MEERA MEMORIAL PAPER PRESENTATION COMPETITION

Why light slows down in transparent media?

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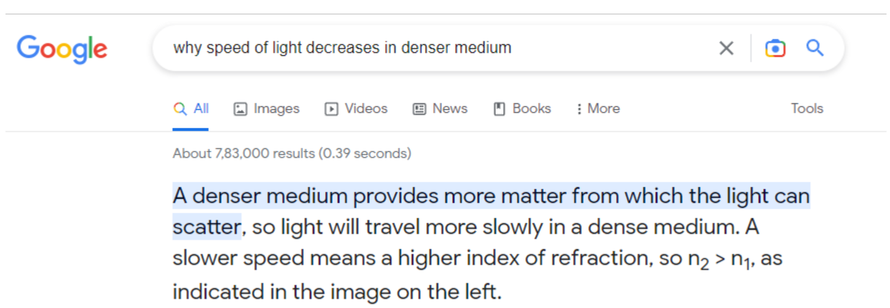
Introduction

Light slows down upon entering transparent media like glass, honey, etc. The "reason" being that refractive index of the medium is larger than the refractive index of air (or vacuum). The rule is that in any medium, speed of light c decreases by a factor of n which is the refractive index of the material:

$$v = \frac{c}{n}$$

The question on my mind for the longest time was **Why exactly light slows down?** I encountered many explanations, for example, A quick search on google shows this:



This explanation brings a good picture to mind. Photons bumping around the lattice, moving at speed c but their path is altered, so they take longer to traverse the same distance. I am not claiming this is an incorrect model. A model is a model, not reality. If the model works, it works. Another justification comes from Electromagnetic theory, using **The Maxwell's** equations....

Contents

1 Maxwell's Explanation

A linear medium is that in which

$$\mathbf{D} = \epsilon\mathbf{E} \quad \text{and} \quad \mathbf{H} = \frac{1}{\mu}\mathbf{B}$$

Maxwell's equation then becomes,

$$\boldsymbol{\nabla}\cdot\mathbf{E} = 0 \quad \text{and} \quad \boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0 \quad \text{and} \quad \boldsymbol{\nabla}\times\mathbf{B} = \mu\epsilon\frac{\partial\mathbf{E}}{\partial t}$$

Some maths...

We can apply the $\boldsymbol{\nabla}\times$ operator on both sides to decouple the equations

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{E}) = \nabla(\boldsymbol{\nabla}\cdot\mathbf{E}) - \nabla^2\mathbf{E} = \boldsymbol{\nabla}\times-\frac{\partial\mathbf{B}}{\partial t}$$

$$= -\frac{\partial}{\partial t}(\boldsymbol{\nabla}\times\mathbf{B}) = -\mu\epsilon\frac{\partial^2\mathbf{E}}{\partial t}$$

But

$$\boldsymbol{\nabla}\cdot\mathbf{E} = 0$$

This gives

$$\nabla^2\mathbf{E} = \mu\epsilon\frac{\partial^2\mathbf{E}}{\partial t^2}$$

Similarly

$$\nabla^2\mathbf{B} = \mu\epsilon\frac{\partial^2\mathbf{B}}{\partial t^2}$$

The two equations we got are **Wave equations** of form

$$\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

where v is the speed of the wave. Hence it is proved that

- Solutions to Maxwell's equation in free linear media are waves.
- These waves travel at a speed given by

$$v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{n} \tag{1}$$

Using the fact that $c = 1/\sqrt{\mu_0\epsilon_0}$

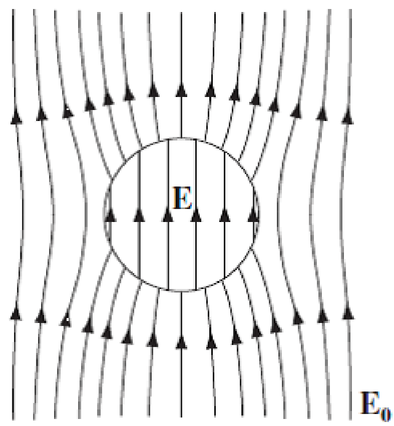
$$\boxed{n \equiv \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}} \approx \sqrt{\epsilon_r} = \sqrt{1+\chi_e}} \tag{2}$$

2 The Pertubative approach

2.1 A simpler example (static fields)

I would like to discuss another model that works. The perturbative approach. An example will help us understand the method.

Consider a sphere of linear dielectric material placed in an external electric field $\mathbf{E_0}$ as shown



Let's say we are interested in the field inside. Idea is that the total field is a sum of many contribution

$$\boxed{\mathbf{E} = \mathbf{E^0} + \mathbf{E^1} + \mathbf{E^2} + ...}$$

We say that the initial zeroth order field $\mathbf{E^0}$ polarised the sphere according to the equation

$$\mathbf{P^1} = \chi_e \epsilon_0 \mathbf{E^0}$$

We know that a sphere with uniform polarisation in one direction acts like a dipole and field inside the sphere is $-\frac{1}{3\epsilon_0}$ times the polarisation vector. Hence, the first order field generated by $\mathbf{P^1}$ is

$$\mathbf{E^1} = -\frac{1}{3\epsilon_0}\mathbf{P^1} = -\frac{\chi_e}{3}\mathbf{E^0}$$

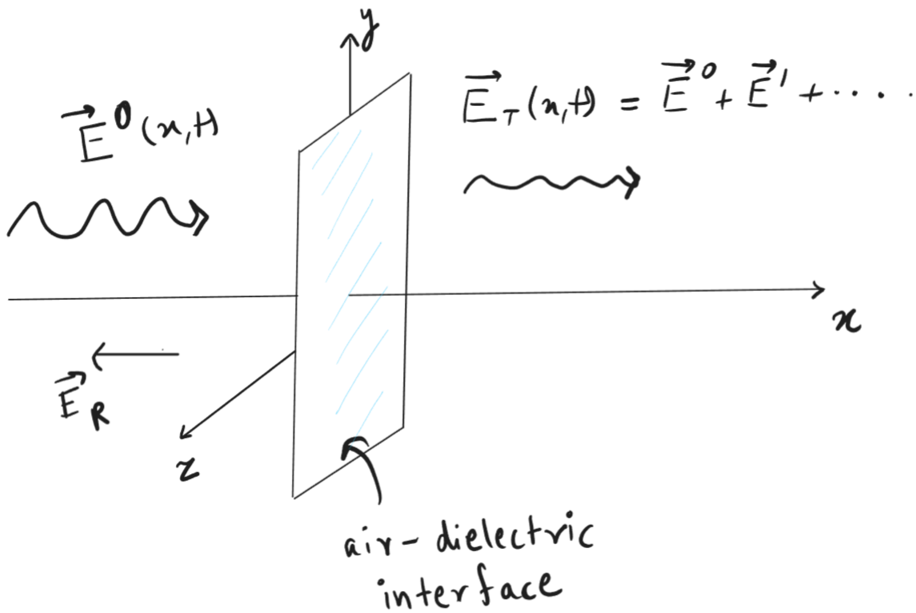
This new first order field $\mathbf{E^1}$ will cause a second order polarisation $\mathbf{P^2}$ that will induce a second order field $\mathbf{E^2}$ which will generate a third order polarisation $\mathbf{P^3}$ which will...

This process will continue up to infinity, but every new ordered field that is produces is weaker than its predecessors. Hence the sum converges to a finite value. In this particular example, we get a geometric series:

$$\mathbf{E} = \sum_{n=0}^{\infty} \left(-\frac{\chi_e}{3}\right) \mathbf{E^0} = \left(\frac{3}{3+\chi_e}\right) \mathbf{E^0}$$

3 Modelling the problem

We want to use this approach to analyse what happens at the air-dielectric interface. The idea is same, we will consider the incident electric field (a monochromatic plane wave) $\mathbf{E}$, which is a time dependent field this time, incident from vacuum ($x < 0$) on a transparent dielectric medium ($x > 0$), which gives rise to a transmitted and reflected wave. We will see that the zeroth order field $\mathbf{E^0}$ will induce oscillating dipoles in the medium, these dipoles will induce a first order field $\mathbf{E^1}$ which again induces second order oscillating dipoles and so on...



As usual, we define the propagation vector and angular frequency by k and w so that, the zeroth order field in the region $x > 0$ is given by

$$\mathbf{E^0}(x,t) = E_0 \exp\left[i(kx - wt)\right]\hat{\jmath}$$

This produces a polarisation in the medium

$$\mathbf{P^1}(x,t) = \epsilon_0 \chi_e \mathbf{E^0} = \epsilon_0 \chi_e E_0 \exp\left[i(kx - wt)\right]\hat{\jmath}$$

The resulting polarisation current is

$$\mathbf{J^1_p}(x,t) = \frac{\partial \mathbf{P}}{\partial t} = -iw\epsilon_0 \chi_e \exp\left[i(kx - wt)\right]\hat{\jmath}$$

To calculate the field generated by this polarisation current, we can chop the dielectric into slabs of infinitesimal thickness dx'. If we know the electric field due to a neutral plate carrying surface current $\mathbf{K}(t)$, we can integrate over all the plates to find the total field inside the dielectric.

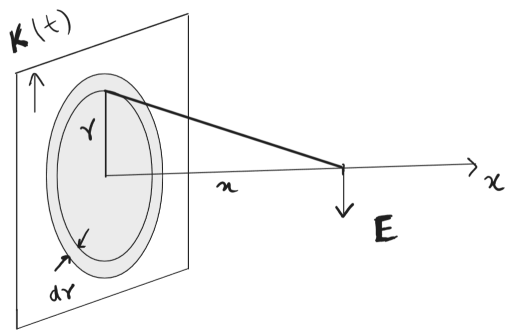
Digression...(Electric field due to a plate carrying surface current $\mathbf{K}(t)$)

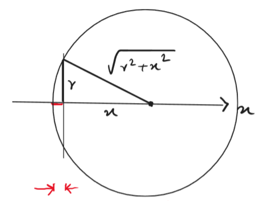
Consider a neutral plate with current density $\mathbf{K(t)}$. The electric field is given by

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

where, ϕ is the scalar potential (which is a constant here because the plate is neutral), and A is vector potential given by

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_S da \frac{\mathbf{K(t_r)}}{|\mathbf{r}-\mathbf{r}'|}$$





$$cu \equiv \sqrt{r^2+x^2} - x$$

$$d(cu) = \frac{r\,dr}{\sqrt{r^2+x^2}}$$

Integral becomes

$$\mathbf{A}(x,t) = \frac{\mu_0}{2}\int_0^\infty \mathbf{K}\left(t - \frac{\sqrt{r^2+x^2}}{c}\right)\frac{r}{\sqrt{r^2+x^2}}dr$$

in terms of u (see diagram)

$$\mathbf{A}(x,t) = \frac{\mu_0 c}{2}\int_0^\infty \mathbf{K}\left(t - \frac{x}{c} - u\right)du$$

Therefore

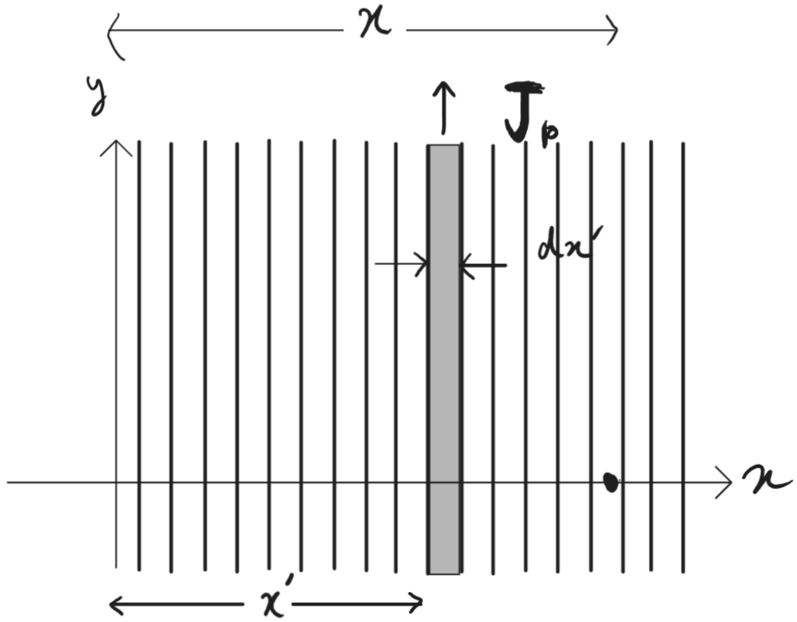
$$\mathbf{E}(x,t) = -\frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 c}{2}\int_0^\infty \left[\frac{\partial}{\partial t}\mathbf{K}\left(t - \frac{x}{c} - u\right)\right]du$$

$$\frac{\mu_0 c}{2}\int_0^\infty \left[\frac{\partial}{\partial u}\mathbf{K}\left(t - \frac{x}{c} - u\right)\right]du = \frac{\mu_0 c}{2}\mathbf{K}\left(t - \frac{x}{c} - u\right)\Big|_0^\infty$$

Hence, Electric field due to a plane neutral plate is

$$\boxed{\mathbf{E}(x,t) = -\frac{\mu_0 c}{2}\mathbf{K}\left(t - \frac{x}{c}\right)}$$

Note that we are dealing with slabs , not plates. But we can multiply the width of the slab dx' by $\mathbf{J_p}$ to make the equations dimensionally consistent. Furthermore, we need to integrate in two parts, one covering all the slabs up to the point x (where the field need to be calculated) and the second part would cover all the slabs up to infinity as shown in the diagram.



Integral becomes

$$\mathbf{E^1} = \left(-\frac{\mu_0 c}{2}\right)(-iw\epsilon_0\chi_0\hat{j})\left(\int_0^x dx' \exp\left[kx' - w\left(t - \frac{(x-x')}{c}\right)\right] + \int_x^\infty dx' \exp\left[kx' - w\left(t - \frac{(x'-x)}{c}\right)\right]\right)$$

$$= \frac{ik}{c}\chi_e E_0\hat{j}\left(e^{i(kx-wt)}\int_0^x dx' + e^{i(-kx-wt)}\int_x^\infty e^{2ikx'}dx'\right)$$

using the fact that $k = w/c$ we can simplify the integral. But the second integral will give us trouble due to its upper limit. Because we added infinite plates, integral diverges. We can either attenuate the wave by a little bit so that contribution from the second integral vanishes and no wave comes back to diverge the integrand. Finally, we get

$$\mathbf{E^1}(x,t) = \frac{\chi_e}{4}(1-2ikx)\mathbf{E^0}(x,t)$$

We can continue in the same fashion and find second order polarisation (and current)

$$\mathbf{J_p^2}(x,t) = \frac{\partial \mathbf{P}^2}{\partial t} = \frac{\partial}{\partial t}(\epsilon_0\chi_e\mathbf{E^1})$$

Second order field can be integrated just as before

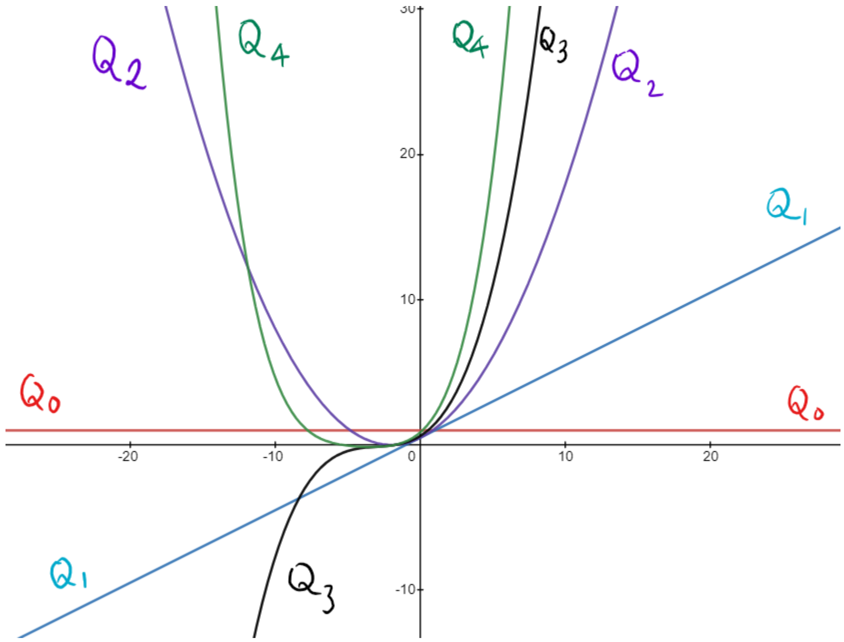
$$\mathbf{E^2}(x,t) = \left(\frac{\chi_e}{2}\right)^2\frac{1}{2}(1-2ikx-k^2x^2)\mathbf{E^0}(x,t)$$

3.1 The Polynomial Q_n

If we continue in the same process we can see a pattern here, we may write the n^{th} order contribution as follows

$$\boxed{\mathbf{E^n}(x,t) = \left(\frac{-\chi_e}{2}\right)^n Q_n(z)\mathbf{E^0}(x,t)}$$

where we have defined $Q_n(z)$ as polynomial of degree n ($z = -2ikx$). First few are plotted below



$$Q_0(z) = 1$$

$$Q_1(z) = \frac{1}{2}(1+z)$$

$$Q_2(z) = \frac{1}{2}\left(1+z+\frac{1}{4}z^2\right)$$

$$Q_3(z) = \frac{5}{8}\left(1+z+\frac{3}{10}z^2+\frac{1}{30}z^3\right)$$

$$Q_4(z) = \frac{7}{8}\left(1+z+\frac{9}{28}z^2+\frac{1}{21}z^3+\frac{1}{336}z^4\right)$$

The recursion relation in the polynomials can be found in terms of the integrals (and in terms of derivatives after differentiation):

$$Q_{n+1}(z) = \frac{1}{2}\left(\int_0^z Q_n(v)dv + e^z \int_z^\infty Q_n(v)e^{-v}dv\right)$$

Therefore

$$\boxed{\frac{d^2}{dz^2}Q_{n+1} - \frac{d}{dz}Q_{n+1} + \frac{1}{2}Q_n = 0}$$

3.2 Summing the series

In our previous simpler example, the final series was geometric, and we were able to sum it up. Here, series can be summed but it is not going to be as straightforward as before. To begin with, we have the net field:

$$\mathbf{E^T}(x,t) = \sum_{n=0}^{\infty} \mathbf{E^n}(x,t) = \left[\sum_{n=0}^{\infty} \left(-\frac{\chi_e}{2}\right)^n Q_n(z)\right] \mathbf{E^0} = \psi(z)\mathbf{E^0} \tag{3}$$

where,

$$\psi(z) \equiv \sum_{n=0}^{\infty} \left(-\frac{\chi_e}{2}\right)^n Q_n(z) \tag{4}$$

Recall that $z = -2ikx$. If we can find $\psi(z)$, our job is complete. We can create differential equation in $\psi(z)$ by using the recursion relation in differential form which was defined above.

$$\frac{d^2\psi}{dz^2} - \frac{d\psi}{dz} + \frac{1}{2}\sum_{n=0}^{\infty} \left(-\frac{\chi_e}{2}\right)^2 Q_{n-1} = 0$$

defining a new summation index : $j \equiv n+1$, we can convert Q_{n-1} into Q_n:

$$\left(\frac{d^2}{dz^2} - \frac{d}{dz} - \frac{\chi_e}{4}\right)\psi(z) = 0$$

This is a trivial differential equation to solve. Solutions are

$$\psi(z) = A(\chi_e)\exp\left[\left(1-\sqrt{1+\chi_e}\right)\frac{z}{2}\right] + B(\chi_e)\exp\left[\left(1+\sqrt{1+\chi_e}\right)\frac{z}{2}\right] \tag{5}$$

Finding $A(\chi_e)$ and $B(\chi_e)$...

We can think of $\psi(z,\chi_e)$ as a function of both z and χ_e, It is immediately clear from the definition of ψ that

$$\psi(z,0) = 1$$

From the solution of ψ

$$1 = A(0) + B(0)e^z \implies B(0) = 0 \quad \text{and} \quad A(0) = 1$$

Whenever we differentiate eq(5) w.r.t χ_e and find the derivatives at $\chi_e = 0$, the second term will always have a factor of e^z which is not there in eq(3) or its derivatives. Hence, ALL derivatives of B must vanish. This leads us to the conclusion that

$$\boxed{B(\chi_e) = 0}$$

To find $A(\chi_e)$, we will consider $\psi(0,\chi_e)$, from eq(5) and (4). This gives

$$\psi(0,\chi_e) = A(\chi_e) = \sum_{n=0}^{\infty} \left(-\frac{\chi_e}{2}\right)^n Q_n(0)$$

$Q_n(0)$ can be found from integral recursion relation given in sect 3.1

$$\implies A(\chi_e) = Q_0(0) + \sum_{n=1}^{\infty} \left(-\frac{\chi_e}{2}\right)^n \frac{1}{2} \int_0^{\infty} e^{-v} Q_{n-1}(v) dv$$

Note that summation is going from 1 to ∞, and we pulled out the n=0 term outside ($Q_0 = 1$). Again, shifting the summation index by 1, we can convert back to ψ:

$$A(\chi_e) = 1 - \frac{\chi_e}{4} \int_0^{\infty} e^{-v} \psi(v, \chi_e) dv = 1 - \frac{\chi_e}{4} A(\chi_e) \frac{2}{1 + \sqrt{1 + \chi_e}}$$

Where the last integral is done by using the solution of ψ i.e., eq(5). We can solve for A from here and we get

$$\boxed{A(\chi_e) = \frac{2}{1 + \sqrt{1 + \chi_e}}}$$

3.3 Combining everything

With the hard work over, we can substitute all values in (3) to find the total transmitted field:

$$\mathbf{E}^{\mathbf{T}}(x,t) = \psi(z, \chi_e) \mathbf{E}^{\mathbf{0}}(x,t) = \frac{2}{1 + \sqrt{1 + \chi_e}} E_0 \exp\left[i(\sqrt{1 + \chi_e} kx - wt)\right] \hat{j} \tag{6}$$

from here it is seen that the velocity of the wave is

$$v = \frac{w}{\sqrt{1 + \chi_e} k} = \frac{(w/k)}{\sqrt{1 + \chi_e}} = \frac{c}{\sqrt{1 + \chi_e}} = \frac{c}{n} \tag{7}$$

SO, speed is reduced by a factor n which is given by

$$\boxed{n = \sqrt{1 + \chi_e} = \sqrt{\epsilon_r}} \tag{8}$$

Which is the same result as in Maxwell's explanation eq(2)

4 Some concluding remarks

Maxwell's explanation was clearly more simple and elegant, hardly requiring any mathematics at all. It is more popular because it is easier to communicate. The approach we used today was more mathematically difficult but it was more rewarding. We could see the mechanism (no pun intended):

As the wave passes through the dielectric, the fields busily polarize and magnetize all the molecules, the resulting oscillating dipoles create their own electric and magnetic fields. These combine with original fields in such a way as to create a single wave with same frequency but a different speed. This conspiracy is reason why things can be **transparent**. Note that linearity of the medium plays an important role (allows us to form recurrence relations that lead to closed form solutions).

4.1 Fresnel equations

If we do the analysis for the other side that is in air , i.e., for $x < 0$, we get

$$\mathbf{E^R}(x,t) = \left(\frac{2}{1+\sqrt{1+\chi_e}}\right) E_0 \exp\left[i(-kx - wt)\right]\hat{j} \tag{9}$$

The speed of light is just $w/k = c$, nothing surprising. But if we note the coefficients from eq(6) and eq(7), we can answer what fraction of incident amplitude transmits through and what fraction gets reflected back.

$$\boxed{\frac{E_R}{E_0} = \left(\frac{1-\sqrt{1+\chi_e}}{1+\sqrt{1+\chi_e}}\right) \quad \text{and} \quad \frac{E_T}{E_0} = \left(\frac{2}{1+\sqrt{1+\chi_e}}\right)} \tag{10}$$

These are called the **Fresnel equations**. Actually, Fresnel equations are more general, we get those when we do the analysis on a wave that is incident at an angle of the interface. Obviously, they reduce to our equations in special case of when incident angle is $\pi/2$

4.2 Perturbation as a problem solving tool

One of the things to learn for us is that perturbation methods are a way of solving general problems in physics and mathematics.

One of the most famous application would be seen in **Quantum mechanics**. When we know the solution to a system in quantum mechanics, we can add a small part to the Hamiltonian as a perturbation and then using the same logic as here, we can consider that the new energy levels and wave functions are updated, and new contributions are present:

$$E_n^T = E_n^0 + E_n^1 + E_n^2 + ... \quad \text{and} \quad \left|\Psi_n^T\right\rangle = \left|\Psi_n^0\right\rangle + \left|\Psi_n^1\right\rangle + \left|\Psi_n^2\right\rangle + ...$$

As the perturbation goes to zero, our new Energies (or wave functions) tend to the old ones.

Other than that, perturbation methods can be used in optics (summing Intensities in bits), in solving differential equations, in developing theories of an harmonic oscillators, or wherever the user can use it. It is a very strong tool to attack general problems in physics and maths, and we should always remember that we have it in our toolkit.

4.3 References

- This presentation is heavily motivated by the paper by **Mary B. James** and **David J. Griffiths**, *Am. J. Phys.* **60**, 309(1992)
- Problem in section 2.1 is taken from the book Introduction to electrodynamics (3rd edition)

END