Partially Ordered Sets:— A relation R on a set A is called Partially ordering if it is Reflexive, Antisymmetric and transitive. i.e if

(i) $aRa, \forall a \in A$

(ii) aRb and $bRa \Rightarrow a=b$

(iii) aRb and $bRc \Rightarrow aRc$

then the set 'A' with partial Order Relation R is a partially ordered set or '<u>Poset</u>' and is denoted by (A,R)

- (Ex) Show that the relation $\geq$ is a partial Ordering on set of Integers Z

Relation R is $\geq$

Sol: (i) Reflexive $\forall a \in Z$; $a \geq a$

$\therefore aRa; \forall a \in Z$

$\therefore$ R is Reflexive

(ii) for $a, b \in Z$, $a \geq b$ and $b \geq a \Rightarrow a=b$

$\therefore aRb$ and $bRa \Rightarrow a=b$

$\therefore$ R is antisymmetric.

- (iii) for $a, b, c \in Z$ Now $a \geq b$ and $b \geq c \Rightarrow a \geq c$

i.e aRb and $bRc \Rightarrow aRc$

$\therefore$ R is Transitive

Since R is Reflexive, Antisymmetric and transitive

$\therefore$ R is partial Ordering

Note:— A partial ordering Relation R is often denoted by the symbol $\preceq$

Now $x \preceq y$ Means x precedes y

$x \prec y$ " x Strictly precedes y

Comparable Two Elements a & b in a poset $(S, \preceq)$ are said to be Comparable if either $a \preceq b$ or $b \preceq a$.
If neither $a \preceq b$ nor $b \preceq a$ then a & b are Called Incomparable.

Ex In Poset $(Z, '/')$ the Integer 3 and 9 are Comparble because 3/9 i.e 3 divides 9 and 5, & 7 are Incomparable since neither 5/7 nor 7/5

Total Ordering Relation or Linear Order or Chain A relation R on a set A is said to be total ordering Relation if R is Partial ordering and satisfy the law of Dichotomy i.e

- for each $a, b \in A$ either $a \preceq b$ or $b \preceq a$ i.e any two elements of A are Comparable.

Immedate Predecessor and Immediate Successor:

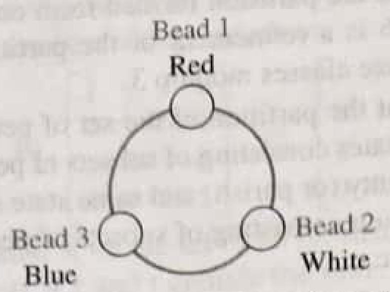
Let $(A, \preceq)$ be a poset and $a, b \in A$,
'a' is said to be Immedate Predecessor of b or
b " " " " Successor of 'a' written as $a \lessdot b$ if $a < b$ and no element of A lies between a & b i.e $\nexists c \in A$ such that $a < c < b$

- **Hasse Diagram / Representation of Poset:**

A graphical representation of a Partial ordering relation in which all arrows heads are understood to be pointing upward is known as Hasse diagram of the Relation.

Steps to Draw Hasse Diagram

(i) Draw the diagraph of given relation
(ii) Delete all loops for graph
(iii) Eleminat all the edges that are Implied by Transitive relation.
(iv) Draw the graph with edge pointing upward so that arrows may be omitted.
(v) Replace the circle Reprensenting the Vertices by Dots.



Define the relation R between bracelets as: (B_1, B_2), where B_1 and B_2 are bracelets, belongs to R if and only if B_2 can be obtained from B_1 by rotating it or rotating it and then reflecting it.

a) Show that R is an equivalence relation.
b) What are the equivalence classes of R?

***59.** Let R be the relation on the set of all colorings of the 2×2 checkerboard where each of the four squares is colored either red or blue so that (C_1, C_2), where C_1 and C_2 are 2×2 checkerboards with each of their four squares colored blue or red, belongs to R if and only if C_2 can be obtained from C_1 either by rotating the checkerboard or by rotating it and then reflecting it.

a) Show that R is an equivalence relation.
b) What are the equivalence classes of R?

60. a) Let R be the relation on the set of functions from $\mathbf{Z}^+$ to $\mathbf{Z}^+$ such that (f, g) belongs to R if and only if f is $\Theta(g)$ (see Section 3.2). Show that R is an equivalence relation.
b) Describe the equivalence class containing $f(n) = n^2$ for the equivalence relation of part (a).

... equivalence relations on a set with three elements by listing them.

62. Determine the number of different equivalence relations on a set with four elements by listing them.

***63.** Do we necessarily get an equivalence relation when we form the transitive closure of the symmetric closure of the reflexive closure of a relation?

***64.** Do we necessarily get an equivalence relation when we form the symmetric closure of the reflexive closure of the transitive closure of a relation?

65. Suppose we use Theorem 2 to form a partition P from an equivalence relation R. What is the equivalence relation R' that results if we use Theorem 2 again to form an equivalence relation from P?

66. Suppose we use Theorem 2 to form an equivalence relation R from a partition P. What is the partition P' that results if we use Theorem 2 again to form a partition from R?

67. Devise an algorithm to find the smallest equivalence relation containing a given relation.

***68.** Let $p(n)$ denote the number of different equivalence relations on a set with n elements (and by Theorem 2 the number of partitions of a set with n elements). Show that $p(n)$ satisfies the recurrence relation $p(n) = \sum_{j=0}^{n-1} C(n-1, j)p(n-j-1)$ and the initial condition $p(0) = 1$. (*Note:* The numbers $p(n)$ are called **Bell numbers** after the American mathematician E. T. Bell.)

69. Use Exercise 68 to find the number of different equivalence relations on a set with n elements, where n is a positive integer not exceeding 10.

7.6 Partial Orderings

Introduction

Links

We often use relations to order some or all of the elements of sets. For instance, we order words using the relation containing pairs of words (x, y), where x comes before y in the dictionary. We schedule projects using the relation consisting of pairs (x, y), where x and y are tasks in a project such that x must be completed before y begins. We order the set of integers using the relation containing the pairs (x, y), where x is less than y. When we add all of the pairs of the form (x, x) to these relations, we obtain a relation that is reflexive, antisymmetric, and transitive. These are properties that characterize relations used to order the elements of sets.

DEFINITION 1 A relation R on a set S is called a *partial ordering* or *partial order* if it is reflexive, antisymmetric, and transitive. A set S together with a partial ordering R is called a *partially ordered set*, or *poset*, and is denoted by (S, R). Members of S are called *elements* of the poset.

We give examples of posets in Examples 1–3.

EXAMPLE 1 Show that the "greater than or equal" relation ($\geq$) is a partial ordering on the set of integers.

Extra Examples

Solution: Because $a \geq a$ for every integer a, $\geq$ is reflexive. If $a \geq b$ and $b \geq a$, then $a = b$. Hence, $\geq$ is antisymmetric. Finally, $\geq$ is transitive because $a \geq b$ and $b \geq c$ imply that $a \geq c$. It follows that $\geq$ is a partial ordering on the set of integers and $(\mathbf{Z}, \geq)$ is a poset. ◀

EXAMPLE 2 The divisibility relation $|$ is a partial ordering on the set of positive integers, because it is reflexive, antisymmetric, and transitive, as was shown in Section 8.1. We see that $(\mathbf{Z}^+, |)$ is a poset. Recall that ($\mathbf{Z}^+$ denotes the set of positive integers.) ◀

EXAMPLE 3 Show that the inclusion relation $\subseteq$ is a partial ordering on the power set of a set S.

Solution: Because $A \subseteq A$ whenever A is a subset of S, $\subseteq$ is reflexive. It is antisymmetric because $A \subseteq B$ and $B \subseteq A$ imply that $A = B$. Finally, $\subseteq$ is transitive, because $A \subseteq B$ and $B \subseteq C$ imply that $A \subseteq C$. Hence, $\subseteq$ is a partial ordering on $P(S)$, and $(P(S), \subseteq)$ is a poset. ◀

Example 4 illustrates a relation that is not a partial ordering.

EXAMPLE 4 Let R be the relation on the set of people such that $x R y$ if x and y are people and x is older than y. Show that R is not a partial ordering.

Extra Examples

Solution: Note that R is antisymmetric because if a person x is older than a person y, then y is not older than x. That is, if $x R y$, then $y \not R x$. The relation R is transitive because if person x is older than person y and y is older than person z, then x is older than z. That is, if $x R y$ and $y R z$, then $x R z$. However, R is not reflexive, because no person is older than himself or herself. That is, $x \not R x$ for all people x. It follows that R is not a partial ordering. ◀

In different posets different symbols such as $\leq$, $\subseteq$, and $|$, are used for a partial ordering. However, we need a symbol that we can use when we discuss the ordering relation in an arbitrary poset. Customarily, the notation $a \preccurlyeq b$ is used to denote that $(a, b) \in R$ in an arbitrary poset (S, R). This notation is used because the "less than or equal to" relation on the set of real numbers is the most familiar example of a partial ordering and the symbol $\preccurlyeq$ is similar to the $\leq$ symbol. (Note that the symbol $\preccurlyeq$ is used to denote the relation in *any* poset, not just the "less than or equals" relation.) The notation $a \prec b$ denotes that $a \preccurlyeq b$, but $a \neq b$. Also, we say "a is less than b" or "b is greater than a" if $a \prec b$.

When a and b are elements of the poset $(S, \preccurlyeq)$, it is not necessary that either $a \preccurlyeq b$ or $b \preccurlyeq a$. For instance, in $(P(\mathbf{Z}), \subseteq)$, $\{1, 2\}$ is not related to $\{1, 3\}$, and vice versa, because neither set is contained within the other. Similarly, in $(\mathbf{Z}^+, |)$, 2 is not related to 3 and 3 is not related to 2, because $2 \nmid 3$ and $3 \nmid 2$. This leads to Definition 2.

DEFINITION 2 The elements a and b of a poset $(S, \preccurlyeq)$ are called *comparable* if either $a \preccurlyeq b$ or $b \preccurlyeq a$. When a and b are elements of S such that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, a and b are called *incomparable*.

EXAMPLE 5 In the poset $(\mathbf{Z}^+, |)$, are the integers 3 and 9 comparable? Are 5 and 7 comparable?

Solution: The integers 3 and 9 are comparable, because $3 \mid 9$. The integers 5 and 7 are incomparable, because $5 \nmid 7$ and $7 \nmid 5$. ◀

The adjective "partial" is used to describe partial orderings because pairs of elements may be incomparable. When every two elements in the set are comparable, the relation is called a **total ordering.**

6.1 Partial Order Set (Poset)

[U.P.T.U. (B.Tech.) 2005, 2007; R.G.P
M.K.U. (B.E.) 2005, 2008; I.G.N.O.U. (M.C.A.) 2001, 2003, 2005, 200

A non empty set A, together with a binary relation R is said to be a partially ordere
following conditions are satisfied.

(P_1) ***Reflexivity:*** $aRa\ \forall\ a \in A$

(P_2) ***Antisymmetry:*** If $a, b \in A$, then

$$aRb \text{ and } bRa \Rightarrow a = b$$

(P_3) ***Transitivity:*** If $a, b, c \in A$ then

$$aRb,\ bRc \Rightarrow aRc$$

The relation R on set A is called ***Partial Order Relation.*** The poset is denoted by (

Remark: We generally use the symbol $\le$ in place of R. We read $\le$ as less than or equal
have nothing to do with the usual less than or equal to relation.)

Example 1: *The set S on any collection of sets. The relation $\subseteq$ read as "is subset of" is pa*

[U.P.T.U. (B.Tech.) 2007; I.G.N.O.U (M.C.A.) 2009; Rohtak

Solution: The set S is poset if it satisfies the following conditions.

(P_1) ***Reflexive:*** Since $A \subseteq A$ for any subset A of S

(P_2) ***Antisymmetric:*** If $A \subseteq B$ and $B \subseteq A\ \forall\ A, B \in S$ then $A = B$

(P_3) ***Transitivity:*** If $A \subseteq B$ and $B \subseteq C$ for any sets $A, B, C \in S$, then $A = C$

Hence, $(S, \le)$ is a poset

Example 2: *A set $S = \{a, b, c\}$ together with the relation of set inclusion $\subseteq$ is a partial*
$P(S)$ is power set of S.

[U.P.T.U. (B.Tech.) 2002; R.G.P.V. (

Solution: The power set of S is $P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$

Then $P(S)$ is partial order relation or poset if it satisfies the following conditions.

(P_1) ***Reflexivity:*** Since every $A \subseteq A\ \forall\ A \in P(S)$. Hence, it is reflexive

(P_2) ***Antisymmetry:*** If $A \subseteq B$ and $B \subseteq A \Rightarrow A = B$. Hence, it is antisymmetric

(P_3) ***Transitivity:*** If $A \subseteq B, B \subseteq C \Rightarrow A \subseteq C$. Hence, it is transitive

$$\therefore \qquad (P(S), \subseteq) \text{ is poset.}$$

Example 3: *Let $A = \{2, 3, 6, 12, 24, 36\}$ and R be the relation in A which is defined by a divides b then ... partial order in A.* [R.G.P.V. (B.E.) Bhopal 2004, 2006, 2008, 2...

Solution: The relation divisor is a partial order if it satisfies the following conditions.

(P_1) Reflexivity: Since a '|' a $\forall a \in A$ $\quad\therefore$ '|' is reflexive

(P_2) Antisymmetric: If a '|' b and b '|' a $\forall a, b \in A$ then $a = b$ $\therefore$ '|' is antisymmetric

(P_3) Transitivity: If a '|' b and b '|' c $\forall a, b, c \in A$ then a '|' c

Illustration: 2 '|' 6, 6 '|' 12 $\Rightarrow$ 2 '|' 6

$\therefore$ '|' is transitive. Hence, (A, '|') is a poset.

Example 4: *The set of integers Z with usual ordering $\leq$ read as "Less than or equal to" is a poset.*

Solution: The set of integer Z is poset if it satisfies the following conditions.

(P_1) Reflexive: Since $a \leq a$ for every integer $a \therefore \leq$ is reflexive

(P_2) Antisymmetric: If $a \leq b$ and $b \leq a$ then $a = b$ $\forall a, b \in Z$. Hence $\leq$ is antisymmetric

(P_3) Transitive: If $a \leq b$ and $b \leq c$, where $a, b, c \in Z$ then $a \leq c \therefore \leq$ is a transitive relation

Hence, $(Z, \leq)$ is a poset.

Example 5: *If R is partially ordered relation on a subset X and $A \subseteq X$, show that*

$R \cap (A \times A)$ is a partial ordering relation on A. [M.K.U. (B.E.) 2...

Solution: Denote $R \cap (A \times A)$ by R' then R' will be partial order relation or poset if.

(P_1) Reflexive: Let $x \in A$ then $(x, x) \in A \times A$.

Since R is reflexive, $(x, x) \in R \Rightarrow x R x$

$$\therefore \quad (x, x) \in R \cap (A \times A) = R'$$

(P_2) Antisymmetric: Suppose $(x, y) \in R'$ and $(y, x) \in R'$

Then $(x, y) \in R \cap (A \times A)$ and $(y, x) \in R \cap (A \times A)$

Since R is antisymmetric, $(x, y) \in R$ and $(y, x) \in R$ then $x = y$ $\quad\therefore$ R' is antisymmetric

(P_3) Transitivity: Suppose $(x, y) \in R' = R \cap (A \times A)$ and $(y, z) \in R' = R \cap (A \times A)$

Then $(x, y), (y, z) \in R$ and $(A \times A)$ is R transitive.

$(x, y) \in R, (y, z) \in R \Rightarrow (x, z) \in R$. Since $R \subset A \times A$

$\therefore$ $(x, z) \in A \times A$ and hence $(x, z) \in R \cap (A \times A) = R'$

Thus $R' = R \cap (A \times A)$ is partial ordering relation on A.

6.1.1 Comparable

Two elements a and b is a poset $(S, \leq)$ are said to be comparable if either $a \leq b$ or $b \leq a$. Thus a and b ... **Incomparable** if neither $a \leq b$ nor $b \leq a$

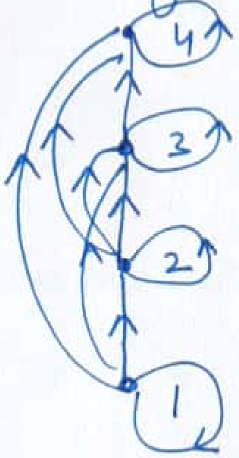
Illustration: ...

Ex Construct the Hasse diagram for $(\{1, 2, 3, 4\} ; \leq)$

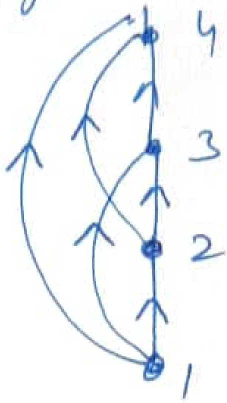
here set $A = \{1, 2, 3, 4\}$ and Relation is $\leq$

Now $R = \{(1,1)\ (2,2)\ (3,3)\ (4,4)\ (1,2)\ (1,3)\ (1,4)$
$(2,3)\ (2,4)\ (3,4)\}$

Clearly R is Partial Ordering Relation



Delete all the loops $\longrightarrow$



Elimate Edges that are Implied by Transitive relation $\longrightarrow$

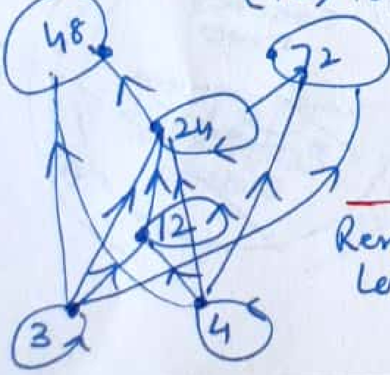
i.e $(1,2)(2,3) \Rightarrow (1,3)$

and also remove the arrows

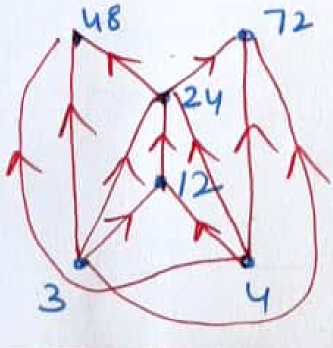
This is required Hasse diagram.

Ex Draw the Hasse diagram of $(A, \leq)$ where $A = \{3, 4, 12, 24, 48, 72\}$ and the relation $\leq$ be such that $a \leq b$ if "a divides b"

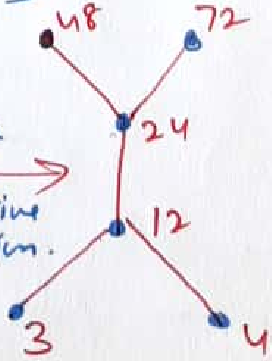
$(A, \leq) = \{(3,3)\ (4,4)\ (12,12)\ (24,24)\ (48,48)\ (72,72)\ (3,12)\ (3,24)$
$(3,48)\ (3,72)\ (4,12)\ (4,24)\ (4,48)\ (4,72)\ (12,24)$
$(12,48)\ (12,72)\ (24,48)\ (24,72)\}$



Remove Loops $\longrightarrow$



Remove Transitive Relation. $\longrightarrow$



Ex Construct the Hasse diagram of the relation S defined as "divides" on set $A = \{2, 3, 4, 6, 12, 36, 48\}$ i.e aSb if "a divides b"

Sol $(A:\preceq) = \{(a,b) ;$ "a divides b"$\}$

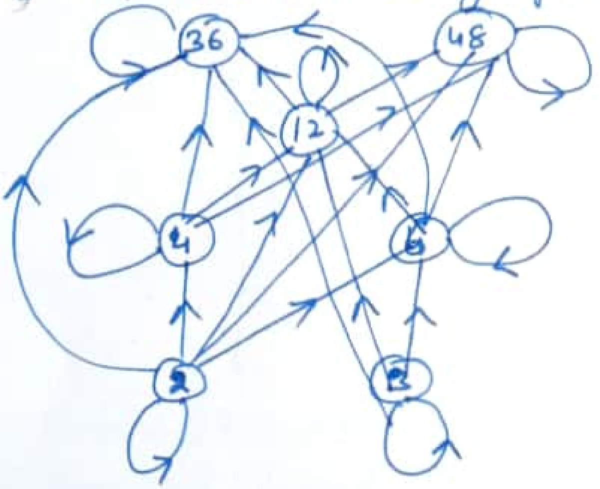
$R = (A, \preceq) = \{(2,4)\,(2,6)\,(2,12)\,(2,36)\,(2,48), (3,6)\,(3,12)\,(3,36)\,(4,12)$
$(4,36)\,(4,48)\,(6,12)\,(6,36)\,(6,48)\,(12,36)\,(12,48)\}$
$(2,2)\,(3,3)\,(4,4)\,(6,6)\,(12,12)\,(36,36)\,(48,48)$

2 divides 4 $\Rightarrow (2,4) \in R$

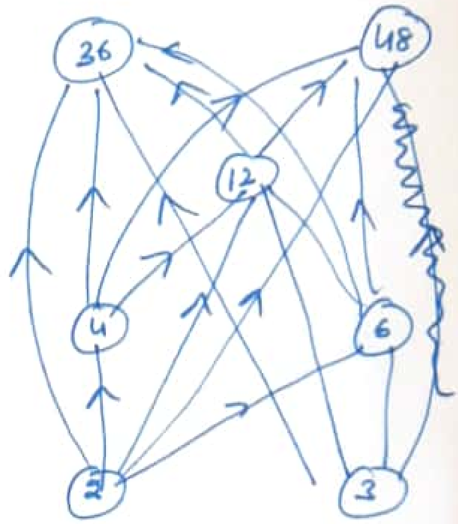
2 ,, 6 $= (2,6) \in R$

- - - - - - - - so on.

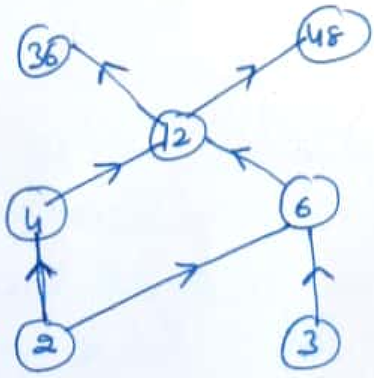
Now. The directed graph.



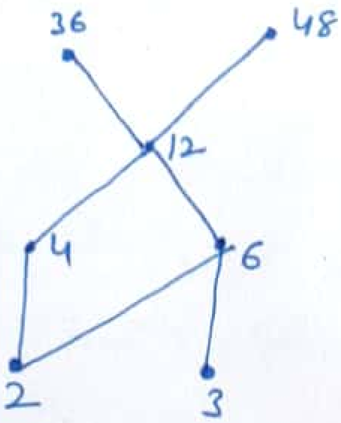
Remove the loop from the graph



Eliminate the edges that Implied by Transitive Relation.



Remove the Arrows



Which is the Required

Solution: (i) The Hasse diagram is given below

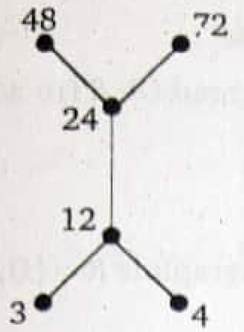


Fig. 6.5

(ii) The Hasse diagram is given below.

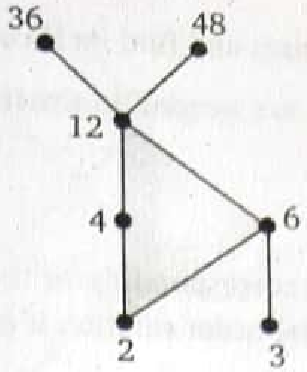


Fig. 6.6

Example 8: *Let A be the set of factors of a particular positive integer m and let ≤ be the relation "divides" i.e.* $\leq = \{(x, y): x \in A, y \in A$ *and* $x \text{ '|' } y\}$ *draw Hasse diagrams for*

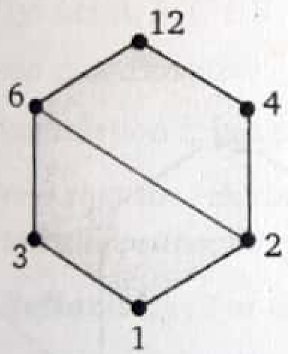
(a) $m = 12$ (b) $m = 30$ (c) $m = 45$

[U.P.T.U. (B.Tech.) 2005, 2008; M.C.A. (Uttranchal) 2007, 2009; M.K.U. (B.E.) 2001, 2003; U.P.T.U. (M.C.A.) 2008-2009]

Solution: (a) $A = \{1, 2, 3, 4, 6, 12\}$

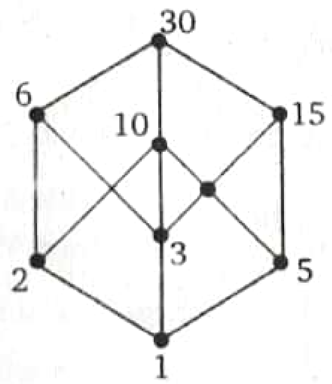
(b) $A = \{1, 2, 3, 5, 6, 10, 15\}$ (c) $A = \{1, 3, 5, 9, 15\}$

The Hasse diagrams is given below



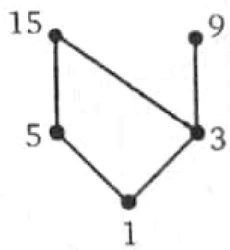
(a)

Fig. 6.7



(b)

Fig. 6.8



(c)

Fig. 6.9

Example 9: *Draw the Hasse diagram for the partial ordering* $\{(A, B): A \subseteq B\}$ *on the power set* $P(S)$ *where* $S = \{a, b, c\}$.

[U.P.T.U. (B.Tech.) 2003; R.G.P.V. (Bhopal) 2005, 2009]

Solution: Let $S = \{a, b, c\}$, then

$$P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

The Hasse diagram of the poset $(P(S), \subseteq)$ is

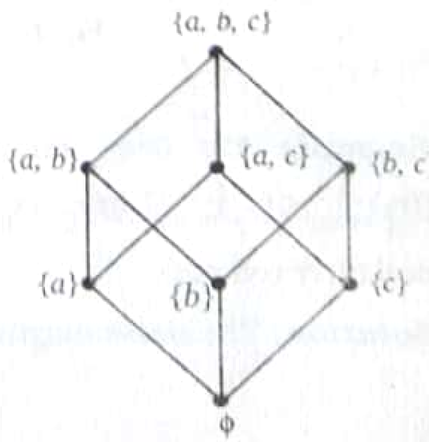


Fig. 6.10

Example 10: *The directed graph G for a relation R on set* $A = \{1, 2, 3, 4\}$ *is shown in Fig. 6.11.*

(i) Verify that (A, R) is a poset and find its Hasse diagram.

(ii) How many more edges are needed in the figure to extend (A, R) to a total order?

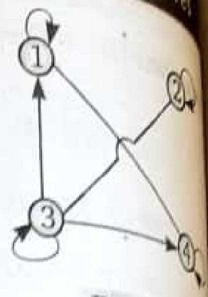


Fig. 6.11

Solution: (i) The relation R corresponding to the given digraph is $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (3, ...$ $(3, 4), (1, 4), (3, 2)\}$ R is a partial order relation if it is

(P_1) Reflexive: Since $aRa \ \forall \ a \in A$. Hence, it is reflexive.

(P_2) Antisymmetric: Since aRb and bRa then we get $a = b$. Hence, it is antisymmetric.

(P_3) Transitive: For every aRb and $bRc \Rightarrow aRc$. Hence, it is transitive.

Therefore (A, R) is poset. Its Hasse diagram is

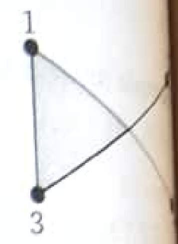


Fig. 6.12

(ii) Only one edge $(4, 2)$ is included to make it total order.

Example 11: *Let* $A = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$ *be ordered by the relation "a divides b". Draw the Hasse diagram.*

[P.T.U. (B.E.) Punjab ...]

Solution: The Hasse diagram is given below

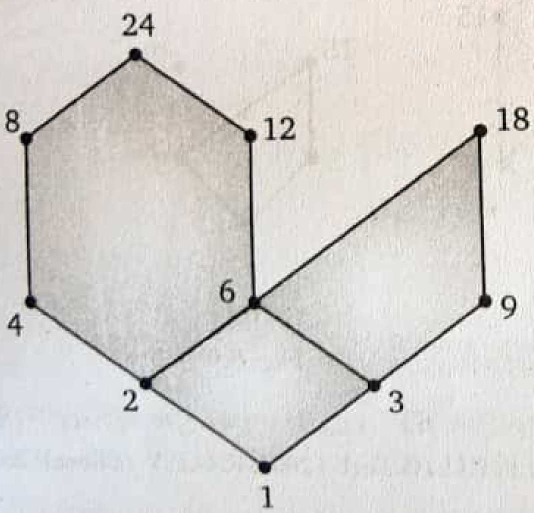


Fig. 6.13

Or

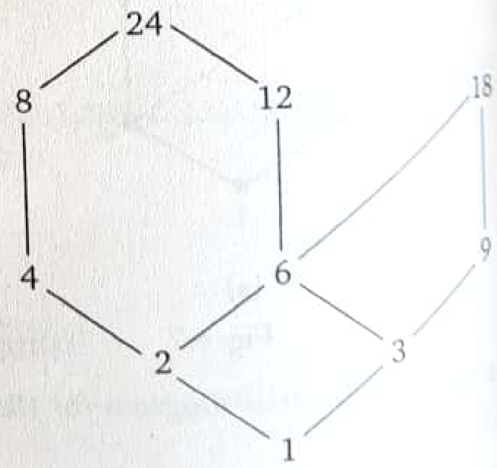


Fig. 6.14

Example 12: *Draw the Hasse diagram of* B^3 *with "≤" as defined for matrices of o's and ...* $[(r_{ij})_{m \times n} \le (s_{ij})_{m \times n}]$ *iff* $r_{ij} \le s_{ij} \ \forall \ i$ *and* j *where* $0 \le 0, 0 \le 1, 1 \le 1, 1 \not\le 0$, *where* B^3 *are matrices with one ... and three column.*

[R.G.P.V. (B.E.) Bhopal 2006; R.G.P.V. (B.E.) Raipur ...]

Solution: The Hasse diagram of R^3 is given below.

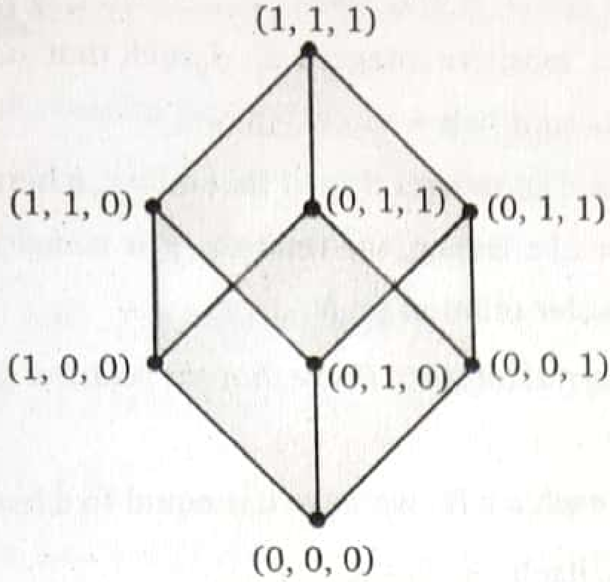


Fig. 6.15

Example 13: *Let S be any class of sets. Prove that the relation of set inclusion $\subseteq$ is a partial order relation on S*

Solution: (P_1) Reflexivity: For each $A \in S$, we have $A \subseteq A$.

Hence, the relation $\subseteq$ is reflexive.

(P_2) **Anti-symmetry:** Let $A, B \in S$. Then $A \subseteq B, B \subseteq A \Rightarrow A = B$.

Hence, the relation $\subseteq$ is anti-symmetric.

(P_3) **Transitivity:** Let $A, B, C \in S$. Then $A \subseteq B, B \subseteq C \Rightarrow A \subseteq C$.

Hence, the relation $\subseteq$ is transitive.

Thus, the inclusion relation $\subseteq$ is a partial order relation on the set S.

Example 14: *Prove that the relation "a divides b", if there exists an integer c such that $ac = b$ and is denoted by $a | b$, on the set of all positive integers **N** is a partial order relation.*

Solution: (P_1) Reflexivity: For each $a \in \mathbf{N}$. $\exists$ an integer 1 such that

$a1 = a \Rightarrow a | a$. Hence, the relation '|' is reflexive.

(P_2) **Anti-symmetry:** Let $a, b \in \mathbf{N}$ and $a | b$, $b | a$. Then

$a | b, b | a \Rightarrow \exists$ integers c_1, c_2 such that $ac_1 = b, bc_2 = a$

$$\Rightarrow \quad c_1 = \frac{b}{a}, c_2 = \frac{a}{b}$$

$$\Rightarrow \quad c_1 c_2 = \frac{b}{a} \cdot \frac{a}{b}$$

$$\Rightarrow \quad c_1 c_2 = 1$$

$$\Rightarrow \quad c_1 = c_2 = 1$$

$$\Rightarrow \quad a = b, \text{ by (1)}.$$

Hence, the relation '|' is anti-symmetric.

(P_3) Transitivity: Let $a, b, c \in \mathbf{N}$ and $a \mid b$, $b \mid c$. Then

$$a \mid b,\ b \mid c \Rightarrow \exists \text{positive integers } d_1, d_2 \text{ such that } ad_1 = b,\ bd_2 = c$$
$$\Rightarrow ad_1d_2 = c$$
$$\Rightarrow \exists \text{ an integer } d \text{ such that } ad = c, \text{ where } d = d_1d_2$$
$$\Rightarrow a \mid c. \text{ Hence, the relation '}\mid\text{' is transitive.}$$

Thus, the relation '|' is a partial order relation on $\mathbf{N}$.

Example 15: *Let* **N** *be the set of partial integers. Prove that the relation $\leq$, where $\leq$ has its usual meaning, is positive order relation on* **N**. [U.P.T.U. (B.Tech.) 200]

Solution: (P_1) Reflexivity: For each $a \in \mathbf{N}$, we have a is equal to a itself

$\Rightarrow a$ is less than or equal to a itself $\Rightarrow \quad a \leq a$.

Hence, the relation $\leq$ is reflexive.

(P_2) Antisymmetry: Let $a, b \in \mathbf{N}$ and $a \leq b$, $b \leq a$. Then $a \leq b$, $b \leq a \Rightarrow a = b$.

Hence, the relation $\leq$ is antisymmetry.

(P_3) Transitivity: Let $a, b, c \in \mathbf{N}$ and $a \leq b$, $b \leq c$. Then $a \leq b$, $b \leq c \Rightarrow a \leq c$.

Hence, the relation $\leq$ is transitive.

Thus, the relation "less than or equal" denoted by $\leq$ is a partial order relation on the set N.

Example 16: *Let X be the set of all 2×2 real matrices. Let*

$$x, y \in X,\ x = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix},\ y = \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix}$$

and the relation $x \leq y$ has the meaning: $x_1 + x_2 + x_3 + x_4 \leq y_1 + y_2 + y_3 + y_4$.

Prove that $\leq$ is not a partial order relation on the set X.

Solution: The given relation $\leq$ is not antisymmetric on the set X because

$$x \leq y, y \leq x \Rightarrow x_1 + x_2 + x_3 + x_4 \leq y_1 + y_2 + y_3 + y_4,\ y_1 + y_2 + y_3 + y_4 \leq x_1 + x_2 + x_3 + x_4$$
$$\Rightarrow x_1 + x_2 + x_3 + x_4 = y_1 + y_2 + y_3 + y_4$$

Always $\quad \not\Rightarrow x = y$.

For example, $\quad x = \begin{bmatrix} 2 & 3 \\ 4 & -5 \end{bmatrix},\ y = \begin{bmatrix} -2 & -3 \\ 8 & 1 \end{bmatrix} \in X,$

where $\quad x_1 + x_2 + x_3 + x_4 = 2 + 3 + 4 + (-5) = 4$

and $\quad y_1 + y_2 + y_3 + y_4 = (-2) + (-3) + 8 + 1 = 4$

i.e., $\quad x_1 + x_2 + x_3 + x_4 = y_1 + y_2 + y_3 + y_4$; nevertheless

$$\begin{bmatrix} 2 & 3 \\ 4 & -5 \end{bmatrix} \neq \begin{bmatrix} -2 & -3 \\ 8 & 1 \end{bmatrix}$$

i.e., $\quad x \neq y$.

Example 17: *Let X be the set of all complex numbers $z = x + iy$. For, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, let the meaning of $z_1 \le z_2$ be: $x_1 \le x_2$ an $y_1 \le y_2$ where $\le$ has usual meaning for the real numbers.*

(i) *Is $\le$ an order relation on X?* (ii) *Is $(X, \le)$ a chain?*

Solution: Let $X = \{z = x + iy \mid x, y \in \mathbf{R}\}$. Then we see that

(i) (P_1) ***Reflexivity:*** For each $z = x + iy \in X$, we have

$$x = x,\ y = y \Rightarrow x \le x,\ y \le y$$

$$\Rightarrow z \le z \text{ i.e., } z \le z \text{ for each } z \in X. \text{ Hence, the relation } \le \text{ is reflexive.}$$

(P_2) ***Antisymmetry:*** Let $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in X$ and $z_1 \le z_2, z_2 \le z_1$.

Then
$$z_1 \le z_2, z_2 \le z_1 \Rightarrow x_1 \le x_2 \text{ and } y_1 \le y_2, x_2 \le x_1 \text{ and } y_2 \le y_1$$
$$\Rightarrow x_1 \le x_2, x_2 \le x_1 \text{ and } y_1 \le y_2, y_2 \le y_1$$
$$\Rightarrow x_1 = x_2 \text{ and } y_1 = y_2$$
$$\Rightarrow x_1 + iy_1 = x_2 + iy_2$$
$$\Rightarrow Z_1 = Z_2$$

Hence, the relation $\le$ is antisymmetric on the set X.

(P_3) ***Transitivity:*** Let $Z_1 = x_1 + iy_1, Z_2 = x_2 + iy_2, Z_3 = x_3 + iy_3 \in X$

and $Z_1 \le Z_2, Z_2 \le Z_3$, then

$$Z_1 \le Z_2, Z_2 \le Z_3 \Rightarrow x_1 \le x_2 \text{ and } y_1 \le y_2 \text{ and } x_2 \le x_3, y_2 \le y_3$$
$$\Rightarrow x_1 \le x_3 \text{ and } y_1 \le y_3$$
$$\Rightarrow Z_1 \le Z_3$$

Hence the relation $\le$ is transitive on the set X. Therefore, $\le$ is partially ordered relation on X.

(ii) The given relation $\le$ is not totally ordered on the set X, because the law of dichotomy is in general are not true in X. If $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in Z$ then it is not necessary true that either

$$z_1 \le z_2 \text{ or } z_2 \le z_1$$

For example, if $z_1 = 3 + 4i, z_2 = 5 - 7i \in X$, then $3 < 5,\ -7 < 4$

$\therefore$ neither $z_1 \le z_2$ nor $z_2 \le z_1$. Hence, $(X, \le)$ is not a chain.

Example 18: *Draw the Hasse diagrams of*

(i) $(D_8, `|\text{'})$ (ii) $(D_6, `|\text{'})$ **[M.C.A. (Uttranchal) 2007]**

(iii) $A = \{2, 3, 5, 30, 60, 120, 180, 360, `|\text{'}\}$ **[I.G.N.O.U. 2004, 2009; R.G.P.V. (B.E.) Bhopal 2005, 2007]**

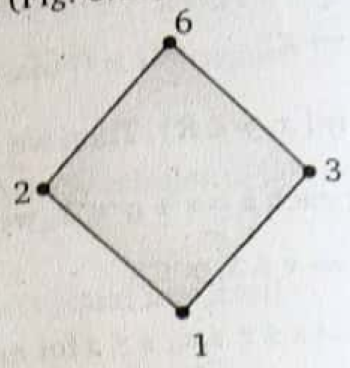
(iv) $h = \{1, 2, 3, 4, 6, 9, `|\text{'}\}$ **[Rohtak (B.E.) 2008]**

Solution: The Hasse diagrams of all these posets are given as

(i) We have $D_8 = \{1, 2, 4, 8\}$, Relation $\leq$ '$|$' = division (Fig. 6.16)

(D_8, '$|$')

Fig. 6.16



D_6, '$|$'

Fig. 6.17

(ii) We have $D_6 = \{1, 2, 3, 6\}$ relation $\leq$ = '$|$' = divisor (Fig. 6.17)

(iii) We have $A = \{2, 3, 5, 30, 60, 120, 180, 360$, '$|$'$\}$ relation is divisor *i.e.* $a | b$ (Fig. 6.18)

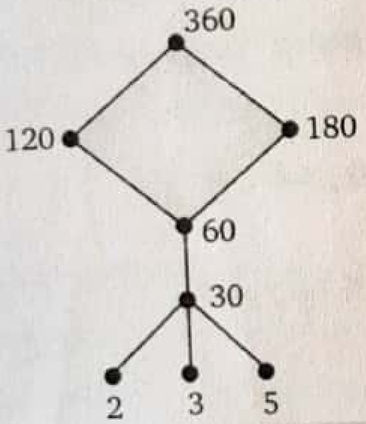


Fig. 6.18

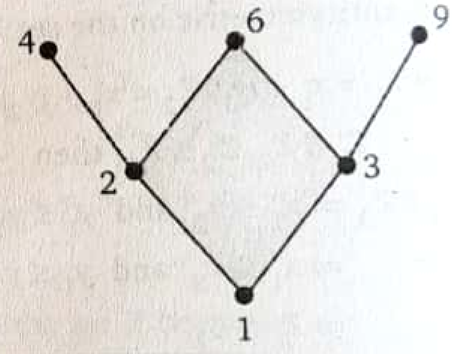


Fig. 6.19

(iv) We have $h = \{1, 2, 3, 4, 6, 9$, '$|$'$\}$ relation is divisor *i.e.* $a | b$ Fig. (6.19)

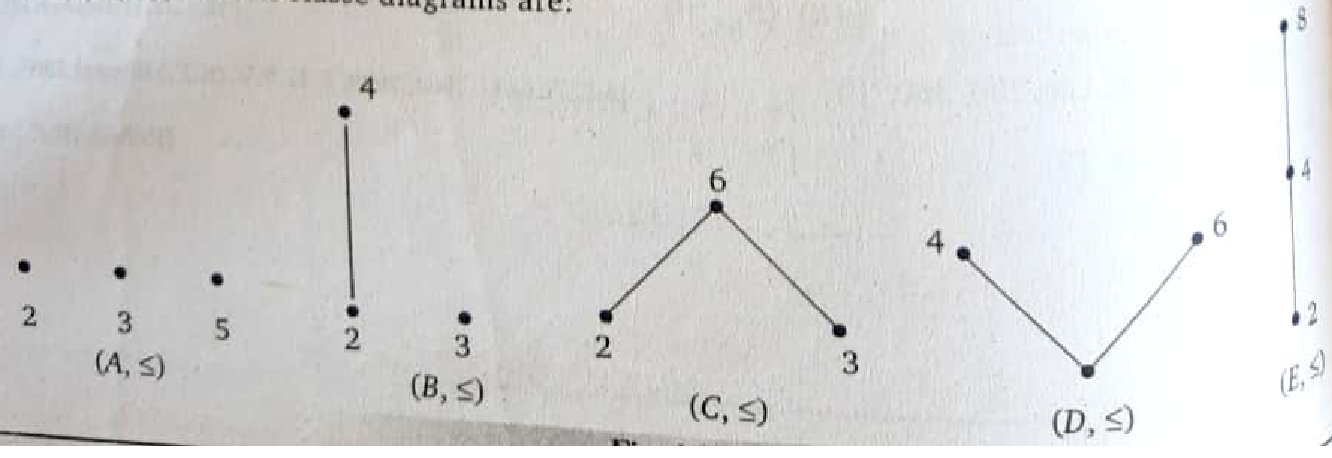
Example 19: *Show that there are only five distinct Hasse diagrams for partially ordered sets that contain three elements.*

Solution: Let us consider that $a \leq b$ if $a | b$ and consider the set

$A = \{2, 3, 5\}$ $\qquad B = \{2, 3, 4\}$

$C = \{2, 3, 6\}$ $\qquad D = \{2, 4, 6\}$

$E = \{2, 4, 8\}$, then its Hasse diagrams are:



Example 20: *Determine the Hasse diagram of the relation R . A = {1, 2, 3, 4}*

$R = \{(1, 1), (1, 2), (2, 2), (2, 4), (1, 3), (3, 3), (3, 4), (1, 4), (4, 4)\}$ **[Nagpur (B.E.) 2008]**

Solution: Diagraph for the given relation set R is

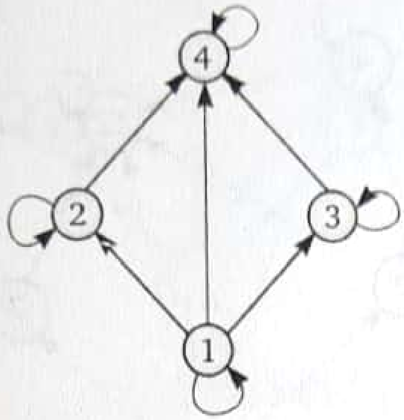


Fig. 6.21

Step 1: Remove Cycles Fig. (6.22)

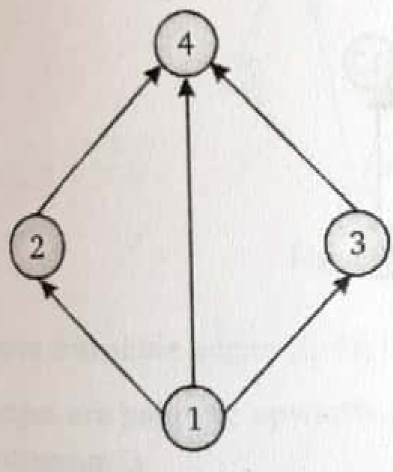


Fig. 6.22

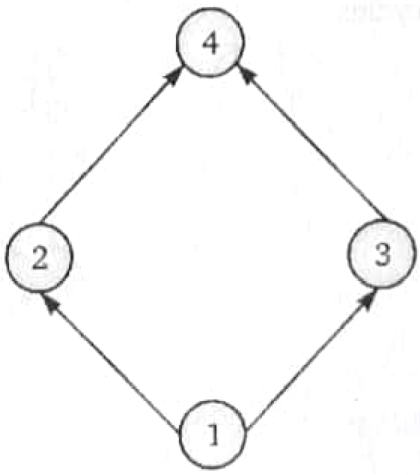


Fig. 6.23

Step 2: Remove transitive edge (Fig. 6.23)

Step 3: All edges are pointing upwards remove arrows from edges, replace circles by dots.

Hence Hasse diagram is

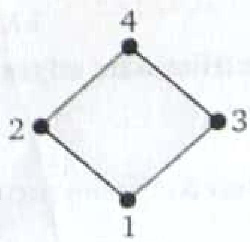


Fig. 6.24

Example 21: *Draw Hasse diagram for the following relations on set*

(i) $A = \{1, 2, 3, 4, 12\}$

$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (12, 12), (1, 2), (4, 12), (1, 3), (1, 4),$
$(1, 12), (2, 4), (2, 12), (3, 12)\}$

(ii) $A = \{1, 2, 3, 4, 5\}$

$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 4), (3, 5), (2, 2), (3, 3), (4, 4), (5, 5)\}$

(*iii*) $A = \{1, 2, 3, 4, 5\}$

$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 2), (4, 5), (1, 3), (1, 4), (1, 5), (2, 4), (2, 5), (3, 5)\}$

Solution: (i) The digraph for the given relation R is

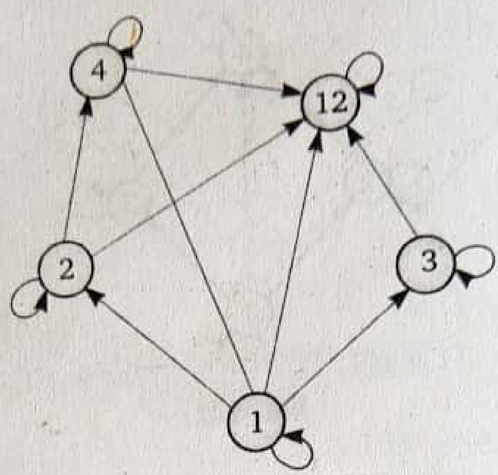


Fig. 6.25

Step 1: Remove cycles

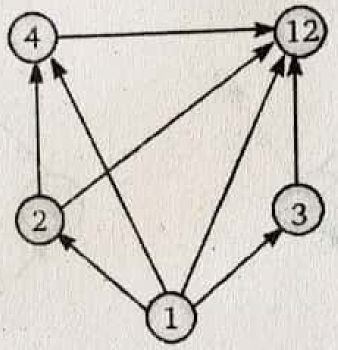


Fig. 6.26

Step 2: Remove transitive edge:

$$1R2,\ 2R4 \Rightarrow 1R4$$
$$2R4,\ 4R12 \Rightarrow 2R12$$
$$1R4,\ 4R12 \Rightarrow 1R12$$

i.e. eliminate edges (1, 4), (2, 12), (1, 12). All arrows are pointing upwards

Step 3: Circles are replaced by dots. Arrows are also removed. Hence Hasse diagram

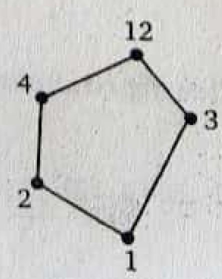


Fig. 6.28

(ii) The diagraph for the given relation R is

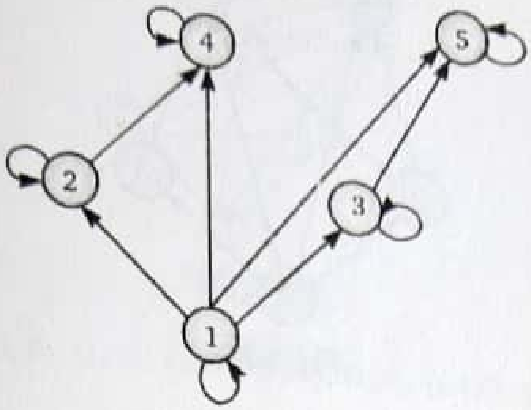


Fig. 6.29

Step 1: Remove Cycles (Fig. 6.30)

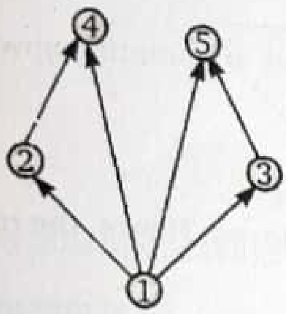


Fig. 6.30

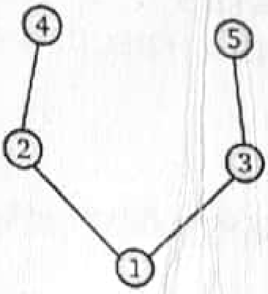


Fig. 6.31

Step 2: Remove transitive edges (1, 4), (1, 5) (Fig. 6.31)

Step 3: All edges are painting upwards, remove arrows from edges replace circles by dots. Hence Hasse diagram is

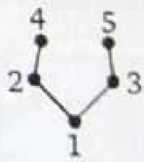


Fig. 6.32

(iii) The diagraph for the given relation set R is

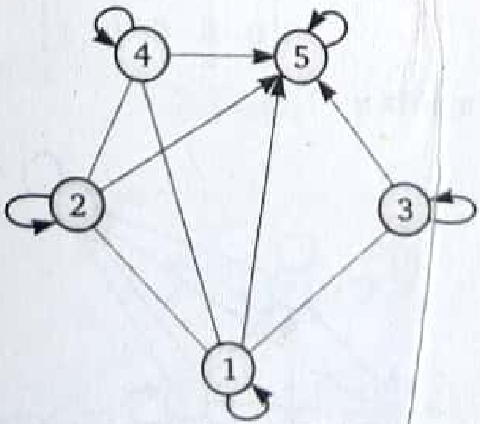


Fig. 6.33

Step 1: Remove Cycles

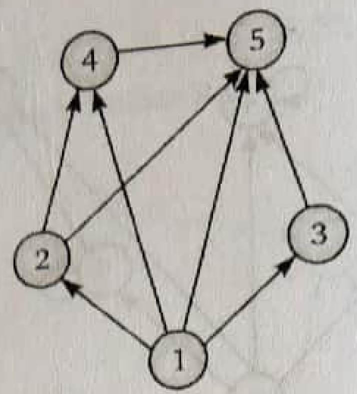


Fig. 6.34

Step 2: Remove transitive edges

$$1\,R\,2,\ 2\,R\,4 \Rightarrow 1\,R\,4$$
$$2\,R\,4,\ 4\,R\,5 \Rightarrow 2\,R\,5$$
$$1\,R\,4,\ 4\,R\,5 \Rightarrow 1\,R\,5$$

i.e. eliminate transitive edges (1, 4), (2, 5), (1, 5). All arrows are pointing upwards.

Fig. 6.35

Step 3: Circles are replaced by dots. Arrows are also removed. Hence, the required Hasse diagram is

Fig. 6.36

Example 22: *Determine the Hasse diagram of the relation on* $A = \{1, 2, 3, 4, 5\}$, *whose matrix is shown.*

(*i*) $M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$ (*ii*) $M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$

Solution: (i) The digraph for given matrix is

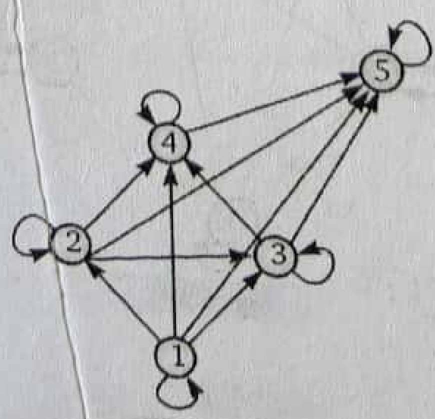


Fig. 6.37

Step 1: Remove Cycles

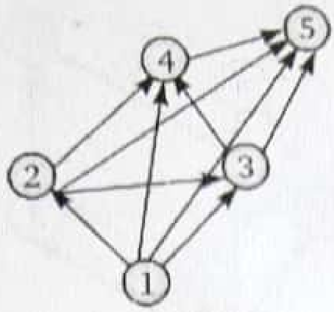


Fig. 6.38

Step 2: Remove transitive edges (2, 5), (1, 3), (2, 4), (1, 5), (1, 4) and (3, 5)

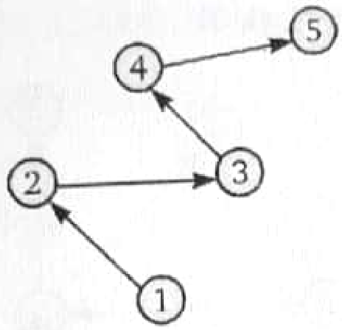


Fig. 6.39

Step 3: Circles are replaced by dots and all edges are pointing upwards. Arrows are removed.

Hence, The required Hasse diagram is

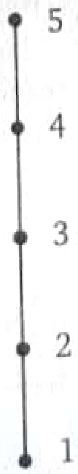
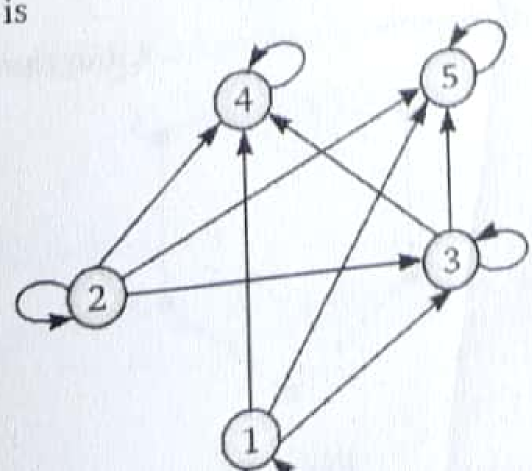


Fig. 6.40

(ii) The digraph for given matrix is



Step 1: Remove Cycles

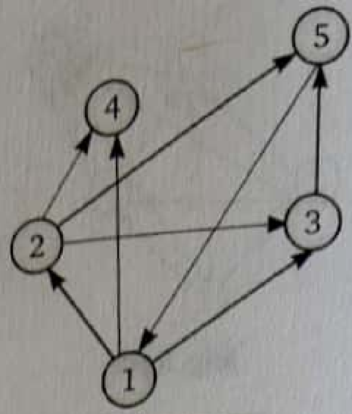


Fig. 6.42

Step 2: Remove transitive edges (2, 4), (1, 4), (1, 5)

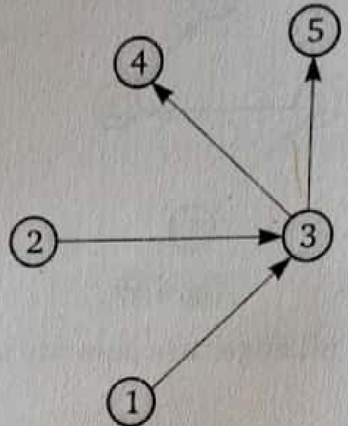


Fig. 6.43

Step 3: Circles are replaced by dots. Arrows are removed. All edges are pointing upwards. Hence required Hasse diagram is

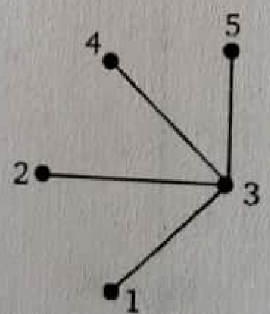


Fig. 6.44

Example 23: *Determine the matrix of the partial order whose diagram is given in fig.* 6.45

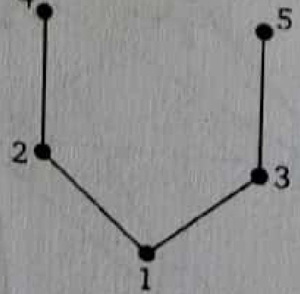


Fig. 6.45

Solution: Step 1: Put arrow on every edge, replace dots by circles

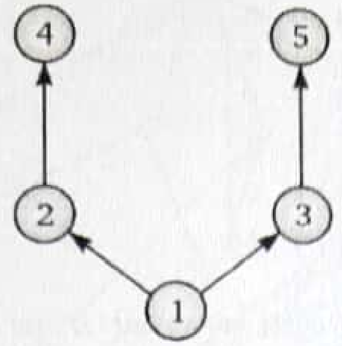


Fig. 6.46

Step 2: Put transitive edges

i.e.
$$1R2 \text{ and } 2R4 \Rightarrow 1R4$$
$$1R3 \text{ and } 3R5 \Rightarrow 1R5$$

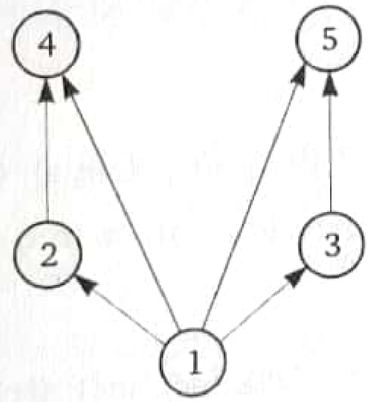


Fig. 6.47

Step 3: Put cycles on all circles

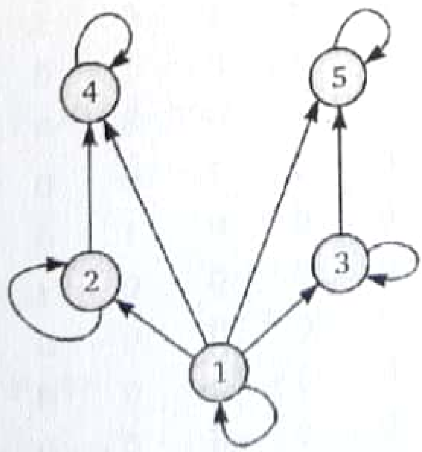


Fig. 6.48

Step 4: Relation set for the above digraph is

$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 2), (2, 4), (1, 4), (1, 3), (3, 5), (1, 5)\}$

Step 5: The matrix for above relation set is

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Example 24: *Let $A = \{a, b, c, d\}$ and $P(A)$ is its power set. Draw diagram for $(P(A), \subseteq)$.*

[R.G.P.V. (B.E.) Bhopal 2003, 20

Solution: We have $A = \{a, b, c, d\}$. Then

$$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\},$$
$$\{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}\}$$

Then $P(A)$ is poset if

The partial order relation R of set $P(A)$

$$R = \{(\phi, \phi), \{\phi, a\}, \{\phi, b\}, \{\phi, c\}, \{\phi, d\}, \{\phi, (a, b)\}, \{\phi, (a, c)\} \{\phi, (a, d)\}, \{\phi, (b, c)\}, \{\phi, b, d\},$$
$$\{\phi, c, d\}, \{\phi, (a, b, c)\}, (\phi, (a, b, d)), \{\phi, (a, c, d)\}, \{\phi, (b, c, d)\}, \{\phi, (a, b, c, d)\} \ldots\}$$

Matrix of Above Relation is given by:

$MR =$

	ϕ	{a}	{b}	{c}	{d}	{ab}	{ac}	{ad}	{bc}	{bd}	{cd}	{abc}	{abd}	{acd}	{bcd}	{abcd}
ϕ	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
{a}	0	1	0	0	0	1	1	1	0	0	0	1	1	1	0	1
{b}	0	0	1	0	0	1	0	0	1	1	0	1	1	0	1	1
{c}	0	0	0	1	0	0	1	0	1	0	1	1	0	1	1	1
{d}	0	0	0	0	1	0	0	1	0	1	1	0	1	1	1	1
{a, b}	0	0	0	0	0	1	0	0	0	0	0	1	1	0	0	1
{a, c}	0	0	0	0	0	0	1	0	0	0	0	1	0	1	0	1
{a, d}	0	0	0	0	0	0	0	1	0	0	0	0	1	1	0	1
{b, c}	0	0	0	0	0	0	0	0	1	0	0	1	0	0	1	1
{b, d}	0	0	0	0	0	0	0	0	0	1	0	0	1	0	1	1
{c, d}	0	0	0	0	0	0	0	0	0	0	1	0	0	1	1	1
{a, b, c}	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1
{a, b, d}	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1
{a, c, d}	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1
{b, c, d}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
{a, b, c, d}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

The Hasse diagram for given relation R is shown in fig. 6.49

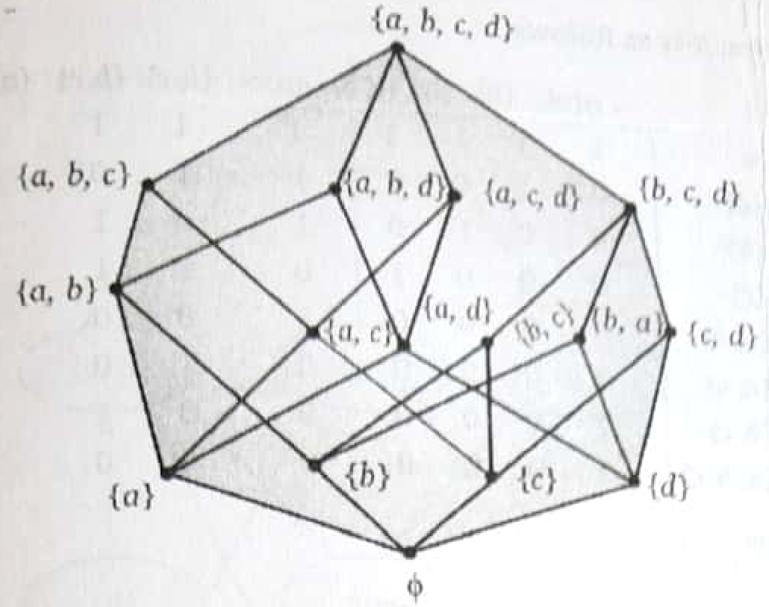


Fig. 6.49

Example 25: *Let $A = \{a, b, c\}$. Show that $(P(A), \subseteq)$ is a poset and draw its Hasse diagram.*

[U.P.T.U. (B.Tech.) 2007]

Solution: We have $A = \{a, b, c\}$. Then

$$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

Then $(P(A), \subseteq)$ will be posets if

(P_1) **Reflexivity:** For each set $B \subseteq P(A)$. We have

$$B \subseteq B \quad i.e. \quad B\,R\,B.\ \text{So} \subseteq \text{is reflexive}$$

(P_2) **Antisymmetry:** For any $B, C \in P(A)$, we have

$$B \subseteq C,\ C \subseteq B \Rightarrow B = C$$

i.e. $\quad B\,R\,B, C\,R\,B \Rightarrow B = C$

So $\quad \subseteq$ is antisymmetric

(P_3) **Transitivity:** For any $B, C, D \in P(A)$, we have

$$B \subseteq C, C \subseteq D \Rightarrow B \subseteq D$$

i.e. $\quad B\,R\,C, C\,R\,D \Rightarrow B\,R\,D$

So $\quad \subseteq$ is transitive on $P(A)$

Thus $\subseteq$ is a partial order relation on $P(A)$

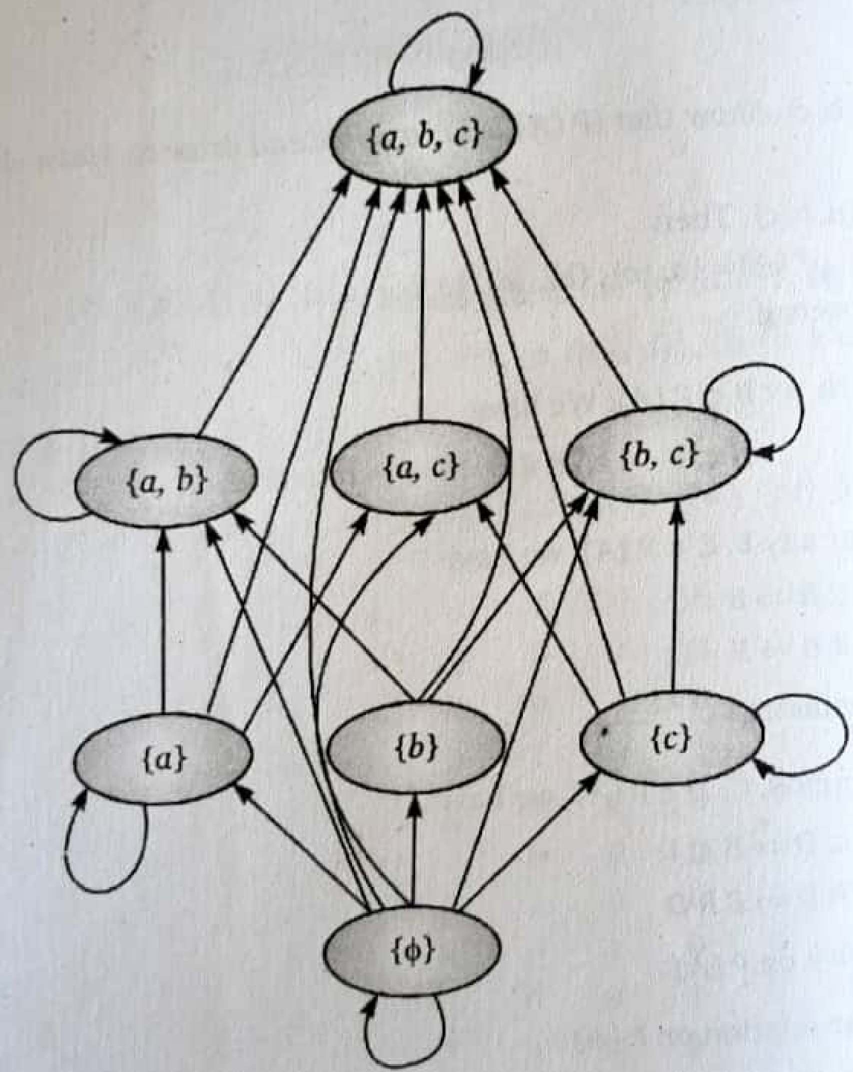
The relation R on $P(A)$ is as

$R = \{(\phi, \phi), (\phi, \{a\}), (\phi, \{b\}), (\phi, \{c\}), (\phi, \{a, b\}), (\phi, \{a, c\}), \{\phi, \{b, c\}, (\phi, \{a, b, c\}, (\{a\}, \{b\}),$
$(\{a\}, \{c\}), (\{a\}, \{a, b\}), (\{a\}, \{a, c\}), (\{a\}, \{b, c\}), (\{a\}, \{a, b, c\}\}, (\{b\}, \{c\}), (\{b\}, \{a, b\}),$
$(\{b\}, \{a, c\}), (\{b\}\ \{b, c\}), (\{b\}, \{a, b, c\}), (\{c\}, \{a, b\}), (\{c\}, \{a, c\}), (\{c\}, \{b, c\}),$
$(\{c\}, \{a, b, d\}), (\{a, b\}, \{a, c\}), (\{a, b\}, \{b, c\}), (\{a, b\}, \{a, b, c\}), (\{a, c\}, \{b, c\}),$
$(\{a, c), \{a, b, c)\}, (\{b, c\}, \{a, b, c\})\}$

The matrix of above relation R is as follows:

$$M_R = \begin{array}{c|cccccccc} & \phi\{a\} & \{b\} & \{c\} & \{a,b\} & \{a,c\} & \{b,c\} & \{b,c\} & \{a,b,c\} \\ \hline \phi & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \{a\} & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ \{b\} & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ \{c\} & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ \{a,b\} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ \{a,c\} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ \{b,c\} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ \{a,b,c\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

Digraph of this matrix M_R is



To convert this digraph into Hasse diagram.

Step 1: Remove Crycles

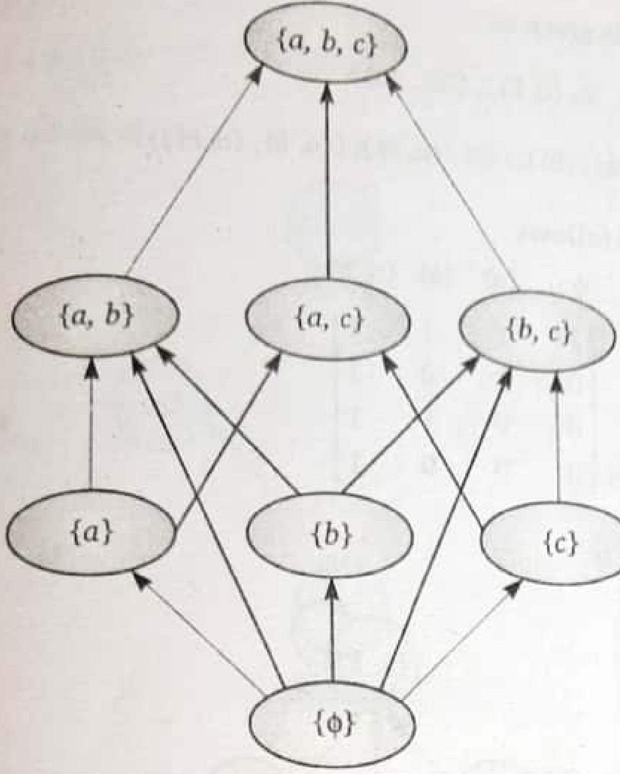


Fig. 6.51

Step 2: Remove transitive edges (ϕ, {a, b}), (ϕ, {a, c}), (ϕ, {b, c}), (ϕ, {a, b, c}), ({a}, {a, b, c}), ({b}, {a, b, c}), ({c}, {a, b, c})

Step 3: All edges are pointing upwards. Now replace circles by dots and remove arrow from edges.

Hence, required Hasse diagram is

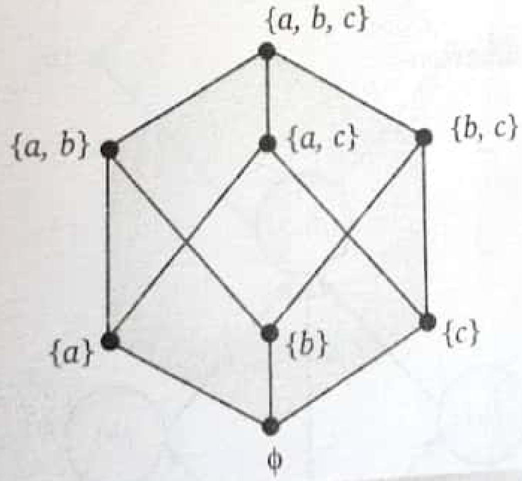


Fig. 6.52

Example 26: *Let A = {a, b}, Show that (P (A), ⊆) is poset and draw its Hasse diagram.*

[Raipur (B.E.) 2008; Rohtak (B.E.) 2007]

Solution: We have A = {a, b}. Then P (A) = {ϕ, {a}, {b}, {a, b}}

Then (P(A) ⊆) will be poset. See example 25.

Hence, (P (A), ⊆) is a poset.

Partial order relation R on $P(A)$ is given as

$$R = \{(\phi, \phi), (\phi, \{a\}), (\phi, \{b\}), (\phi, \{a, b\}), (\{a\}, \{a\})$$

$$(\{a\}, \{b\}), (\{a\}, \{a, b\}), (\{b\}, \{b\}), (\{b\}, \{a, b\}), (\{a, b\}, \{a, b\})\}.$$

Matrix of the above relation is as follows

$$M_R = \begin{array}{c} \\ \phi \\ \{a\} \\ \{b\} \\ \{a, b\} \end{array} \begin{array}{c} \begin{array}{cccc} \phi & \{a\} & \{b\} & \{a, b\} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Diagraph of the above matrix M_R is

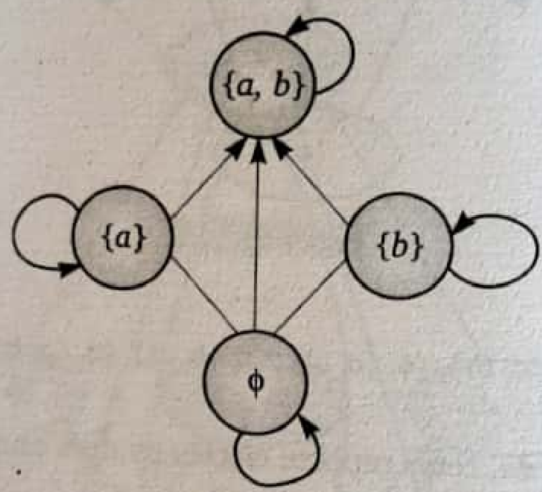


Fig. 6.53

To convert this diagraph into Hasse diagram.

Step 1: Remove Cycles

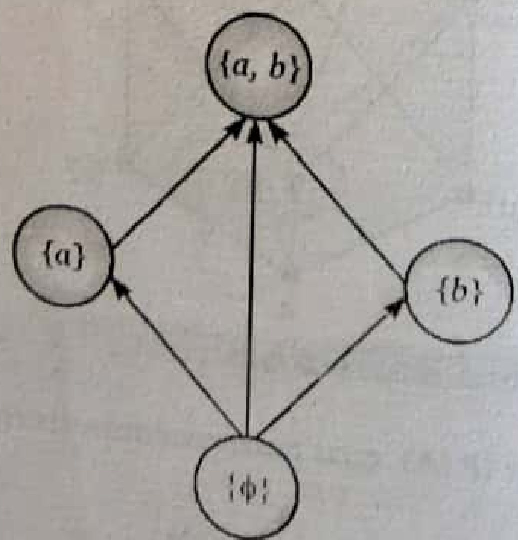


Fig. 6.54

Step 2: Remove transitive edges

$\phi\ R\ \{a\}, \{a\}\ R\ \{a, b\} \Rightarrow \phi\ R\ \{a, b\}$

$\phi\ R\ \{b\}, \{b\}\ R\ \{a, b\} \Rightarrow \phi\ R\ \{a, b\}$

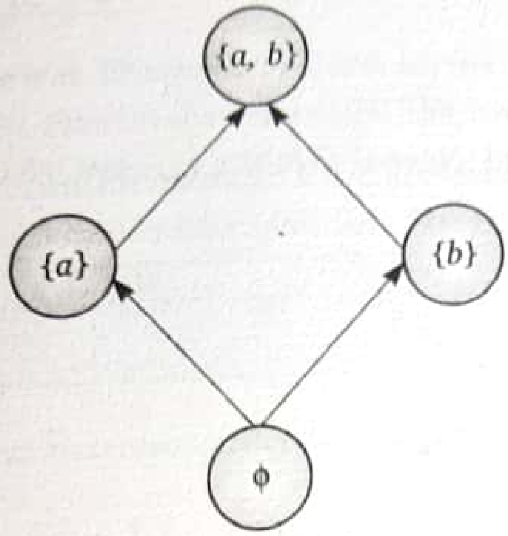


Fig. 6.55

Step 3: All edges are pointing upwards.

Now, replaced circles by dots and remove arrows from edges. Hence, required Hasse diagram is

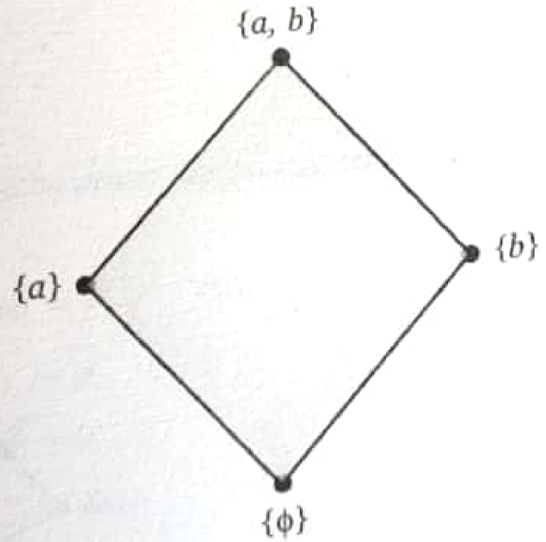


Fig. 6.56

6.5 Combination of Partial Ordered Sets or Components of Poset

6.5.1 Maximal Element, Minimal Element

An element belonging to a point $(a \leq)$ is said to be ***Maximal*** element of A if there is no element c in A such that $a \leq c$. An element $b \in A$ is said to be minimal element of A if there is no element c in A such that $c \leq b$.

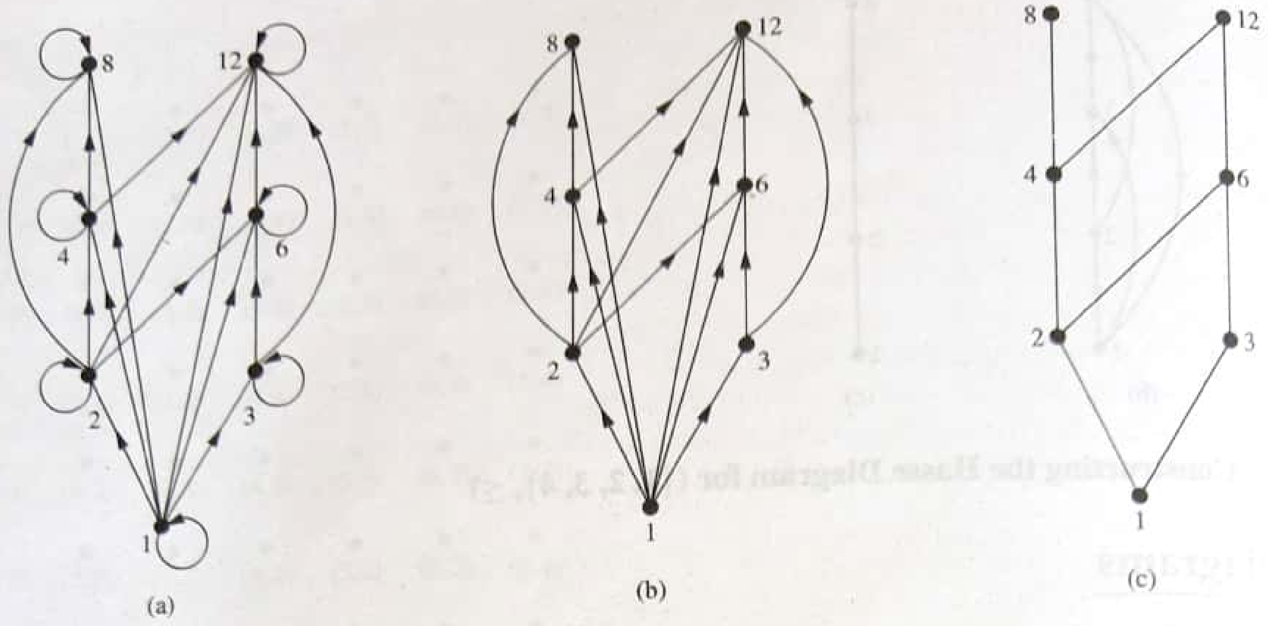


FIGURE 3 **Constructing the Hasse Diagram of ({1, 2, 3, 4, 6, 8, 12}, |).**

the partial ordering. This diagram is called a **Hasse diagram,** named after the twentieth-century German mathematician Helmut Hasse.

EXAMPLE 12 Draw the Hasse diagram representing the partial ordering $\{(a, b) \mid a \text{ divides } b\}$ on $\{1, 2, 3, 4, 6, 8, 12\}$.

Solution: Begin with the digraph for this partial order, as shown in Figure 3(a). Remove all loops, as shown in Figure 3(b). Then delete all the edges implied by the transitive property. These are (1, 4), (1, 6), (1, 8), (1, 12), (2, 8), (2, 12), and (3, 12). Arrange all edges to point upward, and delete all arrows to obtain the Hasse diagram. The resulting Hasse diagram is shown in Figure 3(c). ◀

EXAMPLE 13 Draw the Hasse diagram for the partial ordering $\{(A, B) \mid A \subseteq B\}$ on the power set $P(S)$ where $S = \{a, b, c\}$.

Solution: The Hasse diagram for this partial ordering is obtained from the associated digraph by deleting all the loops and all the edges that occur from transitivity, namely, $(\emptyset, \{a, b\})$, $(\emptyset, \{a, c\})$, $(\emptyset, \{b, c\})$, $(\emptyset, \{a, b, c\})$, $(\{a\}, \{a, b, c\})$, $(\{b\}, \{a, b, c\})$, and $(\{c\}, \{a, b, c\})$. Finally all edges point upward, and arrows are deleted. The resulting Hasse diagram is illustrated in Figure 4. ◀

Maximal and Minimal Elements

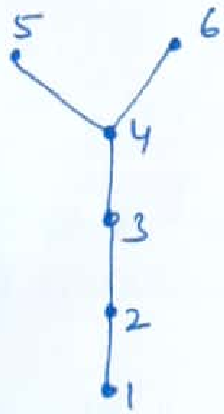
Elements of posets that have certain extremal properties are important for many applications. An element of a poset is called maximal if it is not less than any element of the poset. That is, a is **maximal** in the poset $(S, \preccurlyeq)$ if there is no $b \in S$ such that $a \prec b$. Similarly, an element of a poset is called minimal if it is not greater than any element of the poset. That is, a is **minimal** if there is no element $b \in S$ such that $b \prec a$. Maximal and minimal elements are easy to spot using a Hasse diagram. They are the "top" and "bottom" elements in the diagram.

EXAMPLE 14 Which elements of the poset $(\{2, 4, 5, 10, 12, 20, 25\}, \mid)$ are maximal, and which are minimal?

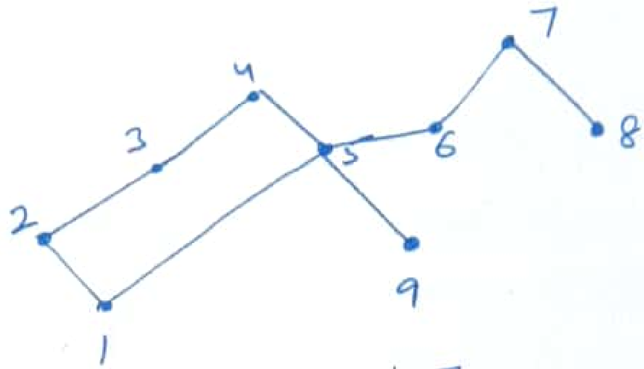
Solution: The Hasse diagram in Figure 5 for this poset shows that the maximal elements are 12, 20, and 25, and the minimal elements are 2 and 5. As this example shows, a poset can have more than one maximal element and more than one minimal element. ◀

Maximal Element and Minimal Element :→ An Element of a poset is called Maximal if it is not less than any element of the poset. That is 'a' is Maximal element in poset $(A, \leq)$ if there is no $b \in A$ such that $a \leq b$ / $a < b$: Similarly an Element elemt of a poset is called Minimal if it is not greater than any element of the poset. ie an element 'a' is minimal if there exist no $b \in A$ such that $b < a$.

Note :- In the Hasse Diagram the "Top" and "Bottom" Elements are the Maximal and Minimal Elements. Ex.



Maximal Element = 5, 6
Minimal Element = 1.



Maximal = 4, 7
Minimal = 1, 9, 8

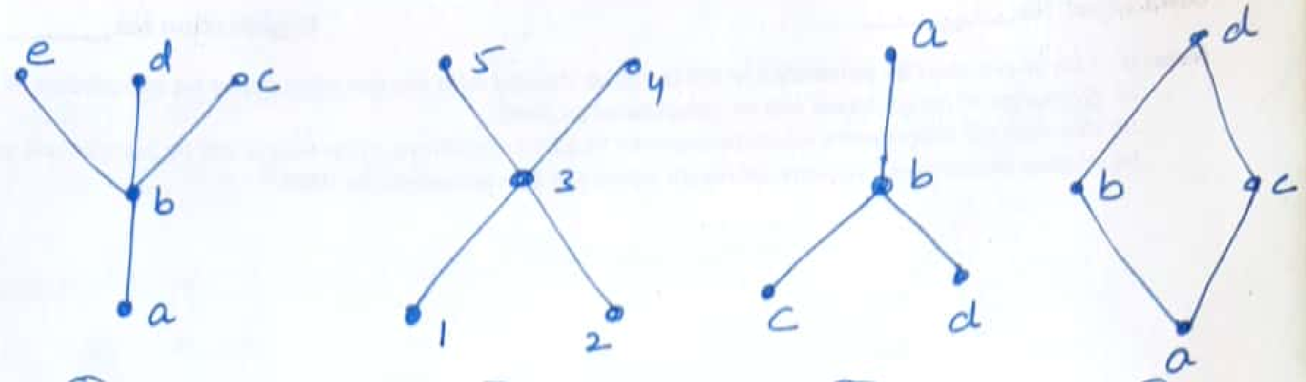
Greatest and Least Element :→ An element $a \in A$ is said to be the Greatest Element of poset $(A, \leq)$ if
$\forall x \in A$; $x \leq a$.
An element $a \in A$ is called the Least Element of Poset if
$a \leq x \ \forall x \in A$

Note : the least element is also called the first Element or Zero element and if it exist is Unique

(2) the Greatest Element is also called the last Element or Unit Element, if it Exist is Unique

(3) Least Element is denoted by 0
Greatest " " " " 1

Ex 2



	①	②	③	④
Least Element	a	No Least Element	No Least Element	a
Greatest Element	No Greatest Element	No Greatest Element	a	d

Upper and Lower bound

① Upper Bound and least upper Bound :- Let $(A, \leq)$ be a poset and let S be a subset of A, then an element $x \in A$ is called an Upper Bound of S if $a \leq x \ \forall a \in S$

An element $x \in A$ is said to be Least upper Bound or lub or Supremum of S if x is an upper bound of S and $x \leq y$ for all upper bounds y of S.

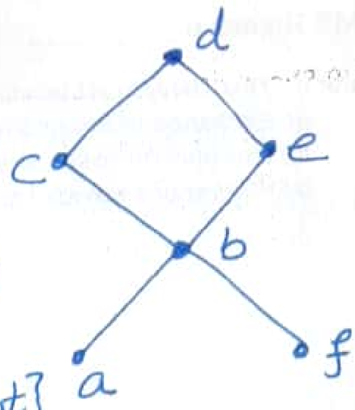
the least upperbound if it exist is Unique

② Lower Bound and Greatest lower Bound :- Let $(A, \leq)$ be a poset and let S be a subset of A $(S \subseteq A)$.

An element $x \in A$ is called the Lower bound of S if $x \leq a, \forall a \in S$.

and $x \in A$ is called Greatest Lower Bound or glb or Infimum if x is an Lower bound of S and $x \geq y$ for all lower bound y of S.

The glb if it Exist is alway Unique.



Ex 2) the Upper Bound of $\{c, e\}$ is 'd'

$\therefore$ lub$\{c, e\}$ or Sup$\{c, e\} = d$

Upper bounds of $\{a, f\}$ are the element b, e, c, d But

lub$\{a, f\}$ or Sup$\{a, f\} = b$ $\{\because b$ is least$\}$

Now Lower Bound of $\{c, e\} = b, a, f$

glb$\{c, e\}$ or Inf$\{c, e\} = b$ $\{\because b$ is greatest$\}$

Lower Bound of $\{a, f\}$ does not Exist.

6.5.5 Greatest Lower Bound or Infimum

Let $(P, \leq)$ be a poset and $A \subseteq P$. An element $x \in P$ is said to be a greatest lower bound or infimum of written as glb (A) or inf (A), if x is a lower bound and $y \leq x$ for all lower bounds y of A.

A greatest lower bound, if it exists is unique.

Illustration: *Consider the poset whose diagram is given by*

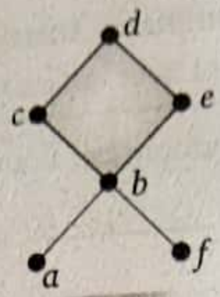


Fig. 6.61

(i) *Upper bound for $\{c, e\}$ is d lub $\{c, e\}$ or sup $\{c, e\} = d$. Lower bound for $\{c, e\}$ are the elements b, a and*

$$\text{glb } \{c, e\} = \inf \{c, e\} = b$$

(ii) *Lower bound for $\{a, f\}$ are the the elements b, c, e, d lub $\{a, f\} = b = \sup \{a, f\}$ lower bounds of a, f do not exist.*

Example 31: *Show that a poset has atmost one greatest and atmost one least element.*

[U.P.T.U. (B.Tech.) 2002, 2...]

Solution: Let a and b are the greatest element of a poset A. Then, since b is the greatest element, we have

$$a \leq b$$

Similarly, since a is the greatest element, we have

$$b \leq a$$

Then from (1) and (2) $b = a$ by antisymmetric property of poset.

Similarly it can be proved for least element. Hence a poset has atmost one greatest and one least element.

Example 32: *Consider the set $A = \{1, 2, 3, 4, 5\}$. Define the relation $<$ on A such that $x < y$ iff*

$$(x \bmod 3) < (y \bmod 3)$$

(i) *Prove that $(A, \leq)$ is a poset.*

(ii) *Draw the Hasse diagram for $(A, \leq)$*

(iii) *What are the maximal elements?*

(iv) *What are the minimal elements?*

[U.P.T.U. (B.Tech.) 2003; Kurukshetra (B.E.) 2009; P.T.U. (B.E.) Punjab 20...]

Solution: (i) $(A, \leq)$ will be poset if

(a) Reflexivity: Since every element of A is related to itself.

i.e. $1 \leq 1 \Rightarrow (1 \bmod 3) \leq (1 \bmod 3)$. *Hence, it is reflexive.*

(b) Antisymmetric: If $a \leq b \Rightarrow (a \bmod 3) \leq (b \bmod 3)$ and $b \leq a \Rightarrow (b \bmod 3) \leq (a \bmod 3)$

Then, we get $a = b$. Hence, it is antisymmetric.

(c) Transitive: If $a \leq b$ and $b \leq c$

Then $(a \bmod 3) \leq (b \bmod 3)$ and $(b \bmod 3) \leq (c \bmod 3)$

$$(a \bmod 3) \le (c \bmod 3)$$

$\therefore$ $$a \le c$$

or

Hence, it is transitive. Therefore $(A, \le)$ is poset.

(ii) The Hasse diagram for $A = \{1, 2, 3, 4, 5\}$

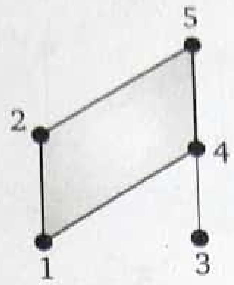


Fig. 6.62

(iii) $\begin{cases} \text{Maximal element} = 5 \\ \text{Minimal element} = 1 \text{ and } 3 \end{cases}$

Example 33: *Consider the partially order set $A = \{2, 4, 6, 8\}$ where $2 | 4$ means 2 divide 4 show with reason whether the following statements are true or false.*

(i) *Every pair of elements in the poset has a greatest lower bound.*

(ii) *Every pair of elements in the poset has a least upper bound.*

(iii) *This poset is a lattice.* **[U.P.T.U. (B.Tech.) 2002]**

Solution: Now first we draw the Hasse diagram under relation of divisor

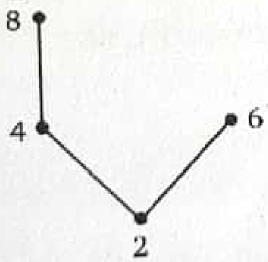


Fig. 6.63

(i) True, its g.l.b. table is

$\wedge$	2	4	6	8
2	2	2	2	2
4	2	4	2	4
6	2	2	6	2
8	2	4	2	8

Here $a \wedge b = \text{H.C.F of } (a, b)$. Hence every pair of elements has g.l.b.

(ii) False, since there exists no l.u.b. of 6 and 8

(iii) False, since 6 and 8 has no l.u.b. This is not a lattice.

Example 34: *Consider the poset A = (1, 2, 3, 4, 6, 9, 12, 18, 36}, '|'). Find the greatest lower bo upper bound of the sets {6, 18} and {4, 6, 9}.*

Solution: Hasse diagram under relation of divisibility

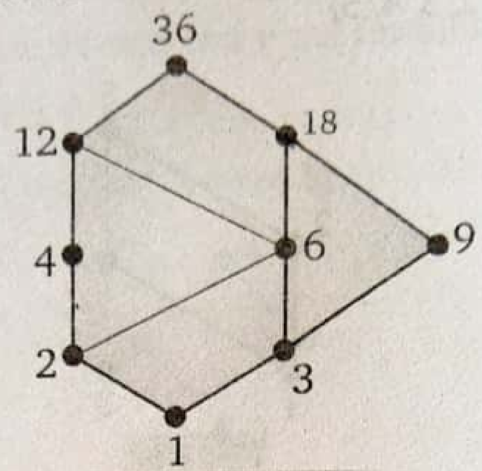


Fig. 6.64

An integer is a lower bound of {6, 18} if 6 and 18 are divisible by this integer. Only such inte 6. Since $1 \mid 6$, 6 is the greatest lower bound of {6, 18}. Hence glb {6, 18} = 6

An integer is an upper bound of {6, 18} if and only if it is divisible by 6 and 18 which is 18.

Hence lub {6, 18} = 18

The only lower bound of {4, 6, 9} = 1. Hence glb {4, 6, 9} = 1.

The only upper bound of {4, 6, 9} = 36 . Hence lub {4, 6, 9} = 36

Remark: $\begin{cases} \text{glb } (a, b) = \text{greatest common divisor (g. c. d)} \\ \text{lub } \{ab\} = \text{lowest common multiple (l. c. m)} \end{cases}$

6.6 Well Ordered Set

A partially ordered set $(A, \leq)$ is said to be will ordered if every non-empty subset of S has a leas

Illustration: *The set of real numbers R and the set of all integer I (with the usual order well-ordered.*

Illustration: *The set of all positive integer I^+ or N is well ordered.*

Theorem 1: *Every well ordered set is totally ordered.*

Proof: Let $x, y \in X$. $(X, \leq)$ being a well ordered set. Then $\{x, y\}$ contains a first element. Hence $x \leq y$ or $y \leq x$. This means that $(X, \leq)$ is totally ordered.

Lattice :→

A poset (Partially ordered Set), $(L, \leq)$ is said to be lattice if every two elements in the set L has Unique least upper bound (lub, Sup) and a Unique greatest lower bound (glb, Inf)

OR

The Poset $(L, \leq)$ is a lattice if for every $a, b \in L$, $Sup\{a,b\}$ and $Inf\{a,b\}$ exist in L i.e

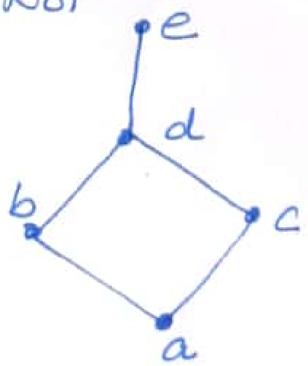
$$Sup\{a,b\} = a \vee b = \text{'a' joint 'b'}$$
$$Inf\{a,b\} = a \wedge b = \text{'a' meet 'b'}$$

Note: The other Notations for $\begin{cases} Sup\{a,b\} = a \cup b \text{ or } a+b \\ Inf\{a,b\} = a \cap b \text{ or } a \cdot b \end{cases}$

Note: The set N of natural numbers Under divisibility relation '/' formed a lattice in which

$$a \vee b = lcm\{a,b\} \in N$$
$$a \wedge b = gcd\{a\,b\} \in N.$$

Ex: Determine whether the following a Lattice or Not



Sol: Construct the Closure table for lub and glb of the given problem.

lub/Sup

	a	b	c	d	e
a	a	b	c	d	e
b	b	b	d	d	e
c	c	d	c	d	e
d	d	d	d	d	e
e	e	e	e	e	e

glb/Inf

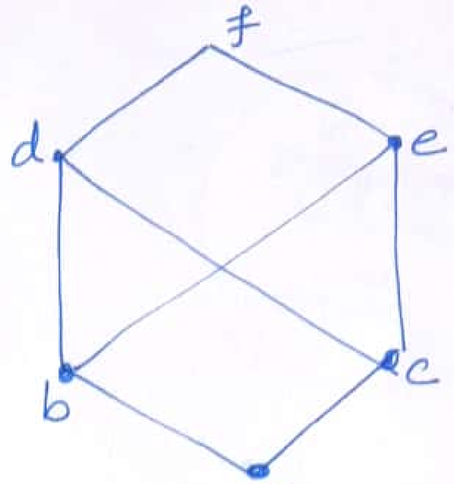
	a	b	c	d	e
a	a	a	a	a	a
b	a	b	a	b	b
c	a	a	c	c	c
d	a	b	c	d	d
e	a	b	c	d	e

Since Each subset of two elements has lub and glb So this is the lattice.

Ex 2

lub | Sup

	a	b	c	d	e	f
a	a	b	c	d	e	f
b	b	b	–	d	e	f
c	c	[illegible]	c	d	e	f
d	d	d	d	d	f	f
e	e	e	e	f	f	f
f	f	f	f	f	f	f



Side 'b' and 'c' has no least upper bound. Since the upper bound of b & c are d, e, f but none of these three element precedes the ~~opest~~ other two with respect to the ordering of this poset.

∴ It is not a Lattice

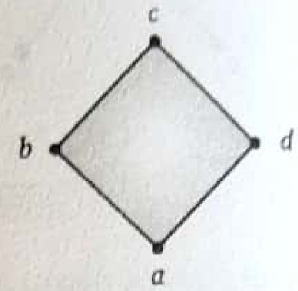
Illustration: *The set N of natural numbers under divisibility relation '|' formed a lattice in which*

$$a \vee b = lcm\,(a, b) \in N$$
$$a \wedge b = gcd\,(a, b) \in N$$

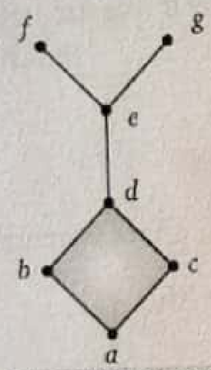
Example 40: *Determine whether the following Hasse diagrams represent lattice or not.*

(i)

(ii)

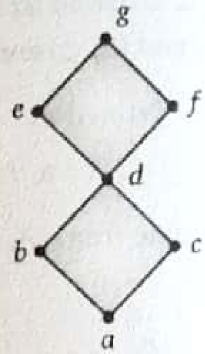


(iii)



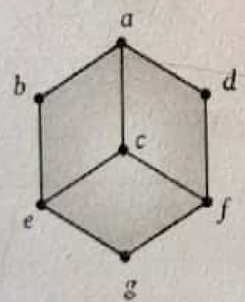
[M.K.U. (B.E.) 2008]

(iv)

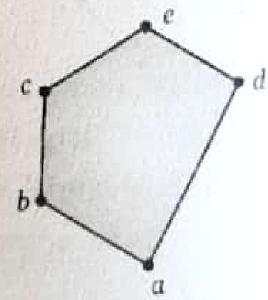


[Kurukshetra (B.E.) 2006]

(v)

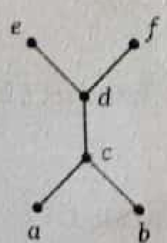


(vi)



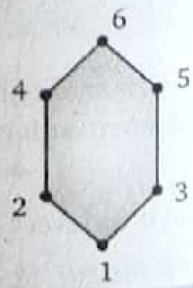
[P.T.U. (Punjab) 2004, 2008]

(vii)



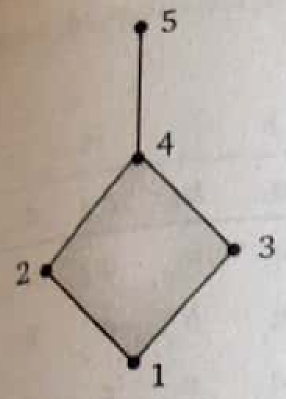
Rohtak (M.C.A.) 2005, 2007]

(viii)



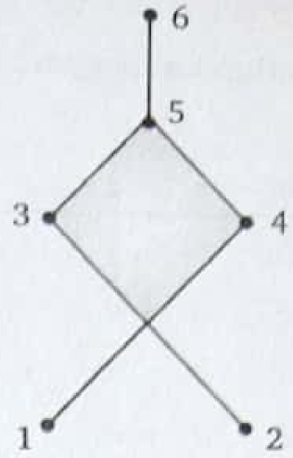
[U.P.T.U. (B.Tech.) 2007]

(ix)



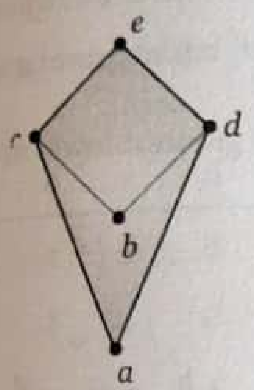
[U.P.T.U. (B.Tech.) 2007]

(x)



[U.P.T.U. (B.Tech.) 2007]

(xi)



[U.P.T.U. (B.Tech.) 2002; R.G.V.P. (B.E.) Bhopal 2005]

Fig. 6.72

on: (i) Construct the closure tables for lub (∨) or supremum or joint and glb (∧)

∨	a	b	c	d
a	a	b	c	d
b	b	b	c	d
c	c	c	c	d
d	d	d	d	d

∧	a	b	c	d
a	a	a	a	a
b	a	b	b	b
c	a	b	c	c
d	a	b	c	d

ch subset of two elements has least upper bound and a greatest lower bound. So this is the lattice.
nstruct the closure tables for lub (∨) and glb (∧)

lub:

∨	a	b	c	d
a	a	b	c	d
b	b	b	c	c
c	c	c	c	c
d	d	c	c	d

glb:

∧	a	b	c	d
a	a	a	a	a
b	a	b	b	a
c	a	b	c	d
d	a	a	d	d

Since each subset of two elements has least upper bound and a greatest lower bound. So this is the lattice.

(iii) Construct the closure tables for lub ($\vee$) and glb ($\wedge$)

lub:

$\vee$	a	b	c	d	e	f	g
a	a	b	c	d	e	f	g
b	b	b	d	d	e	f	g
c	c	d	c	d	e	f	g
d	d	d	d	d	e	f	g
e	e	e	e	e	e	f	g
f	f	f	f	f	f	f	–
g	g	g	g	g	g	–	g

glb:

$\wedge$	a	b	c	d	e	f
a	a	a	a	a	a	a
b	a	b	a	b	b	b
c	a	a	c	c	c	c
d	a	b	c	d	d	d
e	a	b	c	d	e	e
f	a	b	c	d	e	f
g	a	b	c	d	e	e

Since each subset of two elements has not least upper bound but has greatest lower bound. So this is not the lattice.

(iv) Construct the closure tables for lub ($\vee$) and glb ($\wedge$)

lub:

$\vee$	a	b	c	d	e	f	g
a	a	b	c	d	e	f	g
b	b	b	d	d	e	f	g
c	c	d	c	d	e	f	g
d	d	d	d	d	e	f	g
e	e	e	e	e	e	g	g
f	f	f	f	f	g	f	g
g	g	g	g	g	g	g	g

glb:

$\wedge$	a	b	c	d	e	f
a	a	a	a	a	a	a
b	a	b	a	b	b	b
c	a	a	c	c	c	c
d	a	b	c	d	d	d
e	a	b	c	d	e	d
f	a	b	c	d	d	f
g	a	b	c	d	e	f

Since each subset of two elements has least upper bound and a greatest lower bound. So this is the lattice.

(v) Construct the closure tables for lub ($\vee$) and glb ($\wedge$):

($\vee$)	a	b	c	d	e	f	g
a	a	a	a	a	a	a	a
b	a	b	a	a	b	a	b
c	a	a	c	a	c	c	c
d	a	a	a	d	a	d	d
e	a	b	c	a	e	c	e
f	a	a	c	d	c	f	f
g	a	b	c	d	e	f	g

($\wedge$)	a	b	c	d	e	f
a	a	b	c	d	e	f
b	b	b	e	g	e	g
c	c	e	c	f	e	f
d	d	g	f	d	g	f
e	e	e	e	g	e	g
f	f	g	f	f	g	f
g	g	g	g	g	g	g

Since each subset of two elements has least upper bound and a greatest lower bound. So this is the lattice.

Construct the closure tables for lub (∨) and glb (∧).

lub:

	a	b	c	d	e
a	a	b	c	d	e
b	b	b	c	e	e
c	c	c	c	e	e
d	d	e	e	d	e
e	e	e	e	e	e

glb:

	a	b	c	d	e
a	a	a	a	a	a
b	a	b	b	a	b
c	a	b	c	a	c
d	a	a	a	d	d
e	a	b	c	d	e

Since each subset of two elements has least upper bound and a greatest lower bound. So this is the lattice.

(ii) Construct the closure tables for lub (∨) and glb (∧).

lub:

∨	a	b	c	d	e	f
a	a	c	c	d	e	f
b	c	b	c	d	e	f
c	c	c	c	d	e	f
d	d	d	d	d	e	f
e	e	e	e	e	e	–
f	f	f	f	f	–	f

glb:

∧	a	b	c	d	e	f
a	a	–	a	a	a	a
b	–	b	b	b	b	b
c	a	b	c	c	c	c
d	a	b	c	d	d	d
e	a	b	c	d	e	d
f	a	b	c	d	d	f

Here e ∨ f and a ∧ b does not exist. Then given poset is not lattice.

(iii) Construct the closure tables for lub (∨) and glb (∧)

lub:

∨	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	2	6	4	6	6
3	3	6	3	6	6	6
4	4	4	6	6	6	6
5	5	6	5	6	5	6
6	6	6	6	6	6	6

glb:

∧	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	1	2	1	1
3	1	1	3	1	3	3
4	1	2	1	4	1	4
5	1	1	3	1	5	5
6	1	2	3	4	5	6

Since each subset of two elements has least upper bound and a greatest lower bound. So this is the lattice.

(ix) Construct the closure tables for lub $(\vee)$ and glb $(\wedge)$

lub:

$\vee$	1	2	3	4	5
1	1	2	3	4	5
2	2	2	4	4	5
3	3	4	3	4	5
4	4	4	4	4	5
5	5	5	5	5	5

glb:

$\wedge$	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	2
3	1	1	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

Since each subset of two elements has least upper bound and greatest lower bound. So this the lattice.

(x) Since $1 \wedge 2$ does not exist so it is not lattice.

(xi) Construct the closure tables for lub $(\vee)$ and glb $(\wedge)$.

lub:

$\vee$	a	b	c	d	e
a	a	b	c	d	e
b	b	b	c	d	e
c	c	c	c	e	e
d	d	d	e	d	e
e	e	e	e	e	e

glb:

$\wedge$	a	b	c	d	e
a	a	a	a	a	a
b	a	b	b	b	b
c	a	b	c	b	c
d	a	b	b	d	d
e	a	b	c	d	e

Since each subset of two element has least upper bound and a greatest lower bound. So this is the lattice.

Example 41: *Let S be any non empty set and $P(S)$ be its power set. Show that $(P(S), \subseteq)$ is a lattice.*

Solution: We have shown that $(P(S), \subseteq)$ is a poset. Also for any two set $A, B \in P(S)$, we have

$$A \vee B = \sup\{A, B\} = A \cup B$$

and

$$A \wedge B = \inf\{A, B\} = A \cap B$$

Since both $A \cup B$ and $A \cap B$ be the element in $P(S)$, therefore $(P(S), \subseteq)$ is a lattice

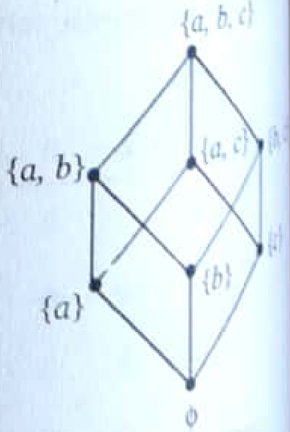


Fig. 6.73

To illustrate this example we consider

$$S = \{a, b, c\}. \text{ Then}$$

$$P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

The lattice is represented by the following Hasse diagram.

Example 42: *For any positive integer m, D_m denote the set of divisors of m ordered of divisibility, then $(D_m, '|')$ is lattice, where*

$$\sup(a, b) = \text{lcm}(a, b)$$
$$\inf(a, b) = \text{gcm}(a, b)$$

for any pair a, b in D_m.

i.e. $D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$ is lattice.

[I.G.N.O.U. (M.C.A.) 2005, 2009; M.K.U. (B.E.) 2008; U.P.T.U. (B.Tech.) 2004]

Solution: The Hasse diagram of $(D_{36}, '|')$ is

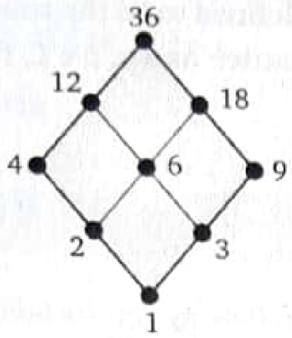


Fig. 6.74

Since each subset of two elements least upper bound and a greatest lower bound. So this is the lattice.

Example 43: *Let L be the set of all factor of 12and let '|' be the divisibility relation on L. Show that (L, '|') is a lattice.*

Solution: Construct the closure table for $\vee$ and $\wedge$

where
$$a \vee b = \sup\{a, b\} = \text{lcm}(a, b)$$
$$a \wedge b = \inf\{a, b\} = \gcd(a, b)$$

$\vee$	1	2	3	4	6	12
1	1	2	3	4	6	12
2	2	2	6	4	6	12
3	3	6	3	12	6	12
4	4	4	12	4	12	12
6	6	6	6	12	6	12
12	12	12	12	12	12	12

$\wedge$	1	2	3	4	6	12
1	1	1	1	1	1	1
2	1	2	1	2	2	2
3	1	1	3	1	3	3
4	1	2	1	4	2	4
6	1	2	3	2	6	6
12	1	2	3	4	6	12

Since each subset of every two elements in L has $\vee$ and $\wedge$. Then $\leq (L, '|')$ is lattice.

Example 44: *Show that every chain is a lattice.*

Solution: Let $(L, \leq)$ be a chain. Let $a, b \in L$. Then, since L is a chain, we have

$$a, b \in L \Rightarrow a \leq b \text{ or } b \leq a$$

We assume that $a \leq b$. Then

$$a \vee b = b \text{ and } a \wedge b = a$$

i.e. $a \vee b$, $a \wedge b$ both exist in L. Hence every chain is a lattice.

6.12 Dual Lattice

Let $(L, \leq)$ be a lattice, for any $a, b \in L$, the converse of relation $\leq$, denoted by $\geq$ is defined as

$$a \geq b \Leftrightarrow b \leq a$$

Then $(L, \geq)$ is also a lattice called ***Dual Lattice*** of $(L, \leq)$.

6.12.1 Dual Statement

The dual of any statement in lattice $(L, \leq)$ is defined to be the statement that is obtained by replacing $\wedge$ by and $\vee$ by $\wedge$, $\leq$ by $\geq$ and $\geq$ by $\leq$. Let L be a lattice and $a, b \in L$, then dual of the statement $a \wedge b \leq b \vee$ $a \vee b \geq b \wedge c$.

6.13 Properties of Lattices

Theorem 4: *Let $(L, \leq)$ be a lattice. Then the following results hold.*

(1) ***Idempotent Law:*** For each $a \in L$

(i) $a \wedge a = a$ (ii) $a \vee a = a$

(2) ***Commutative Law:*** For each $a, b \in L$

(i) $a \wedge b = b \wedge a$ (ii) $a \vee b = b \vee a$ [U.P.T.U. (M.C.A.) 20

(3) ***Associative Law:*** For any $a, b, c \in L$

(i) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ (ii) $(a \vee b) \vee c = a \vee (b \vee c)$

(4) ***Absorption Law:*** For $a, b \in L$

(i) $a \wedge (a \vee b) = a$ (ii) $a \vee (a \wedge b) = a$ [R.G.P.V. (B.E.) Bhopal 2005, 20

Proof: (1) Let $a \in L$. Then

(i) $a \wedge a = \inf\{a, a\} = a$. Then $a \wedge a = a$

(ii) $a \vee a = \sup\{a, a\} = a$. Then $a \vee a = a$

(2) Let $a, b \in L$. Then

(i) $a \wedge b = \inf\{a, b\} = \inf\{b, a\} = b \wedge a$

(ii) $a \vee b = \sup\{a, b\} = \sup\{b, a\} = b \vee a$

(3) Let $a, b, c \in L$. Then show

(i) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ (ii) $(a \vee b) \vee c = a \vee (b \vee c)$

Let $x = a \wedge (b \wedge c)$ and $y = (a \wedge b) \wedge c$

Now, $x = a \wedge (b \wedge c) \Rightarrow x \leq a, x \leq b \wedge c$

$$\Rightarrow x \leq a, x \leq b, x \leq c$$
$$\Rightarrow x \leq a \wedge b, x \leq c$$
$$\Rightarrow x \leq (a \wedge b) \wedge c \leq y$$

$\therefore$ $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

(ii) By principle of duality $a \vee (b \vee c) = (a \vee b) \vee c$

(4) For $a, b, c \in L$. Then show

(i) $a \wedge (a \vee b) = a$ (ii) $a \vee (a \wedge b) = a$

By definition for any $a \in L$, we have

$$a \le a,\ a \le a \vee b \Rightarrow a \le a \wedge (a \vee b) \quad \ldots(1)$$

Bu

$$a \wedge (a \vee b) \le a \quad \ldots(2)$$

Then from (1) and (2)

$$a \wedge (a \vee b) = a$$

By principle of duality $a \vee (a \wedge b) = a$

Theorem 5: *Let $(L, \le)$ be a lattice. For any $a, b, c \in L$, the following hold*

$$a \le c \Leftrightarrow a \vee (b \wedge c) \le (a \vee b) \wedge c$$ **[U.P.T.U. (B.Tech.) 2009]**

Proof: We know that for any $a, b, c \in L$. Then

$$a \le c \Leftrightarrow a \vee c = c \quad \ldots(1)$$

and

$$a \vee (b \wedge c) \le (a \vee b) \wedge (a \vee c) \quad \ldots(2)$$

From (1) and (2) we find

$$a \le c \Leftrightarrow a \vee (b \wedge c) \le (a \vee b) \wedge c$$

Remark: This inequality is known as ***Modular Inequality***.

Theorem 6: *For any $a, b \in L$, prove that*

$$a \le a \vee b \text{ and } a \wedge b \le a$$ **[P.T.U. (Punjab) 2008]**

Proof: We know $a \vee b$ is upper bound of a. Hence $a \le a \vee b$. Also we know $a \wedge b$ is lower bound of a.

Hence

$$a \wedge b \le a.$$

Theorem 7: *Let $(L, \le)$ be a lattice and let $\wedge$ and $\vee$ denote the operations of meet and joint in L. Then for any $a, b \in L$.*

(i) $a \le b \Leftrightarrow a \wedge b = a$ (ii) $a \le b \Leftrightarrow a \vee b = b$ (iii) $a \wedge b \Rightarrow a \Leftrightarrow a \vee b = b$

Proof: (i) Let $a \wedge b = a$. Since $a \wedge b = \inf\{a, b\} \Rightarrow a \wedge b \le b$.

Now

$$a \wedge b \le b \Rightarrow a \le b \quad [a \wedge b = a]$$

Conversly, Let $a \le b$. By reflexivity of $\le$ we have

$$a \le a$$

Now $a \le b$ and $a \le a \Rightarrow a$ is lower bound of $\{a, b\}$

$$\Rightarrow a = \inf\{a, b\} = a \wedge b$$

Since $a \wedge b$ infimum of $\{a, b\}$, we have

$$a \wedge b \le a$$

$\therefore$ $a \le a \wedge b$ and $a \wedge b \le a \Rightarrow a \wedge b = a$ (By antisymmetry)

(ii) Let $a \vee b = b$, then show $a \le b$

Since

$$a \vee b = \sup\{a, b\}$$

$$\Rightarrow a \le a \vee b$$

Now, $a \le a \vee b$ and $a \vee b = b \Rightarrow a \le b$

Conversely, Again let $a \le b$ then reflexivity $b \le b$

Now $\quad a \le b,\ b \le b \Rightarrow b$ is an upper bound of $\{a, b\}$

$$\Rightarrow \sup\{a, b\} \le b$$
$$\Rightarrow a \vee b \le b$$

But from the definition of $a \vee b = \sup\{a, b\} \Rightarrow b \le a \vee b$

Also $\quad a \vee b \le b$ and $b \le a \vee b \Rightarrow a \vee b = b$

(iii) We know $a \wedge b = a \Leftrightarrow a \le b$

$$\Leftrightarrow a \vee b = b$$

Hence, $\quad a \wedge b = a \Leftrightarrow a \vee b = b.$

Theorem 8: *Let $(L, \le)$ be a lattice and $a, b, c \in L$. Then the following implications hold:*

(i) $a \le b$ *and* $a \le c \Rightarrow a \le b \vee c$ $\qquad$ (ii) $a \le b$ *and* $a \le c \Rightarrow a \le b \wedge c.$

Proof: (i) Suppose $a \le b$ and $a \le c$. From the definition of join operation in lattice $(L, \le)$, we have

$$b \vee c = \sup\{b, c\}$$

$\Rightarrow\ b \vee c$ is an upper bound of $\{b, c\}$

$\Rightarrow\ b \le b \vee c.$

Now, by transitivity of the relation $\le$, we have

$$a \le b \text{ and } b \le b \vee c \Rightarrow a \le b \vee c.$$

(ii) Suppose $a \le b$ and $a \le c$. Then

$a \le b$ and $a \le c \Rightarrow a$ is a lower bound of $\{b, c\}$

$$\Rightarrow a \le \text{glb}\{b, c\}$$
$$\Rightarrow a \le b \wedge c \qquad [\because \text{glb}\{b, c\} = b \wedge c]$$

Corollarly: Let $(L, \le)$ be a lattice and $(L, \ge)$ be its dual. Then for $a, b, c \in L$,

(i) $a \ge b$ and $a \ge c \Rightarrow a \ge b \wedge c$ $\qquad$ (ii) $a \ge b$ and $a \ge c \Rightarrow a \ge b \vee c.$

Proof: Applying principle of duality on Theorem 8, we get the results.

Theorem 9: *Let $(L, \le)$ be a lattice and $a, b, c, d \in L$. Then the following implications hold:*

(i) $a \le b$ *and* $c \le d \Rightarrow a \vee c \le b \vee d$ $\qquad$ (ii) $a \le b$ *and* $c \le d \Rightarrow a \wedge c \le b \wedge d.$

Proof: (i) Suppose that $a \le b$ and $c \le d$. By the definition of join operation $\vee$ in lattice $(L, \le)$, we have

$$b \le b \vee d \text{ and } d \le b \vee d.$$

Now, by transitivity of the relation $\le$, we have

$$a \le b \text{ and } b \le b \vee d \Rightarrow a \le b \vee d.$$

Similarly, $\quad c \le d$ and $d \le b \vee d \Rightarrow c \le b \vee d.$

Now, $a \le b \vee d$ and $c \le b \vee d \Rightarrow b \vee d$ is an upper bound of $\{a, c\}$

$$\Rightarrow \text{lub}\{a, c\} \le b \vee d$$
$$\Rightarrow a \vee c \le b \vee d \qquad [\because a \vee c = \text{lub}\{a, c\}]$$

(ii) Suppose that $a \le b$ and $c \le d$. By the definition of meet operation $\wedge$ in lattice $(L, \le)$, we have

$$a \wedge c \le a \text{ and } a \wedge c \le c.$$

By transitivity of the relation $\le$, we have

$$a \wedge c \le a \text{ and } a \le b \Rightarrow a \wedge c \le b.$$

Similarly, $\quad a \wedge c \le c$ and $c \le d \Rightarrow a \wedge c \le d$

$\Rightarrow a \vee b$ is a lo...

We shall show that $a \vee b$ is the greatest lower bound of $\{a, b\}$ in $(L, \geq)$.

Let l be any lower bound of $\{a, b\}$ in $(L, \geq)$. Then

$$l \geq a \text{ and } l \geq b \Rightarrow a \leq l \text{ and } b \leq l$$
$$\Rightarrow l \text{ is an upper bound of } \{a, b\} \text{ in } (L, \leq)$$
$$\Rightarrow \text{lub } \{a, b\} \leq l \text{ in } (L, \leq)$$
$$\Rightarrow a \vee b \leq l \text{ in } (L, \leq)$$
$$\Rightarrow l \geq a \vee b$$
$$\Rightarrow a \vee b \text{ is greatest lower bound of } \{a, b\} \text{ in } (L, \geq).$$

Similarly, we can show that $a \wedge b$ is the least upper bound of $\{a, b\}$ in $(L, \geq)$. Hence $(L, \geq)$ is a lattice.

Theorem 12: *Show that in a lattice $(L, \leq)$ if $a \leq b \leq c$, then*

$$a \vee b = b \wedge c$$

and $$(a \wedge b) \vee (b \wedge c) = b = (a \vee b) \wedge (a \vee c).$$

Solution: We know that

$$a \leq b \Rightarrow a \vee b = b$$

and $$b \leq c \Rightarrow b \wedge c = b.$$

From (1) and (2), we get

$$a \leq b \leq c \Rightarrow a \vee b = b = b \wedge c.$$

Thus $$a \leq b \leq c \Rightarrow a \vee b = b \wedge c.$$

Also, $$a \leq b \Rightarrow a \wedge b = a$$

and $$a \leq b \leq c \Rightarrow a \leq c \quad [\because \leq \text{ is transitive}]$$
$$\Rightarrow a \vee c = c.$$

Now, replace $a \wedge b$ for a, $b \wedge c$ for b, $a \vee b$ for b and $a \vee c$ for c in R.H.S. of (3) we find

$$(a \wedge b) \vee (b \wedge c) = b = (a \vee b) \wedge (a \vee c)$$

Example 45: *Let N denote the set natural numbers. For any $a, b \in N$, show that* max $\{a, \min \{a, b\}\} = a$, min $\{a, \max \{a, b\}\} = a$

Solution: We know $a \vee b = \max \{a, b\}$ and $a \wedge b = \min \{a, b\}$

By absorption law $\quad a \wedge (a \vee b) = a$

$$a \vee (a \wedge b) = a$$

$\therefore$ $\quad$ min $\{a, \max \{a, b\} = \min \{a, a \vee b\} = a \vee (a \wedge b) = a$

max $\{a, \min \{a, b\} = \max \{a, a \wedge b\} = a \vee \{a \wedge b\} = a$

$\forall$ positive integers a and b.

6.14 Sub-lattice

A non empty subset M of lattice $(L, \leq)$ is said to be a sub-lattice of L if M is closed with respect to meet and joint $(\vee)$ *i.e.*

$$x, y \in M \Rightarrow x \vee y \in M \text{ and } x \wedge y \in M$$

or

A non empty subset of M of lattice $(L, \leq)$ is said to be sub-lattice of L if M itself formed lattice with respect to $\vee$ and $\wedge$ operation.

Example 46: *Consider the lattice of all the integer '|' under the operation of divisibility. The lattice D_n of all divisors of $n > 1$ is a sub-lattice of '|'. Determine all the sub-lattices of D_{30} that contain at least four elements.*

$$D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$$

Solution: The sublattice of D_{30} that solution at least four elements are as follows.

(i) $\{1, 2, 6, 30\}$ (ii) $\{1, 2, 3, 30\}$ (iii) $\{1, 5, 15, 30\}$

(iv) $\{1, 3, 6, 30\}$ (v) $\{1, 5, 10, 30\}$ (vi) $\{1, 3, 15, 30\}$

Example 47: *Consider the lattice $L = \{1, 2, 3, 4, 5\}$ as shown in fig. 6.75 Determine all sublattices with three or more elements.*

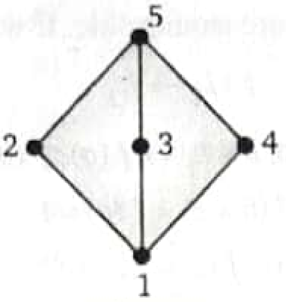


Fig. 6.75

Solution: All the sublattice with three or more elements are those whose least upper bound (lub) and greatest lower bound (glb) exists for every pair of elements which are as.

(i) $\{1, 2, 5\}$ (ii) $\{1, 3, 5\}$ (iii) $\{1, 4, 5\}$ (iv) $\{1, 2, 3, 5\}$

(v) $\{1, 3, 4, 5\}$ (vi) $\{1, 2, 3, 4, 5\}$ (vii) $\{1, 2, 4, 5\}$

6.15 Isomorphic Lattice

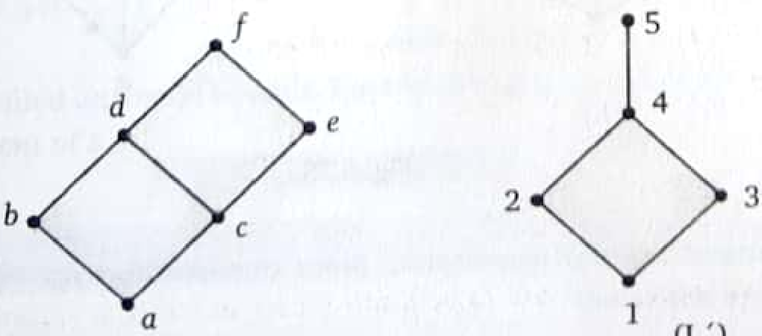
[U.P.T.U. (B.Tech.) 2007, 2008]

Two lattice L_1 and L_2 are isomorphic if there exists a one-to-one correspondence $f : L_1 \to L_2$ such that

$f(a \wedge b) = f(a) \wedge f(b)$

and $f(a \vee b) = f(a) \vee f(b)\ \forall\ a, b \in L_1$ and $f(a), f(b) \in L_2$

Example 48: *Show that the lattice L and L′ given below are not isomorphic?* **[U.P.T.U. (B.Tech.) 2008]**



(L) (L′)

Fig. 6.76

Solution: Consider the mapping

$$f = (a, 1), (b, 2), (c, 3), (d, 4), (e, \text{not defined})$$

Since there is no one-one corresponding between L and L′ so L and L′ are not isomorphic.

Example 49: *Determine whether the lattice shown in Fig. 6.77 are isomorphic.*

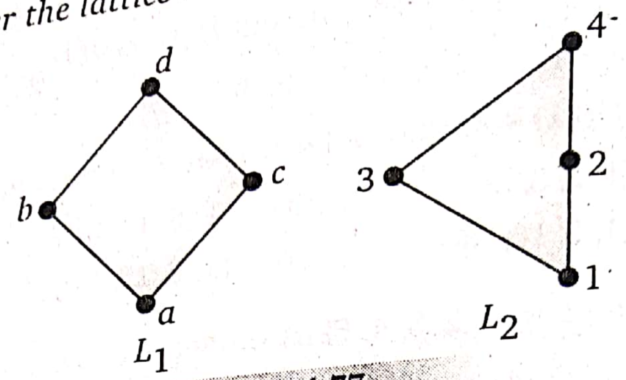


Fig. 6.77

Solution: The lattice shown in fig. 6.77 are isomorphic. If we consider $f = \{(a, 1), (b, 2), (c, 3), (d, 4)\}$

$$f : L_1 \to L_2$$

i.e.

$$a, b \in L_1 \Rightarrow f(a), f(b) \in L_2$$

$$f(b \wedge c) = f(a) = 1$$

Then

$$f(b) \wedge f(c) = 2 \wedge 3 = 1$$

$\because$

$$f(b \wedge c) = f(b) \wedge f(c)$$

Again

$$f(b \vee c) = f(d) = 4$$

$$f(b) \vee f(c) = 2 \vee 3 = 4$$

$\therefore$

$$f(b \vee c) = f(b) \vee f(1)$$

Hence, L_1 and L_2 are isomorphic.

Example 50: *Determine whether the lattice shown in fig.6.78 are isomorphic.* [R.G.P.V. (B.E.) Bhopal 20..]

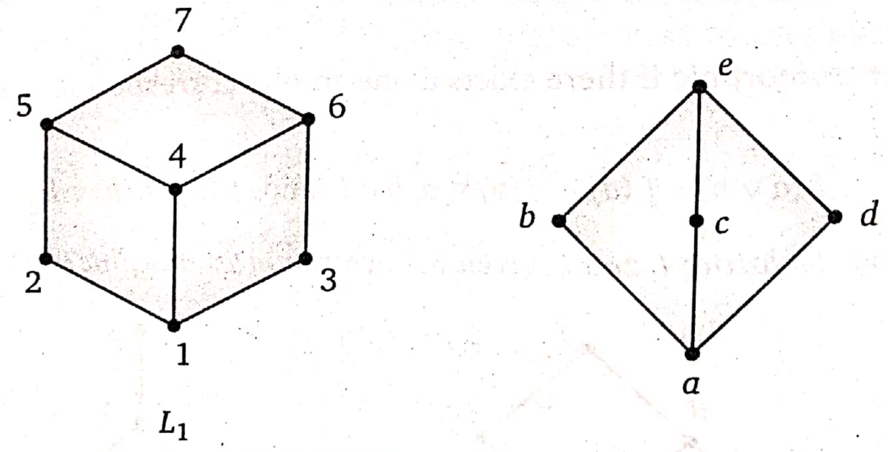


Fig. 6.78

Solution: L_1 and L_2 lattices are not isomorphic. Since one-one corresponding is not possible as the element of two lattices are not same.

6.16 Distributive Lattice

[U.P.T.U. (B.Tech.) 2009; I.G.N.O.U. (M.C.A.) 2001, 2004, 2006, 2009; R.G.P.V. (B.E.) Bhopal 2005, 2008]

A lattice L is called distributive lattice if for any element a, b and c of L, it satisfies the following properties.

(i) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ (ii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

***Theorem* 13:** *Let $a, b, c \in L$, where $(L, \leq)$ is a distributive lattice. Then $a \vee b = a \vee c$ and $a \wedge b = a \wedge c \Rightarrow b = c$*

[Rohtak (B.E.) 2008; Raipur (B.E.) 2007]

***Proof*:** We know that

$$\begin{aligned}
b &= b \vee (b \wedge a) && \text{[absorption]}\\
&= b \vee (a \wedge b) && \text{[commutative]}\\
&= b \vee (a \wedge c) && [a \wedge b = a \wedge c]\\
&= (b \vee a) \wedge (b \vee c) && \text{[distributive]}\\
&= (a \vee b) \wedge (c \vee b) && \text{[commutative]}\\
&= (a \vee c) \wedge (c \vee b) && [a \vee b = a \vee c]\\
&= (c \vee a) \wedge (c \vee b) &&\\
&= c \vee (a \wedge c) && [a \wedge b = a \wedge c]\\
&= c \vee (c \wedge a) &&\\
&= c && \text{[absorption]}
\end{aligned}$$

6.17 Complete Lattice

[U.P.T.U. (B.Tech.) 2005]

Let $(L, \leq)$ be lattice. Then L is said to be complete if every subset A of L, $\wedge A$ and $\vee A$ exist in L. Thus, in every complete lattice $(L, \leq)$ there exist a greatest element g and a least element l.

6.18 Complement of an Element in a Lattice

Let $(L, \leq)$ be a lattice and let 0 and 1 be its lower and upper bounds. If $a \in L$ is an element than an element b is called complement of a if

$$a \vee b = 1 \quad \text{and} \quad a \wedge b = 0$$

Remark: 0 and 1 are called universal bounds. But commutative property we can say if b is complement of a then a is also complement of b.

6.19 Complemented Lattice

Let $(L, \leq)$ be a lattice with universal bounds 0 and 1. The lattice L is said to be complemented lattice if every element in L has a complement *i.e.*

$$\begin{cases} a \vee 1 = 1, \; a \wedge 1 = a \\ a \wedge 0 = 0, \; a \vee 0 = a \end{cases}$$

The complement of a is denoted by a' or $\bar{a}$. Then

$$a \wedge a' = 0, \; a \vee a' = 1$$

6.20 Complemented Complete Lattice

Let $(L, \leq)$ be a complete lattice with greatest and lower elements g and l respectively, then L is called complemented complete lattice. If for each $a \in L$, there exists an element $a' \in L$ such that

$$a \vee a' = g \text{ and } a \wedge a' = l$$

Remark: The complemented distributive lattice is called a ***Boolean Algebra.***

6.21 Bounded Lattice

[U.P.T.U. (B.Tech.) 200...]

Let $(L, \leq)$ be a lattice. Then L is said to be bounded lattice if it has a least element 0 and a greatest element 1.0 is called the identity of joint and 1 is called the identity of meet in a bounded lattice $(L, \vee, \wedge)$.

6.22 Direct Product of Lattices

Let $(L_1, \wedge_1, \vee_1)$ and $(L_2, \wedge_2, \vee_2)$ be two lattices, where $\wedge_1, \vee_1$ are meet and join in L_1 and $\wedge_2, \vee_2$ are meet and join in L_2. The algebraic system $(L_1 \times L_2, \wedge, \vee)$ is called the ***Direct Product*** of the lattices $(L_1, \wedge_1, \vee_1)$ and $(L_2, \wedge_2, \vee_2)$ if for any (a_1, a_2) and (b_1, b_2) in $L_1 \times L_2$, so that $a_1, b_1 \in L_1$ and $a_2, b_2 \in L_2$, define $\wedge$ and $\vee$ in $L_1 \times L_2$

$$(a_1, a_2) \wedge (b_1, b_2) = (a_1 \wedge_1 b_1, a_2 \wedge_2 b_2)$$
$$(a_1, a_2) \vee (b_1, b_2) = (a_1 \vee_1 b_1, a_2 \vee_2 b_2)$$

6.23 Modular Lattices

A lattice L is said to be modular lattice if for all $a, b, c \in L, a \leq c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c$

Theorem 14: *Let $L \{a_1, a_2, \ldots a_n\}$ be finite lattice then L is bounded.*

Proof: Let $L = \{a_1, a_2, a_3 \ldots a_n\}$ be any finite lattice. Then we have to show that L having least and greatest element.

Now,

$$b_1 = a_1$$
$$b_2 = a_2 \wedge b_1$$
$$b_3 = a_3 \wedge b_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$b_n = a_n \wedge b_{n-1}$$

All $\quad b_1, b_2, \ldots b_n \in L$ and $b_n \leq a_i \ \forall i = 1, 2, 3 \ldots n$

$\Rightarrow b_n$ is the least element of L

$\therefore$

$$b_n = a_1 \wedge a_2 \wedge a_3 \ldots \wedge a_n$$

Similarly, $a_1 \vee a_2 \vee a_3 \ldots \vee a_n$ is the greatest element of L. Thus, L is bounded lattice.

Theorem 14: *A lattice in which relative complements are unique is distributive.* **[Delhi (B.E.) 2009]**

Proof: Let L be a lattice in which relative complements are unique. Then L cannot contain a pentagonal sublattice in which relative complements are not unique. Similarly it cannot contain a sublattice isomorphic with M_5. Hence L is distributive.

Theorem 16: *For any a and b in a Boolean algebra B.*

(i) $(a')' = a$ (ii) $(a \vee b)' = a' \wedge b'$

(iii) $(a \wedge b)' = a' \vee b'$ (iv) $\begin{cases} a \vee (a' \wedge b) = a \vee b \\ a \wedge (a' \vee b) = a \wedge b \end{cases}$

[U.P.T.U. (B.Tech.) 2003, 2004, 2005]

Proof: (i) To show $(a')' = a$

Let complement of $a \in B$ be $a' \in B$ then by definition of compliment

$$a \vee a' = 1 \quad \text{and} \quad a \wedge a' = 0 \qquad \text{...(1)}$$

Since the operation $\vee$ and $\wedge$ are commutative then. From (1)

$$a' \vee a = 1 \quad \text{and} \quad a' \wedge a = 0$$

This show $a \in B$ be the complement of a'

i.e. $$(a')' = a$$

(ii) To show $(a \vee b)' = a' \wedge b'$

If $a' \wedge b'$ be complement of $a \vee b$ then show

$$(a \vee b) \vee (a' \wedge b') = 1 \text{ and } (a \vee b) \wedge (a' \wedge b') = 0$$

Now
$$\begin{aligned}(a \vee b) \vee (a' \wedge b') &= [(a \vee b) \vee a'] \wedge [(a \vee b) \vee b'] && \text{[by distributive]}\\ &= [a \vee (b \vee a')] \wedge [a \vee (b \vee b')] && \text{[by associativity]}\\ &= [a \vee (a' \vee b)] \wedge [a \vee 1] && [a \vee b' = 1]\\ &= [(a \vee a') \vee b] \wedge 1 && [a \vee 1 = 1]\\ &= [1 \vee b] \wedge 1\\ &= 1 \wedge 1\\ &= 1\end{aligned}$$

and
$$\begin{aligned}(a \vee b) \wedge (a' \wedge b') &= [a \wedge (a' \wedge b')] \vee [b \wedge (a' \wedge b')]\\ &= [(a \wedge a') \wedge b'] \vee [b \wedge a'] \wedge b' = (0 \wedge b') \vee (a' \wedge b) \wedge b'\\ &= 0 \vee [a' \wedge (b \wedge b')] = 0 \vee [a' \wedge 0] = 0 \vee 0 = 0\end{aligned}$$

Thus, $a' \wedge b'$ is the complement of $a \vee b$ *i.e.*

$$(a \vee b)' = a' \wedge b'$$

(iii) Applying principle of duality on $(a \vee b)' = a' \wedge b'$

We get $$(a \wedge b)' = a' \vee b'$$

(iv) We have $a \vee (a' \wedge b) = (a \vee a') \wedge (a \vee b)$ [by distributive]

$$= 1 \wedge (a \vee b) \quad = a \vee b \qquad [a \vee a' = 1]$$

Again, $$a \wedge (a' \vee b) = (a \wedge a') \vee (a \wedge b)$$
$$= 0 \vee (a \wedge b) = a \wedge b$$

Theorem 17: *Prove that in a distributive lattice, if an element has a complement then this complement is unique.* [U.P.T.U. (B.Tech.) 2003, 2008, 2009]

Proof: Let $(L, \leq)$ be a bounded distributive lattice. Let $a \in L$ having two complement b and c then show $b = c$

Since b and c be complement of a then

$$a \vee b = 1 \qquad a \wedge b = 0$$

$$a \vee c = 1 \qquad a \wedge c = 0$$

Now

$$\begin{aligned} b &= b \wedge 1 \\ &= b \wedge (a \vee c) \\ &= (b \wedge a) \vee (b \wedge c) && \text{[by distributive law]} \\ &= (a \wedge b) \vee (b \wedge c) && [a \wedge b = b \wedge a] \\ &= 0 \vee (b \wedge c) && [a \wedge b = 0] \\ &= (a \wedge c) \vee (b \wedge c) && [0 = a \wedge c] \\ &= (a \vee b) \wedge c && [a \vee b = 1] \\ &= 1 \wedge c = c \end{aligned}$$

Hence, complement of a is unique.

Theorem 18: *If L is a distributive lattice and b' is the complement of b in L and $a \wedge b' = 0$, then show $a \leq b$*

Proof: We have $a \wedge b' = 0$

$$\begin{aligned} &\Rightarrow (a \wedge b') \vee b = 0 \vee b = b \\ &\Rightarrow (a \vee b) \wedge (b' \vee b) = b \\ &\Rightarrow a \vee b = b \\ &\Rightarrow a \leq b \end{aligned}$$

Theorem 19: *The dual of distributive lattice is a distributive lattice.*

Proof: Let $(L, \wedge, \vee)$ be a distributive lattice. Then we have to show its dual $(L, \wedge^*, \vee^*)$ is also distributive where $\wedge^* = \vee$, $\vee^* = \wedge$.

Let $a, b, c \in L$, then we have to show

$a \wedge^* (b \vee^* c) = (a \wedge^* b) \vee^* (a \wedge^* c)$ or $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

This is true because $(L, \wedge, \vee)$ is distributive lattice. Hence $(L, \wedge^*, \vee^*)$ is also distributive.

Theorem 20: *In any lattices $(L, \vee, \wedge)$ the following statements are equivalent*

(i) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ $\forall\, a, b, c \in L$ (ii) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ $\forall\, a, b, c \in L$

Proof: Let (i) $\rightarrow$ (ii) *i.e.* (i) is given than show (ii)

R.H.S. of (ii) $= (a \vee b) \wedge (a \vee c) = a \vee [c \wedge (a \vee b)]$

$$\begin{aligned} &= a \vee [(c \wedge a) \vee (c \wedge b)] && \text{[from (i)]} \\ &= [a \vee (c \wedge a) \vee (c \wedge b) \\ &= a \vee (b \wedge c) = \text{L.H.S. of (ii)} \end{aligned}$$

Again (ii) $\rightarrow$ (i) it can be show in dual number.

Theorem 21: *Dual of a complemented lattice is complemented.*

Proof: Let (L, R) be a complemented lattice with 0 and 1 as least and greatest elements. Let $(L, \bar{R})$ be the dual of (L, R). Then 1 and 0 are the least and greatest elements of $(L, \bar{R})$.

Let a be any element of L. Since (L, R) is complemented, there exists $a' \in L$ such that

$$a \wedge a' = 0 \text{ and } a \vee a' = 1 \text{ in } (L, R)$$

$$0 = \inf\{a, a'\} \text{ and } 1 = \sup\{a, a'\} \text{ in } (L, R)$$

i.e.,

$$0\,R\,a,\ 0\,R\,a' \text{ and } a\,R\,1,\ a'\,R\,1$$

$\Rightarrow$ $$a\,\bar{R}\,0,\ a'\,\bar{R}\,0 \text{ and } 1\,\bar{R}\,a,\ 1\,\bar{R}\,a'$$

$\Rightarrow$

$\Rightarrow$ 0 is an upper bound of $\{a, a'\}$ in $\{L, \bar{R}\}$ and 1 is a lower bound of $\{a, a'\}$ in $\{L, \bar{R}\}$.

Let k be any upper bound of $\{a, a'\}$ in $\{L, \bar{R}\}$. Then $a\,\bar{R}\,k$ and $a'\,\bar{R}\,k$

$\Rightarrow$ $k\,R\,a$ and $k\,R\,a'$

$\Rightarrow$ $k\,R\,0$ as 0 is the inf. of $\{a, a'\}$ in (L, R)

$\Rightarrow$ $0\,\bar{R}\,k$.

This shows that 0 is the least upper bound of $[a, a']$ in $\{L, \bar{R}\}$. Therefore $a \vee a' = 0$ in (L, R).

Similarly, we can show that $a \wedge a' = 1$ in $(L, \bar{R})$. Thus, a is the complement of a' in $(L, \bar{R})$.

Hence $(L, \bar{R})$ is complemented.

Theorem 22: *Two bounded lattices L_1 and L_2 are complemented iff $L_1 \times L_2$ is complemented.*

Proof: Let L_1 and L_2 be two complemented lattices and let 0, 1 and 0′, 1′ are least and greatest elements of L_1 and L_2 respectively. Then $(0, 0')$ and $(1, 1')$ will be the least and greatest elements of $L_1 \times L_2$. Let (a_1, a_2) be any element of $L_1 \times L_2$ then $a_1 \in L_1$ and $a_2 \in L_2$.

Since L_1 and L_2 are complemented, there exists $a_1' \in L_1$ and $a_2' \in L_2$ such that $a_1 \wedge a_1' = 0$, $a \vee a' = 1$ and $a_2 \wedge a_2' = 0'$ and $a_2 \vee a_2' = 1$. We shall show that (a_1', a_2') is the complement of (a_1, a_2) in $L_1 \times L_2$.

We have $(a_1, a_2) \wedge (a_1', a_2') = (a_1 \wedge a_1', a_2 \wedge a_2') = (0, 0')$

and $(a_1, a_2) \vee (a_1', a_2') = (a_1 \vee a_1', a_2 \vee a_2') = (1, 1')$.

This shows that (a_1', a_2') is the complement of (a_1, a_2) in $L_1 \times L_2$.

Hence $L_1 \times L_2$ is complemented.

Conversely, Let $L_1 \times L_2$ be complemented, we have to show that L_1 and L_2 are complemented.

Let $a_1 \in L_1$ and $a_2 \in L_2$ then $(a_1, a_2) \in L_1 \times L_2$.

Since $L_1 \times L_2$ is complemented, therefore exists $(a_1', a_2') \in L_1 \times L_2$ such that

$$(a_1, a_2) \wedge (a_1', a_2') = (0, 0')$$

and $$(a_1, a_2) \vee (a_1', a_2') = (1, 1')$$

$\Rightarrow$ $(a_1 \wedge a_1', a_2 \wedge a_2') = (0, 0')$ and $(a_1 \vee a_1', a_2 \vee a_2') = (1, 1')$

$\Rightarrow$ $a_1 \wedge a_1' = 0$, $a_2 \wedge a_2' = 0'$ and $a_1 \vee a_1' = 1$, $a_2 \vee a_2' = 1'$

$\Rightarrow$ $a_1 \wedge a_1' = 0$, $a_1 \vee a_1' = 1$ and $a_2 \wedge a_2' = 0'$, $a_2 \vee a_2' = 1'$

$\Rightarrow$ a_1' and a_2' are complements of a_1 and a_2 respectively. Hence L_1 and L_2 are complemented.

Theorem 23: *Show that every chain is a distributive lattice.* [U.P.T.U. (B.Tech.)...]

Solution: Let $(L, \leq)$ be a chain and $a, b, c \in L$. We consider the cases:

(i) $a \leq b$ or $a \leq c$ and (ii) $a \geq b$ or $a \geq c$.

Now we shall show that distributive law is satisfied by a, b, c:

For case (i), we have

$$a \wedge (b \vee c) = a \text{ and } (a \wedge b) \vee (a \wedge c) = a.$$

For case (ii), we have

$$a \wedge (b \vee c) = b \vee c \text{ and } (a \wedge b) \vee (a \wedge c) = b \vee c.$$

Thus, we have $a \wedge (b \vee c) = (a \wedge c) \vee (a \wedge c)$.

This shows that a chain is a distributive lattice.

Theorem 24: *Show that every complete lattice is a bounded lattice.*

Proof: Let L be a complete lattice. Then every non-empty subset of L has least upper bound and greatest lower bound. Therefore L itself has least upper bound and greatest lower bound. Hence L is a bounded lattice.

Theorem 25: *In lattice L with least element 0 and greatest element 1, show that 0 is the unique. Complement of 1 and 1 is the unique complements of 0.* [R.G.P.V. (B.E.) Raipur 2005, 2009; Kurukshetra (B.E.) ...]

Proof: We have $0 \wedge 1 = 0$ and $0 \vee 1 = 1$

Therefore 0 and 1 are complements of each other. Now we have to show that each of these two complements are unique. Suppose if possible a be any other complement of 0

Then $0 \wedge a = 0$ and $0 \vee a = 1$

But $0 \vee a = a$ Therefore, $a = 1$

This shows that complement of 0 is unique. Similarly we can show that 0 is the only complement of 1.

Theorem 26: *The pentagonal lattice given below is not modular.*

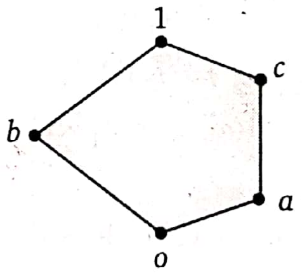


Fig. 6.79

Proof: Let $a \leq c$

and

$$a \vee (b \wedge c) = a \vee 0 = a$$
$$(a \vee b) \wedge c = 1 \wedge c = c$$

$\therefore$

$$a \vee (b \wedge c) \neq (a \vee b) \wedge c$$

Therefore, the pentagonal lattice is not modular.

Theorem 27: *Show that every finite lattice is complete.*

Proof: Let $(L, \wedge, \vee)$ be any finite lattice. And S be any non empty subset of L. Then S is finite set. Let $S = \{a_1, a_2, \ldots a_n\}$. Then $a_1 \wedge a_2 \ldots \wedge a_n$ and $a_1 \vee a_2 \ldots \vee a_n$ are infinum and supremum of S in L. Hence L is complete.

Theorem 28: *Every distributive lattice is modular.* [M.K.U. (B.E.) 2008; P.T.U. (B.E.) Punjab 2008]

Proof: Let L be a distributive lattice and $a, b, c \in L$. Then we have to show

$$a \leq c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c$$

Since L is distributive, then

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \quad \text{...(1)}$$

Also

$$a \leq c \Rightarrow a \vee c = c \quad \text{...(2)}$$

Then from (1) and (2) we find

$$a \leq c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c$$

This show L is modular lattice.

But converse of above theorem is not true.

Theorem 29: *Show that a lattice L is modular iff for any a, b, c ∈ L, the following condition holds*

$$a \vee (b \wedge (a \vee c)) = (a \vee b) \wedge (a \vee c)$$

Proof: Let L is modular then

$$a \leq c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c \quad \text{...(1)}$$

Since $a \leq c$ then from (1)

$$a \vee (b \wedge (a \vee c)) = (a \vee b) \wedge (a \vee c)$$

Conversely, Let for all $a, b, c \in L$

$$a \vee (b \wedge (a \vee c)) = (a \vee b) \wedge (a \vee c) \quad \text{...(2)}$$

Then show L is modular. Let $a, b, c \in L$, $a \leq c$ then $a \leq c$

Hence from (2), we have

$$a \leq c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c$$

$\Rightarrow L$ is modular.

Example 51: *If $A = \{a, b, c\}$. Prove that lattice $(P(A), \cap, \cup)$ (under $\subseteq$) is distributive when $P(A)$ = power set of A.*

Solution: Let $A = \{a, b, c\}$ then power set

$$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

The Hasse diagram is shown in fig. 6.80

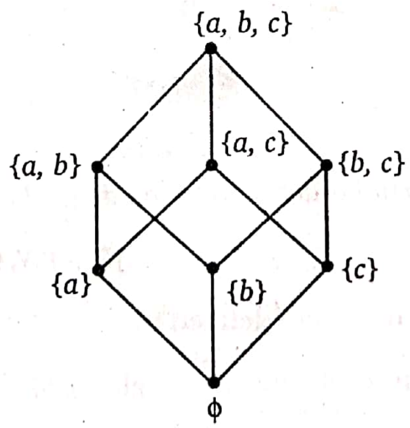


Fig. 6.80

This is distributive lattice under $A \vee B = A \cup B$ and $A \wedge B = A \cap B$, $A, B \in P(A)$

Let $A = \{a\}, B = \{b, c\}, C = \{c, a\}$, then we have

$$B \vee C = \{b, c\} \cup \{c, a\} = \{a, b, c\}$$

$$A \wedge (B \vee C) = \{a\} \cap \{a, b, c\} = a \qquad ...(1)$$

Also
$$A \wedge B = A \cap B = \{a\} \wedge \{b, c\} = \phi$$

$$A \wedge C = A \cap C = \{a\} \cap \{c, a\} = \{a\}$$

$\therefore$
$$(A \wedge B) \vee (A \wedge C) = \phi \cup \{a\} = \{a\} \qquad ...(2)$$

From (1) and (2) we see that

$$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$$

Again
$$B \wedge C = B \cap C = \{b, c\} \wedge \{c, a\} = \{c\}$$

$\therefore$
$$A \vee (B \wedge C) = \{a\} \cup \{c\} = \{a, c\} \qquad ...(3)$$

Also
$$A \vee B = A \cup B = \{a\} \cup \{b, c\} = \{a, b, c\}$$

$$A \vee C = A \cup C = \{a\} \cup \{c, a\} = \{a, c\}$$

$\therefore$
$$(A \vee B) \wedge (A \vee C) = \{a, b, c\} \cap \{a, c\} = \{a, c\} \qquad ...(4)$$

From (3) and (4), we have

$$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$$

Example 52: *The lattice* $(L, \leq)$ *in fig.* 6.81 is not a distributive lattice.

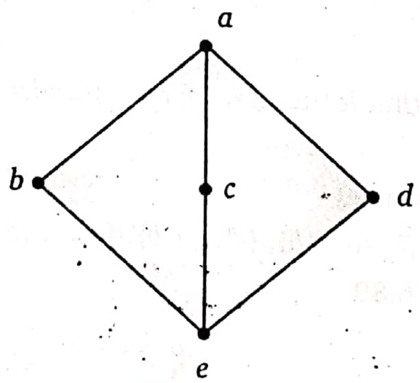


Fig. 6.81

Where $L = \{a, b, c, d, e\}$ *and* $\leq$ *is a partial ordering relation defined on L.*

[R.G.P.V. (B.E.) Bhopal 2009; P.T.U. (B.E.) Punjab 2006]

Solution: The joint and meet operations are defined by

$$a \vee b = \text{lup } \{a, b\} \text{ and } a \wedge b = \text{glb } \{a, b\}$$

The joint and meet operation tables are given as.

$\vee$	a	b	c	d	e
a	a	a	a	a	a
b	a	b	a	a	b
c	a	a	c	a	c
d	a	a	a	d	d
e	a	b	c	d	e

$\wedge$	a	b	c	d	e
a	a	b	c	d	e
b	b	b	e	e	e
c	c	e	c	e	e
d	d	e	e	d	e
e	e	e	e	e	e

$\therefore$ $$b \wedge (c \vee d) = b \wedge a = b$$

and $$(b \wedge c) \vee (b \wedge d) = e \vee e = e$$

Thus, $$b \wedge (c \vee d) \neq (b \wedge c) \vee (b \wedge d)$$

Example 53: *Show that the lattice shown in Fig. 6.82 are non-distributive.*

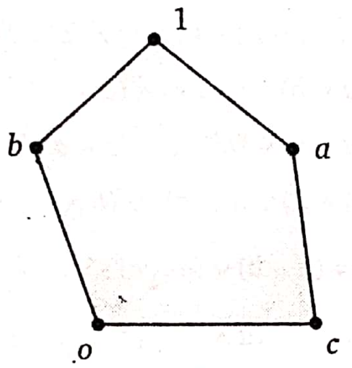


Fig. 6.82

Solution: We have $a \wedge (b \vee c) = a \wedge 1 = a$

and $(a \wedge b) \vee (a \wedge c) = 0 \vee c = c$

$\therefore$ $$a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$$

Hence, the lattice is not distributive.

Theorem 30: *A lattice $(L, \leq)$ is distributive if and only if*

$$(a \vee b) \wedge (b \vee c) \wedge (c \vee a) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \forall\, a, b, c \in L \quad ...(1)$$

Proof: Let $(L, \leq)$ is a distributive lattice, then show

$$(a \vee b) \wedge (b \vee c) \wedge (c \vee a) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$$

L.H.S. $= (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$

$= \{a \vee [(b \vee c) \wedge (c \vee a)]\} \vee \{b \wedge [(b \vee c) \wedge (c \wedge a)]\}$ [L is distributive]

$= [(a \wedge (c \vee a)) \wedge (b \vee c)] \vee [(b \wedge (b \vee c) \wedge (c \vee a)]$

$= [a \wedge (b \vee c)] \vee [b \wedge (c \vee a)]$ [by absorption law]

$= [(a \wedge b) \vee (a \wedge c)] \vee [(b \wedge c) \vee (b \wedge a)]$

$= (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$ = R.H.S.

Conversely: Let the condition (1) hold, then shown $(L, \leq)$ is a distributive lattice.

We first show that $(L, \leq)$ is a modular lattice.

Let x, y, z be any three elements in A, and let $x \leq z$. Then

$$x \vee (y \wedge z) = [x \vee (x \wedge y)] \vee (y \wedge z)$$

$$= [(x \wedge z) \vee (x \wedge y)] \vee (y \wedge z) \qquad [x \leq z \Rightarrow x \wedge z = x]$$

$$= (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$$

$$= (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$$

$$= (x \vee y) \wedge [(y \vee z) \wedge z]$$

$$= (x \vee y) \wedge z$$

Thus, $(L, \leq)$ is a modular lattice.

Now, for any three elements a, b, c in L, we have

$$a \wedge (b \vee c) = [a \wedge (a \vee c)] \wedge (b \vee c) \qquad \text{[By absorption law]}$$

$$= a \wedge (a \vee b) \wedge (a \vee c) \wedge (b \vee c)$$

$$= [(a \vee b) \wedge (b \vee c) \wedge (c \vee a)] \wedge a$$

$$= [(a \wedge b) \vee (b \wedge c) \vee (c \wedge a)] \wedge a$$

$$= [(a \wedge b) \vee (c \wedge a)) \vee (b \wedge c)] \wedge a \qquad \text{...(1)}$$

Since $\quad a \wedge b \leq a, a \wedge c \leq a \Rightarrow (a \wedge b) \vee (a \wedge c) \leq a \vee a$

Then $\quad (a \wedge b) \vee (a \wedge c) \leq a \qquad \text{...(2)}$

$$a \wedge (b \vee c) = [(a \wedge b) \vee (c \wedge a)] \vee [(b \wedge c) \wedge a]$$

$$= [(a \wedge b) \vee (c \wedge a) \vee (b \wedge (c \wedge a))]$$

$$= (a \wedge b) \vee (c \wedge a)$$

Hence $(L, \leq)$ is distributive.

Theorem 31: *Every sublattice of a distributive lattice is distributive.*

Proof: Let M be a sublattic of distributive lattice L.

Let $a, b, c \in M$. Then $a, b, c \in L$.

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \text{ in } L$$

$$\Rightarrow a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \text{ in } M.$$

$$\Rightarrow M \text{ is distributive.}$$

Example 54: *Let $A = \{1, 2, 3, 5, 30\}$ and $a \le b$ iff a divided b. The Hasse diagram is shown in Fig. 6.83 Find complement of 2.* **[U.P.T.U. (B.Tech.) 2002]**

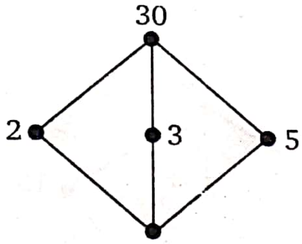


Fig. 6.83

Solution: Since $2 \wedge 3 = 1, \;\; 2 \vee 3 = 30$

$2 \wedge 5 = 1, \;\; 2 \vee 5 = 30$

Hence, 2 has two complement 3 and 5

Example 55: *In the lattice defined by the Hasse diagram given by the following figure.* 6.84.

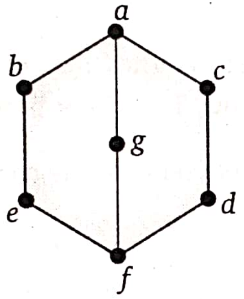


Fig. 6.84

How many complements does the elements 'e' have? Given all **[U.P.T.U. (B.Tech.) 2003, 2006, 2007]**

Solution:

Since $e \wedge g = f, \;\; e \vee g = a$ and $e \wedge d = f \;\; e \vee d = a$

where a is universal upper bound and f be universal lower bound. Hence, d, g be two complement of e.

Example 56: *Homomorphism image of a distributive lattice, is distributive.*

Solution: Let $f : L \to M$ be homomorphism and L be a distributive lattice. Then show M is distributive. Let $x, y, z \in M$ as f is onto $\exists\, a, b, c \in L$ such that $f(a) = x,\; f(b) = y,\; f(c) = z$

$$\begin{aligned} x \wedge (y \vee z) &= f(a) \wedge (f(b) \vee f(c)) \\ &= f(a) \wedge (f(b \vee c)) \\ &= f(a) \wedge f(b \vee c) \\ &= f\{a \wedge (b \vee c)\} \\ &= f\{(a \wedge b) \vee (a \wedge c)\} \\ &= f(a \wedge b) \vee f(a \wedge c) \\ &= [f(a) \wedge f(b)] \vee [f(a) \wedge f(c)] = (x \wedge y) \vee (x \wedge z) \end{aligned}$$

Hence, M is distributive.

Example 57: *Consider the lattice $D_{30} = \{1, 2, 3, 5, 6, 15, 30\}$, the divisor of 30 ordered by divisibility.*

(i) *Draw the Hasse diagram of D_{30}.* (ii) *Find the complement of 2 and 10, if exists.*

Solution: (i) The Hasse diagram of D_{30} is

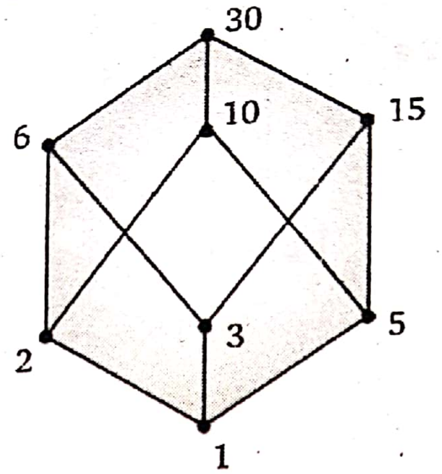


Fig. 6.85

(ii) The elements 2 and 10 has no complement as they do not satisfying the condition of complement $a \vee a' = 30$ and $a \wedge a' = 1$. 30 is upper bound, 1 is lower bound.

Example 58: *Let $(L, \leq)$ be a distributive lattice and let c' be the complement of an element c in L. If $b \wedge c' = 0$ then show that $b \leq c$.*

Solution: Since $b \wedge c' = 0$, then we have

$$(b \wedge c') \vee c = 0 \vee c = c$$

$$\Rightarrow \quad (b \vee c) \wedge (c' \vee c) = c$$

$$\Rightarrow \quad (b \vee c) \wedge 1 = c$$

$$\Rightarrow \quad b \vee c = c$$

$$\Rightarrow \quad b \leq c$$

Example 59: *Let $(L, \leq)$ be a distributive lattice. Show that if $a \wedge x = a \wedge y$ and $a \vee x = a \vee y$ for some $a \in L$ Then $x = y$.* **[U.P.T.U. (M.C.A.) 2005]**

Solution: Let $(L, \leq)$ be a distributive lattice and let

$$a \wedge x = a \wedge y \text{ and } a \vee x = a \vee y \quad \text{...(1)}$$

Now

$$\begin{aligned}
x &= x \wedge (x \vee a) && \text{[by absorption law]}\\
&= x \wedge (a \vee x) && \text{[by commutative]}\\
&= x \wedge (a \vee y) && \text{[by (1)]}\\
&= (x \wedge a) \vee (x \wedge y) && \text{[by distributive]}\\
&= (a \wedge x) \vee (x \wedge y) && \\
&= (a \wedge y) \vee (x \wedge y) && \text{[by (1)]}\\
&= (a \vee x) \wedge y && \\
&= y \wedge (a \vee x) && \text{[by distributive]}\\
&= y \wedge (a \vee y) && \text{[by (1)]}\\
&= y && \text{[by absorption law]}
\end{aligned}$$

Exercise

Posets and Lattices

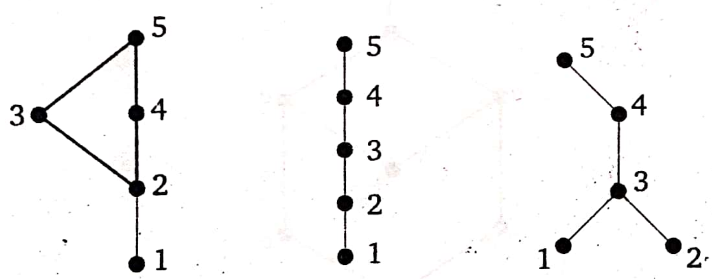
1. Define poset. Give an example of a set X such that $(P(X), \subseteq)$ is totally ordered set.

[U.P.T.U. (B.Tech.) 2007]

2. Show that in a poset least elements, if exists is unique.
3. Show that in a poset greatest elements, if exists is unique.
4. Define the terms
   (i) partially ordered set (ii) linearly ordered set.
5. Let $\leq$ be partial ordering of a set S. Define the dual order on S. How is the dual order related to the inverse of the relation $\leq$
6. What is meant of a "Hasse diagram". Draw the Hasse diagram of the relation R on A where $A = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (1, 2), (2, 2), (2, 4), (1, 3), (3, 3), (3, 4), (1, 4), (4, 4)\}$.
7. Which of the following are partial order.
   (i) The relation $R = \{(a, b) \in Z \times Z \mid |a| \leq |b|$ on Z
   (ii) The relation $R = \{(a, b) \in Z \times Z \mid a - b \leq 0$
   (iii) The relation $R = \{(a, b) \in Z \times Z \mid a$ divide b in Z on Z.
8. Let $A = \{a, b\}$. Describe all partial order relations on A.
9. Find two incomparable elements in the following posets
   (i) $(P\{0, 1, 2\}, \subseteq)$ (ii) $(\{1, 2, 4, 6, 8\})$
10. Let D_m denote the positive divisors of m ordered by divisiblity. Draw the Hasse diagram of
    (a) D_{12} (b) D_{15} (c) D_{16} (d) D_{17}
11. Let A be given finite set and $P(A)$ is a power set. Let $\subseteq$ be inclusion relation on the elements of $P(A)$. Draw Hasse diagrams of $(P(A), \subseteq)$ for
    (i) $A = \{a\}$ (ii) $A = \{a, b\}$ (iii) $A = \{a, b, c\}$

[R.G.P.V. (B.E.) Bhopal 2006; P.T.U. (B.E.) Punjab 2008]

12. Let $A = \{1, 2, 3, 4, 5\}$. Determine the relations represented by the following Hasse diagram.



(i) (ii) (iii)

Fig. 6.86

13. Let $A = \{1, 2, 3, 4, 5\}$ be ordered set as shown in fig. 6.87 below. Find

(i) all minimal and maximal elements of A

(ii) greatest and least element of A

(iii) all linearly ordered subset of A, each of which contains at least three elements.

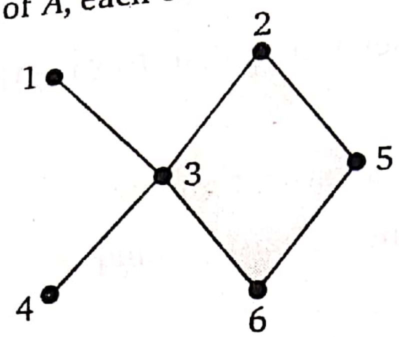


Fig. 6.87

14. Consider the subsets $\{2, 3\}$, $\{4, 6\}$, and $\{3, 6\}$ in the poset ($\{1, 2, 3, 4, 5, 6\}$, '|'). Find for each subsets if exists

(i) upper bound and lower bound

(ii) greatest lower bound and least upper bound.

15. Let S be ordered set in fig. 6.88 Suppose $A = \{1, 2, 3, 4, 5\}$ is isomorphic to S and $f : \{(a, 1), (b, 3), (c, 5), (d, 2), (e, 4)\}$ is similarly mapping from S to A. Draw the Hasse diagram of A.

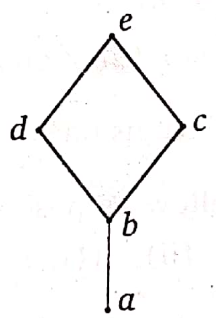


Fig. 6.88

16. Draw the Hasse diagrams of all lattices with upto five elements.

17. Consider the lattice L in fig. 6.89

(i) Find all sublattices with five elements

(ii) Find complements of a and b if they exist

(iii) Is L distributive, complemented?

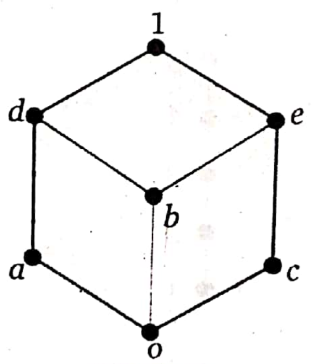


Fig. 6.89

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18. Show that lattice L represented by the diagram is complemented, modular but not distributive.

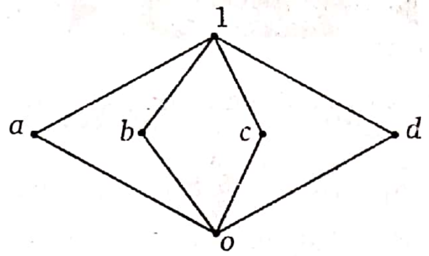

Fig. 6.90

19. Show that lattice L, represented by the diagram is modular, distributive but not complemented.

20. Find the complement of each element of D_{42}.

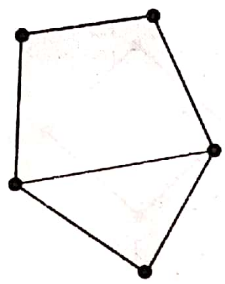

Fig. 6.91

21. Prove that the dual of a lattices is also a lattice. [U.P.T.U. (B.Tech.) 2005]

22. Every finite lattice is bounded. [U.P.T.U. (M.C.A) 2007]

23. Determine the matrix of the partial order whose Hasse diagrams are given below

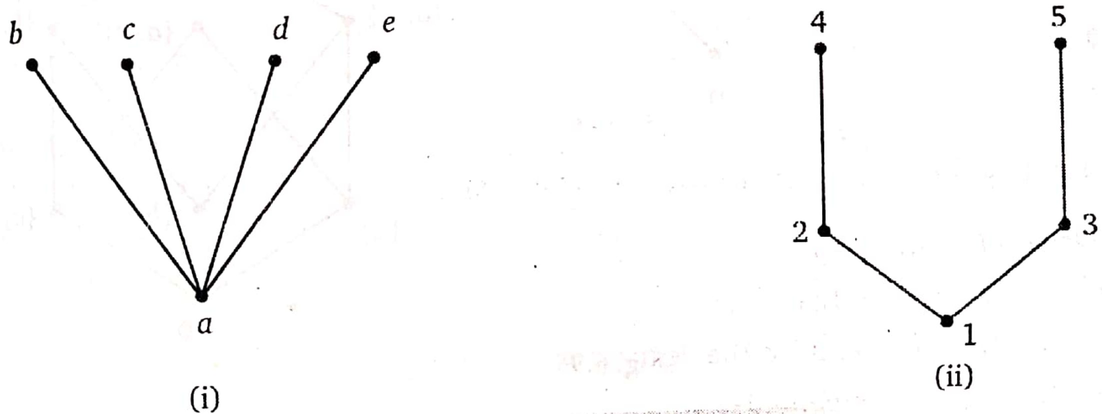

Fig. 6.92

Answer

6.

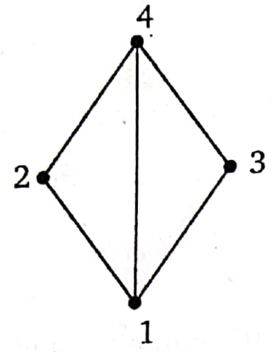

Fig. 6.93

7. (i) No (ii) Yes (iii) No

8. $R_1 = \{(a, a), (b, b)\}$, $R_2 = \{(a, a), (b, b), (a, b)\}$,
$R_3 = \{(a, a), (b, b), (b, a)\}$

9. (i) {0} and 1 {1} (ii) 4 and 6

10.

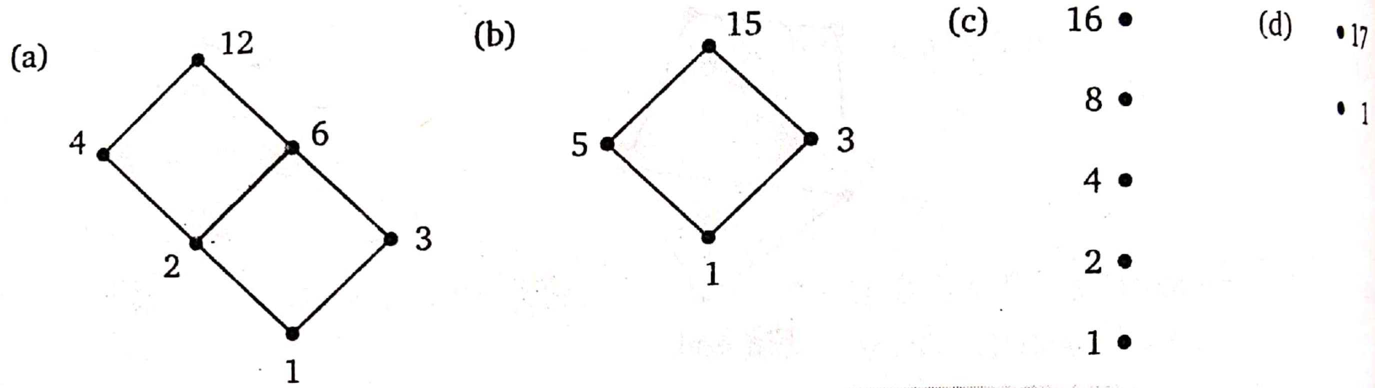

Fig. 6.94

11.

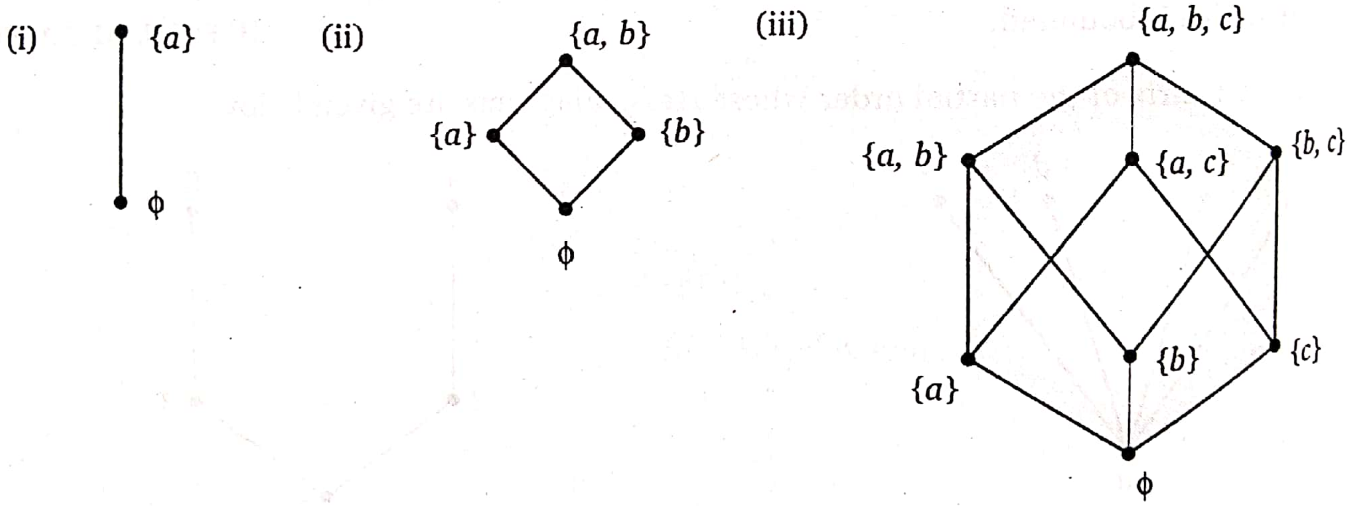

Fig. 6.95

12. (i) {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 2), (2, 3), (2, 4), (2, 5), (3, 3), (3, 5), (4, 4), (4, 5), (5, 5)}
(ii) {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 2), (2, 3), (2, 4), (2, 5), (3, 3), (3, 4), (3, 5), (4, 4), (4, 5), (5, 5)}
(iii) {(1, 1), (1, 3), (1, 4), (1, 5), (2, 2), (2, 3), (2, 4), (2, 5) (3, 3), (3, 4), (3, 5), (4, 4), (4, 5), (5, 5)}

13. (i) Minimal 4 and 6, maximal 1 and 2
(ii) There is no greatest and least elements
(iii) {1, 3, 4}, {1, 3, 6}, {2, 3, 4}, {2, 3, 6}, {2, 5, 6}

14. For {2, 3}, upper bound is 6 and lower bound 1.

lub {2, 3} = 6, glb {2, 3} = 1

For {4, 6}, no upper bound and lower bounds are 2 and 1 glb {4, 6} = 2

For {3, 6} upper bound is 6 and lower bounds are 3 and 1 *i.e.* glb {3, 6} = 3 and lub {3, 6} = 6

15.

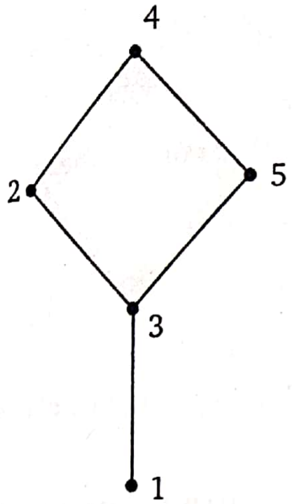

Fig. 6.96

16.

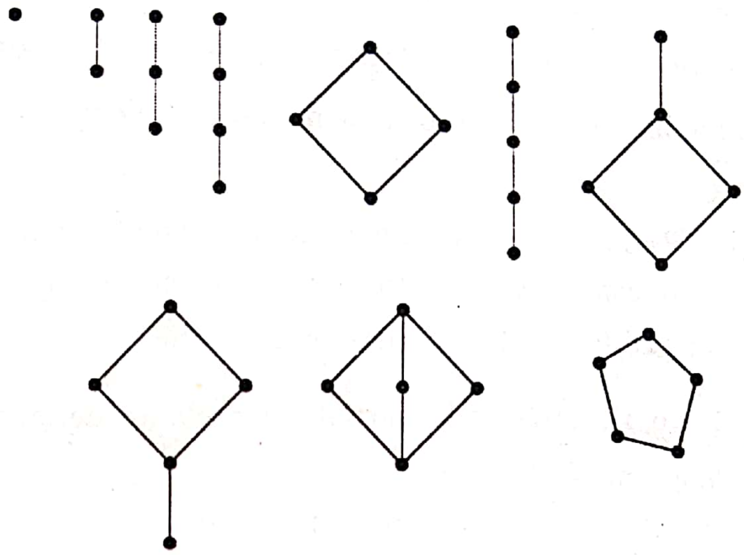

Fig. 6.97

17. (i) since oabd 1, oacd 1, oade 1, obce 1, oace 1, ocde 1.

(ii) c and e are complements of a, b has no complement

(iii) No, No

20. $1' = 42$, $2' = 21$, $3' = 14$, $6' = 7$, $7' = 6$, $14' = 3$, $21' = 2$, $42' = 1$

23. (i) $\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$

(ii) $\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$

Objective Type Quesitons

Multiple Choice Questions

1. The poset $(Z^+, \leq)$ is:
   (a) a lattice (b) not a lattice (c) isomorphic (d) none of these

2. The maximum number of zero element and unit element each in a poset is:
   (a) 0 (b) 1 (c) 2 (d) none of these

3. The lattice $(P(S), \subseteq)$ is bounded with greatest and least elements equal to:
   (a) S and ϕ respectively (b) ϕ and S respectively
   (c) 0 and 1 respectively (d) none of these

4. Let $X = \{2, 3, 6, 12, 24\}$, let $\leq$ be partial order relation defined by $x \leq y$ is x divides y. Number of edges in Hasse diagram of $(X, \leq)$ is:
   (a) 3 (b) 4 (c) 9 (d) none of these [U.P.T.U. (M.C.A.) 2005]

5. Identify which of the following partially ordered sets shown in following figures are lattices:

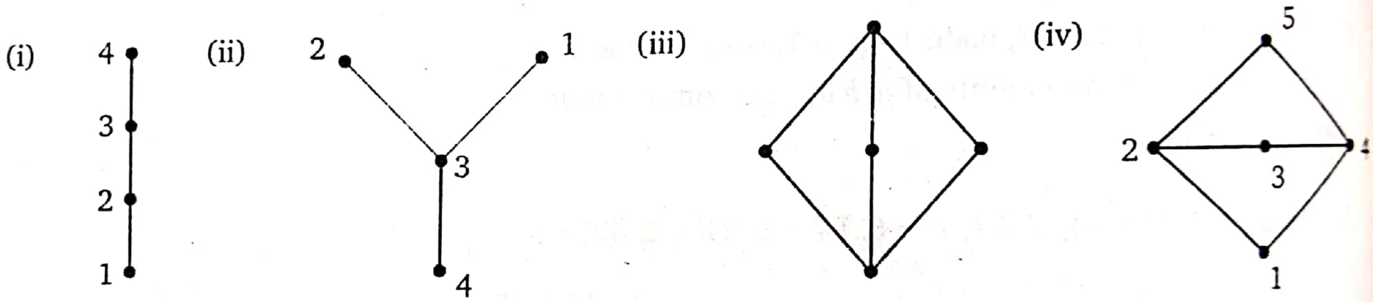

Fig. 6.98

6. From the Hasse diagram, find:
   (i) lub and glb of $A = \{2, 3, 6\}$
   (ii) lub and glb of $B = \{2, 3\}$
   (iii) lub and glb of $C = \{6, 12\}$

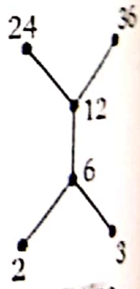

Fig. 6.99

7. In the lattice defined by the Hasse diagram as given. How many complements does the element e have
   (a) 2 (b) 3
   (c) 0 (d) none of these

Fig. 6.100

8. A self complemented, distributive lattice is called:
   (a) Boolean algebra (b) Modular lattice (c) Bounded lattice (d) Complete lattice

9. The maximal and minimal elements of posets are:
   (a) maximal 5, 6 and minimal 2
   (b) maximal 5, 6 and minimal 1
   (c) maximal 3, 5 and minimal 1, 6
   (d) none of these

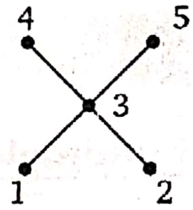

Fig. 6.101

State True or False

1. Every chain is not a distributive lattice.
2. Every chain is lattice.
3. Every distributive lattice is modular.
4. Every poset is lattice.
5. Every lattice is poset.
6. Product of two lattices is lattice.
7. A bounded lattice has two elements then $0 \neq 1$.

Fill in the Blank(s)

1. If L is finite lattice then L is ....
2. The set of natural number is ....

Answer

Multiple Choice Questions

1.	(a)	**2.**	(b)	**3.**	(a)	**4.**	(b)
5.	(i) lattice (ii) not lattice (iii) lattice (iv) not lattice						
6.	(i) lub = 6 glb does not exist (ii) lub = 6 glb does not exist (iii) lub = 12, glb = 6						
7.	2	**8.**	(a)	**9.**	(c)		

(iv) not lattic

State True or False

1.	False	**2.**	True	**3.**	True	**4.**	False	**5.**	True	**6.**	True
7.	True										

Fill in the Blank(s)

1.	bounded	**2.**	well order set