^{-2}+2-2}\,dx$

$= \frac{\left(1+\frac{1}{x^2}\right)}{\left(x-\frac{1}{x}\right)^2+(\sqrt{2})^2}\,dx$

(✱) $\int \frac{x^2+1}{x^4+x^2+1}\,dx = \int \frac{1+x^{-2}}{x^2-2+x^{-2}+3} = \int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+(\sqrt{3})^2}$

(✱) $\int \frac{x^2\,\frac{1}{x}}{x^2\left(\left(x-\frac{1}{x}\right)^2\right)+3}\,dx$ $\quad \int \frac{x}{(x^2)^2+x^2+1}$ $\quad \boxed{t=x^2}$

(✱) $\int \sqrt{\tan x}\,dx$ $\quad \boxed{t^2=\tan x}$

$\Rightarrow \int \frac{2t^2}{1+t^4}\,dt = \int \frac{2t^2-1+1}{t^4+1}\,dt = \int \frac{t^2+1}{t^4+1}\,dt + \int \frac{t^2-1}{t^4+1}\,dt$

$t^2 = \tan x$

$\cancel{\int \left(t + \frac{1}{t}\right) dt}$ $\int \left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx$

$\int \left(t + \frac{1}{t}\right) \frac{2t}{t^4+1} dt = 2 \int \left(\frac{t^2+1}{t^4+1}\right) dt$

$t = \tan x$

$\frac{dt}{dx} = \sec^2 x$

$\int \frac{1}{\sin^4 x + \sin^2 x \cos^2 x + \cos^4 x} dx$

$\int \frac{\sec^4 x}{\tan^4 x + \tan^2 x + 1}$

$\int \frac{1 + t^2}{t^4 + t^2 + 1} dt$

$t = \tan x$

$\int \frac{1}{\sin^4 x + \cos^4 x} dx$

$= \int \frac{\sec^4 x}{\tan^4 x + 1} dx$

$\int \frac{\sec^2 x \sec^2 x}{\tan^4 x + 1}$

Advanced

Eg. $\int \frac{(x^2 + \sin^2 x) \sec^2 x}{1 + x^2} dx$

Sol.) $\int \frac{\left[(x^2+1) - \cos^2 x\right] \sec^2 x}{1 + x^2} = \int \left[\frac{\sec^2 x (x^2+1)}{(1+x^2)} - 1\right] dx$

$\int \sec^2 x \, dx - \int \frac{1}{1+x^2} dx = \tan x - \tan^{-1}(x) + C$

Ex) $\displaystyle\int \frac{dx}{\cos(x-a)\cos(x-b)}$

Sol) $\displaystyle I = \frac{1}{\sin(a-b)}\int \frac{\sin(a-b)}{\cos(x-a)\cos(x-b)} = \frac{1}{\sin(a-b)}\int \frac{\sin[(x-b)-(x-a)]}{\cos(x-a)\cos(x-b)}$

$\displaystyle = \frac{1}{\sin(a-b)}\int \frac{\sin(x-b)\cos(x-a) - \cos(x-b)\sin(x-a)}{\cos(x-a)\cos(x-b)}\,dx.$

$\displaystyle = \frac{1}{\sin(a-b)}\int \left[\tan(x-b) - \tan(x-a)\right] dx$

$\displaystyle = \frac{1}{\sin(a-b)} \times \text{Log}\,\frac{\sec(x-b)}{\sec(x-a)} = \cancel{\frac{1}{\sin(b-a)} \times \text{Log}\,\frac{\cos(x-b)}{\cos(x-a)}}$

Ex) $\displaystyle\int \frac{dx}{\tan x + \cot x + \sec x + \operatorname{cosec} x}$

Sol) $\displaystyle I = \int \frac{dx}{\tan x + \cot x + \sec x + \operatorname{cosec} x} = \int \frac{\sin x \cos x\, dx}{1 + \sin x + \cos x}$

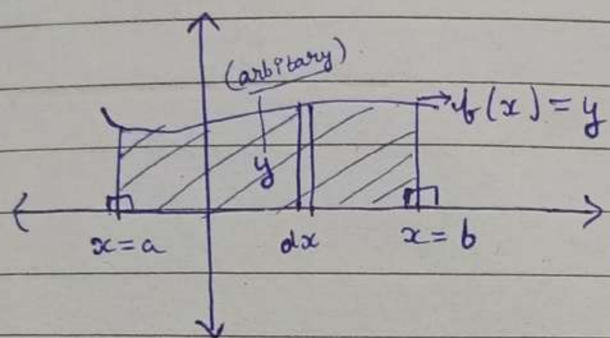
$\displaystyle = \frac{1}{2}\int \frac{\cancel{2\sin}\;(\sin x + \cos x)^2 - 1^2}{1 + \sin x + \cos x} = \frac{1}{2}\int \frac{(\sin x + \cos x + 1)(\sin x + \cos x - 1)}{(1 + \sin x + \cos x)}$

$\displaystyle = \frac{1}{2}\int (\sin x\, \cancel{dx} + \cos x - 1)\,dx = \frac{1}{2} \times \left[-\cos x + \sin x - x\right] + c$

Ch-8 AOI

Area of Simple Curves :-

with the vertical stripe

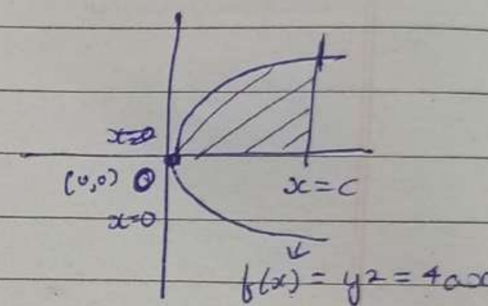


Area :-

$$A = \int_a^b f(x)\,dx$$

$$= [F'(x)]_a^b$$

$$= F'(b) - F'(a)$$

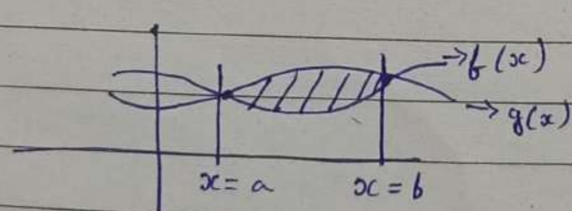


Area:-

$$A = \int_0^c y\,dx$$

$$= \int_0^c \sqrt{4ax}$$

$$= [\sqrt{4ax}]_0^c$$

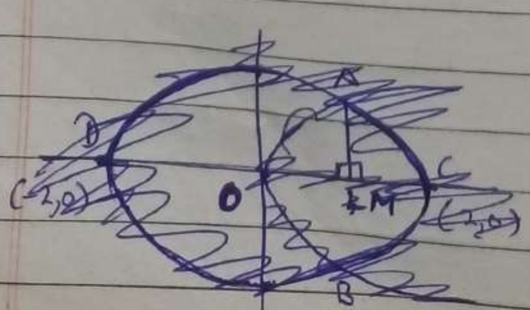


Area :-

$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

x

x^2



~~Circle :-~~

$\cancel{x^2 + y^2 = 4}$

~~if~~ $\cancel{y = 0}$

$\cancel{x^2 = 4}$

$\cancel{\boxed{x = \pm 2}}$

~~Parabola~~

$\cancel{y^2 = 4x}$

~~Solving Both Equations~~

$\cancel{x^2 + 4x - 4 = 0}$

Date:
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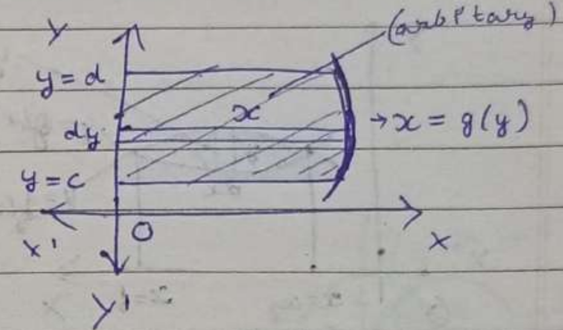
With Horizontal Strips :-

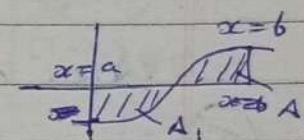
Area :-

$$\Delta = \int_c^d x \, dy$$

$$= \int_c^d g(y) \, dy$$



✱ For ~~taking Area only~~ finding area only consider the numerical value or magnitude.



$$\boxed{A_{total} = |A_1| + A_2}$$

Area in a Circle :-

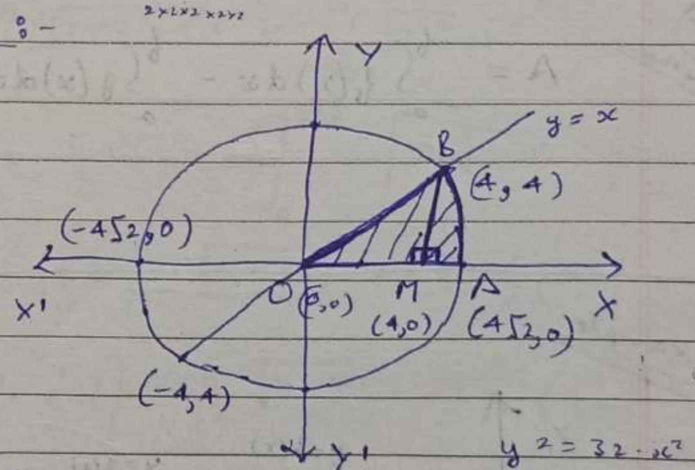
Equation of circle :-

$$x^2 + y^2 = 32$$

if $y = 0$, $\boxed{x = \pm 4\sqrt{2}}$

Equation of line :-

$$y = x$$



✱ Co-ordinates of B :-

$$x^2 + y^2 = 32$$

$$2x^2 = 32$$

$$\Rightarrow \boxed{x = \pm 4}$$

~~$(4)^2 + y^2 = 32$~~

~~$y^2 = 16$~~

~~$y = \pm 4$~~

when $x = 4$,

$$y = x$$

$$\boxed{y = 4}$$

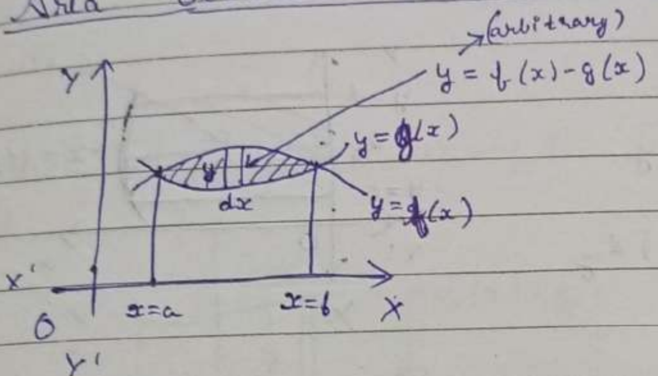
So, $B(4, 4) \Rightarrow M(4, 0)$

$$\text{Area of } \triangle AOB = \int_0^4 x \, dx + \int_4^{4\sqrt{2}} \sqrt{32 - x^2}$$

$\therefore \int_0^a \sqrt{a^2 - x^2} = \frac{a^2}{2} \times 90°$	For Verification :- Area of Circle $= \pi a^2$; Area of Ellipse $= \pi ab$	Area of triangle $\frac{1}{2} \times B \times H$.

Date:
Page:

Area between Two Curves :-



Area of Shaded region :-

(I) Method

$dA = [f(x) - g(x)]\, dx$ {elementary Area}

$A = \int_a^b [f(x) - g(x)]\, dx$ {Total area from a to b}

Note :-

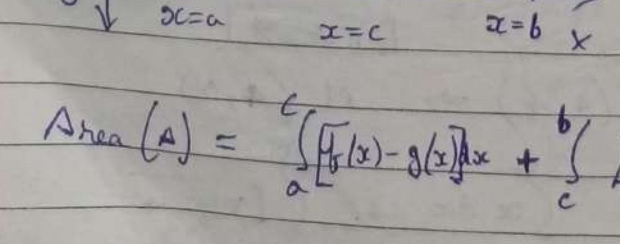
- $f(x) \geq g(x)$ in $[a, b]$

(II) Method

$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b [f(x) - g(x)]\,dx$

$\boxed{\text{where, } f(x) \geq g(x) \text{ in } [a, b]}$

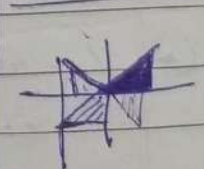
Example



$\text{Area } (A) = \int_a^c [f(x) - g(x)]\,dx + \int_c^b [g(x) - f(x)]\,dx$

Note :-

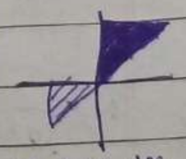
Vertical Stripe



■ ⇒ +ve area

▨ ⇒ −ve area (take mod)

Horizontal Stripe



■ ⇒ +ve area (take mod)

Example [✱ Imp. Formula

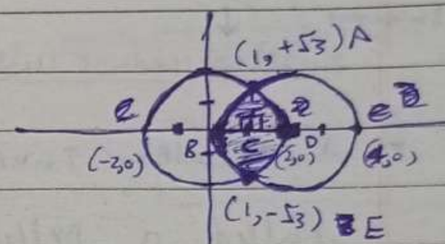
$$\int \sqrt{a^2-x^2} = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$$]

Find the area of the region enclosed between the circles : $x^2+y^2=4$ and $(x-2)^2+y^2=4$.

Sol Equations of circles are

$x^2+y^2=4$ — (i)

$(x-2)^2+y^2=4$ — (ii)



Solving (i) & (ii) we get,

$x=1$, $y=\pm\sqrt{3}$

C (1, 0)
B (0, 0)
D (2, 0)

Area of ABED = 2 × Area of ABD

$$\text{Area of } ABD = \int_0^1 \sqrt{(2)^2-(x-2)^2} + \int_1^2 \sqrt{(2)^2-x^2}$$

$$= \left[\frac{x-2}{2}\sqrt{4-(x-2)^2} + \frac{4}{2}\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^1 + \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_1^2$$

$$= \left[\frac{-1}{2}\sqrt{4-(-1)^2} + 2\sin^{-1}\left(-\frac{1}{2}\right)\right] - \left[-1\sqrt{4-(-2)^2} + 2\sin^{-1}(-1)\right]$$

$$+ \left[1\sqrt{4-(2)^2} + 2\sin^{-1}(1)\right] - \left[\frac{1}{2}\sqrt{4-(1)^2} + 2\sin^{-1}\left(\frac{1}{2}\right)\right]$$

$$= -\frac{\sqrt{3}}{2} - 2\times 30^\circ - (-180^\circ) + (180^\circ) - \left(\frac{\sqrt{3}}{2} + 2\times 30^\circ\right)$$

$$= -\frac{\sqrt{3}}{2} - 60^\circ + 180^\circ + 180^\circ - \frac{\sqrt{3}}{2} - 60^\circ$$

$$= -\sqrt{3} + 240^\circ$$

~~Area of ABED~~ Area of Shaded region = Area of ABED

$= 2(-\sqrt{3}+240^\circ)$

$= -2\sqrt{3}+480^\circ$

Date:
Page:

Ch-9

Differential Equations

$$\underbrace{\frac{dy}{dx}}_{\substack{\text{(3) derivative}\\ \text{(of dependant variable} \\ \text{w.r.t.to independent variable)}}} = x^2 + y^2$$

(1) dependant Variable. (y)

(2) independant Variable (x)

} Their Combination $\Rightarrow$ Differential Equation.

Ex. $\boxed{\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y^2 = 0}$

- A equation involving these three elements is called a Differential equation.

→ Generally, an equation involving ~~derivative (3) derivatives of type (3)~~ derivative/derivatives of type (3) is called a differential Equation.

→ A ~~D.E.~~ ~~from~~ D.E. (Differential Equation) involving only ~~the~~ derivatives of ~~the~~ ~~dependent~~ dependent variable w.r.t only one independent variable is called an ordinary D.E.

- [We shall ~~be~~ only be discussing about ordinary D.E.s]

Note :-

(1) $\frac{dy}{dx} = y'$, $\frac{d^2y}{dx^2} = y''$, $\frac{d^3y}{dx^3} = y'''$.

(2) $y_n = \frac{d^ny}{dx^n}$ $\Rightarrow$ $\left\{ \frac{d^7y}{dx^7} = y_7 \right\}$

~~[illegible]~~

Order of D.E.

→ Order is ~~always~~ defined as the highest no. of times derivative is done.

Eg. $$\frac{d^2y}{dx^2} + x\frac{d^3y}{dx^3} + x\frac{dy}{dx} = 0$$

Note :- order is always defined

Here, order is 3.

Degree of D.E.

→ The Power of the highest derivative is ~~called~~ the Degree of D.E.

Eg. $$\left(\frac{d^2y}{dx^2}\right)^2 + \frac{dy}{dx} = 0$$

Here, Degree is 2.

(*) ~~Degree To find, the degree of a D.E., the equation~~

(*) The equations must be a Polynomial equation in derivatives (y', y'') to define its degree (Power of $\frac{dy}{dx}$ must be whole number & they should be independent)

$$\frac{d^3y}{dx^3} + 2\left(\frac{d^2y}{dx^2}\right)^2 - \frac{dy}{dx} + y = 0$$

$$\left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right) - \sin^2 y = 0$$

} Degree Defined (These are Poly. equation in y, y' etc.)

$$\frac{dy}{dx} + \sin\left(\frac{dy}{dx}\right)$$

} Degree Undefined (These are not Poly. equation in y, y' etc.)

- Order & degree are always Positive.

Date:
Page:

General & Particular Solutions of a D.E.

Consider the Differential Equation -

$$\frac{d^2y}{dx^2} + y = 0 \quad -(i)$$

→ Its solution is a function ϕ that will satisfy it.
→ We have to put $y = \phi(x)$ to get the solution.

Suppose,

$$y = a \sin(x+b)$$

When this function ~~is substituted in equation (i)~~ & its derivatives are substituted in equation (i) ~~we get~~ we get, LHS = RHS.

If, $y = 2\sin(x + 45°)$ are substituted then also we get LHS = RHS.

(*) $y = a\sin(x+b)$ is called General Solution
→ it has two arbitary constants a & b.

(*) $y = 2\sin(x+45°)$ is called Particular Solution

Example 2 ~~Verify that~~

Verify that the function $y = e^{-3x}$ is a solution of D.E.

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0 \quad -(i)$$

Sol

$y = e^{-3x}$

$\frac{dy}{dx} = -3e^{-3x}$

$\frac{d^2y}{dx^2} = 9e^{-3x}$

Putting value of y, $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ in (i)

$9e^{-3x} + (-3e^{-3x}) - 6(e^{-3x}) = 0$

$9e^{-3x} - 9e^{-3x} = 0$

$0 = 0 \Rightarrow$ L.H.S = R.H.S

~~LHS = RHS~~ Therefore given function is a solution of the given D.E.

Date:
Page:

Note :-

* ~~Jitne arbitary constant hote hain~~
* ~~Particular (general) solution mein jitne arbitary constant hote hain utne baar derivate karna hota hai~~

~~first~~

(*) The no. of arbitary constants in the general solution of a D. E. of ~~d~~ n^{th} order is n.

But the no. of arbitary constants in the Particular solution of a D. E of n^{th} order is 0.

FORMATION OF A DIFFERENTIAL EQUATION WHOSE GENERAL SOLUTION IS GIVEN -

Example

Form the D. E. representing the family of curves, $y = mx$ where m is arbitary constant.

~~We have $y = mx$~~

~~$\frac{dy}{dx} = m$ [differentiating w.r.t. x]~~

We have $y = mx$ -(i)

Differentiating w.r.t. x

$$\frac{dy}{dx} = m$$

Putting in (i) the value of m

$$y = \frac{dy}{dx} x \Rightarrow \boxed{x\frac{dy}{dx} - y = 0}$$

This is free from arbitary constant m. Thus it is the required D. E.

Date:
Page:

(*) Jitne arbitary constant hote hain utne baar derivative karna hota hai. (Generally.)

METHOD OF SOLVING Ist Order, Ist Degree D.E.

Example Find General solution of equation (i), $(y \neq 2)$

Q) $\frac{dy}{dx} = \frac{x+1}{2-y}$ — (i)

$(2-y)\,dy = x+1\,dx \Rightarrow \int (2-y)\,dy = \int (x+1)\,dx$

$2y - \frac{y^2}{2} = \frac{x^2}{2} + x + C_1$

$0 = \frac{x^2}{2} + \frac{y^2}{2} + \frac{2x}{2} + \frac{2C_1}{2} - \frac{4y}{2}$

$\boxed{x^2 + y^2 + 2x - 4y + C = 0}$ (where $C = 2C_1$)

is the General Solution of equation (i)

↳ [bcoz it has arbitary constant C]

Example

Q) Find the Particular solution of the D. E. $\frac{dy}{dx} = -4xy^2$ — (i) given that $y=1$, when $x=0$.

Sol $\int \frac{dy}{y^2} = -\int 4x\,dx \Rightarrow -\frac{1}{y} = -\frac{\cancel{4}^{2} x^2}{\cancel{2}_{1}} + C \Rightarrow \boxed{y = \frac{1}{2x^2 - C}}$

{y = 1 when x = 0}

$1 = \frac{1}{-C} \Rightarrow \boxed{C = -1}$ So, $\boxed{y = \frac{1}{2x^2+1}}$ is the Particular Solution of (i) According to given condition

Example

a) In a Bank, Principal increases continuously at the rate of 5% per year. In how many years Rs 1000 double itself?

Sol. Let 'P' be the Principal amount at time t.

then, ~~we have~~ $\frac{dP}{dt} = \frac{5}{100} \times P$

$$\frac{dP}{dt} = \frac{P}{20} \Rightarrow \frac{dP}{P} = \frac{dt}{20}$$

$$\int \frac{dP}{P} = \int \frac{dt}{20} = \log|P| = \frac{t}{20} + c_1 \Rightarrow \boxed{P = e^{\frac{t}{20}} \times C} \text{ --(i)}$$

(where $C = e^{c_1}$)

Now, $P = 1000$, when $t = 0$

~~to~~ $1000 = e^{0} \times C \Rightarrow \boxed{C = 1000}$

So equation (i) ~~becomes~~ gives

$$\boxed{P = e^{\frac{t}{20}} \times 1000}$$

~~Let t years let 'P' Rs double itself~~

Let 't' years be the time required to double the Principal.

then. ~~2000.~~

$$2000 = 1000\, e^{t/20}$$

$$2 = e^{t/20}$$

$$\boxed{20 \log_e|2| = t}$$

Homogeneous D.E. :-

(*) A function $F(x,y)$ is said to be Homogenous function of degree n,

F if $\boxed{F(\lambda x, \lambda y) = \lambda^n F(x,y)}$, (for any non-zero constant λ)

Eg $F_1(x,y) = y^2 + 2xy$

$F_1(\lambda x, \lambda y) = (\lambda y)^2 + 2(\lambda x)(\lambda y) = \lambda^2 (y + 2xy)$

$\{F_1(\lambda x, \lambda y = \lambda^2 F(x,y)\}$ [here $n = 2$ so it is a homogenous function of degree 2.]

Noted Imp. Also Note :-

A function $F(x,y)$ is a homogenous function of degree n if

$$\left[F(x,y) = x^n g\left(\frac{y}{x}\right) \text{ or } y^n h\left(\frac{x}{y}\right)\right]$$

Eg $F_1(x,y) = x^2\left[\frac{y^2}{x^2} + \frac{2y}{x}\right] = x^2 h_1\left(\frac{y}{x}\right)$

$\boxed{F_1(x,y) = x^2 h_1\left(\frac{y}{x}\right)}$. [here $n = 2$ so degree of Homogenous function is 2]

(Or)

$F_1(x,y) = y^2\left[1 + \frac{2x}{y}\right] = y^2 h_2\left(\frac{x}{y}\right)$

$\boxed{F_1(x,y) = y^2 h_2\left(\frac{x}{y}\right)}$ [here $n = 2$ So degree of homogenous function is 2].

(*) A differential equation of the form $\left[\frac{dy}{dx} = F(x,y)\right]$ is said to be Homogeneous if $F(x,y)$ is a homogenous function of degree zero.

i.e. ~~if~~ F

Date:
Page:

i.e if $F(\lambda x, \lambda y) = \lambda^0 F(x, y)$ (or) | if $F(x,y) = x^0 g\left(\frac{y}{x}\right)$ (or) ~~if $F(x,y) = y^0 g\left(\frac{y}{x}\right)$~~ if $F(x,y) = y^0 h\left(\frac{x}{y}\right)$

* ~~So~~ Homogenous

(I) → To Solve a Differential equation of the type

$$\left[\frac{dy}{dx} = F(x, y) = g\left(\frac{y}{x}\right)\right]$$

- We make the Substitution, $\boxed{y = vx}$

(II) → To Solve a D.E of the type.

$$\left[\frac{dx}{dy} = F(x, y) = h\left(\frac{x}{y}\right)\right]$$

- We make the Substitution, $\boxed{x = vy}$

Example

Q) $\frac{dy}{dx} = \frac{x + y}{x}$

Sol.

(I) Method

Let $y = vx$ then,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{x + vx}{x}$$

$$x\frac{dv}{dx} = 1 + v - v$$

$$x\frac{dv}{dx} = 1$$

$$\int dv = \int \frac{dx}{x}$$

$$v = \log|x| + c$$

$$\frac{y}{x} = \log|x| + c$$

$$\boxed{y = x\log|x| + cx}$$

(II) Method

Let $x = vy$ then, $\frac{dx}{dy} = v + y\frac{dv}{dy}$

$$\frac{dx}{dy} = \frac{x}{x + y}$$

$$v + y\frac{dv}{dy} = \frac{vy}{vy + y} = \frac{v}{v+1}$$

$$y\frac{dv}{dy} = \frac{v}{v+1} - \frac{v(v+1)}{v+1}$$

$$y\frac{dv}{dy} = \frac{-v^2}{v+1}$$

$$\int \frac{v+1}{v^2}\,dv = -\int \frac{dy}{y}$$

$$\log|v| - \frac{1}{v} = -\log|y| + c$$

$$\Rightarrow \boxed{y = x\log|x| + Cx}$$

Date:
Page:

Note

(*) Choose the Method carefully to make the calculation Short.

(*) Only Homogenous D.E.s can be solved by these Methods, So first you need to check if the D.E. is Homogenous.

Example

Q) Show that the D.E. $2ye^{x/y}\,dx + (y - 2xe^{x/y})\,dy = 0$ is homogenous and find its Particular solution, given that, $x = 0$ when $y = 1$.

Sol.
$$\frac{dx}{dy} = \frac{2xe^{x/y} - y}{2ye^{x/y}} \quad -(i)$$

Let
$$F(x,y) = \frac{2xe^{x/y} - y}{2ye^{x/y}}$$

$$F(\lambda x, \lambda y) = \frac{2\lambda x e^{x/y} - \lambda y}{2\lambda y e^{x/y}} = \lambda^0 F(x,y)$$

So, the given D.E. is a Homogenous D.E.
To solve it we make the substitution,
$$\boxed{x = vy} \Rightarrow \frac{dx}{dy} = v + y\frac{dv}{dy}$$

Putting these values in equation (i)
$$v + y\frac{dv}{dy} = \frac{2vye^{v} - y}{2ye^{v}} = \frac{2ve^{v} - 1}{2e^{v}}$$

$$y\frac{dv}{dy} = \frac{2ve^{v} - 1}{2e^{v}} - \frac{v(2e^{v})}{2e^{v}} \Rightarrow y\frac{dv}{dy} = \frac{-1}{2e^{v}}$$

$$\int 2e^{v}\,dv = -\int \frac{dy}{y} \Rightarrow 2e^{v} = -\log|y| + c \Rightarrow \boxed{2e^{x/y} + \log|y| = c} \quad -(ii)$$

Substituting $x = 0$ and $y = 1$, we get $\boxed{C = 2}$

So $\left[2e^{x/y} + \log|y| = 2\right]$ is the Particular Solution of the given D.E.

(Here, the Plan of Action)

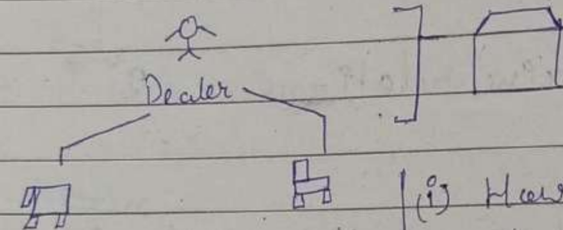
degree of Variable is 1 [Which will determine the optimal Values]

Linear Programming

→ In a nutshell, the technique to optimise an objective function is called Programming.

Q.) ₹ 50,000

Dealer



Storage Capacity: 60 piece

1 piece of chair = 1 piece of table

	Table	Chair
Cost of making	₹ 2500	₹ 500
Profit by selling	₹ 250	₹ 75

(i) How many tables & chairs should he buy to maximize his total profit.

Ans) We have to find no. of chairs and no. of tables in order to maximize the profit

effective utilization of Resources in the order to maximize profit & minimize loss (Optimization)

↓ After optimization, we get

Required Values
(optimal Values)

↓

Mathematical
Formulation

Page No. ____
Date

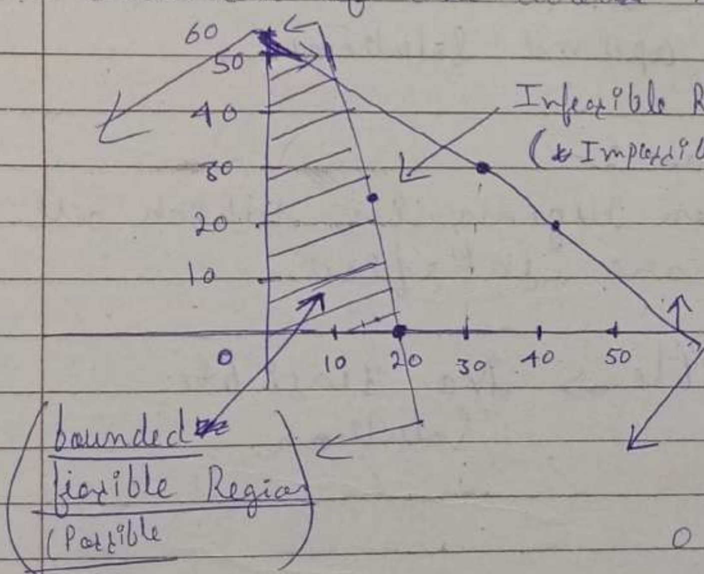
Sol.) Let no. of chairs be y
Let no. of tables be x.

$2500x + 500y \le 50000$ ~~(Input Constraint)~~ ~~Subject to~~
$x + y \le 60$ ~~(Storage Constraint)~~ ~~Constraints.~~

Maximize Profit $Z = 250x + 75y$ (objective function)

Subject to Constraints.
(i) - $5x + y \le 100$ (Investment Constraint)
(ii) $x + y \le 60$ (Storage Constraint)
$x \ge 0, y \ge 0$ (Non-Negative Constraints)

Solution of the above LPP :- (Graphically)



(i)

x	y
20	0
15	25

(ii)

x	y
30	30
40	20

Vertices of Polygon :-
(0,0), (20,0), (0,60), (10,50)

→ Solution Region
∞ Solutions

Vertex	$Z = 250x + 75y$
O (0,0)	0
A (20,0)	5000
B (0,60)	4500
C (10,50)	6250

10 tables 50 chairs.

Page No.
Date

Theorem 1

LPP mein jo bounded region milega uske Corner Points ko rakhne ke baad jo z gie thi Corner Point pe z give max optimal value that values are optimal values.

- Practical Problem → Optimi-sation → Mathematical Formulation → Solve Graphically → Optimal Feasible Solution → Optimal Values

(*) If maximum or minimum found at two points than the line segment joining those two points in the feasible region will contain points that will give us equal value of objective function

↓

∞ optimal solutions.

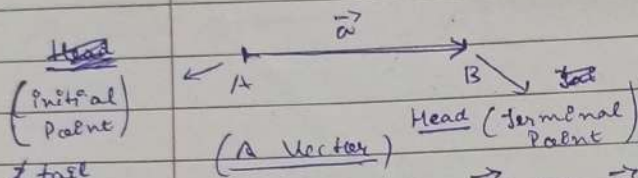
Feasible region :-

→ That common region in which all constraints are satisfied.

If No feasible region ⇒ No Feasible Solution.

Vector Algebra

* A Quantity that has ~~Magnitud~~ Magnitude as well as direction is called a Vector.



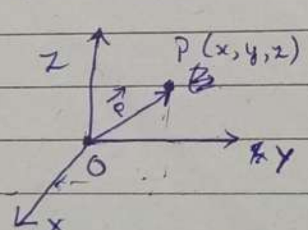
- length of line Segment gives Magnitude.
- Arrow gives Direction.

• denoted by $\vec{AB}$ or $\vec{a}$.

(*) length of is never negative. So, $|\vec{a}| < 0$ is not Possible.

$$\boxed{\text{we can write } |\vec{a}| = a}$$

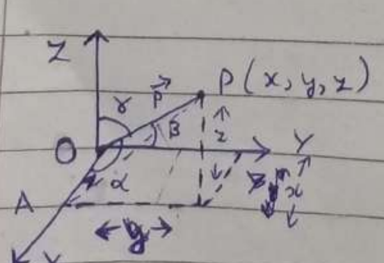
Position Vector



(*) $\vec{OP}$ ($\vec{P}$) is called the Position Vector.
(tail at origin, head at P)

$|\vec{OP}| = \sqrt{x^2+y^2+z^2}$ (Using distance Formula)
($\vec{P}$)

Direction Cosines

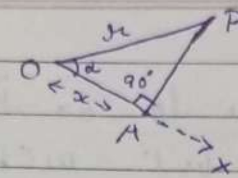


- α, β and γ are called Direction Angles.
- $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called Direction Cosines.

denoted by,

$l = \cos\alpha$ | $m = \cos\beta$ | $n = \cos\gamma$

In ΔOAP



$\cos\alpha = \frac{x}{r}$

(where $r = |\vec{r}|$)

Similarly,

$\boxed{m = \cos\beta = \frac{y}{r}}$ $\boxed{n = \cos\gamma = \frac{z}{r}}$ $\boxed{l = \cos\alpha = \frac{x}{r}}$

and $\boxed{x = l.r}$ $\boxed{y = mr}$ $\boxed{z = nr}$

[These are called Direction ratios of Vector $\vec{P}$.]

- $l^2 + m^2 + n^2 = 1$

Types of Vectors :-

- Zero Vector ($\vec{0}$)

-> Initial & terminal Points Coincide

-> No fixed Direction or have ∞ Directions.

Eg: $\vec{AA}$, $\vec{BB}$

- Unit Vector ($\hat{a}$)

-> Mag. is 1

-> Used to show Direction.

- Coinitial Vectors

-> Two or more Vector having the same initial ~~Shape~~ Point.

- Collinear Vectors.

-> Parallel Vectors.

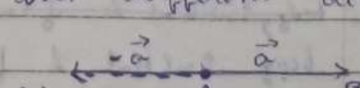
-> ~~Mag & Direction don't matter~~

- Equal Vectors

-> Vectors which have [same Magnitude & same Direction]

- Negative of a Vector

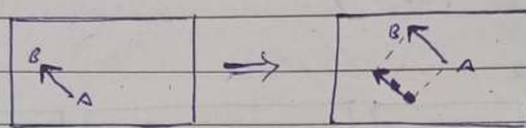
-> A Vector with same mag. but different direction.



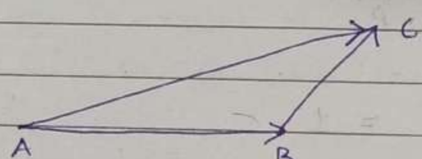
$\boxed{\vec{AB'} = -\vec{AB}}$

Page No. ____
Date ____

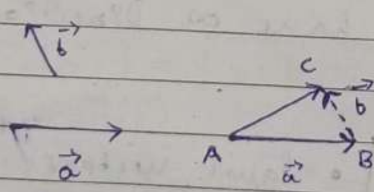
- We are free to do Parallel Shifting of Vectors, but without changing their mag. & Direction.



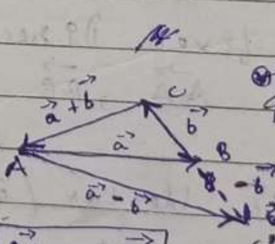
Addition of Vectors



$\boxed{\vec{AC} = \vec{AB} + \vec{BC}}$ — (Triangle law of Vector Addition.)



$\{\vec{AB} + \vec{BC} = \vec{AC}\}$



- $\boxed{\cancel{\vec{BC} = +\vec{BC'}}}$
- $\boxed{\vec{BC'} = -\vec{BC}}$
- $\boxed{\vec{AC} = -\vec{CA}}$

Thus, $\vec{AB} + \vec{BC} + \vec{CA} = \vec{AA} = \vec{0}$

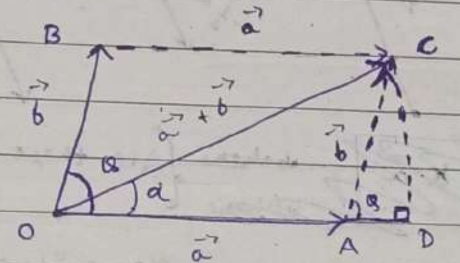
- When the sides of a triangle are taken in order, it leads to Zero Resultant. (Zero Vector)

→ ~~bcoz, initial & final~~

→ bcoz initial & final points get coincided & net vector quantity (displacement etc.) becomes Zero.

- $\vec{AC} = \vec{a} + \vec{b}$ (Sum of Vectors)
- $\vec{AC'} = \vec{a} - \vec{b}$ (Difference of Vectors)

Parallelogram law



~~BE~~ $\cos\theta = \frac{\cancel{BE}}{\cancel{b}}\ \frac{\cancel{AD}}{\cancel{b}}\ \frac{AD}{b}$

~~$BE = b\cos\theta$~~

$\boxed{AD = b\cos\theta}$

$\boxed{CD = b\sin\theta}$

$\vec{OA} + \vec{AC} = \vec{OC}$

$\therefore \vec{OB} = \vec{AC}$

~~$\vec{OA}$~~ $\boxed{\vec{OA} + \vec{OB} = \vec{OC}}$ → (This is Parallelogram law)

~~Mag of~~ ~~$\vec{OC}$~~

Mag. of $\vec{a}+\vec{b}$:-

$|\vec{OC}| = \sqrt{a^2+b^2+2ab\cos\theta}$

Direction of $\vec{a}+\vec{b}$:-

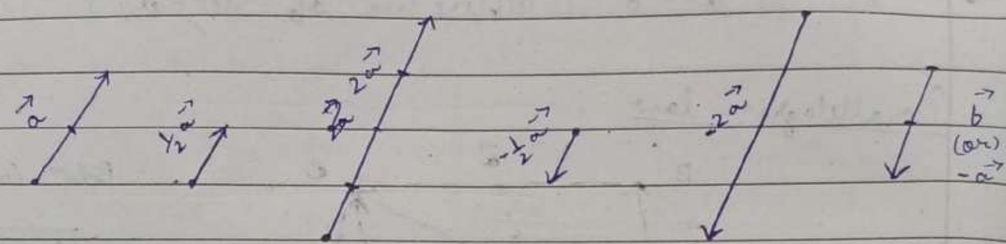
$\tan\alpha = \frac{b\sin\theta}{a+b\cos\theta}$

Properties of Vector Addition

- $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ (Commutative Property)
- $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ (Associative Property)
- $\vec{a} + \vec{O} = \vec{O} + \vec{a} = \vec{a}$ ($\vec{O}$ is the Additive Identity for Vector Addition.)

Page No. ______
Date

Multiplication of a Vector by a Scalar :-



~~Product = λ × a~~ {All these Vectors are Collinear.}

$\boxed{\vec{b} = \lambda \vec{a}}$ ⟶ Magnitude
$|\vec{b}| = |\lambda \vec{a}|$
$\boxed{b = \lambda a}$

↓
Direction

- if λ is +ve, same direction
- if λ is -ve, opposite direction.

Note :-
- $\lambda \vec{0} = \vec{0}$ ✓
- $\vec{a}\, 0 = \vec{0}$

$\vec{0} = 0\hat{i} + 0\hat{j} + 0\hat{k}$

(*) if $\vec{a}$ a vector of mag. a and $\vec{u}$ is a Vector of unit Mag. in the direction of $\vec{a}$ then, we can write $\boxed{\vec{a} = a\vec{u}}$

Note :-

→ when λ = -1, then $\vec{b} = -\vec{a}$ → same Magnitude. → opposite direction.

- $\vec{a} + \vec{b} = \vec{a} - \vec{a}$
  $\vec{a} + \vec{b} = \vec{0}$

($\vec{b}$ is called the Additive inverse of $\vec{a}$)

~~If $\lambda = \frac{1}{|\vec{a}|}$, provided $|\vec{a}| \neq 0$ or $\vec{a}$ is not null vector.~~

~~then, $|\lambda \vec{a}| = |\lambda||\vec{a}| = \frac{1}{|\vec{a}|} \times |\vec{a}| = 1$~~

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} \quad \text{in this case} \left\{ \lambda = \frac{1}{|\vec{a}|} \right\}$$

(Unit Vector)

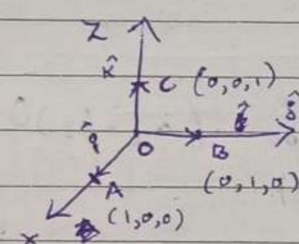
Mag. of $\hat{a}$:-

$$|\hat{a}| = \frac{|\vec{a}|}{|\vec{a}|} = 1$$

Direction of $\hat{a}$:-

→ Direction is same as that of $\vec{a}$.

Component of a Vector

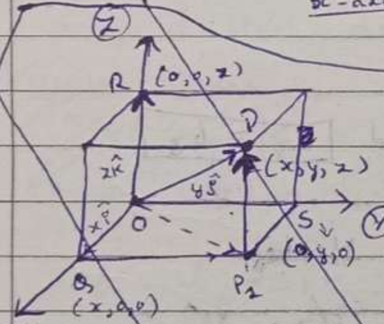


→ The vectors $\vec{OA}$, $\vec{OB}$ and $\vec{OC}$ are called unit vectors along respective axes.

As, $|\vec{OA}| = |\vec{OB}| = |\vec{OC}| = 1$

~~den~~ denoted by,

$\underset{x\text{-axis}}{\vec{OA}\,(\hat{i})}$, $\underset{y\text{-axis}}{\vec{OB}\,(\hat{j})}$, $\underset{z\text{-axis}}{\vec{OC}\,(\hat{k})}$



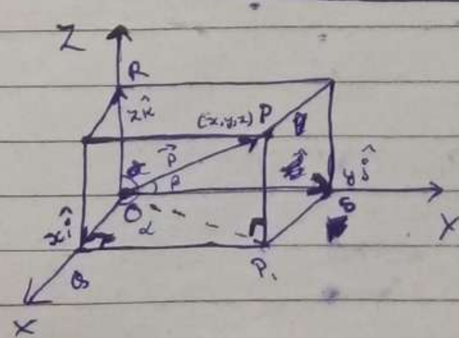
$\vec{z} = z\hat{k}$

- $\vec{P_1P} = \vec{OR} = z\hat{k}$ $(\vec{z} = z\hat{k})$
- $\vec{QP_1} = \vec{OS} = y\hat{j}$ $(\vec{y} = y\hat{j})$
- $\vec{OQ} = \vec{SP_1} = x\hat{i}$ $(\vec{x} = x\hat{i})$

Now, $\vec{OP_1} = \vec{OQ} + \vec{QP_1} = x\hat{i} + y\hat{j}$

$\vec{OP} = \vec{OP_1} + \vec{P_1P} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\boxed{\vec{OP} = x\hat{i} + y\hat{j} + z\hat{k}}$$



$$\boxed{\vec{OP}\ (\text{or } \vec{p}) = x\hat{i} + y\hat{j} + z\hat{k}}$$

↳ This is called Component form.
(Rectangular Components)

- x, y & z are Scalar Components of $\vec{P}$.
- $x\hat{i}$, $y\hat{j}$ & $z\hat{k}$ are Vector Components of $\vec{P}$ along respective axes.

Page No.
Date

$$\vec{OP} = x\hat{i} + y\hat{j} + z\hat{k} = \vec{p}$$

$$\boxed{|\vec{p}| = \sqrt{x^2 + y^2 + z^2}}$$

---

if $\{\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\}$ and $\{\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\}$

then, (i) Sum :-

$$\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$$

(ii) Difference :-

$$\vec{a} - \vec{b} = (a_1 - b_1)\hat{i} + (a_2 - b_2)\hat{j} + (a_3 - b_3)\hat{k}$$

(iii) $\vec{a} = \vec{b}$ if and only if,

$\boxed{a_1 = b_1}$ $\boxed{a_2 = b_2}$ and $\boxed{a_3 = b_3}$

(iv) Multiply of Scalar & Vector.

$$\lambda\vec{a} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$$

→ Let $\vec{a}$ & $\vec{b}$ be any two vectors, and k & m be any scalars, then

(i) $k\vec{a} + m\vec{a} = (k+m)\vec{a}$

(ii) $k(m\vec{a}) = km\vec{a}$

(iii) $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$

Note :-

(i) ~~$\lambda\vec{a}$ is always collinear to the vector $\vec{a}$. such that $\lambda \neq 0$~~

(i) $\lambda \vec{a}$ is always collinear to the vector $\vec{a}$, such that $\boxed{\lambda \neq 0}$

Two vector are collinear if and only if there exists a non-zero scalar λ such that

$$\boxed{\vec{b} = \lambda \vec{a}}$$

if $\left[\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}\right]$ and $\left[\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}\right]$

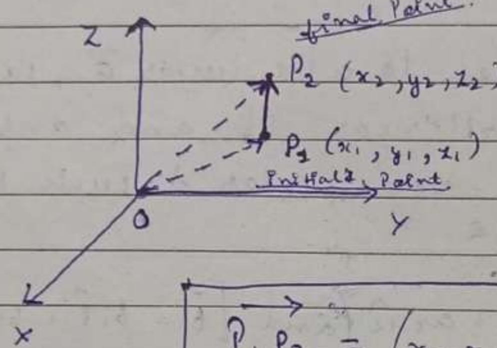
then the two vectors are collinear if & only if

$\boxed{\vec{a} \parallel \vec{b}}$ $\boxed{\vec{a} = \lambda \vec{b}}$ $\boxed{\lambda \neq 0}$ (or) $\boxed{\dfrac{b_1}{a_1} = \dfrac{b_2}{a_2} = \dfrac{b_3}{a_3} = \lambda}$

(ii) If $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$ then, a_1, a_2, a_3 are called direction ratios of $\vec{a}$.

(iii) If l, m, n are direction cosines of a vector, then $l\hat{i} + m\hat{j} + n\hat{k} = (\cos\alpha)\hat{i} + (\cos\beta)\hat{j} + (\cos\gamma)\hat{k}$ is the unit vector in the direction of that vector.

Vector joining two Points.



$$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$$

$$\overrightarrow{P_1P_2} = \left(x_2\hat{i} + y_2\hat{j} + z_2\hat{k}\right) - \left(x_1\hat{i} + y_1\hat{j} + z_1\hat{k}\right)$$

$$\boxed{\overrightarrow{P_1P_2} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}}$$

(initial Point) $\left|\overrightarrow{P_1P_2}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ (final Point)

also,

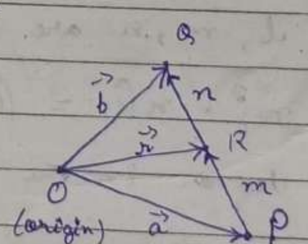
$\overrightarrow{P_2P_1} = (x_1 - x_2)\hat{i} + (y_1 - y_2)\hat{j} + (z_1 - z_2)\hat{k}$

(initial Point) (final Point)

Section Formula

Case I

When R divides PQ internally. $\frac{m}{n} = +ve$



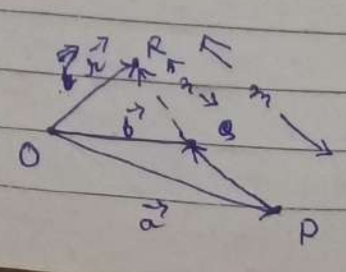
$(\overrightarrow{OR})$ $\boxed{\vec{r} = \dfrac{m\vec{b} + n\vec{a}}{m + n}}$

$$\boxed{\frac{PR}{RQ} = \frac{m}{n}}$$

Case II

When R divides PQ externally. $\frac{m}{n} = -ve.$

$$\boxed{\overrightarrow{OR}\ (\text{or } \vec{r}) = \frac{m\vec{b} - n\vec{a}}{m - n}}$$



~~Note (Section formula) :- {If P initial & Q final Point}~~

~~Vectors :-~~	Ratios :-
~~→ O to P → ($\vec{a}$)~~	~~→ P to R~~ → near P (m)
→ O to Q → ($\vec{b}$)	~~→ Q to R~~ → near Q (n)
~~→ O to R → ($\vec{r}$)~~	

PRODUCT OF TWO VECTORS :-

Dot Product (Scalar Product) :-

$$\boxed{\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta}$$ (angle between $\vec{a}$ and $\vec{b}$)

- where $\vec{a}$ and $\vec{b}$ are two non-zero vectors.

if either $\vec{a}$ or $\vec{b} = 0$ then, $\boxed{\vec{a} \cdot \vec{b} = 0}$ bcoz, $\vec{a} \cdot \vec{0} = a \times 0 \cos\theta$

$\left[\dfrac{\vec{a} \cdot \vec{0}}{a \times 0} = \cos\theta\right]$

So θ becomes Undefined.

- $\vec{a} \cdot \vec{b}$ is a real Number (Scalar).
- if $\left[\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}\right]$, where $\vec{a}$ & $\vec{b}$ are two non-zero Vectors.
- ~~$\vec{a} \cdot \vec{a} = a^2$, where both vectors are equal~~
- If two vectors are equal, then
  $[\vec{a} \cdot \vec{a} = a^2]$

Note :- but $\vec{a} \cdot \vec{a} = |\vec{a}|^2$
$\vec{a} \neq |\vec{a}|$
($\vec{a} \cdot \vec{a} = a^2$)

- If $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors along x, y, z axes respectively, then,

$$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$$

$$\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$$

- $\boxed{\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{ab}}$
- $\boxed{\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}}$

(★) $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$

(★) $(\lambda\vec{a})\cdot\vec{b} = \lambda(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(\lambda\vec{b})$

---

if $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ &

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

then, $\boxed{\vec{a}\cdot\vec{b} = a_1b_2 + a_2b_2 + a_3b_3}$

~~Projection of a Vector on a line~~ :- $\cos\alpha = \frac{a_1}{|\vec{a}|}$ $\beta = a_2$ $\gamma = a_3$

~~Projection of $\vec{a}$ on l~~

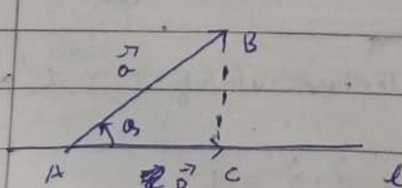
~~• $\vec{p}$ is called the Projection Vector.~~

~~$\cos\theta = \frac{|\vec{p}|}{|\vec{a}|} \Rightarrow |\vec{p}| = a\cos\theta$~~

~~$|\vec{p}| = a\cos\theta$~~

~~This is called Projection vector of $\vec{a}$ on line l.~~

Projection of a Vector :-
(Component)



• $\vec{p}$ is called the Projection Vector.

$\boxed{|\vec{p}| = a\cos\theta}$

→ Direction of $\vec{p}$ may be same or opposite to l

Proof :-

$\cos\theta = \frac{|\vec{p}|}{|\vec{a}|}$

$\boxed{|\vec{p}| = a\cos\theta}$

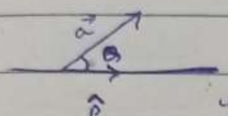
This is called Projection of Vector $\vec{a}$ on line l.

(*) Projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

(*) Projection of $\vec{b}$ on $\vec{a} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

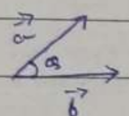
Page No. ______
Date

Note :-

(i) 

$\vec{a}\cdot\hat{p} = a|\hat{p}|\cos\theta$

$\boxed{\vec{a}\cdot\hat{p} = a\cos\theta}$ {component of $\vec{a}$}

(ii) 

$a\cos\theta = \vec{a}\cdot\hat{b} = \vec{a}\cdot\dfrac{\vec{b}}{|\vec{b}|} = \dfrac{1}{|\vec{b}|}(\vec{a}\cdot\vec{b})$

(Projection of $\vec{a}$ on $\vec{b}$) $\boxed{a\cos\theta = \vec{a}\cdot\hat{b} = \dfrac{\vec{a}\cdot\vec{b}}{b}}$

(iii) If $\theta = 0°$ then projection of $\vec{a}$ will be $\vec{a}$ itself.
If $\theta = 180°$ then projection of $\vec{a}$ will be $-\vec{a}$.

(iv) If $\theta = 90°$ or $270°$, then Projection vector of $\vec{a}$ will be $\vec{0}$.

- if α, β & γ are the direction angles of $[\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}]$

$\cos\alpha = \dfrac{\vec{a}\cdot\hat{i}}{|\vec{a}||\hat{i}|} = \dfrac{a_1}{|\vec{a}|}$ | $\cos\beta = \dfrac{a_2}{|\vec{a}|}$ | $\cos\gamma = \dfrac{a_3}{|\vec{a}|}$

Also, $\boxed{a_1 = a\cos\alpha}$ $\boxed{a_2 = a\cos\beta}$ $\boxed{a_3 = a\cos\gamma}$ } These are actually Projections of $\vec{a}$ on respective axes.

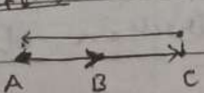
if $\vec{a}$ is a unit vector than, it can be written as -

$\boxed{\hat{a} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}}$

As $|\hat{a}| = 1$

$\boxed{a = 1}$

Note :-



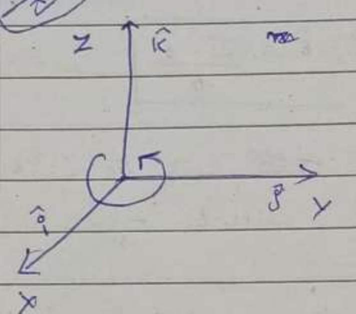
$\vec{AB} + \vec{BC} = \vec{AC}$

$\boxed{\vec{AB} + \vec{BC} + \vec{CA} = \vec{0}}$

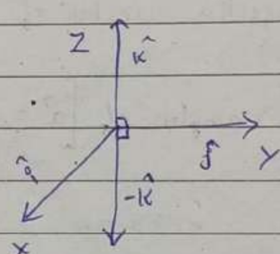
- We get $\vec{0}$ when final & initial points coincide.

Cross Product (Vector Product)

Right hand Rule



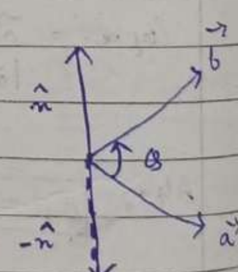
Moving from x to y :-
direction of thumb
Upward.



Moving from Y to X :-
direction of thumb
Downward

~~Vector~~ $\boxed{\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta\, \hat{n}}$

(This is cross Product).



- θ is angle between $\vec{a}$ & $\vec{b}$.
- $\boxed{0 \le \theta \le \pi}$
- $\hat{n}$ is unit vector $\perp$ to both $\vec{a}$ and $\vec{b}$

$\left[\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta\, \hat{n}\right]$ $\left[\vec{b} \times \vec{a} = |\vec{b}|\,|\vec{a}| \sin\theta\, (-\hat{n})\right]$

→ Use Right hand rule to find direction of $\hat{n}$.

If $\vec{a} = 0$ or $\vec{b} = 0$, then $\boxed{\vec{a} \times \vec{b} = \vec{0}}$, As θ become Undefined

- $\vec{a} \times \vec{b}$ is a vector

- If $\vec{a}$ and $\vec{b}$ are non zero vectors, then.

$$\boxed{\vec{a} \times \vec{b} = \vec{0} \Leftrightarrow \vec{a} \parallel \vec{b}}$$ or $\boxed{\theta = 0^\circ}$ & $\{\sin 0^\circ = 0\}$

- $\boxed{\vec{a} \times \vec{a} = \vec{0}}$ and $\boxed{\vec{a} \times (-\vec{a}) = \vec{0}}$ $\{[\vec{a} \times \vec{a} = \vec{a} \times (-\vec{a})]\}$

- If $\theta = 90^\circ$, then $\boxed{\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}|}$

- $[\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0]$

$[\hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i},\ \hat{k} \times \hat{i} = \hat{j}]$

- $$\boxed{\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|}}$$ As, $\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta\, \hat{n}$

$\boxed{|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta}$

- $[\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}]$ [Non Commutative Property]

- $[\hat{j} \times \hat{i} = -\hat{k}]$, $[\hat{k} \times \hat{j} = -\hat{i}]$ and $[\hat{i} \times \hat{k} = -\hat{j}]$

~~Area of triangle ABC~~ $\cancel{= \frac{1}{2} |\vec{a} \times \vec{b}|}$

~~Area of Parallelogram ABCD~~ $\cancel{= |\vec{a} \times \vec{b}|}$

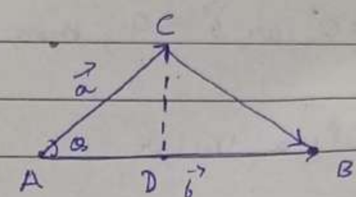
- If θ is angle b/w $\vec{a} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ & $\vec{b} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$

then,
$$\boxed{\cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\ \sqrt{a_2^2 + b_2^2 + c_2^2}}}$$

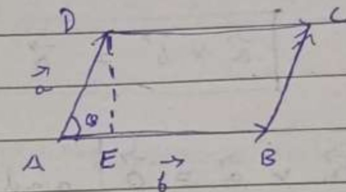
Page No.
Date

(*) ~~Area of ΔABC:~~

Area of ΔABC $= \frac{1}{2}|\vec{a}\times\vec{b}|$



(*) Area of Parallelogram ABCD $= |\vec{a}\times\vec{b}|$



Note :-

(i) $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$

(ii) $\lambda(\vec{a}\times\vec{b}) = (\lambda\vec{a})\times\vec{b} = \vec{a}\times(\lambda\vec{b})$

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ &
$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

then, $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

or

$$\left[\vec{a}\times\vec{b} = (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}\right]$$

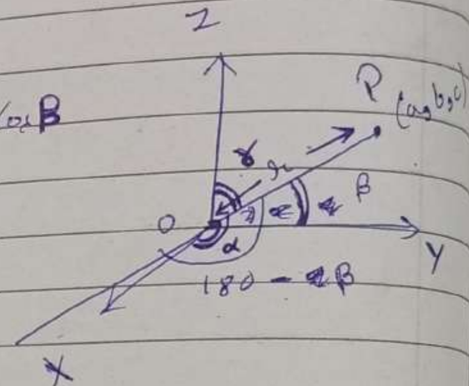
Cauchy Schwartz Inequality $\left[|\vec{a}\cdot\vec{b}| \le |\vec{a}||\vec{b}|\right]$

Also $|\vec{a}+\vec{b}| \le |\vec{a}| + |\vec{b}|$

3-D

Direction cosines of a line

$m = \cos\beta = \cos(\pi-\beta) = -\cos\beta$



$\boxed{l = \pm\cos\alpha} = \pm\frac{a}{r}$

$\boxed{m = \pm\cos\beta} = \pm\frac{b}{r}$

$\boxed{n = \pm\cos\gamma} = \pm\frac{c}{r}$

~~a, b, c are direction ratios of the line &,~~

$r = \sqrt{a^2+b^2+c^2}$

(mag. of line)

$\left[\boxed{l^2+m^2+n^2=1} \right]$

$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = K$ (say)

Direction cosines of a line passing through two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are

$\frac{x_2-x_1}{PQ}, \frac{y_2-y_1}{PQ}, \frac{z_2-z_1}{PQ}$ → Direction ratios.

where $PQ = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

Page No. ______
Date

Section formula :

i) Internal division : $\boxed{\frac{m}{n} = +ve}$

$$x = \frac{mx_2 + nx_1}{m+n}, \quad y = \frac{my_2 + ny_1}{m+n}, \quad z = \frac{mz_2 + nz_1}{m+n}.$$

ii) External division : $\boxed{\frac{m}{n} = -ve}$

$$x = \frac{mx_2 - nx_1}{m-n}, \quad y = \frac{my_2 - ny_1}{m-n}, \quad z = \frac{mz_2 - mz_1}{m+n}$$

iii) Mid point formula :

$$x = \frac{x_1 + x_2}{2}, \quad y = \frac{y_1 + y_2}{2}, \quad z = \frac{z_1 + z_2}{2}$$

Equation of a line :-

1.) Passing through a given point and parallel to a given vector.

i) Vector form : $\boxed{\vec{r} = \vec{a} + \lambda \vec{b}}$

ii) Cartesian form : $\boxed{\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}} = \lambda$

iii) if l, m, n are direction cosines of the line, the equation of the line is

$$\boxed{\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}}.$$

2.) Passing through two given points

i) Vector form : $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$, $\lambda \in R$

(ii) Cartesian equation:

$$\boxed{\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}}$$

Angle between two lines:

(i) Vector form: $\cos\theta = \left| \frac{\vec{b_1} \cdot \vec{b_2}}{|\vec{b_1}|\,|\vec{b_2}|} \right|$

(ii) Cartesian form: $\cos\theta = \left| \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\,\sqrt{a_2^2+b_2^2+c_2^2}} \right|$

Two lines are

(i) $\perp$ if $\boxed{a_1a_2 + b_1b_2 + c_1c_2 = 0}$

(ii) Parallel if $\boxed{\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}}$

Note:- lines must pass through origin

if instead of direction ratios for the line direction cosines are given then.

$$\boxed{\cos\theta = |l_1l_2 + m_1m_2 + n_1n_2|}$$

~~Shortest distance between two lines:~~

(a)

(i)

(ii)

(b)

Page No.
Date

Shortest distance between two lines :

(a) Distance between two skew lines.

(i) Vector form : $$d = \left| \frac{(\vec{b_1} \times \vec{b_2}) \cdot (\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|} \right|$$

(ii) Cartesian form :

$$\frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}$$

(b) Distance between parallel lines

$$d = \left| \frac{\vec{b} \times (\vec{a_2} - \vec{a_1})}{|\vec{b}|} \right|$$

- Two lines are parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ — direction ratios
- Two lines are skew if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

If line intersect / coplanar then $(d = 0)$:-

$$\begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

Page No.
Date

PLANE :- (Definition
If we take any two points and join them, then the line segment joining them must lie within the surface)

1.) Equation of a Plane in normal form.

(i) Vector form: $\vec{r} \cdot \hat{n} = d$
(ii) Cartesian form: $lx + my + nz = d$

(or)

if, $\vec{r} \cdot \vec{n} = d$ then $\boxed{ax + by + cz = d}$ is the Cartesian equation of Plane

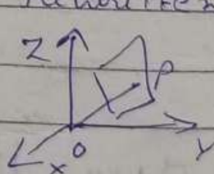
$(\cdot\ \vec{n} = a\hat{i} + b\hat{j} + c\hat{k})$

(This is not normal distance from origin)

Eg 16.) Find the coordinates of the foot of the ⊥ from the origin to the Plane $2x - 3y + 4z - 6 = 0$

Sol.) The Equation of the Plane can be rewritten as

$$\frac{2x}{\sqrt{29}} - \frac{3y}{\sqrt{29}} + \frac{4z}{\sqrt{29}} = \frac{6}{\sqrt{29}} \quad -(i)$$



which is the Cartesian Equation of Plane in Normal form

(*) Let x_1, y_1, z_1 be the coordinates of the foot of the ⊥ from the origin to the Plane. Then the direction ratios of OP are x_1, y_1, z_1. Since dc's and direction ratios are proportional

$$\frac{x_1}{\frac{2}{\sqrt{29}}} = \frac{y_1}{\frac{-3}{\sqrt{29}}} = \frac{z_1}{\frac{4}{\sqrt{29}}} = K$$

i.e. $x_1 = \frac{2k}{\sqrt{29}}, \; y_1 = \frac{-3k}{\sqrt{29}}, \; z_1 = \frac{4k}{\sqrt{29}}$

Substituting in (i)

$$\frac{2}{\sqrt{29}} \times \frac{2k}{\sqrt{29}} - \frac{3}{\sqrt{29}} \times \frac{-3k}{\sqrt{29}} + \frac{4 \times 4k}{\sqrt{29}\sqrt{29}} = \frac{6}{\sqrt{29}}$$

$$4k + 9k + 16k = 6\sqrt{29}$$

$$29k = 6\sqrt{29} \Rightarrow \boxed{k = \frac{6}{\sqrt{29}}}$$

Short cut -

if d is the distance from the origin and l, m, n are the direction cosines of the normal to the plane through the origin, then the foot of the perpendicular is (ld, md, nd).

2) Equation of a Plane ⊥ to a given vector and passing through a given point

(i) Vector form : $(\vec{r} - \vec{a}) \cdot \vec{N} = 0$

$\vec{N}$ → normal to Plane (Not necessarily Normal unit vector.) (Can be any vector i.e. ⊥ to the Plane.)

(ii) Cartesian form :

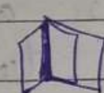
$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

3.) Equation of the Plane Passing through three non collinear points :

(i) Vector form: $(\vec{r} - \vec{a}) \cdot [(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0$

(ii) Cartesian form :

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

Note: If Points Collinear, 
∞ Plane pass through them

INTERCEPT FORM of the equation of a Plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad \Bigg| \quad A = \frac{-D}{a},\ B = \frac{-D}{b},\ C = \frac{-D}{c}$$

When General Equation of Plane is

$$\boxed{Ax + By + Cz + D = 0}$$

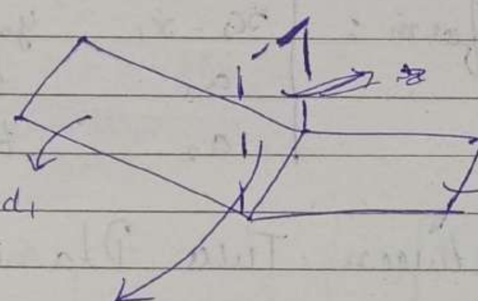
E

Page No. ______
Date

Equation of Plane Passing through the intersection of two given Planes

G. E. :- $[A_1x + B_1y + C_1z = d]$

~~(i) Vector form: $\vec{r} \cdot (\vec{n_1} + \lambda \vec{n_2}) = d_1 + \lambda d_2$~~



$\vec{r} \cdot \hat{n_1} = d_1$ $\qquad$ $\vec{r} \cdot \hat{n_2} = d_2$

(i) Vector form : $\vec{r} \cdot (\vec{n_1} + \lambda \vec{n_2}) = d_1 + \lambda d_2$

(ii) Cartesian form :

$$(A_1x + B_1y + C_1z - d_1) + \lambda(A_2x + B_2y + C_2z - d_2) = 0.$$

Q) Find the equation of the plane through the line of intersection of the planes $x + y + z = 1$ and $2x + 3y + 4z = 5$ which is perpendicular to the plane $x - y + z = 0$.

Sol) Equation of Required Plane :-

$$(x + y + z - 1) + \lambda(2x + 3y + 4z - 5) = 0$$

$\Rightarrow (2\lambda - 1)x + (3\lambda + 1)y + (4\lambda + 1)z - 5(\lambda + 1) = 0 \quad -(i)$

Now, $1(2\lambda - 1) - 1(3\lambda + 1) + 1(4\lambda + 1) = 0 \Rightarrow \boxed{\lambda = -\frac{1}{3}}$

Substituting in (i), we get.

$$\boxed{x - z + 2 = 0}$$

Coplanarity of two lines.

(i) Vector form : $(\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2})=0$

(ii) Cartesian form : $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$

ANGLE between Two Planes :-

- Angle between two planes is the angle between the normal of the planes.

Suppose the equations of plane are,

$[\vec{r}\cdot\vec{n_1}=d_1]$ & $[\vec{r}\cdot\vec{n_2}=d_2]$

then, $\cos\theta=\left|\dfrac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}|\,|\vec{n_2}|}\right|$

~~Vector~~ Cartesian form : {G.E. of planes $Ax+By+Cz+D=0$}

$$\cos\theta=\left|\frac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\,\sqrt{A_2^2+B_2^2+C_2^2}}\right|$$

1.) If the planes are $\perp$ to each other then
$A_1A_2+B_1B_2+C_1C_2=0$.

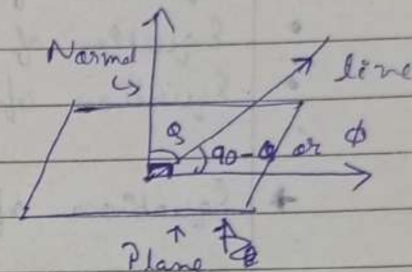
2.) If the planes are $\parallel$ to each other then

$$\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2}$$

$\left\{\frac{x-x_1}{b_1}=\frac{y-y_1}{b_2}=\frac{z-z_1}{b_3}=\lambda\right\}$ $\left\{n_1x+n_2y+n_3z+d=0\right\}$

Angle between a line and a Plane

$$\cos\theta = \left|\frac{\vec{b}\cdot\vec{n}}{|\vec{b}|\,|\vec{n}|}\right|$$



$$\cos(90°-\theta) = \sin\phi$$

$$\sin\phi = \left|\frac{\vec{b}\cdot\vec{n}}{|\vec{b}|\,|\vec{n}|}\right| , \left[\phi \text{ is the Angle b/w the line and Plane}\right]$$

If line & Plane $\parallel$, $\phi = 0$

if line & Plane $\perp$, $\phi = 90°$ (Ratio Proportional)

If lines lies in the plane $\Rightarrow$ $b_1n_1 + b_2n_2 + b_3n_3 = 0$ & $n_1x_1 + n_2y_1 + n_3z_1 + d = 0$

Distance of a Point from a Plane

- Vector form :-

If Equation of Plane is $\vec{r}\cdot\vec{N} = d$, where $\vec{N}$ is normal to the Plane then, distance of a Point P be position vector $\vec{a}$ is.

$$\text{distance} = \left|\frac{\vec{a}\cdot\vec{N} - d}{|\vec{N}|}\right|$$

Note :- The length of the $\perp$ from origin O to the plane $\vec{r}\cdot\vec{N} = d$ is $\frac{d}{|\vec{N}|}$ (since $\vec{a}=0$)

Cartesian ~~Vector~~ form :

When Equ. of Plane = $\boxed{Ax + By + Cz = D}$

$$\text{distance} = \left|\frac{Ax_1 + By_1 + Cz_1 - D}{\sqrt{A^2+B^2+C^2}}\right|$$

Note :-

- Equation of x-y plane is, $z = 0$
- Equation of y-z plane is, $x = 0$
- Equation of z-x plane is, $y = 0$

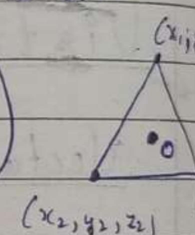
→ Equation of x-axis, $\boxed{\boxed{y=0} \ \& \ \boxed{z=0}}$

$\boxed{x=2}$

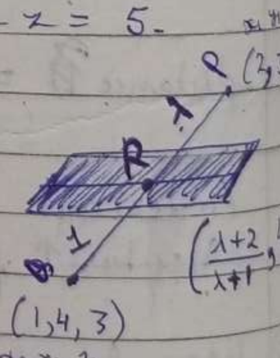
↳ Equation of a Plane, that is parallel to $y-z$ Plane.

- Co-ordinates of Centroid in 3D :-

$$O, \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$



Q.) Find the ratio in which the line segment joining the points $(2, 3, 5)$ and $(1, 4, 3)$ is divided by the Plane $x - 2y + z = 5$.



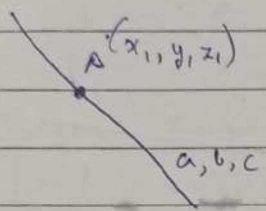
Sol) As P lies on $x - 2y + z = 5$

$$\frac{(\lambda+2) - 2(4\lambda+3) + (3\lambda+5)}{\lambda+1} = 5$$

$$\Rightarrow 9\lambda = -4 \Rightarrow \boxed{\lambda = \frac{-4}{9}}$$

External Division.

Equation of line (Extra)



$\Rightarrow$ equt of line is :-

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \lambda$$

- For question solving take any point on the line as $(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$

Eg $2x = 3y = 6z$ & $x = -y = z$ angle between = ?

Sol $\dfrac{x-0}{3} = \dfrac{y-0}{2} = \dfrac{z-0}{1}$ $\Bigg\}$ $\dfrac{x-0}{1} = \dfrac{y-0}{-1} = \dfrac{z-0}{1}$

line passing through Origin & having d.cs:- 3, 2, 1 $\Big\}$ line passing through Origin & having d.c.s :- 1, -1, 1

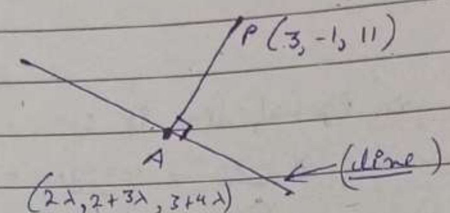
$$\cos\theta = \left| \frac{3\times1 + 2\times-1 + 1\times1}{\sqrt{3^2+2^2+1^2}\,\sqrt{1+1+1}} \right| = \left| \frac{2}{\sqrt{14}\times\sqrt{3}} \right|$$

$$\cos\theta = \frac{2}{\sqrt{42}} \Rightarrow \boxed{\theta = \cos^{-1}\left(\frac{2}{\sqrt{42}}\right)}$$

Page No.
Date

Q.) The length of the $\perp$ drawn from the point $(3,-1,11)$ to the line $\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ is

Sol)



any point A on line is taken as
$A\,(2\lambda,\ 2+3\lambda,\ 3+4\lambda)$

As line AP is $\perp$ to given line :-

$2(3-2\lambda) + 3(-3-3\lambda) + 4(+8-4\lambda) = 0$

$6 - 4\lambda - 9 - 9\lambda + 32 - 16\lambda = 0$

$-29\lambda = -29 \Rightarrow \boxed{\lambda = 1}$

$A\,(2, 5, 7)$

$PA = \sqrt{(1)^2 + 36 + 16} = \sqrt{53}$

Q.) If the lines $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{-k}$ and $\frac{x-1}{k} = \frac{y-4}{2} = \frac{z-5}{1}$ are coplanar then $k = ?$

Sol) Shortest Distance = 0

$$\begin{vmatrix} 2-1 & 3-4 & 4-5 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix}$$

$\Rightarrow 1(1+2k) + 1(1+k^2) + (-1)(2-k) = 0$

$\Rightarrow \boxed{k = 0, -3}$

Page No.
Date

(*) For Two Parallel Planes :-

$$A_1x + B_1y + C_1z + D_1 = 0$$
&
$$A_2x + B_2y + C_2z + D_2 = 0$$

$$\boxed{\perp \text{ distance between them} = \left|\frac{D_1 - D_2}{\sqrt{A^2 + B^2 + C^2}}\right|}$$

Q.) Distance b/w two parallel planes $2x + y + 2z = 8$ and $4x + 2y + 4z + 5 = 0$ is :

Sol.)
$$2x + y + 2z - 8 = 0$$
$$2x + y + 2z + \frac{5}{2} = 0$$

$$\perp \text{ Distance} = \left|\frac{\frac{5}{2} + 8}{\sqrt{4+1+4}}\right| = \frac{7}{2} \text{ ans}$$

Q.) An Equation of a plane parallel to the plane $x - 2y + 2z - 5 = 0$ and at a unit distance from the origin is.

Sol) Equation of a plane parallel to given Plane is :-
$$x - 2y + 2z + K = 0$$

$$1 = \frac{K}{\sqrt{(1)^2 + (2)^2 + (2)^2}} \Rightarrow \boxed{K = \pm 3}$$

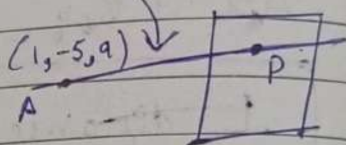
$$\boxed{x - 2y + 2z + 3 = 0} \text{ or } \boxed{x - 2y + 2z - 3 = 0}$$

Page No.
Date

Q.) The distance of the point $(1,-5,9)$ from the plane $x-y+z=5$ measured along a straight line $x=y=z$ is.

Sol.) Equation of line is :-

$\frac{x-1}{1} = \frac{y+5}{1} = \frac{z-9}{1} = \lambda$



any point P on line is :-

$P(1+\lambda, \lambda-5, 9+\lambda)$

as it lies on Plane $x-y+z=5$

$(1+\lambda)-(\lambda-5)+(\lambda+9)=5$

$\boxed{\lambda=-10}$ i.e. $P(-9,-15,-1)$

i.e. $AP = \sqrt{100+100+100} = 10\sqrt{3}$

Q) If the line $\frac{x-3}{2} = \frac{y+2}{-1} = \frac{z+4}{3}$ lies in the Plane $lx+my-z=9$, then $l^2+m^2 = ?$

Sol) line passes through $(3,-2,-4)$ So this point lies on the Plane therefore :-

$3l-2m+4=9 \Rightarrow \boxed{3l-2m=5}$ — (i)

also,

$2(l)+-1(m)+3(-1)=0$

$2l-m=3 \Rightarrow \boxed{2l-m=3}$ — (ii)

Solving (i) & (ii) we get, $\boxed{l=1}$ & $\boxed{m=-1}$

i.e. $l^2+m^2=2$

Q) The plane having $\frac{x-1}{1} =$

Sol) Equation

(or)

$\Rightarrow$

$\perp$ dis

Q) The distance of the point $(1, 3, -7)$ from the plane passing through the point $(1, -1, -1)$ having normal perpendicular to both lines $\frac{x-1}{1} = \frac{y+2}{-2} = \frac{z-4}{3}$ and $\frac{x-2}{2} = \frac{y+1}{-1} = \frac{z+7}{-1}$ is

Sol) Equation of Plane : $(\vec{r} - \vec{a}) \cdot \vec{N} = 0$

(or)
$$\begin{vmatrix} x-1 & y+1 & z+1 \\ 1 & -2 & 3 \\ 2 & -1 & -1 \end{vmatrix} = 0$$

$$\Rightarrow 5x + 7y + 3z + 5 = 0$$

$$\perp \text{dist} = \left| \frac{5(1) + 7(3) + 3(-7) + 5}{\sqrt{5^2 + 7^2 + 3^2}} \right|$$

$$= \frac{10}{\sqrt{83}}$$