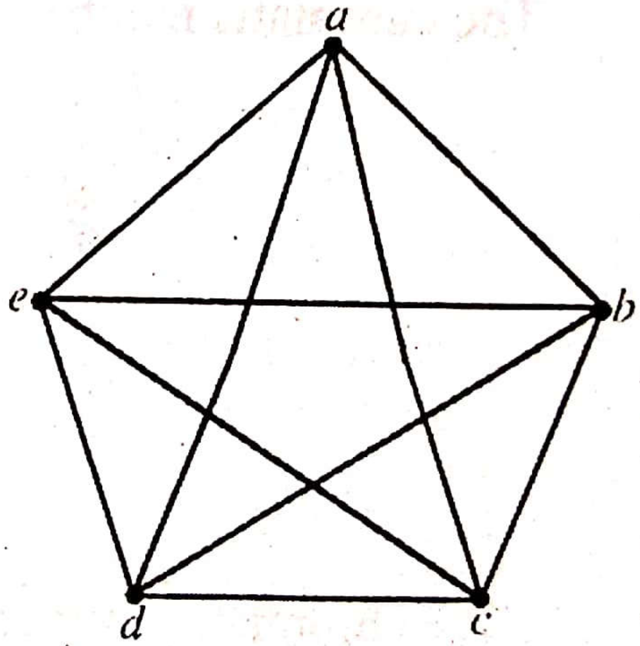
Example 3. Prove that the graph K_5 is not planar.

Sol. Number of vertices in $K_5 = 5$

Number of edges in $K_5 = |E| = 10$



$\because$ for planar graph

$$|E| \leq 3\,|V| - 6$$

$\Rightarrow \quad 10 \leq 3 \times 5 - 6$

$\Rightarrow \quad 10 \leq 9,$

which is contradiction.

$\therefore \quad K_5$ is not planar graph.

Art-10. Coloring

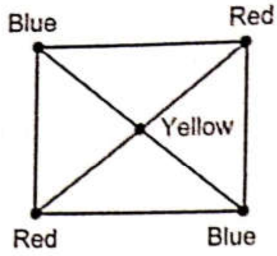
(P.T.U. B.C.A.-I 2007)

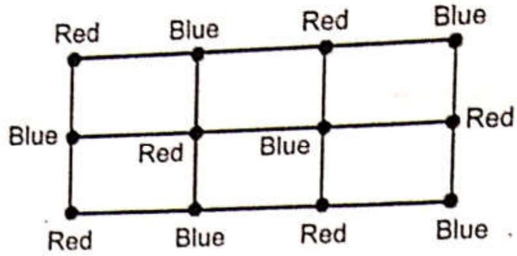
Suppose G be a simple graph with n vertices, we are to paint all its vertices such that no two adjacent vertices have the same colour.

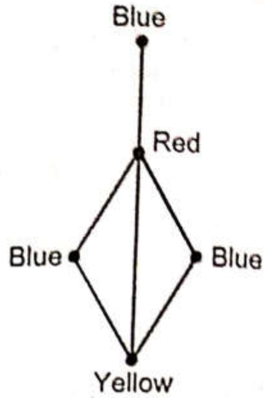
Chromatic Number :

The **minimum number** of colours needed to paint all the vertices of the graph such that no two adjacent vertices have the same colour is called **chromatic number** of G and denoted by C(G).

For example.



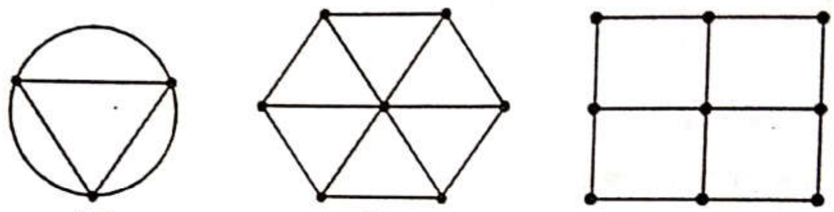




The above graphs are 3-chromatic, 2-chromic and 3-chromatic receptively.

Remark. A complete graph of n vertices is n - chromatic, as all its vertices are adjacent.

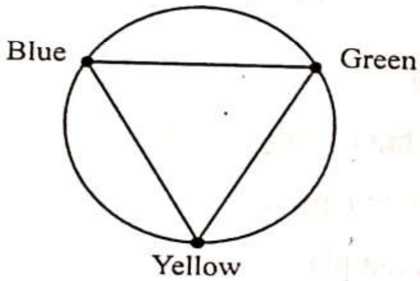
Example. Find the chromatic number for the following graphs.



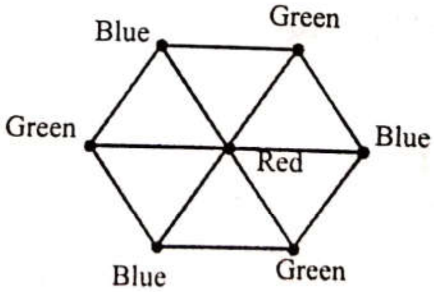
(a) (b) (c)

Sol. Chromatic number of graph G is the minimum number of colour required to paint all the vertices of the graph so that no two adjacent vertices have the same colour.

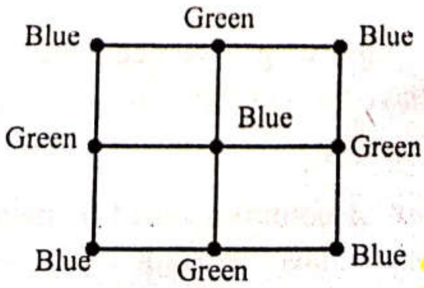
The chromatic colour for the graph (a) is 3 as shown below



The chromatic number for the graph (b) is 3 as shown below



The chromatic number for the graph (c) is 2 as shown below



Scanned by CamScanner

Proper Colouring :

A colouring is proper if any two adjacent vertices u and v have different colours otherwise it is called improper colouring.

Example. The chromatic number of complete bipartite graph $k_{m,n}$ m and n are +ve integers is two.

Sol. The number of colour needed does not depend upon m and n. Only two colours are needed colour the set of m vertices with one colour and the set of n vertices with a second colour. Edges connect only a vertex from the set of m vertices and a vertex from the set of n vertices, no two adjacent vertices have the same colour.

Theorem : Prove that following statements are equivalent for a graph G.

(a) G is 2-colorable.

(b) G is bipartite

(c) G is contains no odd cycle.

Proof : $a \Rightarrow b$.

If G be 2-colorable then graph G has two sets of vertices V_1 and V_2 with different colours say red and blue.

As no vertices of V_1 and V_2 are adjacent

$\therefore$ $\{V_1, V_2\}$ is partition of G.

$\therefore$ G is bipartite.

$b \Rightarrow c$

Let G be bipartite and $\{V_1, V_2\}$ be partition of vertices of G.

Let a vertex $x \in V_1$ and cycle begins at x.

Let it joined to vertex $y \in V_2$ and then to a vertex in V_1 and so on.

When cycle gets completed *i.e.* It returns to x in V_1 then it will be of even length

($\because$ G is bipartite)

$\therefore$ G has no odd cycle.

$c \Rightarrow a$

Let each cycle in G be even let some vertex be coloured while then its adjacent vertex will have different colour black and its adjacent vertex will have colour white because every cycle has even length.

$\therefore$ sequence of vertices of even cycle is WBW ; WBWBW so on.

Only two colours are used to colour the graph.

$\therefore$ G is 2 colourable.

Four Colour Theorem : If G is any planar graph then $C(G) \leq 4$.

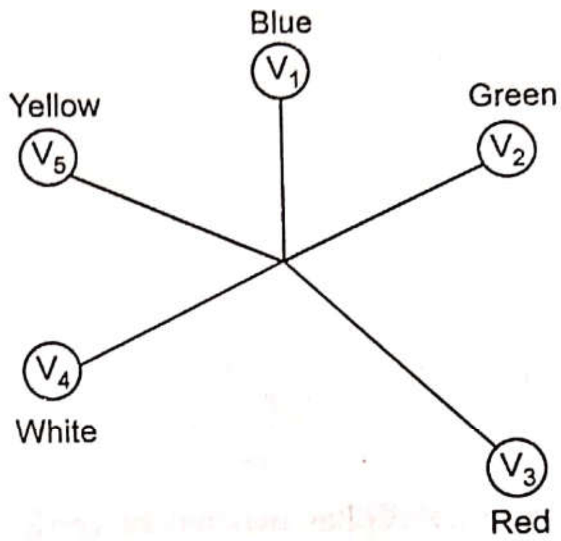
Scanned by CamScanner

Theorem : Five Colour Theorem : If G is planar graph then

$$C(G) \leq 5.$$

Proof : Basis : A graph with one vertex has chromatic number of one.

Induction : Let us assume that all planar graphs with $n - 1$ vertices hav chromatic number of 5 or less. Let G be a planar graph with n vertices.



$\therefore$ $\exists$ a vertex V with degree (V) $\leq$ 5.

Let G-V be the planar obtained by deleting V and all edges that connect V to o vertices in G.

Now by the Induction Hypothesis G-V has a 5-colouring. Let us assume that we the colours red, white, blue, green and yellow.

(*i*) If deg(V) < 5 then we can produce a 5 colouring of G by selecting a colou not used in colouring the vertices that are connected to V with an edge in G.

(*ii*) If deg(V) = 5, then we apply same technique if the five vertices tha adjacent to V are not coloured differently.

Now we have possible condition is that V_1, V_2, V_3, V_4, V_5 are all connected to an edge and they are all coloured differently. Let us assume that they are red colo white, blue, yellow, and green.

If V_1 and V_3 are not connected to one another using only blue and red vertices V. If we take all paths that begin at V_1 and go through only blue and red vertices. we can not reach V_3. When we exchange the colours of the vertices in these including V_1, we still have a 5 colouring of G-V. As V_1 is now red, we can colo blue.

Now we assume that V_1 is connected to V_3 employing only blue and red vertic

Then a path from V_1 to V_3 by employing Blue and red vertices followed edges (V_3, V) and (V,V_1) complete a circuit that either encloses V_2 or encloses V_4 a

$\therefore$ No path from V_2 to V_4 exist employing only green and white vertices. then repeat the same process as in the previous paragraph with V_2 and V_4, whic

Example 1. Determine the chromatic number of the complete graphs k_6, k_{10} and in general k_n.

Sol. It would take six colours to colour a k_6 graph since every vertex is adjacent to every other vertex, we need different colour for every one. Similarly it takes ten colour to colour the graph k_{10} and n-colour to colour the graph k_n.

$$\therefore \quad c(k_6) = 6$$

$$c(k_{10}) = 10$$

$$c(k_n) = n$$

Example 2. A tree with two or more vertices is 2-chromatic. (P.T.U. B.C.A.-I 2007)

Sol. Let T be any tree. Suppose three arbitrary vertices V_1, V_2, V_3 of tree. If V_1 is connected to V_2 and V_3 then V_2 , V_3 are not conntected. (Otherwise cycle will be formed). If V_1 is cloured Red V_2 is coloured Blue then V_3 can be coloured Blue. This is true for all vertices. So maximum colours needed are 2.

Therefore chromatic number of graph is 2.

Again, if T has only two vertices then result is true.

Example 3. What will be chromatic number of complete graph with n-vertices ? Explain. (P.T.U. B.C.A.-I 2007)

Sol. Let G be a graph containing n vertices.

Then a vertex V is connected to exactly $n - 1$ vertices.

So all these vertices must have different colours.

$\therefore$ number of colours required $= n$

i.e. graph is n-chromatic.