If $|R| > 1$, then each region is bounded by at least 3 edges. But in a planar graph, each edge touches at most 2 region. Thus $2|E| \geq 3|R|$

(*b*) From Part (*a*), we have

$$2|E| \geq 3|R| \Rightarrow |R| \leq \frac{2}{3}|E|$$

$$\Rightarrow |V| + |R| \leq \frac{2}{3}|E| + |V|$$

$$\Rightarrow |E| + 2 \leq \frac{2}{3}|E| + |V| \qquad \text{| Euler's formula}$$

$$\Rightarrow 3|E| + 6 \leq 2|E| + 3|V|$$

$$\Rightarrow |E| \leq 3|V| - 6.$$

14. If $K_{3,3}$ is a planar graph, then we must have $2|E| \geq 3|R|$. Where each region is bounded by at least three edges. But for $K_{3,3}$, each region is bounded by at least 4 edges $\therefore$ We have $2|E| \geq 4|R|$

$$\Rightarrow 2|E| \geq 4\{|E| - |V| + 2\} \qquad \text{| Euler's formula}$$

$$\Rightarrow 2 \times 9 \geq 4(9 - 6 + 2) \qquad \text{For } K_{3,3}, |E| = 9, |V| = 6$$

$$\Rightarrow 18 \geq 20, \text{ a contradiction}$$

Hence $K_{3,3}$ is non-planar.

15. No, the graph does not contain Hamiltonian circuit. Since $e \geq \dfrac{n^2 - 3n + 6}{2}$ does not hold.

11.51. GRAPH COLOURING

Suppose that G = (V, E) is a graph with no multiple edges. A vertex colouring of G is an assignment of colours to the vertices of G such that adjacent vertices have different colours. A graph G is M-colourable if there exists a colouring of G which uses M-colours.

Proper Colouring. A colouring is proper if any two adjacent vertices u and v have different colours otherwise it is called improper colouring.

A graph can be coloured by assigning a different colour to each of its vertices. However, for most graphs a colouring can be found that uses fewer colours than the number of vertices in the graph.

11.52. CHROMATIC NUMBER OF G

(P.T.U., B.Tech. Dec. 2013, May 2006, Dec. 2005; M.C.A. May 2007)

The minimum number of colours needed to produce a proper colouring of a graph G is called the chromatic number of G and is denoted by $\chi(G)$.

The graph shown in Fig.11.169 is minimum 3-colourable, hence $\chi(G) = 3$.

Similarly, for the complete graph K_6, we need six colours to colour K_6 since every vertex is adjacent to every other vertex and we need a different colour for each vertex. $\therefore$ The chromatic number for K_6 is $\chi(K_6) = 6$. Similarly, the chromatic number of K_{10} is $\chi(K_{10}) = 10$.

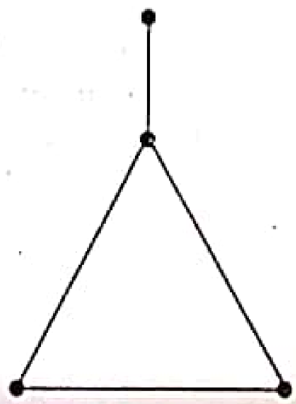

Fig. 11.169

ILLUSTRATIVE EXAMPLES

Example 1. *The chromatic number of K_n is n.* (P.T.U., B.Tech. Dec. 2012)

Sol. A colouring of K_n can be constructed using n colours by assigning a different colour to each vertex. No two vertices can be assigned the same colour, since every two vertices of this graph are adjacent. Hence the chromatic number of $K_n = n$.

Example 2. *The chromatic number of complete bipartite graph $K_{m,n}$, where m and n are positive integers is two.* (P.T.U., B.Tech. Dec. 2013)

Sol. The number of colours needed does not depend upon m and n. However, only two colours are needed to colour the set of m vertices with one colour and the set of n vertices with a second colour. Since, edges connect only a vertex from the set of m vertices and a vertex from the set of n vertices, no two adjacent vertices have the same colour.

Note 1. Every connected bipartite simple graph has a chromatic number of 2 or 1.

2. Conversely, every graph with a chromatic number of 2 is bipartite.

Example 3. *The chromatic number of graph c_n, where c_n is the cycle with n vertices is either 2 or 3.* (P.T.U., B.Tech. Dec. 2013)

Sol. Two colours are needed to colour c_n, when n is even. To construct such a colouring, simply pick a vertex and colour it black. Then move around the graph in clockwise direction colouring the second vertex white, the third vertex black, and so on. The nth vertex can be coloured white since the two vertices adjacent to it, namely the $(n-1)$th and the first are both coloured black as shown in Fig. 11.170.

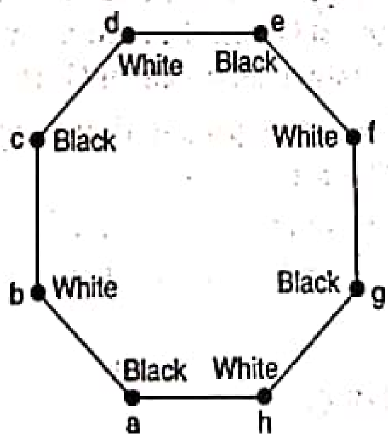

Fig. 11.170

When n is odd and $n > 1$, the chromatic number of c_n is 3. To construct such a colouring, pick an initial vertex. First use only two colours and alternate colours as the graph is traversed in a clockwise direction. However, the nth vertex reached is adjacent to two vertices of different colours, the first and $(n-1)$th. Hence, a third colour is needed. (Fig. 11.171)

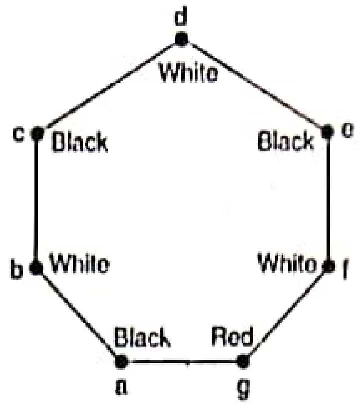

Fig. 11.171

Theorem I. *The following are equivalent for a graph G :*

(*i*) *G is 2-colourable*

(*ii*) *G is bipartite*

(*iii*) *Every cycle of G has even length.*

Proof. (*i*) $\Rightarrow$ (*ii*)

If G is 2-colourable, then G has two sets of vertices V_1 and V_2 with different colours, say, red and blue respectively.

Since no vertices of V_1 or V_2 are adjacent (being of same colour)

$\therefore$ $\{V_1, V_2\}$ is a partition of G $\Rightarrow$ G is bipartite.

(*ii*) $\Rightarrow$ (*iii*)

Let G be bipartite and $[V_1, V_2]$ be partition of vertices of G. Let $x \in V_1$ be any vertex and a cycle begins at x. Join this vertex to another vertex, say, $y \in V_2$ and then to a vertex in V_1 and so on. This cycle will return to $x \in V_1$ after it gets completed and will be of even length. (Since G is a bipartite graph). Hence G has no odd cycle.

(*iii*) $\Rightarrow$ (*i*)

Let each cycle in G is even. Let some vertex, say, x is coloured red, then its adjacent vertex will have different colour, say, blue, and its adjacent vertex will have red colour because every cycle has even length.

$\therefore$ sequence of vertices of even cycles is RBR, RBRBR and so on.

Thus only two colours are used to colour the graph.

$\therefore$ G is 2-colourable.

Example 4. *Determine the chromatic number of the graphs shown in Fig. 11.172.*

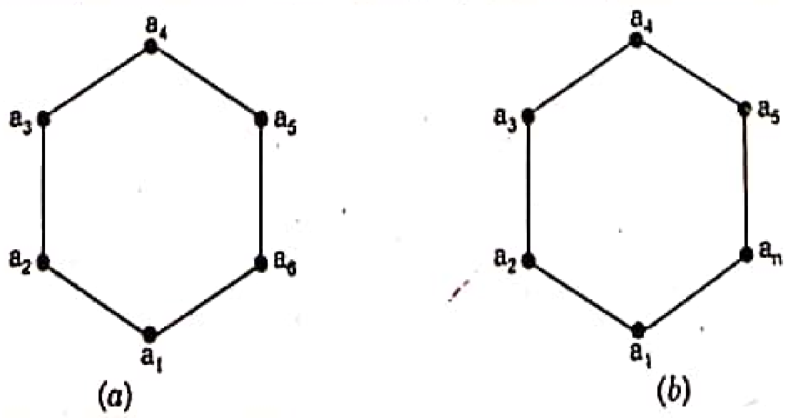

Fig. 11.172

Sol. The graphs shown in Fig. 11.172(*a*), has the chromatic number $\chi(G) = 2$.

The graph shown in Fig. 11.172(*b*) has the chromatic number $\chi(G) = 2$, when n is an even number and $\chi(G) = 3$, where n is odd.

Theorem II. *If an undirected graph has a subgraph K_3, then its chromatic number is at least three.*

Proof. Let G be an undirected graph. As G contains a complete graph K_3, which is 3-colourable. $\therefore$ G cannot be coloured with one or two colours

$\therefore \quad \psi(G) \geq 3$.

Four Colour Theorem. Every planar graph is four colourable.

Five Colour Theorem. Every planar graph has chromatic number ≤ 5.

Theorem III. *The vertices of every planar graph can be properly coloured with five colours.*

Proof. We will prove this theorem by induction. All the graphs with 1, 2, 3, 4 or 5 vertices can be properly coloured with five colours. Now let us assume that every planar graph with $n - 1$ vertices can be properly coloured with five colours. Next, if we prove that any planar graph G with n vertices will require no more than five colours, we have done.

Consider the planar graph G with n vertices.

Since G is planar, it must have at least one vertex with degree five or less as shown in theorem V. Assume this vertex to be 'u'.

Let G_1 be a graph of $n - 1$ vertices obtained from G by deleting vertex 'u'. The G_1 graph requires no more than five colours (Induction hypothesis). Consider that the vertices in G_1 have been properly coloured and now add to it 'u' and all the edges incident on u. If the degree of u is 1, 2, 3, or 4, a proper colour to u can be easily assigned.

Now, we have one case left, in which the degree of u is 5, and all the 5 colours have been used in colouring the vertices adjacent to u, as shown in Fig. 11.173.

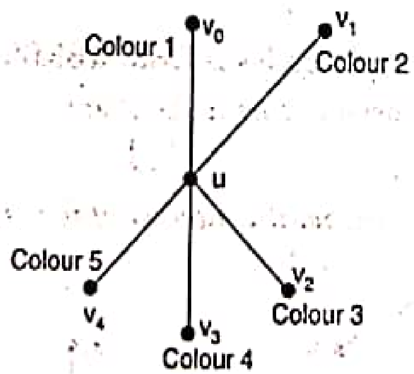

Fig. 11.173

Suppose that there is a path in G_1 between vertices v_0 and v_3 coloured alternately with colours 1 and 4 as shown in Fig. 11.174.

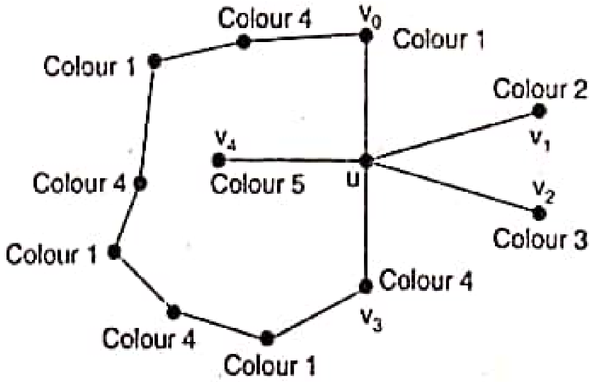

Fig. 11.174

Then a similar path between v_4 and v_2, coloured alternately with colours 5 and 3, can not exist ; otherwise, these two paths will intersect and cause G to be non-planar.

Thus, if there is no path between v_4 and v_2 coloured alternately with colour 5 and 3 of all vertices connected to v_2 through vertices of alternating colours 5 and 3. This interchange will colour vertex v_2 with colour 5 and yet keep G, properly coloured. As vertex v_4 is still with colour 5; the colour 3 is left over with which to colour vertex *u* which proves the theorem.

Example 5. *Consider the following graphs*

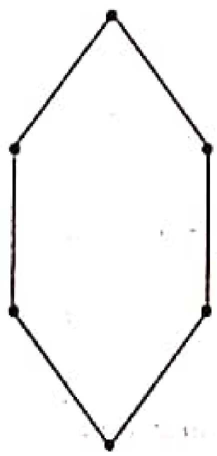

Fig. 11.175.

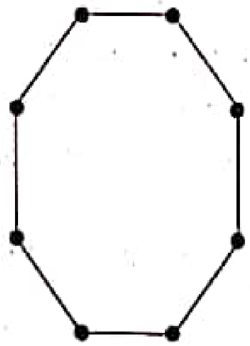

Fig. 11.176

(a) which graph(s) are bipartite graphs ? Colour the vertices of the bipartite graph.

(b) If graph is not bipartite, justify your answer.

Sol. Consider the graph shown in Fig. 11.177 Let V_1 = [R, R, R], V_2 = [B, B, B]

∴ Let V be the set of vertices of the given graph such that $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \phi$

Hence V can be partitioned into two sets V_1 of red colours and V_2 of blue colours.

Sol. The graphs shown in Fig. 11.172(*a*), has the chromatic number $\chi(G) = 2$.

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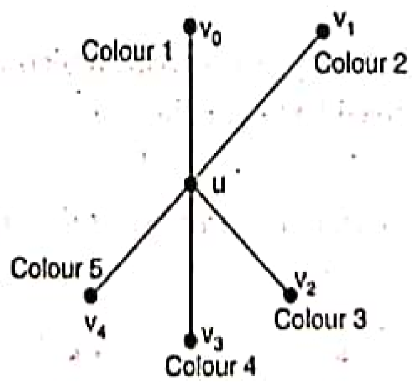

Fig. 11.173

Suppose that there is a path in G_1 between vertices v_0 and v_3 coloured alternately with colours 1 and 4 as shown in Fig. 11.174.

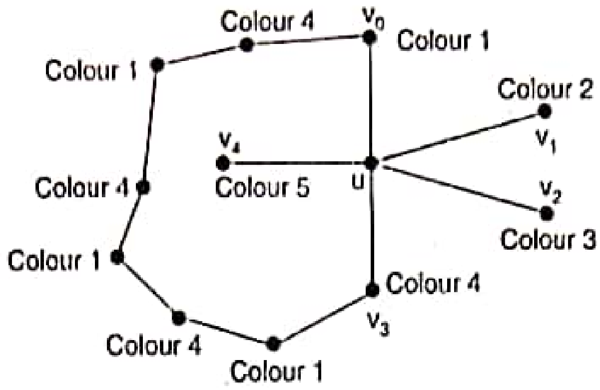

Fig. 11.174

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Thus, if there is no path between v_4 and v_2 coloured alternately with colour 5 and 3 of all vertices connected to v_2 through vertices of alternating colours 5 and 3. This interchange will colour vertex v_2 with colour 5 and yet keep G, properly coloured. As vertex v_4 is still with colour 5, the colour 3 is left over with which to colour vertex u which proves the theorem.

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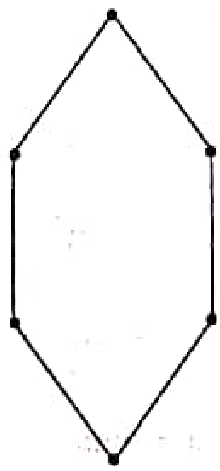

Fig. 11.175.

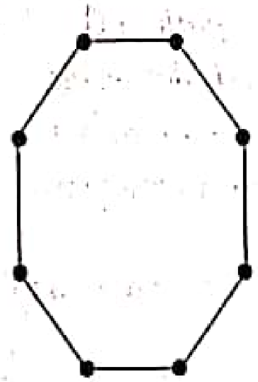

Fig. 11.176

(a) which graph(s) are bipartite graphs ? Colour the vertices of the bipartite graph.

(b) If graph is not bipartite, justify your answer.

Sol. Consider the graph shown in Fig. 11.177 Let $V_1 = [R, R, R]$, $V_2 = [B, B, B]$

$\therefore$ Let V be the set of vertices of the given graph such that $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \phi$

Hence V can be partitioned into two sets V_1 of red colours and V_2 of blue colours.

∴ This graph is a bipartite graph.

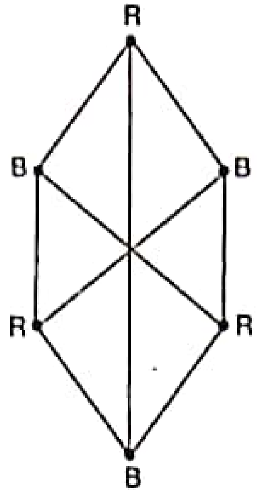

Fig. 11.177

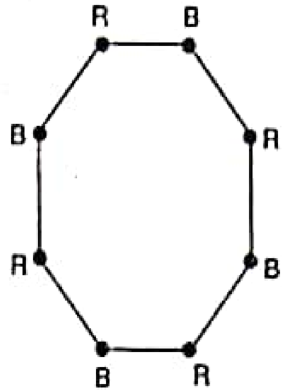

Fig. 11.178

(*b*) However, the graph shown in Fig. 11.178 is not a bipartite graph as it is not 2-colourable. If we try to label the vertices using two colours red (R) and Blue (B), we get the graph (Fig 11.178) with adjacent vertices of same colour.

Example 6. *Consider the graph (Fig. 11.179) G shown below.*

(*a*) *Find the shortest and longest simple path between A and F*

(*b*) *What is the diameter of the graph ?*

(*c*) *Is there Euler's path in G?*

(*d*) *Is G planar and connected?*

(*e*) *Find chromatic number of G.*

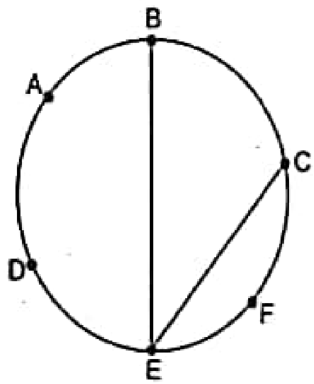

Fig. 11.179

Sol. (*a*) The shortest path between A and F is A, D, E, F. It is of length 3.

The longest path between A and F is A, D, E, B, C, F. It is of length 5.

(*b*) We know that the diameter of a graph is the maximum distance between any two vertices.

∴ diameter (G) = 3

(*c*) Since the vertices B and C are of odd degree.

∴ G has an Euler path between B and C.

(*d*) Since no two edges in the graph intersect and there is a path between each pair of vertices.

∴ G is connected and planar

(*e*) Consider the subgraph EBC(K_3)

∴ The chromatic number of the graph is ≥ 3

But three colours are sufficient to paint the vertices property (see Fig. 11.180). A(Red), B(Blue), C(Red), D(Green), E(Green), F(Blue)

∴ required chromatic number = 3

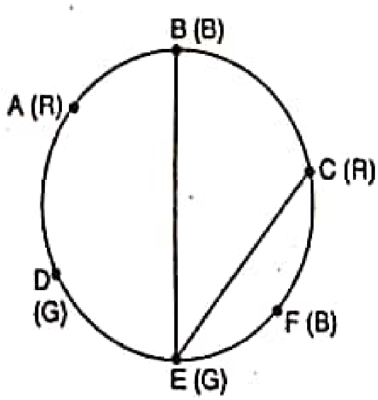

Fig. 11.180

Example 7. *Write any three applications of colouring of graph.* (P.T.U. B.Tech. Dec. 2009)

Sol. (*i*) Scheduling. Vertex coloring models to a number of scheduling problems. In the cleanest form, a given set of jobs need to be assigned to time slots, each job requires one such slot. Jobs can be scheduled in any order, but pairs of jobs may be in conflict in the sense that they may not be assigned to the same time slot, because they both rely on a shared resource. The corresponding graph contains a vertex for every job and an edge for every conflicting pair of jobs. The chromatic number of the graph is exactly the minimum makespan, the optimal time to finish all jobs without conflicts.

Details of the scheduling problem define the structure of the graph. For example, when assigning aircrafts in band with allocation to radio stations, the resulting conflict graph is a unit disk graph, so the coloring problem is 3-approximable.

(*ii*) Register allocation. A compiler is a computer program that translates one computer language into another. To improve the execution time of the resulting code, one of the techniques of compiler optimization is register allocation, where the most frequently used values of the compiled program are kept in the fast processor register. Ideally, values are assigned to registers so that they can all reside in the registers when they are used.

The textbook approach to this problem is to model it as a graph coloring problem. The compiler constructs an interference graph, were vertices are symbolic registers and an edge connects two nodes if they are needed at the same time. If the graph can be colored with *k* color, then the variables can be stored in *k* registers.

(*iii*) Other applications. The problem of coloring a graph has found a number of applications, including pattern matching. The recreational puzzle Sudoku can be seen as completing a 9-coloring on given specific graph with 81 vertices.

11.53. APPLICATIONS OF GRAPH THEORY

11.53.1. Shortest Path in Weighted Graphs (*P.T.U., B.Tech. May 2007*)

Weighted graphs can be used to represent highways connecting the different cities. The weighted edges represent the distance between different cities and the vertices represent the cities. A common problem with this type of graph is to find the shortest path from one city to another city. There are many ways to tackle this problem one of which is as follows :

Shortest Paths from Single Source. We will find shortest paths from a single vertex to all other vertices of the graph. The first algorithm was proposed by **E. Dijkstra** in 1959. Some common terms related with this algorithm are as follows :

Path Length. The length of a path is the sum of the weights of the edges on that path.

Source. The starting vertex of the graph from which we have to start to find the shortest path.

Destination. The terminal or last vertex upto which we have to find the path.

11.54. DIJKSTRA'S ALGORITHM FOR SHORTEST PATH (*P.T.U., M.C.A. Dec. 2005*)

This algorithm maintains a set of vertices whose shortest path from source is already known. The graph is represented by its cost adjacency matrix, where cost being the weight of the edge. In the cost adjacency matrix of the graph, all the diagonal values are zero. If there is no path from source vertex V_s to any other vertex V_i, then it is represented by $+\infty$. In this algorithm, we have assumed all weights are positive.

1. Initially there is no vertex in sets.

2. Include the source vertex V_s in S. Determine all the paths from V_s to all other vertices without going through any other vertex.

Example 7. *Write any three applications of colouring of graph.*

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1. Initially there is no vertex in sets.

2. Include the source vertex V_s in S. Determine all the paths from V_s to all other vertices without going through any other vertex.

3. Now, include that vertex in S which is nearest to V_s and find shortest paths to all the vertices through this vertex and update the values.

4. Repeat the step 3 until $n - 1$ vertices are not included in S if there are n vertices in the graph.

After completion of the process, we get the shortest paths to all the vertices from the source vertex.

Example 8. *Find the shortest path between K and L in the graph shown in Fig. 11.181 by using Dijkstra's Algorithm.*

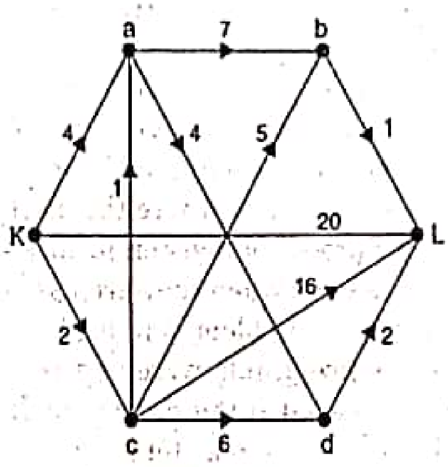


Fig. 11.181

Sol. Step I. Include the vertex K in S and determine all the direct paths from K to all other vertices without going through any other vertex.

S	Distance to all other vertices					
	K	*a*	*b*	*c*	*d*	L
K	0	4(K)	∞	2(K)	∞	20(K)

Step II. Include the vertex in S which is nearest to K and determine shortest paths to all vertices through this vertex and update the values. The nearest vertex is *c*.

S	Distance to all other vertices					
	K	*a*	*b*	*c*	*d*	L
K, *c*	0	3(K, *c*)	7(K, *c*)	2(K)	8(K, *c*)	18(K, *c*)

Step III. The vertex which is 2nd nearest to K is *a*, included in S.

S	Distance to all other vertices					
	K	*a*	*b*	*c*	*d*	L
K, *c*, *a*	0	3(K, *c*)	7(K, *c*)	2(K)	7(K, *c*, *a*)	18(K, *c*)

Step IV. The vertex which is 3rd nearest to K is *b*, is included in S.

S	Distance to all other vertices					
	K	*a*	*b*	*c*	*d*	L
K, *c*, *a*, *b*	0	3(K, *c*)	7(K, *c*)	2(K)	7(K, *c*, *a*)	8(K, *c*, *b*)

Step V. The vertex which is next nearest to K is *d*, is included in S.

S	Distance to all other vertices					
	K	*a*	*b*	*c*	*d*	L
K, *c*, *a*, *b*, *d*	0	3(K, *c*)	7(K, *c*)	2(K)	7(K, *c*, *a*)	8(K, *c*, *b*).

Since, $n - 1$ vertices included in S. Hence we have found the shortest distance from K to all other vertices.

Thus, the shortest distance between K and L is 8 and the shortest path is K, c, b, L.

Example 9. *Find the shortest path between a and z in the graph shown in Fig. 11.182.*

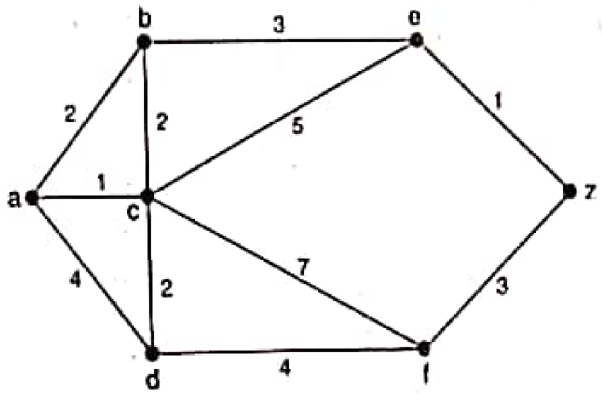


Fig. 11.182

Sol. Step I. Include the vertex a in S and determine all the direct paths from a to all other vertices without going through any other vertices.

S	Distance to all other vertices						
	a	b	c	d	e	f	z
a	0	2(a)	1(a)	4(a)	∞	∞	∞

Step II. Include the vertex in S which is nearest to a and determine shortest path to all the vertices through this vertex. The nearest vertex is c.

S	Distance to all other vertices						
	a	b	c	d	e	f	z
a, c	0	2(a)	1(a)	3(a, c)	6(a, c)	8(a, c)	∞

Step III. Include the vertex in S which is 2nd nearest to S and determine shortest path to all the vertices through this vertex. The 2nd nearest vertex is b.

S	Distance to all other vertices						
	a	b	c	d	e	f	z
a, c, b	0	2(a)	1(a)	3(a, c)	5(a, b)	8(a, c)	∞

Step IV. Next vertex included in S is d.

S	Distance to all other vertices						
	a	b	c	d	e	f	z
a, c, b, d	0	2(a)	1(a)	3(a, c)	5(a, b)	7(a, c)	∞

Step V. Next vertex included in S is e.

S	Distance to all other vertices						
	a	b	c	d	e	f	z
a, c, b, d, e	0	2(a)	1(a)	3(a, c)	5(a, b)	7(a, c)	6(a, b, e)

Step VI. Next vertex included in S is f.

S	Distance to all over vertices						
a, c, b, d, e, f	a	b	c	d	e	f	z
	0	$2(a)$	$1(a)$	$3(a, c)$	$5(a, b)$	$7(a, c)$	$6(a, b, e)$.

This ends our procedure as $n-1$ vertices are included in S. Thus, the shortest distance between a and z is 6 and the shortest path is a, b, e, z.

Example 10. *Using either breadth first search algorithm or Dijkstra's algorithm, find the shortest path from s to t in the following weighted graph. (Fig. 11.183)*

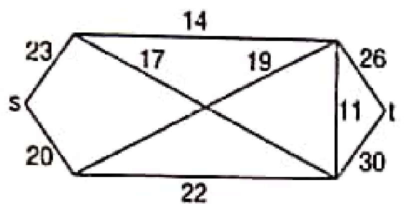


Fig. 11.183

Sol. Let S be the set of vertices of the given weighted graph. *i.e.*, S = {s, a, b, c, d, t} as shown in the Fig. 11.184.

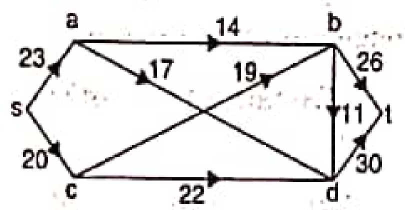


Fig. 11.184

Step I. Include the vertex s in S and determine all the direct paths from s to all other vertices without going through any other vertex.

S	Distance to all other vertices					
	s	a	b	c	d	t
s	0	$23(s)$	∞	$20(s)$	∞	∞

Step II. Include the vertex in S which is nearest to s and determine shortest paths to all vertices through this vertex and update the values. The nearest vertex is C.

S	Distance to all other vertices,					
	s	a	b	c	d	t
s, c	0	$23(s)$	$39\ (s,c)$	$20(s)$	$42(s, c)$	∞

Step III. The vertex which is 2nd nearest to s is a. Include this vertex in S

S	Distance to all other vertices					
	s	a	b	c	d	t
s, c, a	0	$23(s)$	$37(s, a)$	$20(s)$	$42(s, c)$	∞

Step IV. The vertex which is 3rd nearest to s is b. Include this vertex in S.

S	Distance to all other vertices					
	s	a	b	c	d	t
s, c, a, b	0	$23(s)$	$37(s, a)$	$20(s)$	$40(s, a)$	$63(s, a, b)$

Step V. The vertex which is next nearest to s is (d). Include this vertex in S.

S	Distance to all other vertices					
	s	a	b	c	d	t
s, c, a, b, d	0	23(s)	37(s, a)	20(s)	40(s, a)	63(s, a, b)

Since $n - 1$ = S vertices are included in S. Hence we have found the shortest distance from s to all other vertices. Thus, the shortest distance between s and t is 63 and the shortest path is s, a, b, t.

Example 11. *Show that $e \geq 3V - 6$ for the connected planar graphs shown in Figs. 11.185 and 11.186.*

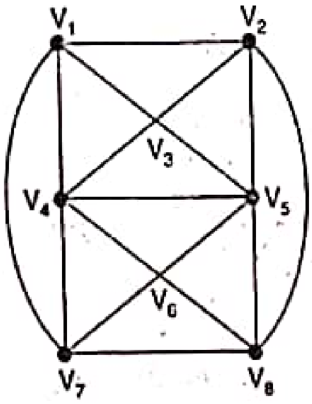


Fig. 11.185

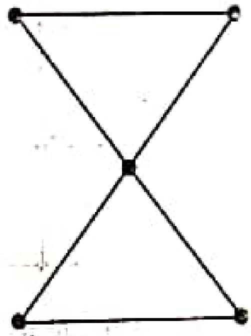


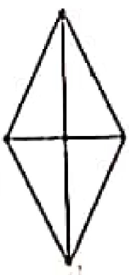
Fig. 11.186

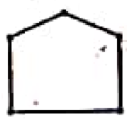
Sol. (*i*) The graph shown in Fig. 11.185 contains vertices V = 8 and edges e = 17. Putting the values we have $e = 3 \times 8 - 6 = 18 \geq 17$. Hence proved.

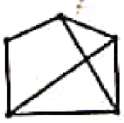
(*ii*) The graph shown in Fig. 11.186 contains vertices V = 5 and edges e = 6. Putting the values, we have $3 \times 5 - 6 = 11 > 6$. Hence proved.

TEST YOUR KNOWLEDGE 11.3

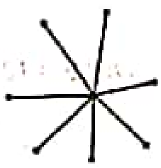
1. Find the chromatic numbers of the following graphs.

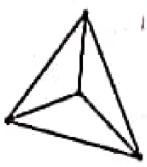
(*a*) 

(*b*) 

(*c*) 

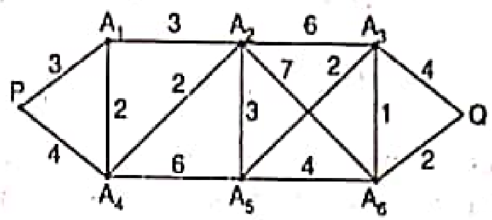
(*d*) 

(*e*) 

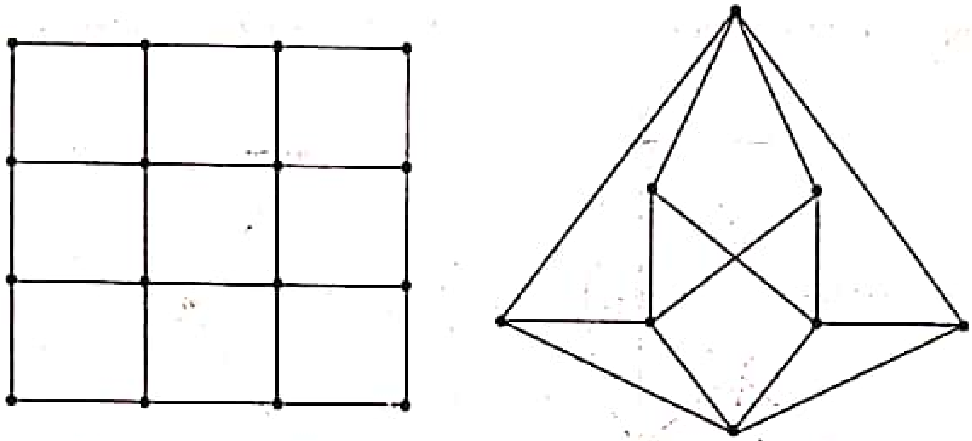
(*f*) 

(*g*) Complete bipartite graph $K_{3,4}$ (*P.T.U., B.Tech. May 2013*)

2. Find the shortest path, by using either Breadth first search or Dijkstra's algorithm, from P to Q in the following weighted graph.

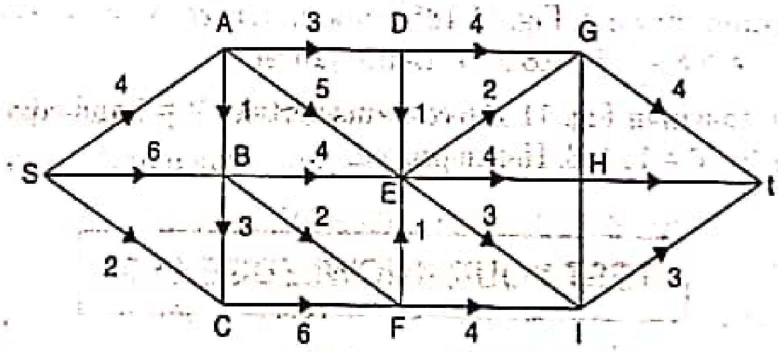


3. Find the chromatic number of the following graphs.

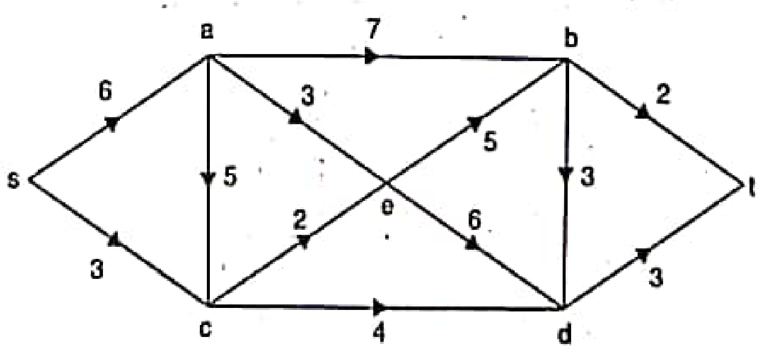


4. Find the shortest path and its length from s to t by using Dijkstra's algorithm in the following graph.

(*a*)



(*b*)



Answers

1. (*a*) 3 (*b*) 3 (*c*) 3 (*d*) 3 (*e*) 2 (*f*) 4 (*g*) 2

2. P A_1 A_2 A_5 A_3 A_6 Q.

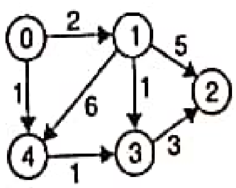
3. 2

4. (*a*) s – A – D – E – H – t, length = 13

(*b*) r – C – d – t.

MULTIPLE CHOICE QUESTIONS (MCQs)

1. Which of the following statement is FALSE about undirected graphs?
   (a) The sum of degrees of all the vertices in a graph is even.
   (b) There is an even number of vertices of odd degree.
   (c) The degree of a vertex is the number of edges incident on it.
   (d) The self loop is counted once, when degree is counted.
2. How many edges do a complete graph contains having n vertices?
   (a) $2n$ (b) $2n/2$
   (c) $n(n-1)/2$ (d) Any number of edges.
3. Which of the following is FALSE statement about planar graphs?
   (a) A complete graph of five or more vertices is not a planar graph.
   (b) A complete bipartite graph having $m \geq 3$ and $n \geq 3$ is a planar graph.
   (c) Every planar graph has at least one vertex of degree 5 or less than 5.
   (d) Every planar graph having e edges and v vertices has $3v - e \geq 6$.
4. The chromatic number of a complete graph K_n is
   (a) $n - 1$ (b) n
   (c) 2 (d) Any number.
5. Which of the following bipartite graph has Hamiltonian circuit?
   (a) $K_{1,2}$ (b) $K_{2,3}$
   (c) $K_{3,3}$ (d) $K_{3,5}$
6. Which of the following statement about the directed graphs is not TRUE?
   (a) The in-degree and the out-degree of the directed graphs are equal.
   (b) The in-degree and out-degree is not same as the number of edges.
   (c) The sum of in-degree and out-degree is even.
   (d) The self loop has one in-degree and one out-degree in the directed graph.
7. Suppose you run Dijkstra's single source shortest path algorithm on the following weighted directed graph with vertex 0 as the source vertex



   In what order do the nodes get included into the set of vertices for which the shortest path distances are finalized?
   (a) 0, 1, 2, 3, 4 (b) 0, 4, 1, 3, 2
   (c) 0, 4, 3, 2, 1 (d) 0, 1, 4, 3, 2
8. How many edges are there in a graph with 20 vertices and the sum of the degrees (in-degree and out-degree) is 100?
   (a) 50 (b) 100
   (c) 20 (d) 40

9. If G is a directed graph with 10 vertices, how many Boolean values will be needed to represent G using an adjacency matrix?
   (a) 50 (b) 100
   (c) 200 (d) 1000
10. Which of the following is Not TRUE about the directed graphs?
   (a) If a digraph is reflexive, then the diagonal elements of the adjacency matrix are 1.
   (b) If G is a simple digraph whose adjacency matrix is A then the adjacency matrix of G^c, is the transpose of A.
   (c) The diagonal elements of $A \cdot A^T$ show the out degree of the vertices.
   (d) The adjacency matrix of a directed graph is a symmetric matrix.

Answers and Explanation

1. (d) The self loop is counted twice, when degree is counted.
2. (c) The number of edges in a complete graph is $n(n-1)/2$
3. (b) A complete bipartite graph having $m \geq 3$ and $n \geq 3$ is not a planar graph.
4. (b) The chromatic number of a complete graph of n vertices is n.
5. (c) $K_{3,3}$ has Hamiltonian circuit.
6. (b) The in-degree and out-degree is same as the number of edges.
7. (b) Apply the algorithm and it will be 0, 4, 1, 3, 2
8. (a) One edge contributes two degree.
9. (b) The adjacency matrix is of N × N dimension, where n is the number of vertices.
10. (d) The adjacency matrix of a directed graph is not a symmetric matrix.