Example 8. *Draw the undirected graphs K_3 and K_5.*

Sol. Undirected graphs K_3 and K_5 are shown in Figs. 11.23 and 11.24.

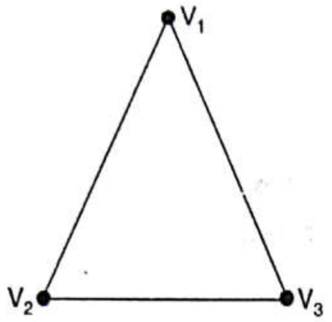


Fig. 11.23. K_3

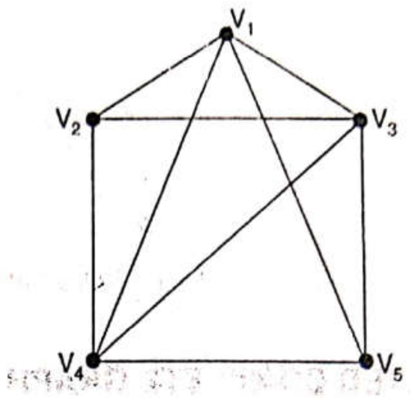


Fig. 11.24. K_5

11.13. CONNECTED GRAPH

(*P.T.U., B.Tech. Dec. 2006*)

A graph is called connected if there is a path from any vertex u to v or vice-versa.

11.14. DISCONNECTED GRAPH

A graph is called disconnected if there is no path between any two of its vertices.

11.15. CONNECTED COMPONENT

A subgraph of graph G is called the connected component of G, if it is not contained in any bigger subgraph of G, which is connected. It is defined by listing its vertices.

Example 9. *Consider the graph shown in Fig. 11.25. Determine its connected components.*

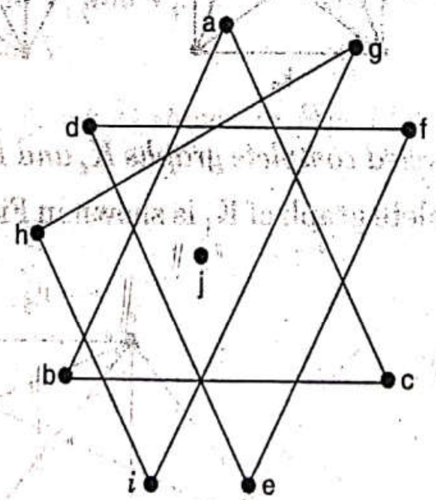


Fig. 11.25

Sol. The connected components of this graph is $\{a, b, c\}$, $\{d, e, f\}$, $\{g, h, i\}$ and $\{j\}$.

Example 10. *Consider the graphs shown in Figs. 11.26, 11.27 and 11.28. Determine whether the graphs are (a) Connected graphs or (b) Disconnected graphs.*

Also write their connected components.

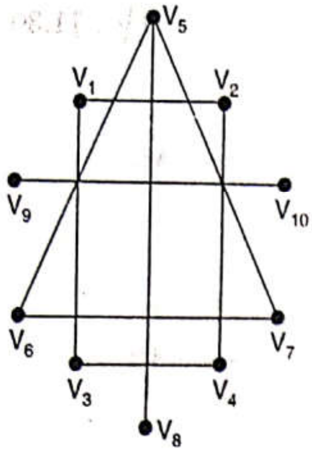


Fig. 11.26

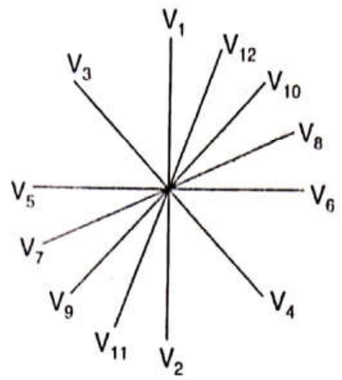


Fig. 11.27

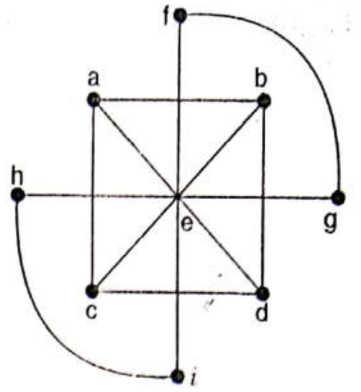


Fig. 11.28

Sol. (*i*) The graph shown in Fig. 11.26 is a disconnected graph and its connected components are

$$\{V_1, V_2, V_3, V_4\}, \{V_5, V_6, V_7, V_8\} \text{ and } \{V_9, V_{10}\}.$$

(*ii*) The graph shown in Fig. 11.27 is a disconnected graph and its connected components are

$$\{V_1, V_2\}, \{V_3, V_4\}, \{V_5, V_6\}, \{V_7, V_8\}, \{V_9, V_{10}\} \text{ and } \{V_{11}, V_{12}\}.$$

(*iii*) The graph shown in Fig. 11.28 is a connected graph.

Theorem IV. *Let G be a connected graph with at least two vertices. If the number of edges in G is less than the number of vertices, then prove that G has a vertex of degree 1.*

Proof. Let G be a connected graph with $n \geq 2$ vertices. Because graph G is connected, G has no isolated vertices. Suppose G has no vertex of degree 1. Then the degree of each vertex is at least 2. This implies that the sum of the degrees of vertices of G is at least $2n$. Hence, it follows that the number of edges is at least n ($\because$ the sum of the degrees of vertices in any graph is twice the number of edges), which is a contradiction. This implies that G contains at least one vertex of degree 1.

11.16. SUBGRAPH

(*P.T.U., B.Tech. Dec. 2006, Dec. 2013*)

A subgraph of a graph $G = (V, E)$ is a graph $G' = (V', E')$ in which $V' \subseteq V$ and $E' \subseteq E$ and each edge of G' has the same end vertices in G' as in graph G.

Note. A single vertex is a subgraph.

Example 11. *Consider the graph G shown in Fig. 11.29. Show the different subgraphs of this graph.*

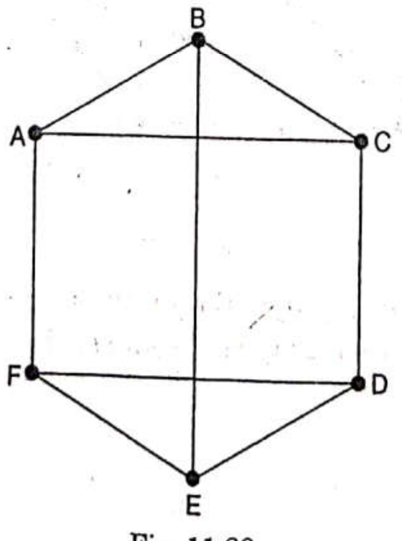


Fig. 11.29

Sol. The following are all subgraphs of the above graph (shown in Figs. 11.30, 11.31, 11.32, 11.33). There may be another subgraphs of this graph.

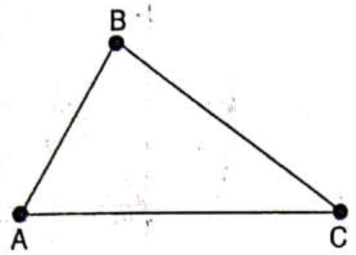


Fig. 11.30

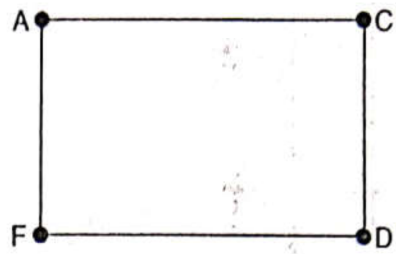


Fig. 11.31

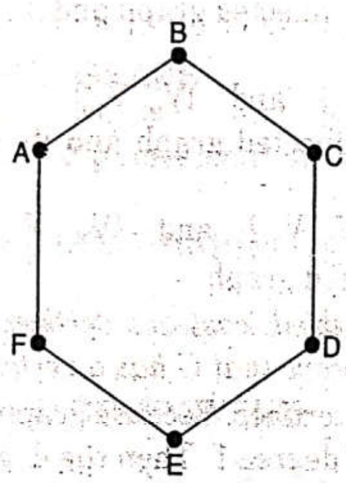


Fig. 11.32

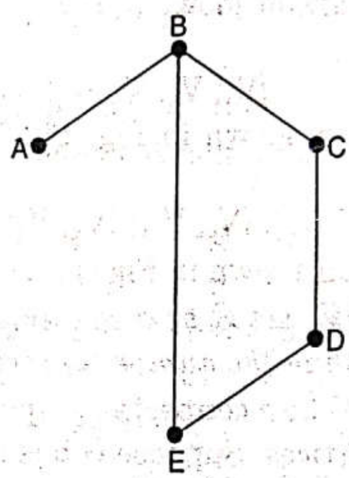


Fig. 11.33

Example 12. *Consider the directed graph as shown in Fig. 11.34. Show the four different subgraphs of this graph having at least four vertices.*

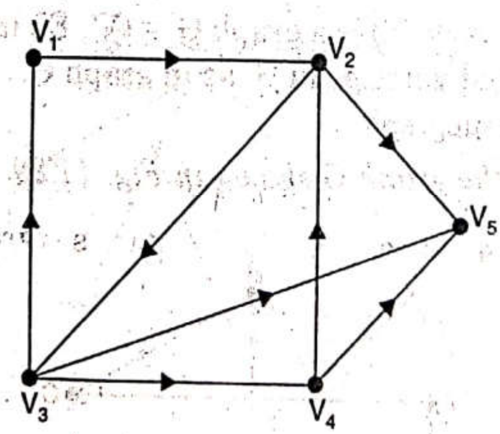


Fig. 11.34

Sol. The four subgraphs of the directed graph are shown in Figs. 11.35, 11.36, 11.37 and 11.38. There may be another subgraphs of this graph.

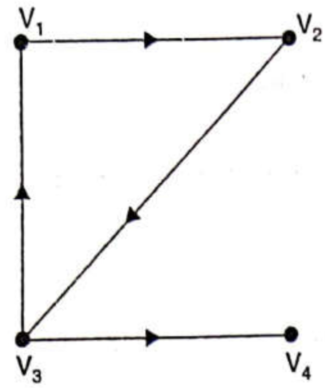


Fig. 11.35

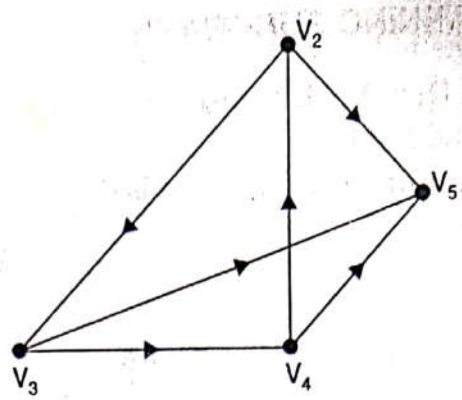


Fig. 11.36

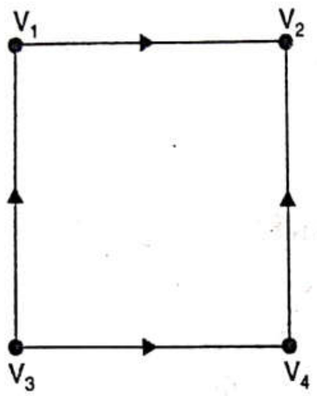


Fig. 11.37

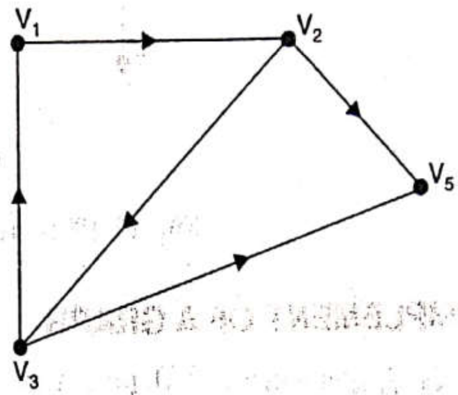


Fig. 11.38

Example 13. *Consider the multigraph shown in Fig. 11.39. Show two different subgraphs of this multigraph which are itself multigraphs.*

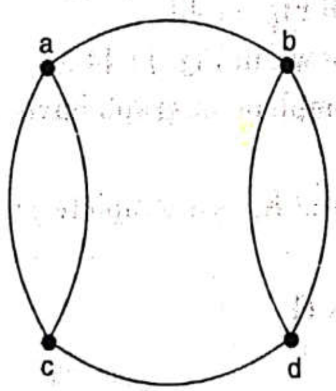


Fig. 11.39

Sol. The two different subgraphs of this multigraph which are itself multigraphs are shown in Figs. 11.40 and 11.41. There may be another subgraphs of this multigraph.

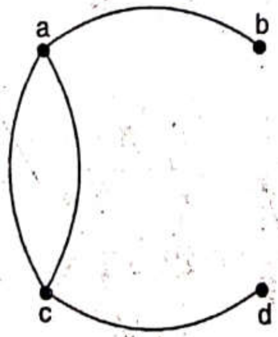


Fig. 11.40

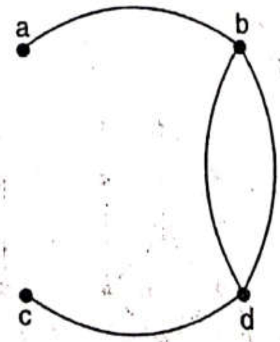


Fig. 11.41

11.17. (a) SPANNING SUBGRAPH

A graph $G_1 = (V_1, E_1)$ is called a spanning subgraph of $G = (V, E)$ if G_1 contains all the vertices of G and $E \neq E_1$.

For example : The Fig. 11.42 is the spanning subgraph of the graph shown in Fig. 11.29.

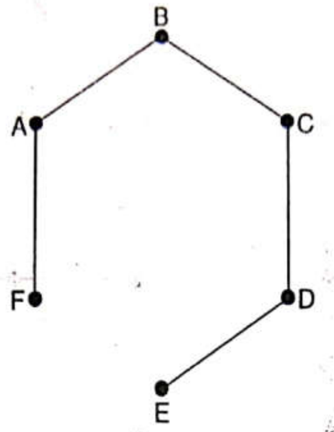


Fig. 11.42. Spanning Subgraph.

11.17. (b) COMPLEMENT OF A GRAPH

Let $G = (V, E)$ be a given graph. A graph $\overline{G} = (\overline{V}, \overline{E})$ is said to be complement of $G = (V, E)$ If $\overline{V} = V$ and $\overline{E}$ does not contain edges of E. *i.e.*, edges in $\overline{E}$ are join of those pairs of vertices which are not joined in G.

Consider the graph shown in Fig. 11.43.

The complement graph is shown in Fig. 11.44 .

Note that a graph and its complement graph have same vertices.

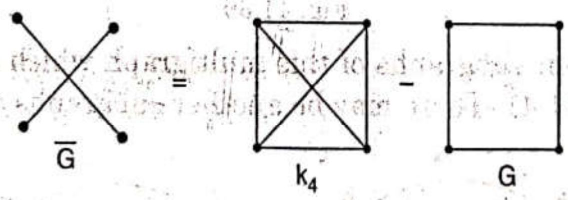
Fig. 11.43

Fig. 11.44

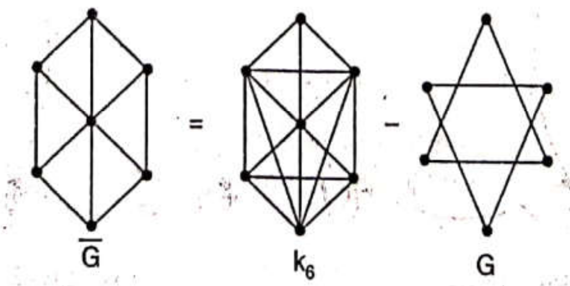
If a graph G has n vertices and K_n is a complete graph with n vertices, then

$$\overline{G} = K_n - G$$

Consider K_4. Then



Consider K_6. Then

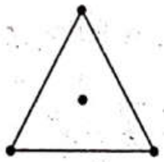


11.17. (c) COMPLEMENT OF A SUBGRAPH

Let G = (V, E) be a graph and S be a subgraph of G. If edges of S be deleted from the graph G, the graph so obtained is complement of subgraph S. It is denoted by $\overline{S}$.

$\therefore\ \overline{S} = G - S$

Consider the graph  and its subgraph  Then the complement of subgraph S is

$\overline{S}$ = 

Note that in a complement of a subgraph, the number of vertices donot change.

11.18. (a) CUT SET

Consider a connected graph G = (V, E). A cut set for G is a smallest set of edges such that removal of the set, disconnects the graph whereas the removal of any proper subset of this set, left a connected subgraph.

For example, consider the graph shown in Fig. 11.45. We determine the cut set for this graph.

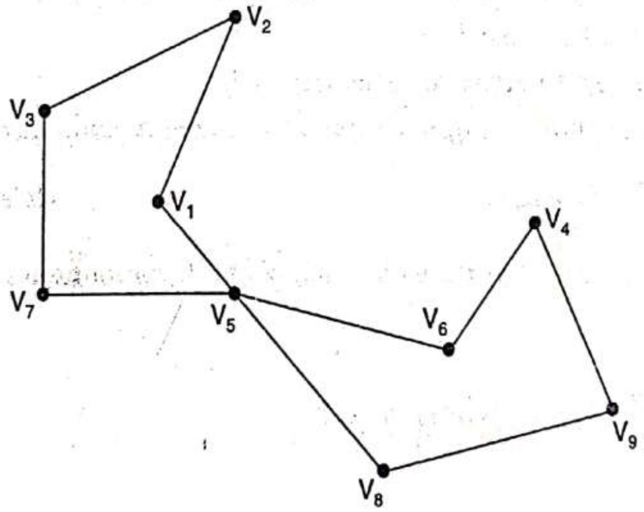


Fig. 11.45

For this graph, the edge set $\{(V_1, V_5), (V_7, V_5)\}$ is a cut set. After the removal of this set, we have left with a disconnected subgraph. While after the removal of any of its proper subset, we have left with a connected subgraph.

11.18. (b) CUT POINTS OR CUT VERTICES

Consider a graph G = (V, E). A cut point for a graph G, is a vertex v such that $G-v$ has more connected components than G or disconnected.

The subgraph $G-v$ is obtained by deleting the vertex v from the graph G and also deleting all the edges incident on v.

11.19. EDGE CONNECTIVITY

Let G = {V, E} be a connected graph, then cardinality of cut set of G is called edge connectivity of graph G.

The edge connectivity of a connected graph cannot be more than the smallest degree of a vertex in the graph. It is denoted as $\lambda(G)$

Vertex connectivity

Let G be a connected graph. Vertex connectivity of a graph is the least number of vertices whose removal disconnects the graph. It is written as K(G) and is given by

$$K(G) = n - 1$$

for a complete graph with n vertices

For example, we find edge and vertex connectivity of following graphs (Figs. 11.46, 11.47)

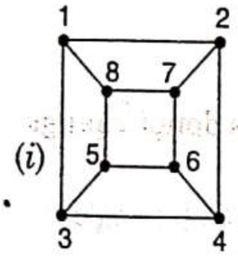


Fig. 11.46

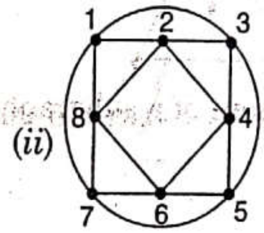


Fig. 11.47

In Fig. 11.47, removal of vertices 1, 2, 6 disconnects the graph while removal of any two vertices does not.

$\therefore$ vertex connectivity is 3.

It is a 3-regular graph. Only all edges incident on a vertex will disconnect it.

$\therefore$ edge connectivity is also 3.

In Fig. 11.48, edge and vertex connectivity is 4.

V. Imp

Theorem V. *Prove that a simple graph with k-components and n vertices can have at the most of* $\dfrac{(n-k)(n-k+1)}{2}$ *edges.* (P.T.U., M.C.A. Dec. 2006)

Proof. Let the number of vertices in each of the k-components of a simple graph G be $n_1, n_2, \ldots\ldots, n_k$. Then

$$\sum_{i=1}^{k} n_i = n, \text{ where } n_i \geq 1$$

We know that maximum number of edge in the i component of G $= \dfrac{n_i(n_i-1)}{2}$

$\therefore$ Maximum number of edges in a graph less $G = \displaystyle\sum_{i=1}^{k} \frac{n_i(n_i-1)}{2} = \frac{1}{2}\left[\sum_{i=1}^{k}(n_i^2 - n_i)\right]$

$$= \frac{1}{2}\left[\sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} n_i\right] = \frac{1}{2}\left[\sum_{i=1}^{k} n_i^2 - n\right] \qquad \ldots(1)$$

Now, we prove that

$$\sum_{i=1}^{k} n_i^2 \le n^2 - (k-1)(2n-k)$$

Since $\sum_{i=1}^{k} n_i^2 = n \Rightarrow \sum_{i=1}^{k} n_i^2 - k = n - k$

$$\Rightarrow \quad \sum_{i=1}^{k} (n_i - 1) = n - k, \text{ squaring both sides}$$

$$\Rightarrow \quad [(n_1-1)+(n_2-1)+ \ldots\ldots + (n_k-1)]^2 = (n-k)^2$$

$$\Rightarrow \quad (n_1-1)^2 + (n_2-1)^2 + \ldots\ldots + (n_k-1)^2 + 2[n_1-1)(n_2-1) + \ldots\ldots + (n_k-1)] = (n-k)^2$$

$$\Rightarrow \quad {n_1}^2 + {n_2}^2 + \ldots\ldots + {n_k}^2 - 2(n_1 + n_2 + \ldots\ldots + n_k) + k + \text{Non-negative terms} = (n-k)^2$$

$$\Rightarrow \quad \sum_{i=1}^{k} n_i^2 - 2\sum_{i=1}^{k} n_i + k + \text{Non-negative terms} = (n-\text{k})^2$$

$$\Rightarrow \quad \sum_{i=1}^{k} n_i^2 - 2n + k + \text{Non-negative terms} = n^2 + k^2 - 2nk$$

$$\Rightarrow \quad \sum_{i=1}^{k} n_i^2 + \text{Non-negative terms} = n^2 + k^2 - 2kn + 2n - k$$

$$= n^2 - 2n(k-1)(2n-k) - n$$

$$\Rightarrow \quad \sum_{i=1}^{k} n_i^2 \le n^2 - (k-1)(2n-k)$$

$\therefore$ From (1), we have

Maximum number of edges in the graph G.

$$= \frac{1}{2}[n^2 - (k-1)(2n-k) - n]$$

$$= \frac{1}{2}[n^2 - 2nk + k^2 + n - k]$$

$$= \frac{1}{2}[(n-k)^2 + (n-k)] = \frac{1}{2}(n-k)(n-k+1)$$

Hence the theorem.

Example 14. *Give an example of a graph with six vertices that has exactly two cut points.*

Sol. The graph with exactly two cut points is shown in Fig. 11.48.

Scanned with CamScanner

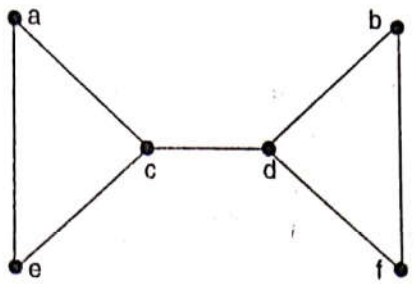


Fig. 11.48

The two cut points in this graph are c and d. The other vertices are not cut points since removal of them does not divide the graph into more than one connected component.

Example 15. *Give an example of a graph with six vertices that has no cut points.*

Sol. The graph with no cut points is shown in Fig. 11.49. This graph does not contain any cut point since removal of any vertex and the edges incident on it does not divide it into more than one connected components.

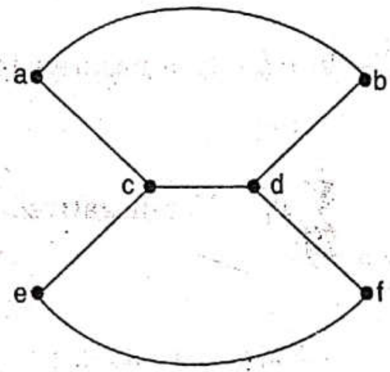


Fig. 11.49

Example 16. *Consider the graph shown in Fig. 11.50. Determine the subgraphs*
(*i*) $G - v_1$ (*ii*) $G - v_3$ (*iii*) $G - v_5$.

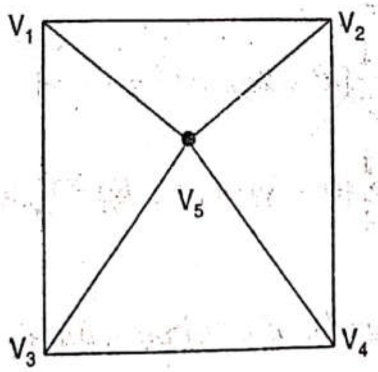


Fig. 11.50

Sol. (*i*) The subgraph $G - v_1$ is shown in Fig. 11.51.
(*ii*) The subgraph $G - v_3$ is as shown in Fig. 11.52.
(*iii*) The subgraph $G - v_5$ is as shown in Fig. 11.53.

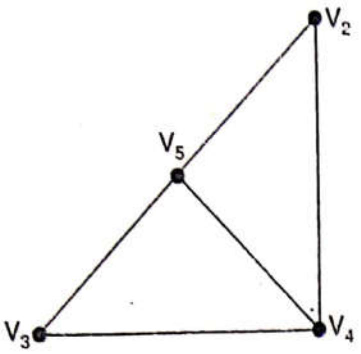


Fig. 11.51

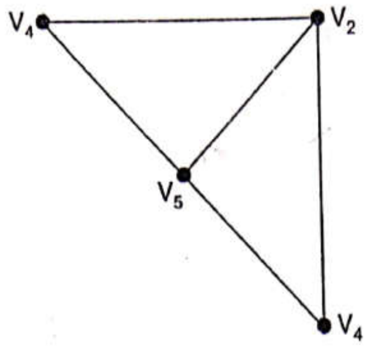


Fig. 11.52

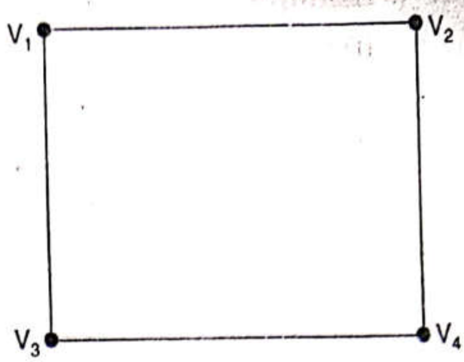


Fig. 11.53

Example 17. *Consider the graph G shown in Fig. 11.54. Determine all the cut points of G.*

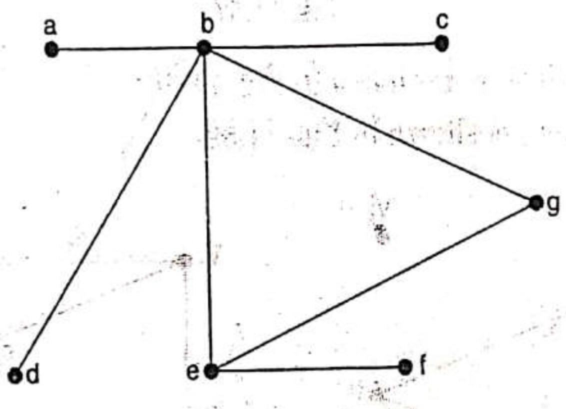


Fig. 11.54

Sol. (*a*) The vertex *b* is cut point for G. Since, G–*b* has more than one connected components as shown in Fig. 11.55.

(*b*) The vertex *e* is also a cut point for G. Since G–*e* has more than one connected components as shown in Fig. 11.56.

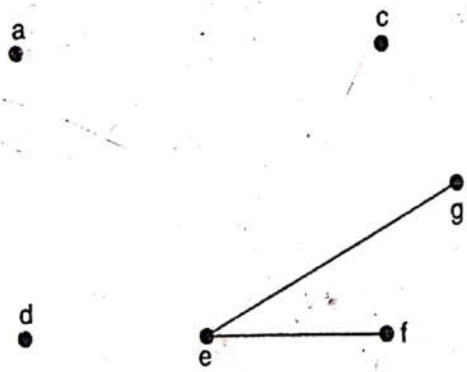


Fig. 11.55

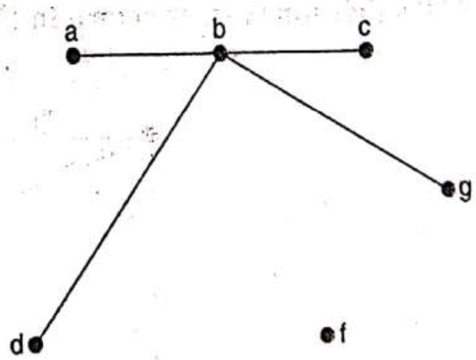


Fig. 11.56

11.20. BRIDGE (Cut Edges)

Consider a graph G = (V, E). A bridge for a graph G, is an edge *e* such that G–*e* has more connected components than G or disconnected.

Example 18. *Consider the graph shown in Fig. 11.57. Determine the subgraphs*
(i) $G - e_1$ *(ii)* $G - e_3$ *(iii)* $G - e_4$.

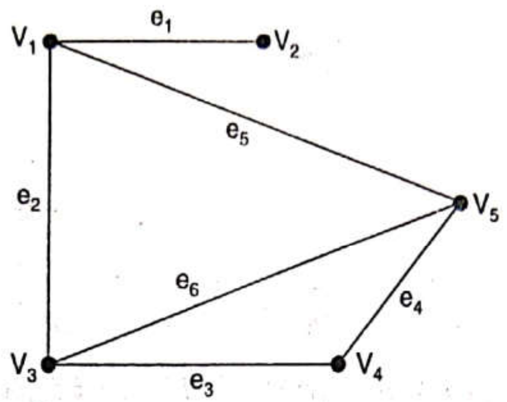


Fig. 11.57

Sol. *(i)* The subgraph $G-e_1$ is shown in Fig. 11.58.
(ii) The Subgraph $G-e_3$ is shown in Fig. 11.59.

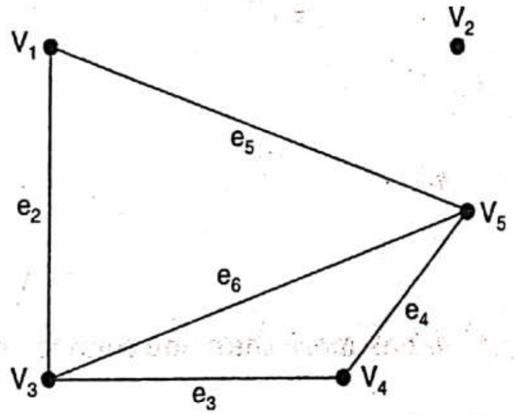


Fig. 11.58

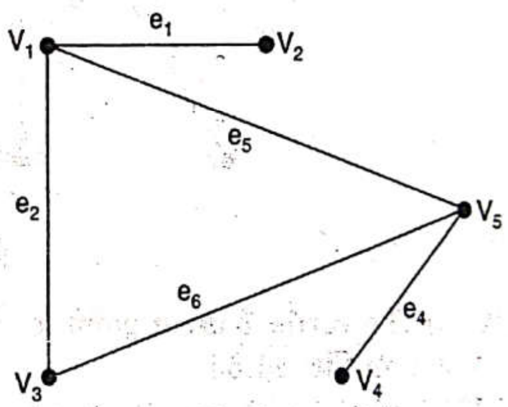


Fig. 11.59

(iii) The subgraph $G-e_4$ is shown in Fig. 11.60.

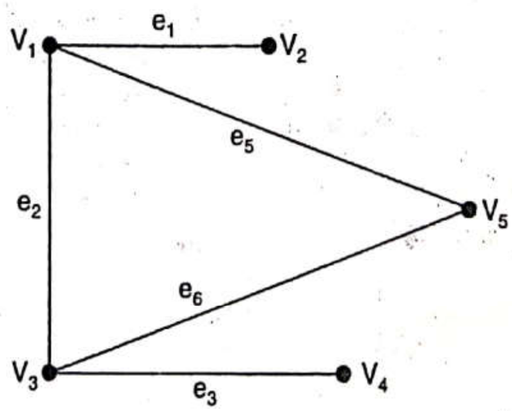


Fig. 11.60

Example 19. *Consider the graph G shown in Fig. 11.61. Determine all the bridges of G.*

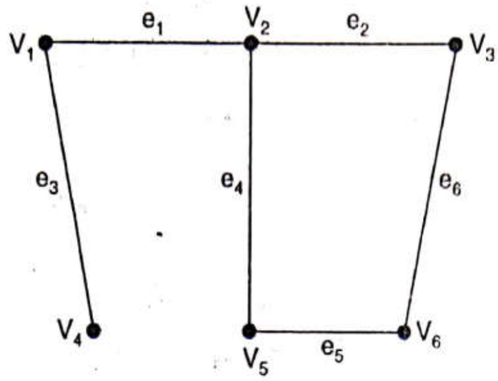


Fig. 11.61

Sol. (*a*) The edge e_1 is a bridge for G. Since G–e_1 has more than one connected components as shown in Fig. 11.62.

(*b*) The edge e_3 is a bridge for G. Since G–e_3 has more than one connected components as shown in Fig. 11.63.

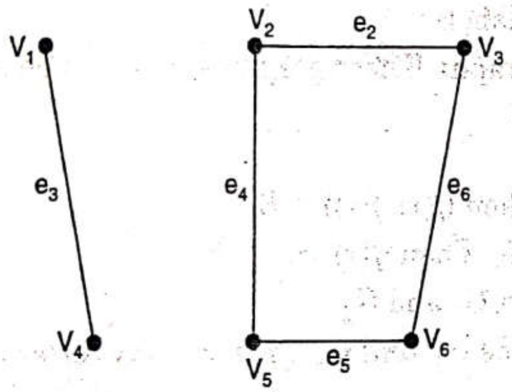


Fig. 11.62

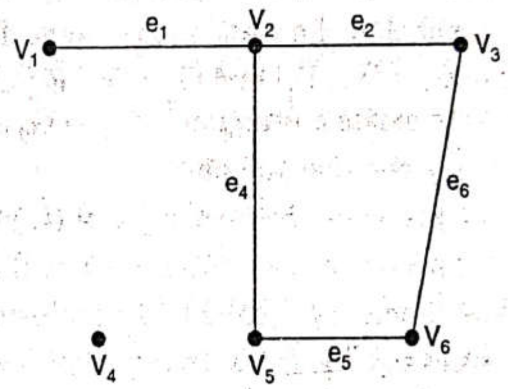


Fig. 11.63

Example 20. *Give an example of a graph with six vertices for the following :*

(*a*) *that has exactly two bridges.*

(*b*) *that has no bridges.*

Sol. (*a*) The graph that has exactly two bridges is shown in Fig. 11.64.

The two bridges are e_1 and e_2.

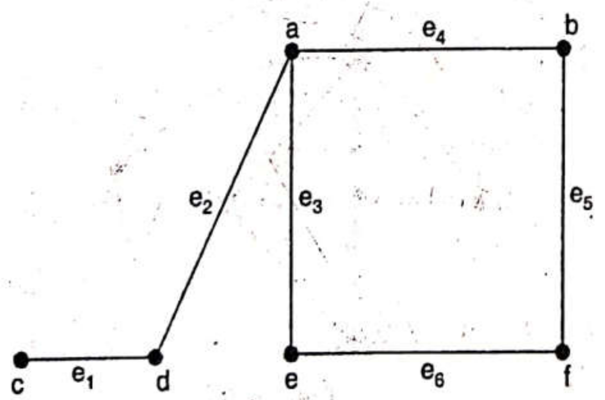


Fig. 11.64

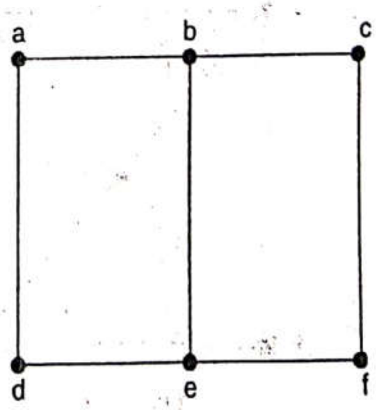


Fig. 11.65

(*b*) The graph with no bridges is shown in Fig. 11.65. This graph does not contain any bridge since removal of any edge does not divide it into more than one connected components.

Example 21. *Draw a graph whose every edge is a bridge.*

Sol. The graph shown in Fig. 11.66 is a graph whose every edge is a bridge because if any edge is removed from the graph *h*, we got two components or a disconnected graph.

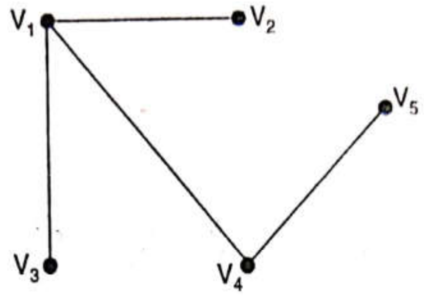


Fig. 11.66

11.21. ISOMORPHIC GRAPHS

Two graphs G_1 and G_2 are called isomorphic graphs if there is a one-to-one correspondence between their vertices and between their edges *i.e.*, the graphs have identical representation except that the vertices may have different labels.

Let $G_1 = [V_1, E_1]$ and $G_2 = [V_2, E_2]$ are two graphs. These graphs are said to be isomorphic if there exists a function $f : G_1 \to G_2$ such that

(*i*) *f* is one-one and onto

(*ii*) *f* preserves adjancies *i.e.*, If $(x, y) \in E_1$, Then $(f(x), f(y)) \in E_2$

(*iii*) *f* preserves non-adjancies *i.e.*, If $(x, y) \notin E_1$, Then $(f(x), f(y)) \notin E_2$

The function *f* is called isomorphism between G_1 and G_2.

Theorem VI. *If f is an isomorphism of graphs G_1 and G_2, then, for any vertex v in G_1, the degrees of v and f(v) are equal.*

Proof. Let $\deg(v) = m$: we can find exactly *m* vertices $v_1, v_2, ..., v_m$ adjacent to *v*.

Since *f* is an isomorphism, $f(v_1), f(v_2), ... f(v_m)$ are adjacent to $f(v)$.

Also as there is no other vertex adjacent to the vertex *v* in G_1, there is no other vertex adjacent to $f(v)$ in $G_2 \therefore \deg f(v) = m$. Hence the Theorem.

For example, consider the following graphs shown in Fig. 11.67 and Fig. 11.68. They are isomorphic graphs.(use above theorem)

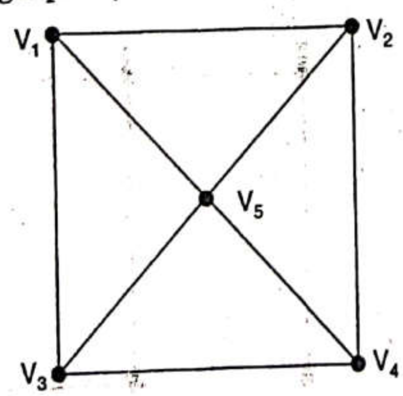


Fig. 11.67

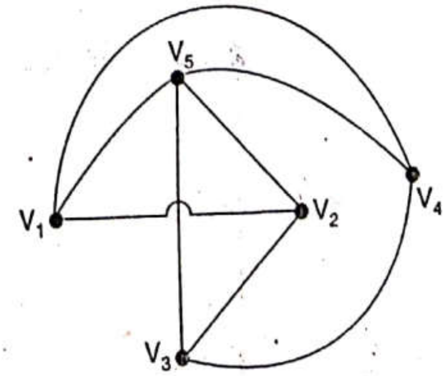


Fig. 11.68

Example 22. *Show that the graphs shown in Fig. 11.69 and Fig. 11.70 are isomorphic.*

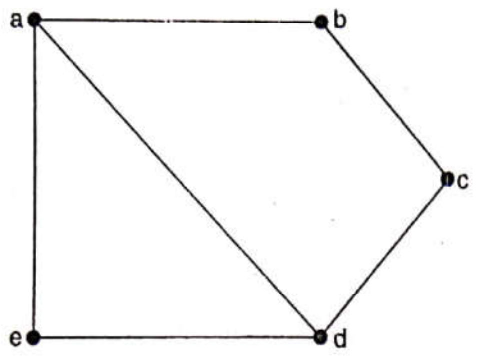


Fig. 11.69

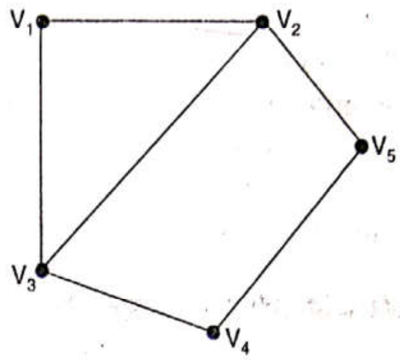


Fig. 11.70

Sol. Compare the degrees of vertices of two graphs and find the vertices from both the graphs having same degrees and make the pairs of the vertices in decreasing order of degree. If both the graphs contain vertices having same degree, then they are isomorphic otherwise not. The pairs of vertices in decreasing order of degree are as follows :

$$d(a) \leftrightarrow d(v_2),\ d(d) \leftrightarrow d(v_3),\ d(b) \leftrightarrow d(v_1),\ d(e) \leftrightarrow d(v_4),\ d(c) \leftrightarrow d(v_5).$$

Since, both the graphs contain vertices having same degrees, hence they are isomorphic.

Example 23. *Show that the graphs shown in Fig. 11.71 and Fig. 11.72 are not isomorphic.*

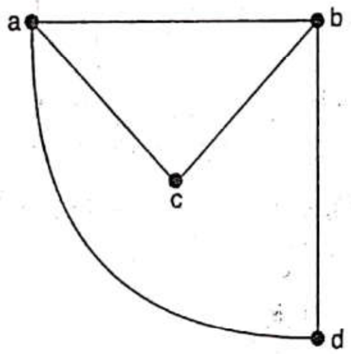


Fig. 11.71

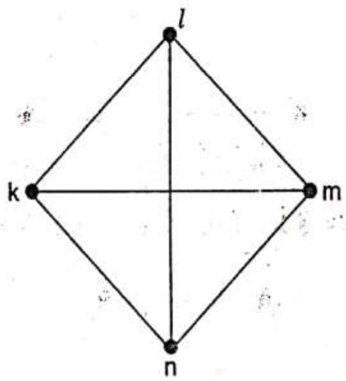


Fig. 11.72

Sol. The graphs are not isomorphic because the vertices of the graph shown in Fig. 11.72 is having degree 3 but the graph shown in Fig. 11.71 contains two vertices having degree less than three.

Example 23. (*a*) *List any five properties of a graph which are invariant under graph isomorphism.* (P.T.U., B.Tech. May 2013)

Sol. The five properties are as follows:

1. Order : The number of vertices.

2. Size : The number of edges

3. Vertex Connectivity: The smallest number of vertices whose removal disconnects the graph.

4. Edge Connectivity: The smallest number of edges whose removal disconnects the graph.

5. Vertex Covering Number: The minimal number of vertices needed to cover all edges.

6. Edge Covering Number: The minimal number of edges needed to cover all vertices.

11.22. ORDER AND SIZE OF GRAPH

Let G be a graph. The number of vertices in a graph G is called order of G. The number of edges in a graph G is called size of G.

For example, Consider the graph G shown in Fig. 11.70

Here order of G = 3

size of G = 4

($\because$ number of edges in G = 4.)

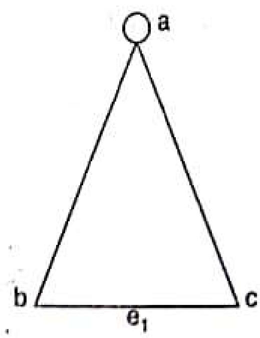


Fig. 11.73

11.23. HOMEOMORPHIC GRAPHS

Two graphs G_1 and G_2 are called homeomorphic graphs if G_2 can be obtained from G_1 by a sequence of subdivisions of the edges of G_1. In other words, we can introduce vertices of degree two in any edge of graph G_1. For example,

Consider the graph shown in Fig. 11.74 and Fig. 11.75. They are homeomorphic graphs.

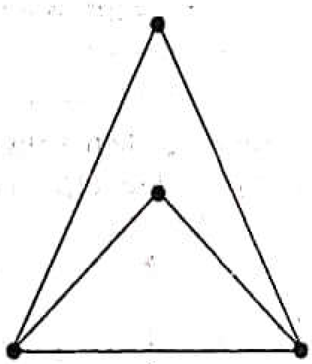


Fig. 11.74. G_1.

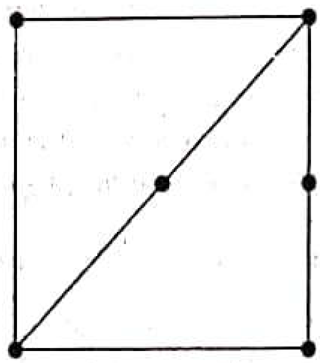


Fig. 11.75. G_2.

Example 24. *Show that the graphs shown in Figs. 11.76 and 11.77 are homeomorphic.*

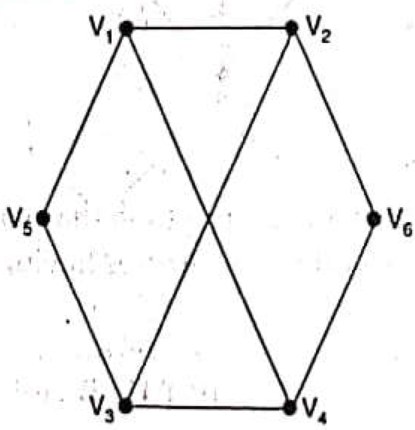


Fig. 11.76. G_1.

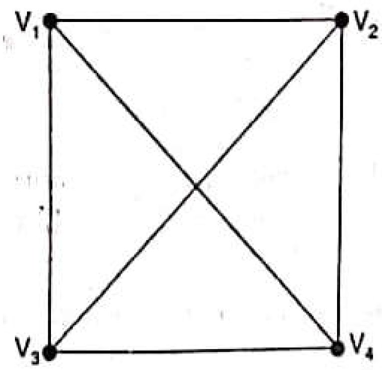


Fig. 11.77. G_2.

Sol. The two graphs are homeomorphic because G_1 can be obtained from G_2 by introducing vertices of degree 2 on edges (V_1, V_3) and (V_2, V_4).

Example 25. *Consider the directed graph G = (V, E) as shown in Fig. 11.78. Determine the vertex set and edge set of graph G.*

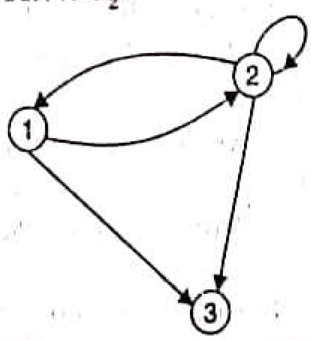


Fig. 11.78

Sol. The vertex and edge set of graph G = (V, E) is as follows

$$G = \{\{1, 2, 3\}, \{(1, 2), (2, 1), (2, 2), (2, 3), (1, 3)\}\}.$$

Example 26. *Let* $G = \{\{a, b, c, d\}\ \{(a, b), (b, c), (c, c), (d, d), (d, a)\}\}$. *Draw the graph G.*

Sol. The graph of G = (V, E) is shown in Fig. 11.79.

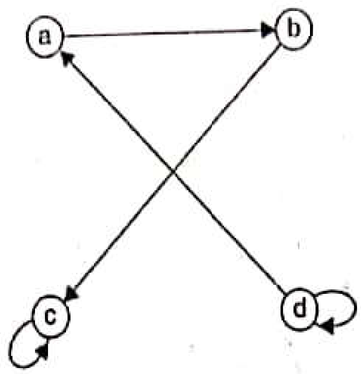


Fig. 11.79

Example 27. *Consider the directed graph shown in Fig. 11.80. Determine the indegree and outdegree of each of vertices of the graph.*

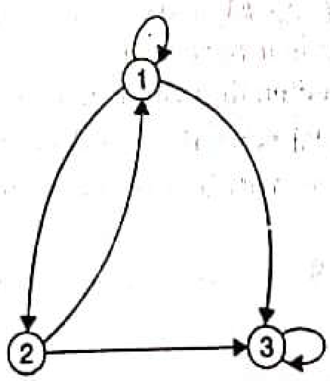


Fig. 11.80

Sol. The indegree of digraph is indeg (1) = 2, indeg (2) = 1, indeg (3) = 3

The outdegree of digraph is outdeg (1) = 3, outdeg (2) = 2, outdeg (3) = 1.

11.24. WEAKLY CONNECTED

(P.T.U., B.Tech. Dec. 2005)

A directed graph is called weakly connected if its undirected graph is connected *i.e.*, the graph obtained after neglecting the direction.

11.25. UNILATERALLY CONNECTED DIGRAPH

A directed graph is called unilaterally connected if there is a directed path from any node *u* to *v* or vice-versa, for any pair of nodes of the graph.

11.26. STRONGLY CONNECTED DIGRAPH

A directed graph is called strongly connected if there is a directed path from any node *u* to *v* and vice-versa, for any pair of nodes of the graph.

11.27. DISCONNECTED DIGRAPH

A directed graph is called disconnected if its undirected graph is disconnected.

Example 28. *Consider the graphs shown in Figs. 11.81, 11.82 and 11.83 which of the graphs are*

(i) *Unilaterally connected digraph* (ii) *Weakly connected digraph*

(iii) *Strongly connected digraph*

(iv) *Disconnected digraph (also find its connected components).*

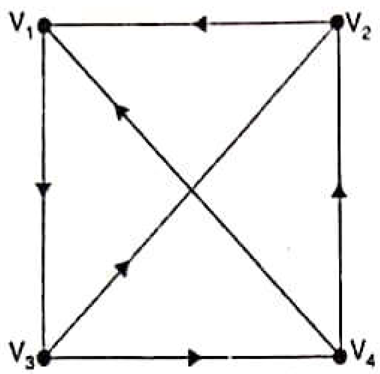


Fig. 11.81

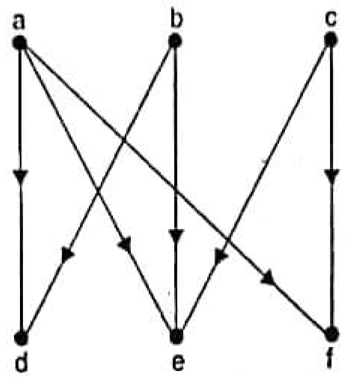


Fig. 11.82

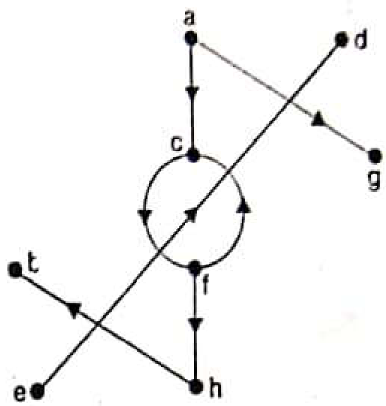


Fig. 11.83

Sol. The graph shown in Fig. 11.81 is strongly connected because there is a path from every vertex u to v and also there is a path from v to u. It is also weakly and unilaterally connected because a strongly connected digraph is both weakly and unilaterally connected.

The graph shown in Fig. 11.82 is weakly connected but not unilaterally connected because there is no directed path from vertex a to b or a to c etc. but its undirected graph is connected.

The graph shown in Fig. 11.83 is a disconnected graph. The components of this graph are $\{g, a, c, f, h, b\}$ and $\{e, d\}$.

11.28. DIRECTED COMPLETE GRAPH

A directed complete graph G = (V, E) on n vertices is a graph in which each vertex is connected to every other vertex by an arrow. It is denoted by K_n.

Example 29. *Draw directed complete graphs K_3 and K_5.*

Sol. Place the number of vertices at appropriate place and then draw an arrow from each vertex to every other vertex as shown in Figs. 11.84 and 11.85.

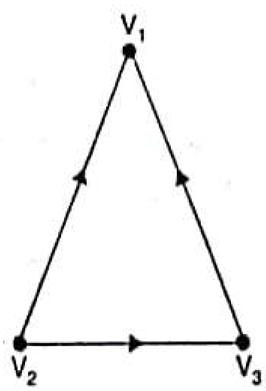


Fig. 11.84. K_3.

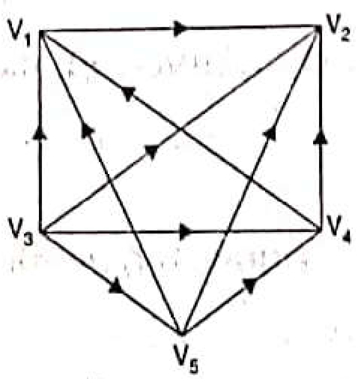
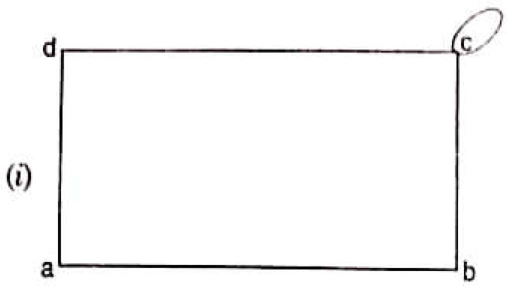


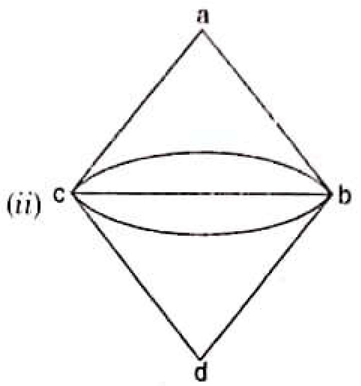
Fig. 11.85. K_5.

TEST YOUR KNOWLEDGE 11.1

1. (*a*) If V = {1, 2, 3, 4, 5} and E = {(1, 2), (2, 3), (3, 3), (3, 4), (4, 5)}. Find the number of edges and size of graph G = (V, E)

   (*b*) Find the order and size of the graph G shown in the figure below :





   (*c*) What is the difference between directed and undirected graph?

   (*d*) Differentiate between paths and circuits. (*P.T.U., B. Tech. May 2007*)

2. (*a*) A graph G has 16 edges and all vertices of G are of degree 2. Find the number of vertices.

   (*b*) A graph G has 21 edges, 3 vertices of degree 4 and other vertices are of degree 3. Find the number of vertices in G.

   (*c*) A graph G has 5 vertices, 2 of degree 3 and 3 of degree 2. Find the number of edges.

3. (*a*) How many nodes (vertices) are required to construct a graph with exactly 6 edges in which each node is of degree 2?

   (*b*) Show that there does not exist a graph with 5 vertices with degrees 1, 3, 4, 2, 3 respectively.

   (*c*) Can there be a graph with 8 vertices and 29 edges?

   (*d*) How many vertices are there is a graph with 10 edges if each vertex has degree 2?

   (*e*) Does there exist a graph with two vertices each of degree 4? If so, draw it.

4. (*a*) Draw a simple graph with 3 vertices

   (*b*) Draw a simple graph with 4 vertices

   (*c*) Give an example for

   (*i*) simple graph (*ii*) non-simple graph (*iii*) Multigraph, with suitable diagrams

5. Show that the maximum number of edges in a graph with n vertices and no multiple edges are $\frac{n(n-1)}{2}$.

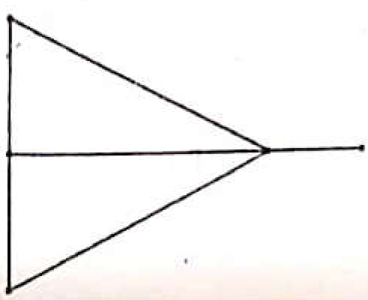
6. Prove Handshaking theorem which states that the sum of degree of the vertices of a graph is equal to twice the number of edges.

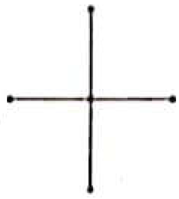
7. (*a*) Determine whether it is possible to construct a graph with 12 edges such that 2 of the vertices have degree 3 and the remaining vertices have degree 4.

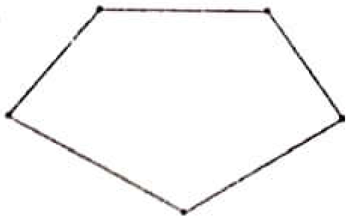
   (*b*) Give an example of each of multigraph, weighted graph, simple graph, non-simple graph, directed graph with suitable diagrams. (*P.T.U., B.Tech. May 2005*)

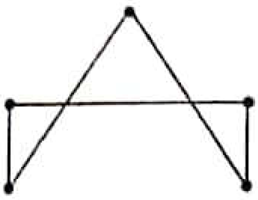
8. (*a*) Which of the graphs in the given figures are isomorphic ?

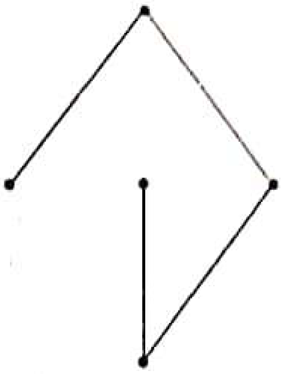
(*i*)

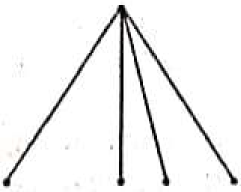
(*ii*)


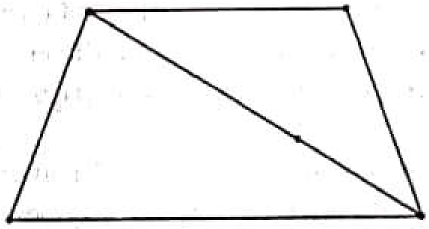
(iii)


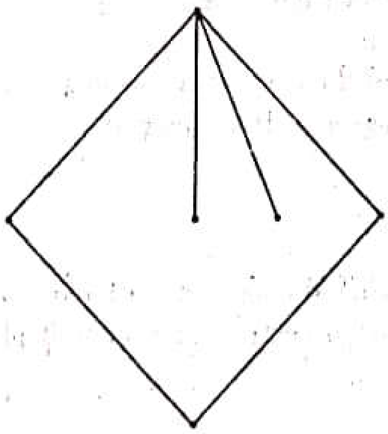
(iv)


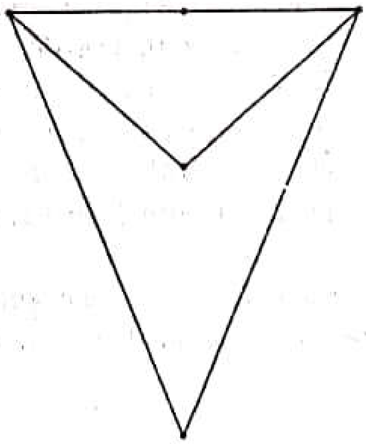
(v)


(vi)


(vii)


(viii)


(ix)


(x)


(b) Show that the following graphs are not isomorphic.

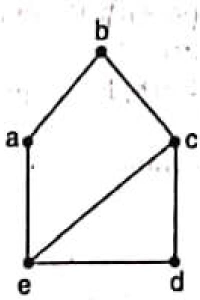


Fig. I

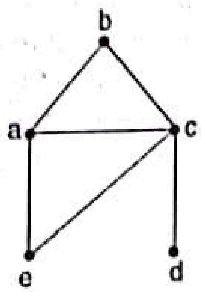


Fig. II

(c) Determine whether the following graphs are isomorphic or not.

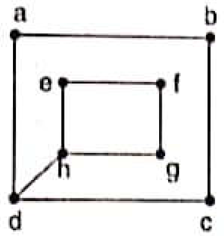


Fig. I

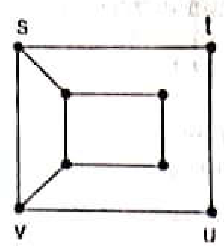


Fig. II

(d) Determine whether the following graphs are isomorphic or not

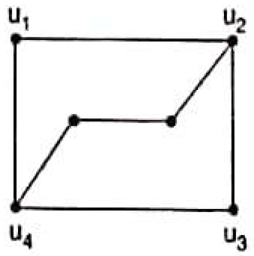


Fig. I

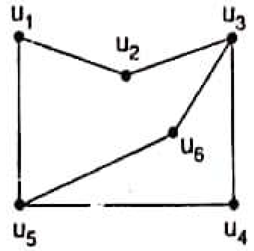


Fig. II

9. (a) Draw the graph $\overline{G}$ (complement of G) of the graph shown below. Also show that G and $\overline{G}$ are isomorphic. (*P.T.U., B.Tech. Dec. 2002*)

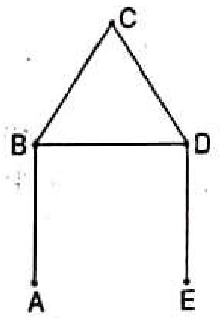


Fig. I

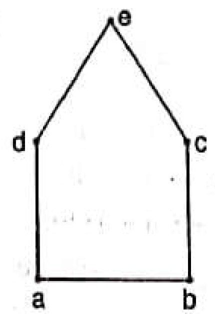


Fig. II

(b) Draw the complement of the graph shown in the fig. II

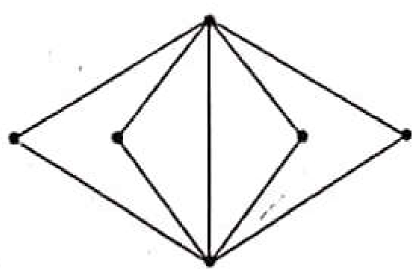
10. Draw (a) a graph in which no edge is a cut edge.

(b) a graph in which every edge is a cut edge.

(c) only one cut vertex

11. Find k, if a k-regular graph with 8 vertices has 12 edges. Also draw k-regular graph.

12. Consider the graph G shown below:



(a) Is G simple?

(b) What is order and size of incidence matrix for G?

(c) Find minimum and maximum degree for G.

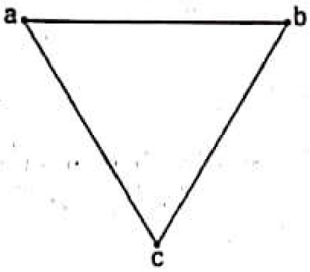
13. Prove that in a simple graph with n vertices, each vertex has maximum degree $(n - 1)$.
14. Prove that maximum degree of edges in a graph G with n vertices and no multiple edges are $\dfrac{n(n-1)}{2}$.
15. Suppose a directed graph has m vertices. Show that if there is a path from vertex u to v, then there is a path p' of length $m - 1$ or less from u to v.
16. Construct a graph that has six vertices and five edges but is not a tree

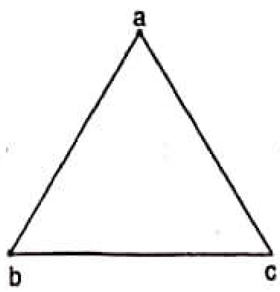
(P.T.U., B.Tech. Dec. 2012)

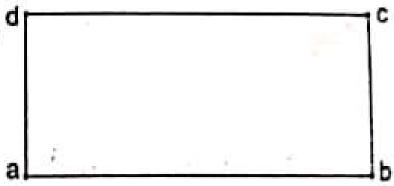
Answers

1. (*a*) Number of edges = 4, Size of graph = 5 (*b*) (*i*) Order = 4, Size = 6, (*ii*) Order = 4, Size = 7
2. (*a*) 6 (*b*) 13
3. (*a*) 6 (*c*) No

   (*d*) 10. (*e*) 

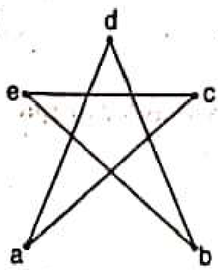
4. (*a*) 

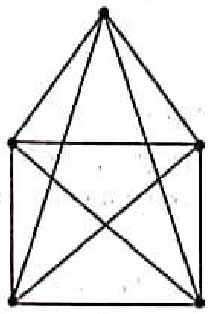
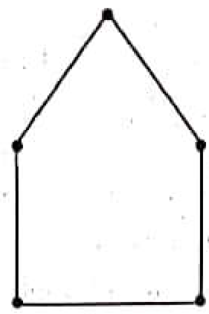
or 

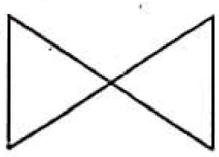
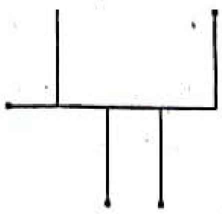
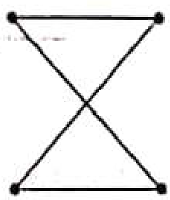
(*b*) 

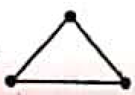
7. (*a*) Not possible
8. (*a*) (*i*) and (*vi*) are isomorphic (*ii*) and (*ix*) are isomorphic
   (*iii*) and (*vii*) are isomorphic (*iv*) and (*v*) are isomorphic
   (*viii*) and (*ix*) are isomorphic

   (*c*) Not isomorphic (*d*) Isomorphic

9. (*b*) 

, Since $\overline{G} = K_5 - G =$  $-$ 

10. (*a*)  (*b*)  (*c*) 

11. 3, $k_3 =$ 
12. (*a*) Yes (*b*) 6, 9 (*c*) 5, 2

Hints

2. (*a*) Let $v_1, v_2 \dots v_n$ be n vertices such that deg $(v_i) = 2$, $1 \le i \le n$.

Since sum of degree of all vertices is equal to twice the number of edges, *i.e.*,

$$\therefore \quad \Sigma \deg(v_i) = 2 \text{ (Number of edges)}$$
$$\Rightarrow \quad \deg(v_1) + \deg(v_2) + \dots + \deg(v_n) = 2 \times 16$$
$$\Rightarrow \quad 2 + 2 + \dots + 2 \text{ (}n\text{ times)} = 32$$
$$\Rightarrow \quad 2n = 32$$
$$\Rightarrow \quad n = 16$$

(*b*) Let n be the number of vertices in G.

Since, sum of degree of all vertices is equal to twice the number of edges *i.e.*,

$$\sum_{i=1}^{n} \deg(v_i) = 2 \times \text{number of edges}$$
$$\Rightarrow \quad \deg(v_1) + \deg(v_2) + \dots + \deg(v_n) = 2 \times 21$$
$$\Rightarrow \quad \underbrace{(4 + 4 + 4)}_{3 \text{ times}} + \underbrace{(3 + 3 + \dots + 3)}_{(n-3) \text{ times}} = 42$$
$$\Rightarrow \quad 12 + 3(n - 3) = 42$$
$$\Rightarrow \quad 3(n - 3) = 30$$
$$\Rightarrow \quad n = 13.$$

3. (*a*) Let n be the number of vertices

Since, $\Sigma \deg(v_i) = 2$ (number of edges)

$$\Rightarrow \quad \deg(v_1) + \deg(v_2) + \dots + \deg(v_n) = 2 \times 6$$
$$\Rightarrow \quad 2 + 2 + \dots + 2 \text{ (}n\text{ times)} = 12$$
$$\Rightarrow \quad 2n = 12$$
$$\Rightarrow \quad n = 6$$

(*b*) Using, $\Sigma \deg(v_i) = 2 \times$ number of edges

$$\Rightarrow \quad 1 + 3 + 4 + 2 + 3 = 2 \text{ (number of edges)}$$
$$\Rightarrow \quad \text{number of edges} = \frac{13}{2}, \text{ which is not possible.}$$

(*c*) $n = 8$, $e = 29$. Maximum number of edges $= \dfrac{n(n-1)}{2} = \dfrac{8.7}{2} = 28$

(*d*) e = number of edges = 10. Sum of degree of all vertices = $2n$

Also $\quad 2e = 2n \quad \Rightarrow \quad n = e = 10$

(*e*) Here $n = 2$. Let e be the number of edges. Each vertex is of degree 4.

$\therefore$ Sum of degree of two vertices = 4 + 4 = 8

$$\Rightarrow \quad 2e = 8 \quad \Rightarrow \quad e = 4$$

5. Let n be the number of vertices of a graph G. Then degree of each vertices $\le n - 1$

$$\Rightarrow \quad \text{sum of degree of } n \text{ vertices} \le n(n-1)$$
$$\Rightarrow \quad 2 \times \text{(number of edges)} \le n(n-1)$$
$$\Rightarrow \quad \text{number of edges} \le \frac{n(n-1)}{2}.$$

7. Let n be the number of vertices, then

$$\sum_{i=1}^{n} \deg(v_i) = 2 \times \text{number of edges}$$
$$\Rightarrow \quad (v_1 + v_2) + (v_3 + v_4 + \dots (n-2)) = 2 \times 12$$
$$\Rightarrow \quad (3 + 3) + (4 + 4 + \dots (n-2)) = 24$$

$$\Rightarrow \quad 6 + 4(n-2) = 24$$
$$\Rightarrow \quad 4(n-2) = 18$$
$$\Rightarrow \quad n - 2 = \frac{18}{4} = \frac{9}{2}$$
$$\Rightarrow \quad n = \frac{9}{2} + 2 = \frac{13}{2}, \text{ not possible.}$$

8. (*b*) Both graphs have five vertices and six edges. However the graph (Fig. II) has a vertex of degree 1 namely, *e*, but the graph (Fig. I) has no vertex of degree 1. Hence the given graphs are not isomorphic.

(*c*) Both graphs (Fig. I and Fig. II) have 8 vertices and 10 edges. Also both graphs have four vertices of degree 2 and four vertices of degree 3. But deg(a) = 2, it must correspond to either x, y, z, t or 4 in Fig. II. All these vertices in Fig. II is adjacent to another vertex of degree 2. But this is not true for the vertex a in Fig. I. Hence they cannot be isomorphic.

11. Given, V = 8, E = 12

$\therefore$ Sum of degree of all vertices = 2 × 12 = 24

$$\Rightarrow \text{ degree of each vertex} = \frac{24}{8} = 3$$

13. A simple graph is a graph without parallel edges or self loops.

If G = [V, E] is a simple graph with only one vertex, then the number of edges in G is zero

$\therefore$ maximum degree of a vertex in G = 1 – 1 = (0)

If G = [V, E] be a simple graph with 2 vertices, then the number of edges in G is 1 = 2 – 1 and degree of each vertex is (2 – 1).

....

If G = [V, E] be a simple graph with n vertices, then maximum degree of each vertex = $n - 1$.

14. Let G be a simple graph with n vertices, then degree of each vertex in G is $\leq n - 1$

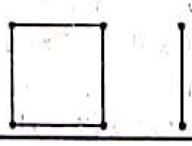
$\therefore$ sum of degrees of n vertices in G $\leq n(n-1)$

$$\Rightarrow \quad 2e \leq n(n-1) \;\Rightarrow\; e \leq \frac{n(n-1)}{2} \text{ where } e \text{ is the number of edges in G.}$$

15. Suppose there is a path from the vertex u to the vertex v. Let $[u_1, u_2, ..., u_i, u_{i+1}, ... v]$ be the sequence of vertices that the path meets when it is traced from u to v. If there are k edges in the path. There are $k + 1$ vertices in the sequence. Choose a number $k > m - 1$ such that the vertex, say, u_l, should appear more than once in the sequence, that is, $(u, ... u_i, ... u_e, ... u_l, ... v)$.

Deleting the edges in the path that leads u_e back to u_l, we have a path from u to v that has fewer edges than the original one. Repeating this process until we have a path that has $(m - 1)$ or fewer edges. Hence we have proved that in a directed graph with m vertices, if there is a path from vertex u to v, then there is also a path from u to v of length $m - 1$ or less edges.

16. The required graph is



11.29. LABELLED GRAPHS

A graph G = (V, E) is called a labelled graph if its edges are labelled with some name or data. So, we can write these labells in place of an ordered pair in the edge set. For *e.g.*,