$\therefore \quad V_1 \cup V_2 = [a, b, c, d] = V$ and $V_1 \cap V_2 = \phi$

$\therefore \quad [V_1, V_2]$ is a partition of V.

Consider another graph shown in Fig. 11.116

This graph is also a bipartite graph.

Here $\quad V = [a, b, x, y, z]$

Let $\quad V_1 = [a, b], V_2 = [x, y, z]$

$\therefore \quad V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \phi$

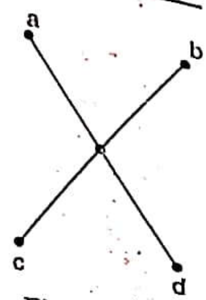



Fig. 11.115

Bipartite graphs are also called 2-colourable graphs as one can think vertices in $V_1$ of one colour and in $V_2$ of another colour and vertex is joined by an edge to a vertex of same colour.

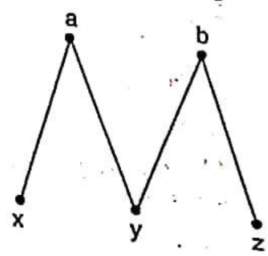



Fig. 11.116

(*b*) **Complete Bipartite Graph.** A graph G = (V, E) is called a complete bipartite graph if its vertices V can be partitioned into two subsets $V_1$ and $V_2$, such that each vertex of $V_1$ is connected to each vertex of $V_2$. The number of edges in a complete bipartite graph is $m.n$ as each of the $m$ vertices is connected to each of the $n$ vertices. It is denoted by $K_{m,n}$ and $m \leq n$.

*For example,* the graphs shown in Fig. 11.117 are complete bipartite graphs.

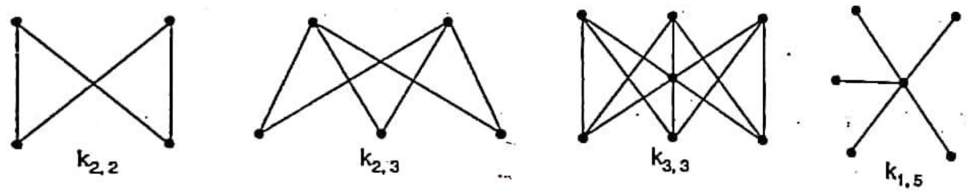



Fig. 11.117

A complete Bipartite graph $K_{1,n}$ is called a **star graph.**

**Example 6.** *Draw the bipartite graphs $K_{2,4}$ and $K_{3,4}$. Assuming any number of edges.*

**Sol.** First draw the appropriate number of vertices on two parallel columns or rows and connect the vertices in one column or row with the vertices in other column or row. The bipartite graphs $K_{2,4}$ and $K_{3,4}$ are shown in Fig. 11.118 and Fig. 11.119 respectively.

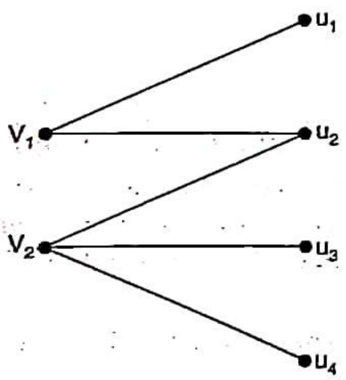



Fig. 11.118. Bipartite Graph $K_{2,4}$

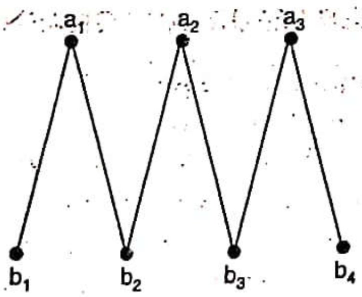



Fig. 11.119 Bipartite Graph $K_{3,4}$

**Example 7.** *Draw the complete bipartite graphs $K_{3,4}$ and $K_{1,5}$.*

**Sol.** First draw the appropriate number of vertices in two parallel columns or rows and connect the vertices in first column or row with all the vertices in second column or row. The graphs $K_{3,4}$ and $K_{1,5}$ are shown in Fig. 11.120 and Fig. 11.121 respectively.

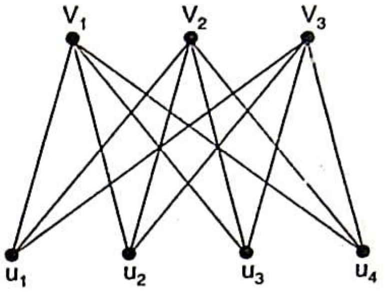



Fig. 11.120 $K_{3,4}$.

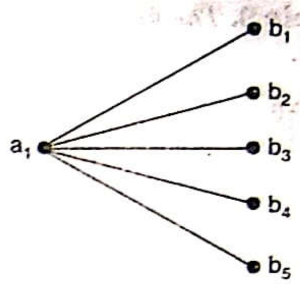



Fig. 11.121 $K_{1,5}$.

**Example 8.** *Draw the complete bipartite graphs $K_{2,3}$, $K_{2,4}$ and $K_{2,5}$.*

**Sol.** The complete bipartite graphs $K_{2,3}$, $K_{2,4}$ and $K_{2,5}$ are shown in Figs. 11.122, 11.123 and 11.124 respectively.

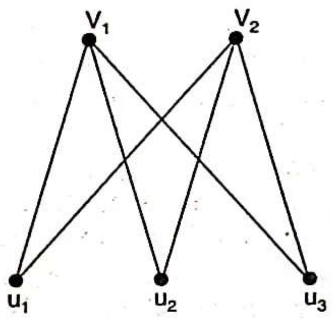



Fig. 11.122 $K_{2,3}$.

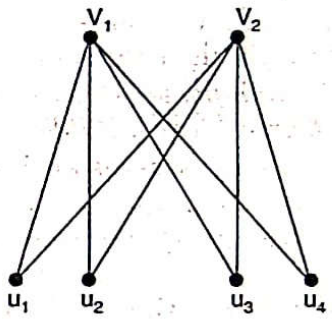



Fig. 11.123 $K_{2,4}$.

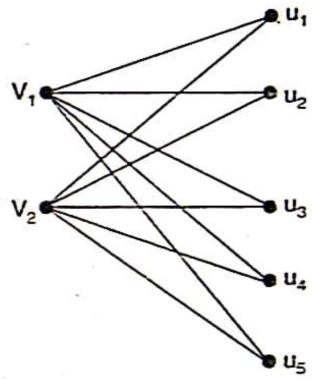



Fig. 11.124. $K_{2,5}$

## 11.36. EULER PATH (OR CHAIN) *(P.T.U., B.Tech. Dec. 2007, Dec. 2006)*

An Euler path (or chain) through a graph is a path whose edge list contains each edge of the graph exactly once.

**Example.** *If a graph G has more than two vertices of odd degree, then there can be no Euler path in G.* (P.T.U. B.Tech. May 2008)

**Sol.** Given, the graph G has more than two vertices of odd degree. We are required to prove that there can be no Euler path in G.

Let G has three vertices say $v_1$, $v_2$ and $v_3$ of odd degree. As these three vertices are of odd degree, therefore, any possible Euler path in G must arrive at each of $v_1$, $v_2$, $v_3$ with no way to return. One vertex of these three vertices $v_1$, $v_2$, $v_3$ may be the beginning of the Euler path and another the end, but this leaves the third vertex at one end of an untravelled edge. Hence there is no Euler path.

## 11.37. EULER CIRCUIT (OR CYCLE) *(P.T.U., M.C.A. May 2007, P.T.U. B.Tech. Dec. 2006)*

An Euler circuit (or cycle) is a path through a graph, in which the initial vertex appears second time as the terminal vertex.

## 11.38. EULER GRAPH

An Euler graph is a graph that possesses an Euler circuit. An Euler circuit uses every edge exactly once but vertices may be repeated.

**Example 9.** *The graph shown in Fig. 11.125 is an Euler graph. Determine Euler circuit for this graph.*

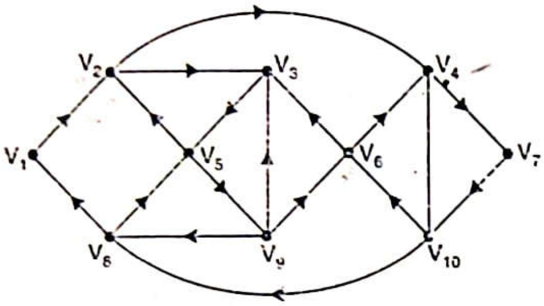



Fig. 11.125 Euler Graph.

**Sol.** The Euler circuit for this graph is

$V_1, V_2, V_3, V_5, V_2, V_4, V_7, V_{10}, V_6, V_3, V_9, V_6, V_4, V_{10}, V_8, V_5, V_9, V_8, V_1.$

We can produce an Euler circuit for a connected graph with no vertices of odd degree. The following algorithm called Fleury's algorithm is helpful to construct an Euler's Path.

**Theorem I.** *An undirected graph possesses an Eulerian path iff it is connected and has either zero or two vertices of odd degree. Give suitable example.*

**Proof.** Let the undirected graph possesses an Eulerian path. Then, by definition, the graph must be connected. Now, since the graph has Eulerian path, it means that every time the path meets a vertex, it goes through two edges which are incident with the vertex and have not been traced before.

Thus, except for the two vertices at the two ends of the path, the degree of all other vertices in the graph must be even.

If the two vertices at the two ends of the Eulerian path are distinct, then there are only two vertices with odd degree.

**Converse.** Let the undirected graph is connected and two of its vertices are of odd degree. We show the graph possesses an Eulerian path. Since the graph is connected, no edge will be traced more than once. For a vertex of even degree, whenever the path 'enters' the vertex through an edge, it can always 'leave' the vertex through another edge that has not been traced before. Therefore, when the construction is completed, we must have reached the other vertex of odd degree. Tracing all the edges in this way, we will get an Eulerian path. If not all of the edges in the graph were traced, we shall remove those edges that have been traced and obtain a subgraph formed by the remaining edges. The degrees of the vertices of this subgraph will be even. Starting from one of these vertices, we can again construct a path that passes through the edges. Because the degrees of the vertices are all even, this path must return to the vertex at which it starts. Combining this path with the path we have constructed to obtain one which starts and ends at the two vertices of odd degree, the path so obtained is Eulerian path.

*For example,* consider the graph (Fig. 11.126) as shown below.

This graph has only two vertices of odd degree, by above theorem, it has an Eulerian path.

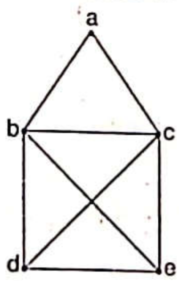



Fig. 11.126

**Theorem II.** *An undirected graph possesses an Eulerian circuit iff it is connected and its vertices are all of even degree. Prove with the help of suitable example.*

**Proof.** Let the undirected graph possess an Eulerian circuit. Then, by definition, the graph must be connected. Now since the graph has Eulerian circuit, it means that every time the graph meets a vertex, it goes through edges which are incident with the vertex and have not been traced before. Thus, the degree of all vertices in the graph must be even.

**Converse.** Let the undirected graph is connected and all its vertices are of even degree. We show that the graph possesses an Eulerian circuit. Since the graph is connected, no edge will be traced more than once. As all the vertices are of even degree, whenever the circuit 'enters' the vertex through an edge, it can always 'leave' the vertex through another edge that has not been traced before and also the circuit must return to the vertex at which it starts. Hence the circuit is an Eulerian circuit.

*For example,* consider the graph shown in the Fig. 11. 127.

The graph is connected and all vertices are of even degree. Therefore, it has an Eulerian circuit.

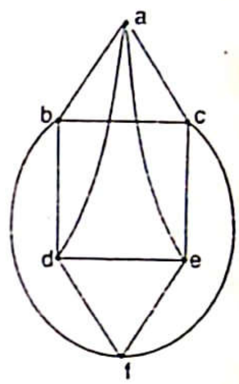


Fig. 11.127

**Theorem III.** *Let G be a connected graph such that each vertex is of degree 2. Prove that G is a cycle.*

**Proof.** Because G is a connected graph such that every vertex is of even degree, it follows that G has an Euler circuit. This circuit contains all the vertices and all the edges of G. Because the degree of each vertex is 2, it follows that in the above circuit, a vertex, except the starting vertex, cannot appear more than once. Hence, the above circuit is a cycle. This shows that graph G is a cycle.

## 11.39. FLEURY'S ALGORITHM

**Input**

Let G = (V, E) be a connected graph with every vertex of even degree.

**Step 1.** Select a vertex $v_0$ of V as the starting vertex to construct the circuit and designate P : $v_0$ as the starting of the path to be constructed.

**Step 2.** Consider that P : $v_0, v_1, v_2, \ldots, v_k$ as been constructed so far. If at $v_k$ there is only one edge say $\{v_k, v_{k+1}\}$, then extend P to P : $v_0, v_1, v_2 \ldots, v_{k+1}$. Remove $\{v_k, v_{k+1}\}$ from edge set E and $v_{k+1}$ from V. If at $v_k$ there are several edges, choose one that is not a bridge to the remaining graph, say $\{v_k, v_{k+1}\}$. Extend P to P : $v_0, v_1, v_2, \ldots, v_k, v_{k+1}$ and remove $\{v_k, v_{k+1}\}$ from edge set E.

**Step 3.** Repeat step 2 until no edges left in E.

**Example 10.** *Use Fleury's algorithm to construct an Euler circuit for the graph shown in Fig. 11.128.*

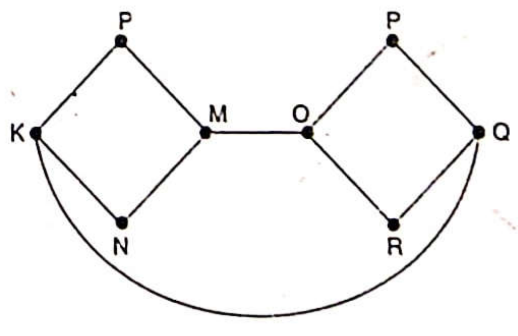


Fig. 11.128

**Sol.** Start from any vertex, as per step 1. So, choose vertex $k$ as the starting vertex. The summary of the results applying step 2 repeatedly is shown in the table—

| *Current path* | *Next edge to traverse* | *Reason* |
|---|---|---|
| P : $k$ | $\{k, l\}$ | No edge from $k$ is a bridge. Choose any one. |
| P : $k, l$ | $\{l, m\}$ | Only one edge from $l$. |
| P : $k, l, m$ | $\{m, n\}$ | No edge from $m$ is a bridge. Choose any one. |
| P : $k, l, m, n$ | $\{n, k\}$ | Only one edge from $n$. |
| P : $k, l, m, n, k$ | $\{k, m\}$ | No edge from $k$ is a bridge. Choose any one. |
| P : $k, l, m, n, k, m$ | $\{m, o\}$ | Only one edge from $m$ remains. |
| P : $k, l, m, n, k, m, o$ | $\{o, p\}$ | No edge from $k$ is a bridge. Choose any one. |
| P : $k, l, m, n, k, m, o, p$ | $\{p, q\}$ | Only one edge from $k$ remains. |
| P : $k, l, m, n, k, m, o, p, q$ | $\{q, r\}$ | No edge from $k$ is a bridge. Choose any one. |
| P : $k, l, m, n, k, m, o, p, q, r$ | $\{r, o\}$ | Only one edge from $r$ remains. |
| P : $k, l, m, n, k, m, o, p, q, r, o$ | $\{o, q\}$ | Only one edge from $o$ remains. |
| P : $k, l, m, n, k, m, o, p, q, r, o, q$ | $\{q, k\}$ | Only one edge from $q$ remains. |
| P : $k, l, m, n, k, m, o, p, q, r, o, q, k$ | | |

## 11.40. HAMILTONIAN PATH (OR CHAIN)

*(P.T.U., B.Tech. Dec. 2007)*

A Hamiltonian path (or chain) through a graph is a path whose vertex list contain each vertex of the graph exactly once, except if path is a circuit.

## 11.41. HAMILTONIAN CIRCUIT (OR CYCLE)

*(P.T.U., M.C.A. May 2007 ; B.Tech. Dec. 2007)*

A Hamiltonian circuit (or cycle) is a path in which the initial vertex appears a second time as the terminal vertex.

## 11.42. HAMILTONIAN GRAPH

*(P.T.U., B.Tech. May 2007)*

A Hamiltonian graph is a graph that possesses a Hamiltonian path. A Hamiltonian path uses each vertex exactly once but edges may not be included.

**Theorem IV.** *A graph G has a Hamilton circuit if* $e \geq \dfrac{n^2 - 3n + 6}{2}$ *where n is the number of vertices and e the number of edges in G.*

(P.T.U., M.C.A. May 2007)

**Proof.** Let, if possible, the graph G is non-Hamilton and we show $e \leq \dfrac{n^2 - 3n + 6}{2}$.

By **Dirac's theorem** which states that the connected graph G with $|v| \geq n$ vertices has a Hamiltonian circuit if $\deg(v_i) \geq n/2$ for each vertex $n_i$) there, exists a pair of non-adjacent vertices $u$ and $v$ such that

$$\deg(u) + \deg(v) \leq n - 1 \quad \ldots(1)$$

Let H be the subgraph of G obtained by deleting vertices $u$ and $v$ from G. So graph H has $n - 2$ vertices and $e - [\deg(u) + \deg(v)]$ edges. Thus maximum number of edges in H will be ${}^{n-2}C_2$ and hence

$$e - [\deg(u) + \deg(v)] \le {}^{n-2}C_2$$

$$= \frac{(n-2)!}{2!(n-2-2)!} = \frac{(n-2)!}{2(n-4)!} = \frac{(n-2)(n-3)(n-4)!}{2(n-4)!} = \frac{1}{2}(n^2 - 5n + 6)$$

$\therefore$
$$e \le \frac{1}{2}(n^2 - 5n + 6) + [\deg(u) + \deg(v)]$$

$\Rightarrow$
$$e \le \frac{1}{2}(n^2 - 5n + 6) + (n - 1)$$

$\Rightarrow$
$$e \le \frac{n^2 - 5n + 6 + 2n - 2}{2} = \frac{n^2 - 3n + 4}{2}$$

$$\le \frac{n^2 - 3n + 6}{2}$$

Thus $e \le \frac{1}{2}(n^2 - 3n + 6)$. Hence the theorem.

**Remark.** The converse of above theorem, however, is not true.

## 11.43. RULES FOR CONSTRUCTING HAMILTON PATHS (OR CHAINS) AND HAMILTON CIRCUITS (OR CYCLES) IN A GRAPH

**Rule I.** If a graph G has $n$ vertices, then a Hamilton path in G must contain exactly $(n - 1)$ edges and a Hamilton circuit in G must contain exactly $n$ edges.

**Rule II.** In a Hamilton circuit, there cannot be more than three or more edges incident with one vertex. *i.e.*, every vertex V in a Hamilton circuit will contain exactly 2 edges incident on V.

Also, If V is a vertex in G, then a Hamilton path must contain atleast one edge incident on V and atmost 2 edges incident on V.

**Remarks**

(*i*) Multigraph cannot have Hamilton circuit.
(*ii*) Hamilton path, if it exists, is the longest simple path in a graph.
(*iii*) Each graph which has Hamilton cycle will have Hamilton path, the converse, however, is not true.
(*iv*) Every complete graph $K_n$ is Hamilton for $n \ge 3$.
(*v*) A Hamilton graph with $n$ vertices must have atleast $n$ edges.

**Example 11.** *The graph shown in Fig. 11.129 is a Hamiltonian graph. Determine Hamiltonian circuit for this graph.*

**Sol.** The Hamiltonian circuit is shown in Fig. 11.130.

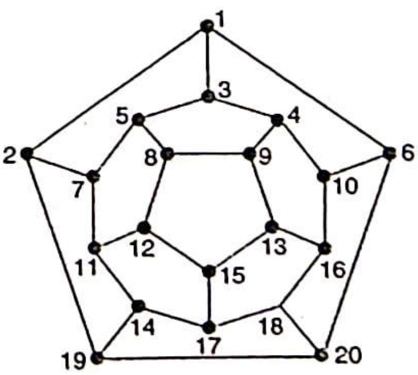



Fig. 11.129 Hamiltonian Graph.

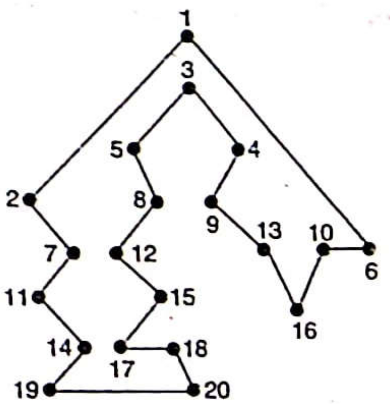



Fig. 11.130 Hamiltonian Circuit.

**Example 12.** *Give an example of a graph that has an Euler circuit which is also a Hamiltonian circuit.*

**Sol.** The graph having an Euler circuit which is also a Hamiltonian circuit is shown in Fig. 11.131.

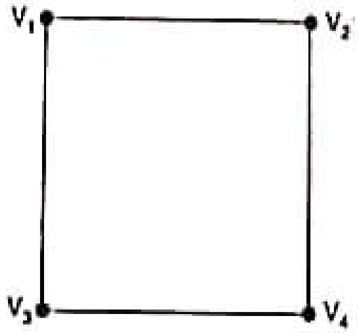



Fig. 11.131

In this graph $V_1, V_2, V_3, V_4, V_1$ is both an Euler circuit as well as Hamiltonian circuit. Since using this path, we can traverse both vertices and edges exactly once.

**Example 13.** *Give an example of a graph that has an Euler circuit and a Hamiltonian circuit, which are distinct.* (P.T.U., M.C.A. Dec. 2006)

**Sol.** The graph having an Euler circuit and a Hamiltonian circuit which are distinct is shown in Fig. 11.132.

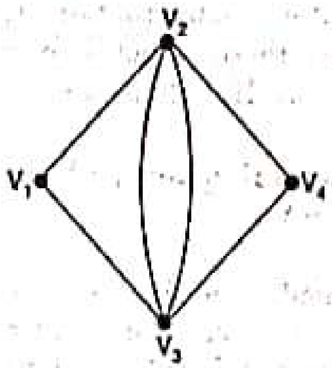



Fig. 11.132

The Euler circuit is $V_1, V_3, V_2, V_3, V_4, V_2, V_1$, which visits each edge exactly once.

The Hamiltonian circuit is $V_1, V_2, V_4, V_3, V_1$, which visits each vertex exactly once.

**Example 14.** *Give an example of a graph which has an Euler circuit but not a Hamiltonian circuit.*

**Sol.** The graph having an Euler circuit but not a Hamiltonian circuit is shown in Fig. 11.133.

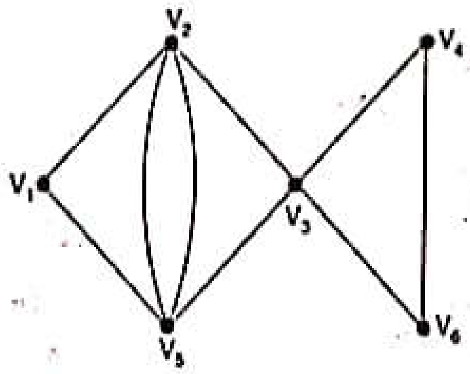



Fig. 11.133

The Euler circuit is $V_1, V_5, V_2, V_5, V_3, V_4, V_6, V_3, V_2, V_1$.

There is no Hamiltonian circuit. Since it is not possible to traverse each vertex of this graph exactly once.

**Example 15.** *Give an example of a graph which has a Hamiltonian circuit but not an Euler circuit.* *(P.T.U., M.C.A. May 2008)*

**Sol.** The graph having a Hamiltonian circuit but not an Euler circuit is shown in Fig. 11.134.

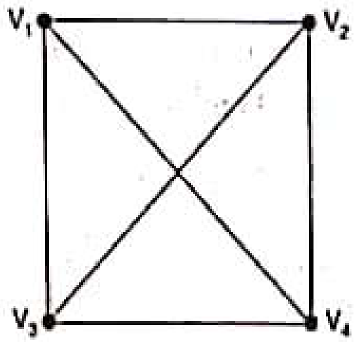



Fig. 11.134

The Hamiltonian circuit is $V_1, V_2, V_4, V_3, V_1$. There is no Euler circuit. Since it is not possible to traverse each edge of this graph exactly once.

**Example 16.** (*a*) *Give an example of a graph that has neither an Euler circuit nor a Hamiltonian circuit.*

(*b*) *Show that the graphs in Fig. 11.135 has a Hamiltonian circuit where as the graph in Fig. 11.136 has no Hamiltonian circuit.*

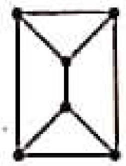


Fig. 11.135

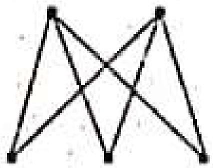


Fig. 11.136

**Sol.** (*a*) The graph having neither an Euler circuit nor a Hamiltonian circuit is shown in Fig. 11.137. It does not contain Euler circuit since each vertex is not of even degree.

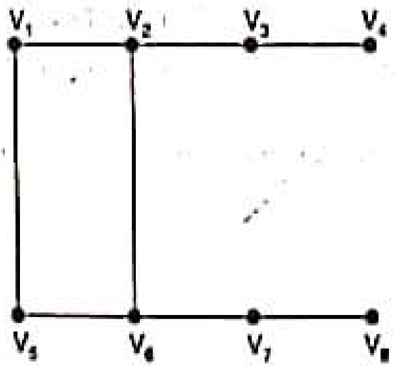



Fig. 11.137

(*b*) We know that if there is a path in a graph G that uses each vertex of the graph exactly once, except initial vertex that appears twice as the terminal vertex, then such a path is called a **Hamiltonian circuit.**

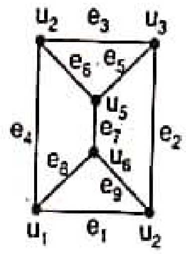



Fig. 11.138

The graph in Fig. 11.138 has Hamiltonical circuit given as

$$u_1, e_1, u_2, e_2, u_3, e_3, u_2, e_6,, u_5, e_7, u_6, e_8, u_1$$

Also, we know that a connected graph G with $|v| \geq n$ vertices has a Hamiltonian circuit if $\deg(v_i) \geq n/2$ for each vertex $n_i$.

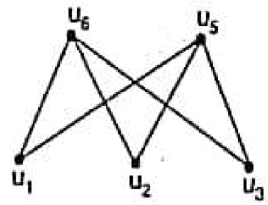



Fig. 11.139

The graph in Fig. 11.139 has 5 vertices. Here $\deg(u_1) = 2 \ngeq 5/2$

The graph in Fig. 11.139 has no Hamiltonian circuit.

**Example 17.** *Consider the graph G shown in the following Fig. 11.140.*

*(a) Find the Euler's path if it exists.*

*(b) Is the graph G Eulerian?*

*(c) Is the graph bipartite?*

*(d) Is the graph hamiltonian?*

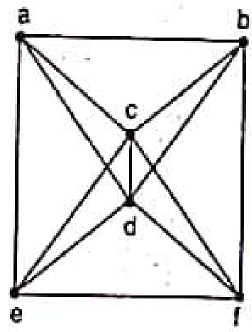



Fig. 11.140

**Sol.** (*a*) Consider the vertices '*c*' and '*d*'. The degree of these vertices is 5(odd)

$\therefore$ G has Euler's path between *c* and *d*.

The Euler's path is *c, a, b, d, f, b, d, f, e, a, d, e, c, d*

(*b*) A graph G can have Euler's circuit if each vertices of G are of even degree. Since the vertices '*c*' and '*d*' are of odd degree, therefore, G cannot have Euler's circuit.

(*c*) The graph G has odd cycle *a, c, d, a*

$\therefore$ it cannot be a bipartite graph

(*d*) Yes, the Hamiltonian circuit is *a, b, f, e, d, c, a*.

**Example 18.** *Consider the graph G shown below in Fig. 11:141.*

*(a) Is it a complete graph?*

*(b) Is G connected and regular?*

*(c) Is it a planar graph? If so, find the number of regions.*

*(d) Is G Eulerian?*

**Sol.** (*a*) Since the edge between *a* and *d* is not present in the given graph. It cannot be complete graph. (Every pair of vertices must be joined by an edge.)

(*b*) Given graph is a 4-regular graph. Also it is connected because there is a path from every vertex to other.

(*c*) The given graph is planar graph as it can be re-drawn as shown in Fig. 11.142 in which no two edges cross

Here $\qquad V = 6, E = 12$

By Euler's theorem, $V - E + R = 2$

$\Rightarrow \qquad 6 - 12 + R = 2$

$\Rightarrow \qquad R = 8$

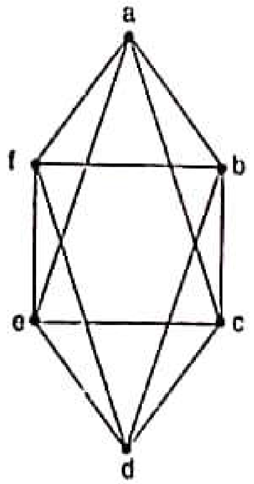



Fig. 11.141

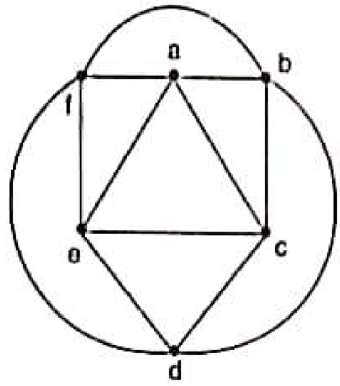



Fig. 11.142

(*d*) Given graph is 4-regular. $\therefore$ every vertex is of degree 4 (even)

$\therefore$ G has an Euler's circuit and hence G is an Eulerian graph.

**Example 19.** *State and prove Eulerian theorem on graph to show that Königsberg's graph is not proved to a solution.*

**Sol.** The word Königsberg is the name of a town, situated on the bank of a river, Pregel in Germany. This city has seven bridges. In 1736, L. Euler, the father of graph theory, proved that *it was not possible to cross each of the seven bridges once and only once in a walking tour.* A map of the Königsberg is shown in the following Fig. 11.143.

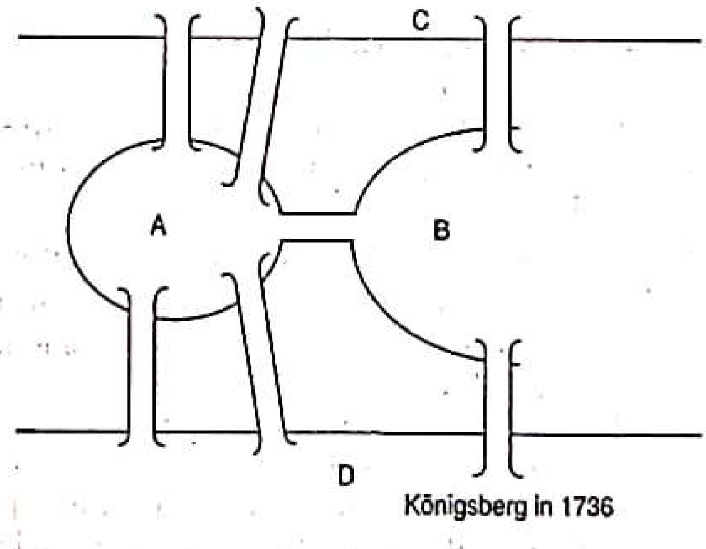



Fig. 11.143

Euler replaced the islands and the two sides of the river by points and the bridges by curves as shown in Fig. 11.144.

Figure 11.144 is a multigraph. A multigraph is a said to be **traversable** if it can be drawn without any breaks in the curve and without repeating any edges. *i.e.*, if there is a path which includes all vertices and uses each edge exactly once and such a path is called Travesable Trial.

According to Euler, the walk in Königsberg is possible iff the multigraph in Fig. 11.144 is traversable. But Euler proved that the multigraph in Fig. 11.144 is not traversable and hence the walk in Königsberg is impossible. We prove it.

We know that a vertex is even or odd according as its degree is even or odd. Suppose a multigraph is travesable and that a traversable trail does not begin or end at a vertex, say, P. We claim that P is an even vertex. For whenever the traversable trail enters P by an edge, there must always be an edge not previously used by which the trail can leave P. Thus, the edges in the trail incident with P must appear in pairs and so P is an even vertex. Further, if a vertex, Q is odd, the traversable trail must begin or end at Q. Hence, a multigraph with more than two odd vertices cannot be traversable.

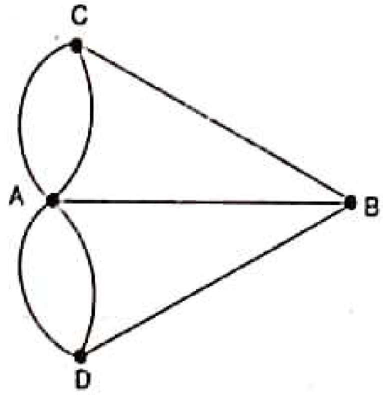


Fig. 11.144

Now the multigraph corresponding to the Königsberg bridge problem has four odd vertices. Thus, one cannot walk through Königsberg so that each bridge is crossed exactly once.

Euler actually proved the converse of the above statement, which is contained in the following theorem, called Euler theorem.

**Theorem V.** *A finite connected graph is Eulerian iff each vertex has even degree.*

(P.T.U., M.C.A. May 2008, B.Tech. Dec. 2012)

**Proof.** We know that a graph G is called an Eulerian graph if there exists a closed traversable Trial, called an Eulerian Trial. Suppose G is Eulerian and T is a closed Eulerian trial. Let $v$ be any vertex of G. We show the vertex $v$ is of even degree. Since the trail T enters and leaves the vertex $v$ the same number of times without repeating any edge. $\therefore$ $v$ is of even degree.

Conversly, Let each vertex of G has even degree. We construct an Eulerian Trial. Start with a trial $T_1$ at any edge $e$. Extend $T_1$ by adding one edge after the other. If $T_1$ is not closed *i.e.*, If $T_1$ begins at $u$ and ends at $v \neq u$, then only an odd number of edges incident on $v$ appear in $T_1$. Hence we can extend $T_1$ by another edge incident on $v$. Thus, we can continue to extend $T_1$ until $T_1$ returns to its initial vertex $u$. *i.e.*, until $T_1$ is closed.

If $T_1$ includes all the edges in G, then $T_1$ is the required Eulerian Trial.

If $T_1$ does not include all edges of G, consider the graph H obtained by deleting all edges of $T_1$ from G. Now H has each vertex of even degree (since $T_1$ contains an even number of the edges incident on any vertex). Since G is connected, there is an edge $e'$ of H which has an end point $u'$ in $T_1$. We construct a trail $T_2$ in H begining at $u'$ and using $e'$. Since all the vertices in H have even degree, we can continue to extent $T_2$ in H until $T_2$ returns to $u'$ as shown in the following Fig. 11.145. We can clearly put $T_1$ and $T_2$ together to form a larger closed trial in G. Proceeding the above process until all the edges of G are used, we finally obtain an Eulerian trial and hence G is Eulerian.

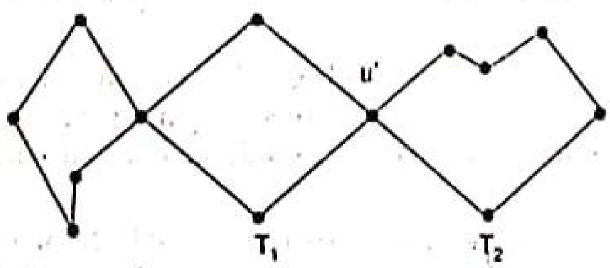


Fig. 11.145

## 11.44. REGULAR GRAPH

A graph is said to be regular or K-regular if all its vertices have the same degree K. A graph whose all vertices have degree 2 is called 2-regular graph. A complete graph $K_n$ is regular of degree $n-1$.

**Example 20.** *Draw regular graphs of degree 2 and 3.*

**Sol.** The regular graphs of degree 2 and 3 are shown in Figs. 11.146 and 11.147.

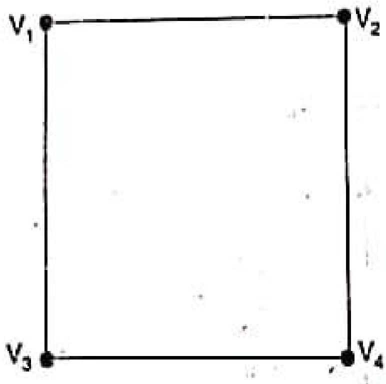



Fig. 11.146. 2-regular Graph.

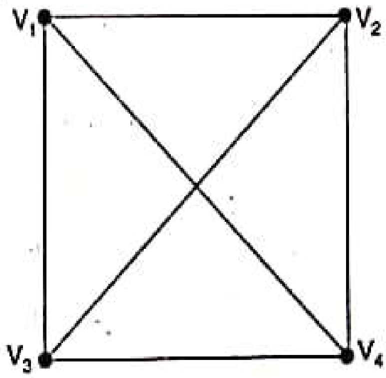



Fig. 11.147. Regular Graph.

**Example 21.** *Draw a 2-regular graph of five vertices.*

**Sol.** The 2-regular graph of five vertices is shown in Fig. 11:148

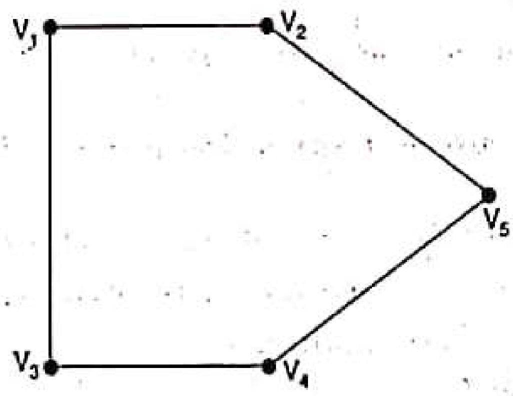



Fig. 11.148

**Example 22.** *Draw a 3-regular graph of five vertices.*

**Sol.** It is not possible to draw 3-regular graph of five vertices. The 3-regular graph must have an even number of vertices.

**Theorem VI.** *Prove that K-regular graph must have even number of vertices when the value of K is odd.*

**Proof.** Consider a graph with $n$ vertices. Let T is the sum of degrees of all the $n$ vertices of a K-regular graph. Then, we have

$$T = K \cdot n$$

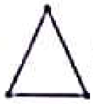
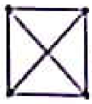
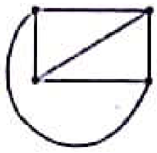
The sum T must be even (from the theorem V).

Now, suppose that K is odd, so the value of $n$ must be even.

## 11.45. PLANAR GRAPH

(*P.T.U. B.Tech. Dec. 2013*)

A graph is said to be planar if it can be drawn in a plane so that no edges cross.

For *e.g.*, the graph  is a planar graph. Also $K_4$ =  is a planar graph because it can be re-drawn as  in which edges do not cross each other.

*For example*: The graphs shown in Fig. 11.149 and Fig. 11.150 are planar graphs.

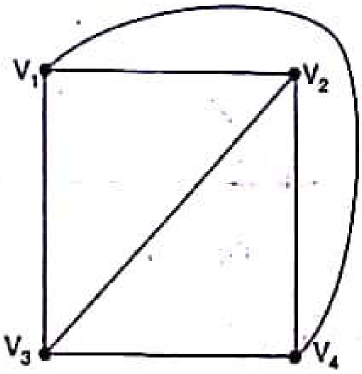



Fig. 11.149

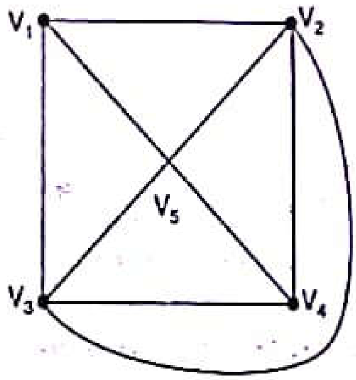



Fig. 11.150

**Theorem VII.** *A planar and connected graph has a vertex of degree less than or equal to 5.*

**Proof.** Let G be connected and planar and suppose, if possible, degree of each vertex $x \in G$ is greater than 5.

*i.e.*, $\deg x > 5 \Rightarrow \deg x \geq 6$ *i.e.*, sum of degree of all vertices $\geq 6v$

$\Rightarrow$ $2e \geq 6v$, where $e$ and $v$ are the number of edges and vertices respectively.

$\Rightarrow$ $e \geq 3v$, which contradicts $e \leq 3u - 6 < 3v$.

Hence $\deg x \leq 5$.

## 11.46. REGION OF A GRAPH

Consider a planar graph G = (V, E). A region is defined to be an area of the plane that is bounded by edges and cannot be further subdivided. A planar graph divides the plane into one or more regions. One of these regions will be infinite.

(*a*) **Finite Region.** If the area of the region is finite, then that region is called finite region.

(*b*) **Infinite Region.** If the area of the region is infinite, that region is called infinite region. A planar graph has only one infinite region.

**Example 23.** *Consider the graph shown in Fig. 11.151. Determine the number of regions, finite regions and an infinite region.*

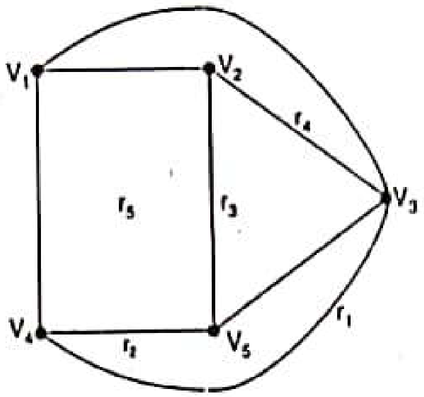



Fig. 11.151

**Sol.** There are five regions in the above graph *i.e.*, $r_1$, $r_2$, $r_3$, $r_4$ and $r_5$.

There are four finite regions in the graph *i.e.*, $r_2$, $r_3$, $r_4$ and $r_5$.

There is only one infinite region *i.e.*, $r_1$.

**Example 24.** *Draw a planar representation of graphs shown in Figs. 11.152 and 11.153.*

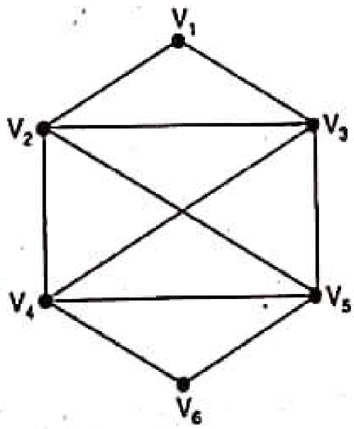



Fig. 11.152

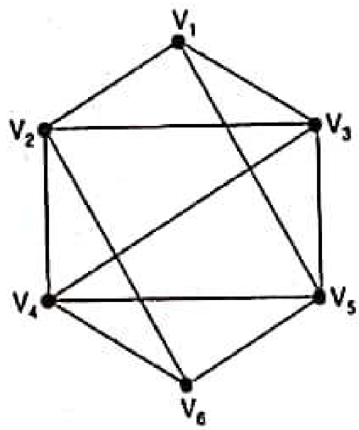



Fig. 11.153

**Sol.** The planar representation of graph shown in Fig. 11.154 is shown in Fig. 11.155.

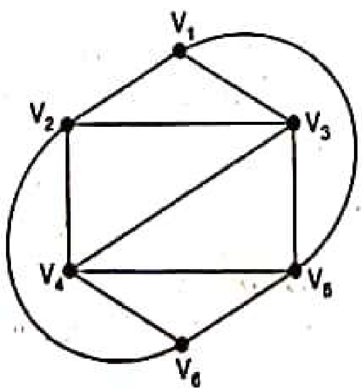



Fig. 11.154

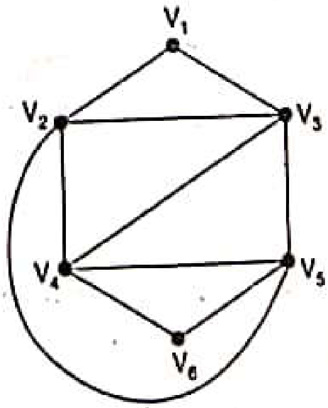



Fig. 11.155

## 11.47. PROPERTIES OF PLANAR GRAPHS

(*P.T.U. B.Tech. Dec. 2013*)

**Theorem I.** *If a connected planar graph G has e edges and r regions, then* $r \le \frac{2}{3}e$.

**Theorem II.** *If a connected planar graph G has e edges and v vertices, then* $3v - e \ge 6$.

**Theorem III.** *A complete graph* $K_n$ *is planar if and only if* $n < 5$.

**Theorem IV.** *A complete bipartite graph* $K_{m,n}$ *is planar if and only if* $m < 3$ *or* $n > 3$.

**I. Proof.** In a connected planar graph, each region is bounded by at least 3 regions

$\therefore$ $r$ regions are bounded by minimum $3r$ edges

$\Rightarrow$ Number of edges in graph $\ge 3r$

But number of edge in the graph $= 2e$ (as each edge belongs to two regions)

$$\therefore \quad 2e \ge 3r$$

$$\Rightarrow \quad r \le \frac{2e}{3}$$

**II.** Let $r$ be the no. of regions in a planar representation of G. $\therefore$ By Euler formula

$$v + r - e = 2 \qquad \ldots(1)$$

Now sum of degrees of the regions $= 2e$. But each region has degree 3 or more.

$$\therefore \quad 2e \ge 3r \quad \Rightarrow \quad r \le \frac{2e}{3}.$$

From (1) we get $\quad 2 = v + r - e \le v + \frac{2e}{3} - e = v - \frac{e}{3}$

$$6 \le 3v - e$$

$$\Rightarrow \quad e \le 3v - 6 \quad \text{Hence proved.}$$

**III.** If G has one or two vertices, then the result is true. If G has at least 3 vertices then

$$e \le 3v - 6 \quad \text{or} \quad 2e \le 6v - 12 \qquad \ldots(1)$$

If degree of every vertex were at least 6, then using $2e = \sum_{v \in V} \deg v$, we would have $2e \ge 6v$, which contradicts the inequality (1), Hence there must be a vertex with degree not greater than 5.

**IV.** Proof is beyond the scope of the book.

**Example 25.** *Prove that complete graph* $K_4$ *is planar.*

**Sol.** The complete graph $K_4$ contains 4 vertices and 6 edges.

We know that for a connected planar graph $3v - e \ge 6$. Hence for $K_4$, we have

$$3 \times 4 - 6 = 6 \text{ which satisfies the property (3).}$$

Thus $K_4$ is a planar graph. Hence proved.

## 11.48. STATE AND PROVE EULER'S THEOREM ON GRAPHS

(*P.T.U., B.Tech. Dec. 2007, 2006, May 2006*)

**Statement.** *Consider any connected planar graph G = (V, E) having R regions, V vertices and E edges. Then*

$$V + R - E = 2. \qquad \text{(P.T.U., B.Tech. May 2012, May 2005; M.C.A. Dec. 2005)}$$

**Proof.** Use induction on the number of edges to prove this theorem.

Assume that the edges $e = 1$. Then we have two cases, graphs of which are shown in Figs. 11.156 and 11.157.

Fig. 11.156

Fig. 11.157

In Fig. 11.156 we have $V = 2$ and $R = 1$. Thus $2 + 1 - 1 = 2$

In Fig. 11.157 we have $V = 1$ and $R = 2$. Thus $1 + 2 - 1 = 2$. Hence, the result holds for $e = 1$.

Let us assume that the formula holds for connected planar graphs with K edges.

Let G be a graph with K + 1 edges.

Firstly, we suppose that G contains no circuits. Now, take a vertex $v$ and find a path starting at $v$. Since G is circuit free, whenever we find an edge, we have a new vertex. At last we will reach a vertex $v$ with degree 1. So we cannot move further as shown in Fig. 11.158.

Now remove vertex $v$ and the corresponding edge incident on $v$. So, we are left with a graph G* having K edges as shown in Fig. 11.159.

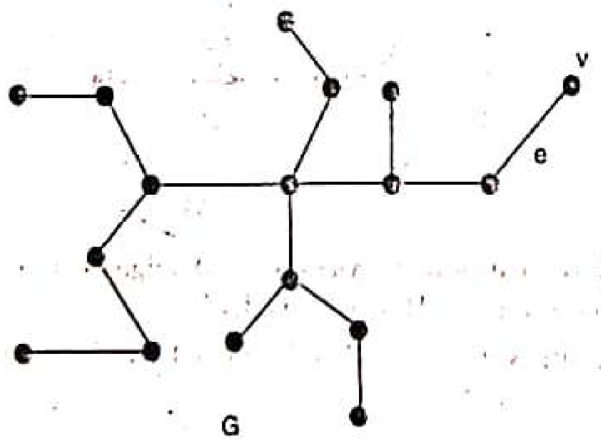



Fig. 11.158. G.

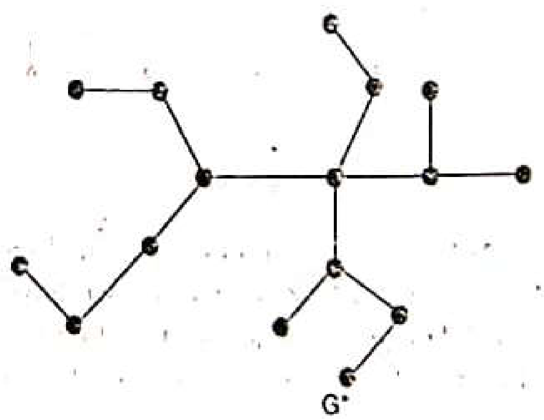



Fig. 11.159. G*.

Hence, by inductive assumption, Euler's formula holds for G*.

Now, since G has one more edge than G*, one more vertex than G* with same number of regions as in G*. Hence, the formula also holds for G.

Secondly, we assume that G contains a circuit and $e$ is an edge in the circuit shown in Fig. 11.160.

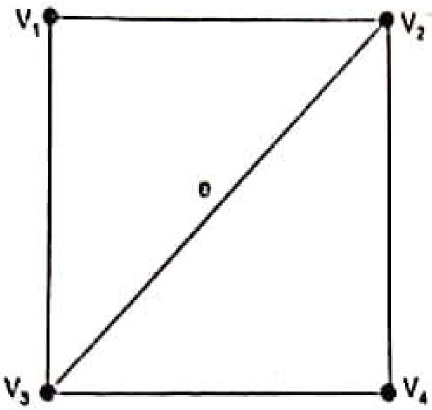



Fig. 11.160

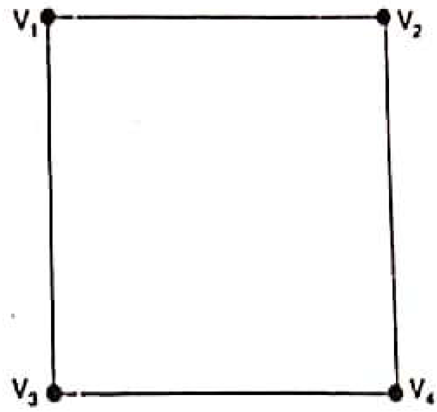



Fig. 11.161

Now, as $c$ is the part of a boundary for two regions. So, we only remove the edge and we are left with graph $G^*$ having K edges (Fig. 11.161).

Hence, by inductive assumption, Euler's formula holds for $G^*$.

Now, since G has one more edge than $G^*$, one more region than $G^*$ with same number of vertices as $G^*$. Hence the formula also holds for G which, verifies the inductive step and hence proves the theorem.

**Example 26.** *Show that $V - E + R = 2$ for the connected planar graphs shown in Figs. 11.162 and 11.163.*

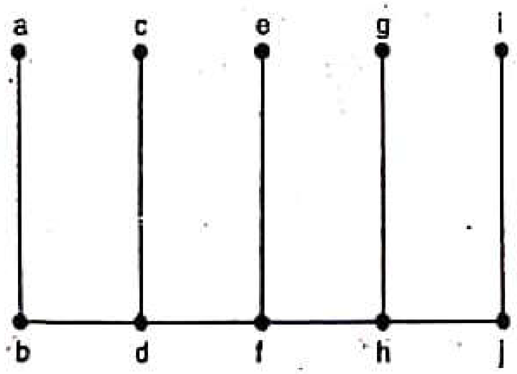



Fig. 11.162

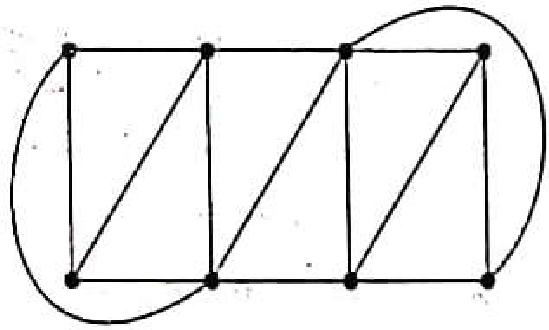


Fig. 11.163

**Sol.** (*i*) The graph shown in Fig. 11.162 contains vertices V = 10, edges E = 9 and regions R = 1. Putting the values, we have $10 - 9 + 1 = 2$. Hence proved.

(*ii*) The graph shown in Fig. 11.163 contains vertices V = 8, edges E = 15 and regions R = 9. Putting the values, we have $8 - 15 + 9 = 2$. Hence proved.

## 11.49. NON PLANAR GRAPHS

A graph is said to be non planar if it cannot be drawn in a plane so that no edges cross.

**For example :** The graphs shown in Figs. 11.164 and 11.165 are non planar graphs.

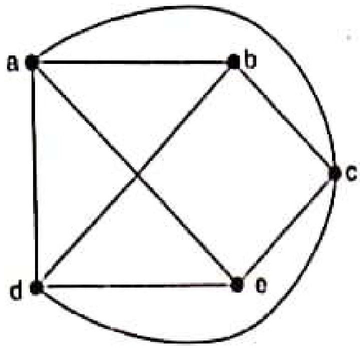



Fig. 11.164

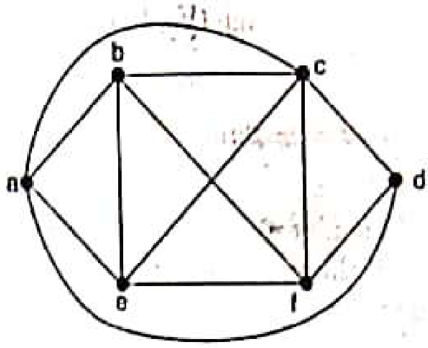



Fig. 11.165

These graphs cannot be drawn in a plane so that no edges cross hence they are non planar graphs.

## 11.50. PROPERTIES OF NON PLANAR GRAPHS

A graph is non-planar if and only if it contains a subgraph homeomorphic to $K_5$ or $K_{3,3}$ [KURATOWSKI'S THEOREM].

Example 27. *Show that $K_5$ is non-planar. Fig. 11.166.*

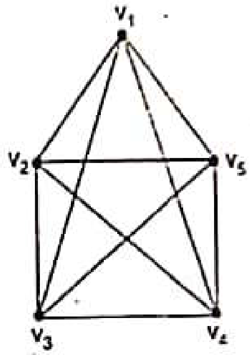



Fig. 11.166

Sol. Clearly $K_5$ is a connected. Also we show $K_5$ is non planar.

For,

$$v = 5, e = 10$$

If, $K_5$ is planar then, $\quad e \le 3v - 6$

$\Rightarrow \quad 10 \le 3(5) - 6$

$\Rightarrow \quad 10 \le 15 - 6$

$\Rightarrow \quad 10 \le 9$, a contradiction

$\therefore$ The graph $K_5$ is non planar.

Remark. If $e \le 3v - 6$ does not hold, then G is always non planar. But if this condition holds, then we can not conclude that G is planar.

Example 28. *Show that the graphs shown in Figs. 11.167 and 11.168 are non-planar by finding a subgraph homeomorphic to $K_5$ or $K_{3,3}$.*

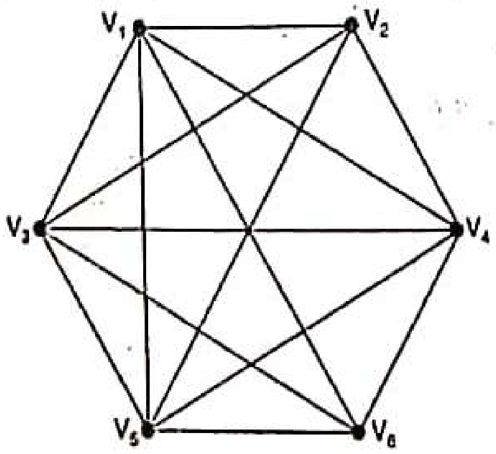



Fig. 11.167. $G_1$.

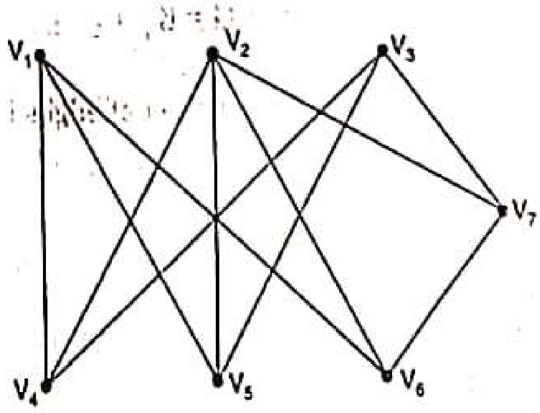



Fig. 11.168. $G_2$.

Sol. If we remove the edges $(V_1, V_4)$, $(V_3, V_4)$ and $(V_5, V_4)$, the graph $G_1$ becomes homeomorphic to $K_5$. Hence it is non planar.

If we remove the edge $(V_2, V_7)$, the graph $G_2$ becomes homeomorphic to $K_{3,3}$. Hence it is non-planar.

**Theorem.** *Prove that every planar graph has at least one vertex of degree 5 or less than 5.*

**Proof.** Consider a graph G, whose all vertices are of degree 6 or more, then the sum of the degrees of all the vertices would be greater than or equal to $6v$. We know that the sum of the degrees of the vertices is twice the number of edges. Therefore, we have

$$6v \leq 2e$$

or $$v \leq \frac{e}{3} \quad \text{...(1)}$$

But, any planar graph have the property,

$$r \leq \frac{2e}{3} \quad \text{...(2)}$$

Also, from Euler's formula, we have

$$2 = v - e + r \quad \text{...(3)}$$

Now, putting the value of $v$ and $r$ from (1) and (2) in (3), we have

$$2 \leq \frac{e}{3} - e + \frac{2e}{3} = 0$$

Since, the statement $2 \leq 0$ is not true, hence we conclude that there must exist some vertex in G with degree 5 or less than 5.

**Example 29.** *Let $G = [V, E]$ be a graph having at least 11 vertices. Prove that G or its complement $\overline{G}$ is non planar.*

**Sol.** Let if possible, G and $\overline{G}$ are planar. We know that G and $\overline{G}$ have same number of vertices say, $n$. $\therefore$ we have $n \geq 11$

Now $$\overline{G} = K_n - G$$

$$\Rightarrow \quad \overline{G} + G = K_n$$

$$\Rightarrow \quad |\overline{E}| + |E| = \text{set of edges in } K_n = \frac{n(n-1)}{2} \quad \text{...(1)}$$

Since G and $\overline{G}$ are planar

$$\therefore \quad |E| \leq 3n - 6, \quad |\overline{E}| \leq 3n - 6$$

$$\Rightarrow \quad |E| + |\overline{E}| \leq 3n - 6 + 3n - 6$$

$$\Rightarrow \quad \frac{n(n-1)}{2} \leq 6n - 12 \quad | \text{ Using (1)}$$

$$\Rightarrow \quad n(n-1) \leq 12n - 24, \text{ which is not true for } n \geq 11$$

Hence G or $\overline{G}$ is non-planar.

## TEST YOUR KNOWLEDGE 11.2

1. Consider the graph shown in the given Fig. I.
   (a) Find all simple paths from A to F.
   (b) All Trials (distinct edges) from A to F.
   (c) $d(A, F)$, the distance from A to F.
   (d) Diam (G), the diameter of G.
   (e) All cycles which include vertex A.
   (f) All cycles in G.

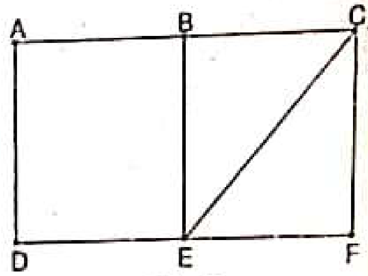


Fig. I

2. Consider the graph shown in the given Fig. II Find.
   (a) All sample paths from A to G.
   (b) All trials (distinct edges) from B to C.
   (c) $d(A, C)$, the distance from A to C.
   (d) Diam G, the diameter of G.

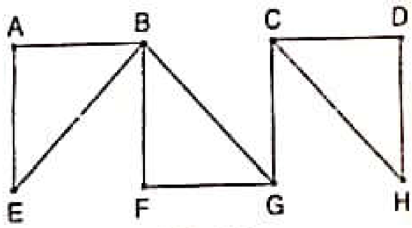
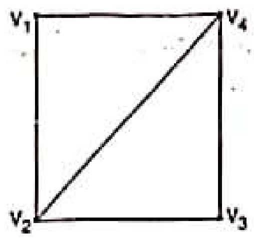
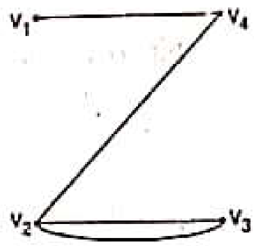


Fig. II

3. Find the adjacency matrix $A = \{a_{ij}\}$ of the graphs shown below :



(a)



(b)

4. Draw the graph G corresponding to each adjacency matrix.

$$(a)\ A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \qquad (b)\ A = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

5. (a) Consider the graph (Fig. I) G show in the given figure. Verify Euler Theorem *i.e.*, $V + R - E = 2$.

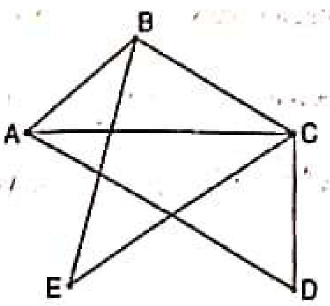


Fig. I.

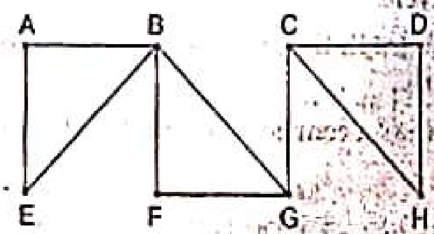
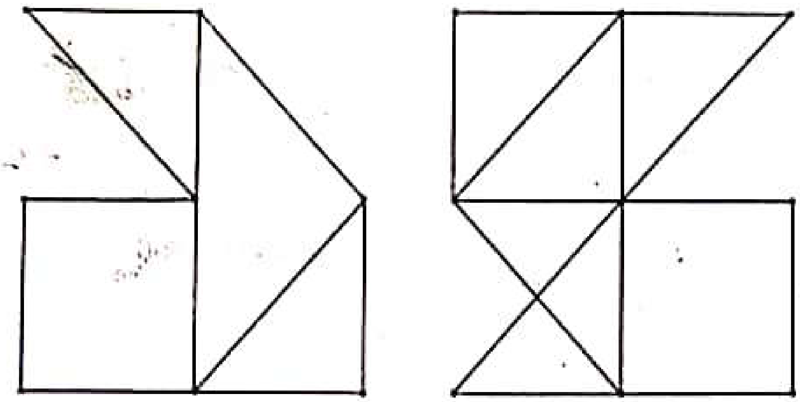


Fig. II.

(b) Verify Euler Theorem *i.e.*, $V + R - E = 2$ for the graph Fig. II.

6. Consider each graph as shown below :



(a) (b) (c)

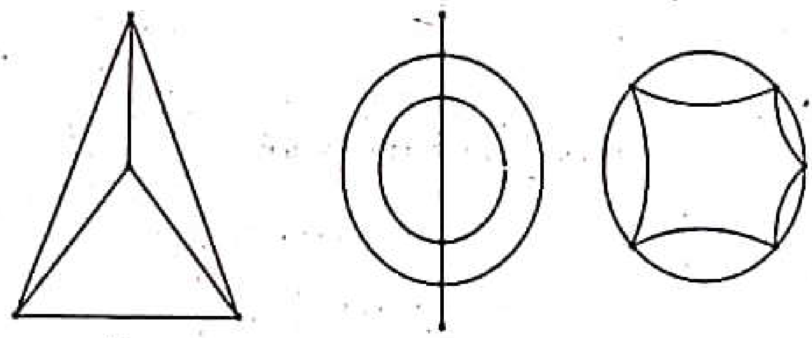
(i) Which of the graphs (a), (b), (c) has Euler path ? Which have Euler circuit. If not, explain why ?

(ii) Which of the graphs (a), (b), (c) have a Hamiltonian circuit ? If not, explain why ?

7. (a) Verify Euler's formula for the following graphs :

(b) Show that if G is a bipartite simple graph with $u$ vertical and $e$ edges, then $e = \frac{u}{4}$.

*(P.T.U., M.C.A. Dec 2006)*



(a) (b) (c)

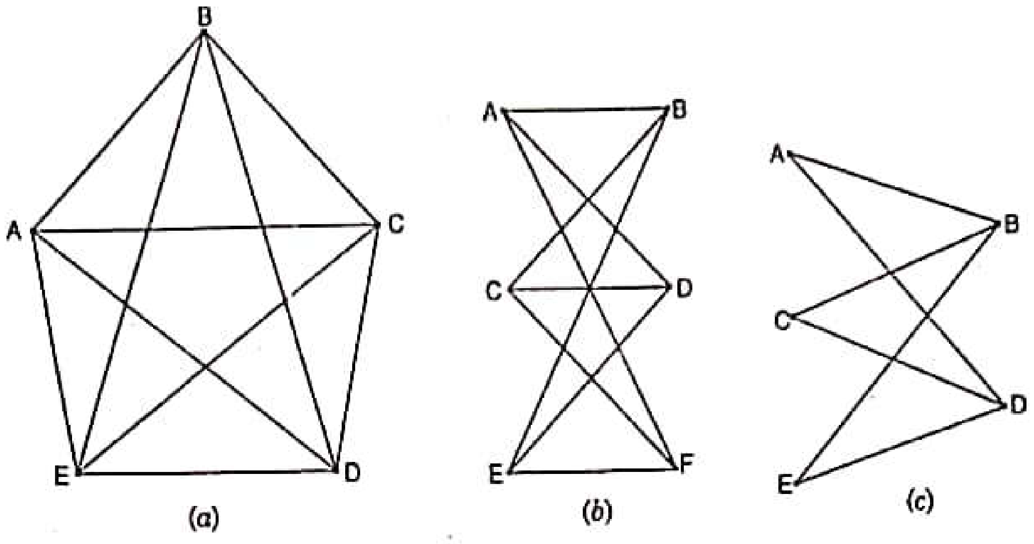
8. Let G be a finite connected planner graph with at least three vertices. Show that G has at least one vertex of degree 5 or less.

9. (a) Suppose a graph G contains two distinct paths from a vertex $a$ to a vertex $b$. Show that G has a cycle.

(b) If a graph G has more than two vertices of odd degree, then prove that there can be no Euler Path. *(P.T.U., B.Tech. May 2008)*

10. Show that a connected graph G with $n$ vertices must have atleast $(n - 1)$ edges.

11. Consider each graph G in the given figures:




(i) Find an Euler path or Euler circuit, if it exists. If not, explain why ?

(ii) Find a Hamilton path or a Hamilton circuit, if it exists. If not, explain why ?

12. Draw the following graphs

(a) $K_{2, 5}$ (b) $K_4$ (c) $K_{2, 3}$

13. If G is a simple, connected and planner graph with more than one edge, then

(i) $2 \mid E \mid \geq 3 \mid R \mid$

(ii) $\mid E \mid \leq 3 \mid V \mid - 6$, where $\mid E \mid$ denotes the number of edges, $\mid R \mid$, the number of regions and $\mid V \mid$, the number of vertices.

14. Show that $K_{3, 3}$ is non-planar graph.

15. Does the graph shown below has a Hamiltonian circuit? (P.T.U., B.Tech. Dec. 2012)

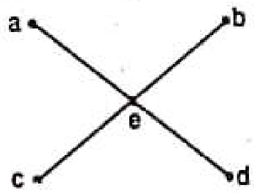



Fig. Q.15

## Answers

1. (a) There are seven simple paths from A to F.

$A \to B \to C \to F$, $A \to B \to C \to E \to F$, $A \to B \to E \to F$,
$A \to B \to E \to C \to F$, $A \to D \to E \to F$,
$A \to D \to E \to B \to C \to F$, $A \to D \to E \to C \to F$

(b) There are nine trials. The seven simple paths of part (a) and

$A \to D \to E \to B \to C \to E \to F$; $A \to D \to E \to C \to B \to E \to F$

(c) $d(A, F) = 3$ (d) $d(G) = 3$

(e) There are three cycles including the vertex A;

$A \to B \to E \to D$; $A \to B \to C \to E \to D \to A$, $A \to B \to C \to F \to E \to D \to A$

(f) There are six cycles in G. The three cycles are of part (e) and

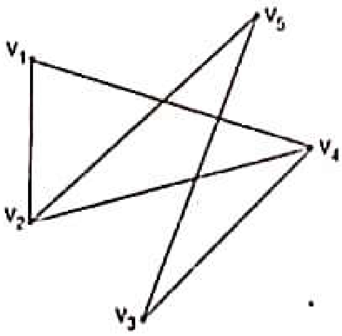
$B \to C \to E \to B$; $C \to F \to E \to C$; $B \to C \to F \to E \to B$

2. (a) ABG, ABFG, AEBG, AEBFG (b) BGC, BFGC, BAEBGC, BAEBFGC

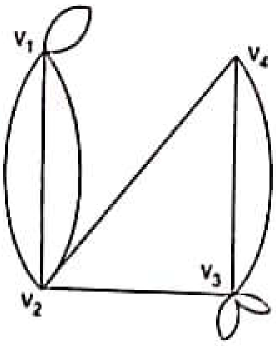
(c) $d(A, C) = 3$ (d) $\dim(G) = 4$

3. (a) $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

4. (*a*)



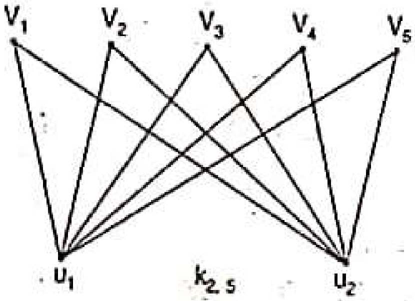

(*b*)



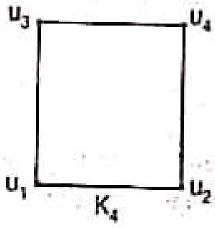

6. (*a*) Has Euler path (*b*) Has Euler circuit
   (*c*) Has not Euler path

11. (*i*) (*a*) Is Eulerian since all vertices are even. The Eulerian path is ABCDEACEBDA
    (*b*) Not an Eulerian Path, not an Eulerian circuit
    (*c*) Has Eulerian path, BADCBED
    (*ii*) (*a*) ABCDEA (*b*) ABCDEFA
    (*c*) has not Hamilton path and Hamilton circuit

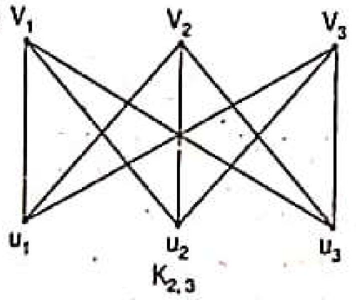
12. (*a*)




(*b*)




(*c*)




## Hints

5. Here deg(A) = 3, deg(B) = 3, deg(C) = 4, deg(D) = 2, deg(E) = 2

   $\therefore$ Sum of degrees = 3 + 3 + 4 + 2 + 2 = 14

   Also number of edges = 7

6. A graph G has an Euler path iff 0 or 2 vertices have odd degree. A graph G has an Euler circuit if all the vertices are of even degree.

13. (*a*) Assume $|E| > 1$. If G has only one region (unbounded), then $|R| = 1$.

    Since $|E| > 1 \Rightarrow |E| \geq 2$

    $\therefore$ $2|E| \geq 3|R|$ is true

If $|R| > 1$, then each region is bounded by at least 3 edges. But in a planar graph, each edge touches at most 2 region. Thus $2|E| \geq 3|R|$

(b) From Part (a), we have

$$2|E| \geq 3|R| \Rightarrow |R| \leq \frac{2}{3}|E|$$

$$\Rightarrow \quad |V| + |R| \leq \frac{2}{3}|E| + |V|$$

$$\Rightarrow \quad |E| + 2 \leq \frac{2}{3}|E| + |V| \qquad | \text{ Euler's formula}$$

$$\Rightarrow \quad 3|E| + 6 \leq 2|E| + 3|V|$$

$$\Rightarrow \quad |E| \leq 3|V| - 6.$$

14. If $K_{3,3}$ is a planar graph, then we must have $2|E| \geq 3|R|$. Where each region is bounded by at least three edges. But for $K_{3,3}$, each region is bounded by at least 4 edges $\therefore$ We have

$2|E| \geq 4|R|$

$$\Rightarrow \quad 2|E| \geq 4\{|E| - |V| + 2\} \qquad | \text{ Euler's formula}$$

$$\Rightarrow \quad 2 \times 9 \geq 4(9 - 6 + 2) \qquad \text{For } K_{3,3}, |E| = 9, |V| = 6$$

$$\Rightarrow \quad 18 \geq 20, \text{ a contradiction}$$

Hence $K_{3,3}$ is non-planar.

15. No, the graph does not contain Hamiltonian circuit. Since $e \geq \frac{n^2 - 3n + 6}{2}$ does not hold.

## 11.51. GRAPH COLOURING

Suppose that G = (V, E) is a graph with no multiple edges. A vertex colouring of G is an assignment of colours to the vertices of G such that adjacent vertices have different colours. A graph G is M-colourable if there exists a colouring of G which uses M-colours.

**Proper Colouring.** A colouring is proper if any two adjacent vertices $u$ and $v$ have different colours otherwise it is called improper colouring.

A graph can be coloured by assigning a different colour to each of its vertices. However, for most graphs a colouring can be found that uses fewer colours than the number of vertices in the graph.

## 11.52. CHROMATIC NUMBER OF G

*(P.T.U., B.Tech. Dec. 2013, May 2006, Dec. 2005; M.C.A. May 2007)*

The minimum number of colours needed to produce a proper colouring of a graph G is called the chromatic number of G and is denoted by $\chi(G)$.

The graph shown in Fig.11.169 is minimum 3-colourable, hence $\chi(G) = 3$.

Similarly, for the complete graph $K_6$ we need six colours to colour $K_6$ since every vertex is adjacent to every other vertex and we need a different colour for each vertex. $\therefore$ The chromatic number for $K_6$ is $\chi(K_6) = 6$. Similarly, the chromatic number of $K_{10}$ is $\chi(K_{10}) = 10$.

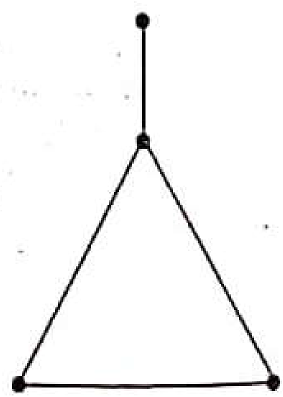


Fig. 11.169