Chapter 2 | Random Variable

Random variable is a real valued function which associated with each outcomes of a random experiment.

In simple words, R.V. (Random Variable) is defined as the outcomes associated with a random experiment.

It is denoted by Capital Letters. Generally, we use X, Y, Z, -- etc.

The value taken by a random variable is denoted by small letters like x, y, z, -- etc.

There are two types of random variable.

- → Discrete Random Variable
  - that takes discrete (i.e. integer or countable) values
- → Continuous Random Variable

07/08/09

Friday

• Discrete Random Variable

→ A random variable is said to be discrete random variable if it takes only integer values (countable values).

→ In other words, it is defined as real valued function defined on discrete sample space of a random experiment.

→ Examples:

(i) No. of students enrolled every year

(ii) No. of errors per page

(iii) No. of accidents per week, etc.

• Continuous Random Variable

→ A random variable is said to be continuous random variable if it takes all possible values within a given range.

→ Examples:

(1) Height of the individuals

(2) Size of compact disk

(3) temperature recorded for a particular day

Probability Distribution of Random Variable

→ It is a mathematical function or mathematical expression which provides the probabilities for all possible outcomes of a random experiment.

→ There are 2 types of probability distribution:

(i) Discrete probability distribution

(ii) Continuous probability distribution

• Probability Distribution of Discrete Random Variable

Probability function →

Let X be a discrete random variable with the probability function $P(X=x)$ or $P(x)$; $\forall x$

Then the probability function $P(X=x)$ or $P(x)$ is called Probability Mass Function (pmf) if it satisfies the following conditions:

(i) $P(X=x)$ or $P(x) \geq 0$; $\forall x$

$$\boxed{\text{(ii) } \sum_{x} P(X=x) = 1 \quad (\text{Total prob.} = 1)}$$

Distribution Function →

Let X be a discrete random variable with pmf $P(X=x)$ or $P(x)$; $\forall x$. Then the distribution function of X, is denoted by $F(x)$ and is

defined as:

$$\boxed{F(x) = P(X \leq x) = \sum_{-\infty}^{x} P(X=x)}$$

It is also called Cummilative Distribution Function.

Properties of Distribution Function

(i) $0 \leq F(x) \leq 1 \; ; \forall x$

(ii) $F(+\infty) = \lim_{x \to +\infty} F(x) = 1$

(iii) $P(a < X \leq b) = F(b) - F(a)$

(iv) $F(-\infty) = \lim_{x \to -\infty} F(x) = 0$

(v) $F(x) > F(y)$ if $x > y$

Example 1: A random variable X has the following probability distribution.

$X = x$	0	1	2	3	4	5	6
$P(X=x)$	K	$3K$	$2K$	$5K$	$7K$	$4K$	$8K$

(i) Find the value of K

(ii) Find $P(X<2)$, $P(X \geq 4)$ and $P(2 < X < 5)$

(iii) Find the distribution function of X

(iv) Find $F(2)$ and $F(4)$

Soln (1)

Since the total probability is 1.

i.e. $\sum P(X=x) = 1$

i.e. $P(X=0) + P(X=1) + \cdots + P(X=6) = 1$

or, $K + 3K + 2K + 5K + 7K + 4K + 8K = 1$

or, $30K = 1$

$\therefore K = \frac{1}{30}$

now the prob. distribution of X is,

$X=x$	$P(X=x)$
0	1/30
1	3/30
2	2/30
3	5/30
4	7/30
5	4/30
6	8/30

(ii) Soln $P(X<2), P(X \geq 4), P(2<X<5) = ?$

so,

$P(X<2) = P(X=0) + P(X=1) \cancel{+ P(X=2)}$

$= 1/30 + 3/30$

$= \frac{4}{30}$

$$P(X \geq 4) = P(X=4) + P(X=5) + P(X=6)$$

$$= \frac{7}{30} + \frac{4}{30} + \frac{8}{30} = \frac{19}{30}$$

$$P(2 < X < 5) = P(X=3) + P(X=4)$$

$$= \frac{5}{30} + \frac{7}{30} = \frac{12}{30}$$

(iv) Soln $F(2) = ?$

$F(4) = ?$

Then,

$X = x$	$P(X=x)$	$F(x)$
0	1/30	1/30
1	3/30	4/30
2	2/30	6/30
3	5/30	11/30
4	7/30	18/30
5	4/30	22/30
6	8/30	$30/30 = 1$

$\rightarrow F(0) \rightarrow P(X=0)$

$\rightarrow F(1) \rightarrow P(X=0) + P(X=1)$

$\rightarrow F(2) \rightarrow P(X=0) + P(X=1) + P($

$\rightarrow F(3)$ and so on for all $F(x)$

$\rightarrow F(4)$

$\rightarrow F(5)$

$\rightarrow F(6)$

Example 2: If a random variable X takes the values 1, 2, 3 and 4. Such that $P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$

Then, find the probability distribution of X.

Soln

given,

$X = 1, 2, 3, 4$

$P(X=1) = 3P(X=2)$

$P(X=1) = P(X=3)$

$P(X=1) = 5P(X=4)$

Let, $P(X=1) = P(X=3) = K$

Then, $P(X=2) = K/3$

$P(X=4) = K/5$

So,

$K + K/3 + K + K/5 = 1$ (since, total probability is 1)

$\Rightarrow 2K + \frac{5K+3K}{15} = 1$

$\Rightarrow 30K + 8K = 15 \Rightarrow 38K = 15 \therefore K = 15/38$

now,

$X = x$	$P(X=x)$ → probability distn. of X
1	15/38
2	5/38
3	15/38
4	3/38

ans

- Probability Distribution of a Continuous R.V.

Probability Function →

Let X be a continuous random variable with probability function $f(x)$; $\forall x$

Then, the probability function $f(x)$ is called Probability Density Function (pdf) which satisfies the following conditions:

(i) $f(x) \geq 0$; $\forall x$

(ii) $\int_x f(x)\,dx = 1$ (Total prob. is 1)

Example 1: Verify whether the given probability function is pdf or not.

(for being pdf, we have to show that the prob of given function is equals to 1)

$$f(x) = \frac{3}{4}x(2-x) \; ; \quad 0 < x < 2$$

$$= 0 \; ; \quad \text{otherwise}$$

Proof:

It is obvious that $f(x) \geq 0$ for all $0 < x < 2$

then, for pdf, total prob. = 1

therefore,

$$\int_x f(x) \cdot dx = \int_0^2 \left[\frac{3}{4} x (2-x)\right] \cdot dx$$

$$= \frac{3}{4}\left[\left(\int_0^2 2x - \int_0^2 x^2\right) \cdot dx\right]$$

$$= \frac{3}{4}\left\{2\left[\frac{x^2}{2}\right]_0^2 - \left[\frac{x^3}{3}\right]_0^2\right\}.$$

$$= \frac{3}{4}\left[2 \times \frac{4}{2} - \frac{8}{3}\right]$$

$$= \frac{3}{4}\left[4 - 8/3\right] = \frac{3}{4}\left[\frac{12-8}{3}\right]$$

$$= \frac{\cancel{3}}{\cancel{4}} \times \frac{\cancel{4}}{\cancel{3}} = 1$$

Here, $\int_0^2 \left\{\frac{3}{4} x (2-x)\right\} \cdot dx = 1$ which shows

that the given probability function is pdf.

Example 2:

Given pdf is:

$$f(x) = Kx^3 \; ; \; 0 < x < 1$$
$$= 0 \; ; \text{ otherwise}$$

(i) Find K

(ii) Find $P(X < 0.4)$

(iii) Find $P(0.2 < X < 0.6)$

Soln given,

(i)
$$f(x) = Kx^3 \; ; \; 0 < x < 1$$
$$= 0 \; ; \text{ otherwise}$$

Since the given function is pdf.

So,
$$\int_0^1 Kx^3 = 1$$

or,
$$K\left[\frac{x^4}{4}\right]_0^1 = 1$$

or,
$$K \times \frac{1}{4} = 1$$

$$\therefore K = 4$$

So, the given pdf becomes,

$$f(x) = 4x^3 \; ; \; 0 < x < 1$$
$$= 0 \; ; \text{ otherwise}$$

) soln $P(X < 0.4) = ?$

we have,

$$\int_0^{0.4} (4x^3) \cdot dx \Rightarrow 4\left[\frac{x^4}{4}\right]_0^{0.4}$$

$$\Rightarrow 4 \cdot \frac{(0.4)^4}{4} \Rightarrow 0.0256 \text{ ans}$$

iii) soln

$P(0.2 < X < 0.6) = ?$

we have,

$$\int_{0.2}^{0.6} (4x^3) \cdot dx \Rightarrow 4\left[\frac{x^4}{4}\right]_{0.2}^{0.6}$$

$$\Rightarrow 4\left\{\frac{(0.6)^4}{4} - \frac{(0.2)^4}{4}\right\}$$

$$\Rightarrow 4(0.032)$$

$$\Rightarrow 0.128 \text{ ans}$$

2079/08/11

Sunday

Assignment : 6

Let X be a random variable with following pdf.

$$f(x) = Kx(1-x); \quad 0<x<1$$
$$= 0; \quad \text{otherwise}$$

(i) Find K

(ii) Find $P(X<0.4)$ and $P(0.3 < X < 0.5)$

Soln given,

$$f(x) = Kx(1-x); \quad 0<x<1$$
$$= 0; \quad \text{otherwise}$$

Since the given function is pdf,

so, $\int_0^1 f(x).dx = 1$ (∵ being pdf by the question)

or, $\cancel{\int_0^1 f(x).dx = 1}$

or $\int_0^1 \{Kx(1-x)\}.dx = 1$

or, $K \int_0^1 (x-x^2).dx = 1$

$$\text{or, } K\left[\int_0^1 (x).dx - \int_0^1 (x^2).dx\right] = 1$$

$$\text{or, } K\left\{\left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1\right\} = 1$$

$$\text{or, } K\left[\left(\tfrac{1}{2}-0\right) - \left(\tfrac{1}{3}-0\right)\right] = 1$$

$$\text{or, } K\left[\frac{1}{2} - \frac{1}{3}\right] = 1$$

$$\text{or, } K\left[\frac{3-2}{6}\right] = 1$$

$$\therefore K = 6$$

Hence, the value of K is 6.

again $P(X<0.4) = ?$

we have,

the range of X is from 0 to 0.4.

so, $\int_0^{0.4} f(x).dx \Rightarrow \int_0^{0.4} \{Kx(1-x)\}.dx$

$$= \int_0^{0.4} \{6x(1-x)\}.dx \quad (\because K=6)$$

$$= 6\int_0^{0.4} (x-x^2).dx$$

$$= 6\left[\int_0^{0.4} (x).dx - \int_0^{0.4} (x^2).dx\right]$$

$$= 6\left\{\left[\frac{x^2}{2}\right]_0^{0.4} - \left[\frac{x^3}{3}\right]_0^{0.4}\right\}$$

$$= 6\left[(0.08-0)-(0.022-0)\right]$$

$$= 0.348$$

$$\therefore P(X<0.4) = 0.348$$

<u>again</u> $P(0.3<X<0.5) = ?$

we have,
the range of X is from 0.3 to 0.5

so,
$$\int_{0.3}^{0.5} f(x).dx$$

$$= \int_{0.3}^{0.5} \{6x(1-x)\}.dx \quad (\because K=6)$$

$$= 6\left[\int_{0.3}^{0.5} (x).dx - \int_{0.3}^{0.5} (x^2).dx\right]$$

$$= 6\left\{\left[\frac{x^2}{2}\right]_{0.3}^{0.5} - \left[\frac{x^3}{3}\right]_{0.3}^{0.5}\right\}$$

$$= 6\left[(0.125-0.045)-(0.042-0.009)\right]$$

$$= 6(0.08-0.033) = 0.282$$

Hence, $P(0.3<X<0.5) = 0.282$ <u>ans</u>

Distribution Function of Continuous random variable

Let X be a continuous random variable with pdf $f(x)$; $\forall x$. Then, the probability distribution function of X is denoted by $F(x)$ and is defined as:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x) \cdot dx$$

It is also called Cumulative Distribution Function.

Properties of Distributive Function

(i) $0 \leq F(x) \leq 1$; $\forall x$

(ii) $F'(x) = f(x)$; where $f(x)$ is pdf

(iii) $F(-\infty) = 0$

(iv) $F(+\infty) = 1$

(v) $P(a \leq X \leq b) = P(a < X < b) = F(b) - F(a)$

(vi) $F(x) > F(y)$; if $x > y$

Example 1:

Given pdf is

$$f(x) = Ae^{-x/5} \; ; x \geq 0$$
$$= 0 \; ; \text{otherwise}$$

(i) Find A

(ii) Find $P(X \geq 5)$ and $P(X < 10)$

(iii) Find the distribution function of X

Soln (i) Since the given function is pdf

$$\therefore \int_x f(x).dx = 1$$

or, $$\int_0^\infty f(x).dx = 1$$

or, $$\int_0^\infty Ae^{-x/5}.dx = 1$$

or, $$A\left[\frac{e^{-x/5}}{-1/5}\right]_0^\infty = 1$$

or, $$A\left[-5e^{-x/5}\right]_0^\infty = 1$$

or $$A[0 + 5 \times 1] = 1$$

or, $$5A = 1$$

$$\therefore A = 1/5$$

therefore, the given pdf becomes,

$$f(x) = \frac{1}{5} e^{-x/5} \; ; \; x \geq 0$$

(ii) Soln

$P(X \geq 5) = ?$ and $P(X < 10) = ?$

then,

$$P(X \geq 5) = \int_5^\infty f(x).dx \Rightarrow \int_5^\infty \frac{1}{5} \cdot e^{-x/5} . dx$$

$$\Rightarrow \frac{1}{\cancel{5}} \left[\frac{e^{-x/5}}{-1/\cancel{5}} \right]_5^\infty \Rightarrow \left[-e^{-x/5} \right]_5^\infty$$

$$\Rightarrow e^{-1} \Rightarrow \frac{1}{e} \text{ ans}$$

Mathematically,
$e = 2.7183$

similarly,

$$P(X < 10) = \int_0^{10} f(x).dx \Rightarrow \int_0^{10} \frac{1}{5} e^{-x/5} . dx$$

$$\Rightarrow \frac{1}{5} \cdot \left[\frac{e^{-x/5}}{-1/5} \right]_0^{10} \Rightarrow \left[-e^{-x/5} \right]_0^{10}$$

$$\Rightarrow -e^{-2} + 1 \Rightarrow 1 - e^{-2} \text{ ans}$$

(iii) Soln

Distribution function of X :-

$$F(x) = P(X \le x) \Rightarrow \int_0^x f(x).dx$$

$$\Rightarrow \int_0^x \frac{1}{5} \cdot e^{-x/5} \cdot dx \Rightarrow \frac{1}{5}\left[\frac{e^{-x/5}}{-1/5}\right]_0^x$$

$$\Rightarrow \left[-e^{-x/5}\right]_0^x \Rightarrow -e^{-x/5} + e^0$$

$$\Rightarrow 1 - e^{-x/5}$$ ans

Bivariate Distribution (= Joint Prob. distn of two random variables)

→ The probability distribution of two random variables (either both discrete or both continuous random variables) is called joint prob. distribution or Bivariate distribution.

→ Examples:

(i) Distribution of height and weight of a group

(ii) Height of fathers and sons

Discrete joint Probability distribution

Let X and Y are two discrete random variables
th joint prob. function $P(X=x, Y=y)$ or
$x,y)$; $\forall x$ and y. Then the joint prob.
nction $P(X=x, Y=y)$ is called joint probabi-
y mass function if it satisfies the
ollowing conditions:

) $P(X=x, Y=y) \geq 0$; $\forall x$ and y

) $\sum_x \sum_y P(X=x, Y=y) = 1$ (Total prob. is 1)

Marginal Probability Distribution

Let X and Y are the two discrete random
riables with joint pmf $P(X=x, Y=y)$ or
(x,y); $\forall x$ and y.

Then the marginal distribution of X is
enoted by $P(X=x)$ or $P(x)$ and is defined
as :

$$P(X=x) = \sum_y P(X=x, Y=y)$$

Similarly,

the marginal distribution of Y is denoted by $P(Y=y)$ or $P(y)$ and is defined as:

$$P(Y=y) = \sum_{x} p(X=x, Y=y)$$

- <u>Conditional Probability Distribution</u>

→ Let X and Y are two discrete random variables with joint pmf $P(X=x, Y=y)$; $\forall x$ and y

Then the conditional probability distribution of X given that $Y=y$ is denoted by $P(X=x \mid Y=y)$ or $P(X \mid Y)$ and is defined as:

$$P(X=x \mid Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)}$$ provided that $P(Y=y) \neq 0$

Similarly,

the conditional probability distribution of Y given that $X=x$ is denoted by $P(Y=y \mid X=x)$ or $P(Y \mid X)$ and is defined as:

$$P(Y=y \mid X=x) = \frac{P(X=x, Y=y)}{P(X=x)}$$ provided that $P(X=x) \neq 0$

- Condition for Independency

→ Two discrete random variables X and Y are said to be independent if;

$$P(X=x, Y=y) = P(X=x) \cdot P(Y=y)$$

otherwise, it is not independent

- Joint Distribution Function of Discrete Random Variable

→ Let X and Y are two discrete random variables with joint pmf $P(X=x, Y=y)$; $\forall x$ and y.

Then the joint distribution function of X and Y is denoted by $F(x,y)$ and is given by,

$$F(x,y) = P(X \leq x, Y \leq y) = \sum_x \sum_y P(X=x, Y=y)$$

Example 1: Let X and Y are two random variables with joint probability mass function.

Y \ X	2	3
1	0.1	0.15
2	0.2	0.3
3	0.1	0.15

(i) Find the marginal prob. distribution of X and Y.

(ii) Find the conditional probability of X given $Y=2$.

(iii) Find the cond. prob. of Y given $X=3$.

(iv) Find $F(2,2)$ and $F(3,2)$

(v) Are X and Y independent?

Sol^n

(i) Marginal distribution of X:

$$P(X=x) = \sum_{y=1}^{3} P(X=x, Y=y)$$

For $x=2$

$$P(X=2) = \sum_{y=1}^{3} P(X=2, Y=y)$$

$$= P(X=2, Y=1) + P(X=2, Y=2) + P(X=2, Y=3)$$

$$= 0.1 + 0.2 + 0.1 = 0.4$$

For $x=3$

$$P(X=3) = \sum_{y=1}^{3} P(X=3, Y=y)$$

$$= P(X=3, Y=1) + P(X=3, Y=2) + P(X=3, Y=3)$$

$$= 0.15 + 0.3 + 0.15$$

$$= 0.6$$

(ii) Marginal distribution of Y:

$$P(Y=y) = \sum_{x=2}^{3} P(X=x, Y=y)$$

For $y=1$

$$P(Y=1) = \sum_{x=2}^{3} P(X=x, Y=1)$$

$$= P(X=2, Y=1) + P(X=3, Y=1)$$

$$= 0.1 + 0.15 = 0.25$$

For $Y=2$

$$P(Y=2) = \sum_{X=2}^{3} P(X=x, Y=2)$$

$$= P(X=2, Y=2) + P(X=3, Y=2)$$

$$= 0.2 + 0.3 = 0.5$$

For $Y=3$

$$P(Y=3) = \sum_{X=2}^{3} P(X=x, Y=3)$$

$$= P(X=2, Y=3) + P(X=3, Y=3)$$

$$= 0.1 + 0.15 = 0.25$$

Hence, the marginal distribution of X is given by,

$X=x$	2	3
$P(X=x)$	0.4	0.6

Likewise, the marginal distribution of Y is given by,

$Y=y$	1	2	3
$P(Y=y)$	0.25	0.5	0.25

ans

wednesday

(ii) Soln

the conditional probability of X given Y = 2 is given by,

$$P(X=x \mid Y=2) = \frac{P(X=x, Y=2)}{P(Y=2)}$$

for $x = 2$

$$P(X=2 \mid Y=2) = \frac{P(X=2, Y=2)}{P(Y=2)}$$

$$= \frac{0.2}{0.5} = 0.4$$

Similarly,

for $x = 3$

$$P(X=3 \mid Y=2) = \frac{P(X=3, Y=2)}{P(Y=2)}$$

$$= \frac{0.3}{0.5} = 0.6$$ ans

(iii) soln.

the cond. prob. of Y given X = 3 is given by,

$$P(Y=y \mid X=3) = \frac{P(X=3, Y=y)}{P(X=3)}$$

For $Y=1$

$$P(Y=1 \mid X=3) = \frac{P(X=3, Y=1)}{P(X=3)}$$

$$= \frac{0.15}{0.6} = 0.25$$

For $Y=2$

$$P(Y=2 \mid X=3) = \frac{P(X=3, Y=2)}{P(X=3)}$$

$$= \frac{0.3}{0.6} = 0.5$$

For $Y=3$

$$P(Y=3 \mid X=3) = \frac{P(X=3, Y=3)}{P(X=3)}$$

$$= \frac{0.15}{0.6} = 0.25$$ ans

(v) Soln

here, $P(X=2, Y=3) = 0.1$

any joint prob. value can be considered from the table

Also,

$P(X=2) = 0.4$

and, $P(Y=3) = 0.25$

Therefore,

$P(X=2) \cdot P(Y=3) = 0.4 \times 0.25 = 0.1$

here, $P(X=2, Y=3) = P(X=2) \cdot P(Y=3)$

Hence, X and Y are independent. ans

(iv) Soln $F(2,2) = ?$

we have,

$F(2,2) = P(X \le 2, Y \le 2)$ (from the defn of joint distn function of two discrete r.v.)

$$= \sum_{x=2}^{2} \sum_{y=1}^{2} P(X=2, Y \le 2)$$

$$= \sum_{y=1}^{2} P(X=2, Y=y)$$

$$= P(X=2, Y=1) + P(X=2, Y=2)$$

$$= 0.1 + 0.2 = 0.3$$

Similarly, $F(3,2) = ?$

we have,

$F(3,2) = P(X \le 3, Y \le 2)$

$$= \sum_{x=2}^{3} \sum_{y=1}^{2} P(X \le 3, Y \le 2)$$

$$= P(X=2, Y=1) + P(X=2, Y=2) + P(X=3, Y=1) + P(X=3, Y=2)$$

$$= 0.1 + 0.2 + 0.15 + 0.3 = 0.75$$ ans

ssignment 7:

X \ Y	1	2
0	0	1/8
1	3/8	0
2	3/8	0
3	0	1/8

(i) Find the marginal distribution of X and Y.

(ii) Are X and Y independent?

oln

(i) For marginal distn. of X:

$$P(X=x) = \sum_{y=1}^{3} P(X=x, Y=y)$$

For X=0

$$P(X=0) = \sum_{y=1}^{2} P(X=0, Y=y)$$

$$= P(X=0, Y=1) + P(X=0, Y=2)$$

$$= 0 + \frac{1}{8} = 0.125$$

For X=1

$$P(X=1) = \sum_{y=1}^{2} P(X=1, Y=y)$$

$$= P(X=1, Y=1) + P(X=1, Y=2)$$

$$= 3/8 + 0 = 0.375$$

For X=2

$$P(X=2) = \sum_{y=1}^{2} P(X=2, Y=y)$$

$$= P(X=2, Y=1) + P(X=2, Y=2)$$

$$= \frac{3}{8} + 0 = 0.375$$

For $X=3$

$$P(X=3) = \sum_{y=1}^{2} P(X=3, Y=y)$$

$$= P(X=3, Y=1) + P(X=3, Y=2)$$

$$= 0 + \frac{1}{8} = 0.125$$

Hence, the marginal distribution of X is given by,

$X=x$	0	1	2	3
$P(X=x)$	0.125	0.375	0.375	0.125

For marginal distn of Y

$$P(Y=y) = \sum_{x=0}^{3} P(X=x, Y=y)$$

For $Y=1$

$$P(Y=1) = \sum_{x=0}^{3} P(X=x, Y=1)$$

$$= P(X=0, Y=1) + P(X=1, Y=1) + P(X=2, Y=1) + P(X=3, Y=1)$$

$$= 0 + \frac{3}{8} + \frac{3}{8} + 0$$

$$= 0.75$$

For $Y=2$

$$P(Y=2) = \sum_{x=0}^{3} P(X=x, Y=2)$$

$$= P(X=0, Y=2) + P(X=1, Y=2) + P(X=2, Y=2) + P(X=3, Y=2)$$

$$= \frac{1}{8} + 0 + 0 + \frac{1}{8} = 0.25$$

Hence, the marginal distn of Y is given by,

$Y=y$	1	2
$P(Y=y)$	0.75	0.25

ii) Soln here, $P(X=3, Y=2) = \frac{1}{8} = 0.125$.

Also, $P(X=3) = 0.125$

and, $P(Y=2) = 0.25$

Then, $P(X=3) \cdot P(Y=2) = 0.125 \times 0.25$

$= 0.03125$

here,

$$P(X=3, Y=2) \neq P(X=3) \cdot P(Y=2)$$

hence, X and Y are not independent.

Joint Probability Distribution of Continuous Random Variables

→ Let X and Y are two continuous random variables with joint prob. distⁿ function $f(x,y)$; $\forall x$ and y. Then, the joint probability function $f(x,y)$ is called joint probability density function if it satisfies the following conditions:

(i) $f(x,y) \geq 0, \ \forall x$ and y

(ii) $\int_x \int_y f(x,y)\,dx\,dy = 1$ (Total probability = 1)

For convenience:

$$\int_x \left[\int_y f(x,y)\,dy \right] dx = 1$$

- Marginal Distribution

→ Let X and Y are two continuous random variables with joint probability random function $f(x,y)$; $\forall x$ and y.

Then the marginal distⁿ of X is denoted by $f(x)$ and is defined as:

$$f(x) = \int_y f(x,y)\,dy$$

Similarly,

The marginal distribution of Y is denoted by $f(y)$ and is defined as:

$$f(y) = \int_x f(x,y) \cdot dx$$

- <u>Conditional Probability Distribution</u>

→ Let X and Y are two continuous random variables with joint prob. density function $f(x,y)\ \forall\ x$ and y. Also, the marginal distribution of X and Y are respectively $f(x)$ and $f(y)$.

Then,

the conditional density function of X given Y is denoted by $f(X|Y)$ and is defined as:

$$f(X|Y) = \frac{f(x,y)}{f(y)} \text{ ; provided that } f(y) \neq 0$$

Similarly,

the conditional distribution of Y given X is denoted by $f(Y|X)$ and is defined as:

$$\cancel{f(X|Y)}\ f(Y|X) = \frac{f(x,y)}{f(X)} \text{ ; provided that } f(X) \neq 0$$

- Condition for independency

→ Two continuous random variables X and Y are said to be independent if:

$$f(x,y) = f(x) \cdot f(y)$$

Otherwise, they are not independent.

- Joint distribution function of Continuous random variable

→ Let X and Y are two continuous random variables with joint pdf $f(x,y)$, $\forall$ x and y.

Then,

The joint distribution function of X and Y is denoted by $F(x,y)$ and is given by,

$$F(x,y) = P(X \leq x, Y \leq y)$$

$$= \int_{-\infty}^{x} \int_{-\infty}^{y} f(x,y)\, dx \cdot dy$$

Marginal distribution function

→ Let X and Y are two continuous random variables with joint pdf $f(x,y)$, $\forall x$ and y.

Also, the joint prob. distn. function is given by $F(x,y)$.

Then the marginal distn. function of X is denoted by $F(x)$ and is defined as:

$$F(x) = P(X \leq x, Y \leq \infty)$$

$$= \int_{-\infty}^{x} \int_{-\infty}^{\infty} f(x,y)\, dx\, dy$$

Similarly, the marginal distn. function of Y is denoted by $F(y)$ and is defined as:

$$F(y) = P(X \leq \infty, Y \leq y)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{b} f(x,y).\, dx\, dy$$

Friday

Example 1:

Suppose that the joint probability function of random variables X and Y is given by

$f(x,y) = 4xy \; ; \; 0<x<1$ and $0<y<1$

(i) Verify that the given function is pdf

(ii) Find $P(X<0.5, Y<0.8)$

(iii) Find the marginal distn. of X and Y

(iv) Find the conditional density function of X given Y and Y given X.

(v) Are X and Y independent?

Soln

(i) $\int_0^1 \int_0^1 4xy \, dx.dy$

$= \int_0^1 4 \int_0^1 xy \, dy$

$= \int_0^1 4x \left\{ \int_0^1 y.dy \right\}.dx$

$= 4 \int_0^1 x \left\{ \left[\frac{y^2}{2} \right]_0^1 \right\}.dx$

$$= 4 \int_0^1 \left\{ x \left(\tfrac{1}{2} \right) \right\} . dx$$

$$\cancel{= 4 \times \tfrac{1}{2} \left[\frac{x^2}{2} \right]_0^1 \Rightarrow 2 \times \tfrac{1}{2}}$$

$$= \frac{4}{2} \cdot \left[\frac{x^2}{2} \right]_0^1 \Rightarrow \frac{4}{2} \cdot \frac{1}{2} \Rightarrow \frac{4}{4} = 1$$

hence pdf

(ii) Soln

$$P(X < 0.5,\ Y < 0.8) = ?$$

we have,

$$\int_0^{0.5} \int_0^{0.8} (4xy) . dy\, dx$$

$$\Rightarrow \int_0^{0.5} 4x \left\{ \left[\frac{y^2}{2} \right]_0^{0.8} \right\} . dx$$

$$\Rightarrow 4 \int_0^{0.5} \left\{ x \left(\frac{8}{25} \right) \right\} . dx \Rightarrow 4 \times \frac{8}{25} \int_0^{0.5} (x) . dx$$

$$\Rightarrow \frac{32}{25} \times \left[\frac{x^2}{2} \right]_0^{0.5} \Rightarrow \frac{32}{25} \times \frac{1}{8} \Rightarrow 0.16$$ ans

(iii) Soln

Marginal distn. of X is;

$$f(x) = \int_0^1 4xy \cdot dy$$

$$= 4\int_0^1 xy \cdot dy$$

$$= 4x \left[\frac{y^2}{2}\right]_0^1$$

$$= \frac{4x}{2} = 2x$$

Similarly,

Marginal distn of Y is,

$$f(y) = \int_0^1 4xy \cdot dx$$

$$= 4y \int_0^1 x \cdot dx$$

$$= 4y \left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{4y}{2}$$

$$= 2y$$

Ans

(iv) Soln

$$P(X|Y) = \frac{f(x,y)}{f(Y)} = \frac{4xy}{2y} = 2x$$

$$P(Y|X) = \frac{f(x,y)}{f(X)} = \frac{4xy}{2x} = 2y$$ ans

(v) Soln here, $f(x,y) = 4xy$

and, $f(x) = 2y$

$f(y) = 2x$

so, $f(x) \cdot f(y) = 2y \cdot 2x = 4xy$

Since, $f(x,y) = f(x) \cdot f(y)$. so, they are independent

Example 2: Suppose that the joint prob. density function of random variables X and Y is given by:

$$f(x,y) = e^{-(x+y)}; \quad 0 < x < \infty \text{ and } 0 < y < \infty$$

(i) Find the Marginal distn of X and Y.

(ii) Find the conditional density function of X given Y and Y given X.

(iii) Are X and Y independent?

Soln given,

(i) $f(x,y) = e^{-(x+y)}$; $0<x<\infty$ and $0<y<\infty$

For marginal distn of X

$$f(X) = \int_0^\infty e^{-(x+y)} \, dy$$

$$= \int_0^\infty (e^{-x} \cdot e^{-y}) \, dy$$

$$= e^{-x} \left[\frac{e^{-y}}{-1} \right]_0^\infty$$

$$= e^{-x} [0+1] = e^{-x}$$

For marginal distn of Y

$$f(Y) = \int_0^\infty e^{-(x+y)} \, dx$$

$$= \int_0^\infty (e^{-x} \cdot e^{-y}) \, dx$$

$$= e^{-y} \left[\frac{e^{-x}}{-1} \right]_0^\infty$$

$$= e^{-y}(0+1) = e^{-y}$$

ans

(ii) Soln

$$p(x|y) = \frac{f(x,y)}{f(y)} = \frac{e^{-(x+y)}}{e^{-y}}$$

$$= \frac{e^{-x} \cdot \cancel{e^{-y}}}{\cancel{e^{-y}}} = e^{-x}$$

$$p(y|x) = \frac{f(x,y)}{f(x)} = \frac{\cancel{e^{-x}} \cdot e^{-y}}{\cancel{e^{-x}}} = e^{-y}$$

(iii) Soln here, $f(x,y) = e^{-(x+y)}$

and, $f(x) = e^{-x}$

$f(y) = e^{-y}$

so, $f(x) \cdot f(y) = e^{-x} \cdot e^{-y}$

$= e^{-x-y}$

$= e^{-(x+y)}$

Since, $f(x,y) = f(x) \cdot f(y) = e^{-(x+y)}$,

So, they are independent.

Example 3:

Suppose that the joint probability density f of random variables X and Y is given by,

$f(x,y) = K(4-x-y); \quad 0<x<2$ and $0<y<$

(i) Find the marginal distn of X and Y.

(ii) Find the conditional density function of X given Y and Y given X.

(iii) Are X and Y independent?

Soln

Since the given function is pdf.

So,

$$\int_0^2 \int_0^2 \{K(4-x-y)\}\, dy.dx = 1$$

or,

$$\int_0^2 K\left[\int_0^2 (4)\, dy - \int_0^2 (x).dy - \int_0^2 (y).dy\right] dx =$$

or,

$$\int_0^2 K\left\{4[y]_0^2 - x[y]_0^2 - \left[\frac{y^2}{2}\right]_0^2\right\} dx =$$

or,

$$\int_0^2 \left\{K\left[8 - 2x - 2\right]\right\} dx = 1$$

or, $K \int_0^2 (6-2x) \cdot dx = 1$

or, $2K \int_0^2 (3-x) \cdot dx = 1$

or, $2K \left[\int_0^2 (3) \cdot dx - \int_0^2 (x) \cdot dx \right] = 1$

or, $2K \left\{ 3 [x]_0^2 - \left[\frac{x^2}{2} \right]_0^2 \right\} = 1$

or, $2K (6 - 2) = 1$

~~or, $4K = 1$~~

~~$\therefore K = 1/4$~~

or, $8K = 1$

$\therefore K = 1/8$

Then, the pdf becomes,

$f(x,y) = \frac{1}{8} (4-x-y)$; $0 < x < 2$ and $0 < y < 2$

(1) Soln

Marginal distn of X :

$$f(x) = \int_0^2 \left[\frac{1}{8}(4-x-y) \right] dy$$

$$= \frac{1}{8}\left[\int_0^2 (4)\,dy - \int_0^2 (x)\,dy - \int_0^2 (y)\,dy \right]$$

$$= \frac{1}{8}\left\{ 4[y]_0^2 - x[y]_0^2 - \left[\frac{y^2}{2}\right]_0^2 \right\}$$

$$= \frac{1}{8}(8 - 2x - 2)$$

$$= \frac{1}{8}(6-2x)$$

$$= \frac{2(3-x)}{8}$$

$$= \frac{3-x}{4}$$

Marginal distn of Y :

$$f(y) = \int_0^2 \left[\frac{1}{8}(4-x-y) \right] . dx$$

$$= \frac{1}{8}\left[\int_0^2 (4)\, dx - \int_0^2 (x) . dx - \int_0^2 (y)\, dx \right]$$

$$= \frac{1}{8}\left\{ 4[x]_0^2 - \left[\frac{x^2}{2}\right]_0^2 - y[x]_0^2 \right\}$$

$$= \frac{1}{8}(8 - 2 - 2y)$$

$$= \frac{1}{8}(6 - 2y)$$

$$= \frac{2(3-y)}{8}$$

$$= \frac{3-y}{4}$$

ans

(ii) Soln

$$\cancel{f(x|y)} \quad f(X|Y) = \frac{f(x,y)}{f(Y)}$$

$$= \frac{\frac{1}{8}(4-x-y)}{\frac{3-y}{4}}$$

$$= \frac{4-x-y}{8} \times \frac{4}{3-y} \Rightarrow \frac{4-x-y}{6-2y}.$$

Similarly,

$$f(Y|X) = \frac{f(x,y)}{f(X)}$$

$$= \frac{1/8\,(4-x-y)}{\frac{3-x}{4}}$$

$$= \frac{4-x-y}{8} \times \frac{4}{3-x} \Rightarrow \frac{4-x-y}{6-2x}$$ ans

(iii) Soln here, $f(x,y) = 1/8\,(4-x-y)$

and, $f(X) = \frac{3-x}{4}$, $f(Y) = \frac{3-y}{4}$

So, $f(X)\cdot f(Y) = \frac{(3-x)(3-y)}{16} = \frac{9-3y-3}{16}$

here, $f(x,y) \neq f(x)\cdot f(Y)$. So, they are not indepe

Example 4: Suppose that the joint prob. density function of random variable X and Y is given by,

$$f(x,y) = Axy \; ; \quad 0<x<1 \text{ and } 0<y<x$$

(i) Find A

(ii) Find the marginal distn. of X and Y

(iii) Find the conditional density function of X given Y and Y given X.

(iv) Are X and Y independent ?

Soln

(i) $$\int_0^1 \int_0^x (Axy)\, dy . dx = 1$$ (since the given function is pdf)

or, $$\int_0^1 Ax\left\{\left[\frac{y^2}{2}\right]_0^x\right\} . dx = 1$$

or, $$\int_0^1 \left(Ax \times \frac{x^2}{2}\right) . dx = 1$$

or, $$\frac{A}{2}\int_0^1 x^3 . dx = 1$$

or, $$\frac{A}{2}\left[\frac{x^4}{4}\right]_0^1 = 1$$

or, $$\frac{A}{2} \times \frac{1}{4} = 1$$

or, $$\frac{A}{8} = 1$$

$\therefore A = 8$ ans

(iii) Marginal distn of X

~~$f(x) = f(x, y)$~~

$$f(x) = \int_0^x (8xy) \cdot dy \quad (\because A = 8)$$

$$= 8x \left[\frac{y^2}{2}\right]_0^x \Rightarrow 8x \times \frac{x^2}{2} \Rightarrow 4x^3$$

similarly,

Marginal distn of Y

~~$f(y) =$~~

here,

$0 < x < 1$

and, $0 < y < x$

so, with these two ranges, the new range becomes $0 < y < x < 1$

This means the y has range from 0 to x and x has range from y to 1.

so,

$$f(y) = \int_y^1 (8xy) \cdot dx$$

$$= 8y\left[\frac{x^2}{2}\right]_y^1 \Rightarrow 8y\left(\frac{1}{2}-\frac{y^2}{2}\right)$$

$$= \frac{8y}{2} - \frac{8y^3}{2} \Rightarrow 4y - 4y^3 \Rightarrow 4y(1-y^2)$$ ans

iii) soln

$$f(X/Y) = \frac{f(x,y)}{f(Y)} \Rightarrow \frac{2\cancel{8}x\cancel{y}}{\cancel{4y}(1-y^2)} \Rightarrow \frac{2x}{1-y^2}$$

similarly, $f(Y/X) = \frac{f(x,y)}{f(X)} = \frac{2\cancel{8}\cancel{x}y}{\cancel{4}x^{\cancel{3}}} = \frac{2y}{x^2}$ ans

iv) soln

here, $f(x,y) = 8xy$

and, $f(x) = 4x^3$

$f(y) = 4y(1-y^2)$

so, $f(x)\cdot f(y) = 4x^3 \times 4y(1-y^2)$

$= 16x^3y(1-y^2)$

here, $f(x,y) \neq f(x)\cdot f(y)$

so, they are not independent ans

Assignment : 8

Suppose that the ~~given~~ joint probability density function of random variable X and Y is given by

~~$f(x,y) = kx(x-y)$~~ ;

$$f(x,y) = \frac{2}{3}(x+2y) \; ; \; 0 < x < 1 \text{ and } 0 < y < 1$$

(i) Find the marginal distn of X and Y.

(ii) Find the conditional distn of Y given X and X given Y.

(iii) Are X and Y independent ?

Soln

(i) Marginal distn of X :

$$f(x) = \int_0^1 \left[\frac{2}{3}(x+2y)\right] dy$$

$$= \frac{2}{3}\left\{\int_0^1 (x)\, dy + \int_0^1 (2y)\, dy\right\}$$

$$= \frac{2}{3}\left\{x\,[y]_0^1 + 2\left[\frac{y^2}{2}\right]_0^1\right\}$$

$$= \frac{2}{3}\left[x + \frac{2}{2}\right] = \frac{2}{3}(1+x)$$

Marginal distn. of Y:

$$f(y) = \int_0^1 \left[\frac{2}{3}(x+2y)\right] dx$$

$$= \frac{2}{3}\left[\int_0^1 (x).dx + \int_0^1 (2y)\, dx\right]$$

$$= \frac{2}{3}\left\{\left[\frac{x^2}{2}\right]_0^1 + 2y\,[x]_0^1\right\}$$

$$= \frac{2}{3}\left(\frac{1}{2} + 2y\right) \Rightarrow \frac{\cancel{2}}{3} \times \frac{1+4y}{\cancel{2}}$$

$$\Rightarrow \frac{1}{3}(1+4y)$$

ans

1) Soln

$$f(X/Y) = \frac{f(x,y)}{f(y)} = \frac{\frac{2}{3}(x+2y)}{\frac{1}{3}(1+4y)} = \frac{2}{\cancel{3}}(x+2y) \times \frac{\cancel{3}}{(1+4y)}$$

$$= \frac{2x+4y}{1+4y}$$

Similarly,

$$f(Y/X) = \frac{f(x,y)}{f(X)} = \frac{\frac{2(x+2y)}{3}}{\frac{2(1+x)}{3}} \Rightarrow \frac{\cancel{2}(x+2y)}{\cancel{3}} \times \frac{\cancel{3}}{\cancel{2}(1+x)}$$

$$\Rightarrow \frac{x+2y}{1+x}$$

here, $f(x,y) = \frac{2}{3}(x+2y)$

Also, $f(x) = \frac{2}{3}(1+x)$

and, $f(y) = \frac{1}{3}(1+4y)$

So, $f(x) \cdot f(y) = \frac{2}{3}(1+x) \times \frac{1}{3}(1+4y)$

$$= \frac{2(1+x)(1+4y)}{9}$$

Here, $f(x,y) \neq f(x) \cdot f(y)$.
So, they are not independent.

Assignment : 9

Suppose that the joint probability density function of random variable X and Y is given by : $f(x,y) = Kx(x-y)$; $0 < x < 2$ and $-x < y < x$.

$= 0$; otherwise

(i) Find the value of K.

(ii) Find the marginal distn of X and Y

(iii) Find the conditional distn of Y given X and X given Y.

(iv) Are X and Y independent ?

Soln given,

$f(x,y) = Kx(x-y)$; $0 \le x < 2$ and $-x < y < x$

Since the function is pdf, so;

$$\int_0^2 \int_{-x}^{x} f(x,y).\,dy\,dx = 1$$

or, $$\int_0^2 \int_{-x}^{x} \{Kx(x-y)\}\,dy.dx = 1$$

or, $$\int_0^2 Kx \left[\int_{-x}^{x} (x).\,dy - \int_{-x}^{x} (y).\,dy \right].dx = 1$$

or, $$\int_0^2 Kx \left\{ x\,[y]_{-x}^{x} - \left[\frac{y^2}{2}\right]_{-x}^{x} \right\}.dx = 1$$

or, $$\int_0^2 Kx \left\{ x\,(x+x) - \left(\frac{x^2}{2} - \frac{x^2}{2}\right) \right\}.dx = 1$$

or, $$\int_0^2 (Kx \times 2x^2).\,dx = 1$$

or, $$2K \int_0^2 (x^3).\,dx = 1$$

or, $2K\left[\frac{x^4}{4}\right]_0^2 = 1$

or, $2K \times \frac{16}{4} = 1 \Rightarrow 8K = 1$

$\therefore K = \frac{1}{8}$

with this value of K, the pdf becomes

$f(x,y) = \frac{1}{8}x(x-y)$; $0<x<2$ and $-x<y<x$

$= 0$; otherwise

As (ii) Soln Marginal distn of X

$$f(X) = \int_{-x}^{x} \left\{\frac{1}{8}x(x-y)\right\}.dy$$

$$= \frac{1}{8}x\left[\int_{-x}^{x}(x).dy - \int_{-x}^{x}(y).dy\right]$$

$$= \frac{x}{8}\left\{x[y]_{-x}^{x} - \left[\frac{y^2}{2}\right]_{-x}^{x}\right\}$$

$$= \frac{x}{8}\left\{x(x+x) - \left(\frac{x^2}{2} - \frac{x^2}{2}\right)\right\}$$

$$= \frac{x}{8}(2x^2 - 0) = \frac{2x^3}{8} = \frac{x^3}{4}$$

Similarly,

Marginal distn. of Y

$$f(Y) = \int_y^2 \left\{\frac{1}{8}x(x-y)\right\}.dx$$

$$= \frac{1}{8}\int_y^2 (x^2 - xy).dx$$

$$= \frac{1}{8}\left[\int_y^2 (x^2).dx - \int_y^2 (xy).dx\right]$$

$$= \frac{1}{8}\left\{\left[\frac{x^3}{3}\right]_y^2 - y\left[\frac{x^2}{2}\right]_y^2\right\}$$

$$= \frac{1}{8}\left[\left(\frac{8}{3} - \frac{y^3}{3}\right) - y\left(\frac{4}{2} - \frac{y^2}{2}\right)\right]$$

$$= \frac{1}{8}\left[\frac{8-y^3}{3} - \frac{y(4-y^2)}{2}\right]$$

$$= \frac{1}{8}\left[\frac{2(8-y^3) - 3(4y-y^3)}{6}\right]$$

$$= \frac{1}{8}\left(\frac{16-2y^3-12y+3y^3}{6}\right)$$

$$= \frac{1}{8}\left(\frac{16-12y+y^3}{6}\right) \; ; \; -2<y<2$$

(ii) Sol

$$f(x/y) = \frac{f(x,y)}{f(y)}$$

$$= \frac{x(x-y)}{\cancel{8}} \times \frac{6\times\cancel{8}}{16-12y+y^3}$$

$$= \frac{6x(x-y)}{16-12y+y^3}$$

$$f(y/x) = \frac{f(x,y)}{f(x)} = \frac{\cancel{x}(x-y)}{\cancel{8}2} \times \frac{\cancel{4}}{x\cancel{8}}$$

$$\Rightarrow \frac{x-y}{2x^2}$$

(iv) Soln

here,

$$f(x,y) = \frac{x(x-y)}{8}$$

Also, $f(X) = x^3/4$

and, $f(Y) = \frac{1}{8}\left(\frac{16-12y+y^3}{6}\right)$

So, $f(X).f(Y) = \frac{x^3}{4} \times \frac{1}{8}\left(\frac{16-12y+y^3}{6}\right)$

$$= \frac{x^3(16-12y+y^3)}{192} \neq f(x,y)$$

Hence, they are not independent.

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Sunday

Mathematical Expectation

→ It is the mean value of a random variable.

→ Let X be a random variable, then the expected value or mean of random variable X is denoted by E(X), called expectation of X and is given by,

$$E(X) = \sum_{x} x \cdot P(X=x) \quad \text{(if X is discrete random variable)}$$

Also,

$$E(X) = \int_{x} x f(x)\, dx \quad \text{(if X is continuous random variable)}$$

Properties:

(i) $E(a) = a$; where a is constant

(ii) $E(aX) = aE(X)$; where 'a' is constant

(iii) $E(a+bX) = a + bE(X)$; where 'a' and 'b' are constants

(iv) $E(X \pm Y) = E(X) \pm E(Y)$

(v) $E(XY) = E(X) \cdot E(Y)$; if X and Y are independent

(vi) Let $g(X)$ be the function of X.

Then,

$$E[g(X)] = \sum g(x) \cdot P(X=x); \text{ if } X \text{ is discrete}$$

$$= \int_x g(x) \cdot f(x) \cdot dx; \text{ if } X \text{ is continuous}$$

① Let $g(X) = x^2$

Then, $E[g(X)] = E(X^2)$

$$= \sum_x x^2 \cdot P(X=x); \text{ if } X \text{ is discrete}$$

$$= \int_x x^2 f(x)\, dx; \text{ if } X \text{ is continuous}$$

② Let $g(X) = X + 1$

Then,

$$E(X+1) = \sum_x (x+1)\, P(X=x)$$

$$= \int_x (x+1)\, f(x) \cdot dx$$

③ Let $g(X) = (X+1)^2$

Then,

$$E(X+1)^2 = \sum_x (x+1)^2\, P(X=x)$$

Example 1 :

Let X be a random variable with following probability distribution.

$X=x$	-2	-1	0	1	2	3
$P(X=x)$	0.1	K	0.2	$2K$	0.3	K

Find $E(X)$, $E(X^2)$, $E(X^2+2)$, $E(2X+3)$

Soln Since the total prob. $= 1$

so,

$$\sum P(X=x) = 1$$

Then,

$$0.1 + K + 0.2 + 2K + 0.3 + K = 1$$

or, $0.6 + 4K = 1$

or, $K = \frac{0.4}{4} = 0.1$

$X=x$	$P(X=x)$	$x \cdot P(X=x)$
-2	0.1	-0.2
-1	0.1	-0.1
0	0.2	0
1	0.2	0.2
2	0.3	0.6
3	0.1	0.3

$$\sum x \cdot P(X=x) = 0.8$$

$$\therefore E(X) = \sum_x x \cdot P(X=x) = 0.8$$

Similarly,

$$E(X^2) = \sum_x x^2 \cdot P(X=x)$$

$X=x$	$P(X=x)$	$x \cdot P(X=x)$	$x^2 \cdot P(X=x)$
-2	0.1	-0.2	0.4
-1	0.1	-0.1	0.1
0	0.2	0	0
1	0.2	0.2	0.2
2	0.3	0.6	1.2
3	0.1	0.3	0.9

$$\sum x^2 \cdot P(X=x) = 2.8$$

$$\therefore E(X^2) = \sum_x x^2 \cdot P(X=x) = 2.8$$

again

$$E(X^2+2) = E(X^2) + 2 = 2.8 + 2 = 4.8$$

Also,

$$E(2X+3) = 3 + 2\,E(X)$$

$$= 3 + 2 \times 0.8$$

$$= 4.6$$ ans

Example 2:

Let X be a continuous random variable with probability density function

$$f(x) = 2x \; ; \quad 0 < x < 1$$

Then find $E(X)$ and $E(X^2)$

Soln

$$E(X) = \int_0^1 \{(2x).x\} dx$$

$$= \int_0^1 (2x^2) . dx \Rightarrow 2 \left[\frac{x^3}{3}\right]_0^1$$

$$= \frac{2}{3} \Rightarrow 0.67$$

Similarly,

$$E(X^2) = \int_0^1 (x^2 . 2x) . dx$$

$$= 2 \left[\frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{2} \Rightarrow 0.5$$ ans

Variance

- It measures the variability or variation
- Let X be a random variable with the expectation $E(X)$.
- Then the variance of X is denoted by $V(X)$ or $Var(X)$ and is given by;

$$Var(X) = E[X - E(X)]^2$$

On simplification we get,

$$Var(X) = E(X^2) - [E(X)]^2$$

Properties:

(i) $Var(a) = 0$; where 'a' is constant

(ii) $Var(aX) = a^2 Var(X)$

(iii) $Var(aX + b) = a^2 Var(X)$

(iv) $Var(X \pm Y) = Var(X) + Var(Y) \pm 2\,\underbrace{cov}_{\text{covariance}}(X, Y)$

But if X and Y are independent,

$$Var(X \pm Y) = Var(X) + Var(Y)$$

Since, $cov(X, Y) = 0$

Assignment 10 :

A coin is tossed until a head appears. What is the average number of tosses required?

Soln

Let X = no. of Tosses required to get first head.

Also,
Let H and T are the events that represents the occurance of Head and Tail respectively in each toss.

Then, the no. of tosses required are obtained in the following ways.

Outcomes	x	$P(X=x)$
H	1	$\frac{1}{2}$
TH	2	$\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$
TTH	3	$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{8}$
TTTH	4	$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{16}$
⋮	⋮	⋮

Then,

$$E(X) = \sum_{x=1}^{\infty} x \cdot P(X=x)$$

$$\therefore E(X) = 1\cdot\frac{1}{2} + 2\cdot\frac{1}{4} + 3\cdot\frac{1}{8} + 4\cdot\frac{1}{16} + 5\cdot\frac{1}{32} + \cdots \quad \text{——— (i)}$$

Similarly,

$$\frac{E(X)}{2} = 1\cdot\frac{1}{4} + 2\cdot\frac{1}{8} + 3\cdot\frac{1}{16} + 4\cdot\frac{1}{32} + \cdots \quad \text{——— (ii)}$$

subtracting eqn. (ii) from eqn (i),

$$E(X) - \frac{E(X)}{2} = 1\cdot\frac{1}{2} + 1\cdot\frac{1}{4} + 1\cdot\frac{1}{8} + \cdots$$

or, $$\frac{E(X)}{2} = 1\cdot\frac{1}{2} + 1\cdot\frac{1}{4} + 1\cdot\frac{1}{8} + \cdots$$

here, $a = 1/2$, $r = \frac{1/4}{1/2} \Rightarrow 1/2$

then, $$\frac{E(X)}{2} = \frac{a}{1-r} \Rightarrow \frac{1/2}{1-1/2} \Rightarrow 1$$

$$\therefore E(X) = 2$$

Hence, the required average number of tosses to obtain a head is 2. ans

Covariance

→ it measures the variation of two random variables simultaneously.

→ let X and Y are two random variables with expectation E(X) and E(Y) respectively. Then, the covariance of X and Y is denoted by Cov(X, Y) and is defined as:

$$\text{Cov}(X, Y) = E(X - E(X))(Y - E(Y))$$

On simplification we get,

$$\boxed{Cov(X,Y) = E(XY) - E(X).E(Y)}$$

here,

$$E(XY) = \sum_x \sum_y xy\, P(X=x, Y=y)$$; if X and Y are discrete r.v.

Also,

$$E(XY) = \int_x \int_y xy\, f(x,y)\, dx\, dy$$; if X and Y are continuous r.v.

Note:

If X and Y are independent then,

$$Cov(X,Y) = 0$$

Example 1: Let X and Y are two random variables with joint probability distribution.

Y \ X	2	3
1	0.1	0.15
2	0.2	0.3
3	0.1	0.15

(1) Find the marginal distribution of X and Y.

(2) Are X and Y independent?

(3) Find E(X), E(Y), V(X) and V(Y)

(4) Find cov(X,Y)

Soln

For Marginal distribution of X

$$P(X=x) = \sum_{y} P(X=x, Y=y)$$

i.e. $$P(X=x) = \sum_{1}^{3} P(X=x, Y=y)$$

For $X=2$

$$P(X=2) = P(X=2, Y=1) + P(X=2, Y=2) + P(X=2, Y=3)$$
$$= 0.1 + 0.2 + 0.1 \Rightarrow 0.4$$

For $X=3$

$$P(X=3) = P(X=3, Y=1) + P(X=3, Y=2) + P(X=3, Y=3)$$
$$= 0.15 + 0.3 + 0.15 \Rightarrow 0.6$$

For Marginal distribution of Y

$$P(Y=y) = \sum_{x} P(X=x, Y=y)$$

i.e. $$P(Y=y) = \sum_{2}^{3} P(X=x, Y=y)$$

For $Y=1$

$$P(Y=1) = P(X=2, Y=1) + P(X=3, Y=1)$$
$$= 0.1 + 0.15 \Rightarrow 0.25$$

For $Y=2$

$$P(Y=2) = P(X=2, Y=2) + P(X=3, Y=2)$$
$$= 0.2 + 0.3 \Rightarrow 0.5$$

For $Y=3$

$$P(Y=3) = P(X=2, Y=3) + P(X=3, Y=3)$$
$$= 0.1 + 0.15 \Rightarrow 0.25$$

(ii) Soln

$P(X=x, Y=y) = P(X=x) \cdot P(Y=y)$ (To be independent)

here, $P(X=x, Y=y) = P(X=2, Y=3) = 0.1$

Also, $P(X=2) = 0.4$

and, $P(Y=3) = 0.25$

so, $P(X=2) \cdot P(Y=3) = 0.4 \times 0.25 = 0.1$

here, $P(X=2, Y=3) = P(X=2) \cdot P(Y=3)$. so, they are independent.

(iii) Soln

For $E(X)$

$X=x$	$P(X=x)$	$x \cdot P(X=x)$	$x^2 \cdot P(X=x)$
2	0.4	0.8	1.6
3	0.6	1.8	5.4
		$\Sigma x \cdot P(X=x) = 2.6$	$\Sigma x^2 \cdot P(X=x) = 7$

now,

$$E(X) = \Sigma x \cdot P(X=x) = 2.6$$

similarly,

$$E(X^2) = \Sigma x^2 \cdot P(X=x) = 7$$

For $E(Y)$

$Y=y$	$P(Y=y)$	$y \cdot P(Y=y)$	$y^2 \cdot P(Y=y)$
1	0.25	0.25	0.25
2	0.5	1	4
3	0.25	0.75	2.25

$\Sigma y \cdot P(Y=y) = 2$ $\qquad$ $\Sigma y^2 \cdot P(Y=y) = 6.5$

now,

$E(Y) = \Sigma y \cdot P(Y=y) = 2$

similarly,

$E(Y^2) = \Sigma y^2 \cdot P(Y=y) = 6.5$

So, $Var(X) = E(X^2) - [E(X)]^2 \Rightarrow 7 - (2.6)^2 = 0.24$

And, $Var(Y) = \Sigma(Y^2) - [E(Y)]^2 \Rightarrow 6.5 - (2)^4 \Rightarrow 6.5 - 4$
$\Rightarrow 2.5$

(iv) soln

Since, $cov(X,Y) = E(XY) - E(X) \cdot E(Y)$

So,

$$E(XY) = \sum_x \sum_y (xy)\, P(X=x, Y=y)$$

$$\Rightarrow \sum_{2}^{3} \sum_{1}^{3} (xy)\, P(X=x, Y=y)$$

$$\Rightarrow (2\times1\times0.1) + (2\times2\times0.2) + (2\times3\times0.1) + (3\times1\times0.15) + (3\times2\times0.3) + (3\times3\times0.15) \Rightarrow 5.2$$

Now,

$cov(X,Y) = 5.2 - (2.6 \times 2) \Rightarrow 5.2 - 5.2 \Rightarrow 0$

ans.

Example 2: Let X and Y are two random variables joint pdf

$$f(x,y) = Axy \; ; \quad 0<x<1 \text{ and } 0<y<1$$

Find $E(X)$, $E(Y)$, $V(X)$, $V(Y)$ and $cov(X,Y)$

Soln

Since the given function is pdf, so;

$$\int_x \int_y f(x,y)\, dy\, dx = 1$$

or, $$\int_0^1 \int_0^1 (Axy)\, dy \cdot dx = 1$$

or, $$\int_0^1 Ax \left[\frac{y^2}{2}\right]_0^1 \cdot dx = 1$$

or, $$\frac{A}{2} \int_0^1 (x) \cdot dx = 1$$

or, $$\frac{A}{2} \left[\frac{x^2}{2}\right]_0^1 = 1$$

or, $$\frac{A}{2} \cdot \frac{1}{2} = 1$$

$\therefore A = 4$

So, the function becomes:

$$f(x,y) = 4xy \; ; \quad 0<x<1 \text{ and } 0<y<1$$

Since,

$$E(X) = \int_x x f(X).dx$$

and, $$E(Y) = \int_y y f(Y).dy$$; where $f(X)$ and $f(Y)$ are marginal distribution of X and Y respectively.

So, <u>For Marginal Distribution of X</u>

$$f(X) = \int_y f(x,y).dy \Rightarrow \int_0^1 (4xy).dy$$

$$= 4x\left[\frac{y^2}{2}\right]_0^1 \Rightarrow 2x$$

<u>Similarly, For Marginal Distribution of Y</u>

$$f(Y) = \int_x f(x,y).dx \Rightarrow \int_0^1 (4xy).dy$$

$$= 4y\left[\frac{x^2}{2}\right]_0^1 \Rightarrow 2y$$

Now,

$$\underline{E(X)} = \int_0^1 (x \cdot 2x).dx \Rightarrow \int_0^1 (2x^2).dx$$

$$= 2\left[\frac{x^3}{3}\right]_0^1 \Rightarrow \underline{2/3}$$

~~Similarly~~ And,

$$E(X^2) = \int_0^1 (x^2 \cdot 2x).dx \Rightarrow \int_0^1 (2x^3).dx$$

$$= 2\left[\frac{x^4}{4}\right]_0^1 \Rightarrow 1/2$$

Similarly,

$$\underline{E(Y)} = \int_0^1 (y \cdot 2y)\, dy \Rightarrow \int_0^1 (2y^2) \cdot dy$$

$$\Rightarrow 2\left[\frac{y^3}{3}\right]_0^1 \Rightarrow \underline{\frac{2}{3}}$$

and, $$E(Y^2) = \int_0^1 (y^2 \cdot 2y)\, dy \Rightarrow \int_0^1 (2y^3) \cdot dy$$

$$= 2\left[\frac{y^4}{4}\right]_0^1 \Rightarrow \frac{2}{4} \Rightarrow \frac{1}{2}$$

Again,

$$\underline{V(X)} = E(X^2) - [E(X)]^2 \cancel{\Rightarrow \frac{2}{3} - \left(\frac{1}{2}\right)^2}$$

$$\cancel{\Rightarrow \frac{2}{3} - \frac{1}{4}}$$

$$= \frac{1}{2} - \left(\frac{2}{3}\right)^2 \Rightarrow \underline{\frac{1}{18}}$$

Similarly,

$$\underline{V(Y)} = E(Y^2) - [E(Y)]^2 \Rightarrow \frac{1}{2} - \left(\frac{2}{3}\right)^2 \Rightarrow \underline{\frac{1}{18}}$$

Since, $\mathrm{cov}(X,Y) = E(XY) - E(X) \cdot E(Y)$

So,

$$E(XY) = \int_x \int_y xy\, f(x,y) \cdot dy \cdot dx$$

$$= \int_0^1 \int_0^1 (xy \cdot 4xy)\, dy\, dx$$

$$= \int_0^1 \left\{ 4x^2 \left[\frac{y^3}{3}\right]_0^1 \right\} \cdot dx$$

$$= \int_0^1 \left(\frac{4x^2}{3}\right).dx \Rightarrow \frac{4}{3}\left[\frac{x^3}{3}\right]_0^1 \Rightarrow \frac{4}{3} \times \frac{1}{3} \Rightarrow \frac{4}{9}$$

so,

$$\underline{cov(X,Y)} = \frac{4}{9} - \left(\frac{2}{3} \times \frac{2}{3}\right) \Rightarrow \frac{4}{9} - \frac{4}{9} \Rightarrow \underline{0} \text{ ans}$$

2079/08/28
wednesday

Short note
Moments of Random Variables

→ The mean of different powers of the deviation of the given observations for any arbitary constant is called moments.

→ Let X be a random variable. Let r be any positive integer and c be any arbitary constant. Then the rth moments of random variable X from the arbitary constant c is defined as

$$E[X-c]^r$$

or,

$$E[(X-c)^r] \quad \text{where } r = 1, 2, 3, \ldots$$

→ There are two types of moments:

1) Raw moments
   or,
   Moments about the origin

2) Central moments
   or,
   Moments about the mean

Raw moments (= moments about the origin)

→ Let X be a random variable. Then the rth raw moments (or, moments about the origin) of random variable X is denoted by μ_r' and given by

$$\mu_r' = E[X^r] \quad (\text{since } c = 0)$$

where, $r = 1, 2, 3, \ldots$

→ If X is a discrete random variable, then

$$\mu_r' = E[X^r] = \sum_x x^r P(X = x)$$

here, $P(X = x)$ = prob. mass function

→ Similarly, if X is a continuous random variable then

$$\mu_r' = E(X^r) = \int_x x^r f(x) \cdot dx$$

here, $f(x)$ = prob. density function

Note:

(i) $\mu_0' = 1$

(ii) $\mu_1' = E(X) = \mu$; the mean of X

Central moments (= moments about the ~~origin~~ mean)

→ Let X be a random variable. Then the rth central moments of random variable X is denoted by μ_r and is given by

$$\mu_r = E[X - E(X)]^r \text{ or } E(X-\mu)^r; \quad r = 1, 2, 3, \ldots$$

→ If X is a discrete r.v. Then,

$$\mu_r = E[X-\mu]^r = \sum (x-\mu)^r \, p(X=x)$$

here, $p(X=x) =$ pmf

and, $\mu = E(X)$

→ Similarly, if X is a continuous r.v. Then,

$$\mu_r = E[X - E(X)]^r$$

$$= \int (x - E(x))^r \cdot f(x) \cdot dx$$

here, $f(x) =$ pdf

<u>Note:</u>

(1) $\mu_0 = 1$

(2) $\mu_1 = 0$

(3) $\mu_2 = E(X - E(X))^2 = \sigma^2$; the variance of X.

- Relation between Raw moments

1) $\mu_0 = \mu_0' = 1$

2) $\mu_1 = \mu_1' - \mu_1' = 0$

3) $\mu_2 = \mu_2' - (\mu_1')^2$; the variance

4) $\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3$

5) $\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_2')^4$

Moment Generating Function (MGF)

imp

→ The functions which is used to find the moment of a random variable.

→ However, moment generating function does not exists always for all kind of distribution

→ Let X be a random variable. Then the moment generating function (MGF) of random variable X is denoted by $M_X(t)$ and is given by:

$$M_X(t) = E(e^{tx}); \; t \in R$$ where t is any arbitary constant that lies in real number

→ If X is a discrete r.v.

then, $M_X(t) = E(e^{tx}) = \sum_x e^{tx} P(X=x)$

Here, $P(X=x) =$ pmf (prob. mass function)

Similarly,

→ If X is a continuous r.v. then,

$$M_X(t) = E(e^{tx}) = \int_x e^{tx} f(x) \cdot dx$$

here, $f(x)$ = prob. density function

Now, the rth raw moment can be obtained from $M_X(t)$ as:

$$\mu_r' = \left. \frac{d^r M_X(t)}{dt^r} \right|_{t=0}$$

<u>Example:</u>

$$\mu_1' = \left. \frac{dM_X(t)}{dt} \right|_{t=0}$$

$$\mu_2' = \left. \frac{d}{dt} \left[\frac{dM_X(t)}{dt} \right] \right|_{t=0}$$

<u>Example 1:</u>

Let X be a random variable with pdf ~~f~~

$$f(x) = \theta e^{-\theta x} \; ; \; x \geq 0$$

(i) Find MGF of X

(ii) Find first four raw moments

(iii) Find first four central moments

(iv) Find mean and variance

Soln

given,

$$f(x) = \theta e^{-\theta x} \; ; \quad x \geq 0$$

(i) Soln

Since,

$$M_x(t) = E(e^{tx}) = \int_x e^{tx} \cdot f(x) \cdot dx$$

so,

$$M_x(t) = \int_0^\infty e^{tx} \cdot \theta e^{-\theta x} \cdot dx$$

$$= \theta \int_0^\infty (e^{-\theta x} \cdot e^{tx}) \cdot dx$$

$$= \theta \int_0^\infty (e^{-\theta x + tx}) \cdot dx$$

$$= \theta \int_0^\infty e^{-(\theta - t)x} \cdot dx$$

$$= \theta \left[\frac{e^{-(\theta - t)x}}{-(\theta - t)} \right]_0^\infty$$

$$= \theta \left[\frac{e^{-\infty} - e^{0}}{-(\theta - t)} \right]$$

$$= \theta \left[\frac{0-1}{-\theta+t}\right] \Rightarrow \frac{-\theta}{-\theta+t} \Rightarrow \frac{\theta}{\theta-t}$$

$$\therefore M_x(t) = \frac{\theta}{\theta-t}$$

(ii) Soln

Since the rth raw moments is given by,

$$\mu_r' = \left.\frac{d^r M_x(t)}{dt^r}\right|_{t=0}$$

So, For $r=1$

$$\mu_1' = \left.\frac{d}{dt}\left(\frac{\theta}{\theta-t}\right)\right|_{t=0} \Rightarrow \theta \left.\frac{d}{dt}(\theta-t)^{-1}\right|_{t=0}$$

$$= \theta.(-1).(\theta-t)^{-2}.(-1)\Big|_{t=0}$$

$$= \left.\frac{\theta}{(\theta-t)^2}\right|_{t=0} \Rightarrow \frac{\theta}{\theta^2} \text{ (when } t=0)$$

$$\therefore \mu_1' \Rightarrow 1/\theta$$

For $r=2$

$$\mu_2' = \left.\frac{d^2}{dt^2} M_x(t)\right|_{t=0} \Rightarrow \left.\frac{d}{dt}\left[\frac{d}{dt} M_x(t)\right]\right|_{t=0}$$

$$\Rightarrow \left.\frac{d}{dt}\frac{\theta}{(\theta-t)^2}\right|_{t=0} \Rightarrow \left.\theta.\frac{d}{dt}(\theta-t)^{-2}\right|_{t=0}$$

$$\Rightarrow \theta.(-2)(\theta-t)^{-3}(-1)\Big|_{t=0} \Rightarrow \left.\frac{2\theta}{(\theta-t)^3}\right|_{t=0}$$

$$= \frac{2\theta}{\theta^3} \text{ (when } t=0)$$

$$\therefore \mu_2' = \frac{2}{\theta^2}$$

<u>For $r=3$</u>

$$\mu_3' = \frac{d^3}{dt^3} M_x(t)\Big|_{t=0} \Rightarrow \frac{d}{dt}\left[\frac{d^2}{dt^2} M_x(t)\right]\Big|_{t=0}$$

$$\Rightarrow \frac{d}{dt} \frac{2\theta}{(\theta-t)^3}\Big|_{t=0} \Rightarrow 2\theta \frac{d}{dt}(\theta-t)^{-3}\Big|_{t=0}$$

$$\Rightarrow 2\theta(-3)(\theta-t)^{-4}(-1)\Big|_{t=0} \Rightarrow \frac{6\theta}{(\theta-t)^4}\Big|_{t=0}$$

$$= \frac{6\theta}{\theta^4} \text{ (when } t=0)$$

$$\therefore \mu_3' = \frac{6}{\theta^3}$$

<u>For $r=4$</u>

$$\mu_4' = \frac{d^4}{dt^4} M_x(t)\Big|_{t=0} \Rightarrow \frac{d}{dt}\left[\frac{d^3}{dt^3} M_x(t)\right]\Big|_{t=0}$$

$$= \frac{d}{dt} \frac{6\theta}{(\theta-t)^4}\Big|_{t=0} \Rightarrow 6\theta(-4)(\theta-t)^{-5}(-1)\Big|_{t=0}$$

$$\Rightarrow \frac{24\theta}{\theta^5} \Rightarrow \frac{24}{\theta^4}$$

v) Soln

Mean = ? and variance = ?

We know that,

$$\text{Mean} = E(X) = \mu_1'$$

and, $$\text{Variance} = E(X^2) - [E(X)]^2$$

i.e. $$\text{variance} = \mu_2' - (\mu_1')^2$$

so,

$$E(X) = \mu_1' = \frac{d}{dt}\left(\frac{\theta}{\theta - t}\right)\Big|_{t=0} \Rightarrow \theta \cdot \frac{d}{dt}(\theta - t)^{-1}\Big|_{t=0}$$

$$\Rightarrow \theta \cdot (-1) \cdot (\theta - t)^{-2} \cdot (-1)\Big|_{t=0} \Rightarrow \frac{\theta}{(\theta - t)^2}\Big|_{t=0}$$

$$= \frac{\theta}{\theta^2} \quad (\text{when } t = 0)$$

$$\therefore E(X) = \mu_1' = \underline{\text{mean}} = \underline{\frac{1}{\theta}}$$

Similarly,

$$\mu_2' = \frac{d^2}{dt^2}\left(\frac{\theta}{\theta - t}\right)\Big|_{t=0} \Rightarrow \frac{d}{dt}\left[\frac{d}{dt}\left(\frac{\theta}{\theta - t}\right)\right]\Big|_{t=0}$$

$$= \frac{d}{dt}\frac{\theta}{(\theta - t)^2}\Big|_{t=0} \Rightarrow \theta(-2)(\theta - t)^{-3}(-1)\Big|_{t=0}$$

$$= \frac{2\theta}{(\theta - t)^3}\Big|_{t=0} \Rightarrow \frac{2\theta}{\theta^3} \quad (\text{when } t = 0)$$

$$\therefore \mu_2' = \frac{2}{\theta^2}$$

now,

variance $= \mu_2' - (\mu_1')^2 \Rightarrow \frac{2}{\theta^2} - \left(\frac{1}{\theta}\right)^2$

$= \frac{2}{\theta^2} - \frac{1}{\theta^2} \Rightarrow \frac{1}{\theta^2}$

Assignment 11: Let X be a random variable with pmf $P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$; $x = 0, 1, 2, \ldots$

(i) Find MGF

(ii) Find Mean and Variance

Soln

given pmf is,

$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$; $x = 0, 1, 2, \ldots$

we have the formula,

the MGF for a discrete random variable is,

$M_X(t) = E(e^{tx}) \Rightarrow \sum_x e^{tx} P(X=x)$

$= \sum_0^\infty e^{tx}\left(\frac{e^{-\lambda}\lambda^x}{x!}\right) \Rightarrow e^{-\lambda}\sum_0^\infty \frac{e^{tx}.\lambda^x}{x!}$

$= e^{-\lambda}\sum_0^\infty \frac{(e^t\lambda)^x}{x!}$

since by the formula,

$e^x = \sum_{n=0}^\infty \frac{x^n}{n!}$

So,

$M_X(t)$ becomes,

$\cancel{M_X(t) = e^{-\lambda} e^{et}}$

$$M_X(t) = e^{-\lambda} e^{e^t \lambda} = e^{-\lambda(1-e^t)}$$

now Mean = ? and variance = ?

we know that,

$$\text{Mean} = E(X) = \mu_1'$$

And, $\text{Variance} = E(X^2) - [E(X)]^2$

i.e. $\text{Variance} = \mu_2' - (\mu_1')^2$

So,

$$E(X) = \mu_1' = \frac{d}{dt}\left[e^{-\lambda(1-e^t)}\right]\Big|_{t=0}$$

$$= \frac{d}{dt} \frac{e^{-\lambda(1-e^t)}}{(1-e^t)} \times \frac{d(1-e^t)}{dt}\Big|_{t=0}$$

$$= -\lambda e^{-\lambda(1-e^t)} \left[\frac{d.1}{dt} - \frac{d\,e^t}{dt}\right]\Big|_{t=0}$$

$$= -\lambda e^{-\lambda(1-e^t)} . [0 - e^t]\Big|_{t=0}$$

$$= -\lambda e^{-\lambda(1-e^t)} . (-e^t)\Big|_{t=0}$$

$$= e^{-\lambda(1-e^t)} . (\lambda e^t)\Big|_{t=0}$$

$$= e^{-\lambda(1-1)} (\lambda . 1) = 1 . \lambda$$

$$\therefore E(X) = \mu_1' = \text{Mean} = \lambda$$

Similarly,

$$E(X^2) = \mu_2' \Rightarrow \frac{d^2}{dt^2}\left[M_X(t)\right]\Big|_{t=0}$$

$$= \frac{d}{dt}\left\{\frac{d}{dt}\left[M_X(t)\right]\right\}\Big|_{t=0}$$

$$= \frac{d}{dt}\left[e^{-\lambda(1-e^t)}(\lambda e^t)\right]\Big|_{t=0}$$

$$= e^{-\lambda(1-e^t)}(\lambda e^t).(\lambda e^t) + e^{-\lambda(1-e^t)}.(\lambda e^t) \text{ at } t=0$$

$$= e^{-\lambda(1-e^t)}.(\lambda e^t)^2 + e^{-\lambda(1-e^{-t})}.(\lambda e^t) \text{ at } t=0$$

$$= e^{-\lambda(1-e^0)}(\lambda e^0) + e^{-\lambda(1-e^0)}(\lambda e^0)^2$$

$$= 1.\lambda + 1.\lambda^2$$

$$\therefore E(X^2) = \mu_2' = \lambda + \lambda^2$$

so,

$$\text{Variance} = E(X^2) - [E(X)]^2$$

$$= \mu_2' - (\mu_1')^2$$

$$= \lambda + \lambda^2 - \lambda^2$$

$$= \lambda$$

$$\therefore \text{variance} = \lambda$$ ans

2079/09/01

Friday

Characteristic Function

→ The MGF does not exist for all distribution. Thus characteristic function is used to find the moments of the distribution. Since characteristic function exists for all distribution.

→ Let X be a random variable. Then the characteristic function of X is denoted by $\phi_X(t)$ and is given by

$$\phi_X(t) = E(e^{itx}); \quad t \in R$$

→ If X is discrete r.v. Then,

$$\phi_X(t) = \sum_x e^{itx} P(X=x) = E(e^{itx})$$

→ And if X is continuous random variable then,

$$\phi_X(t) = E(e^{itx}) = \int_x e^{itx} f(x)\,dx$$

Now, the rth raw moment of random variable X is obtained as:

$$\mu_r' = \frac{1}{i^r} \frac{d^r \phi_X(t)}{dt^r} \Big|_{t=0}$$

Example : 1

If a random variable X has following pdf

$f(x) = \theta e^{-\theta x}$; $x \geq 0$

(i) Find the characteristic function

(ii) Find mean and variance

Soln given pdf is,

$f(x) = \theta e^{-\theta x}$; $x \geq 0$

then,

the characteristic function of X is,

$$\phi_X(t) = E(e^{itx}) = \int_x e^{itx} f(x).dx$$

$$= \int_0^\infty e^{itx}.\theta.e^{-\theta x}.dx$$

$$\therefore \phi_X(t) \Rightarrow \frac{\theta}{\theta - it}$$

For $r = 1$

$$\mu_1' = \frac{1}{i} \cdot \frac{d\phi_X(t)}{dt}\Big|_{t=0}$$

$$= \frac{1}{i}\left[\frac{d}{dt}\left(\frac{\theta}{\theta - it}\right)\right]\Big|_{t=0}$$

$$= \frac{1}{i}\left[\theta(-1)(\theta - it)^{-2}(-i)\right]\Big|_{t=0}$$

$$= \frac{1}{i}\left[i \cdot \frac{\theta}{(\theta - it)^2}\right]\Big|_{t=0} \Rightarrow \frac{\theta}{(\theta - it)^2}\Big|_{t=0}$$

$$= \frac{\theta}{\theta^2} \Rightarrow \frac{1}{\theta}$$

$\therefore$ Mean $= E(X) = \mu_r' = \frac{1}{\theta}$

Again,

For $r = 2$

$$\mu_2' = \frac{1}{i^2} \cdot \frac{d^2\phi_X(t)}{dt^2}\Big|_{t=0}$$

$$= \frac{1}{i^2}\left[\frac{d}{dt}\frac{i\theta}{(\theta - it)^2}\right]\Big|_{t=0}$$

$$= \frac{1}{i^2}\, i\left[\frac{d}{dt}\frac{\theta}{(\theta - it)^2}\right]\Big|_{t=0} \Rightarrow \frac{1}{i}\left[2i\frac{\theta}{(\theta - it)^2}\right]\Big|_{t=0}$$

$$= \frac{1}{i}\left[2i \cdot \frac{\theta}{\theta^3}\right] \Rightarrow \frac{2}{\theta^2}$$

now Variance $= \mu_2' - (\mu_1')^2 \Rightarrow \frac{2}{\theta^2} - \left(\frac{1}{\theta}\right)^2$

$\Rightarrow \frac{2-1}{\theta^2} \Rightarrow \frac{1}{\theta^2}$ ans

Example 2: If a random variable X has following pdf :

$$f(x) = \frac{1}{b-a} \; ; \quad a < x < b$$

Then,

(i) Find the characteristic function

(ii) Find Mean and variance

soln

(i) given pdf is,

$$f(x) = \frac{1}{b-a} \; ; \quad a < x < b$$

Then the characteristic function of X is,

$$\phi_x(t) = E(e^{itx}) = \int_x e^{itx} f(x).dx$$

$$= \int_a^b e^{itx}\left(\frac{1}{b-a}\right).dx \Rightarrow \frac{1}{b-a}\int_a^b (e^{itx}).dx$$

$$= \frac{1}{b-a}\left|\frac{e^{itx}}{it}\right|_a^b \Rightarrow \frac{1}{b-a}\left[\frac{e^{itb}}{it} - \frac{e^{ita}}{it}\right]$$

$$= \frac{1}{b-a}\left(\frac{e^{itb} - e^{ita}}{it}\right)$$

(ii) Mean $= E(X) = \int_a^b x \cdot f(x) \cdot dx$

$$= \int_a^b x \cdot \left(\frac{1}{b-a}\right) \cdot dx \Rightarrow \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{1}{b-a}\left(\frac{b^2}{2} - \frac{a^2}{2}\right) \Rightarrow \frac{1}{b-a}\left(\frac{b^2-a^2}{2}\right) \Rightarrow \frac{(b+a)\cancel{(b-a)}}{2\cancel{(b-a)}}$$

$\therefore E(X) = \frac{a+b}{2}$ = Mean

Similarly,

$$E(X^2) = \int_a^b x^2\left(\frac{1}{b-a}\right) \cdot dx \Rightarrow \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b$$

$$= \frac{1}{b-a}\left[\frac{b^3}{3} - \frac{a^3}{3}\right] \Rightarrow \frac{1}{b-a}\left(\frac{b^3-a^3}{3}\right)$$

$$= \frac{\cancel{(b-a)}(b^2+ba+a^2)}{3\cancel{(b-a)}} \Rightarrow \frac{b^2+ba+a^2}{3} = E(X^2)$$

Now

$$\text{Variance} = E(X^2) - [E(X)]^2 \Rightarrow \frac{b^2+ba+a^2}{3} - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{a^2+ab+b^2}{3} - \frac{a^2+2ab+b^2}{4} \Rightarrow \frac{4a^2+4ab+4b^2-3a^2-6ab-3b^2}{12}$$

$$= \frac{a^2-2ab+b^2}{12} \Rightarrow \therefore \text{variance} = \frac{(a-b)^2}{12}$$ ans

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Sunday

Short notes

Binomial Distribution

↳ This concept of Binomial Distribution was given by James Bernoulli.

- Conditions for Binomial Distribution :-

1) The number of trials (sample size) i.e. n, is fixed and finite.

2) Each trial has only two possible outcomes, such as success or failure, Yes or No, defective or non-defective, etc.

3) The probability of success (p) is constant for each trial such that,

$$p + q = 1$$

so, $q = 1 - p$ (probability of Failure)

4) Every trials are independent.

Note:

Success represents the presence of certain characteristic or desired outcomes of an particular experiment.

- **Probability Function (pmf)**

→ Let X be a random variable that represents num of success which follows Binomial Distribution with parameters n and p.

Then, the probability of getting x success in n number of trials is given by,

$$P(X=x) = {}^nC_x \cdot p^x \cdot q^{n-x} \; ; x = 0, 1, 2, \ldots, n$$

where,

$n \rightarrow$ number of trials

$x \rightarrow$ required no. of success

$p \rightarrow$ prob. of success (constant)

$q \rightarrow 1-p =$ prob. of failure

imp

- **Properties**

1) It is a discrete probability distribution.

2) The mean of Binomial distribution

i.e. $E(X) = np$

3) The variance of Binomial distn. is npq

or, $np(1-p)$

i.e. $V(X) = npq$

Note:

(i) Mean is greater than variance

(ii) Standard deviation of binomial distn is $\sqrt{npq}$

(iii) If X follows Binomial distribution with parameters n and p, then we can write

$$X \sim B(n, p)$$

Example 1: Determine the Binomial distribution which has mean 4 and variance 3.

Soln Let $X \sim B(n, p)$

Then we have,

$$E(X) = np = 4 \quad \text{—(i)}$$

$$Var(X) = npq = 3 \quad \text{—(ii)}$$

dividing (i) by (ii),

$$\frac{np}{npq} = \frac{4}{3} \Rightarrow \boxed{q = \frac{3}{4}}$$

$$\therefore p + q = 1$$

$$\text{or, } p = 1 - q \Rightarrow 1 - 3/4 \Rightarrow \boxed{\therefore p = \frac{1}{4}}$$

Since, $npq = 3$

$$\text{or, } n = \frac{3}{pq} \Rightarrow \frac{3}{\frac{1}{4} \times \frac{3}{4}} \Rightarrow \frac{\cancel{3} \times 16}{\cancel{3}}$$

$$\Rightarrow \therefore n = 16$$

Hence,

$$X \sim B(n, p) \Rightarrow X \sim B(16, \tfrac{1}{4})$$ ans

Example 2: A fair coin is tossed
probability of getting
(i) 5 heads
(ii) 0 heads
(iii) at least 2 heads
(iv) at most 1 head
(v) at least 9 heads
(vi) at most 9 heads

Soln given, $n = 10$ = no. of tosses
prob. of getting head $(p) = 1/2$
$\Rightarrow q = 1 - p = 1 - 1/2 = 1/2$

let X be number of heads
therefore, $X \sim B(n, p)$
Then, $p(X = x) = {}^nC_x \, p^x q^{n-x}$; $x = 0, 1, 2, \ldots, n$

(i) soln
getting 5 heads means $X = 5$.
and, $p(X = 5) = ?$
we know that,

$$p(X = 5) = {}^{10}C_5 \left(\tfrac{1}{2}\right)^5 \left(\tfrac{1}{2}\right)^{10-5}$$

$$= 252 \times \frac{1}{32} \times \frac{1}{32}$$

$\therefore p(X = 5) \Rightarrow 0.24$ ans

(ii) Soln

here, $X = 0$

and, $P(X=0) = ?$

we know that,

$$P(X=0) = {}^{10}C_0 \left(\tfrac{1}{2}\right)^0 \left(\tfrac{1}{2}\right)^{10-0}$$

$$= 1 \times 1 \times \left(\tfrac{1}{2}\right)^{10} \Rightarrow 9.76 \times 10^{-4} = P(X=0)$$

ans

(iii) Soln

here,

$X = 0, 1, 2, 3, \ldots, 10$

$$P(0) + P(1) + P(2) + P(3) + \ldots + P(10) = 1$$

or, $P(X \le 1) + P(X \ge 2) = 1$

or, $P(X \ge 2) = 1 - P(X \le 1)$

$$= 1 - [P(X=0) + P(X=1)]$$

$$= 1 - \left[{}^{10}C_0 \left(\tfrac{1}{2}\right)^0 \left(\tfrac{1}{2}\right)^{10-0} + {}^{10}C_1 \left(\tfrac{1}{2}\right)^1 \left(\tfrac{1}{2}\right)^{10-1}\right]$$

$$= 1 - \left[1 \times 1 \times \left(\tfrac{1}{2}\right)^{10} + 10 \times \tfrac{1}{2} \times \left(\tfrac{1}{2}\right)^9\right]$$

$$= 0.75$$

$\therefore P(X \ge 2) = 0.75$ ans

(iv) Soln here, $x \leq 1$

so, $P(X \leq 1) = P(X=0) + P(X=1)$

$= {}^{10}C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{10-0} + {}^{10}C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^9$

$= \frac{1}{1024} + \frac{5377}{512} \Rightarrow 10.50$

$\therefore P(X \leq 1) = 10.50$

(vi) Soln $P(X \leq 9) = ?$

we know,

$P(X \leq 9) = 1 - P(X = 10)$

$= 1 - \left[{}^{10}C_{10} \left(\frac{1}{2}\right)^{10} \left(\frac{1}{2}\right)^{10-10}\right]$

$= 1 - \frac{1}{1024} \Rightarrow \therefore P(X \leq 9) = 0.999$ ans

(v) Soln

022 Fall)

Q The normal rate of covid virus infection in certain city is known to be 35%. In an experiment with 10 persons, calculate the probability that none of them are infected.

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Wednesday

Class Test's Question Discussion

(Q.no.3.b) The group of children contain 3 girls and 1 boy; 2 girls and 2 boys and 1 girl and 3 boys. One child is selected at random from each group. Find the prob. that the selected three consists of 1 girl and 2 boys.

Hint:

Group 1	Group 2	Group 3	
3 G	2 G	1 G	
1 B	2 B	3 B	
1	1	1	= 3 are selected

(i) G and B and B $\Rightarrow \frac{3}{4} \times \frac{2}{4} \times \frac{3}{4} \Rightarrow 9/32$

(ii) B and G and B $\Rightarrow \frac{1}{4} \times \frac{2}{4} \times \frac{3}{4} \Rightarrow 3/32$

(iii) B and B and G $\Rightarrow \frac{1}{4} \times \frac{2}{4} \times \frac{1}{4} \Rightarrow 1/32$

Then, $P(1G + 2B) = P(i) + P(ii) + P(iii)$

Sol^n For 1 Girl and 2 Boys selecting each child from each group.

Then, we have the following mutually exclusive cases:

(i) G B B

(ii) B G B

(iii) B B G

Therefore,

$p(i) = P(G).\ P(B).\ P(B) = \frac{3}{4} \times \frac{2}{4} \times \frac{3}{4} \Rightarrow 9/32$

$p(ii) = P(B).\ P(G).\ P(B) = 1/4 \times \frac{2}{4} \times 3/4 \Rightarrow 3/32$

$p(iii) = P(B).\ P(B).\ P(G) = 1/4 \times 2/4 \times 1/4 \Rightarrow 1/32$

Hence, the probability that group contains 1G and 2B is,

$P(1G \text{ and } 2B) = P(i) + P(ii) + P(iii) \Rightarrow \frac{9}{32} + \frac{3}{32} + \frac{1}{32}$

$\Rightarrow 13/32$

$\Rightarrow 0.40625$ ans

Q.no. 2. b) A first step towards ~~test~~ identifying the spam is to create a list of words that are more likely to appear in spam than in normal message. Suppose a specified list of words is available and your data base of 5000 message contains 1700 that are spam. Among the spam messages, 1343 contain words in the list of the 3300 normal messages, only 297 contain words in the list. Obtain the probability

that a message is spam given that the message contains words in the list.

Soln defining the events:

E_1 = spam messages

and, E_2 = normal messages

Then,

$$P(E_1) = \frac{1700}{5000} \Rightarrow 0.34$$

$$P(E_2) = \frac{3300}{5000} \Rightarrow 0.66$$

Also,

let A = message contains words in the list

$$\therefore P(A|E_1) = \frac{1343}{1700} \Rightarrow 0.79$$

and $$P(A|E_2) = \frac{297}{3300} \Rightarrow 0.09$$

now, $P(E_1|A) = ?$

we know that,

$$P(E_1|A) = \frac{P(E_1) \cdot P(A|E_1)}{P(E_1) \cdot P(A|E_1) + P(E_2) \cdot P(A|E_2)}$$

$$= \frac{0.34 \times 0.79}{(0.34 \times 0.79) + (0.66 \times 0.09)} \Rightarrow \frac{0.2686}{(0.2686 + 0.0594)}$$

$\Rightarrow$ 0.819 ans

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Friday

Poisson Distribution

- This concept was given by French Mathematician Siemeon Denis Poisson.
- This concept is used for rare case events and the events are defined in the interval of time.
- Poisson Distribution is the limiting case of Binomial distribution under the following conditions:

(i) number of trials (sample size) i.e. n is indefinitely large.

i.e. $n \to \infty$

i.e. when $n \geq 20$ and $p \leq 0.05$

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(ii) the probability of success (p) is very small

(iii) $np = \lambda$, is finite

Some examples of Poisson Distribution

1) No. of air accidents per year
2) No. of phone calls per minute
3) No. of breakdowns of machine per week.
4) No. of typing error per page, etc.

Probability Function

→ Let X be a random variable that represents the number of success which follows Poisson Distribution with parameter λ. Then the prob. of getting x success is given by,

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \quad ; x = 0, 1, 2, \ldots$$

Here, x = required number of success

λ = parameter of the distribution (Mean or Variance)

e = Exponential constant

Properties of Poisson distribution

(i) It is a discrete probability distribution.

(ii) Random variable of poisson distribution is no. of success.

(iii) Mean of Poisson distribution is λ.

ie. $E(X) = \lambda$

(iv) Variance of Poisson distn. is also λ.

ie. $V(X) = \lambda$

Note:

1) here, Mean and Variance in Poisson distribution are equal.

2) Standard deviation of Poisson distn is $\sqrt{\lambda}$

3) If X follows Poisson distn with parameter λ, then we can write

$$X \sim P(\lambda)$$

Example 1: At a checkout counter, the customers arrive at an average rate of 1.5 customers per minute. Find the probability that during any given minutes, there will be

(i) 3 customers

(ii) at least 3 customers

(iii) at most 3 customers

(iv) 4 customers during an interval of 2 minutes.

Soln given,

average rate of arrival of customers per minute $(\lambda) = 1.5$

Let, X = number of customers arriving at checkout counter per minute

$\therefore X \sim P(\lambda)$

Then, $P(X=x) = \dfrac{e^{-\lambda}\lambda^{x}}{x!}$; $x = 0, 1, 2, \ldots$

(i) Soln $\lambda = 3 = ?$

we have,

$$P(X=3) = \frac{e^{-1.5}(1.5)^3}{3!} \Rightarrow 0.125$$ ans

(ii) Soln

P(at least 3 customers) = $P(X \geq 3) = ?$

we have,

$$P(X=0) = \frac{e^{-1.5}(1.5)^0}{0!} \Rightarrow 0.223$$

$$P(X=1) = \frac{e^{-1.5}(1.5)^1}{1!} \Rightarrow 0.334$$

$$P(X=2) = \frac{e^{-1.5}(1.5)^2}{2!} \Rightarrow 0.251$$

now,

$$P(X \geq 3) = 1 - [P(X=0) + P(X=1) + P(X=2)]$$

$$= 1 - (0.223 + 0.334 + 0.251)$$

$$\therefore P(X \geq 3) = 0.192$$ ans

(iii) Soln P(at most 3 customers) = $P(X \leq 3) = ?$

we have,

$$P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

$$= 0.223 + 0.334 + 0.251 + \frac{e^{-1.5}(1.5)^3}{3!}$$

$$\therefore P(X \leq 3) = 0.934$$ ans

(iv) soln

given rate is : 4 customers during 2 mins

here,

average rate per min = 1.5

so, " " per 2 min = $1.5 \times 2 \Rightarrow 3 = \lambda$

therefore,

$$p(X=4) = \frac{e^{-3}(3)^4}{4!} \Rightarrow 0.168 \text{ ans}$$

Imp

Example 2: During the hours of 2 P.M. and 4 P.M., the average number of calls per minute come to ~~be~~ a switch board of a company is 2.5. Find the probability that during any given minutes, there is:

(i) no calls at all

(ii) 3 calls

(iii) at least 2 calls

(iv) at most 4 calls

(v) during an interval of 30 second, what is the probability that there is only one call?

Soln given,

average no. of calls per minute (λ) = 2.5

Let,

X = no. of calls per minute

(i) prob. of no calls at all $[P(X=0)]$ = ?

we know that,

$$P(X=0) = \frac{e^{-2.5}(2.5)^0}{0!} \Rightarrow 0.082 \text{ ans}$$

(ii) Prob. of 3 calls $[P(X=3)]$ = ?

we know that,

$$P(X=3) = \frac{e^{-2.5}(2.5)^3}{3!} \Rightarrow 0.213 \text{ ans}$$

(iii) prob of at least 2 calls $[P(X \geq 2)]$ = ?

we know that,

$$P(X \geq 2) = 1 - [P(X=0) + P(X=1)]$$

Then,

$$P(X=1) = \frac{e^{-2.5}(2.5)^1}{1!} \Rightarrow e^{-2.5} \times 2.5$$

$$\Rightarrow 0.205$$

So,

$$P(X \geq 2) = 1 - (0.082 + 0.205) \Rightarrow 0.713 \text{ ans}$$

(iv) prob. of at most 4 calls $[P(X \leq 4)] = ?$

since,

$$P(X \leq 4) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)$$

so,

$$P(X=2) = \frac{e^{-2.5}(2.5)^2}{2!} \Rightarrow 0.256$$

$$P(X=4) = \frac{e^{-2.5}(2.5)^4}{4!} \Rightarrow 0.133$$

$$\therefore P(X \leq 4) = 0.082 + 0.205 + 0.256 + 0.213 + 0.133$$

$$\Rightarrow 0.889 \text{ ans}$$

(v) Soln

rate per 60 sec = 2.5

" " 30 sec = $\frac{2.5}{60} \times 30 \Rightarrow \frac{2.5}{2} \Rightarrow 1.25$

since, no. of call = 1

so, $x = 1$

then,

$$P(X=1) = \frac{e^{-1.25}(1.25)^1}{1!} \Rightarrow 0.358 \text{ ans}$$

Example 6: The probability of an item produced by a certain machine will be defective is 0.05. If the produced items are sent to the market in packets of 20. Find the probability that a certain packet contains:

(i) exactly 2 defectives
(ii) at least 2 defectives
(iii) at most 2 defectives.

Use Poisson approximation.

Soln

given,

the probability of item to be defective $(p) = 0.05$

and, no. of trials $(n) = 20$

Let, $X \rightarrow$ no. of defective items.

so,

$$\lambda = np \Rightarrow 20 \times 0.05 \Rightarrow 1$$

(i) Soln

$P(\text{exactly 2 defectives}) = P(X=2) = ?$

we know that,

$$P(X=2) = \frac{e^{-1}(1)^2}{2!} \Rightarrow 0.183$$

Note:

Expected Frequency = $\underbrace{N}_{\text{population size}} \times \underbrace{P(X=x)}_{\text{required probability}}$

(ii) Soln

P(at least 2 defectives) = $P(X \geq 2) = ?$

since, $P(X \geq 2) = 1 - [P(X=0) + P(X=1)]$

so, $P(X=0) = \dfrac{e^{-1}(1)^0}{1!} \Rightarrow 0.367$

and, $P(X=1) = \dfrac{e^{-1}(1)^1}{1!} \Rightarrow 0.3678$

so, $P(X \geq 2) = 1 - (0.367 + 0.367) \Rightarrow 0.266$ ans

(iii) Soln

P(at most 2 defectives) = $P(X \leq 2) = ?$

we know that,

$P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$

so, $P(X \leq 2) = 0.367 + 0.367 + 0.183$

$\Rightarrow 0.913$ ans

Assignment 12: In a component manufacturing industry, there is a small probability of $\frac{1}{500}$ for any component to be defective. The components are supplied in packets of 10. Use Poisson distribution to calculate the approximate number of packets containing

(i) at least 1 defective

(ii) at most 1 defective, in a consignment of 1000 packets.

Soln

Let, $X \rightarrow$ components found to be defective

given, number of packets $(n) = 10$

probability of component found to be defective $(P) = \frac{1}{500} = 2 \times 10^{-3}$

$\therefore \lambda = np \Rightarrow 10 \times 2 \times 10^{-3} \Rightarrow 0.02$

(i) Soln

Probability of at least 1 component defective is given by,

$$P(X \geq 1) = 1 - P(X=0) \Rightarrow 1 - \frac{e^{-0.02}(0.02)^0}{0!}$$

$$\Rightarrow 1 - 0.98 \Rightarrow 0.019$$

Now,

the number of packets containing at least 1 defective (i.e. Expected frequency) $= N \times P(X \geq 1)$

$= 10 \times 0.019$

$= 0.19 \approx 1$ packets

(ii) Soln

probability of at most 1 defective is given by,

$$P(X \le 1) = P(X=0) + P(X=1)$$

$$= \frac{e^{-0.02}(0.02)^0}{0!} + \frac{e^{-0.02}(0.02)^1}{1!}$$

$$\cancel{= 0.98 + 0.019}$$

$$= 0.98 + 0.019 \Rightarrow 0.99$$

so, the number of packets containing at most 1 defective in a consignment of 1000 packets (i.e. Expected frequency) $= N \times P(X \le 1) \Rightarrow 1000 \times 0.99$

$\Rightarrow 990$ packets

ans

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Tuesday

Normal Distribution

→ it is continuous probability distribution.

→ also called Gaussian distribution

→ this concept is the important finding of statistics because most of the data follows normal distribution such as height, weight, age, income, etc.

Probability Function

→ A continuous random variable X is said to have normal distribution if its probability density function is given by :-

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$\boxed{\begin{array}{c} -\infty < x < \infty \\ -\infty < \mu < \infty \\ \sigma^2 > 0 \end{array}}$$

here,
μ and σ^2 are the parameters of normal distribution.

Note:
If X follows Normal Distribution with parameters μ (Mean) and σ^2 (variance).
Then, we can write

$$X \sim N(\mu, \sigma^2)$$

Standard Normal Variate

Let $X \sim N(\mu, \sigma^2)$. Then

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} ; \quad -\infty < x < \infty$$

$-\infty < \mu < \infty$
and, $\sigma^2 > 0$

Define a new variable,

$$\boxed{Z = \frac{X-\mu}{\sigma}}$$

here, μ = Mean
σ = standard deviation

→ This variable is called standard normal variate and its distribution is called standard normal distribution.

Standard normal distribution is a special case of normal distribution with mean 0 and variance 1.

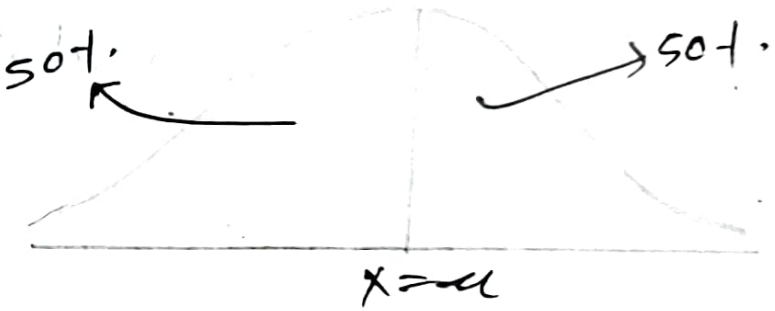
i.e. $Z \sim \underbrace{N(0,1)}_{\text{standard normal distribution}}$

and, its probability density function is given by,

$$\boxed{f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}} \; ; \quad -\infty < z < \infty$$

Characteristics of Normal Distribution

The curve of normal distribution i.e. the normal curve is bell shaped and symmetrical at $x = \mu$.



) The total area (i.e. probability) under the curve is 1.

) The normal curve is asymptomatic to x-axis.

) The values of Mean, Median and Mode are same.

5) Area property

(i) if $X = \mu \pm \sigma$, then $z = \pm 1$

$P(\mu - \sigma \le X \le \mu + \sigma)$

$\Rightarrow P(-1 \le z \le +1) \Rightarrow 0.6826$

(ii) if $X = \mu \pm 2\sigma$, $z = \pm 2$

$P(\mu - 2\sigma \le X \le \mu + 2\sigma) \Rightarrow P(-2 \le z \le +2)$

$\Rightarrow 0.9544$

(iii) if $X = \mu \pm 3\sigma$, $z = \pm 3$

$P(\mu - 3\sigma \le X \le \mu + 3\sigma)$

$\Rightarrow P(-3 \le z \le +3) \Rightarrow 0.997$

Calculating Area (prob.) under the normal curve

(1) $P(X > b)$

(a) If $b > \mu$

when $X = b$, $z = \dfrac{b - \mu}{\sigma} = z_1$ (say)

$X = \mu$ $X = b$

$z = 0$ $z = z_1$

$P(X > b) = P(z > z_1)$

$= 0.5 - P(0 < z < z_1)$

it's value is found from the table.

Example:

Let $X \sim N(\mu = 15, \sigma^2 = 16)$

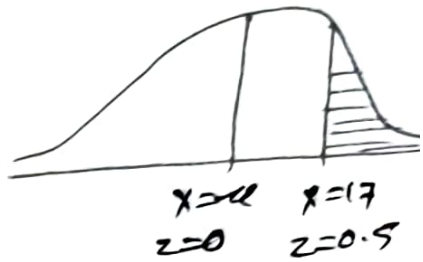
then,

$\mu = 15$

$\sigma^2 = 16$

$\Rightarrow \sigma = 4$; $P(X > 17) = ?$

Define: $z = \frac{X - \mu}{\sigma} \Rightarrow \frac{X - 15}{4}$



when $X = 17$, then $z = \frac{17-15}{4} \Rightarrow 0.5$

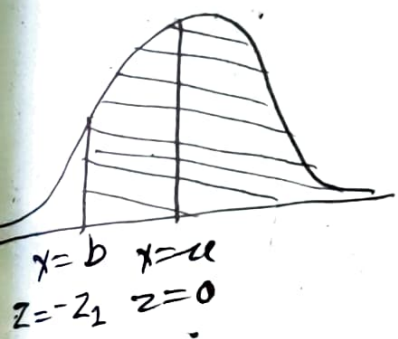
so, $P(X > 17) = P(Z > 0.5)$

$= 0.5 - P(0 < z < 0.5)$

$= 0.5 - 0.1915 \Rightarrow 0.3085$ ans

(b) If $b < \mu$

when $X = b$, $z = \frac{b - \mu}{\sigma} \Rightarrow -z_1$ (say)



so,

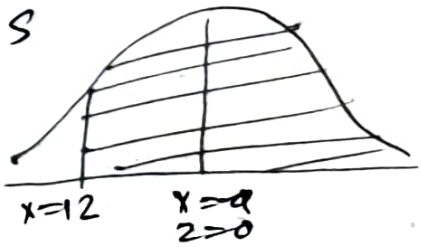
$P(X > b) = P(z > -z_1)$

$= 0.5 + P(-z_1 < z < 0)$

$= 0.5 + P(0 < z < z_1)$ ($\because$ due to symmetric)

Example: Find $P(X > 12)$

when, $X = 12$, $z = \frac{12 - 15}{4} \Rightarrow 0.75$



so, $P(X > 12) = P(z > -0.75)$

$= 0.5 + P(-0.75 < z < 0)$

$= 0.5 + P(0 < z < 0.75)$ (due to symmetric)

$= 0.5 + 0.2734$

$= 0.7734$ ans

Example

Q. Let $X \sim N(100, 25)$

Find (i) $P(X>110)$ (ii) $P(X>96)$ (iii) $P(X>112)$

(iv) $P(X>86)$

Soln. Let $X \sim N(100, 25)$

so, $\mu = 100$

$\sigma^2 = 25$

$\Rightarrow \sigma = 5$

Define: $z = \frac{x-\mu}{\sigma} \Rightarrow \frac{x-100}{5}$

(i) $P(X>110) = ?$

when $x=110$, $z = \frac{110-100}{5} = 10/5 \Rightarrow 2.00$

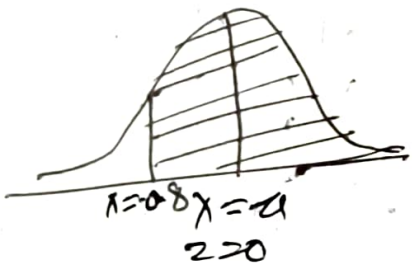
$\therefore P(X>110) = P(z>2.00)$

$= 0.5 - P(0<z<2)$

$= 0.5 - 0.4772 \Rightarrow 0.0228$ ans

(ii) $P(X>96)$

when $x=96$, $z = \frac{96-100}{5} \Rightarrow -0.8$



so,

$P(X>96) = P(z>-0.8)$

$= 0.5 + P(-0.8<z<0)$

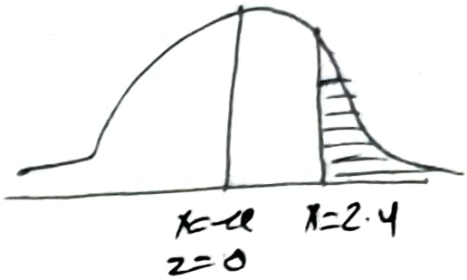
$= 0.5 + P(0<z<0.8)$ (∵due to symmetric)

$= 0.5 + 0.2881$

$= 0.7881$ ans

(ii) $P(X > 112)$

when $x = 112$, then $z = \frac{112-100}{5} \Rightarrow 2.4$



$\therefore P(X > 112) = P(z > 2.4)$

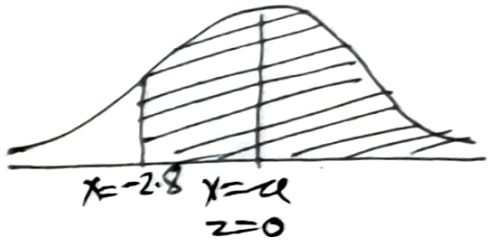
$= 0.5 - P(0 < z < 2.4)$

$= 0.5 - 0.4918$

$= 0.0082$ ans

(iv) $P(X > 86)$

when $x = 86$, then $z = \frac{86-100}{5} \Rightarrow -2.8$



$\therefore P(X > 86) = P(z > -2.8)$

$= 0.5 + P(-2.8 < z < 0)$

$= 0.5 + P(0 < z < 2.8)$

($\because$ due to symmetric)

$= 0.5 + 0.4974$

$= 0.9974$ ans

079/09/13

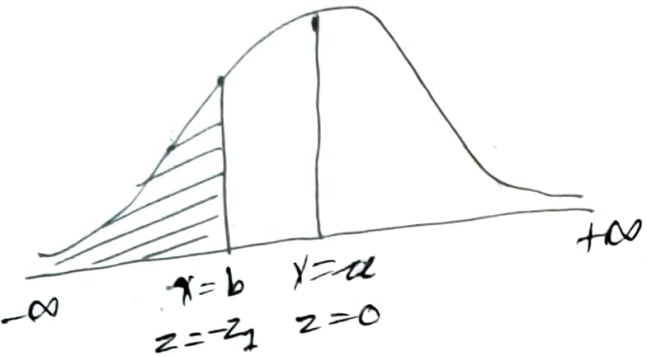
Wednes

2) For $P(X<b)$

(a) If $b<\mu$

when $x=b$, $z=\frac{b-\mu}{\sigma}$

$= -z_1$ (sa



then,

$P(X<b) = P(z<-z_1)$

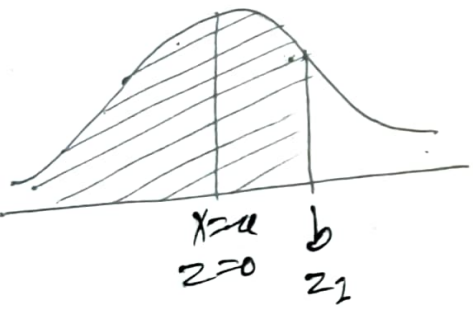
$= 0.5 - P(-z_1<z<0)$

$= 0.5 - P(0<z<z_1)$

($\because$ due to symmet

(b) If $b>\mu$

when $x=b$, $z=\frac{b-\mu}{\sigma} = z_1$ (sa



Then,

$P(X<b) = P(z<z_1)$

$= 0.5 + P(0<z<z_1)$

Q. Let $X \sim N(\mu=15, \sigma^2=16)$

(i) Find $P(X<12)$

(ii) Find $P(X<19)$

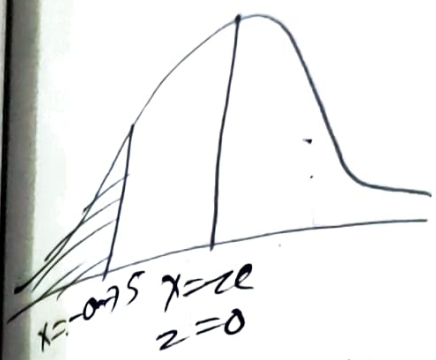
Soln $P(X<12) = ?$

given, $X \sim N(\mu=15, \sigma^2=16)$

so, $\mu=15$,

$\sigma=4$

when $X = 12$ then $z = \frac{X-\mu}{\sigma} = \frac{12-15}{4} \Rightarrow -0.75$



$\therefore P(X<12) = P(Z<-0.75)$

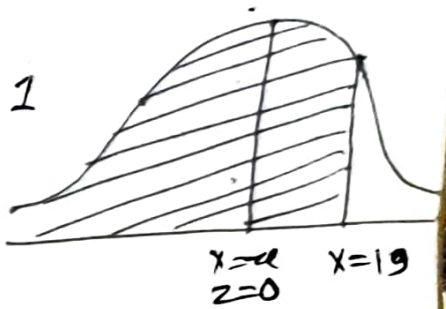
$= 0.5 - P(-0.75 < Z < 0)$

$= 0.5 - P(0 < Z < 0.75)$ ($\because$ due to symmetric)

$= 0.5 - 0.2734 \Rightarrow 0.2266$ ans

again, $P(X<19) = ?$

when $x = 19$, $z = \frac{X-\mu}{\sigma} \Rightarrow \frac{19-15}{4} \Rightarrow 1$



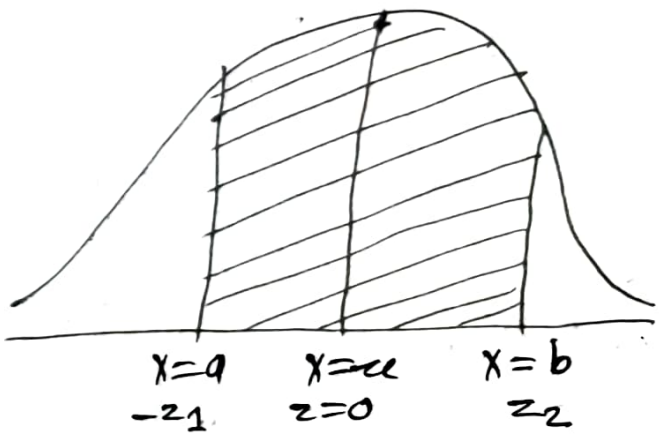
$\therefore P(X<19) = P(Z<1)$

$= 0.5 + P(0<Z<1)$

$= 0.5 + 0.3413 \Rightarrow 0.8413$ ans

(3) <u>For $P(a<X<b)$</u>

(i) <u>if $a<\mu<b$</u>



when $x=a$, $z = \frac{a-\mu}{\sigma} = -z_1$ (say)

and,

when $x=b$, $z = \frac{b-\mu}{\sigma} = z_2$ (say)

$\therefore P(a<X<b) = P(-z_1 < X < z_2)$

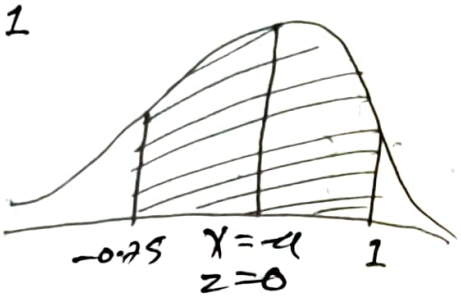
$= P(-z_1 < Z < 0) + P(0 < Z < z_2)$

$= P(0 < Z < z_1) + P(0 < Z < z_2)$ ($\because$ due to symmetrical distribution)

Example: Find $P(12 < X < 19)$ ~~when X = 12~~ ; $\mu = 15$, $\sigma^2 = 16$

Sol^n when $X = 12$, $z = \frac{12-15}{4} \Rightarrow -0.75$

when $X = 19$, $z = \frac{19-15}{4} \Rightarrow 1$



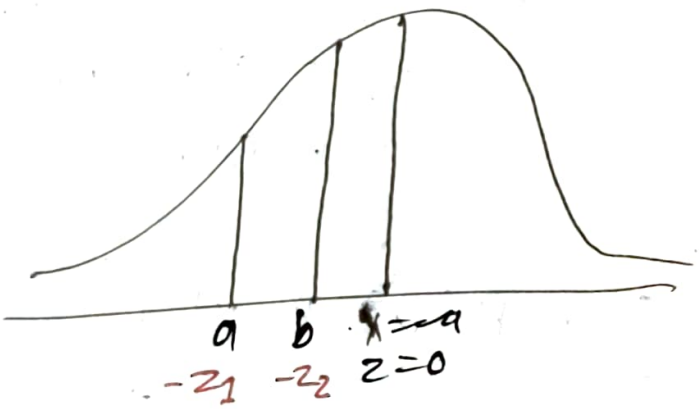
$\therefore P(12 < X < 19) = P(-0.75 < z < 1)$

$= P(-0.75 < z < 0) + P(0 < z < 1)$

$= P(0 < z < 0.75) + P(0 < z < 1)$ ($\because$ due to symmetry)

$= 0.2734 + 0.3413 \Rightarrow 0.6147$ ans

(ii) If $a < b < \mu$



when $X = a$,

$z = \frac{a-\mu}{\sigma} = -z_1$ (say)

when $X = b$,

$z = \frac{b-\mu}{\sigma} = -z_2$ (say)

$\therefore P(a < b < \mu) = P(-z_1 < z < -z_2)$

$= P(-z_1 < z < 0) - P(-z_2 < z < 0)$

$= P(0 < z < z_1) - P(0 < z < z_2)$

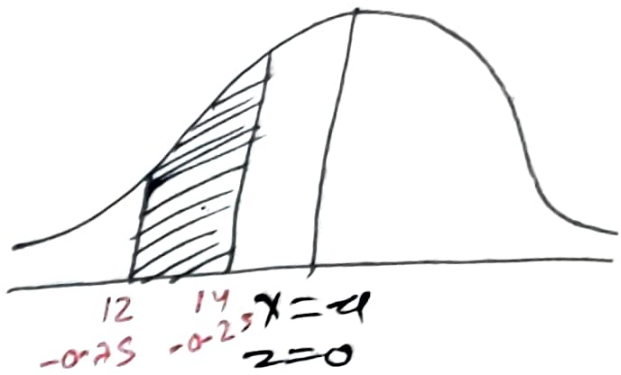
(due to symmetry)

Example: Find $P(12 < X < 14)$; $\mu = 15$, $\sigma^2 = 16$.

Soln

When $x = 12$, $z = \frac{12-15}{4} \Rightarrow -0.75$

When $x = 14$, $z = \frac{14-15}{4} \Rightarrow -0.25$



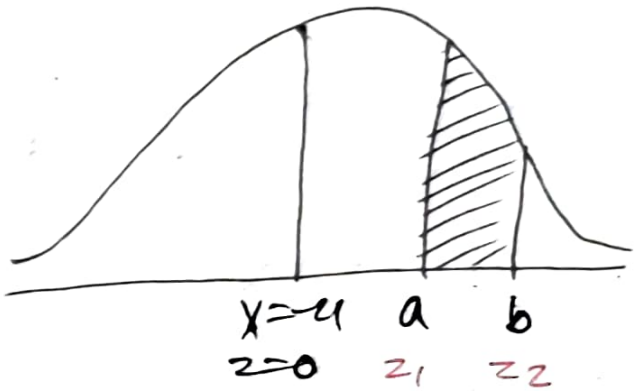
$\therefore P(12 < X < 14) = P(-0.75 < X < -0.25)$

$= P(-0.75 < X < 0) - P(-0.25 < X < 0)$

$= P(0 < X < 0.75) - P(0 < X < 0.25)$ ($\because$ due to symmetry)

$= 0.2734 - 0.0987 \Rightarrow 0.1747$ ans

iii) if $\mu < a < b$



when,

$x = a$, $z = \frac{a-\mu}{\sigma} = z_1$ (say)

when,

$x = b$, $z = \frac{b-\mu}{\sigma} = z_2$ (say)

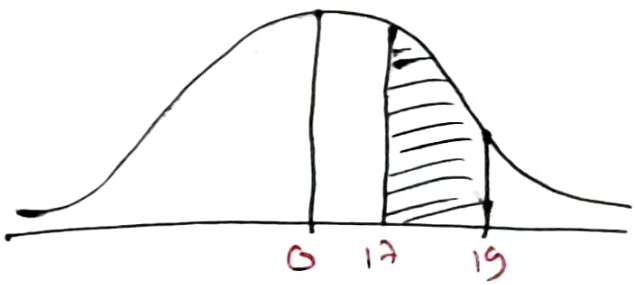
$\therefore P(a < X < b) = P(z_1 < z < z_2)$

$= P(0 < z < z_2) - P(0 < z < z_1)$

Example: Find $P(17 < X < 19)$; $\mu = 15$; $\sigma^2 = 16$

Soln when $x = 17$, $z = \frac{17-15}{4} \Rightarrow 0.5$

When $x = 19$, $z = \frac{19-15}{4} \Rightarrow 1$



$\therefore P(17 < X < 19) = P(0.5 < Z < 1)$

$= P(0 < z < 1) - P(0 < z < 0.5)$

$= 0.3413 - 0.1915 \Rightarrow 0.1498$ ans

The customer accounts of certain department store have an average balance of Rs. 120 and a standard deviation of Rs. 40. Assuming that the account balance are normally distributed find the proportion of the account which is

(i) over Rs. 150

(ii) between Rs. 100 and Rs. 150

(iii) less than Rs. 150

(iv) between Rs. 60 and Rs. 90

(v) between Rs. 130 and Rs. 160

Soln Let X = account balance

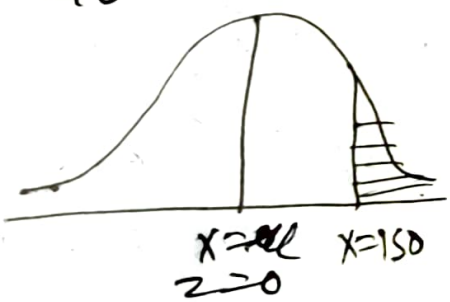
then, $X \sim N(\mu, \sigma^2)$

average balance $(\mu) = 120$

standard deviation $(\sigma) = 40$

(i) $P(X > 150)$

when $X = 150$ then $z = \frac{X-\mu}{\sigma} = \frac{150-120}{40} \Rightarrow 0.75$



$\therefore P(X > 150) = P(Z > 0.75)$

$= 0.5 - P(0 < Z < 0.75)$

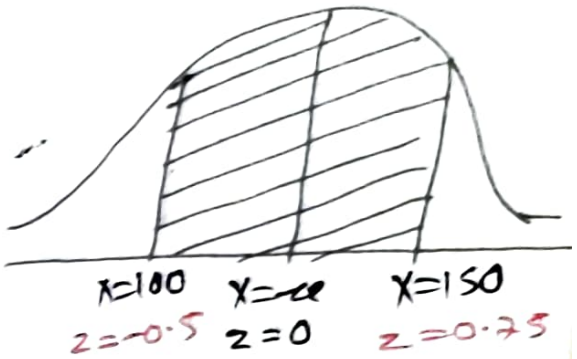
$= 0.5 - 0.2734$

$= 0.2266$ ans

(ii) $P(100 < X < 150) = ?$

when $X = 100$ then $z = \frac{100-120}{40} \Rightarrow -0.5$

when $X = 150$, then $z = \frac{150-120}{40} \Rightarrow 0.75$



$\therefore P(100 < X < 150) = P(-0.5 < z < 0.75)$

$= P(-0.5 < X < 0) + P(0 < X < 0.75)$

$= P(0 < X < 0.5) + P(0 < X < 0.75)$

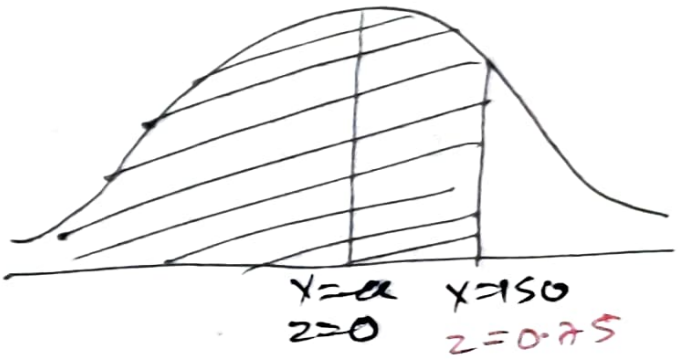
(due to symmetry)

$= 0.1915 + 0.2734 \Rightarrow 0.4649$ ans

(iii) $P(X < 150) = ?$

when $X = 150$ then $z = \frac{150-120}{40}$

$\Rightarrow 0.75$



$\therefore P(X < 150) = P(z < 0.75)$

$= 0.5 + P(0 < z < 0.75)$

$= 0.5 + 0.2734 \Rightarrow 0.7734$ ans

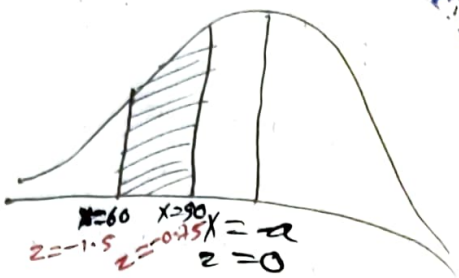
(iv) Soln

$P(60 < z < 90) = ?$

when $X = 60$, $z = \frac{60-120}{40} \Rightarrow -1.5$

when $X = 90$, $z = \frac{90-120}{40} \Rightarrow -0.75$

$\therefore P(60 < X < 90) = P(-1.5 < X < -0.75)$



~~$= P(60$~~

$= P(-1.5 < z < 0) - P(-0.75 < z < 0)$

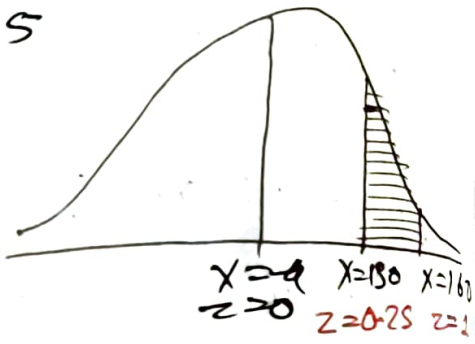
$= P(0 < z < 1.5) - P(0 < z < 0.75)$ (due to symmetrical distn)

$= 0.3531 - 0.2734 \Rightarrow 0.0797$ <u>ans</u>

(v) $P(130 < X < 160) = ?$

when $X = 130$, $z = \dfrac{130 - 120}{4} \Rightarrow 0.25$

when $X = 160$, $z = \dfrac{160 - 120}{4} \Rightarrow 1$



$\therefore P(130 < X < 160) = P(0.25 < X < 1)$

$= P(0 < z < 1) - P(0 < z < 0.25)$

$= 0.3413 - 0.0987$

$= 0.2426$ <u>ans</u>

07/09/17

Sunday

Q. In an intelligence test, the average score of the students is 60 and ~~student~~ standard deviation is 10. Assuming that the test score follows normal distribution, find the probability that:

(i) the student scores more than 50

(ii) between 44 and 74

(iii) the value of score exceeded by top 10% student.

(iv) the maximum value of score obtained by poorest 15% student.

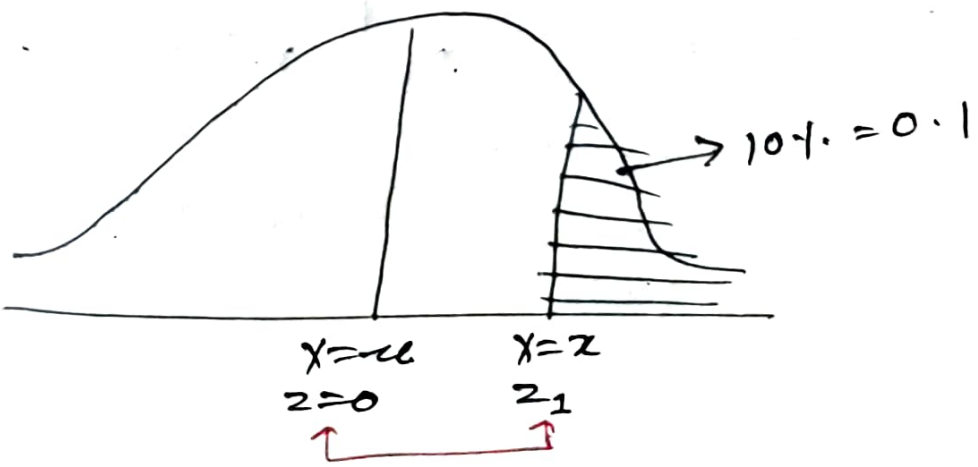
Soln Let X = test score

$$X \sim N(\mu, \sigma^2)$$

Then we have,

$$\mu = 60$$
$$\sigma = 10$$

Define $z = \dfrac{X-\mu}{\sigma} = \dfrac{X-60}{10}$



Let X be the ~~initial~~ value of score exceeded by top 10% students.

i.e. $P(X > x) = 0.1$

when $X = x$, $z = \frac{x-60}{10} = z_1$ (say) —— (i)

$\therefore P(X > x) = 0.1$

$\Rightarrow P(z > z_1) = 0.1$

$\Rightarrow 0.5 - P(0 < z < z_1) = 0.1$

$\Rightarrow P(0 < z < z_1) = 0.40$

$\Rightarrow z_1 = 1.28$

substituting $z_1 = 1.28$ in eqn. (i),

$\frac{x-60}{10} = 1.28 \Rightarrow x - 60 = 12.8$

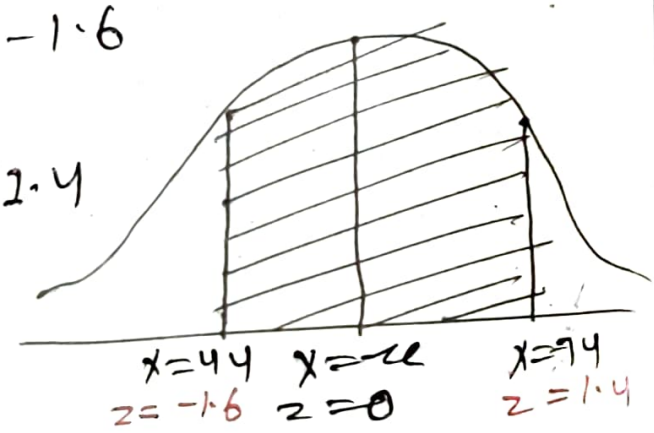
$\therefore x = 72.8$ ans

again

$P(44 < X < 74) = ?$

when $X = 44$, $z = \frac{44-60}{10} \Rightarrow -1.6$

when $x = 74$, $z = \frac{74-60}{10} \Rightarrow 1.4$



$\therefore P(44 < X < 74) = P(-1.6 < z < 1.4)$

$= P(-1.6 < z < 0) + P(0 < z < 1.4)$

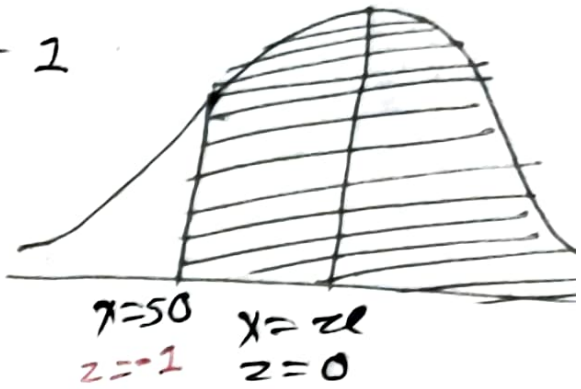
$= P(0 < z < 1.6) + P(0 < z < 1.4)$ ($\because$ due to symmetry)

$= 0.4452 + 0.4192$

$= 0.8644$ ans

$P(X>50) = ?$

when $x=50$, $z = \frac{50-60}{10} \Rightarrow -1$



$P(X>50) = P(z>-1)$

$= 0.5 + P(-1<z<0)$

$= 0.5 + P(0<z<1)$ ($\because$ due to symmetry)

$= 0.5 + 0.3413 \Rightarrow 0.8413$ ans

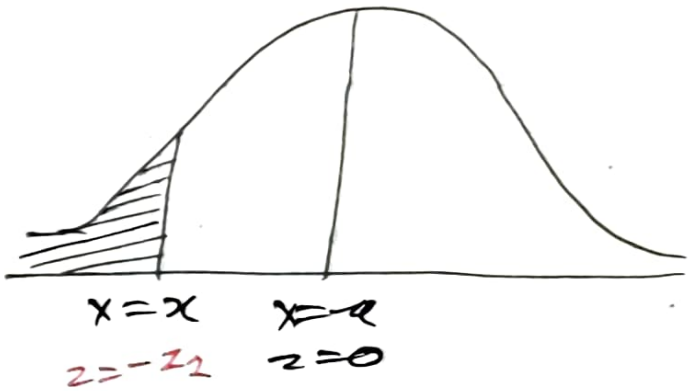
again

Let x be the value of score obtained by poorest 15% student.

i.e. $P(X<x) = 0.15$

when $X=x$, $z = \frac{x-60}{10}$

$= -z_1$ (say) —(i)



$\therefore P(X<x) = 0.15$

$\Rightarrow P(z<-z_1) = 0.15$

$\Rightarrow 0.5 - P(-z_1<z<0) = 0.15$

$\Rightarrow P(-z_1<z<0) = 0.35$

$\Rightarrow P(0<z<z_1) = 0.35$

$\Rightarrow \therefore z_1 = 1.04$

substituting this value of z_1 in eqn (1) we get,

$\frac{x-60}{10} = z_1 \Rightarrow \frac{x-60}{10} = 1.04 \Rightarrow x-60 = -10.4$

$\therefore x = 49.6$ ans

Q. In an intelligence test administrative to 1000 students. The average score of the students in the Test is 42 and standard deviation of 24. Find the number of the students who obtained the scores:

(i) over 50

(ii) between 30 and 54

(iii) The minimum value of scores obtained by top 100 students.

(iv) the maximum value of scores obtained by poorest 200 students.

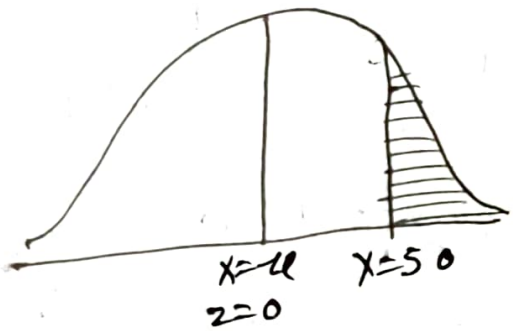
Soln Given,

average score of the student $(\mu) = 42$

standard deviation $(\sigma) = 24$

number of total students $(N) = 1000$

(i) over 50 i.e. $P(X>50) = ?$



when $X=50$, $Z = \frac{50-42}{24} = 0.33$

$\therefore P(X>50) = P(Z>0.33)$

$= 0.5 - P(0<Z<0.33)$

$= 0.5 - 0.1293 = 0.3707$

So,

the number of students = N × required probability

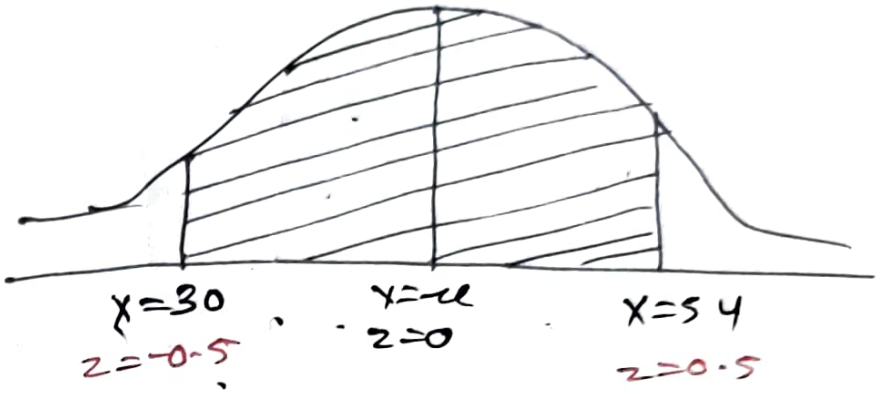
$= 1000 \times 0.3707$

≈ 371 Ans

(ii) $P(30 < X < 54)$

when $X = 30$, $z = \frac{30-42}{24} \Rightarrow -0.5$

when $X = 54$, $z = \frac{54-42}{24} \Rightarrow 0.5$



$\therefore P(30 < X < 54) = P(-0.5 < z < 0.5)$

$= P(-0.5 < z < 0) + P(0 < z < 0.5)$

$= P(0 < z < 0.5) + P(0 < z < 0.5)$ ($\because$ due to symmetry)

$= 0.1915 + 0.1915 \Rightarrow 0.383$

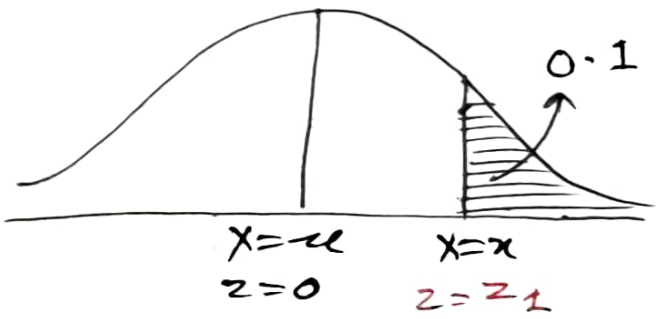
So, no. of students = N x required probability

$= 1000 \times 0.383 \Rightarrow 383$ ans

(iii) $\left(\frac{100}{1000} \times 100\right)\% = 10\% \Rightarrow 0.1$

i.e. $P(X > x) = 0.1$

when $X = x$, $z = \frac{x-42}{24} = z_1$ (say)



So, $P(X > x) = P(z > z_1) = 0.1$

$\Rightarrow 0.5 - P(0 < z < z_1) = 0.1$

$\Rightarrow P(0 < z < z_1) = 0.4$

$\Rightarrow z_1 = 1.28$

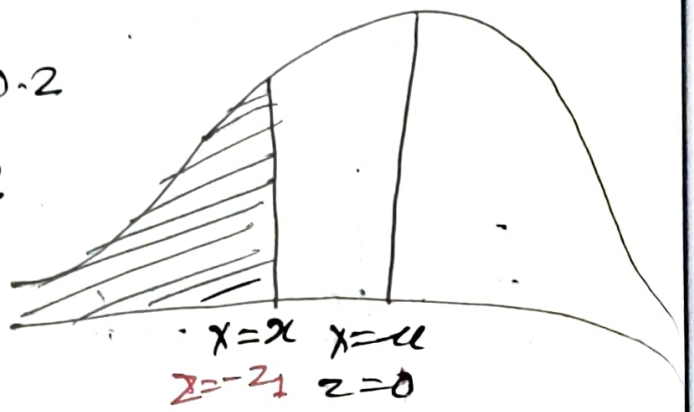
again,

$$\frac{x-42}{24} = 1.28 \Rightarrow x-42 = 30.72$$

$\therefore x = 72.72$ which is the required minimum value of scores obtained by top 100 students.

(iv) $\left(\frac{200}{1000} \times 100\right)\% = 20\% \Rightarrow 0.2$

i.e. $P(X<x) = 0.2$



when,

$$X=x,\ z = \frac{x-42}{24} = -z_1 \text{ (say)}$$

so, $P(X<x) = P(Z<-z_1) = 0.2$

$\Rightarrow 0.5 - P(-z_1 < Z < 0) = 0.2$

$\Rightarrow 0.5 - P(0 < Z < z_1) = 0.2$ ($\because$ due to symmetry)

$\Rightarrow P(0 < Z < z_1) = 0.3$

$\Rightarrow z_1 = 0.84$

Then, $\frac{x-42}{24} = -0.84$

$\Rightarrow x-42 = -20.16$

$\therefore x = 21.84$ which is the required max^m value of scores obtained by ~~top 2~~ poorest 200 students.