Chapter 4

Indeterminate Forms

L'Hospital's Rule

Statement:

Let f(x) and g(x) are two functions such that f(a) = g(a) = 0 and also their derivatives f '(x) and g' (x) are continuous at x = a and g '(a) ≠ 0. Then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}=\frac{f'(a)}{g'(a)}.$$

Exercise 4.1

1. Show that the following limits:

(i) $\lim_{x\to 2}\frac{x^3-2x^2+2x-4}{x^2-5x+6}=-6$

Solution : Here,

$$\lim_{x\to 2}\frac{x^3-2x^2+2x-4}{x^2-5x+6} \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$=\lim_{x\to 2}\frac{3x^2-4x+2}{2x-5}=\frac{3.4-4.2+2}{2\times 2-5}=-6.$$

(ii) $\lim_{x\to 0}\frac{\tan x - x}{x-\sin x}=2$ **[Short, 2004, Spring]**

Solution : Here,

$$\lim_{x\to 0}\frac{\tan x - x}{x-\sin x} \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$=\lim_{x\to 0}\frac{\sec^2 x-1}{1-\cos x}$$

$$=\lim_{x\to 0}\frac{\tan^2 x}{2\sin^2\frac{x}{2}}$$

$$=\lim_{x\to 0}\frac{\frac{\tan^2 x}{x^2}\times x^2}{\frac{2\sin^2\frac{x}{2}}{\frac{x^2}{4}}\times\frac{x^2}{4}}=2.$$

(iii) $\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

Solution : Here,

$$\lim_{x\to a} \frac{x^n - a^n}{x - a} \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to a} \frac{nx^{n-1} - 0}{1} = na^{n-1}.$$

(iv) $\lim_{x\to 0} \frac{x - \sin^{-1}x}{\sin^{3x}} = -\frac{1}{6}$

Solution : Here,

$$\lim_{x\to 0} \frac{x - \sin^{-1}x}{\sin^{3x}} \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{x - \sin^{-1}x}{\left(\frac{\sin^{3x}}{x}\right)^3 . x^3}$$

$$= \lim_{x\to 0} \frac{x - \sin^{-1}x}{x^3} \qquad \left[\text{in } \frac{0}{0} \text{ form}\right] \left(\lim_{x\to 0} \frac{\sin x}{x} = 1\right)$$

$$= \lim_{x\to 0} \frac{1 - \frac{1}{\sqrt{1 - x^2}}}{3x^2} \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{-\frac{1}{2}(1 - x^2)^{-3/2} . 2x}{6x} = \lim_{x\to 0} \frac{-1}{6}(1 - x^2)^{-3/2} = -\frac{1}{6}$$

(v) $\lim_{x\to 0} \frac{2 \sin x - \sin 2x}{\tan^3 x} = 1$

Solution : Here,

$$\lim_{x\to 0} \frac{2 \sin x - \sin 2x}{\tan^3 x}$$

$$= \lim_{x\to 0} \frac{2 \sin x - \sin 2x}{x^3} \qquad \left[\text{in } \frac{0}{0} \text{ form}\right] \left(\lim_{x\to 0} \frac{\tan x}{x} = 1\right)$$

$$= \lim_{x\to 0} \frac{2 \cos x - 2 \cos 2x}{3x^2} \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{-2\sin x + 4\sin 2x}{6x} \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{-2\cos x + 8\cos 2x}{6} = -\frac{2 \cos 0° + 8 \cos 0°}{6} = \frac{6}{6} = 1.$$

(vi) $\lim_{x\to 1} \frac{1 + \log x - x}{1 - 2x + x^2} = -\frac{1}{2}.$

Solution : Here,

$$\lim_{x\to 1} \frac{1 + \log x - x}{1 - 2x + x^2} \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 1} \frac{\frac{1}{x} - 1}{-2 + 2x} \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 1} \frac{-\frac{1}{x^2}}{2} = -\frac{1}{2}.$$

(vii) $\lim_{x\to 0} \frac{x e^x - \log(1 + x)}{x^2} = \frac{3}{2}$

Solution : Here,

$$\lim_{x\to 0} \frac{x e^x - \log(1 + x)}{x^2} \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{e^x + x e^x - \frac{1}{1 + x}}{2x} \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{e^x + e^x + xe^x + \frac{1}{(1 + x)^2}}{2} = \frac{e^0 + e^0 + 0 . e^0 + \frac{1}{(1 + 0)^2}}{2} = \frac{3}{2}$$

(vii) $\lim_{x\to 0} \frac{\cosh x - \cos x}{x \sin x} = 1$

Solution : Here,

$$\lim_{x\to 0} \frac{\cosh x - \cos x}{x \sin x} \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{\sin hx + \sin x}{2x} \qquad \left[\text{in } \frac{0}{0} \text{ form}\right] \left(\lim_{x\to 0} \frac{\sin x}{x} = 1\right)$$

$$= \lim_{x\to 0} \frac{\cos hx + \cos x}{2} = \frac{\cos h0 + \cos 0}{2} = \frac{2}{2} = 1.$$

(ix) $\lim\limits_{t\to 0} \dfrac{\sin t^2}{t} = 0$

Solution : Here,

$$\lim_{t\to 0} \frac{\sin t^2}{t} \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{t\to 0} \frac{\cos t^2 \, . \, 2t}{1} = \cos 0.(2)\, 0 = 0.$$

(x) $\lim\limits_{x\to 0} \dfrac{\sin 5x}{x} = 5.$

Solution : Here,

$$\lim_{x\to 0} \frac{\sin 5x}{x} \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} 5 \cos 5x = 5 \, . \cos 0^\circ = 5.$$

(xi) $\lim\limits_{\theta\to \pi} \dfrac{\sin \theta}{\pi - \theta} = 1$

Solution : Here,

$$\lim_{\theta\to \pi} \frac{\sin \theta}{\pi - \theta} \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{\theta\to \pi} \frac{\cos \theta}{-1} = \frac{\cos \pi}{-1} = \frac{-1}{-1} = 1.$$

(xii) $\lim\limits_{x\to \frac{\pi}{4}} \dfrac{\sin x - \cos x}{x - \frac{\pi}{4}} = \sqrt{2}$ [2007, Fall (Short)]

Solution : Here,

$$\lim_{x\to \frac{\pi}{4}} \frac{\sin x - \cos x}{x - \frac{\pi}{4}} \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to \frac{\pi}{4}} \frac{\cos x + \sin x}{1} = \cos \frac{\pi}{4} + \sin \frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

2. Prove the following:

(i) $\lim\limits_{x\to 0^+} \dfrac{\log \sin x}{\cot x} = 0$

Solution : Here,

$$\lim_{x\to 0^+} \frac{\log (\sin x)}{\cot x} \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to 0^+} \frac{\frac{\cos x}{\sin x}}{(-\operatorname{cosec}^2 x)}$$

$$= \lim_{x\to 0^+} \frac{-\cos x}{\operatorname{cosec} x} = \lim_{x\to 0^+} (-\cos x \sin x)$$

$$= -\cos 0 \, . \sin 0 = 0.$$

(ii) $\lim\limits_{x\to 0^+} x \log x = 0$

Solution : Here,

$$\lim_{x\to 0^+} x \log x \qquad [0 \times \infty]$$

$$= \lim_{x\to 0^+} \frac{\log x}{1/x} \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} \frac{-1}{x^{-1}} = \lim_{x\to 0^+} (-x) = 0.$$

(iii) $\lim\limits_{x\to 0^+} \log_{\tan x} \tan 2x = 1$

Solution : Here,

$$\lim_{x\to 0^+} \log_{\tan x} \tan 2x$$

$$= \lim_{x\to 0^+} \frac{\log \tan 2x}{\log \tan x} \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to 0^+} \frac{\frac{2 \sec^2 2x}{\tan 2x}}{\frac{\sec^2 x}{\tan x}}$$

> Let $\log_x(y) = u$.
> $\Rightarrow \quad y = x^u$.
> So, $\log(y) = u \log(x)$.
> $\Rightarrow u = \dfrac{\log(y)}{\log(x)}$.
> Thus,
> $\log_x(y) = \dfrac{\log(y)}{\log(x)}$.

$$= \lim_{x\to 0^+} \frac{2\sec^2 2x \cdot \dfrac{\cos 2x}{\sin 2x}}{\sec^2 x \cdot \dfrac{\cos x}{\sin x}}$$

$$\frac{2\sec 2x \cdot \frac{1}{\sin 2x}}{\sec x \cdot \frac{1}{\sin x}}$$

$$= \lim_{x\to 0^+} \frac{\sin 2x}{\cos 2x \cdot \sin 2x}$$

$$= \frac{1}{\cos 0^\circ} = 1.$$

(iv) $\displaystyle \lim_{x\to a} (a-x)\cdot\tan\left(\frac{\pi x}{2a}\right) = \frac{2a}{\pi}$

Solution : Here,

$$\lim_{x\to a} (a-x)\tan\left(\frac{\pi x}{2a}\right) \quad (\text{in } 0\times\infty \text{ form})$$

$$= \lim_{x\to a} \frac{\tan\left(\frac{\pi x}{2a}\right)}{\frac{1}{(a-x)}} \quad \left(\text{in } \frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x\to a} \frac{\sec^2\left(\frac{\pi x}{2a}\right)\cdot\left(\frac{\pi}{2a}\right)}{\frac{1}{(a-x)^2}}$$

$$= \left(\frac{\pi}{2a}\right) \lim_{x\to a} \frac{(a-x)^2}{\cos^2\left(\frac{\pi x}{2a}\right)} \quad \left(\text{in } \frac{0}{0} \text{ form}\right)$$

$$= \left(\frac{\pi}{2a}\right)\cdot \lim_{x\to a} \frac{-2(a-x)}{-2\cos\left(\frac{\pi x}{2a}\right)\sin\left(\frac{\pi x}{2a}\right)\cdot\left(\frac{\pi}{2a}\right)}$$

$$= 2\cdot \lim_{x\to a} \frac{(a-x)}{\sin\left(\frac{\pi x}{a}\right)} \quad \left(\text{in } \frac{0}{0} \text{ form}\right) \quad [2\sin A\cos A = \sin 2A]$$

$$= 2 \lim_{x\to a} \frac{-1}{\cos\left(\frac{\pi x}{a}\right)\cdot\frac{\pi}{a}} = 2\cdot\frac{-1}{\cos\pi\cdot\frac{\pi}{a}} = \frac{2a}{\pi}.$$

(v) $\displaystyle \lim_{x\to\infty} \frac{x^n}{e^x} = 0$

Solution : Here,

$$\lim_{x\to\infty} \frac{x^n}{e^x} \quad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to\infty} \frac{n.x^{n-1}}{e^x} \quad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to\infty} \frac{n(n-1)\,x^{n-2}}{e^x} \quad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

Continuing the process up to n^{th} times then,

$$= \lim_{x\to\infty} \frac{n(n-1)\ldots 2.1\,x^0}{e^x} = \frac{n!}{e^\infty} = 0 \quad \left[\text{because } \infty \approx \frac{1}{0}\right]$$

(vi) $\displaystyle \lim_{x\to 0} \frac{\log(x^2)}{\log(\cot^2 x)} = -1.$

Solution : Here,

$$\lim_{x\to 0} \frac{\log(x^2)}{\log(\cot^2 x)} \quad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{\frac{2x}{x^2}}{\frac{1}{\cot^2 x}\cdot 2\cot x\,(-\operatorname{cosec}^2 x)}$$

$$= \lim_{x\to 0} -\frac{\cot x\cdot\sin^2 x}{x}$$

$$= \lim_{x\to 0} \frac{-\cos x.\sin x}{x} = \lim_{x\to 0}(-\cos x).1 \left(\lim_{x\to 0}\frac{\sin x}{x} = 1\right)$$

$$= -1.$$

(vii) $\displaystyle \lim_{x\to 0} x\log(\sin^2 x) = 0.$

Solution : Here,

$$\lim_{x\to 0} x\log(\sin^2 x) \quad (\text{in } 0\times\infty \text{ form})$$

$$= \lim_{x\to 0} \frac{\log(\sin^2 x)}{(1/x)} \quad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{\frac{2\sin x.\cos x}{\sin^2 x}}{-1/x^2}$$

$$= \lim_{x\to 0} \frac{-2\cos x.\, x^2}{\sin x}$$

$$= \lim_{x\to 0} \frac{-2x^2}{\tan x}$$

$$= -2 \lim_{x\to 0} x \qquad \left(\lim_{x\to 0} \frac{\tan x}{x} = 1\right)$$

$$= -2\times 0 \quad = 0.$$

(viii) $\lim_{x\to \frac{\pi}{2}} \sec\left(x\sin x - \frac{\pi}{2}\right) = -1.$

Solution : Here,

$$\lim_{x\to \frac{\pi}{2}} \sec\left(x\sin x - \frac{\pi}{2}\right) \qquad [\text{in } \infty\times 0 \text{ form}]$$

$$= \lim_{x\to \frac{\pi}{2}} \frac{x\sin x - \frac{\pi}{2}}{\cos x} \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to \frac{\pi}{2}} \frac{\sin x + x\cos x}{-\sin x} = \frac{\sin\frac{\pi}{2} + 0\,.\cos\frac{\pi}{2}}{-\sin\frac{\pi}{2}} = -1.$$

(ix) $\lim_{x\to 0^+} x^m (\log x)^n = 0,$ m and n being position.

Solution : Here,

$$\lim_{x\to 0^+} \frac{(\log x)^n}{\frac{1}{x^m}} \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to 0^+} \frac{n(\log x)^{n-1}\cdot\frac{1}{x}}{-m\,x^{-m-1}}$$

$$= \lim_{x\to 0^+} \frac{n(\log x)^{n-1}}{-m\,x^{-m}} \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to 0^+} \frac{n(n-1)(\log x)^{n-2}\cdot\frac{1}{x}}{(-m)(-m)\,x^{-m-1}}$$

$$= \lim_{x\to 0^+} \frac{n(n-1)(\log x)^{n-2}}{(-m)^2 x^{-m}} \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

Continuing the process upto n times then,

$$= \lim_{x\to 0^+} \frac{n(n-1)\ldots.2.1(\log x)^0}{(-m)^n x^{-m}}$$

$$= \frac{n!}{(-m)^n} \lim_{x\to 0^+} \frac{1}{x^{-m}}$$

$$= \frac{n!}{(-m)^n} \lim_{x\to 0^+} x^{+m} = \frac{n!}{(-m)^n}\cdot 0 \quad = 0.$$

3. Prove the following limits:

(i) $\lim_{x\to 0}\left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right) = -\frac{1}{3}$

Solution : Here,

$$\lim_{x\to 0}\left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right) \qquad (\text{in } \infty - \infty \text{ form})$$

$$= \lim_{x\to 0}\left(\frac{\sin^2 x - x^2}{x^4\left(\frac{\sin^2 x}{x^2}\right)}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sin^2 x - x^2}{x^4}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right] \quad \left(\lim_{x\to 0} \frac{\sin x}{x} = 1\right)$$

$$= \lim_{x\to 0}\left(\frac{\sin 2x - 2x}{4x^3}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{2\cos 2x - 2}{12x^2} \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{-4\sin 2x}{24x} \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{-8\cos 2x}{24} = -\frac{1}{3}$$

(ii) $\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{e^x - 1}\right)$

Solution: Here,

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right) \qquad (\text{in } \infty-\infty \text{ form})$$

$$=\lim_{x\to 0}\left(\frac{e^x-1-x}{xe^x-x}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$=\lim_{x\to 0}\frac{e^x-1}{e^x+xe^x-1} \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$=\lim_{x\to 0}\frac{e^x}{e^x+e^x+xe^x} = \frac{e^0}{e^0+e^0+0\,.\,e^0} = \frac{1}{1+1} = \frac{1}{2}$$

(iii) $\lim_{x\to 0}\left(\frac{1}{x^2}-\cot^2 x\right)=\frac{2}{3}$

Solution : Here,

$$\lim_{x\to 0}\left(\frac{1}{x^2}-\cot^2 x\right) \qquad (\text{in } \infty-\infty \text{ form})$$

$$=\lim_{x\to 0}\left(\frac{\tan^2 x-x^2}{x^2\tan^2 x}\right)$$

$$=\lim_{x\to 0}\left(\frac{\tan^2 x-x^2}{x^4}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right] \left(\lim_{x\to 0}\frac{\tan x}{x}=1\right)$$

$$=\lim_{x\to 0}\left(\frac{2\tan x\,.\,\sec^2 x-2x}{4x^3}\right)$$

$$=\lim_{x\to 0}\left(\frac{2\tan x\,(1+\tan^2 x)-2x}{4x^3}\right)$$

$$=\lim_{x\to 0}\frac{2\tan x+2\tan^3 x-2x}{4x^3} \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$=\lim_{x\to 0}\frac{2\sec^2 x+6\tan^2 x\,.\,\sec^2 x-2}{12x^2}$$

$$=\lim_{x\to 0}\frac{2+2\tan^2 x+6\tan^2 x+6\tan^4 x-2}{12x^2}$$

$$=\lim_{x\to 0}\frac{\tan^2 x}{x^2}\left(\frac{8+6\tan^2 x}{12}\right)$$

$$=\lim_{x\to 0}\left(\frac{8+6\tan^2 x}{12}\right) \qquad \left(\lim_{x\to 0}\frac{\tan x}{x}=1\right)$$

$$=\frac{8+6\tan 0}{12}$$

$$=\frac{8}{12}=\frac{2}{3}.$$

(iv) $\lim_{x\to 0}\left[\frac{1}{x}-\frac{1}{x^2}\log(1+x)\right]=\frac{1}{2}$

Solution : Here,

$$\lim_{x\to 0}\left[\frac{1}{x}-\frac{1}{x^2}\log(1+x)\right] \qquad (\text{in } \infty-\infty \text{ form})$$

$$=\lim_{x\to 0}\left[\frac{x-\log(1+x)}{x^2}\right] \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$=\lim_{x\to 0}\left(\frac{1-\frac{1}{1+x}}{2x}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$=\lim_{x\to 0}\left(\frac{\frac{1}{(1+x)^2}}{2}\right) = \frac{1}{2}.$$

(v) $\lim_{x\to 0^+} x^x = 1$

Solution : Let $y = x^x$

Taking log on both sides, we get

$\log y = x \log x$

Taking limit $\lim_{x\to 0^+}$ on both sides, we get

$$\lim_{x\to 0^+}\log y = \lim_{x\to 0^+} x\log x \qquad (\text{in } 0\times\infty \text{ form})$$

$$=\lim_{x\to 0^+}\frac{\log x}{\left(\frac{1}{x}\right)} \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$=\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

$$=\lim_{x\to 0^+}(-x) = 0.$$

Hence, $\lim_{x\to 0^+}\log y = 0 \Rightarrow \lim_{x\to 0^+} y = e^0 = 1.$

$\therefore \lim_{x\to 0^+} x^x = 1.$

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(vi) $\lim\limits_{x\to\frac{\pi}{2}} (\sin x)^{\tan x} = 1$

Solution : Let $y = (\sin x)^{\tan x}$

Taking log on both sides, we get

$$\log y = \log (\sin x)^{\tan x}$$

$$\log y = \tan x \,.\, \log (\sin x)$$

Taking $\lim\limits_{x\to\frac{\pi}{2}}$ on both sides then we get,

$$\lim_{x\to\frac{\pi}{2}} \log y = \lim_{x\to\frac{\pi}{2}} \tan x \log (\sin x) \quad (\text{in } 0\times\infty \text{ form})$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\log(\sin x)}{\cot x} \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\frac{\cos x}{\sin x}}{-\operatorname{cosec}^2 x} = \lim_{x\to\frac{\pi}{2}} (-\cos x \sin x) = 0.$$

Now, $\lim\limits_{x\to\frac{\pi}{2}} \log y = 0 \Rightarrow \lim\limits_{x\to\frac{\pi}{2}} y = e^0 = 1.$

$\therefore \lim\limits_{x\to\frac{\pi}{2}} (\sin x)^{\tan x} = 1.$

(vii) $\lim\limits_{x\to 0} (\cot x)^{\sin 2x} = 1$

Solution : Let $y = (\cot x)^{\sin 2x}$

Taking log both sides, we get

$$\log y = \sin 2x \log (\cot x)$$

Taking $\lim\limits_{x\to 0}$ on both sides

$$\lim_{x\to 0} \log y = \lim_{x\to 0} \sin 2x \log (\cot x) \quad (\text{in } 0\times\infty \text{ form})$$

$$= \lim_{x\to 0} \frac{\log(\cot x)}{\operatorname{cosec} 2x} \quad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{\frac{1}{\cot x}(-\operatorname{cosec}^2 x)}{-2 \operatorname{cosec} 2x \,.\, \cot 2x}$$

$$= \lim_{x\to 0} \frac{\frac{\sin x}{\cos x} \times \operatorname{cosec}^2 x}{2 \operatorname{cosec} 2x \,.\, \cot 2x}$$

$$= \lim_{x\to 0} \frac{1}{2 \cos x \sin x \,.\, \operatorname{cosec} 2x \,.\, \cot 2x}$$

$$= \lim_{x\to 0} \frac{+1}{\sin 2x \,.\, \operatorname{cosec} 2x \,.\, \cot 2x}$$

$$= \lim_{x\to 0} \frac{+1}{\cot 2x} = \frac{1}{\infty} = 0.$$

Now, $\lim\limits_{x\to 0} \log y = 0 \Rightarrow \lim\limits_{x\to 0} y = e^0 = 1.$

$\therefore \lim\limits_{x\to 0} (\cot x)^{\sin 2x} = 1.$

(viii) $\lim\limits_{x\to\pi^-} (\sin x)^{\tan x} = 1.$

Solution : Let $y = (\sin x)^{\tan x}$

Taking log both sides, we get

$$\log y = \tan x \,.\, \log (\sin x)$$

Taking $\lim\limits_{x\to\pi^-}$ on both sides,

$$\lim_{x\to\pi^-} \log y = \lim_{x\to\pi^-} \tan x \,.\, \log (\sin x) \quad (\text{in } 0\times\infty \text{ form})$$

$$= \lim_{x\to\pi^-} \frac{\log(\sin x)}{\cot x} \quad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to\pi^-} \frac{\frac{\cos x}{\sin x}}{-\operatorname{cosec}^2 x}$$

$$= \lim_{x\to\pi^-} (-\cos x \sin x) = \lim_{x\to\pi^-} -\frac{1}{2} \,.\, \sin 2x$$

$$= -\frac{1}{2} \,.\, \sin 2\pi = -\frac{1}{2} \times 0 = 0.$$

Now,

$$\lim_{x\to\pi^-} \log y = 0 \Rightarrow \lim_{x\to\pi^-} y = e^0 = 1.$$

$$\therefore \lim_{x\to\pi^-} (\sin x)^{\tan x} = 1.$$

(ix) $\lim_{x\to 0} (\cot^2 x)^{\sin x} = 1$

Solution : Let $y = (\cot^2 x)^{\sin x}$

Taking log both sides, we get

$$\log y = \sin x \,.\, \log(\cot^2 x)$$

Taking $\lim_{x\to 0} \log y = \lim_{x\to 0} (\sin x \,.\, \log(\cot^2 x))$ (in $0\times\infty$ form)

$$= \lim_{x\to 0} \frac{\log(\cot^2 x)}{\operatorname{cosec} x} \quad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{\frac{1}{\cot^2 x} \,.\, 2\cot x \,.\, -\operatorname{cosec}^2 x}{-\operatorname{cosec} x \,.\, \cot x}$$

$$= \lim_{x\to 0} \frac{2}{\cot^2 x} = \frac{2}{\infty} = 0.$$

Now,

$$\lim_{x\to 0} \log y = 0 \Rightarrow \lim_{x\to 0} y = e^0 = 1.$$

$$\therefore \lim_{x\to 0} (\cot^2 x)^{\sin x} = 1.$$

(x) $\lim_{x\to 0} \left(\frac{1}{x^2}\right)^{\tan x} = 1.$ **[2001] [1999]**

Solution : Let $y = \left(\frac{1}{x^2}\right)^{\tan x}$

Taking log both sides, we get

$$\log y = \tan x \log\left(\frac{1}{x^2}\right)$$

Taking limit $\lim_{x\to 0}$ on both sides

$$\lim_{x\to 0} \log y = \lim_{x\to 0} \tan x \,.\, \log\left(\frac{1}{x^2}\right) \quad (\text{in } 0\times\infty \text{ form})$$

$$= \lim_{x\to 0} \frac{\log\left(\frac{1}{x^2}\right)}{\cot x} \quad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{\frac{1}{1/x^2} \,.\, \frac{-2}{x^3}}{-\operatorname{cosec}^2 x}$$

$$= \lim_{x\to 0} \frac{2\sin^2 x}{x} = \lim_{x\to 0} 2\sin x \,.\, \frac{\sin x}{x}$$

$$= 2\sin 0^\circ = 0.$$

Now,

$$\lim_{x\to 0} \log y = 0 \Rightarrow \lim_{x\to 0} y = e^0 = 1.$$

$$\therefore \lim_{x\to 0} \left(\frac{1}{x^2}\right)^{\tan x} = 1.$$

(xi) $\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x} = 1$ **[2007, Spring]**

Solution : Let $y = \left(\frac{\tan x}{x}\right)^{1/x}$

Taking log both sides

$$\log y = \frac{1}{x} \log\left(\frac{\tan x}{x}\right)$$

Taking $\lim_{x\to 0}$ both sides, we get

$$\lim_{x\to 0} \log y = \lim_{x\to 0} \left[\frac{1}{x} \log\left(\frac{\tan x}{x}\right)\right] \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{x}{\tan x} \frac{(x\sec^2 x - \tan x)}{x^2}$$

$$= \lim_{x\to 0} \frac{x\sec^2 x - \tan x}{x^2} \left[\text{in } \frac{0}{0} \text{ form}\right] \left(\lim_{x\to 0} \frac{\tan x}{x} = 1\right)$$

$$= \lim_{x\to 0} \frac{\sec^2 x + 2x\sec^2 x \,.\, \tan x - \sec^2 x}{2x}$$

$$= \lim_{x\to 0} \frac{2x\sec^2 x \,.\, \tan x}{2x} = \lim_{x\to 0} \sec^2 x \,.\, \tan x$$

$$= 1 \,.\, 0 = 0.$$

Now,

$$\lim_{x\to0}\log y = 0 \Rightarrow \lim_{x\to0} y = e^0 = 1.$$

$$\Rightarrow \lim_{x\to0}\left(\frac{\tan x}{x}\right)^{1/x} = 1.$$

(xii) $\lim_{x\to0}\left(\frac{\sin x}{x}\right)^{1/x^2} = e^{-1/6}$ **[2008, Spring] [2006, Spring] [2005, Spring] [2004, Fall] [2009, Fall]**

Solution : Let $y = \lim_{x\to0}\left(\frac{\sin x}{x}\right)^{1/x^2}$

Taking log both sides

$$\log y = \log\left(\frac{\sin x}{x}\right)^{1/x^2}$$

$$\log y = \frac{1}{x^2}\log\left(\frac{\sin x}{x}\right)$$

Taking $\lim_{x\to0}$ on both sides,

$$\lim_{x\to0}\log y = \lim_{x\to0}\frac{1}{x^2}\log\left(\frac{\sin x}{x}\right) \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to0}\frac{\frac{x}{\sin}\left(\frac{x\cos x - \sin x}{x^2}\right)}{2x}$$

$$= \lim_{x\to0}\left(\frac{x\cos x - \sin x}{2x^3}\right)\left[\text{in } \frac{0}{0} \text{ form}\right]\left(\lim_{x\to0}\frac{\sin x}{x} = 1\right)$$

$$= \lim_{x\to0}\frac{\cos x - x\sin x - \cos x}{6x^2}$$

$$= \lim_{x\to0}\frac{-x\sin x}{6x^2}$$

$$= \lim_{x\to0}\frac{-\sin x}{6x}$$

$$= -\frac{1}{6} \qquad \left(\lim_{x\to0}\frac{\sin x}{x} = 1\right)$$

Now,

$$\lim_{x\to0}\log y = -\frac{1}{6} \Rightarrow \lim_{x\to0} y = e^{-1/6}$$

$$\therefore \lim_{x\to0}\left(\frac{\sin x}{x}\right)^{1/x^2} = e^{-1/6}$$

(xii) $\lim_{x\to0^+}(\cot x)^{1/\log x} = e^{-1}$

Solution : Let $y = (\cot x)^{1/\log x}$

Taking log both sides,

$$\log y = \frac{1}{\log x}\log(\cot x)$$

Taking $\lim_{x\to0^+}$ on both sides, we get

$$\lim_{x\to0^+}\log y = \lim_{x\to0^+}\frac{1}{\log x}\log(\cot x) \quad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to0^+}\frac{\frac{1}{\cot x}(-\operatorname{cosec}^2 x)}{1/x}$$

$$= \lim_{x\to0^+}\frac{-x}{\cos x.\sin x}$$

$$= \lim_{x\to0^+}\frac{-1}{\cos x}\left(\lim_{x\to0}\frac{\sin x}{x} = 1\right)$$

$$= -\frac{1}{\cos 0^\circ} = -1.$$

Now,

$$\lim_{x\to0^+} y = e^{-1} \Rightarrow \lim_{x\to0^+}(\cot x)^{1/\log x} = e^{-1}.$$

(xiv) $\lim_{x\to\frac{\pi}{2}}(\sec x - \tan x) = 0$

Solution : Here,

$$\lim_{x\to\frac{\pi}{2}}(\sec x - \tan x) \qquad (\text{in } \infty - \infty \text{ form})$$

$$= \lim_{x\to\frac{\pi}{2}}\left(\frac{1-\sin x}{\cos x}\right) \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{-\cos x}{-\sin x} = \lim_{x \to \frac{\pi}{2}} \cot x = 0.$$

(xv) $\lim_{x \to 0} (x^{-1} - \cot x) = 0$

Solution : Here,

$$\lim_{x \to 0} (x^{-1} - \cot x) \qquad (\text{in } \infty - \infty \text{ form})$$

$$= \lim_{x \to 0} \frac{1 - x\cot x}{x}$$

$$= \lim_{x \to 0} \frac{\tan x - x}{x^2 \frac{\tan x}{x}}$$

$$= \lim_{x \to 0} \frac{\tan x - x}{x^2} \qquad \left[\text{in } \frac{0}{0} \text{ form}\right] \left(\lim_{x \to 0} \frac{\tan x}{x} = 1\right)$$

$$= \lim_{x \to 0} \frac{\sec^2 x - 1}{2x} = \lim_{x \to 0} \frac{\tan^2 x}{2x^2} . x = \lim_{x \to 0} \frac{x}{2} = 0.$$

(xvi) $\lim_{x \to 1} \left[\frac{x}{x-1} - \frac{1}{\log x}\right] = \frac{1}{2}$

Solution: Here,

$$\lim_{x \to 1} \left[\frac{x}{x-1} - \frac{1}{\log x}\right] \qquad (\text{in } \infty - \infty \text{ form})$$

$$= \lim_{x \to 1} \left[\frac{x\log x - x + 1}{(x-1)\log x}\right] \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 1} \left[\frac{\log x + 1 - 1}{\log x + (x-1) \times \frac{1}{x}}\right]$$

$$= \lim_{x \to 1} \frac{x\log x}{x\log x + x - 1} \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 1} \frac{\log x + 1}{\log x + 1 + 1} = \frac{1}{2}.$$

(xvii) $\lim_{x \to 2} \left[\frac{4}{x^2 - 4} - \frac{1}{x-2}\right] = -\frac{1}{4}$

Solution: Here,

$$\lim_{x \to 2} \left[\frac{4}{x^2 - 4} - \frac{1}{x-2}\right] \qquad (\text{in } \infty - \infty \text{ form})$$

$$= \lim_{x \to 2} \left[\frac{4}{(x-2)(x+2)} - \frac{1}{x-2}\right]$$

$$= \lim_{x \to 2} \left[\frac{4 - (x+2)}{(x-2)(x+2)}\right]$$

$$= \lim_{x \to 2} \left[\frac{4 - x - 2}{x^2 - 4}\right]$$

$$= \lim_{x \to 2} \left(\frac{2 - x}{x^2 - 4}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 2} \frac{-1}{2x} = -\frac{1}{4}.$$

(xviii) $\lim_{x \to \infty} (x - \sqrt{x^2 - 9}) = 0.$

Solution: Here,

$$\lim_{x \to \infty} (x - \sqrt{x^2 - 9})$$

$$= \lim_{x \to \infty} \left[\left(x - \sqrt{x^2 - 9} \times \frac{x + \sqrt{x^2 + 9}}{(x + \sqrt{x^2 - 9})}\right)\right]$$

$$= \lim_{x \to \infty} \frac{x^2 - x^2 + 9}{x + \sqrt{x^2 - 9}}$$

$$= \lim_{x \to \infty} \frac{9}{x + x\left(\sqrt{1 - \frac{9}{x^2}}\right)}$$

$$= \lim_{x \to \infty} \frac{9}{x\left(1 + \sqrt{1 - \frac{9}{x^2}}\right)} = \frac{9}{\infty\left(1 + \sqrt{\frac{1-9}{\infty^2}}\right)} = 0.$$

(xix) $\lim_{x \to \infty} [\sqrt{x^2 + 2x} - x] = 1.$

Solution: Here,

$$\lim_{x \to \infty} [\sqrt{x^2 + 2x} - x]$$

$$= \lim_{x \to \infty} \left[\sqrt{x^2 + 2x} - x \times \frac{(\sqrt{x^2 + 2x} + x)}{(\sqrt{x^2 + 2x} + x)}\right]$$

$$= \lim_{x\to\infty} \frac{x^2+2x-x^2}{\sqrt{x^2+2x}+x}$$

$$= \lim_{x\to\infty} \frac{2x}{\sqrt{x^2+2x}+x} = \lim_{x\to\infty} \frac{2x}{x\left(\sqrt{1+\frac{2}{x}}+1\right)}$$

$$= \frac{2}{\sqrt{1+\frac{2}{\infty}}+1} = \frac{2}{1+1} = 1.$$

(xxi) $\lim_{x\to\infty} \frac{\sqrt{3x+4}}{\sqrt{2x+3}} = \sqrt{\frac{3}{2}}$

Solution : Here,

$$\lim_{x\to\infty} \frac{\sqrt{3x-4}}{\sqrt{2x-3}} = \lim_{x\to\infty} \frac{\sqrt{x}\sqrt{3-\frac{4}{x}}}{\sqrt{x}\sqrt{2-\frac{3}{x}}} = \frac{\sqrt{3-\frac{4}{\infty}}}{\sqrt{2-\frac{3}{\infty}}} = \sqrt{\frac{3}{2}}$$

(xxii) $\lim_{x\to 0} x^{2x} = 1.$

Solution : Let $y = x^{2x}$

Taking log both side, we get

$$\text{Log}(y) = 2x \log x$$

Taking $\lim_{x\to 0}$ on both sides,

$$\lim_{x\to 0} \log y = \lim_{x\to 0} 2x \log x \quad [\text{ in } 0\times\infty \text{ form}]$$

$$= 2 \lim_{x\to 0} \frac{\log x}{(1/x)} \quad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= 2 \lim_{x\to 0} \left[\frac{1/x}{-1/x^2}\right] = -2 \lim_{x\to 0} \frac{1}{x} = 2\times 0 = 0.$$

Now, $\lim_{x\to 0} \log y = 0 \Rightarrow \lim_{x\to 0} y = e^0 = 1$

$\therefore \lim_{x\to 0} x^{2x} = 1.$

(xxii) $\lim_{x\to 0} x^{2\sin x} = 1$

Solution : Let $y = x^{2\sin x}$

Taking log both sides, we get

$$\log y = 2 \sin x \log x$$

Taking $\lim_{x\to 0}$ on both sides, we get

$$\log y = 2 \sin x \log$$

Taking $\lim_{x\to 0}$ on both sides

$$\lim_{x\to 0} \log y = \lim_{x\to 0} 2 \sin x \log x \quad [0\times\infty]$$

$$= 2 \lim_{x\to 0} \frac{x \sin x}{x} . \log x$$

$$= 2 \lim_{x\to 0} \frac{\log x}{(1/x)} \quad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right] \left(\lim_{x\to 0} \frac{\sin x}{x} = 1\right)$$

$$= 2 \lim_{x\to 0} \frac{1/x}{-1/x^2}$$

$$= 2 \lim_{x\to 0} (-x) = 0.$$

Now,

$$\lim_{x\to 0} \log y = 0 \Rightarrow \lim_{x\to 0} y = e^0 = 1$$

$\therefore \lim_{x\to 0} x^{2\sin x} = 1.$

(xxiii) $\lim_{x\to 0} (\sin x)^{2\tan x} = 1$

Solution : Let $y = (\sin x)^{2\tan x}$

Taking log both sides, we get

$$\log y = 2 \tan x \log (\sin x)$$

Taking $\lim_{x\to 0} \log y = \lim_{x\to 0} 2 \tan x \log (\sin x) \quad [\text{ in } 0\times\infty \text{ form}]$

$$= 2 \lim_{x\to 0} \frac{\log (\sin x)}{\cot x} \quad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= 2\lim_{x\to 0}\frac{\frac{\cos x}{\sin x}}{(-\operatorname{cosec}^2 x)} = 2\lim_{x\to 0}(-\cos x \sin x)$$
$$= -2\times 0 = 0.$$

Now, $\lim_{x\to 0}\log y = 0 \Rightarrow \lim_{x\to 0} y = e^0 = 1.$

$\therefore \lim_{x\to 0}(\sin y)^{2\tan x} = 1.$

(xxiv) $\lim_{x\to 1} x^{(1/1-x)} = \frac{1}{e}$

Solution : Let $y = x^{1/1-x}$

Taking log both sides, we get

$$\log y = \frac{1}{1-x}\log x$$

Taking $\lim_{x\to 1}$ on both sides, we get

$$\lim_{x\to 1}\log y = \lim_{x\to 1}\left(\frac{1}{1-x}\right)\log x \quad [\text{ in } 0\times\infty \text{ form}]$$
$$= \lim_{x\to 1}\left(\frac{\log(x)}{1-x}\right) \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$
$$= \lim_{x\to 1}\frac{1/x}{-1} = -1.$$

Now,

$$\lim_{x\to 0}\log y = -1 \Rightarrow \lim_{x\to 0} y = e^{-1} = \frac{1}{e}.$$

$$\therefore \lim_{x\to 0} x^{1/(1-x)} = \frac{1}{e}.$$

(xxv) $\lim_{x\to\infty}(\log x)^{1/x} = 1.$

Solution : Let $y = (\log x)^{1/x}$

Taking log both sides, we get

$$\log y = \frac{\log x}{x}$$

Taking $\lim_{x\to\infty}$ on both sides,

$$\lim_{x\to\infty}\log y = \lim_{x\to\infty}\frac{\log x}{x} \quad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$
$$= \lim_{x\to\infty}\frac{1/x}{1} = 0.$$

Now,

$$\lim_{x\to\infty}\log y = 0 \Rightarrow \lim_{x\to\infty} y = e^0 = 1.$$

$$\therefore \lim_{x\to\infty}\frac{\log x}{x} = e.$$

(xxvi) $\lim_{x\to 0}(e^x + x)^{1/x} = e^2$ **[2009 Spring]**

Solution : Put $y = (e^x + x)^{1/x}$

Taking log on both sides then,

$$\log y = \frac{1}{x}\log(e^x + x) = \frac{\log(e^x + x)}{x}$$

Taking $\lim_{x\to 0}$ on both sides then,

$$\log\left(\lim_{x\to 0} y\right) = \lim_{x\to 0}(\log y) = \lim_{x\to 0}\frac{\log(e^x + x)}{x} \quad \left(\text{in } \frac{0}{0} \text{ form}\right)$$
$$= \lim_{x\to 0}\frac{1}{e^x + x}\cdot(e^x + 1)$$
$$= \frac{1+1}{1+0} = \frac{2}{1} = 2.$$

Thus

$$\log\left(\lim_{x\to 0} y\right) = \lim_{x\to 0}(\log y) = 2.$$
$$\Rightarrow \left(\lim_{x\to 0} y\right) = e^2.$$
$$\Rightarrow \lim_{x\to 0}(e^x + x)^{1/x} = e^2.$$

(xvii) $\lim_{x\to\infty}\left[1 + \frac{1}{x^2}\right]^x = 1$

Solution :

Let $y = \left[1 + \frac{1}{x^2}\right]^x$

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Taking log both sides, we get

$$\log y = x \log\left(1 + \frac{1}{x^2}\right)$$

Taking $\lim\limits_{x\to\infty}$ on both sides

$$\lim_{x\to\infty} \log y = \lim_{x\to\infty} x \log\left(1 + \frac{1}{x^2}\right) \quad [\text{in } \infty \times 0 \text{ form}]$$

$$= \lim_{x\to\infty} \frac{\log\left(1 + \frac{1}{x^2}\right)}{\left(\frac{1}{x}\right)} \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x^2}} \cdot \left(-\frac{2}{x^3}\right)}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{2}{x\left(1 + \frac{1}{x^2}\right)}$$

$$= \lim_{x\to\infty} \frac{2x}{x^2+1} \quad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to\infty} \frac{2}{2x} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

Now,

$$\lim_{x\to\infty} \log y = 0 \Rightarrow \lim_{x\to\infty} y = e^0 = 1.$$

$$\therefore \lim_{x\to\infty} \left[1 + \frac{1}{x^2}\right]^x = 1.$$

(xxviii) $\lim\limits_{x\to 0} \left(\frac{\sin x}{x}\right)^{1/x} = 1$ **[2003, Spring]**

Solution :

Let $y = \left(\frac{\sin x}{x}\right)^{1/x}$

Taking log both sides, we get

$$\text{Log}(y) = \frac{1}{x} \log\left(\frac{\sin x}{x}\right)$$

Taking $\lim\limits_{x\to 0}$ on both sides

$$\lim_{x\to 0} \log y = \lim_{x\to 0} \frac{\log\left(\frac{\sin x}{x}\right)}{x} \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{4}{\sin x}\left(\frac{x\cos x - \sin x}{x^2}\right)$$

$$= \lim_{x\to 0} \frac{x\cos x - \sin x}{x^2} \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{\cos x - x\sin x - \cos x}{2x}$$

$$= \lim_{x\to 0} \frac{-\sin x}{2} = 0.$$

Now,

$$\lim_{x\to 0} \log y = 0 \Rightarrow \lim_{x\to 0} y = e^0 = 1$$

$$\therefore \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{1/x} = 1.$$

(xxix) $\lim\limits_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x^2} = e^{1/3}$ **[2002]**

Solution :

Let $y = \left(\frac{\tan x}{x}\right)^{1/x^2}$

Taking log both sides, we get

$$\log y = \frac{1}{x^2} \log\left(\frac{\tan x}{x}\right)$$

Taking $\lim\limits_{x\to 0}$ on both sides

$$\lim_{x\to 0} \log y = \lim_{x\to 0} \frac{\log\left(\frac{\tan x}{x}\right)}{x^2} \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{\frac{x}{\tan x} \cdot \left[\frac{x\sec^2 x - \tan x}{x^2}\right]}{2x}$$

$$= \lim_{x\to 0} \left[\frac{x\sec^2 x - \tan x}{2x^3}\right] \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$\left(\lim_{x\to0}\frac{x}{\tan x}=1\right)$$

$$=\lim_{x\to0}\frac{\sec^2x+2\sec^2x\,.\,\tan x\,.\,x-\sec^2x}{6x^2}$$

$$=\lim_{x\to0}\frac{2\sec^2x\,.\,\tan x.\,x}{6x^2}$$

$$=\lim_{x\to0}\frac{\sec^2x}{3}\cdot\frac{\tan x}{x}$$

$$=\lim_{x\to0}\frac{\sec^2x}{3}\,.\qquad\left(\lim_{x\to0}\frac{\tan x}{x}=1\right)$$

$$=\frac{1}{3}\,.$$

Now,

$$\lim_{x\to0}\log y=\frac{1}{3}\ \Rightarrow\ \lim_{x\to0}y=e^{1/3}$$

$$\therefore\ \lim_{x\to0}\left(\frac{\tan x}{x}\right)^{1/x^2}=e^{1/3}\,.$$

4. Show that

(i) $\lim\limits_{x\to\infty}(1+2x)^{1/2\log(x)}=\infty$

Solution: Let $y=(1+2x)^{1/2\log(x)}$

Taking log both sides, we get

$$\log y=\frac{1+2x}{2\log(x)}$$

Taking $\lim\limits_{x\to\infty}$ on both sides,

$$\lim_{x\to\infty}\log y=\lim_{x\to\infty}\frac{1+2x}{2\log(x)}\qquad\left[\text{in }\frac{\infty}{\infty}\text{ form}\right]$$

$$=\lim_{x\to\infty}\frac{2}{2/x}=\infty.$$

Now,

$$\lim_{x\to\infty}\log y=\infty\ \Rightarrow\ \lim_{x\to\infty}y=e^{\infty}=\infty.$$

$$\therefore\ \lim_{x\to\infty}(1+2x)^{1/2\log(x)}=\infty.$$

Answer to be corrected in the book.

(ii) $\lim\limits_{x\to\infty}x^{1/x}=1.$

Solution : Let $y=x^{1/x}$

Taking log both sides, we get

$$\log y=\frac{1}{x}\log x$$

Taking $\lim\limits_{x\to\infty}$ on both sides,

$$\lim_{x\to\infty}\log y=\lim_{x\to\infty}\frac{\log(x)}{x}\qquad\left[\text{in }\frac{\infty}{\infty}\text{ form}\right]$$

$$=\lim_{x\to\infty}\frac{1}{x}=0.$$

Now,

$$\lim_{x\to\infty}\log y=0\ \Rightarrow\ \lim_{x\to\infty}y=e^0=1.$$

$$\therefore\ \lim_{x\to\infty}x^{1/x}=1.$$

(iii) $\lim\limits_{x\to0}\frac{x\,e^x-(1+x)\log(1+x)}{x^2}=\frac{1}{2}$

Solution : Here,

$$\lim_{x\to\infty}\frac{x\,e^x-(1+x)\log(1+x)}{x^2}\qquad\left[\text{in }\frac{0}{0}\text{ form}\right]$$

$$=\lim_{x\to\infty}\frac{e^x+xe^x-\log(1+x)-1}{2x}\qquad\left[\text{in }\frac{0}{0}\text{ form}\right]$$

$$=\lim_{x\to\infty}\frac{e^x+e^x+xe^x-\frac{1}{1+x}}{2}$$

$$=\frac{2e^0+0\,.\,e^0-\frac{1}{1+0}}{2}=\frac{1}{2}\,.$$

(iv) $\lim\limits_{x\to0}(\cos x)^{1/x^2}=e^{-1/2}$

Solution :

Put, $y = \lim_{x\to 0} (\cos x)^{1/x^2}$

Taking log both sides, we get

$$\log y = \frac{1}{x^2} \log(\cos x)$$

Taking $\lim_{x\to 0}$ on both sides,

$$\lim_{x\to 0} \log y = \lim_{x\to 0} \frac{\log(\cos x)}{x^2} \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{-\frac{\sin x}{\cos x}}{2x}$$

$$= \lim_{x\to 0} -\frac{\tan x}{2x}$$

$$= -\frac{1}{2} \qquad \left(\lim_{x\to 0} \frac{\tan x}{x} = 1\right).$$

Now,

$$\lim_{x\to 0} \log y = -\frac{1}{2} \Rightarrow \lim_{x\to 0} y = e^{-1/2}.$$

$$\therefore \lim_{x\to 0} (\cos x)^{1/x^2} = e^{-1/2}.$$

(v) $\lim_{x\to a} \left(2 - \frac{x}{a}\right)^{\tan \pi x/2a} = e^{2/\pi}$

Solution : Let $y = \left(2 - \frac{x}{a}\right)^{\tan \pi x/2a}$

Taking log both sides, we get

$$\log y = \tan \frac{\pi x}{2a} \log\left(2 - \frac{x}{a}\right)$$

Taking $\lim_{x\to a}$ on both sides

$$\lim_{x\to a} \log y = \lim_{x\to a} \frac{\tan \pi x}{2a} \log\left(2 - \frac{x}{a}\right) \qquad [\text{in } 0 \times \infty \text{ form}]$$

$$= \lim_{x\to a} \frac{\log\left(2 - \frac{x}{a}\right)}{\cot \frac{\pi x}{2a}} \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to a} \frac{\left(\frac{1}{2 - x/a}\right) \cdot \left(-\frac{1}{a}\right)}{\left(-\operatorname{cosec}^2 \frac{\pi x}{2a}\right) \cdot \frac{\pi}{2a}}$$

$$= \lim_{x\to a} \frac{-\left(\frac{a}{2a - x}\right)}{-\operatorname{cosec}^2 \frac{\pi x}{2a}} \times \frac{2}{\pi}$$

$$= \frac{\left(\frac{a}{2a - a}\right) \cdot \frac{2}{\pi}}{\operatorname{cosec}^2 \frac{\pi}{2}} = \frac{2}{\pi}.$$

Now,

$$\lim_{x\to a} \log y = \frac{2}{\pi} \Rightarrow \lim_{x\to a} y = e^{2/\pi}$$

$$\therefore \lim_{x\to a} \left(2 - \frac{x}{a}\right)^{\tan \pi x/2a} = e^{2/\pi}.$$

(vi) $\lim_{x\to 0} \frac{x \cos x - \log(1 + x)}{x^2} = \frac{1}{2}$

Solution : Here,

$$\lim_{x\to 0} \frac{x \cos x - \log(1 + x)}{x^2} \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{-x \sin x + \cos x - \frac{1}{1 + x}}{2x} \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{-\sin x - x \cos x - \sin x + \frac{1}{(1 + x)^2}}{2}$$

$$= \frac{0 - 0 - 0 + 1}{2} = \frac{1}{2}.$$

(vii) $\lim_{x\to 0} \frac{e^x - e^{-x} + 2 \sin x - 4x}{x^5} = \frac{1}{30}$

Solution : Here,

$$\lim_{x\to 0} \frac{e^x - e^{-x} + 2 \sin x - 4x}{x^5} \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{e^x + e^{-x} + 2\cos x - 4}{5x^4} \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{e^x - e^{-x} - 2\sin x}{20x^3} \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{e^x + e^{-x} - 2\cos x}{60x^2} \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{e^x - e^{-x} + 2\sin x}{120x} \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{e^x + e^{-x} + 2\cos x}{120} = \frac{e^0 + e^{-0} + 2\cos 0^\circ}{120} = \frac{4}{120} = \frac{1}{30}$$

(viii) $\displaystyle \lim_{x\to 0} \frac{\cos x - \log(1+x) + \sin x - 1}{e^x - (1+x)} = 0.$

Solution : Here,

$$\lim_{x\to 0} \frac{\cos x - \log(1+x) + \sin x - 1}{e^x - (1+x)} \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{-\sin x - \frac{1}{1+x} + \cos x}{e^x - 1} \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{-\cos x + \frac{1}{(1+x)^2} - \sin x}{e^x}$$

$$= \frac{-\cos 0^\circ + \frac{1}{(1+0)^2} - 0}{e^0} = \frac{-1+1}{1} = 0.$$

(ix) $\displaystyle \lim_{x\to 2} \frac{x^2 - 4}{\sqrt{x+2} - \sqrt{3x-2}} = -8$

Solution : Here,

$$\lim_{x\to 2} \frac{x^2 - 4}{\sqrt{x+2} - \sqrt{3x-2}}$$

$$= \lim_{x\to 2} \frac{x^2 - 4}{(\sqrt{x+2} - \sqrt{3x+2})} \times \frac{(\sqrt{x+2} + \sqrt{3x-2})}{(\sqrt{x+2} + \sqrt{3x-2})}$$

$$= \lim_{x\to 2} \frac{(x^2-4)(\sqrt{x+2} + \sqrt{3x-2})}{x+2-3x+2}$$

$$= \lim_{x\to 2} \frac{(x^2-4)(\sqrt{x+2} + \sqrt{3x-2})}{-2(x-2)}$$

$$= \lim_{x\to 2} \left(-\frac{1}{2}\right)(x+2)(\sqrt{x+2} + \sqrt{3x-2})$$

$$= \left(-\frac{1}{2}\right) \times 4 \times (2+2) = -8.$$

(x) $\displaystyle \lim_{x\to\infty} \left[x - x^2 \log\left(1 + \frac{1}{x}\right)\right] = \frac{1}{2}$

Solution : Here,

$$\lim_{x\to\infty} \left[x - x^2 \log\left(1 + \frac{1}{x}\right)\right] \qquad (\text{in } \infty - \infty \text{ form})$$

$$= \lim_{x\to\infty} x^2 \left[\frac{1}{x} - \log\left(1 + \frac{1}{x}\right)\right] \qquad [\text{in } \infty \times 0 \text{ form}]$$

$$= \lim_{x\to\infty} \frac{\left[\frac{1}{x} - \log\left(1 + \frac{1}{x}\right)\right]}{\left(\frac{1}{x^2}\right)} \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to\infty} \frac{\left[-\frac{1}{x^2} - \frac{1}{\left(1+\frac{1}{x}\right)}\left(-\frac{1}{x^2}\right)\right]}{-\frac{2}{x^3}}$$

$$= \lim_{x\to\infty} \frac{\left[-\frac{1}{x^2} + \frac{1}{x(1+x)}\right]}{-\frac{2}{x^3}}$$

$$= \lim_{x\to\infty} \left[\frac{-(1+x)+x}{-2x^2(1+x)} \cdot x^3\right]$$

$$= \lim_{x\to\infty} \left[\frac{(-1-x+x)\,x}{-2(1+x)}\right]$$

$$= \lim_{x\to\infty} \left[\frac{x}{2(1+x)}\right] \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to\infty} \cdot \frac{1}{2}$$

$$= \frac{1}{2}.$$

$\dfrac{1}{\frac{2(1+x)}{x}}$

$\dfrac{1}{\frac{2}{x}+2}$

$0 \quad \dfrac{1}{2}$

5. If $\lim\limits_{x\to 0} \frac{a\sin x - \sin 2x}{\tan^3 x}$ is finite show that a = 2 and limit is 1.

[2006, Fall]

Solution : Here,

$$\lim_{x\to 0} \frac{a\sin x - \sin 2x}{\tan^3 x}$$

$$= \lim_{x\to 0} \frac{a\sin x - \sin 2x}{x^3 . \left(\frac{\tan x}{x}\right)^3}$$

$$= \lim_{x\to 0} \frac{a\sin x - \sin 2x}{x^3} \quad \left[\text{in } \frac{0}{0} \text{ form}\right] \left(\lim_{x\to 0} \frac{\tan x}{x} = 1\right)$$

$$= \lim_{x\to 0} \frac{a\cos x - 2\cos x}{3x^2} \quad \left[\frac{a-2}{0}\right]$$

[Givan that the limit is finite than $a - 2 = 0 \Rightarrow a = 2$]. Then,

$$= \lim_{x\to 0} \frac{2\cos x - 2\cos 2x}{3x^2} \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{-2\sin x + 4\sin 2x}{6x} \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{-2\cos x + 8\cos 2x}{6} = \frac{-2+8}{6} = 1.$$

Thus, the value of a is 2 and limit of the form is 1.

6. If $\lim\limits_{x\to 0} \frac{x(1 + a\cos x) - b\sin x}{x^3} = 1$. Show that $a = -\frac{5}{2}$, and $b = -\frac{3}{2}$.

Solution : Here,

$$\lim_{x\to 0} \frac{x(1 + a\cos x) - b\sin x}{x^3} \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{1 + a\cos x - ax\sin x - b\cos x}{3x^2}$$

$$\left(\text{has } \frac{1+a-b}{0} = \frac{\text{finite}}{0} \text{ form}\right)$$

As the limit exist we must have $a - b + 1 = c$

$$\Rightarrow a - b = -1 \quad \text{...... (i)}$$

So, suppose that the above form has $\frac{0}{0}$ form. Then,

$$= \lim_{x\to 0} \frac{-a\sin x - a\sin x - ax\cos x + b\sin x}{6x} \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{-a\cos x - a\cos x - a\cos x + ax\sin x + b\cos x}{6}$$

$$= \frac{1}{6}(-a.\cos 0° - a\cos 0° - a\cos 0° + 0 + b\cos 0°)$$

$$= \frac{-3a+b}{6}$$

We have,

$$\lim_{x\to 0} \frac{x(1 + a\cos x) - b\sin x}{x^3} = 1 \Rightarrow \frac{-3a+b}{6} = 1$$

From (i), $a = b - 1$

So, $\quad -3(b-1) + b = 6. \Rightarrow -3b + 3 + b = 6$

$$\Rightarrow -2b = 3 \Rightarrow b = -\frac{3}{2}.$$

Then,

$$a = -\frac{3}{2} - 1 = -\frac{5}{2} \Rightarrow a = -\frac{5}{2}.$$

Thus, $a = -\frac{5}{2}$ and $b = -\frac{3}{2}$.

7. Show that a = 1, b = 2, c = 1, when $\lim\limits_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x\sin x} = 2.$

Solution : Here,

$$\lim_{x\to 0} \left[\frac{ae^x - b\cos x + ce^{-x}}{x\sin x}\right]$$

$$= \lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x^2\left(\frac{\sin x}{x}\right)} \quad \left[\because \lim_{x\to 0} \frac{\sin x}{x} = 1\right]$$

$$= \lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x^2} \quad \left[\text{in } \frac{a-b+c}{0} \text{ form}\right]$$

As the limit exists, we must have

$$a - b + c = 0 \quad \text{...... (i)}$$

then the above form becomes as $\frac{0}{0}$ form. So,

$$= \lim_{x\to 0} \frac{ae^x + b\sin x - ce^{-x}}{2x} \quad \left[\text{in } \frac{a-c}{0} \text{ form}\right]$$

As the limit exist, we must have get

$a - c = 0, \quad a = c$ .... (ii)

then the above form becomes as $\frac{0}{0}$ form. So,

$$= \lim_{x\to 0} \frac{ae^x + b\cos x + ce^{-x}}{2}$$

$$= \frac{a+b+c}{2} \quad ....(iii)$$

Given that,

$$\lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x \sin x} = 2.$$

$$\Rightarrow \frac{a+b+c}{2} = 2 \Rightarrow a+b+c = 4 \quad ......(iv) \quad [\text{by (iii)}]$$

From (i) and (ii), we get

$2a - b = 0 \Rightarrow b = 2a$ ... (v)

From equations (ii), (iv) and (v) we get,

$a + 2a + a = 4 \Rightarrow a = 1,$

Therefore, $b = 2a = 2$ and $c = a = 1$.

Thus, $a = 1$, $b = 2$ and $c = 1$.

OTHER IMPORTANT QUESTIONS FROM FINAL EXAM

Long Questions

a. State L' Hospital's rule and evaluate: $\lim_{x\to\frac{\pi}{2}} \left(\frac{\tan x}{x}\right)^{1/x}$ **[2000]**

Solution : See statement of L' Hospital's rule and question is mistake. We should replace $\frac{\pi}{2}$ by 0 then see Q. 3(xi).

b. State the L' Hospital Rule and evaluate the limit:

$$\lim_{x\to 0} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right).$$ **[2007, Fall] [2003, Fall]**

Solution : See statement of L' Hospital's rule and see Q. 3(i).

c. Evaluate: $\lim_{x\to\infty} \left(\frac{\pi}{2} - \tan^{-1} x\right)^{1/x}$ **[2005, Fall]**

Solution : Put $y = \left(\frac{\pi}{2} - \tan^{-1} x\right)^{1/x}$

So, $\log(y) = \frac{1}{x} \log\left(\frac{\pi}{2} - \tan^{-1} x\right)$

$$= \frac{\log\left(\frac{\pi}{2} - \tan^{-1} x\right)}{x}$$

So, taking $\lim_{x\to\infty}$ on both sides then,

$$\lim_{x\to\infty} \log(y) = \lim_{x\to\infty} \frac{\log\left(\frac{\pi}{2} - \tan^{-1} x\right)}{x} \quad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to\infty} \left(\frac{1}{\frac{\pi}{2} - \tan^{-1} x}\right) \cdot \frac{-1}{x^2+1} \cdot \frac{1}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{x^2}{\left(\frac{\pi}{2} - \tan^{-1} x\right)(x^2+1)} \quad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to\infty} \frac{2x}{\left(\frac{\pi}{2} - \tan^{-1} x\right) 2x + \left(-\frac{1}{x^2+1}\right) \cdot (x^2+1)}$$

$$= \lim_{x\to\infty} \frac{2x}{\left(\frac{\pi}{2} - \tan^{-1} x\right) 2x - 1} \quad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to\infty} \frac{2}{2\left(\frac{\pi}{2} - \tan^{-1} x\right) + 2x\left(-\frac{1}{x^2+1}\right)}$$

$$= \frac{2}{2\left(\frac{\pi}{2} - \frac{\pi}{2}\right) - \frac{2 \cdot \infty}{\infty^2 + 1}} = \frac{-2}{\frac{2\infty}{\infty^2}} = -\infty.$$

Thus,

$$\lim_{x\to\infty} \log(y) = \log\left(\lim_{x\to\infty} y\right) = -\infty$$

$$\Rightarrow \lim_{x\to\infty} (y) = \lim_{x\to\infty} \left(\frac{\pi}{2} - \tan^{-1} x\right)^{1/x} = e^{-\infty} = 0..$$

d. Evaluate $\lim_{x\to 0} (\cos x)^{\cot^2 x}$ **[2008, Fall]**

Solution: Let $y = (\cos x)^{\cot^2 x}$

Taking log on both sides we get,

$$\log y = \log (\cos x)^{\cot^2 x} = \cot^2 x \log \cos x$$

i.e. $\log y = \dfrac{\log (\cos x)}{\tan^2 x}$

Taking $\lim_{x\to 0}$ on both sides, we get

$$\lim_{x\to 0} (\log y) = \lim_{x\to 0} \frac{\log (\cos x)}{\tan^2 x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{1}{\cos x} \frac{(-\sin x)}{2 \tan x . \sec^2 x}$$

$$= \lim_{x\to 0} \frac{-\tan x}{2 . \tan x \sec^2 x} = -\frac{1}{2}$$

Hence,

$$\lim_{x\to 0} (\log y) = -\frac{1}{2} \Rightarrow \log\left(\lim_{x\to 0} y\right) = -\frac{1}{2}$$

$$\Rightarrow \lim_{x\to 0} y = e^{-1/2}$$

$$\therefore \lim_{x\to 0} (\cos x)^{\cot^2 x} = e^{-1/2}$$

Short Questions

a. Evaluate: $\lim_{x\to 0} \dfrac{x}{|x|}$. **[2000]**

Solution : Since we have |x| in denominator. So, the limit exist only right (i.e. positive) side.

Now,

$$\lim_{x\to 0} \frac{x}{|x|} \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{1}{1} = 1.$$

b. Evaluate $\lim_{x\to 0} \dfrac{x - \sin x}{x^3}$. **[2002]**

Solution : Here,

$$\lim_{x\to 0} \frac{x - \sin x}{x^3} \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{1 - \cos x}{3x^2} \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{\sin x}{6x} \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{1}{6} \quad \left(\lim_{x\to 0} \frac{\sin x}{x} = 1\right)$$

$$= \frac{1}{6}.$$

c. Evaluate: $\lim_{x\to 0} \dfrac{\tan x - x}{x^3}$. **[2005, Spring] [2004, Fall] [2002]**

Solution : Here,

$$\lim_{x\to 0} \frac{x - \tan x}{x^3} \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{1 - \sec^2 x}{3x^2} \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{-2 \sec^2 x \cdot \tan x}{6x} \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{-2 \sec^2 x}{6} \quad \left(\lim_{x\to 0} \frac{\tan x}{x} = 1\right)$$

$$= -\frac{2}{6} = -\frac{1}{3}.$$