Lecture - 3 (Tree)

Result :→ A full m-ary tree with

i) n vertices has $i = \frac{(n-1)}{m}$ internal vertices and
$l = \frac{[(m-1)n+1]}{m}$ leaves

ii) i internal vertices has $n = mi+1$ vertices and
$l = (m-1)i+1$ leaves

iii) l leaves has $n = \frac{(ml-1)}{m-1}$ vertices and $i = \frac{(l-1)}{(m-1)}$
internal vertices.

Proof :→ Let n be the no. of vertices, i the no. of internal vertices and l be the no. of leaves

As we already know that a full m-ary tree with i internal vertices has $\boxed{n = mi+1}$ vertices —— (1)

and also $\boxed{n = i+l}$ —— (2)

So

i) from (1) $n-1 = mi \Rightarrow \boxed{i = \frac{(n-1)}{m}}$

and from (2) $l = n-i = n - \frac{(n-1)}{m}$

$$l = \frac{nm-n+1}{m} = \frac{n(m-1)+1}{m}$$

$$\Rightarrow \boxed{l = \frac{n(m-1)+1}{m}}$$

(ii) $n = mi + 1$ which is obvious.

and $l = n - i$ from (2)

$\Rightarrow l = (mi+1) - i = mi + 1 - i = (m-1)i + 1$

$\Rightarrow \boxed{l = (m-1)i + 1}$

(iii) Again from (1) and (2) we have

$n = mi + 1$ and $n = i + l$

So $\boxed{i = n - l}$

$$n = m(n-l) + 1$$

$$n = mn - ml + 1$$

$$\Rightarrow n - mn = -ml + 1$$

$$\Rightarrow n(1-m) = -ml + 1$$

$$\Rightarrow n = \frac{-ml+1}{1-m} = \frac{ml-1}{m-1}$$

and $i = n - l = \frac{ml-1}{m-1} - l = \frac{\cancel{ml} - 1 - \cancel{lm} + l}{m-1}$

$$i = \frac{(l-1)}{(m-1)}$$

Level of a vertex :→ The level of a vertex v in a rooted tree is the length of the unique path from the root to this vertex.

Height (Depth) of a tree :→ The height of a rooted tree is the maximum of the levels of the vertices. It is the length of the longest path from the root to any vertex.

Thm :→ There are at most m^h leaves in an m-ary tree of height h.

Proof :→ By mathematical Induction.

Step 1 Consider m-ary tree of height 1.

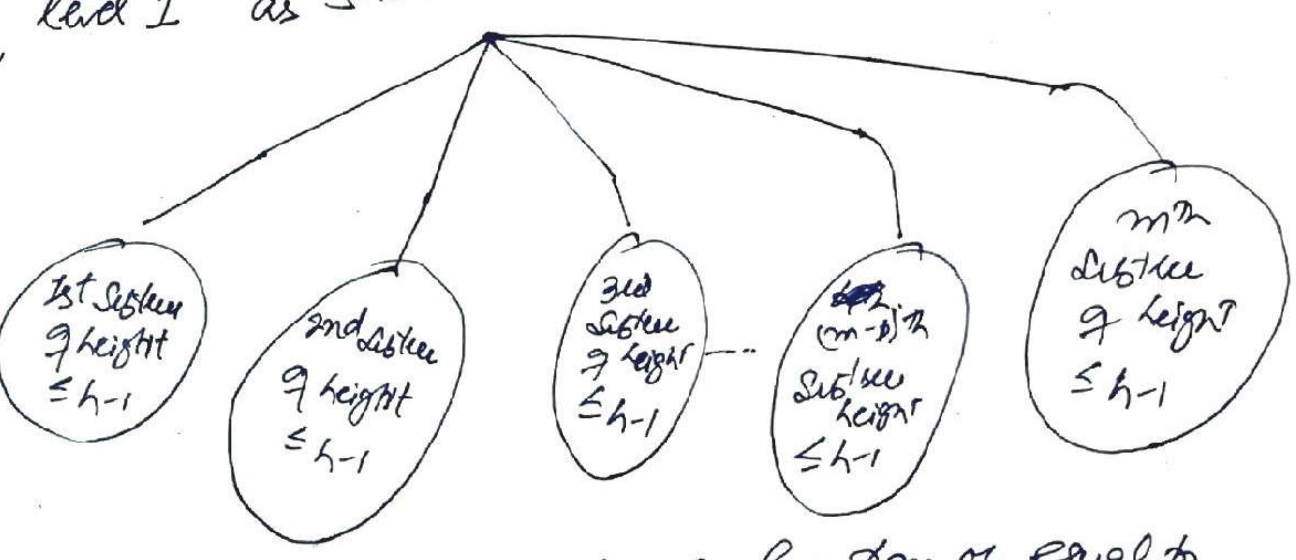
The trees here consists of a root with no more than m-children each of which is a leaf.

Hence there are no more than $m^1 = m$ leaves in an m-ary tree of height 1.

⇒ This proves the step-1

Step 2 Assume that the result is true for all m-ary trees of height less than h.

Step 3 Let T be an m-ary tree with height h. The leaves of T are the leaves of the subtrees of T obtained by deleting the edges from the roots to each of the vertices of level 1 as shown.



Each of these subtree has height less than or equal to h-1. So from step 2. each of these rooted tree has at most m^{h-1} leaves.

Because there are at most m such subtrees, each with a maximum of m^{h-1} leaves, there are at most $m \cdot m^{h-1} = m^h$ leaves in the rooted tree.

$\Rightarrow$ the ~~st~~ which proof the step 3