$$\Rightarrow \quad 6 + 4(n-2) = 24$$

$$\Rightarrow \quad 4(n-2) = 18$$

$$\Rightarrow \quad n - 2 = \frac{18}{4} = \frac{9}{2}$$

$$\Rightarrow \quad n = \frac{9}{2} + 2 = \frac{13}{2}, \text{ not possible.}$$

8. (*b*) Both graphs have five vertices and six edges. However the graph (Fig. II) has a vertex of degree 1 namely, *e*, but the graph (Fig. I) has no vertex of degree 1. Hence the given graphs are not isomorphic.

(*c*) Both graphs (Fig. I and Fig. II) have 8 vertices and 10 edges. Also both graphs have four vertices of degree 2 and four vertices of degree 3. But deg(a) = 2, it must correspond to either x, y, z, *t* or 4 in Fig. II. All these vertices in Fig. II is adjacent to another vertex of degree 2. But this is not true for the vertex *a* in Fig. I. Hence they cannot be isomorphic.

11. Given V = 8, E = 12

$\therefore$ Sum of degree of all vertices = $2 \times 12 = 24$

$\Rightarrow$ degree of each vertex = $\frac{24}{8} = 3$

13. A simple graph is a graph without parallel edges or self loops.

If G = [V, E] is a simple graph with only one vertex, then the number of edges in G is zero

$\therefore$ maximum degree of a vertex in G = 1 – 1 = (0)

If G = [V, E] be a simple graph with 2 vertices, then the number of edges in G is 1 = 2 – 1 and degree of each vertex is (2 – 1)

....

If G = [V, E] be a simple graph with *n* vertices, then maximum degree of each vertex = $n - 1$.

14. Let G be a simple graph with *n* vertices, then degree of each vertex in G is $\leq n - 1$

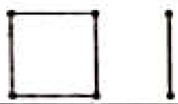
$\therefore$ sum of degrees of *n* vertices in G $\leq n(n-1)$

$$\Rightarrow \quad 2e \leq n(n-1) \quad \Rightarrow \quad e \leq \frac{n(n-1)}{2} \text{ where } e \text{ is the number of edges in G.}$$

15. Suppose there is a path from the vertex *u* to the vertex *v*. Let $[u_1, u_2, \ldots, u_i, u_{i+1}, \ldots v]$ be the sequence of vertices that the path meets when it is traced from *u* to *v*. If there are *k* edges in the path. There are $k + 1$ vertices in the sequence. Choose a number $k > m - 1$ such that the vertex, say, u_i, should appear more than once in the sequence, that is, $(u, \ldots u_i, \ldots u_e, \ldots u_i, \ldots v)$.

Deleting the edges in the path that leads u_e back to u_i, we have a path from *u* to *v* that has fewer edges than the original one. Repeating this process until we have a path that has $(m - 1)$ or fewer edges. Hence we have proved that in a directed graph with *m* vertices, if there is a path from vertex *u* to *v*, then there is also a path from *u* to *v* of length $m - 1$ or less edges.

16. The required graph is



11.29. LABELLED GRAPHS

A graph G = (V, E) is called a labelled graph if its edges are labelled with some name or data. So, we can write these labells in place of an ordered pair in the edge set. For *e.g.*,

The graphs shown in Figs. 11.86 and 11.87 are labelled graphs.

G = {{1, 2, 3, 4, 5}, $\{e_1, e_2, e_3, e_4, e_5\}\}$. G = {{*a*, *b*, *c*, *d*}, $\{e_1, e_2, e_3, e_4\}\}$.

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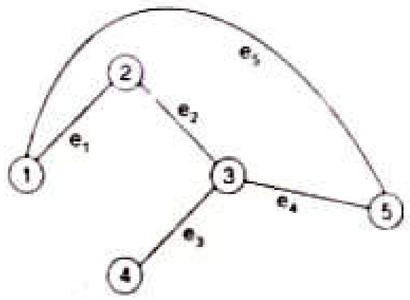

Fig. 11.86. Undirected Labelled Graph.

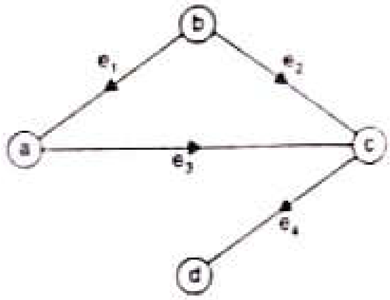

Fig. 11.87. Directed Labelled Graph.

11.30. WEIGHTED GRAPHS

(P.T.U., B.Tech. May 2007, Dec. 2006, May 2005)

A graph G = (V, E) is called a weighted graph if each edge of graph G is assigned a positive number w called the weight of the edge e. For example,

The graph shown in Figs. 11.88 and 11.89 is a weighted graph.

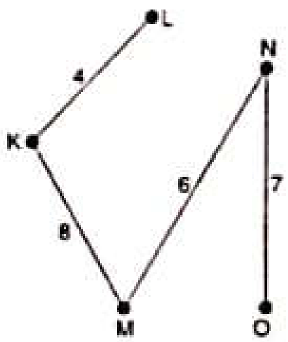

Fig. 11.88

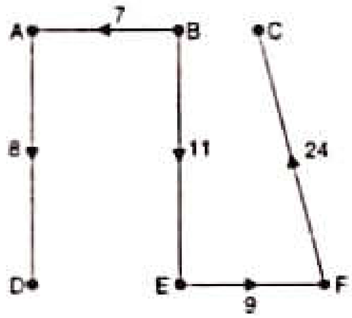

Fig. 11.89

11.31. MULTIPLE EDGES

Two edges e_1 and e_1' which are distinct are said to be multiple edges if they connect the end points *i.e.*, if $e_1 = (u, v)$ and $e_1' = (u, v)$ then e_1 and e_1' are multiple edges.

11.32. MULTIGRAPH

(P.T.U., B.Tech. May 2007, Dec. 2006, May 2005)

A multigraph G = (V, E) consists of a set of vertices V and a set of edges E such that edge set E may contain multiple edges and self loops. For example,

Consider the following graph shown in Fig. 11.90.

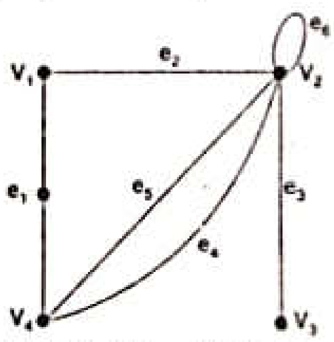

Fig. 11.90. Undirected Multigraph.

In the above Fig. 11.91, e_4 and e_5 are multiple edges, e_6 is a self-loop.

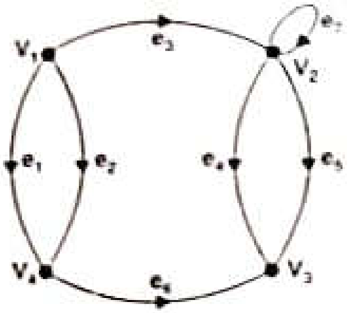

Fig. 11.91. Directed Multigraph.

In the graph shown in Fig. 11.91, the edges e_1, e_2 and e_4, e_5 are multiple edges e_7 is a loop.

11.33. TRAVERSABLE MULTIGRAPHS

Consider a multigraph G = (V, E). If the multigraph G consists of a path which includes all vertices and whose edge list contains each edge of graph exactly once. Then, the multigraph G is called a traversable multigraph.

The sufficient and necessary condition for a multigraph to be traversable is that it should be connected and have either zero or two vertices of odd degree.

Consider the multigraph shown in Fig. 11.92.

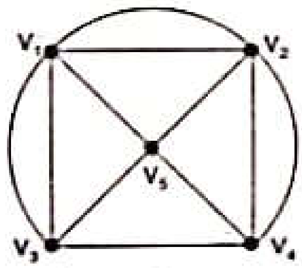

Fig. 11.92

The multigraph has three even degree vertices *i.e.*, V_3, V_4 and V_5 and two odd degree vertices *i.e.*, V_1 and V_2. Hence it is a traversable multigraph.

11.34. REPRESENTATION OF GRAPHS

There are two important ways to represent a graph G with the matrices *i.e.*,

I. Adjacency matrix representation.

II. Incidence matrix representation.

(a) Representation of Undirected Graph

(i) Adjacency matrix representation. If an undirected graph G consists of n vertices, then the adjacency matrix of graph is an $n \times n$ matrix $A = [a_{ij}]$ and defined by

$$a_{ij} = \begin{cases} 1, \text{ if } \{v_i, v_j\} \text{ is an edge } i.e., v_i \text{ is adjacent to } v_j \\ 0, \text{ if there is no edge between } v_i \text{ and } v_j \end{cases}$$

If there exists an edge between vertex v_i and v_j, where i is a row and j is a column then value of $a_{ij} = 1$.

If there is no edge between vertex v_i and v_j, then value of $a_{ij} = 0$.

Note that adjacency matrix of G is a symmetric matrix. Since simple graph does not contain any self loop, so diagonal entries of adjacency matrix are all zero. Further, as adjacency matrix contains 0 or 1, so it is also known as Boolean matrix.

Note. Degree of a vertex v_i in G is equal to sum of entries in the ith row or ith column of the adjacency matrix.

For example, *we find the adjacency matrix M_A of graph G shown in Fig. 11.93.*

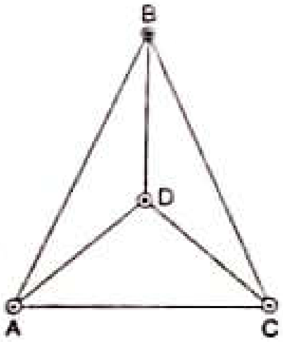

Fig. 11.93

Since the graph G consists of four vertices. Therefore, the adjacency matrix will be a 4 × 4 matrix. The adjacency matrix is as follows in Fig. 11.94.

$$M_A = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

degree of vertex 'c' is 3 which is equal to sum of entries in third row 6/4 mm of adjacency matrix

Fig 11.94

Adjacency List. In a adjacency list of a graph, we list each vertex followed by the vertices adjacent to it. First write vertices of graph in a vertical column, then after each vertex, write the vertices adjacent to it.

Consider the graph shown in Fig. 11.95 the adjacency list is given below :

v_1: v_2, v_3

v_2: v_1, v_3

v_3: v_1, v_2, v_4

v_4: v_3

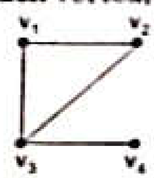

Fig. 11.95

(*ii*) **Incidence matrix or Binary matrix representation.** If an undirected graph consists of n vertices and m edges, then the incidence matrix is an $n \times m$ matrix C = $[c_{ij}]$ defined by

$$c_{ij} = \begin{cases} 1, \text{ if the vertex } v_i \text{ incident by edge } e_j \\ 0, \text{ otherwise} \end{cases}$$

There is a row for every vertex and a column for every edge in the incidence matri•

Note that incidence matrix of a graph need not be a square matrix. Entries in a row added to give degree of corresponding vertex.

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For example :

Consider the graph $G = [V, E]$

where $V = [v_1, v_2, v_3, v_4]$, $E = [e_1, e_2, e_3]$ as shown in Fig. 11.96.

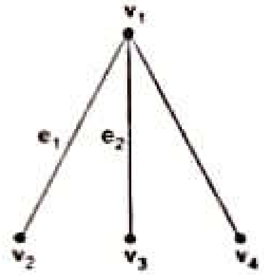

Fig. 11.96

The incidence matrix M_1 for G is shown below :

$$M_1 = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Since each edge in the graph is incident on v_1,

$\therefore$ first row for v_1 has all entries 1.

$\therefore$ degree $v_1 = 1 + 1 + 1 = 3$

Also v_2, v_3, v_4 are pendant vertices.

In incidence matrix of a graph, sum of entries in column is not degree of vertex. As an edge is incident on two vertices in a graph, therefore, each column of incidence matrix will have two 1's.

The number of one's in an incidence matrix of undirected graph (without loops) is equal to the sum of degrees of all the vertices of the graph.

For example : *Consider the undirected graph G as shown in Fig. 11.97. We find its incidence matrix M_I.*

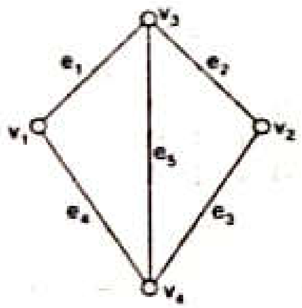

Fig. 11.97

Sol. The undirected graph consists of four vertices and five edges. Therefore, the incidence matrix is a 4 × 5 matrix, which is shown in Fig. 11.98

$$M_I = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Fig. 11.98

(b) Representation of Directed Graph

(i) Adjacency matrix representation. If a directed graph G consists of n vertices, then the adjacency matrix of graph is an $n \times n$ matrix $A = [a_{ij}]$ defined by

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$$a_{ij} = \begin{cases} 1, \text{ if } v_i, v_j \text{ is an edge } i.e., \text{ if } v_i \text{ is initial vertex and } v_j \text{ is final vertex} \\ 0, \text{ if there is no edge between } v_i \text{ and } v_j \end{cases}$$

If there exists an edge between vertex v_i and v_j with v_i as initial vertex and v_j as final vertex, then value of $a_{ij} = 1$.

If there is no edge between vertex v_i and v_j then value of $a_{ij} = 0$.

The number of one's in the adjacency matrix of a directed graph is equal to the number of edges.

For example : *Consider the directed graph shown in Fig. 11.99. We determine its adjacency matrix M_A.*

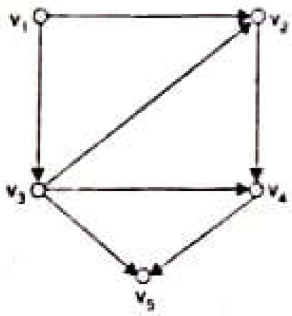

Fig. 11.99

Sol. Since the directed graph G consists of five vertices. Therefore, the adjacency matrix will be a 5 × 5 matrix. The adjacency matrix of the directed graph is as follows in Fig. 11.100.

$$M_A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Fig. 11.100

(*ii*) **Incidence matrix representation.** If a directed graph consists of n vertices and m edges then the incidence matrix is an $n \times m$ matrix $C = [c_{ij}]$, defined by

$$c_{ij} = \begin{cases} 1, & \text{if } v_i \text{ is initial vertex of edge } e_j \\ -1, & \text{if } v_i \text{ is final vertex of edge } e_j \\ 0, & \text{if } v_i \text{ is not incident on edge } e_j \end{cases}$$

The number of one's in the incidence matrix is equal to the number of edges in the graph.

For example, Consider the directed graph G shown in Fig. 11.101. Find its incidence matrix M_I.

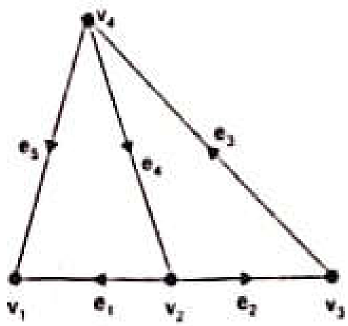

Fig. 11.101

Sol. The directed graph consists of four vertices and five edges. Therefore, the incidence matrix is a 4 × 5 matrix which is shown in Fig. 11.102.

$$M_I = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{bmatrix} \end{array}$$

Fig. 11.102.

(c) Representation of Multigraph

Represented only by adjacency matrix representation.

(*i*) **Adjacency matrix representation of multigraph.** If a multigraph G consists of n vertices, then the adjacency matrix of graph is an $n \times n$ matrix $A = [a_{ij}]$ and is defined by

$$a_{ij} = \begin{cases} \text{N, If there are one or more than one edges between vertex } v_i \text{ and } v_j \text{, where} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{N is the number of edges.} \\ 0, \text{ otherwise.} \end{cases}$$

If there exists one or more than one edges between vertex v_i and v_j then $a_{ij} = N$, where N is the number of edges between v_i and v_j.

If there is no edge between vertex v_i and v_j then value of $a_{ij} = 0$. For *e.g.*,

For example : *Consider the multigraph shown in Fig. 11.103. We determine its adjacency matrix.*

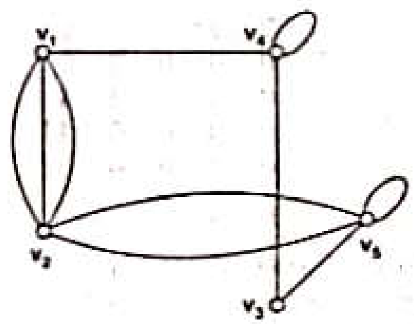

Fig. 11.103

Sol. Since the multigraph consists of five vertices. Therefore, the adjacency matrix will be an 5 × 5 matrix. The adjacency matrix of the multigraph is as follows in Fig. 11.104.

$$M_A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 3 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

Fig. 11.104.

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ILLUSTRATIVE EXAMPLES

Example 1. *Draw the undirected graph represented by adjacency matrix M_A shown in Fig. 11.105.*

$$M_A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Fig. 11.105.

Sol. The graph represented by adjacency matrix M_A is shown in Fig. 106.

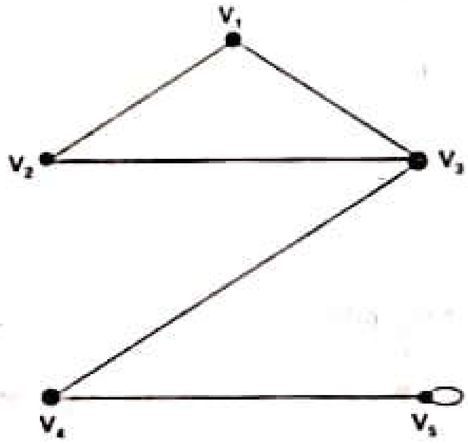

Fig. 11.106.

Example 2. *Draw the undirected graph represented by incidence matrix M_I shown in Fig. 11.107.*

$$M_I = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Fig. 11.107.

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Sol. The graph represented by incidence matrix M_I is shown in Fig. 11.108.

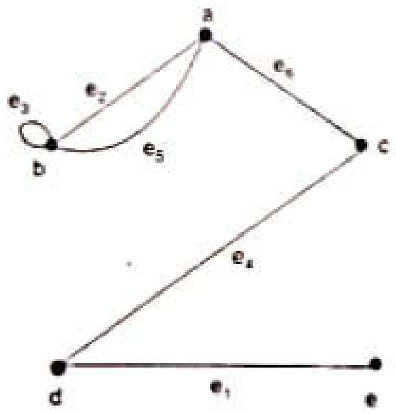

Fig. 11.108.

Example 3. *Draw the multigraph G whose adjacency matrix M_A is shown in Fig. 11.109.*

$$M_A = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

Fig. 11.109.

Sol. The multigraph corresponding to the adjacency matrix M_A is shown in Fig. 11.110.

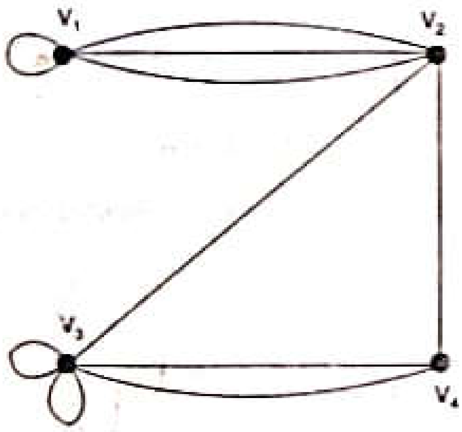

Fig. 11.110.

Example 4. *Draw the directed graph G whose adjacency matrix M_A is shown in Fig. 11.111.*

$$M_A = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Fig. 11.111

Sol. The directed graph corresponding to the adjacency matrix M_A is shown in Fig. 11.112.

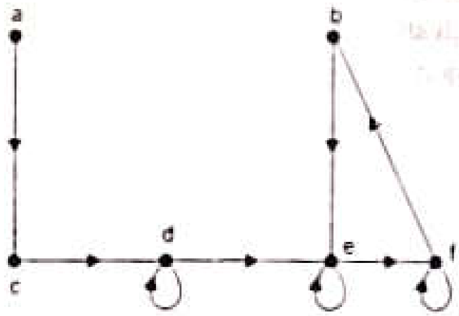

Fig. 11.112

Example 5. *Draw the directed graph G whose incidence matrix M_I is shown in Fig. 11.113.*

$$M_I = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 \end{array} \\ \begin{bmatrix} -1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & +1 & -1 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 \end{bmatrix} \end{array}$$

Fig. 11.113

Sol. The directed graph corresponding to the incidence matrix M_I is shown in Fig. 11.114

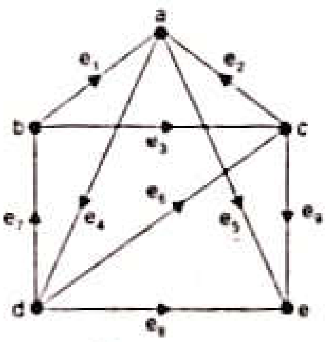

Fig. 11.114

11.35. OTHER IMPORTANT GRAPHS

(a) Bipartite Graph *(P.T.U., M.C.A. May 2008, 2007)*

A graph G = (V, E) is called a bipartite graph if its vertices V can be partitioned into two subsets V_1 and V_2 such that each edge of G connects a vertex of V_1 to a vertex of V_2. In other words, no edge joining two vertices in V_1 or two vertices in V_2. It is denoted by $K_{m,n}$, where m and n are number of vertices in V_1 and V_2 respectively.

For *e.g.*; consider the graph shown in Fig. 11.115. This graph is a bipartite graph.

Here $\qquad V = \{a, b, c, d\}$

Let $\qquad V_1 = \{a, b\}, V_2 = \{c, d\}$

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