CUT THE KNOT

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Probability Riddles

Alexander Bogomolny

Cut the Knot: Probability Riddles

Mathematics / Recreations & Games
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Foreword
by Nassim Nicholas Taleb

Maestro Bogomolny

How do you learn a language?

There are two routes; the first is to memorize imperfect verbs, grammatical rules, future vs. past tenses, recite boring context-free sentences, and pass an exam. The second approach is by going to a bar, struggling a little bit and, out of the need to blend-in and integrate with a fun group of people, then suddenly find yourself able to communicate. In other words, by playing, by being alive as a human being. I personally have never seen anyone learn to speak a language properly by the first route. Also, I have never seen anyone fail to do so by the second one.

It is a not well-known fact that mathematics can also be learned by playing—just watch the private correspondence, discussions and pranks of the members of the august Bourbaki circle. Some of us (and it includes this author) do not perform well on tasks via "cold" approaches, unable to muster the motivation to do boring things. But, somehow we upregulate when stimulated or when there is play (or money) involved. This may disturb many people married to cookie-cutter pedagogical methods that require things to be drab, boring, and bureaucratic for them to be effective—but that's reality.

It is thanks to Maestro Alexander B. that numerous people have learned mathematics by the second route, by playing, just for the sake of entertainment. He helped many to make it their hobby. His mathematical website cut-the-knot has trained a generation—many seemingly approached the problem as hobbyists then got stuck with it. For, if you liked mathematics just a little bit, Maestro Bogomolny made it impossible for you to not love it. Mathematics was turned into a frolic.

I discovered his riddles on social media. (Alexander B. does not like the word "problems". I now understand why.)

Social media brings out the best and the worst in people. He was rigorous yet open-minded, allowing people like me who did some mathematical economics to cheat with inequalities by using the various canned methods for finding minima and maxima. He even tolerated computerized mathematics, provided of course there was some rigor in the process. I initially knew nothing about him but could observe rare attributes: an extraordinary amount of patience and a remarkable sense of humor. One summer, as he was in Israel, I informed him that I was vacationing in Lebanon. His answer: "Walking distance". He always had a short comment that makes you smile, not laugh, which is a social art.

Alexander B. created a vibrant community around his Twitter account. He would pose a question, collect answers and patiently explain to people where they were wrong.

I, for myself, started almost every day with a puzzle, with the excitement of unpredictability, as it took from 5 minutes to 4 hours to complete—and it was usually impossible to tell from the outset. For a couple of years, it was the first thing I looked at with the morning coffee. There was some mild competition, mild enough to be entertaining but not too intense to resemble an academic rat race. Once someone got a proof, we had to look for another approach so it paid to wake up early and beat those with a time zone advantage.

In the two years since he left us there has been no Saturday morning—104 of them—that I did not solve a riddle randomly selected on the web in his memory. But, without him, it is not the same.

—

How did Alexander Bogomolny get there?

I met him in an Italian restaurant in New Jersey. I was surprised to see a mathematician who looked much more like a maturing actor than someone in a technical specialty: tall, athletic, jovial, and with a charismatic presence. But, as he had warned me, he had a severe hearing problem, the result of a medical treatment for the flu.

This explained to me his veering away from an academic career to get involved in computer pedagogy. His hearing was worsening with time. It is hard to imagine being a professor with reduced auditory function in one ear (in spite of a hearing aid) and none in the other.

There was something fresh and entertaining about him. He was happy. One could talk and laugh with him without much communication.

He was neither interested in money nor rank—something refreshing as I was only exposed to academics who, whether they admit it or not, are obsessed with both. When I asked him about commercializing his website cut-the-knot his answer was "I have two pensions. Next year I turn seventy". He wasn't interested in poisoning his life for more money.

Why did I start nicknaming him Maestro? Because it was pretty much literal: he played math like a master would with a musical instrument—and mostly to himself. He was physically bothered by a sloppy derivation or an error, as if he heard a jarring note in the middle of a sonata. It was a joy to see someone so much in sync with his subject matter—and totally uncorrupted by the academic system.

—

Now, probability. In one conversation, I mentioned to him that probability riddles would be very useful for people who want to get into the most scientifically applicable scientific subject in the world (my very, very biased opinion). What I said earlier about play is even more applicable to probability, a field that really started with gamblers, used by traders and adventurers, and perfected by finance and insurance mathematicians. Probability applies to all empirical fields: gambling, finance, medicine, engineering, social science, risk, linguistics, genetics, car accidents. Let's play with it by adding to his feed some probability riddles.

His eyes lit up. Hence this book.

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I thank Marcos Carreira, Amit Itagi, Mike Lawler, Salil Mehta, and numerous others who supported us in this project.

And a special gratitude to Stephen Wolfram and Jeremy Sykes for ensuring that Cut-the-Knot stays alive and that this book sees the day. Additional thanks to Paige Bremner, Glenn Scholebo, and other members of Wolfram Media for editing the manuscript.

Preface

> On a superficial view, we may seem to differ very widely from each other in our reasoning, and no less in our pleasures: but notwithstanding this difference, which I think to be rather apparent than real, it is *probable* that the standard both of reason and Taste is the same in all human creatures.
>
> E. Burke, 1757, *A Philosophical Inquiry into the Origin of Our Ideas of the Sublime and Beautiful*

The American Heritage Dictionary defines probability theory as the branch of mathematics that studies the likelihood of occurrence of random events in order to predict the behavior of defined systems. (Of course "what is random?" is a question that is not all that simple to answer.)

Starting with this definition, it would (probably) be right to conclude that probability theory, being a branch of mathematics, is an exact, deductive science that studies uncertain quantities related to random events. This might seem to be a strange marriage of mathematical certainty and the uncertainty of randomness.

This collection of probability problems is a collaborative effort born on the web and tested in social networks. Even though meeting over the wires is a far cry from a face-to-face encounter, it has a clear positive side. Web acquaintances are drawn to each other due to mutual interests, mostly without seeking to learn or attaching importance to unrevealed personal details.

Some of the problems are original but most have been gathered from a variety of sources: books, magazine articles and online resources. The collection is reasonably comprehensive, though its value is mostly in that many if not most of the problems come with multiple solutions. So much so that quite often it was difficult to assign problems to specific chapters.

Receiving solutions from people, many of whom would confess to not being mathematicians or having forgotten all the math they studied in school or college, was an exhilarating experience. One of the ideas we pursued in writing this book was to share that experience with a broader audience.

We prefer the term *riddle* to *problem*.

Chapter 1

Intuitive Probability

> One sees in this essay that the theory of probabilities is basically only common sense reduced to a calculus. It makes one estimate accurately what right-minded people feel by a sort of instinct, often without being able to give a reason for it.
>
> Laplace, 1814, *Philosophical Essay on Probabilities*

The theory of probability supplies a good deal of counterintuitive results (see Chapter 3, page 15). However, the theory of probability arose from practical applications and is, in essence, a formal encapsulation of the intuitive view of chance.

This chapter collects a few probability problems whose solutions are based on intuition and common sense and do not require any theoretical knowledge.

Be warned, though, that even great mathematical minds sometimes get beguiled by intuition. For example, when asked for the probability that heads appears at least once in two throws of a fair coin, Jean le Rond d'Alembert (1717–1783), replied $\frac{2}{3}$ (the correct answer is $\frac{3}{4}$) because in his opinion there was no reason to continue tossing the coin after heads showed up on the first toss. Thus, he judged events H, TH, TT as equiprobable.

> The bracketed items under the section and subsection heads are citations to items listed in the bibliography. The page numbers refer to the exact portion of the cited item. Some riddles and solutions were created for the Cut The Knot project and are not published elsewhere.

Riddles

1.1 Bridge Hands

[45, p. 39]

A game of bridge is played by two pairs of players, with players on a team sitting opposite each other. Which situation is more likely after four bridge hands have been dealt: you and your partner hold all the clubs or you and your partner have no clubs?

1.2 Secretary's Problem

[45, p. 39]

A secretary types four letters to four people and addresses the four envelopes. If the letters are inserted at random, each in a different envelope, what is the probability that exactly three letters will go into the right envelope?

1.3 Pencil Logo

[102, p. 1]

A pencil with pentagonal cross-section has a maker's logo imprinted on one of its five faces. If the pencil is rolled on the table, what is probability that it stops with the logo facing up?

1.4 Balls of Two Colors I

[54, pp. 1–2], [65, Problem 319]

A box contains p white balls and q black balls, and beside the box lies a large pile of black balls. Two balls are drawn at random (with equal likelihood) out of the box. If they are of the same color, a black ball from the pile is put into the box; otherwise, the white ball is put back into the box. The procedure is repeated until the last two balls are removed from the box and one last ball is put in.

What is the probability that this last ball is white?

1.5 Division by 396

[91, Problem 56]

Find the probability that if the digits 0, 1, 2, …, 9 are placed in random order in the blank spaces of 5_383_8_2_936_5_8_203_9_3_76 the resulting number will be divisible by 396.

1.6 Fair Dice

Michel Paul

Toss two dice. What is the probability of having 2 or 5 on at least one die?

1.7 Selecting a Medicine

[58, pp. 97–98]

Half the people who contracted a certain disease which is spreading across the country have died and half got better on their own.

Two medicines, A and B, have been developed but not actually tested. A was administered to three patients and all survived. B was administered to eight patients and seven survived.

In the unfortunate event of your contracting the disease, which medicine would you choose?

1.8 Probability of Random Lines Crossing

[62, Problem 8]

An urn contains the numbers $1, 2, 3, \ldots, 2018$. We randomly draw, without replacement, four numbers in order from the urn which we will denote a, b, c, d. What is the probability that the following system will have a solution strictly inside (i.e., not on the axes) the first quadrant?

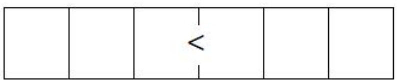
$$\left\{\begin{array}{rcl} ax + by & = & ab \\ cx + dy & = & cd. \end{array}\right.$$

1.9 Six Numbers, One Inequality

A. Bogomolny. Inspired by Donald Knuth's puzzles in [75].

Six distinct digits, 1, 2, 3, 4, 5, 6, are placed randomly into the boxes below. What is the probability that the indicated inequality holds between two digits adjacent to the inequality symbol?

Answer the same question concerning another placement of the inequality sign:



1.10 Metamorphosis of a Quadratic Function

A.A. Berzin'sh [94, Problem 383].
The problem was offered at the XVII All-Union Mathematical Olympiad for grade 8.

A math teacher in a special math school wrote the quadratic polynomial $x^2 + 10x + 20$ on the blackboard and asked the students in a class of 24 to modify either the free coefficient or the one by x, every time by ± 1. The polynomial $x^2 + 20x + 10$ was the result of the experiment.

What is the probability that, along the way, there was a quadratic polynomial with integer roots?

1.11 Playing with Balls of Two Colors

[45, p. 2.16], [71, Challenge Problem 10], [101, pp. 79–80]

An urn contains balls of two colors—black and white—at least one of each. You draw balls at random. In a round of ball drawings, you draw a ball, discharge it but note its color. You continue drawing and discharging the balls as long as they are of the same color. The first ball of a different color is placed back into the urn. This ends the round. The next and all successive rounds are played until the urn becomes empty. You win if the last ball to be drawn is black.

How many balls of each color in the urn should you start with to maximize your probability of winning?

1.12 Linus Pauling's Argument

[98]

On the occasion of his receiving a second Nobel prize, Dr. Linus Pauling, the chemist, remarked that, while the chances of any person in the world receiving his first Nobel prize were one in several billion (the population of the world), the chances of receiving the second Nobel prize were one in several hundred (the total number of living people who had received the prize in the past), and that it was therefore less remarkable to receive one's second prize than one's first.

Is there a flaw in Pauling's argument?

1.13 Galton's Paradox

[36, p. 2.1.10]

This question from Francis Galton (1894) concerns the chance of three (fair) coins turning up alike (all heads or else all tails).

At least two of the coins must turn up alike, and, as it is an even chance whether a third coin is heads or tails, the chance of (all three) being alike is 1 to 2.

Where does the fallacy lie?

Solutions

Bridge Hands

If my partner and I have no clubs then all the clubs are in the hands of our competition. If we do have all the clubs, they have none of the clubs. Both pairs of players are equally likely to get all the clubs or to get none of them.

It follows that the events of us getting all the clubs or getting none of them are equally likely.

Secretary's Problem

Since there were just four letters and four envelopes, the fact that some three letters were placed into the right envelopes implies that the remaining letter has also been placed into the right envelope. Therefore, the probability is 0 that exactly three letters went into the right envelopes.

A similar problem was published in [91] under the caption *The Careless Mailing Clerk*:

After a typist had written 10 letters and addressed the 10 corresponding envelopes, a careless mailing clerk inserted the letters in the envelopes at random, one letter per envelope. What is the probability that exactly nine letters were inserted in the proper envelopes?

The solution is left as an exercise.

Pencil Logo

It is impossible for the pencil to stop on an edge as to make a face—with or without the maker's logo—show on top. The probability of the event in question is 0.

Balls of Two Colors I

The key to solving the riddle is to notice that the parity of the number of white balls never changes. Thus, the last ball will be white if at the outset the number of white balls was odd; if it was even, the last ball will be black.

Thus, the answer is 1 if the box originally contained an odd number of white balls and 0 otherwise.

Notes

Over time, the riddle has appeared in several disguises. For example, [39] offers a game with circles and squares where, on each step, two different figures are replaced by a square and two equal ones, by a circle.

Division by 396

The riddle is obviously contrived. The last two digits of the longish number are 76, implying that the number is divisible by 4 and $\dfrac{396}{4} = 99 = 9 \times 11$. We need to check that, however the 10 digits are distributed over the empty spaces, the resulting number is divisible by 9 and also by 11.

A number is divisible by 9 if and only if the sum of its digits is divisible by 9. The sum of the missing digits is

$$0 + 1 + 2 + \cdots + 9 = 45,$$

which is divisible by 9. The sum of the present digits is

$$5 + 3 + 8 + 3 + 8 + 2 + 9 + 3 + 6 + 5$$
$$+ 8 + 2 + 0 + 3 + 9 + 3 + 7 + 6 = 90$$

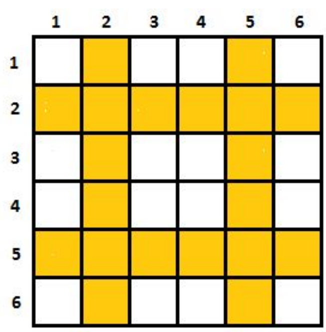
which is also divisible by 9. It follows that the resulting number is divisible by 9 regardless of the placement of the missing digits.

To check whether a number is divisible by 11 we compute two sums: that of the evenly placed digits and that of the oddly placed digits. All the missing digits come together along with 8, 3, 0 and 6, which add up to $45 + 17 = 62$. The sum of the remaining digits is the $90 - 17 = 73$. The difference of the two sums is $73 - 62 = 11$ which is divisible by 11, and we are finished.

Fair Dice

For a single die, all six outcomes are equally likely. So, on average, one gets 2 or 5 in two cases out of six, making the probability $P(2 \text{ or } 5) = \frac{2}{6} = \frac{1}{3}$. This is less than the fair half.

For two dice, the situation is different and may be represented graphically:



There are $6 \times 6 = 36$ equiprobable squares/events. The event "2 or 5" occurs in 20 cases, making the probability $P(2 \text{ or } 5) = \frac{20}{36} = \frac{5}{9}$ which is more than fair.

Selecting a Medicine

From the available data we may assume that, without any medicine, your chances of survival are 50:50. We shall never know whether either medicines does more good than harm. What we can estimate and compare are the chances of survival with and without taking any medicine.

There were three patients treated with A. Were they not, the three of them would survive with the probability of $\left(\frac{1}{2}\right)^3 = \frac{1}{2^3}$.

The probability of having seven survivors out of eight is $8 \cdot \frac{1}{2^8}$ which is $\frac{1}{2^5} = \frac{1}{32}$.

Thus, the probability of having seven survivors out of eight is four times smaller than that of having three survivors out of three. There is a good chance that medicine B proved to be more useful than medicine A.

Probability of Random Lines Crossing

The system is best rewritten as

$$\begin{cases} \dfrac{x}{b} + \dfrac{y}{a} &= 1 \\ \dfrac{x}{d} + \dfrac{y}{c} &= 1. \end{cases}$$

In this form it is obvious that we deal with two straight lines: the first intersecting x and y axes at $(b, 0)$ and $(0, a)$, respectively, and the second intersecting the axes at points $(d, 0)$ and $(0, c)$, respectively. From a geometric view, the system never has a solution on the axes because the four numbers are distinct, although random. Thus, the extra condition is a harmless red herring [19].

The two lines intersect within the first quadrant when $(a - c)(b - d) < 0$. This condition is as probable as its complement $(a - c)(b - d) > 0$. Thus, both have the probability of $\frac{1}{2}$.

Six Numbers, One Inequality

The answer for both questions is $\frac{1}{2}$ and this can be obtained without any calculations. Indeed, any placement of the digits that satisfies the inequality is paired with another one that does not. The two are obtained from each other by swapping the two digits adjacent to the inequality sign. Nicu Mihalache was the first to suggest that approach on Twitter.

We can do the counting as well. The sample space consists of 6! elements. There are $\binom{6}{2}$ ways to select a pair adjacent to the inequality symbol and only one way to place them so the inequality is satisfied. The remaining four numbers can be placed in 4! ways, giving the total number of "successful" combinations as $\binom{6}{2} \cdot 4!$ and the corresponding probability as

$$\frac{\binom{6}{2} \cdot 4!}{6!} = \frac{1}{2}.$$

Metamorphosis of a Quadratic Function

Consider the polynomial $f_{a,b}(x) = x^2 + ax + b$ and its value at $x = -1$: $f_{a,b}(-1) = b - a$. Every time a or b changes by ± 1 so does the expression $b - a$. Now $f_{10,20}(-1) = 11$ whereas $f_{20,10}(-1) = -9$. Thus, it follows that in the process of getting one from the other there ought to be a stage with $f_{a,b}(-1) = 0$, meaning that the integer -1 is the root of a provisional polynomial. The other root is naturally also an integer. The probability of this happening is 1.

Playing with Balls of Two Colors

Solution 1

Imagine that prior to every round, the remaining balls are lined up in a random order and you simply pick the balls from left to right. A round corresponds to a sequence of balls of the same color. The balls before the last round are all of one color; the balls before the penultimate round are in two groups of different colors, say black balls on the left, followed by white balls on the right. It could be the other way around. Regardless of the size of each group, these two orders are equally likely. But the order of the groups does impact the result: in one case, the last ball will be white, in the other it will be black. Thus, the probability of either is $\frac{1}{2}$.

Solution 2

An excerpt from "A Sampling Process," B.E. Oakley and R.L. Perry, *The Mathematical Gazette*, February 1965, pp. 42–44.

Let an urn contain $m > 0$ black balls and $n > 0$ white balls. Let $u_{mn}(b)$ be the probability that the sequence of balls being discarded should begin and end with a black ball. Let $u_{mn}(w)$ be the probability that it should begin with a white ball and end with a black ball. We require $u_{mn} = u_{mn}(b) + u_{mn}(w)$. We have

$$u_{mn}(b) = \frac{m}{m+n}\left(u_{m-1,n}(b) + \frac{n}{m+n-1}u_{m-1,n}\right). \tag{1.1}$$

Equation 1.1 also holds if $m = 1$ and $n \geq 1$ since both sides are zero.

Similarly, for $m \geq 1$, $n \geq 2$,

$$u_{mn}(w) = \frac{m}{m+n}\left(u_{m,n-1}(w) + \frac{m}{m+n-1}u_{m,n-1}\right) \tag{1.2}$$

and again, since $u_{m1}(w) = \frac{1}{m+1}$, $u_{m0}(w) = 0$ and $u_{m0} = 1$, equation 1.2 holds if $m \geq 1$, $n = 1$.

Solution 3

This is like Riddle 7.8, The Lost Boarding Pass on page 158.

Balls come in runs of the same color. The first passenger gets in somebody's seat, some after him get in their seats until comes a fellow in whose seat sits the first passenger. The round starts anew. The last fellow to enter the plane has the probability of $\frac{1}{2}$ to get into his own seat. This probability is independent of how many people sit in their assigned seats and how many are in somebody else's.

The analogy is in the sequence of runs/rounds. The passenger runs can be looked at as the balls of the same color.

Example 1

Joshua B. Miller

Consider the case of one white (W) ball and two black (B) balls. You have four possible sequences in which the balls are drawn from the urn (a dash designates an interruption due to a color change so that the ball before a dash is put back into the urn):

Action	Result	Probability
$WB - BB$	win	$\frac{1}{3}$
$BW - WB - B$	win	$\frac{1}{6}$
$BW - BW - W$	loss	$\frac{1}{6}$
$BBW - W$	loss	$\frac{1}{3}$

Example 2

Joshua B. Miller

Two white balls and one black ball:

Action	Result	Probability
$WB - BW - W$	loss	$\frac{1}{6}$
$WB - WB - B$	win	$\frac{1}{6}$
$BW - WW$	loss	$\frac{1}{3}$
$WWB - B$	win	$\frac{1}{3}$

Example 3

Joshua B. Miller

Three white balls and one black ball:

Action	Result	Probability
$WWWB - B$	win	$\frac{1}{4}$
$WWB - WB - B$	win	$\frac{1}{8}$
$WWB - BW - W$	loss	$\frac{1}{8}$
$WB - WWB - B$	win	$\frac{1}{12}$
$WB - WB - WB - B$	win	$\frac{1}{24}$
$WB - WB - BW - W$	loss	$\frac{1}{24}$
$WB - BW - WW$	loss	$\frac{1}{12}$
$BW - WWW$	loss	$\frac{1}{4}$

Solution 4

Joshua B. Miller, Thamizh Kudimagan

The penultimate round will begin with an urn with composition of black and white balls. For each of these urn compositions (not equally likely as penultimate rounds), there are just two ways to have a penultimate round draw, and they are equally likely from that urn. The two ways will have different pay offs.

Solution 5

Alejandro Rodríguez

Let $P(B,W)$ be the probability of winning given that the urn contains B black balls and W white balls. We can compute $P(B,W)$ as follows:

$$P(B,W) = \frac{B}{B+W}V(B-1,W,\text{black}) + \frac{W}{B+W}V(B,W-1,\text{white})$$

where $V(B,W,x)$ is the probability of winning given that there are B black balls and W white balls remaining and the last ball drawn is of color x. $V(B,W,x)$ can be calculated as:

$$V(B,W,\text{black}) = \frac{B}{B+W}V(B-1,W,\text{black}) + \frac{W}{B+W}P(B,W)$$

$$V(B,W,\text{white}) = \frac{B}{B+W}P(B,W) + \frac{W}{B+W}V(B,W-1,\text{white})$$

Boundary conditions are given by:

$$V(B,0,x) = P(B,0) = 1$$

$$V(0,W,x) = P(0,W) = 0.$$

Plug the formula into a spreadsheet and get $P(B,W) = 0.5$ for B and W greater than zero and less than 30 (my guess is that it is always the case).

Notes

This is a problem from P. Winkler [101] and also from M. Gardner [45].

Gardner announces the result and refers to the paper “A Sampling Process” by B.E. Oakley and R.L. Perry in *The Mathematical Gazette* (February 1965, pp. 42–44), see Solution 2 on page 8. Winkler refers to Gardner and gives a solution by Sergiu Hart from the Hebrew University of Jerusalem. Solution 1 is a paraphrase of that in [101].

Winkler mentions Hart's observation that the game is isomorphic, in a sense, to Riddle 7.8, The Lost Boarding Pass on page 158. This was discussed in the *Analogous to Other Puzzles* section.

P. Nahin offers that as Challenge Problem 10 in [71] where he presents computer simulation that confirms the probability of $\frac{1}{2}$. Like Gardner and Winkler, he refers to the long proof by Oakley and Perry.

Joshua B. Miller was instrumental in shaping the present section.

Linus Pauling's Argument

The chance of receiving a second Nobel prize only depends on the total number of living Nobel prize winners if you know for a fact that the Nobel prize committee have decided to honor, again, one of the selected group. However, Nobel prizes are awarded quite independently of past awards, and so the actual chance of being awarded two Nobel prizes is one chance in a billion times a billion (if you suppose, which is implausible, that every human being on earth has an equal chance of winning the award).

Linus Pauling's Nobel prizes, incidentally, were the prize for Chemistry in 1954 and the Peace Prize in 1962.

Galton's Paradox

First, this is a fallacy. An experiment with three fair coins has eight equiprobable outcomes, so each comes up with the probability of $\frac{1}{8}$. Two of them—HHH or TTT—combine into an event of the probability $\frac{1}{8} + \frac{1}{8} = \frac{1}{4}$, not $\frac{1}{2}$. Thus the question: Where does the fallacy lie?

The argument is valid to a point. There are indeed always two coins that come up the same way: both heads up or both tails up. (Pigeonhole Principle, [18].) It is also true that the remaining coin comes up either way with the probability of $\frac{1}{2}$. The conclusion is, however, false because there is no certainty of which two coins get involved in the first part of the argument. If the two coins were fixed, say, #1 and #2, then certainly the third one—#3—could come up either way with the probability of $\frac{1}{2}$. But, in that case, #1 and #2 may come up alike with the probability of $2 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$. All four possible outcomes are equiprobable: HH, HT, TH, TT. Thus, there is no certainty that #1 and #2 will come up alike.

Among eight equiprobable outcomes of three tosses of a coin, exactly two come up alike with the probability of $\frac{3}{4}$, but there is no knowing which pair it is:

$$HHH, \underline{HHT}, \underline{HTH}, \underline{HTT}, \underline{THH}, \underline{THT}, \underline{TTH}, TTT.$$

Chapter 2

What Is Probability?

> "Probability is the bane of the age," said Moreland, now warming up. "Every Tom, Dick, and Harry thinks he knows what is probable. The fact is most people have not the smallest idea what is going on round them. Their conclusions about life are based on utterly irrelevant—and usually inaccurate—premises."
>
> Anthony Powell, 1960, *Casanova's Chinese Restaurant*

Intuitively, the mathematical theory of probability deals with patterns that occur in random events. For the theory of probability, the nature of randomness is inessential. (Note for the record, that according to the eighteenth century French mathematician Marquis de Laplace, randomness is a perceived phenomenon explained by human ignorance, while late twentieth century mathematics came with a realization that chaos may emerge as the result of deterministic processes.) An *experiment* is a process—natural or set up deliberately—that has an observable outcome. In the deliberate setting, the words experiment and trial are synonymous. An experiment has a random outcome if the result of the experiment cannot be predicted with absolute certainty. An *event* is a collection of possible outcomes of an experiment. An event is said to occur as a result of an experiment if it contains the actual outcome of that experiment. Individual outcomes comprising an event are said to be *favorable* to that event. Events are assigned a measure of certainty which is called *probability* (of an event).

Quite often the word experiment describes an experimental setup, while the word trial applies to actually executing the experiment and obtaining an outcome.

A formal theory of probability was developed in the 1930s by the Russian mathematician A.N. Kolmogorov.

The two fundamental notions of probability theory are *sample space* and *equiprobable* events.

A sample space is where the events take place. The nature of a sample space depends on the kind of experiment under consideration.

When tossing a coin, two outcomes are possible: heads and tails. The sample space for a single toss of a coin is $\{H, T\}$. Two events are possible and, provided the coin is fair, these two events are equiprobable. If $P(X)$ denotes the probability of event X, then, for an experiment with a fair coin, $P(H) = P(T)$. On the other hand, a coin always comes to rest with one of the sides up. We interpret that with the formula $P(H \text{ or } T) = 1$.

1 is the probability of certain truth.

0 is the probability of certain falsity.

It follows that, since events H and T are equiprobable and cannot happen simultaneously then $P(H) = P(T) = \dfrac{1}{2}$.

Similarly, when tossing a fair die, the sample space consists of six equiprobable events $\{1, 2, 3, 4, 5, 6\}$ such that each of them comes with the probability of $\frac{1}{6}$.

In addition to sample space and equiprobability, elementary probability theory employs a few basic notions—definitions and theorems. These are discussed at greater length in the appendices.

Chapter 3

Likely Surprises

> It is remarkable that a science which began with the consideration of games of chance should have become the most important object of human knowledge.
>
> Pierre Simon Laplace, 1812, *Théorie Analytique des Probabilités*

Riddles

3.1 Balls of Two Colors II

[36], [71]

An urn contains w white balls and b black balls, $w > 0$ and $b > 0$. The balls are thoroughly mixed and two are drawn, one after the other, without replacement. Let W_i and B_i denote the respective outcomes "white on the i^{th} draw" and "black on the i^{th} draw," for $i = 1, 2$.

Prove that $P(W_2) = P(W_1) = \dfrac{w}{w+b}$. (Which clearly implies a similar identity for B_1 and B_2).

Furthermore, $P(W_i) = \dfrac{w}{w+b}$, for any i not exceeding the total number of white balls w.

3.2 Coin Tossing Contest

[8, Problem 99], [103]

$\mathcal{A}$ tosses a fair coin 20 times; $\mathcal{B}$ tosses the coin 21 times.

What is the probability that $\mathcal{B}$ will have more heads than $\mathcal{A}$?

3.3 Probability of Visiting Grandparents

Mathematical folklore

A boy, who lives in Moscow, visits his grandparents every Thursday. Both his maternal and paternal grandparents are alive and live at the two ends of the same metro line. The boy lives close to a metro station in between on the same line. When he gets down to the platform, he takes the first train that arrives, regardless of the direction.

Short of "things happen," what could be a rational explanation of the fact that after a (long) period of time he observes that he has made twice as many visits to one pair of grandparents than to the other?

The trains go in both directions at the same fixed interval, and the boy gets down to the station at about the same time every Thursday. The trains are so scheduled that they never arrive at the same time from both directions.

3.4 Probability of Increasing Sequence

[28, Problem 1]

Slips of paper with the numbers from 1 to 99 are placed in a hat. Five numbers are randomly drawn out of the hat one at a time (without replacement). What is the probability that the numbers are chosen in increasing order?

What is the probability that the first number drawn is a 1?

3.5 Probability of Two Integers Being Coprime

What is the probability that two randomly selected integers are coprime, i.e., have no common prime factors?

3.6 Overlapping Random Intervals

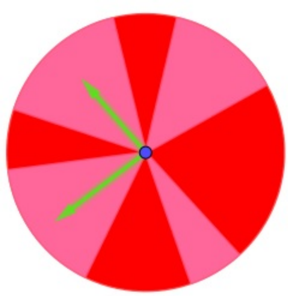
[21, Problem 102], [59]

$2n \geq 4$ points are chosen uniformly randomly on the interval $[0, 1]$, say, $X_1, X_2, \ldots, X_{2n}$ for $1 \leq j \leq n$, where J_j is the closed interval with endpoints X_{2j-1} and X_{2j}. Find the probability that one of the intervals J_j has overlaps with all other intervals.

3.7 Random Clock Hands

Mathematical folklore

A clock face is split randomly into alternating red and pink sectors. This question is about the relative probabilities of each of the clock hands being in the red

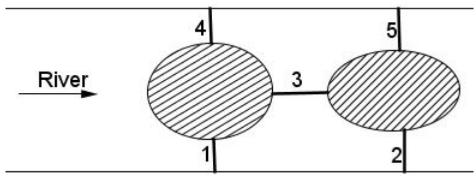


- when the watch is functioning properly.
- when the watch is broken and the hands rotate at a fixed angle (and, of course, at a fixed rate).
- when the hands jump uniformly randomly all over the dial.

3.8 Crossing River after Heavy Rain

[40, Marcus Moore]

A traveler approaches a river spanned by bridges that connect its shores and islands. There has been a great storm the night before, and each of the bridges was as likely as not to be washed away. How probable is it that the traveler can still cross?



3.9 The Most Likely Position

[62, Problem 6]

A standard deck has 52 cards, of which four are aces. When the deck is shuffled, what is the most likely position of the first ace?

3.10 Coin Tossing Surprises I

[44, p. 303]

A fair coin is tossed repeatedly.

1. What is the expected number of tosses before HT shows up for the first time?
2. What is the expected number of tosses before TT shows up for the first time?
3. What is the probability that HT shows up before TT?
4. What is the probability that HT shows up before HH?

Solutions

Balls of Two Colors II

Solution 1

[36], [71]

We must remember the formula for the **total** probability:

$$P(W_2) = P(W_2|W_1) \cdot P(W_1) + P(W_2|B_1) \cdot P(B_1),$$

from which

$$\begin{aligned}P(W_2) &= \frac{w-1}{b+w-1} \cdot \frac{w}{b+w} + \frac{w}{b+w-1} \cdot \frac{b}{b+w} \\ &= \frac{w \cdot (w-1+b)}{(b+w-1) \cdot (b+w)},\end{aligned}$$

which is simplified to $\dfrac{w}{w+b}$.

The wonderful thing here is that the initial probability $\dfrac{w}{w+b}$ is being maintained even though the balls are not returned to the urn.

Solution 2

N.N. Taleb

$P(W_1) = \dfrac{w}{b+w}$, $P(B_1) = \dfrac{b}{b+w}$. And further,

$$\begin{aligned}P(W_2) &= P(B_1)\frac{w}{b+w-1} + P(W_1)\frac{(w-1)}{b+w-1} \\ &= \frac{(w-2)\left(\frac{w}{b+w}\right)^2}{b+w-2} + \frac{(w-1)w\left(1-\frac{w}{b+w}\right)}{(b+w-2)(b+w)} \\ &\qquad + \frac{(w-1)w\left(1-\frac{w}{b+w}\right)}{(b+w-2)(b+w)} + \frac{w\left(1-\frac{w}{b+w}\right)^2}{b+w-2} \\ &= \frac{w}{b+w}.\end{aligned}$$

More generally,

$$P(W_n) = \sum_{k=0}^{n} \frac{\left(1-\frac{w}{b+w}\right)^k \binom{n}{k}(k-n+w)\left(\frac{w}{b+w}\right)^{n-k}}{b-n+w} = \frac{w}{b+w}.$$

Solution 3

Joshua B. Miller

If you empty the urn one ball at a time you will generate a sequence of length $n = w+b$. By exchangeability, each sequence is equally likely. What is the probability that, on the i^{th} draw, you draw w? It is the number of unique sequences with w in the i^{th} position, $\binom{n-1}{w-1}$, divided by the number of unique sequences $\binom{n}{w}$, i.e.,

$$\frac{\binom{n-1}{w-1}}{\binom{n}{w}} = \frac{\dfrac{(n-1)!}{(w-1)!(n-w)!}}{\dfrac{n!}{w!(n-w)!}}$$

$$= \frac{(n-1)!w!(n-w)!}{n!(w-1)!(n-w)!}$$

$$= \frac{w}{n}$$

$$= \frac{w}{w+b}.$$

Coin Tossing Contest

Solution 1

Amit Itagi

We shall consider several events concerning the possible total outcomes:

- $H_{B>A}$—the event of $\mathcal{B}$ having more heads than $\mathcal{A}$,
- $H_{B\leq A}$—the event of $\mathcal{B}$ having at most as few heads as $\mathcal{A}$,
- $T_{B>A}$—the event of $\mathcal{B}$ having more tails than $\mathcal{A}$.

We are interested in the probability of the first of these events. Observe that $H_{B\leq A} = T_{B>A}$. On the other hand, when it comes to probabilities, by symmetry, $P(H_{B>A}) = P(T_{B>A})$.

If X is the total number of outcomes, then

$$X = H_{B>A} \cup H_{B\leq A} \text{ and } H_{B>A} \cap H_{B\leq A} = \emptyset.$$

Finally then,

$$1 = P(H_{B>A}) + P(H_{B\leq A})$$

$$= P(H_{B>A}) + P(T_{B>A})$$

$$= 2P(H_{B>A}),$$

implying $P(H_{B>A}) = \dfrac{1}{2}$.

Solution 2

Long Huynh Huu

Choose A from $1, \ldots, 20$, following a symmetric distribution. Choose B from $0.5, 1.5, \ldots, 20.5$, similarly. By symmetry, $P(A < B) = P(A > B) = 0.5$.

(By "symmetric" I mean $A - 10.5$ and $B - 10.5$ have same distribution as $10.5 - A$ and $10.5 - B$.)

Solution 3

Faryad D. Sahneh

There are two possibilities for getting more heads in experiment B with 21 toss trials than experiment A with 20 toss trials.

- Event 1: The number of heads in the first 20 trials of B is already more than that of A.
- Event 2: The number of heads in the first 20 trials of B is exactly equal to that of A, and trial 21 is a head.

Let $p = P$(number of heads in the first 20 trials of B is equal to that of A). Then, $P(\text{Event 1}) = \frac{1}{2}(1-p)$ and $P(\text{Event 2}) = p \times \frac{1}{2}$.

Therefore,

$$\begin{aligned} &P(\text{number heads in experiment } B > \text{number of heads in experiment } A) \\ &\quad = P(\text{Event 1}) + P(\text{Event 2}) = \frac{1}{2}. \end{aligned}$$

Solution 4

[8]

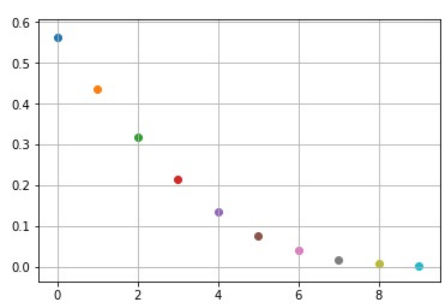
If, in general $\mathcal{A}$ tosses a fair coin n times and $\mathcal{B}$ tosses the coin $n+1$ times, then, due to symmetry, the probability of $\mathcal{B}$ having more heads than $\mathcal{A}$ is the same as him/her having more tails and this with equal probabilities. $\mathcal{B}$ must either have more heads or more tails than $\mathcal{A}$. Hence, both probabilities are $\frac{1}{2}$.

Notes

[78]

Interestingly, even when both players toss a fair coin an equal number of times, say $(m = n = 20)$, there's a 40% chance that one of them ends up with more heads than the other. Equal distributions do not collapse to zero.

Following is the result of a simulation where both players have the same number $(m = n = 20)$ of tosses, but we count the difference of heads between them. We plot the probability as a function of the difference of heads:



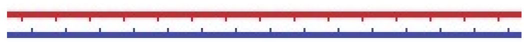
We note that, given two players toss a fair coin the same number of times, there's a 44% chance one of them has at least one more heads than the other, so a fairly high chance of a 1 heads lead.

As we insist on a lead of more than 1 heads, the probability drops off almost linearly. Chances of a 2 heads lead are about 30%, 4 heads lead about 15%, 6 heads about 5% and practically zero chance of an 8 heads lead.

Probability of Visiting Grandparents

Solution 1

The schedules of the trains in the two directions are shifted relative to each other:



If, e.g., on the red line, the trains arrive at 3, 6, 9, etc. minutes to an hour, while on the blue line that's 1, 4, 7, 10 minutes, etc., then the chances of the boy catching the red direction are 2 to 1.

Solution 2

Amit Itagi

The probability distribution function of his arrival is such that the ratio of the areas under the probability distribution function for time intervals corresponding to the two directions is ~ 2.

Solution 3

Elia Noris

Probability of Increasing Sequence

Solution to Question 1

It does not at all matter whether there are 99 slips of paper or just five. The best way to approach the problem is to consider drawing just two numbers. So you've drawn two distinct numbers: what is the probability that the first one will be less than the second? Well, there are just two ways to draw two chosen numbers: the smallest either comes first or last, meaning that the probability of drawing two numbers in their natural order is $\dfrac{1}{2}$.

When it comes to five numbers. There are 120 permutations of five objects and thus 120 ways to draw a given set of five numbers. Only in one permutation out of 120 all five numbers follow in their natural order of magnitude.

Solution 1 to Question 2

Josh Jordan

$$\frac{\#\text{ of ways to choose four other numbers when one has already been chosen}}{\#\text{ of ways to choose five numbers}}$$
$$= \frac{\binom{98}{4}}{\binom{99}{5}} = \frac{5}{99}.$$

Solution 2 to Question 2

Michael Weiner

Let A be the event that a 1 is drawn. Let B be the event that the numbers come out in increasing order. I asked for $P(A|B)$ but A and B are independent, so $P(A|B) = P(A)$. Five numbers are drawn; the probability that one of them is 1 is $\dfrac{5}{99}$.

Probability of Two Integers Being Coprime

Notes

A short solution of this statement by Aaron Abrams and Matteo Paris [1] in the *College Mathematics Journal* was introduced by a letter from Henry L. Adler where he tells the story of the proof. He found the question "natural to ask" so it was no surprise when students in his introductory number theory class indeed raised the question. He answered it with a "traditional proof" that required "considerable prior knowledge of number theory," but then the two authors (16- and 17-year-olds at the time) came up with a short and intuitively clear proof. He then encouraged the boys to share their proof with the readers of the *College Mathematics Journal* "who might be asked the same question in their classes."

A few lines of the proof made a reference to [103, pp. 202–204] for the justification of the concepts involved (the formalism of selecting a random integer). Yagloms's book also poses the question at hand and provides a more or less elementary but very detailed solution. Yagloms refers to the question as "Chebyshev's problem" and outlines the contribution of Pafnuty L. Chebyshev in a short footnote, although there is no indication that the solution in the book originated with the famous Russian mathematician. It is rather straightforward and does not require "considerable prior knowledge of number theory."

One that does can be found in the classic, *An Introduction to the Theory of Numbers* by Hardy and Wright [49]. Without doubt, Chebyshev could have authored either of the two proofs, but Hardy and Wright do not mention his name either as the question poser or a solver.

In any event, I find it amusing that this "natural to ask question" might have been first asked by a mathematician of Chebyshev's stature.

Following are all three proofs in this order: Yagloms's, Abrams and Paris's and then Hardy and Wright's; the latter will be extended shortly with relevant information from number theory. A simple intuitive proof precedes the three.

Aaron Abrams and Matteo Paris's proof can be found in [2, pp. 230–233], where the authors also refer to Yagloms's book but do not mention Chebyshev's name, the boys' ages, or Adler's story.

Solution 1, à la Euler

The common convention for selecting a random integer is selecting one uniformly randomly from the set $\{1, 2, \ldots, n\}$ and then letting $n \to \infty$.

For a prime p, every p^{th} integer is divisible by p. Assuming that divisiblity by p for two random numbers constitute independent events, the probability that two numbers are not divisible by p is $\dfrac{1}{p^2}$ so that the probability that at least one of them is divisible by p is $1 - \dfrac{1}{p^2}$.

Further, divisibility of a number by one prime is independent of its divisibility by another. Thus, rather informally, the sought probability is

$$\prod_{p \text{ is prime}} \left(1 - \frac{1}{p^2}\right) = \frac{6}{\pi^2}$$

as proved by L. Euler in 1735. For a more rigorous proof, see [49, Theorem 332].

Solution 2, Straightforward

If two numbers are coprime, then, in particular, no prime is their common factor. So we start with a simpler problem:

> What is the probability that 2 is not a common factor of two random integers?

Given two random numbers a and b, for each, the probability of being divisible by 2 is $\frac{1}{2}$ such that the probability of them both being divisible by 2 is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. Therefore, the probability that 2 is not their common factor is $1 - \frac{1}{4}$.

Similarly, for any prime p, the probability that p does not divide both selected numbers is $1 - \frac{1}{p^2}$.

There is no basis to assume that divisibility by one prime number is in anyway related to divisibility by any other prime number. We therefore assume that events of divisibility by prime numbers (or their complements) are independent such that the probability that two numbers are not simultaneously divisible by any prime should be the product

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right) \cdots = \prod_p \left(1 - \frac{1}{p^2}\right),$$

where the product is taken over the whole sequence of prime numbers. This needs to be understood as the limit of finite products:

$$\prod_p \left(1 - \frac{1}{p^2}\right) = \lim_{n\to\infty} \prod_{k=1}^{n} \left(1 - \frac{1}{p_k^2}\right),$$

where $\{p_k\}$ is the sequence of prime numbers $p_1 = 2$, $p_2 = 3$, $p_3 = 5$ and so on.

We have to check that the limit exists:

$$\begin{aligned}\left[\prod_{k=1}^{n} \left(1 - \frac{1}{p_k^2}\right)\right]^{-1} &= \prod_{k=1}^{n} \frac{1}{1 - \frac{1}{p_k^2}} \\ &= \prod_{k=1}^{n} \left(1 + \frac{1}{p_k^2} + \frac{1}{p_k^4} + \frac{1}{p_k^6} + \cdots\right)\end{aligned}$$

because each fraction $\dfrac{1}{1 - \frac{1}{p_k^2}}$ is the sum of the geometric series $1 + \frac{1}{p_k^2} + \frac{1}{p_k^4} + \frac{1}{p_k^6} + \ldots$.

But, if we multiply out, we get a surprising result:

$$\prod_{k=1}^{n}\left(1+\frac{1}{p_k^2}+\frac{1}{p_k^4}+\frac{1}{p_k^6}+\cdots\right)=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}\cdots$$

where the sum is over all integers whose prime factors do not exceed p_k. As k grows, the gaps in the series are filled in so that at the limit

$$\prod_{p}\left(1+\frac{1}{p_k^2}+\frac{1}{p_k^4}+\frac{1}{p_k^6}+\cdots\right)=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}\cdots$$

where on the right, each integer appears exactly once due to the *Fundamental Theorem of Arithmetic*, i.e., the fact that every integer has a unique (up to the order of the factors) representation as the product of powers of distinct prime numbers. The sum on the right is known to converge and its sum, as was first found by L. Euler, equals

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}\cdots=\frac{\pi^2}{6},$$

from which it follows that the probability of two random integers being coprime is $\dfrac{6}{\pi^2}$.

Solution 3, Shortest Option

Let q denote the sought probability of two random integers being coprime. Pick an integer k and two random numbers a and b. The probability that k divides a is $\dfrac{1}{k}$, and the same holds for b. Therefore, the probability that both a and b are divisible by k equals $\dfrac{1}{k^2}$. The probability that a and b have no other factors, i.e., that $\gcd\left(\dfrac{a}{k},\dfrac{b}{k}\right)=1$, equals q by our initial assumption. But $\gcd\left(\dfrac{a}{k},\dfrac{b}{k}\right)=1$ is equivalent to $\gcd(a,b)=k$. Assuming independence of the events, it follows that the probability that $\gcd(a,b)=k$ equals $\dfrac{1}{k^2}\cdot q=\dfrac{q}{k^2}$.

Now, k was just one possibility for the greatest common divisor of two random numbers. Any number could be the $\gcd(a,b)$. Furthermore, since the events $\gcd(a,b)$ are mutually exclusive (the gcd of two numbers is unique) and the total probability of having a gcd at all is 1, we are led to

$$1=\sum_{k=1}^{\infty}\frac{q}{k^2},$$

implying that

$$q=\left[\sum_{k=1}^{\infty}\frac{1}{k^2}\right]^{-1}=\frac{6}{\pi^2}.$$

Solution 4, Number-Theoretic

Notations first:

$\phi(n)$—Euler's totient function, the number of integers not greater than n and coprime to n:

$$\Phi(n) = \sum_{k=1}^{n} \phi(n).$$

$\mu(n)$—the Möbius function:

$$\begin{cases} \mu(1) = 1; & \\ \mu(n) = 0, & \text{if } n \text{ has a squared factor;} \\ \mu(p_1 p_2 \dots p_k) = (-1)^k, & \text{if all primes } p_k \text{ are different.} \end{cases}$$

Both functions ϕ and μ are multiplicative, meaning that for coprime a and b, $\phi(ab) = \phi(a)\phi(b)$ and $\mu(ab) = \mu(a)\mu(b)$.

The two functions are related in the following manner:

$$\phi(n) = n \sum_{d|n} \mu(d) = \sum_{d|n} \frac{n}{d}\mu(d) = \sum_{d|n} d\mu\left(\frac{n}{d}\right) = \sum_{dd'=n} d'\mu(d).$$

μ has an important property:

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Finally, remember the "Big O" notation: to say that $f = O(g)$ is equivalent to saying that $|f| \leq C|g|$, for C not dependent on g.

$$\begin{aligned} \Phi(n) &= \sum_{m=1}^{n} m \sum_{d|m} \frac{\mu(d)}{d} = \sum_{dd' \leq n} d'\mu(d) \\ &= \sum_{d=1}^{n} \mu(d) \sum_{d'=1}^{[n/d]} d' = \frac{1}{2} \sum_{d=1}^{n} \mu(d) \left(\left[\frac{n}{d}\right]^2 + \left[\frac{n}{d}\right] \right) \\ &= \frac{1}{2} \sum_{d=1} \mu(d) \left\{ \frac{n^2}{d^2} + O\left(\frac{n}{d}\right) \right\} \\ &= \frac{1}{2} n^2 \sum_{d=1}^{n} \frac{\mu(d)}{d^2} + O\left(n \sum_{d=1}^{n} \frac{1}{d} \right) \\ &= \frac{1}{2} n^2 \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} + O\left(n^2 \sum_{n+1}^{\infty} \frac{1}{d^2} \right) + O(n \log n) \\ &= \frac{n^2}{2\zeta(2)} + O(n) + O(n \log n) = \frac{3n^2}{\pi^2} + O(n \log n), \end{aligned}$$

where $\zeta(s)$ is the famous ζ function: $\zeta(s) = \displaystyle\sum_{n=1}^{\infty} n^{-s}$. As mentioned above, it was established by L. Euler that $\zeta(2) = \dfrac{\pi^2}{6}$.

Noteworthy is the fact that $\Phi(n)+1$ is the number of terms in the n^{th} Farey series [15] which consists of an ordered list of irreducible fractions from the interval $[0,1]$ whose denominators do not exceed n.

Now, assume n is given and consider fractions $\dfrac{q}{p}$ that satisfy

$$q > 0 \text{ and } 1 \le q \le p \le 1.$$

Say ξ_n is the number of those fractions in lowest terms out of the total $\psi_n = \dfrac{n(n+1)}{2} \sim \dfrac{n^2}{2}$ fractions. Thus, the probability that q and p are coprime is given by

$$\lim_{n\to\infty} \frac{\xi_n}{\psi_n} = \frac{6}{\pi^2}.$$

Overlapping Random Intervals

[21, Graham Brightwell]

Instead of the unit interval, we drop $2n+1$ points on a circle. One, say X, is the starting point for traversing the circle. All other points are labeled in successive pairs $a, b, \ldots$. By assumption, all cyclic orderings of $2n$ points are equally likely. Given such point placement, start with X and move clockwise until you meet one symbol, say a, for the second time. Turn around, and move counterclockwise until you meet another symbol, say b, twice. Necessarily, you have gone beyond X. Now look at the string of symbols between b and a at the extremes of this trajectory but running clockwise from the (second) b to the (second) a. This string includes X, the other a and the other b. Look at the order in which these three occur. There are just three possibilities:

1. bXa,
2. Xba,
3. Xab.

Claim

In cases 2 and 3, the interval b intersects all others, whereas in case 1 there is no interval that intersects all others.

Proof of the Claim

In cases 2 and 3, if the b-interval does not intersect some c-interval, then the relevant symbols occur either as $bccXb$ or $bXccb$, contradicting the choice of b in the first case and the choice of a in the second.

On the other hand, in case 1, if a c-interval intersects both the b-interval and the a-interval, then both c symbols must lie in the string, contradicting the choice of b. This completes the proof.

Since all distributions are equally likely, each of the three cases has probability of $\frac{1}{3}$, meaning that the existence of an interval that meets all other intervals is $\frac{2}{3}$, independent of n.

Notes

Justicz, Scheinerman and Winkler proved a more general result in [59].

For any $k < n$, the probability that in a family of n random intervals there are at least k which intersect all others is

$$\frac{2^{k+1}}{\binom{2k+1}{k}}$$

which is independent of n.

Random Clock Hands

As hard as it may be to believe, the events of the two hands hitting red are independent even when they are rigidly attached at a certain angle. Imagine throwing the first hand randomly, attaching the second hand at a certain angle but only drawing the latter. Would you think that the second hand (which actually appears alone) is less random or has a different probability of hitting red than the lone first hand?

To add a little more rigor to the above argument, let $\mathbf{R}$ be the set of red points on the unit circle. (Selecting a hand is equivalent to picking a point on the unit circle.) We may also think of it as a set $\mathbf{R}_o$ in the interval $[0, 2\pi)$. Let α be the angle between the two hands. Selecting the second hand is equivalent to selecting a point in $[0, 2\pi)$ and then shifting it by α, i.e., selecting a random point in $[0, 2\pi)+\alpha$, which is the same $[\alpha, 2\pi+\alpha)$. The set $\mathbf{R}_o$ undergoes a shift into $\mathbf{R}_o+\alpha$, which has the same relative size as the set $\mathbf{R}_o$. Furthermore, the original set $\mathbf{R}$ is clearly left unchanged by the shift. What does change is the point at which the circle is cut before being unfolded into a segment.

Crossing River after Heavy Rain

The configuration of bridges has a remarkable symmetric structure, Figure 3.1.

Imagine that there is a boat floating down the stream. Any bridge that withstood the storm will block the passage of the boat. The boat will pass through if a sufficient number of bridges have been washed away. What is the probability that the boat will get through?

The key observation that brings a solution to the problem is that it has the same abstract structure for the traveler and the boat, Figure 3.2.

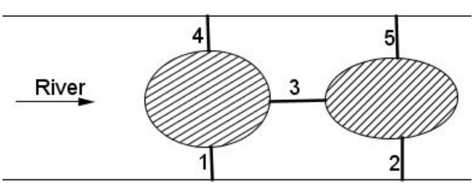


Figure 3.1: Crossing river after heavy rain.

Figure 3.2: Crossing river after heavy rain: abstract structure.

For the traveler, bridges 1 and 2 are the entries, whereas bridges 4 and 5 are exits. For the boat, the entries are at bridges 1 and 4 and the exits are at bridges 2 and 5.

A bridge that has withstood the storm is good for the traveler but bad for the boat and vice versa. But each individual event has the probability of 50%. Having an identical abstract structure suggests (by the Principle of Symmetry, Appendix B on page 285) the following observation (writing $P(X)$ for the probability of event X):

1. $P(\text{traveler crosses}) = P(\text{boat gets through})$.

   But, beyond that, as we just observed, what is good for one is bad for the other such that

2. $P(\text{boat gets through}) = P(\text{traveler does not cross})$.

   As a consequence of 1 and 2, we get

3. $P(\text{traveler crosses}) = P(\text{traveler does not cross})$,

with the inevitable conclusion that both probabilities evaluate to 50%.

The Most Likely Position

Notes

The formulation of the problem seems to be open to interpretations. There are at least two:

1. On many, many trials, what is the expected position of the first ace?

2. For what position of the first ace is the probability the highest?

The first of these is discussed elsewhere (Problem 8.10 on page 194). The second question may appear obvious to experts, with an immediate answer. The answer is still surprising to nonexperts, so this interpretation will be solved:

> A standard deck has 52 cards, of which four are aces. When the deck is shuffled, for what position of the first ace is the probability the highest?

Solution 1

@BKKeconcode on Twitter

Let $p(k)$ be the probability for the first ace to appear at position k. Then

$$\begin{aligned}
p(1) &= \frac{4}{52}\\
p(2) &= \frac{48}{52}\cdot\frac{4}{51}\\
p(3) &= \frac{48}{52}\cdot\frac{47}{51}\cdot\frac{4}{50}\\
p(4) &= \frac{48}{52}\cdot\frac{47}{51}\cdot\frac{46}{50}\cdot\frac{4}{49}\\
\ldots\\
p(k) &= \left(\prod_{j=0}^{j=k-2}\frac{48-j}{52-j}\right)\cdot\frac{4}{53-k}.
\end{aligned}$$

Note that

$$\begin{aligned}
\frac{p(k+1)}{p(k)} &= \left(\frac{49-k}{53-k}\cdot\frac{4}{52-k}\right)\cdot\left(\frac{53-k}{4}\right)\\
&= \frac{49-k}{52-k} < 1,
\end{aligned}$$

implying $p(k+1) < p(k)$ so that $p(1)$, however small in itself, is the largest of all the probabilities.

Solution 2

Amit Itagi

If the first ace is at position n, the number of ways of placing the remaining three aces is $\binom{52-n}{3}$. Having chosen these four positions, the number of permutations of cards is $4!48!$. Thus, the probability of the first ace being at position n is

$$P(n) = \binom{52-n}{3}\cdot\frac{4!48!}{52!} = \frac{\binom{52-n}{3}}{\binom{52}{4}}.$$

$\binom{52-n}{3}$ is a decreasing function of n. Thus, the probability is maximized for $n=1$.

Coin Tossing Surprises I

A. Bogomolny

Notes

Since all four possible outcomes of two coin tosses are equiprobable, a couple of facts may seem surprising and, as such, have been discussed in a number of publications, e.g., by Martin Gardner in his *The Colossal Book of Mathematics* [44].

First, as the transition diagram (Markov chain) shows, except for an appearance at the first two tosses, TT may only show up after HT, Figure 3.3.

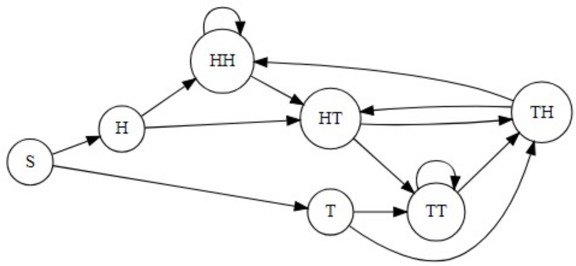


Figure 3.3: Coin tossing: whole picture.

So perhaps the diagram makes it less surprising that the expected number of tosses to the first appearance of HT is less than that for TT. This phenomenon will acquire a more accurate estimate in these solutions.

Solution to Question 1

In Figure 3.4 we only focus on achieving HT. Assign a variable to each of the three nodes, say x, y, z, that express the expected number of tosses in reaching our present destination, namely, HT. Naturally, $z = 0$ and was only added for the sake of uniformity. The three are combined into two equations:

$$y = 1 + \frac{1}{2}y + \frac{1}{2}z$$
$$x = 1 + \frac{1}{2}x + \frac{1}{2}y.$$

The inclusion of 1 in the equations is due to the fact that they describe the number of coin tosses, and each tells us how these expectations are connected after one toss. The first equation yields $y = 2$ so that, from the second equation, $x = 4$.

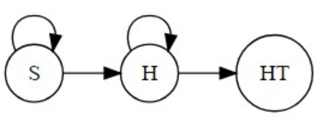


Figure 3.4: Getting HT.

Solution to Question 2

As in question 1, we introduce the variables x, y, z, with $z = 0$, again:

$$y = 1 + \frac{1}{2}x + \frac{1}{2}z$$
$$x = 1 + \frac{1}{2}x + \frac{1}{2}y.$$

The second equation reduces to $x = y + \frac{1}{2}y$ so that $y = \frac{2}{3}x$. Then the first equation becomes $\frac{2}{3}x = 1 + \frac{1}{2}x$, from which $x = 6$.

Solution to Question 3

Each of the four possible outcomes of two coin tosses is equiprobable and comes with the probability of $\frac{1}{4}$. This is the only opportunity for TT to come before HT (Figure 3.5). It follows that HT is three times more likely than TT to show up first. In other words, the probability of HT showing first is $\frac{3}{4}$.

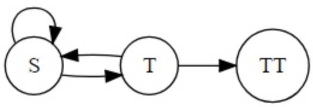


Figure 3.5: Getting TT.

Solution to Question 4

Both HT and HH start with H but then split with equal probabilities of $\frac{1}{2}$. It follows that the probability of HT preceding HH and the complementary one of HH preceding HT are both $\frac{1}{2}$.

Zhuo Xi suggested that question while pointing out that HH, like TT, has 6 tosses as the expected time to first appearance. This shows that the probability of $\frac{3}{4}$ from question 3 is unrelated to the ratio 6:8 of the "time to first appearance" computed in questions 1 and 2.

We may summarize similar results for all pairs of outputs of two tosses:

	HH	HT	TH	TT
HH		$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$
HT	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{3}{4}$
TH	$\frac{3}{4}$	$\frac{1}{2}$		$\frac{1}{2}$
TT	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	

Chapter 4

Basic Probability

> The very name calculus of probabilities is a paradox. Probability opposed to certainty is what we do not know, and how can we calculate what we do not know?
>
> Henri Poincare, 1913, *Science and Hypothesis*

Riddles

4.1 Admittance to a Tennis Club

[45, pp. 2–10], [51, pp. 4–6]

As a condition for the acceptance to a tennis club, a novice player N is set to meet two members of the club, G (**g**ood) and T (**t**op), in three games. In order to be accepted, N must win against both G and T in two successive games. N must choose one of the two schedules: playing G, T, G or T, G, T. Which one should he choose?

4.2 Black Boxes in a Chain

[71]

N black boxes are chained so that the output of one is the input of the next. The inputs (and the outputs) are binary, 0 or 1.

Each box passes its input on with probability p and changes it with probability $1 - p$.

What is the probability of having the last output the same as the original input?

4.3 A Question of Checkmate

[88, Problem 2.5]

The white king is randomly placed in the third row, the black king in the first row and the white queen either in the second row or elsewhere in the first row. What is the probability that the pieces are in a checkmate position?

4.4 Concerning Even Number of Heads

[71]

Flipping a coin, heads comes up with the probability of p, $0 < p < 1$.

- What is the probability of having an even number of heads in n flips?
- What is the expected number of flips before getting an even number of heads the first time?

4.5 Getting Ahead by Two Points

[12, Proof 3]

A and B are playing ping-pong with the agreement that the winner of a game will get 1 point and the loser 0 points; the match ends as soon as one of the players is ahead by 2 points or the number of games reaches 6. Suppose that the probabilities of A and B winning a game are $\frac{2}{3}$ and $\frac{1}{3}$, respectively, and each game is independent. Find the expectation $E(\xi)$ for the match ending with ξ games.

4.6 Recollecting Forgotten Digit

[103]

You forget a digit of a phone number and make random guesses. What is the probability that you'll guess correctly in at most two attempts? Three attempts?

4.7 Two Balls of the Same Color

[26, Problem 65284]

In an urn there are black and white balls, 121 balls in all. The probability of randomly drawing two balls of the same color equals 0.5.

How many balls of each color are there?

4.8 To Bet or Not to Bet

[8, p. 317]

I invite you to play the following card game:

Shuffle an ordinary pack of cards and deal them in pairs, face up. If both cards of a pair are black, you get them; if both are red, I get them. Otherwise, the cards are ignored.

You pay \$1 for the privilege of playing the game. When the deck is exhausted, the game is over, and you pay nothing if you have no more cards than I have. On the other hand, I will pay you \$3 for every card that you have more than I have.

Would you care to play with me?

4.9 Lights on a Christmas Tree

[26, Problem 65277]

For the holidays, a Christmas tree was adorned with 100 lights. The lights were numbered from 1 through 100. A controller was programmed to apply a switching algorithm: on every step the light that was on is turned off and vice versa. The algorithm worked in a sequence. First it turned all lights on. It then applied to the even numbers, then to the multiples of 3, next to the multiples of 4, and so on.

What was the probability that a randomly selected light would be on after 100 steps of the algorithm?

4.10 Drawing Numbers and Summing Them Up

[3, Problem 24]

A bag contains four pieces of paper, each labeled with one of the digits, 1, 2, 3, 4, with no repeats. Three of these pieces are drawn, one at a time without replacement, to construct a three-digit number. What is the probability that the three-digit number is a multiple of 3?

4.11 Numbers in a Square

[86]

Each of the 49 entries in a square 7×7 table is filled with an integer, $1, 2, \ldots, 7$, so that each column contains all of the seven integers and the table is symmetric with respect to its diagonal D going from the upper left corner to the lower right corner. What is the probability that the diagonal D has all the integers $1, 2, \ldots, 7$?

4.12 Converting Temperature from °C to °F

[64, Problem 108]

A two-digit street temperature display flashes back and forth between temperature in degrees Fahrenheit and degrees Centigrade (Celsius). Suppose that over the course of a week in the summer, the temperature was uniformly distributed between 15°C and 25°C.

What is the probability that, at any given time, the rounded value in degrees F of the converted temperature (from degrees C) is not the same as the value obtained by first rounding the temperature in degrees C, then converting to degrees F, then rounding once more.

4.13 Probability of No Distinct Positive Roots

Christopher D. Long, Riddle 10.5 on page 261

An ordered pair (b, c) of real numbers, each with an absolute value less than or equal to five, is chosen at random with each such ordered pair having an equal likelihood of being chosen.

What is the probability that the equation $x^2 + bx + c = 0$ will not have distinct positive roots?

4.14 Playing with Integers and Limits

[64, Problem 68]

A game starts with a random integer. Two players take turns modifying the integer present on their turn. A legal move depends on whether the integer is odd or even. If it is even, the player can either subtract 1 or divide by 2. If the number is odd, the player may subtract 1 or subtract 1 and subsequently divide by 2. The game ends when the number reaches 0, and the player who made the last move wins.

Assuming both players use the best available strategies, if the starting number ranges from 1 to some n, with $n \to \infty$, what is the probability of a win for the second player?

4.15 Given Probability, Find the Sample Space

[82, Chapter: From Sex to Quadratic Forms]

A school teacher is in charge of a group of students. She wants to select two of them at random and observes that it is exactly an even chance (50%) that they are of the same sex. What can be said about the number of children of each sex in the group?

4.16 Gladiator Games

[84, pp. 76–77]

There are two teams of gladiators, each of which has been assigned a strength attribute (a positive number). When two gladiators from different teams meet, their chances to win the duel are proportional to their strengths. A fellow who loses a duel, dies and takes no part in further combats. The winner—who returns to the bench and is immediately ready for another fight—has his strength augmented by that of the loser. The last team standing wins.

Let there be two teams with individual strengths (in, say, non-decreasing order) $\{a_1, a_2, \ldots, a_m\}$ and $\{b_1, b_2, \ldots, b_n\}$. Find an optimal strategy for a team, i.e., the order of sending team members to duel opponents from the other team, to maximize its chances of winning the competition.

4.17 In Praise of Odds

[56, Chapter 12]

Teams A and B play a series of games, each with three possible outcomes: A wins, B wins or they tie. The probability that A wins is p, the probability that B wins is q and the probability that they tie is $r = 1 - p - q$. The series ends when one team has won two more games than the other, that team being declared the winner of the series.

What is the probability that team A wins the series?

4.18 Probability in Scoring

[26, Problem 65264]

A boy plays a game of Scoring with a computer without yet learning to play optimally, but the computer makes no mistakes. They start with a pile of 14 stones and remove 1, 2, 3, 4 or 5 stones at a time. The player to remove the last stone wins. The boy goes first.

What is the probability of his winning the game?

4.19 Probability of 2^n Beginning with Digit 1

[103]

What is the probability that a power of 2 begins with the digit 1?

For an event A that could be defined for all $n \in \mathbb{N}$, $A(N) = A \cap [1, 2, \ldots, N]$. The probability $P(A)$ is defined as $\lim\limits_{N\to\infty} \dfrac{|A(N)|}{N}$.

4.20 Probability of First Digits in a Sequence of Powers

[90]. The riddle is an extension of the previous one, Riddle 4.19

Let N be an r-digit number. What is the probability that the first r digits of 2^n represent N?

4.21 Probability of Four Random Integers Having a Common Factor

[103]

What is the probability of four random integers having a common factor?

4.22 Probability of a Cube Ending with 11

[46]

What is the probability that the cube of a positive integer chosen at random ends with the digits 11.

4.23 Odds and Chances in Horse Race Betting

[65, Problem 313]

If the odds against a horse winning a race are $a : b$, then we may call $\dfrac{b}{a+b}$ the apparent chance of that horse winning the race.

Prove that, if the sum of all apparent chances of winning of all the horses in the race is less than 1, one can arrange bets so as to make sure of winning the same sum of money whatever the outcome of the race.

4.24 Acting As a Team

[50, Chapter 6], [102, pp. 120, 123–126]

Three hats (each of which is either red or blue) are placed on the heads of three people facing each other. Each has to guess the color of his hat, if possible. There are some rules to the task:

1. The players act as a team. The team wins or loses, not the individuals.
2. After the hats have been placed on the heads there must be no communication between team members.
3. Each player may guess only once and all those who choose to make a guess do so simultaneously on a signal.
4. Each player is allowed to pass rather than make a guess.
5. The team wins only if at least one player makes a correct guess and no player guesses incorrectly.

Devise a strategy (that can be discussed before the game starts) and compute the resulting probability of success. Try to make the latter as large as you can imagine.

4.25 Sum of Two Outcomes of Tossing Three Dice

[79, AFSME 1996 47-16]

A fair standard six-sided die is tossed three times. Given that the sum of the first two tosses equals the third, what is the probability that at least one 2 is tossed?

4.26 Chess Players Truel

Players A, B and C play a series of chess games (a truel). Assume that A is the strongest player and C is the weakest one. Assume that there is no tie for each game. The winner of each game will play with the third player. The player who first gets two wins is the winner of the series. Player B determines who will play the first game. Find the best choice for B. In general, if probability of A to win against B is $p > .5$, probability of B to win against C is $q > .5$, probability of A to win against C is r and $r > p$, evaluate the chances of B.

4.27 Two Loaded Dice

[48, Problem 8.7]

Show that if two dice are loaded with the same probability distribution, the probability of doubles is always at least $\frac{1}{6}$.

4.28 Crossing Bridge in Crowds

[31]

It takes 5 minutes to cross a certain bridge and 1000 people cross it in a day of 12 hours, all times of day being equally likely.

Find the probability that there will be nobody on the bridge at noon.

Solutions

Admittance to a Tennis Club

Let g and t denote the probabilities of N beating G and T, respectively. The possibilities for the sequence TGT can be summarized in the following table:

T	G	T	Probability
W	W	W	tgt
W	W	L	$tg(1-t)$
L	W	W	$(1-t)gt$

Pertinent to the previous discussion is the observation that the first two rows naturally combine into one: the probability of the first two wins is

$$P(WW) = tgt + tg(1-t) = tg,$$

which is simply the probability of beating both T and G (in the first two games in particular).

Since winning the first two games and losing the first game but winning the second and the third are mutually exclusive events, the sum rule applies. Gaining acceptance playing the TGT sequence has the total probability of

$$P_{TGT} = tg + tg(1-t) = tg(2-t).$$

Similarly, the probability of acceptance for the GTG schedule is based on the following table:

G	T	G	Probability
W	W		gt
L	W	W	$(1-g)tg$

The probability in this case is found to be

$$P_{GTG} = gt + gt(1-g) = gt(2-g).$$

This is a curiosity. Do you see why?

Assuming that the top member T is a better player than just the good one G, $t < g$. But then $gt(2-g) < tg(2-t)$. In other words,

$$P_{GTG} < P_{TGT}.$$

The novice N has a better chance of being admitted to the club by playing the apparently more difficult sequence TGT than the easier one GTG. Perhaps there is a moral to the story/riddle: the more difficult tasks offer greater rewards.

Black Boxes in a Chain

Solution 1

The original input and the final output will coincide if the number of input/output reversals is even; they will differ if the number of reversals is odd.

For each box, imagine a coin toss that comes up heads with a probability p decides the behavior of the box. We know that after N transmissions the probability of having an even number of heads is $P(N) = \dfrac{1}{2} + \dfrac{1}{2}(1-2p)^N$.

If N is even, then the probability of having an even number of tails, i.e., an even number input/output reversals, is $Q(N) = P(N) = \dfrac{1}{2} + \dfrac{1}{2}(1-2p)^N$. But, if N is odd then getting an even number of tails is the same as getting an odd number of heads, i.e.,

$$Q(N) = 1 - P(N) = 1 - \left(\frac{1}{2} + \frac{1}{2}(1-2p)^N\right) = \frac{1}{2} - \frac{1}{2}(1-2p)^N.$$

The two expressions for $Q(N)$ can be combined into a single formula:

$$Q(N) = \frac{1}{2} + \frac{(-1)^N}{2}(1-2p)^N = \frac{1}{2} + \frac{1}{2}(2p-1)^N.$$

Solution 2

With the same reasoning as solution 1, if $q = 1 - p$ then the probability of getting an even number of tails is

$$\begin{aligned} Q(N) &= \frac{1}{2} + \frac{1}{2}(1-2q)^N \\ &= \frac{1}{2} + \frac{1}{2}(1-2(1-p))^N \\ &= \frac{1}{2} + \frac{1}{2}(2p-1)^N. \end{aligned}$$

Solution 3

Amit Itagi

Any number of flips will do. Let $x = p$ and $y = 1 - p$. Then the required probability is

$$\sum_{k \text{ is even}}^{N} = \frac{1}{2}\left[(y+x)^N + (y-x)^N\right] = \frac{1}{2}\left[1 + (2p-1)^N\right].$$

A Question of Checkmate

This is clearly a counting problem in which the probability of an event is determined in the classical manner, as the ratio of the number of successful outcomes to the number of all possible outcomes. The latter is immediate: The white king may be in one of eight positions in the 3rd row. Similarly, the black king may be in one of eight positions in the first row. The white queen is either in one of the eight positions in the 2nd row or in one of the remaining seven in the first. By the product rule, the total number of possibilities is $8 \times 8 \times (8 + 7) = 960$.

The task is to count the number of the successful outcomes, i.e., the number of positions in which the black king is checkmated. There are several cases: two for the black king in a corner, and three for the black king elsewhere.

The Black King in a Corner

For each corner, the white pieces may be in 16 positions, with the total of $2 \times 16 = 32$ possible configurations, Figure 4.1.

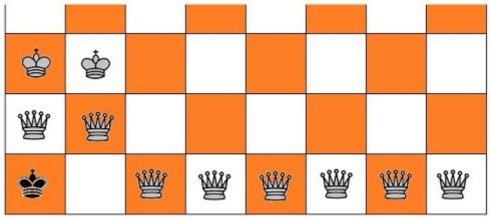


Figure 4.1: The black king in a corner.

For each corner, there is one extra position (+2), Figure 4.2.

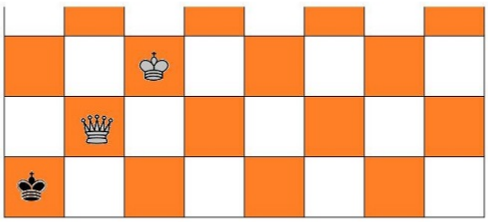


Figure 4.2: The black king in a corner, extra position.

The Black King Not in a Corner

There are five positions when the kings are in opposition ($+6 \times 5 = 30$), Figure 4.3.

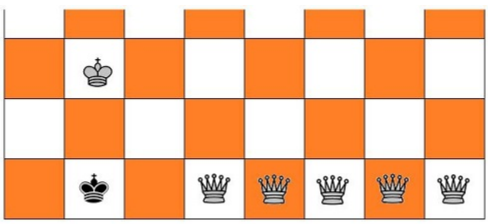


Figure 4.3: The kings are in opposition.

There are three positions with the white queen in the second row ($+6 \times 3 = 18$), Figure 4.4.

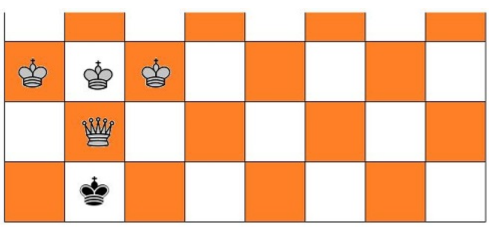


Figure 4.4: The white queen in the second row.

There are five positions where the kings are one vertical position apart and the black king is blocked by the queen from moving into the second row. This could happen in either direction ($+2 \times 5 = 10$), Figure 4.5.

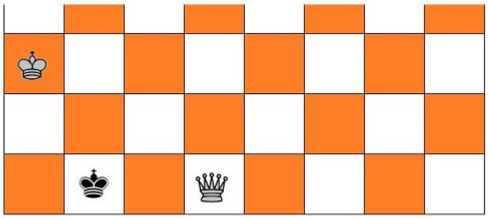


Figure 4.5: The kings are one vertical position apart.

All in all, the number of checkmate positions is $32 + 2 + 30 + 18 + 10 = 92$. The probability of this happening is $\dfrac{92}{960} = \dfrac{23}{240}$.

Concerning Even Number of Heads

Let $P(n)$ be the probability of having an even number of heads after n coin tosses. Obvioiusly, $P(0) = 1$, as 0 is an even number. Also, $1 - P(n)$ is the probability of having an odd number of heads after n tosses.

If the number of heads was even after n tosses, it will remain even next time with probability $1-p$; if the number of heads was odd it will become even with probabiliy p. Thus, we can establish that $P(n)$ satisfies the following recurrence relation:

$$P(n+1) = p(1-P(n)) + (1-p)P(n) = p + (1-2p)P(n),$$

with the initial condition $P(0) = 1$. To make sure that we are on the right track, $P(1) = p + (1-2p)P(0) = 1-p$, the probability of having a tails and, in this case, keeping the number of heads even. To solve the recurrence, assume $P(n) = A + Ba^n$, for some constants A, B and a:

$$A + Ba^{n+1} = p + (1-2p)(A + Ba^n).$$

Handling the expressions by A and B separately gives

$$\begin{aligned} A &= p + (1-2p)A \\ Ba^{n+1} &= (1-2p)Ba^n. \end{aligned}$$

We find $A = \dfrac{1}{2}$, independent of p, and B arbitrary, provided $a = 1-2p$. Thus, the solution to the recurrence relation appears to be

$$P(n) = \frac{1}{2} + B(1-2p)^n.$$

To detemine B, we will use the initial condition: $1 = P(0) = \dfrac{1}{2} + B$, which gives $B = \dfrac{1}{2}$, with the final form of the solution $P(n) = \dfrac{1}{2} + \dfrac{1}{2}(1-2p)^n$.

For a fair coin, i.e., if $p = \dfrac{1}{2}$, $P(n) = \dfrac{1}{2}$, independent of n.

To answer the second question, for the first appearance of an even number of heads in n tosses,

- the last one needs to be heads,

- the total number of heads needs to be 2 (in the limit the rare event of no heads will play no role),
- the first heads could be in any of the first $n-1$ tosses.

Thus, we get the expectation of the first occurrence of an even number of heads:

$$E(p) = \sum_{n=2}^{\infty} n(n-1)(1-p)^{n-2}p^2 = \frac{2}{p}.$$

(Denote, say, $f(x) = \sum_{n=2}^{\infty} n(n-1)x^{n-2}$. Integrate twice from 0 to x to get $\sum_{n=2} x^n = \frac{x^2}{1-x}$. Differentiate twice to find $f(x) = \frac{2}{(1-x)^3}$. Then substitute $x = 1-p$ and multiply by p^2 that was left over.)

Thus, $E\left(\frac{1}{4}\right) = 8$, $E\left(\frac{1}{2}\right) = 4$, $E\left(\frac{3}{4}\right) = \frac{8}{3}$.

Getting Ahead by Two Points

Solution 1

It is easy to see that ξ can only be 2, 4 or 6. So, we divide six games into three rounds of two consecutive games each. If one of the players wins two games in the first round, the match ends with the probability of

$$\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 = \frac{5}{9}.$$

Otherwise, the players tie with each other, earning 1 point each, and the match enters the second round; the probability of this happening is $\frac{4}{9}$.

We have similar discussions for the second and the third rounds. Thus,

$$P(\xi = 2) = \frac{5}{9},$$

$$P(\xi = 4) = \frac{4}{9} \cdot \frac{5}{9} = \frac{20}{81},$$

$$P(\xi = 6) = \left(\frac{4}{9}\right)^2 = \frac{16}{81}.$$

Finally,

$$E\xi = 2 \cdot \frac{5}{9} + 4 \cdot \frac{20}{81} + 6 \cdot \frac{16}{81} = \frac{266}{81}.$$

Solution 2

Amit Itagi

Let A_k denote the event that A wins the k^{th} game, and $\bar{A}_k$ denote that B wins the k^{th} game. Since A_k and $\bar{A}_k$ are incompatible and are independent of the other events, we have

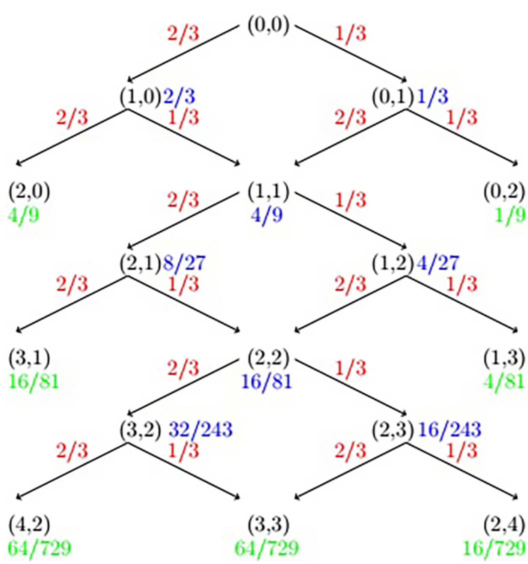
$$\begin{aligned}
P(\xi=2) &= P(A_1A_2)+P(\bar{A}_1\bar{A}_2)=\frac{5}{9},\\
P(\xi=4) &= P(A_1\bar{A}_2A_3A_4)+P(A_1\bar{A}_2\bar{A}_3\bar{A}_4)+P(\bar{A}_1A_2A_3A_4)+P(\bar{A}_1A_2\bar{A}_3\bar{A}_4)\\
&= 2\left[\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)+\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)\right]\\
&= \frac{20}{81},\\
P(\xi=6) &= P(A_1\bar{A}_2A_3\bar{A}_4)+P(A_1\bar{A}_2\bar{A}_3A_4)+P(\bar{A}_1A_2A_3\bar{A}_4)+P(\bar{A}_1A_2\bar{A}_3A_4)\\
&= 4\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2\\
&= \frac{16}{81}.
\end{aligned}$$

It follows that,

$$E\xi = 2\cdot\frac{5}{9}+4\cdot\frac{20}{81}+6\cdot\frac{16}{81}=\frac{266}{81}.$$

Following is a graphical representation of the solution.

Scores in parentheses; Branch probabilities in red; Node cumulative probabilities in blue and green (green for terminal nodes)



$$\begin{aligned}
E[\xi] &= P(1)+2\cdot P(2)+3\cdot P(3)+4\cdot P(4)+5\cdot P(5)+6\cdot P(6)\\
&= 0+2\cdot(4/9+1/9)+3\cdot 0+4\cdot(16/81+4/81)+\\
&\quad 5\cdot 0+6\cdot(64/729+64/729+16/729)\\
&= 266/81\sim 3.284
\end{aligned}$$

Recollecting Forgotten Digit

Solution 1

The riddle is a variant of drawing balls of two colors (Riddle 3.1 on page 15), where, say $w = 1$, $b = 0$ and $w + b = 10$. $P(W_1) = P(W_2) = P(W_3) = \frac{1}{10}$. An important observation is that successive guesses are independent events, Appendix A on page 281.

The (first) event we are looking for is $W = W_1 \cup W_2$ which comes with the probability $P(W) = \frac{1}{10} + \frac{1}{10} = \frac{1}{5}$.

For three attempts the probability is $\frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{3}{10}$.

Solution 2

The probability of a correct guess is $p = \frac{1}{10}$. Two attempts will suffice if we guess correctly the first time or if we fail and guess correctly on a second attempt. This gives the probability as

$$\frac{1}{10} + \left(1 - \frac{1}{10}\right) \cdot \frac{1}{9} = \frac{2}{10}.$$

For three guesses,

$$\begin{aligned} P(\text{Success}) &= \frac{1}{10} + \left(1 - \frac{1}{10}\right) \cdot \frac{1}{9} + \left(1 - \frac{1}{10}\right)\left(1 - \frac{1}{9}\right) \cdot \frac{1}{8} \\ &= \frac{1}{10} + \frac{9}{10} \cdot \frac{1}{9} + \frac{9}{10} \cdot \frac{8}{9} \cdot \frac{1}{8} \\ &= \frac{3}{10}. \end{aligned}$$

Solution 3

Imagine that we try two times, regardless of whether the first guess was correct. Then there are 10×9 ordered pairs of numbers that could be "recollected" with equal chances. Of these, nine include the right guess on the first attempt and nine on the second. In all, there are $9 + 9 = 18$ pairs that realize a successful recollection. The probability of this event is $\frac{18}{90} = \frac{1}{5}$.

For the second part, we make three guesses regardless of whether either the first or the second was successful. There are $10 \times 9 \times 8 = 720$ ordered triples to choose from. If the first guess was a hit, there are $9 \times 8 = 72$ variants for the second and the third guesses. With the second correct guess, there are nine first guess failures times eight distinct but irrelevant third guesses, $9 \times 8 = 72$ in all. The right third guess comes after $9 \times 8 = 72$ failures in the first and second guesses, giving the probability of the success in three guesses as

$$\frac{72 + 72 + 72}{720} = \frac{3}{10}.$$

Two Balls of the Same Color

Solution 1

Let there be b black and w white balls. We have $b + w = 121$ and

$$\begin{aligned}\frac{1}{2} &= \frac{w}{121} \cdot \frac{w-1}{120} + \frac{b}{121} \cdot \frac{b-1}{120}\\ &= \frac{w(w-1) + b(b-1)}{121 \cdot 120}.\end{aligned}$$

It follows that b and w satisfy another equation:

$$121 \cdot 60 = w^2 + b^2 - (w + b) = w^2 + b^2 - 121,$$

so that $w^2 + b^2 = 121 \cdot 61$. Now, $(w + b)^2 = 121^2$ from which $wb = 121 \cdot 30$. Thus, w and b are the roots of the quadratic equation:

$$x^2 - 121x + 121 \cdot 30.$$

Applying the quadratic formula,

$$\begin{aligned}x_{1,2} &= \frac{121 \pm \sqrt{121^2 - 4 \cdot 121 \cdot 30}}{2}\\ &= \frac{121 \pm \sqrt{121(121 - 120)}}{2} = \frac{121 \pm 11}{2}.\end{aligned}$$

Then, e.g., $x_1 = 55$ and $x_2 = 66$ which could be the numbers of black and white balls or vice versa.

55 and 66 are successive triangular numbers and sum up to 121, a square.

Solution 2

Using the same framework from solution 1,

$$\begin{aligned}\frac{1}{2} &= \frac{w}{w+b} \cdot \frac{w-1}{w+b-1} + \frac{b}{w+b} \cdot \frac{b-1}{w+b-1}\\ &= \frac{w(w-1) + b(b-1)}{(w+b) \cdot (w+b-1)},\end{aligned}$$

and multiplying out, we get

$$(w + b)(w + b - 1) = 2w(w - 1) + 2b(b - 1) = 2w^2 + 2b^2 - 2 \cdot (w + b),$$

or

$$(w - b)^2 = w + b = 121,$$

meaning that, assuming $w \geq b$, $w - b = 11$. Combining this with $w + b = 121$, we get the same result as above.

Solution 3

N.N. Taleb

We have four possible outcomes:

$$S = \begin{pmatrix} W & W \\ W & B \\ B & W \\ B & B \end{pmatrix}.$$

We have

$$P(WW||BB) = \frac{W}{121} \cdot \frac{W-1}{120} + \frac{B}{121} \cdot \frac{B-1}{120} = \frac{1}{2}$$
$$P(WB||BW) = \frac{W}{121} \cdot \frac{B}{120} + \frac{B}{121} \cdot \frac{W}{120} = \frac{1}{2}.$$

Solving, we get $B^2 + W^2 = 7260 + b + W$ and $BW = 3630$, from which one root is 66 and the other is 55.

To Bet or Not to Bet

As a matter of fact, we always end up with the same number of cards. Thus, you always lose $1.

Lights on a Christmas Tree

Solution

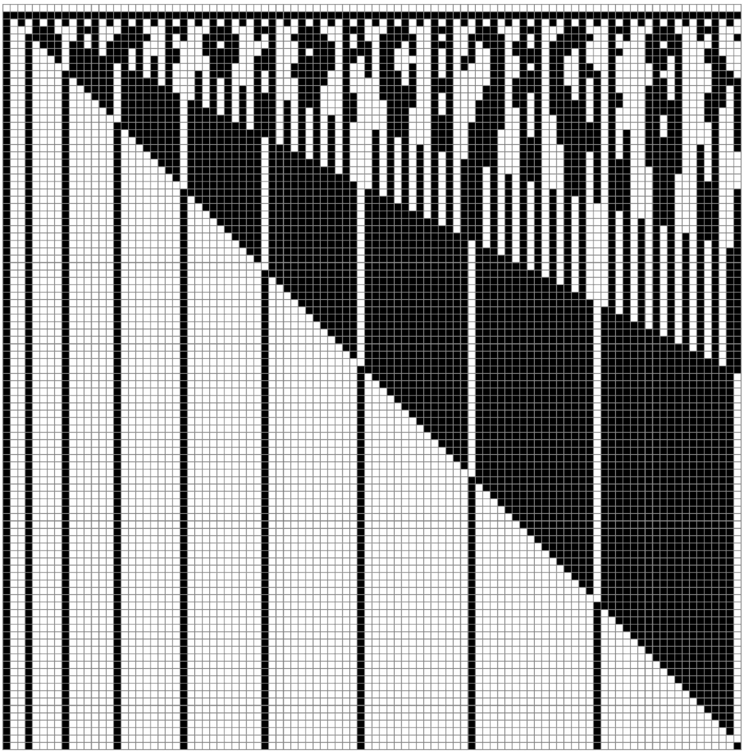
The state of a light depends on the number of factors of the corresponding number. All integers but squares have an even number of distinct factors. For every factor, a light is switched on/off. After an even number of such operations the light is in its original state—off.

The last light to which the algorithm applies is #100. At this point, the only lights that are on correspond to the squares between 1 and 100, inclusive. There are 10 of them. Thus, the probability of a random light being on is $\frac{10}{100} = \frac{1}{10}$.

Illustration

Marcos Carreira

A step-by-step of the lights' on/off state is well represented by the following diagram.



Drawing Numbers and Summing Them Up

Notes

We will give two solutions to the riddle. The starting point for both is the criteria of divisibility by 3. A number is divisible by 3 if and only if the sum of its digits is divisible by 3. The digits in the box add up to 10. The three digits drawn may add up to $2+3+4=9$, $1+3+4=8$, $1+2+4=7$ or $1+2+3=6$. Two of these (6 and 9) are divisible by 3. The second observation is that the sum of the digits of a number does not depend on the order in which the digits are added because addition of the integers is commutative. This means that the order in which the digits are drawn from the box has no effect on the divisibility by 3 of the final number.

Solution 1

We are looking into the probability of drawing the numbers composed either of the digits 1, 2, 3 or of the digits 2, 3, 4. Let us first consider the probability of drawing the digits 1, 2, 3. The probability of drawing one of these on the first draw is $\frac{3}{4}$. The probability of drawing the second one out of the remaining three is $\frac{2}{3}$ and that of getting the third on the last drawing is $\frac{1}{2}$. Since the order of the digits' appearance from the box is not important, the three drawings are independent and the probability of getting the three particular digits 1, 2, 3 is then the product of the three: $p = \frac{3}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{4}$.

So, the probability of getting the number composed of the digits 1, 2, 3 is $\frac{1}{4}$. The same reasoning applies to the digits 2, 3, 4 so that the probability of getting the latter three is also $\frac{1}{4}$. Finally, the two events—one of getting the digits 1, 2, 3 and the other of getting the digits 2, 3, 4—do not occur at the same time. You get either one or the other (the two are mutually exclusive) meaning that the combined event has the probability which is the sum of the two probabilities: $\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$.

Solution 2

Let us focus on the number that remains in the box after three drawings. It may be any one of the digits 1, 2, 3, 4. When this is 1 or 4 that is left in the box the three drawn digits add up to a number divisible by 3. When the left-over digit is either 2 or 3, the number drawn is not divisible by 3. Thus, in half the cases (two out of four) the number drawn is divisible by 3. The two possibilities are equiprobable and mutually exclusive, implying that each of the possibilities has the probability of $\frac{1}{2}$.

Numbers in a Square

The total number of "1" entries in the table is odd. However, due to the symmetry condition, the number of 1s off the diagonal D is even. It follows that there is a 1 on the diagonal. The same holds for other entries and other numbers. Therefore, all seven integers 1 through 7 appear on the diagonal, implying that the probability in question is 1.

Converting Temperature from °C to °F

Amit Itagi

The conversion from °C to °F is given by

$$F = \frac{9}{5}C + 32.$$

The 32 is just an integral constant, the rounding steps do not affect it, and it can be dropped in the context of this riddle. Also, the riddle is periodic in C with a period of 5. So without loss of generality, we can map all the C values modulo 5 to a range $[0, 5]$. In fact, the range $[2.5, 5.0]$ can also be mapped to $[0, 2.5]$ by using a map $f(x) = 5.0 - x$ without changing the considerations of the riddle. This can be seen from the following argument. Consider two intervals $p \in [0, 2.5)$ and $q \in (2.5, 5]$ for C (modulo 5) (the second and third columns in the following table). The notation $R(\cdot)$ is for rounding. Whenever p is different from $R(p)$ or $9p/5$ is different from $R(9p/5)$, q is symmetrically different from $R(q)$ and $9q/5$ from $R(9q/5)$.

C	$5K + p$	$5K + q$ $(= 5K + 5 - p)$
$R(C)$	$5K + R(p)$	$5K + 5 - R(p)$
$9C/5$	$9K + 9p/5$	$9K + 9q/5$ $(= 9K + 9 - 9p/5)$
$R(9C/5)$	$R(9p/5)$	$9 - R(9p/5)$

Thus, without loss of generality, let us assume $0 \le C \le 2.5$ and $F = \frac{9}{5}C$. Hence, with or without any of the rounding steps, the converted F will lie in $[0, 4.5]$. The notation $R(\cdot)$ is used for rounding. The different intervals are tabulated in the following:

C	$R(C)$	$9C/5$	$9\ R(C)/5$	$9C/5$ Intervals with Differences in Rounded Values
$0.0 - 0.5$	0	$0.0 - 0.9$	0.0	$0.5 - 0.9$
$0.5 - 1.0$	1	$0.9 - 1.8$	1.8	$0.9 - 1.5$
$1.0 - 1.5$	1	$1.8 - 2.7$	1.8	$2.5 - 2.7$
$1.5 - 2.0$	2	$2.7 - 3.6$	3.6	$2.7 - 3.5$
$2.0 - 2.5$	2	$3.6 - 4.5$	3.6	N/A

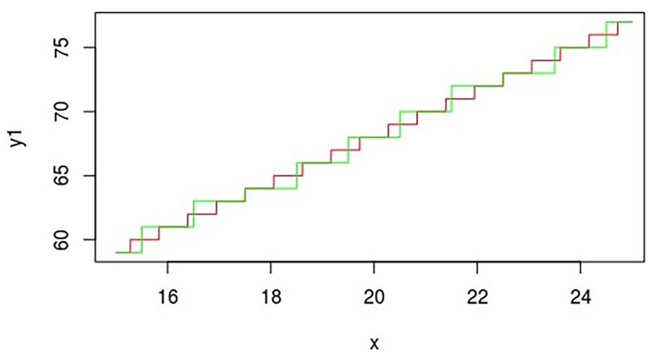
Thus, the required probability is

$$\frac{(0.9 - 0.5) + (1.5 - 0.9) + (2.7 - 2.5) + (3.5 - 2.7)}{4.5} = \frac{2}{4.5} = \frac{4}{9}.$$

Illustration

Attila Kun

The following graph lets one visually compare the two kinds of conversion:



The graph was produced by this small piece of code:

```
x <- seq(15, 25, 0.001)

y1 <- round(1.8*x + 32)
y2 <- round(32 + 1.8*(round(x)))

plot(x, y1, type="l", col="red")
lines(x, y2, col="green")
sum((y1 - y2)^2 > 0) / length(x) # p ~ 0.447555
```

Notes

Keith Dawid

The discrepancies by range are listed in the following table:

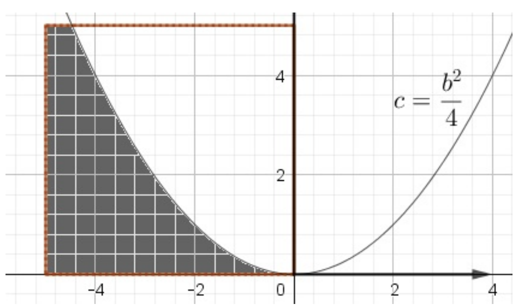
$15° - 16°$	–	$1/9$
$16° - 17°$	–	$2/9$
$17° - 18°$	–	$3/9$
$18° - 19°$	–	$4/9$
$19° - 20°$	–	$5/9$
$20° - 21°$	–	$1/9$
$21° - 22°$	–	$2/9$
$22° - 23°$	–	$3/9$

Probability of No Distinct Positive Roots

As in the original Riddle 10.5 on page 261, the basis for the solution is naturally the quadratic formula: $x_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4c}}{2}$.

Now, it may be easier to determine when the equation does have distinct positive roots. This happens when $c > 0$, $b < 0$ and $b^2 - 4c > 0$.

With b on the horizontal axis and c on the vertical one, we have to compute the area under the parabola $c = \dfrac{b^2}{4}$ inside the square $[-5, 5] \times [-5, 5]$, relative to the square itself.



The parabola crosses $c = 5$ for $b = -\sqrt{20}$. The area in question is

$$S = 5(5 - \sqrt{20}) + \int_{-\sqrt{20}}^{0} = 5(5 - \sqrt{20}) + 5 = 5(6 - \sqrt{20}),$$

so that the probability of having distinct positive roots is

$$\frac{5(6 - \sqrt{20})}{100} = \frac{3 - \sqrt{5}}{10},$$

with the complementary probability

$$1 - \frac{3 - \sqrt{5}}{10} = \frac{7 + \sqrt{5}}{10} \approx 0.923607.$$

This is a little more than in the original riddle $\left(\dfrac{111}{121} \approx 0.917355\right)$.

Playing with Integers and Limits

Try playing the game with small starting numbers.

If the starting number is 1, the player whose turn it is to make a move (i.e., the first player) wins.

If the starting number is 2, regardless of the move, the first player leaves 1, meaning that the second player wins.

If the starting number is 3, the first player may leave either 1 or 2 and become the second player on the next move. A good player would leave 2 and win the game.

Here is a proposition:

> If the starting integer N has an even number of factors of 2 (in particular when N is odd), the first player should win. If the starting integer N has an odd number of factors of 2, the second player should win.

Solution 1

We will prove the above statement by strong mathematical induction.

Assume it is true for any $N \le k$, for some $k \ge 1$. Consider $N = k+1$. There are three cases:

Case 1

$k+1$ is odd.

The first player may move to either k or $\frac{k}{2}$. One of k or $\frac{k}{2}$ has an odd number of factors of 2. According to the proposition, this is the number the first player should leave on the first move, after which he becomes the second player and wins.

Case 2

$k+1$ is even and has an odd number of factors of 2.

k is odd and, if left odd, the other player wins. If the first player leaves $\frac{k+1}{2}$, which has an even number of factors 2, the other player still wins.

Case 3

$k+1$ is even and has an even number of factors of 2.

The first player loses if he leaves the odd k. If he leaves $\frac{k+1}{2}$, then, according to case 2, he will win after becoming the second player.

Thus, in cases 1 and 3, the first player should win; in case 2, the second player wins. In other words, the second player wins if the starting number has an odd number of factors of 2. How many such numbers are there? Let us count:

- Every number in the form $2n$ where n is odd: these are numbers $2, 6, 10, \ldots$. Every fourth integer is in this form. So, the probability of falling into that group is $\frac{1}{4}$.
- Every number in the form $8n$ where n is odd: these are numbers $8, 24, 40, \ldots$. These numbers differ by $16 = 4^2$. So every sixteenth integer falls into that group.

We may continue in this manner, but it becomes clear that the proportion of integers having an odd number of factors 2 is the sum of the series

$$\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\cdots=\frac{1}{4}\cdot\frac{1}{1-\frac{1}{4}}=\frac{1}{3}.$$

This is (more or less) the probability of a win for the second player.

Solution 2

Amit Itagi

We claim that the starting number n for the first player uniquely determines the outcome of the game. If this claim is true, then we can define a function f:

$$f(n)=\begin{cases}1, & \text{if player 1 wins}\\ 0, & \text{if player 2 wins.}\end{cases}$$

Clearly, $f(1)=1$ and $f(2)=0$. $f(2)=0$ because if the first player starts at 2, the only choice available is 1 and then the second player wins by going from 1 to 0. We validate our claim by showing that the function gets recursively defined for all n in the following way: if the possible two outcomes of the current move lead to a place where the person making the next move is guaranteed to win, then the person making the current move is guaranteed to lose. Hence,

$$f(2k)=\begin{cases}0, & \text{if } f(2k-1)=1 \text{ and } f(k)=1\\ 1, & \text{otherwise;}\end{cases}$$

and

$$f(2k+1)=\begin{cases}0, & \text{if } f(2k)=1 \text{ and } f(k)=1\\ 1, & \text{otherwise.}\end{cases}$$

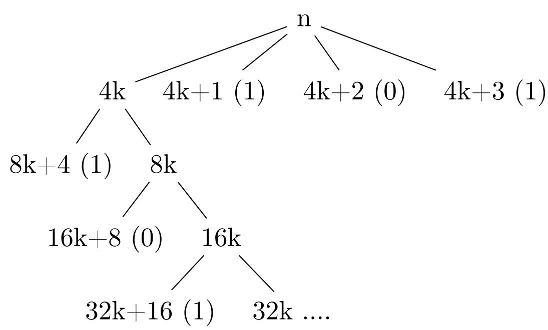
The definition of $f(2k)$ leads to the following implication:

$$f(k)=0\Rightarrow f(2k)=1.$$

Thus, the first condition in the definition of $f(2k+1)$ can never be satisfied and $f(2k+1)=1$ unconditionally. This in turn proves the converse of the preceding implication:

$$f(k)=0 \iff f(2k)=1.$$

Thus, from the definition of $f(2k)$, $f(4k+2)=0$ because $f(4k+1)=1$ and $f(2k+1)=1$. Thus, the only numbers left to investigate are the numbers of the form $4k$. Because multiplying by 2 flips the function, we end up with the following tree like structure:



The nodes show the form of the number and the number in the parentheses shows the value of $f(n)$. Thus, the probability that the second player wins is the probability of getting $f(n) = 0$. The required probability is

$$\begin{aligned}P &= \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots \\ &= \sum_{m=1}^{\infty} \frac{1}{4^m} = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3}.\end{aligned}$$

Given Probability, Find the Sample Space

Solution 1

Simon Norton

From a group of b boys and g girls a teacher can form $\dfrac{(b+g)(b+g-1)}{2}$ pairs of which bg pairs are of different sexes. To ensure the 50% chance of the same sex selection, the former number needs to be twice the latter:

$$\frac{(b+g)(b+g-1)}{2} = 2bg$$

which reduces to

$$(b-g)^2 = b+g.$$

Letting $b - g = n$ leads to a system,

$$\begin{aligned}&b - g = n \\ &b + g = n^2,\end{aligned}$$

from which $b = \dfrac{n(n+1)}{2}$, $g = \dfrac{n(n-1)}{2}$. Cases where $n = 0$ (no kids at all) or $n = \pm 1$ (one sex is missing) can be disregarded. Other than those, n could be any integer. When n is negative, $g > b$; when it is positive, $b > g$.

In any event, b and g need to be two consecutive triangular numbers.

Solution 2

Amit Itagi

Suppose there are b boys and g girls. The number of ways of choosing two boys is $C(b,2)$, the number of ways of choosing two girls is $C(g,2)$ and the number of ways of choosing a boy and a girl is bg. Thus, the condition of the riddle implies

$$\begin{aligned} C(b,2)+C(g,2) &= bg \\ b(b-1)+g(g-1) &= 2bg \\ b &= g+\frac{1}{2}\pm\frac{1}{2}\sqrt{8g+1} \end{aligned}$$

Note, $b=g$ results in $b=g=0$. Thus, $b \neq g$. Noting that the riddle is symmetric in b and g, let us find solutions with $b>g$ and get the other set of solutions by swapping b and g from the first set. For $b>g$, we have to drop the negative square root.

Noting that any odd square is of the form $(8k+1)$, let $8g+1=(2n+1)^2$ for $n>0$. Thus, the first set of solutions is

$$\begin{aligned} g &= \frac{(2n+1)^2-1}{8} = \frac{n(n+1)}{2} \\ b &= \frac{(2n+1)^2-1}{8}+(n+1) = \frac{(n+1)(n+2)}{2}, \end{aligned}$$

and the alternate set of solutions is

$$\begin{aligned} b &= \frac{(n+1)(n+2)}{2} \\ g &= \frac{n(n+1)}{2}, \end{aligned}$$

where n is any positive integer.

Solution 3

N.N. Taleb

The binomial distribution with n trials and success probability p is

$$\mathbb{P}(X=x)=f(x)=\begin{cases}(1-p)^{n-x}p^x\binom{n}{x} & 0\le x\le n\\ 0 & \text{otherwise.}\end{cases}$$

Here we have $f(2)=\frac{1}{2}$. Let $n=n_B+n_G$ be the total number of pupils, with n_B the initial number of boys, etc. Let $p=\frac{n_B}{n}=\frac{n_B}{n_A+n_B}$. We have the following tuples in a draw of 2:

$$\begin{pmatrix} B & B & \frac{n_B}{n}\frac{n_B-1}{n-1} \\ B & G & \frac{n_B}{n}\frac{n_G}{n-1} \\ G & B & \frac{n_G}{n}\frac{n_B}{n-1} \\ G & G & \frac{n_G}{n}\frac{n_G-1}{n-1} \end{pmatrix}.$$

We have to solve for

$$\frac{(n_B-1)\,n_B}{(n_B+n_G-1)\,(n_B+n_G)}+\frac{(n_G-1)\,n_G}{(n_B+n_G-1)\,(n_B+n_G)}=\frac{1}{2},$$

which reduces to

$$\frac{4n_Bn_G}{(n_B+n_G-1)\,(n_B+n_G)}=1,$$

leading to two solutions, depending if $n_G > n_B$ or the reverse:

$$n_G=\frac{1}{2}\left(\pm\sqrt{8n_B+1}+2n_B+1\right).$$

It follows that $n_G, n_B = \frac{1}{2}(n \pm \sqrt{n})$. To make all the quantities integer we need to assume that n is a square: $n = k^2$, simplifying the above to $n_G, n_B = \frac{1}{2}k(k \pm 1)$.

Notes

A. Bogomolny

Rob Eastaway posted a modification on Twitter (@robeastaway):

> In a group of children, there are three boys. If I pick two children at random, there is a 50% chance both are boys. How many girls are in the group?

This can be better generalized. Let $T_n = \frac{n(n+1)}{2}$ be the n^{th} triangular number. Then we can pose the following problem:

> In a group of children, there are T_n boys. If I pick two children at random, there is a 50% chance both are boys. How many girls are in the group?

Repeating the derivation above will lead to the answer $g = \frac{n(n-1)}{2} = T_{n-1}$.

To answer Rob's question, $3 = 2 \cdot \frac{3}{2} = T_2$. Therefore, in his case, the number of girls is $T_1 = \frac{1 \cdot 2}{2} = 1$.

This riddle (under the guise of drawing socks from a drawer) is also included in [69, Problem 1].

Gladiator Games

There is no optimal strategy. No strategy effects the chances of a team to win the competition. Let $A = \sum_{i=1}^{m} a_i$ and $B = \sum_{i=1}^{n} b_i$. Then the chances of a team to win are proportional to its total strength, i.e., for the first team, it is $\frac{A}{A+B}$; for the second, it is $\frac{B}{A+B}$.

The proof is by induction on $n+m$, assuming both n and m are greater than zero.

The claim is certainly true for $m=n=1$. Assume it is true for $m+n=N$, and now that one team still has m members, let the membership of the other grow to $n+1$. So now $A=\sum_{i=1}^{m} a_i$ and $B=\sum_{i=1}^{n+1} b_i$.

For the first duel, the first team sends a gladiator of strength a, the second a gladiator of strength b. The first wins with the probability of $\frac{a}{a+b}$, the second with the probability of $\frac{b}{a+b}$.

Depending on who wins that duel, there will be two possible distributions of strength:

1. a wins: $\{a_1,\dots,a+b,\dots,a_m\}$ and $\{b_1,\dots,\overline{b}\dots,b_{n+1}\}$,
2. b wins: $\{a_1,\dots,\overline{a},\dots,a_m\}$ and $\{b_1,\dots,b+a\dots,b_{n+1}\}$,

where overline means the absence of the fighter.

According to the induction assumption, in the first case, the chances of the first team to win equal $\frac{A+b}{A+B}$; in the second case they equal $\frac{A-a}{A+B}$, making the overall probability as claimed:

$$
\begin{aligned}
P_a &= \frac{a}{a+b}\cdot\frac{A+b}{A+B}+\frac{b}{a+b}\cdot\frac{A-a}{A+B}\\
&= \frac{aA+bA}{(a+b)(A+B)}\\
&= \frac{A}{A+B}.
\end{aligned}
$$

In Praise of Odds

The following solutions allow for a comparison of two approaches: a conventional approach with lengthy calculations of probabilities and the one based on estimating the odds. As the reader may see, odds provide a significant shortcut for computing probabilities.

Solution Setup 1

The tied games are of no consequence in determining the winner of the series and can be ignored. We know, that the odds of A winning a game are $p:q$ so that a step toward A winning a series has the probability of $p'=\frac{p}{p+q}$. For B the probability is $q'=\frac{q}{p+q}$. Naturally, $p'+q'=1$ and $p':q'=p:q$.

Solution Setup 2

Let A denote the game won by A, B the game won by B and T the game that ended up tied. The latter is of no consequence in determining the winner of the series and can be ignored. Then the series might look like this:

$$TATTBATBTTTBTTA\ldots.$$

We shall split the sequence into "consequential" pieces by grouping a won game with the preceding ties:

$$(TA)(TTB)(A)(TB)(TTTB)(TTA)\ldots.$$

What matters at the end of the day is the last game in a group. The groups that end with A contribute to the advancement of A. These are:

$$A, TA, TTA, TTTA, \ldots,$$

each exclusive of any other. The corresponding probabilities are $p, rp, r^2p, \ldots$ that add up to

$$p + rp + r^2p + \cdots = p \cdot \frac{1}{1-r} = \frac{p}{p+q}$$

which we shall denote as p'. q' is defined similarly: $q' = \dfrac{q}{p+q}$.

Solution 1

OSWEGO Problems Group

Either way, the riddle reduces to a win/lose sequence with the probabilities p' and q'.

The last two games in a series are bound to be won by the same team and the games beforehand come in pairs of wins (or losses) for both teams. Thus, those pairs have the probability $p'q' + q'p' = 2p'q'$. A sequence of n such pairs comes with the probability of $(2p'q')^n$. It follows that A wins a series of $2n+2$ games with the probability $(2p'q')^n(p')^2$. The total probability of A being declared a winner is

$$\begin{aligned} P(A) &= \sum_{k=0}^{\infty}(2p'q')^n(p')^2 = (p')^2\sum_{k=0}^{\infty}\frac{1}{1-2p'q'} \\ &= \frac{p^2}{(p+q)^2} \cdot \frac{(p+q)^2}{(p+q)^2 - 2pq} = \frac{p^2}{p^2+q^2}. \end{aligned}$$

Solution 2

The last two games in a series are bound to be won by the same team and the games beforehand come in pairs of wins (or losses) for both teams. The odds of A versus B winning two games in a row are $p^2 : q^2$. It follows that A wins the series with the probability of $\dfrac{p^2}{p^2+q^2}$.

Probability in Scoring

Solution 1

A. Bogomolny

The game of Scoring is a great tool for learning the modulo arithmetic. It is best analyzed starting from the end. The last move was the removal of 1, 2, 3, 4 or 5 stones. Thus, the previous player has left less than six stones. If he left six he would win. If he left more than six, the other player would be able to leave six stones and win.

So, whoever manages to leave six stones wins the game. Put another way, whoever takes the seventh stone wins. We can now apply the previous argument, but replace "first" with "seventh" and continue backward. The right strategy is then to leave a multiple of six after each move. If you do not, the other player will.

The game starts with 14 stones; the boy should take two. If he does not, he loses right away.

If he does, whichever move the computer makes next, the boy has one in six chance to make the only right move. The same applies to the next two moves, computer/boy.

However they play, the right strategy is to leave a multiple of six. On the first move the boy does that with the probability of $\frac{1}{5}$ (he needs to remove two and leave 12 stones). From there on there remain two pairs of moves, meaning that the boy yet has two opportunities to make a wrong move. As before, there is a one in five chance to make a right move. Thus, the boy does that with the probability of $\frac{1}{5} \cdot \frac{1}{5} \cdot \frac{1}{5} = \frac{1}{125}$.

We may observe that the boy may reach the state of six stones with the probability of $\frac{1}{5^2}$ (hitting two on the first move and whatever it takes to get to six, depending on the computer's move). After the computer's move (probably random), the boy may be left with 1, 2, 3, 4 or 5 stones. Assuming that even the fellow who has yet to master the necessary skills to play Scoring is still clever enough to remove all the remaining stones in order to win the game, at this point, he wins it with certainty, making the final probability of the win equal to $\frac{1}{5^2}$.

Solution 2

Amit Itagi

If player A gets the stone count to six, that player optimally forces a win. Player B can only leave 1–5, and in the final move, player A picks all the stones.

With this observation, we note that if in the first move the boy gets the stone count to 13, 11, 10 or 9, the computer can force a win. In the cases of 9–11, the computer can bring the stone count down to six in a single move. For the case of 13, the computer can bring the stone count to 12 making 7–11 possible moves for the boy. The computer can then force a 6 irrespective of the stone count that the boy leaves.

Thus, we focus on the case where the boy picks two stones in the first step and gets the count down to 12. This first move has probability $P_1 = \frac{1}{5}$. In this case, the boy always has an optimal path (not necessarily the chosen path) to victory. No matter what the computer chooses, the boy has only one in five ways of getting to six. In all other cases, the boy either has to leave a count greater than six with six accessible to the opponent in a single move or a count less than six such that all the stones can be picked up in one move. Thus, the probability of the boy winning in this case if $P_2 = \frac{1}{5}$.

Hence, the probability of the boy winning is $P_1 \cdot P_2 = \frac{1}{25}$.

Probability of 2^n Beginning with Digit 1

Solution 1

A. Bogomolny

Observe that, for a given length (number of digits) there exists at most one power of 2 that begins with digit 1. Moreover, for a given $k > 1$, there is always a power of 2 of length k. Exactly one of these begins with 1. Examples: $2^4 = 16$, $2^7 = 128$, $2^{10} = 1024$, $2^{14} = 16384$.

Indeed, for a given k, consider the largest power of 2 that has length k, say 2^p. Then 2^{p+1} is bound to have $(k+1)$ digits as it is less than $10 \cdot 2^p$ of $(k+1)$ digits and is larger than the largest among the powers of 2 with k digits. So, if there is a power of 2 with k digits, there is at least one with $k+1$ digits.

On the other hand, for a given k, the least power of 2 of lengths k is bound to start with 1. Otherwise, it could be divided by 2 to obtain a smaller power of 2 of the same length.

It follows that for any $k > 1$ there exists exactly one power of 2 with k digits that begins with 1. Let $q(K)$ denote the number of powers of 2 that begin with 1 and are less than 10^K, i.e., have no more than K digits. Then, what we found is that $q(K) = K - 1$. The total number N of powers of 2 below 10^K comes from $2^N < 10^K < 2^{N+1}$, so $N \log_{10} 2 < K < (N+1)\log_{10} 2$, or $K = N \log_{10} 2 + \alpha$, where $0 < \alpha < (N+1)\log_{10} 2 - N \log_{10} 2 = \log_{10} 2$. Finally,

$$\lim_{K\to\infty} \frac{q(K)}{N} = \lim_{K\to\infty} \frac{(K-1)\log_{10} 2}{K-\alpha} = \log_{10} 2.$$

Solution 2

Amit Itagi

If a positive number k is written in scientific notation $m \times 10^n$ where $m \in (0, 10)$ and $n \in \mathbb{Z}^+$, then the mantissa is defined by $f(k) = \log_{10} m = \log_{10} k - [\log_{10} k]$. Noting that $\log_{10} 2$ is irrational, from the equidistribution theorem [99], the set $\{f(2^n) : n \in \mathbb{Z}^+\}$ is uniformly distributed on $[0, 1)$. For the leading integer of k to be 1, its mantissa has to be less than $\log_{10} 2$. Thus, the required probability is

$$p = \frac{\log_{10} 2}{1} \approx 0.301.$$

Probability of First Digits in a Sequence of Powers

Léo Sauvé

A more general theorem holds where 2 can be replaced with any other integer, except for 10, and there is even a more general formulation. Given an r-digit number N, we may inquire about appearance of N as the first digits in the decimal representation of the powers of .2. From here, the riddle can be stated for an irrational α, not a rational power of 10:

> For a given r-digit number N, and an irrational α, as above, what is the probability that the decimal representation of α^n begins with N.

Define, for $x > 0$, $\mu(x)$ as the mantissa of $\log x$, that is,

$$\mu(x) = \log x - \lfloor \log x \rfloor = \log x (\text{mod } 1).$$

Let $S = \{\alpha^n\}_{i=1}^{\infty}$. Then $\mu(S)$ is dense in the open interval $(0,1)$ which is proved with a relatively standard invocation of the pigeonhole principle [18]. See also another example: Riddle 4.19 on page 36.

Due to the equidistribution theorem [99], the probability that the decimal representation of α^n begins with N is the probability that $\mu(\alpha^n)$ lies in the interval $I = [\mu(N), \mu(N+1))$ whose length is

$$\mu(N+1) - \mu(N) = \log(N+1) - \log N = \log\left(1 + \frac{1}{N}\right).$$

Note that that probability is independent of α.

Note also that, because the density of $\mu(S)$ is in $(0,1)$, it follows that, for any N, there are infinitely many powers of α whose decimal representation starts with N.

Probability of Four Random Integers Having a Common Factor

Solution 1

As in Riddle 3.5 on page 16, the probability of an integer to be even, i.e., to be divisible by 2, is $\frac{1}{2}$. The probability of four numbers being even is $\left(\frac{1}{2}\right)^4$. Thus, the probability that at least one of the four numbers is odd is $1 - \left(\frac{1}{2}\right)^4$.

Similarly, the probability that at least one of four random integers is not divisible by 3 is $1 - \left(\frac{1}{3}\right)^4$. For any other prime, that is $1 - \left(\frac{1}{p}\right)^4$.

Since divisibility by one prime number is independent of divisibility by any other prime number, four random numbers have no common factor with the probability of $\prod_{p \text{ prime}} \left[1 - \left(\frac{1}{p}\right)^4\right]$. It follows that the probability of four random numbers to have a common factor is

$$1 - \prod_{p \text{ prime}} \left[1 - \left(\frac{1}{p}\right)^4\right].$$

Solution 2, Rational Estimate

The sum $\sum_{k=1}^{\infty} \frac{1}{k^s}$ is known as a zeta-function $\zeta(s)$, many of whose values are known specifically. In particular, $\zeta(4) = \frac{\pi^4}{90}$. In the following, we will find some rational estimates directly. We will start with a special case of the Euler–Kronecker formula:

$$\frac{1}{(1-2^{-4})(1-3^{-4})(1-5^{-4})(1-7^{-4})\cdots} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \cdots = \sum_{k=1}^{\infty} \frac{1}{k^4}.$$

Now, we group the terms on the right by the powers of 2:

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{k^4} &= 1+\left(\frac{1}{2^4}+\frac{1}{3^4}\right)+\left(\frac{1}{4^4}+\frac{1}{5^4}+\frac{1}{6^4}+\frac{1}{7^4}\right)+\cdots\\ &< 1+\left(\frac{1}{2^4}+\frac{1}{2^4}\right)+\left(\frac{1}{4^4}+\frac{1}{4^4}+\frac{1}{4^4}+\frac{1}{4^4}\right)+\cdots\\ &= 1+2\cdot\frac{1}{2^4}+4\cdot\frac{1}{4^4}+8\cdot\frac{1}{8^4}+\cdots\\ &= 1+\frac{1}{2^3}+\frac{1}{4^3}+\frac{1}{8^3}+\cdots=\frac{1}{1-\dfrac{1}{8}}=\frac{8}{7}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{k^4} &= 1+\frac{1}{2^4}+\left(\frac{1}{3^4}+\frac{1}{4^4}\right)+\left(\frac{1}{5^4}+\frac{1}{6^4}+\frac{1}{7^4}+\frac{1}{8^4}\right)+\cdots\\ &> 1+\frac{1}{2^4}+\left(\frac{1}{4^4}+\frac{1}{4^4}\right)+\left(\frac{1}{8^4}+\frac{1}{8^4}+\frac{1}{8^4}+\frac{1}{8^4}\right)+\cdots\\ &= 1+\frac{1}{2^4}+\frac{1}{2}\left(4\cdot\frac{1}{4^4}+8\cdot\frac{1}{8^4}+\dots\right)\\ &= 1+\frac{1}{2^4}+\frac{1}{2}\left(\frac{1}{4^3}+\frac{1}{8^3}+\cdots\right)=1+\frac{1}{2^4}+\sum_{k=2}^{\infty}\frac{1}{2^{3k+1}}\\ &= 1+\sum_{k=1}^{\infty}\frac{1}{2^{3k+1}}=1+\frac{1}{16}\cdot\frac{1}{1-\dfrac{1}{8}}=\frac{15}{14}.\end{aligned}$$

We found that

$$\frac{8}{7}<\left(\prod_{p\text{ prime}}\left[1-\left(\frac{1}{p}\right)^4\right]\right)^{-1}<\frac{15}{14}$$

so that

$$1-\frac{14}{15}<1-\prod_{p\text{ prime}}\left[1-\left(\frac{1}{p}\right)^4\right]<1-\frac{7}{8},$$

i.e.,

$$\frac{1}{15}<1-\prod_{p\text{ prime}}\left[1-\left(\frac{1}{p}\right)^4\right]<\frac{1}{8}.$$

So, the sought probability of four random integers having a common factor is between $\dfrac{1}{15}$ and $\dfrac{1}{8}$.

Solution 3
Amit Itagi

The probability that the i^{th} prime p_i divides an integer is $\dfrac{1}{p_i}$. Thus, the probability that a p_i divides all four integers chosen is $\dfrac{1}{p_i^4}$, and the probability that p_i is not a common factor of the four integers is $1 - \dfrac{1}{p_i^4}$. Hence, the desired probability is

$$\lim_{n\to\infty}\left[1 - \prod_{i=1}^{n}\left(1 - \frac{1}{p_i^4}\right)\right] = 1 - \frac{1}{\zeta(4)} = 1 - \frac{90}{\pi^4} \sim 0.076.$$

Probability of a Cube Ending with 11

Solution 1
If a^3 ends in the digits 11, then $a^3 \equiv 11 \pmod{100}$. This is equivalent to the following two conditions:

1. $a^3 \equiv 11 \equiv 3 \pmod 4$ and
2. $a^3 \equiv 11 \pmod{25}$.

Easily, 3 is the only residue modulo 4 that satisfies $a^3 \equiv 3 \pmod 4$.

Next, $a^3 \equiv 11 \pmod{25}$ implies $a^3 \equiv 1 \pmod 5$. Checking the residues, 1 is the only one that satisfies this condition. From here, suffice it to check 1, 6, 11, 16 and 21 modulo 25:

$$1^3 = 1 \equiv 1 \pmod{25}$$

$$6^3 = 200 + 16 \equiv 16 \pmod{25}$$

$$11^3 = 121 \cdot 11 \equiv 21 \cdot 11 = 231 \equiv 6 \pmod{25}$$

$$16^3 = 256 \cdot 16 \equiv 6 \cdot 16 \equiv 21 \pmod{25}$$

$$21^3 = 441 \cdot 21 \equiv 16 \cdot 21 = 210 + 126 \equiv 10 + 1 = 11 \pmod{25}.$$

Thus, we are looking for an integer a such that $a \equiv 3 \pmod 4$ and $a \equiv 21 \pmod{25}$. By the Chinese remainder theorem [14] we find that the only number below 75 satisfying the two equations is 71. The only candidate for $a \equiv 21 \pmod{25}$ below 100 is 96 and that fails $a \equiv 3 \pmod 4$. It follows that 71 is the only solution to $a^3 \equiv 11 \pmod{100}$. We conclude that the probability of randomly hitting on a solution to $a^3 \equiv 11 \pmod{100}$ is $\dfrac{1}{100}$.

Solution 2
Sriram Natarajan

The number ends with 1. The form is $(10x + 1)^3$ or $1000x^3 + 300x^2 + 30x + 1$. $30x$ fixes the tenth place and $x = 7$ is the unique integer solution. So, $71^3 \equiv 11 \pmod{100}$

and is the only number below 100 with this property. The probability of hitting it equals $\frac{1}{100}$.

Solution 3

Jim Henegan

Since $x^3 \equiv 11 \pmod{100}$ implies $x^3 \equiv 1 \pmod{10}$, we may begin by narrowing our attention to $x \in \{1, 11, 21, \ldots, 91\}$.

Now make the substitution $a = x - 1$. Then we must solve

$$(a+1)^3 \equiv 11 \pmod{100}$$

for $a \in \{10, 20, \ldots, 90\}$. Note that, for any such a, this congruence becomes

$$a^3 + 3a^2 + 3a + 1 \equiv 11 \pmod{100} \implies$$

$$3a^2 + 3a \equiv 10 \pmod{100}.$$

Now make the substitution $b = \frac{a}{10}$. Then, for $b \in \{1, 2, \ldots, 9\}$, we must solve

$$3(10b)^2 + 3(10b) \equiv 10 \pmod{100} \implies$$

$$30b \equiv 10 \pmod{100}.$$

By inspection, this yields $b = 7$, which gives $a = 70$, which gives $x = 71$.

Solution 4

Amit Itagi

$$x^3 \equiv 11 \pmod 4 \;\Rightarrow\; x^3 \equiv 3 \pmod 4 \;\Rightarrow\; x \equiv 3 \pmod 4$$

$$x^3 \equiv 11 \pmod 5 \;\Rightarrow\; x^3 \equiv 1 \pmod 5 \;\Rightarrow\; x \equiv 1 \pmod 5.$$

The only numbers less than or equal to 100 that satisfy these two conditions belong to the set $P = \{11, 31, 51, 71, 91\}$. These numbers are respectively 11, 6, 1, 21, 16 (mod 25). The only case that satisfies $x^3 \equiv 11 \pmod{25}$ is $x \equiv 21 \pmod{25}$. Thus, $x \equiv 71 \pmod{100}$.

The required probability is 1 part in 100.

Note 1

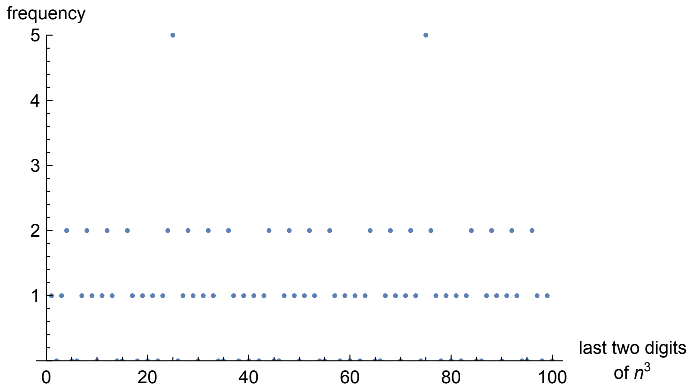
Samuel Walters, Jim Henegan

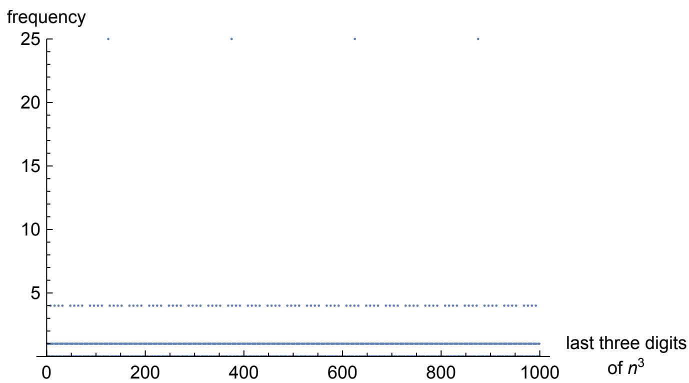
Interestingly, not every residue modulo 100 belongs to an integer cube. Being able to select only among those that do will increase the probability of hitting on the solution. Indeed, there are only 62 distinct residues modulo 100.

There are 40 residues that are attained by unique numbers below 100. These are obtained by numbers ending with 1, 3, 7, 9. There are 20 residues with the second digit even, non-zero. These are attained by pairs of numbers with the difference of 50 between them. E.g., $32^3 = 327\mathbf{68}$ and $82^3 = 5513\mathbf{68}$. Finally, the ending of 25 is attained by 5, 25, 45, 65, 85 and the ending of 75 by 15, 35, 55, 75, 95.

The first plot shows frequencies at which the last two digits of n^3 are x, for $n \in \{1, 2, \ldots, 100\}$. Note the two 5-tuples, 20 pairs and 40 singles. Here the two 5-tuples occur when $x = 25$ and $x = 75$.

The second plot shows the frequencies at which the last three digits of n^3 are x, for $n \in \{1, 2, \ldots, 999, 1000\}$. Here the four 25-tuples occur when $x = 125$, $x = 375$, $x = 625$ and $x = 875$.





Note 2

Alexandre Borovik

Luckily, in this riddle we have a natural probability measure. Interestingly, for the 7^{th}, 11^{th} and 13^{th} powers, the answer is the same.

Illustration

N.N. Taleb

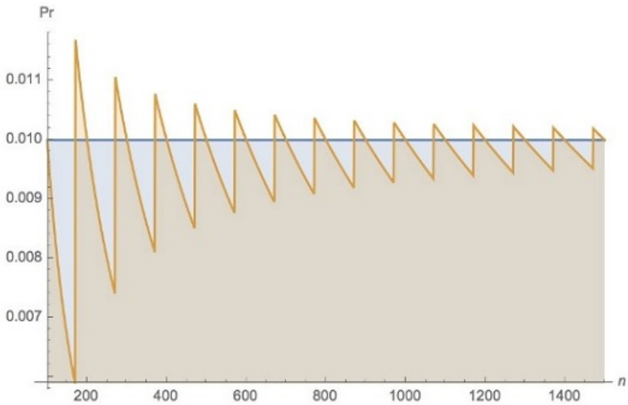
For the probability p to be $\frac{1}{100}$, one has to further assume that the support $[0, n]$ is either very large or n is a multiple of 100. Now, if the sample space is $[0, 150]$, the probability is $\frac{1}{150}$, not $\frac{1}{100}$. If the sample space is $[0, 171]$, $p = \frac{2}{171}$. Hence, in general,

$$p = \frac{n + 100 \cdot \mathbb{1}_{[n \bmod 100] > 71} - (n \bmod 100)}{100n}, \qquad n \geq 100.$$

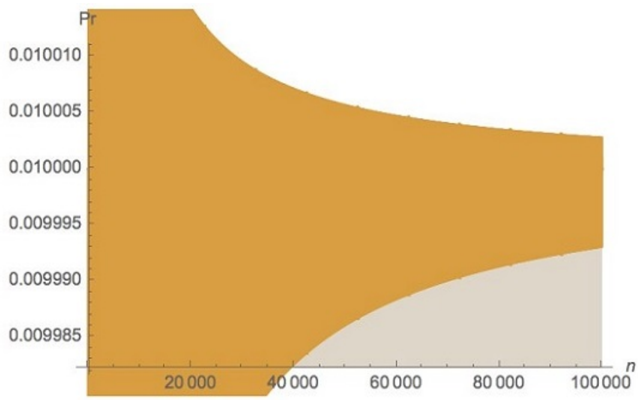
Let us visualize:

$$\boldsymbol{p = \frac{n + 100\ \text{Boole}[\text{Mod}[m, 100] \geq 71] - \text{Mod}[n, 100]}{100n}}$$

DiscretePlot[{.01, p}, {n, 100, 1500, 1}, PlotRange → All, AxesLabel → {n, "Pr"}]



DiscretePlot[{.01, p}, {n, 100, 10^5, 1}, PlotRange → All, AxesLabel → {n, "Pr"}]



Odds and Chances in Horse Race Betting

Let there be n horses in the race. If the better bets $\dfrac{a_i}{a_i+b_i}:\dfrac{b_i}{a_i+b_i}$, for $(i=1,2,\ldots,n)$, against the i^{th} horse winning, then, if the k^{th} horse wins, the better wins $\dfrac{a_k}{a_k+b_k}$ and loses

$$\sum_{i=1, i\neq k}^{n}\frac{b_i}{a_i+b_i}=\sum_{i=1}^{n}\frac{b_i}{a_i+b_i}-\frac{b_k}{a_k+b_k}.$$

Therefore, the better's winnings are

$$\frac{a_k}{a_k+b_k}-\left(\sum_{i=1}^{n}\frac{b_i}{a_i+b_i}-\frac{b_k}{a_k+b_k}\right)=1-\sum_{i=1}^{n}\frac{b_i}{a_i+b_i},$$

which is positive and the same whichever horse wins.

Acting As a Team

Solution 1

Each player chooses a color randomly. The probability of success is $P = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$.

Solution 2

There are eight possible combinations of hat colors. Each player has to choose between $R(ed)$, $B(lue)$ or $P(ass)$, making 27 possible team answers. Each player makes a random choice between three possibilities.

If all pass, the team loses. With two passes, there are six possible answers, each with probability of $\frac{1}{2}$ of a correct guess. With one pass, there are 12 possible answers, each with probability of $\frac{1}{4}$ of a correct guess. With no passes, there are eight possible answers, each with probability of $\frac{1}{8}$ of being right. The total comes to

$$\begin{aligned} P &= \frac{1}{27}\left(0 \cdot 1 + \frac{1}{2} \cdot 6 + \frac{1}{4} \cdot 12 + \frac{1}{8} \cdot 8\right) \\ &= \frac{3+3+1}{27} = \frac{7}{27} > \frac{7}{28} = \frac{1}{4}. \end{aligned}$$

Solution 3

Strategies 1 and 2 are stupid: the fewer people make a guess, the higher the probability of success. However, in order to win, at least one guess ought to be made. The probability of success is $\frac{1}{2}$.

Solution 4

Elwin Berlekamp

1. If you see two hats of the same color, guess the other color.

2. If you see two hats of different colors, pass.

The results of the strategy are summarized in the following table:

A	B	C	A	B	C	Outcome
R	R	R	GBW	GBW	GBW	lose
R	R	B	pass	pass	GBC	win
R	B	R	pass	GBC	pass	win
R	B	B	GRC	pass	pass	win
B	R	R	GRC	pass	pass	win
B	R	B	pass	GRC	pass	win
B	B	R	pass	pass	GRC	win
B	B	B	GRW	GRW	GRW	lose

In the table, GBW = guess blue: wrong, GBC = guess blue: correct, GRW = guess red: wrong, GRC = guess red: correct. The strategy gives six team wins out of eight: $\frac{6}{8} = \frac{3}{4}$.

This remarkable strategy has roots in the coding theory, especially in the field of protective codes [50, Chapter 6], [102, pp. 123–126].

Sum of Two Outcomes of Tossing Three Dice

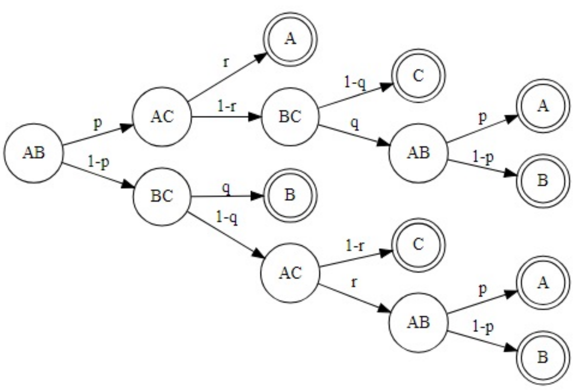
2 on the third toss could be the sum of the outcomes of the first tosses in one way: $2 = 1 + 1$; 3 in two ways: $2 + 1 = 3$ and $1 + 2 = 3$ and so on. In all there are 15 combinations of three tosses that satisfy this condition:

$\boxed{(1,1,2)}$	$\boxed{(1,2,3)}$	$(1,3,4)$	$(1,4,5)$	$(1,5,6)$
$\boxed{(2,1,3)}$	$\boxed{(2,2,4)}$	$\boxed{(2,3,5)}$	$\boxed{(2,4,6)}$	
$(3,1,4)$	$\boxed{(3,2,5)}$	$(3,3,6)$		
$(4,1,5)$	$\boxed{(4,2,6)}$			
$(5,1,6)$				

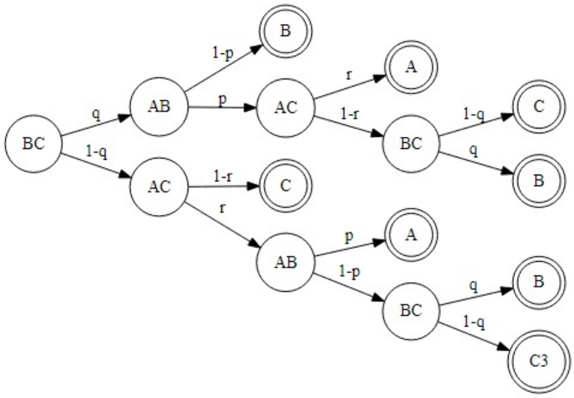
The three tosses are independent, implying that all 15 outcomes in the table have equal probabilities. Those that include at least one 2 are boxed; there are eight of them. Thus, the sought probability is $\frac{8}{15}$.

Chess Players Truel

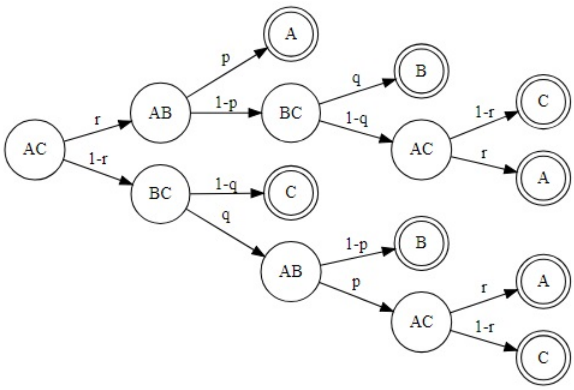
The simplest way to approach the riddle is by constructing an *event tree*. We are going to have three of them. The trees grow away from the root, where the root (first game) is one of the pairs AB, BC or AC. Each node is a game, except for the terminal ones shown in double circles. These are the winners of the tournament who collected two wins first. For the first two trees we will find the probability of B being the winner. These will be denoted P_{AB} and P_{BC}. We assume the outcomes of all games are independent.



$$P_{AB} = p \cdot (1 - r) \cdot q \cdot (1 - p) + (1 - p) \cdot q + (1 - p) \cdot (1 - q) \cdot r \cdot (1 - p).$$



$$P_{BC} = q \cdot (1-p) + q \cdot p \cdot (1-r) \cdot q + (1-q) \cdot r \cdot (1-p) \cdot q.$$



The task is to compare the two expressions P_{AB} and P_{BC} assuming $p, q, r > .5$ and $r > p$. In the following we will prove that, always, $P_{AB} < P_{BC}$. The result is imminently plausible as it seems quite reasonable for B to start with playing the weakest player C first.

So here are the two expressions to compare:

1. $P_{AB} = p \cdot (1-r) \cdot q \cdot (1-p) + (1-p) \cdot q + (1-p) \cdot (1-q) \cdot r \cdot (1-p)$,
2. $P_{BC} = q \cdot (1-p) + q \cdot p \cdot (1-r) \cdot q + (1-q) \cdot r \cdot (1-p) \cdot q$.

In the difference $P_{BC} - P_{AB}$, the common term $(1-p) \cdot q$ cancels out, leading to

$$P_{BC} - P_{AB} = [p \cdot q \cdot (1-r) + (1-p) \cdot (1-q) \cdot r] \cdot (p + q - 1)$$

which, since $p + q > .5 + .5 = 1$, is positive. Note that this is true even without the condition $r > p$.

Two Loaded Dice

Solution 1

We want to show that, if p_k is the probability of getting k on top, $k = 1, \ldots, 6$, then

$$p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2 + p_6^2 \geq \frac{1}{6}.$$

What we know is that $p_1 + \cdots + p_6 = 1$.
By the Cauchy–Schwarz inequality [87],

$$\begin{aligned}1 &= (p_1 + p_2 + p_3 + p_4 + p_5 + p_6)^2 \\ &= (1 \cdot p_1 + 1 \cdot p_2 + 1 \cdot p_3 + 1 \cdot p_4 + 1 \cdot p_5 + 1 \cdot p_6)^2 \\ &\le (1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2)(p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2 + p_6^2)\end{aligned}$$

and the required inequality follows.

Solution 2

Kriteesh Parashar

Let $\{p_1, p_2, p_3, \ldots, p_6\}$ be the probability of each number on the die for both dice. Then $p_1 + p_2 + \cdots + p_6 = 1$, and $P(\text{doubles}) = p_1^2 + p_2^2 + \cdots + p_6^2$. By Jensen's inequality [17], since x^2 is convex,

$$\begin{aligned}\frac{P(\text{doubles})}{6} &= \left(\frac{p_1 + p_2 + \cdots + p_6}{6}\right)^2 \\ \frac{P(\text{doubles})}{6} &\ge \left(\frac{1}{6}\right)^2 \\ P(\text{doubles}) &= \frac{1}{6}.\end{aligned}$$

Solution 3

Amit Itagi

Let p_i be the probability that a die turns up i:

$$\begin{aligned}1 &= \left(\sum_i p_i\right)\left(\sum_j p_j\right) = \sum_i p_i^2 + 2 \sum_{j,k,j \neq k} p_j, p_k \\ &\le \sum_i p_i^2 + \sum_{j,k,j \neq k} (p_j^2 + p_k^2) = \sum_i p_i^2 + 5 \sum_j p_j^2 = 6 \sum_i p_i^2.\end{aligned}$$

Thus, the probability of a double is $\displaystyle\sum_i p_i^2 \ge \frac{1}{6}$.

Solution 4

Jacob Janssen

You can as well throw one die twice. Let p_x be the probability of getting x with one die. Then $\displaystyle\sum_x p_x^2$ is minimal when the largest p_x^2 is minimal as x^2 convex, which is when $p_x = \dfrac{1}{6}$ for all x. So $\displaystyle\sum_x p_x^2 \ge \frac{1}{6}$.

Solution 5
Ian on Twitter

Let $\bar{e} = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)^T$. Note that $\bar{e}^T\bar{e} = \frac{1}{6}$. Let $\bar{x} = \bar{e} + \bar{\eta}$ where $\sum_i \eta_i = 0$. Then

$$\begin{aligned}\bar{x}^T\bar{x} &= \bar{e}^T\bar{e} + 2\bar{e}^T\bar{\eta} + \bar{\eta}^T\bar{\eta} \\ &= \bar{e}^T\bar{e} + \bar{\eta}^T\bar{\eta} \\ &\geq \bar{e}^T\bar{e} = \frac{1}{6}.\end{aligned}$$

Notes

For $n \geq 1$ there are 6^n equiprobable outcomes of which six are n-tuples of equal tops. Thus, the probability—in case of fair dice—of all dice showing the same number is $\frac{6}{6^n} = \frac{1}{6^{n-1}}$.

For loaded dice with the same probability distribution, we may invoke Hölder's inequality [9]. With $k = \frac{n}{n-1}$,

$$\begin{aligned}1 = \sum_{i=1}^{6} p_i &\leq \left(\sum_{i=1}^{6} 1^k\right)^{\frac{1}{k}} \left(\sum_{i=1}^{6} p_i^n\right)^{\frac{1}{n}} \\ &= 6^{\frac{n-1}{n}} \left(\sum_{i=1}^{6} p_i^n\right)^{\frac{1}{n}},\end{aligned}$$

or

$$1 \leq 6^{n-1} \cdot \left(\sum_{i=1}^{6} p_i^n\right),$$

and, finally,

$$\sum_{i=1}^{6} p_i^n \geq \frac{1}{6^{n-1}}.$$

Crossing Bridge in Crowds
G.D. Kaye

If there is nobody on the bridge at noon, no one has entered it in the five-minute interval before noon. Since there are 144 intervals of 5 minutes in 12 hours, the probability that an individual enters the bridge in a specific one is $\frac{1}{144}$. The probability that none of the 1000 individuals enters in that interval is

$$\left(1 - \frac{1}{144}\right)^{1000} = \left[\left(1 - \frac{1}{144}\right)^{144}\right]^{\frac{1000}{144}} \approx e^{-\frac{125}{18}} \approx 0.00096.$$

Chapter 5

Geometric Probability

> Whatever way uncertainty is approached, probability is the only sound way to think about it.
>
> Dennis Lindley, 2006, *Understanding Uncertainty*

Random events that take place in continuous sample space may invoke geometric imagery for at least two reasons: the nature of the problem or the nature of the solution.

Problems that fall under the category of *Geometric Probabilities* mostly relate geometric measurements of one set to those of another. Sometimes the measurements come as a result of advanced calculations, but often comparison of the measurements submits to reasonably educated intuition.

Riddles

5.1 Three Numbers

Mathcount 2018

What is the probability that three randomly drawn real numbers between 0 and 1 have a sum less than 1?

Express your answer as a common fraction.

5.2 Three Points on a Circle I

[8, Problem 244], [88, Problem 3.12]

Three points A, B, C are placed at random on a circle of radius 1. What is the probability for $\triangle\,ABC$ to be acute?

5.3 Three Points on a Circle II

[61, Problem A6 (1992)]

Three points are randomly chosen on a circle. What is the probability that the triangle with vertices at the three points has the center of the circle in its interior?

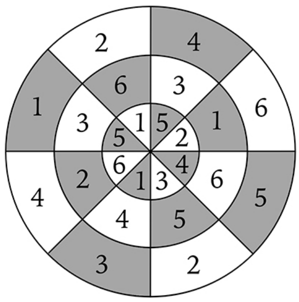
5.4 Two Friends Meeting

Two friends who take the metro to their jobs from the same station arrive to the station uniformly randomly between 7 and 7:20 in the morning. They are willing to wait for one another for 5 minutes, after which they take a train whether together or alone. What is the probability of their meeting at the station?

5.5 Hitting a Dart Board

Mathcount 2018

Setzer throws a dart that lands within one of the 24 numbered regions on the dartboard shown and then rolls a standard six-sided die.



What is the probability that the number of the region his dart hits is the same as the number he rolls on the die? What is the most likely coincidence? Express your answers as common fractions.

Assume that the probability of hitting a region on the dartboard is proportional to its area and that the small circle and the rings have the radii 1, 2, 3 and 4.

5.6 Circle Coverage

Given a circle and n of its semicircles chosen at random with no chosen semicircles sharing their endpoints, what is the probability that the circle is completely covered by the semicircles?

5.7 Points in a Semicircle

[29, Problem 10]

Choose n points at random (uniformly and independently) on the circumference of a circle. Find the probability p_n that all the points lie on a semicircle. (For instance, $p_1 = p_2 = 1$.)

More generally, fix ϕ, $0 < \phi < 2\pi$, and find the probability that the n points lie on an arc subtending an angle ϕ.

5.8 Flat Probabilities on a Sphere

[83, Problem 305], Randy Schwartz of Schoolcraft College, MN

You are to select two points on a sphere and offer a friend to select a third one. If the triangle (plane) so obtained is acute, you win; otherwise, you lose.

Which selection of the first two points maximizes your chances of winning?

5.9 Four Random Points on a Sphere

[61, Problem A6 (1992)]

Four points are chosen at random on the surface of a sphere. What is the probability that the center of the sphere lies inside the tetrahedron whose vertices are at the four points? (It is understood that each point is independently chosen relative to a uniform distribution on the sphere.)

5.10 Random Numbers and Obtuse Triangle

[55, Chapter 1]

Two numbers x and y are uniformly distributed on $[0, 1]$. What is the probability that sides of length x, length y and length 1 form an obtuse triangle?

5.11 Random Intervals with One Dominant

N.N. Taleb, as a follow-up to Riddle 3.6 on page 16

$2n \geq 4$ points are chosen uniformly randomly on the interval $[0, 1]$, say, $X_1, X_2, \ldots, X_{2n}$ and, for $1 \leq j \leq n$, let J_j be the closed interval with endpoints X_{2j-1} and X_{2j}. Find the probability that one of the intervals J_j is dominant, i.e., includes all other intervals as subsets.

5.12 Distributing Balls of Two Colors in Two Bags

[36, Problem 2.3.9]

10 black and 10 white balls are placed into two bags. A bag is chosen at random and a ball is drawn from that bag.

How should one distribute the balls between the bags as to maximize the probability of drawing a white ball?

5.13 Hemisphere Coverage

[32, Section 5.10]

Given a sphere and n of its hemispheres chosen at random, what is the probability that the sphere is completely covered by the hemispheres?

5.14 Random Points on a Segment

[69, Problems 42–43]

1. A point is dropped randomly on a unit segment. It divides the segment into two pieces. What is the average length of the shortest piece? The longest one? Their ratio? What is the expected value of the ratio of the lengths of the longest segment to the shortest one?

2. Two random points, chosen uniformly and independently on a unit segment, split the segment into three pieces. What is the average length of the left segment?

3. Two points are dropped randomly on a unit segment. They divide the segment into three pieces. What is the average length of the shortest piece? The longest one? Their ratio?

5.15 Probability of First Digit in Product

[64, Problem 165]

Two d-digit numbers (whose first digits are not zero) are chosen randomly and independently, then multiplied one by the other.

1. Let P_d be the probability that the first digit of the product is 9. Find $\lim_{d\to\infty} P_d$.

2. More generally, if $n \neq 0$ is a decimal digit and P_d^n is the probability that the first digit of the product is n, what is $\lim_{d\to\infty} P_d^n$?

5.16 Birds on a Wire

Mark Galecki, Marcin Kuczma

Take a wire stretched between two posts, and have a large number of birds land on it at random. Take a bucket of yellow paint, and for each bird, paint the interval from it to its closest neighbor. The question is: what proportion of the wire will be painted? More strictly: as the number of birds goes to infinity, what is the limit of the expected value of the proportion of painted wire, assuming a uniform probability distribution of birds on the wire?

5.17 Lucky Times at a Moscow Math Olympiad

[37, Problems A4 and A5]

1. A moment of time is lucky if the hour, the minute and the second hands are in this order clockwise starting with the hour hand. A moment is unlucky if the order of the hands clockwise is hour, second, minute. Which is more probable?

2. A moment of time is lucky if the hour, the minute and the second hands all lie on the same side of some diameter of the clock face and unlucky otherwise. Which is more probable?

5.18 Probability of a Random Inequality

[46, Problem 1985–7]

Let a and b be positive constants with $b > 1$. Given that $x + y = 2a$ and that x is uniformly distributed on $[0, 2a]$, find the probability that

$$xy > \frac{(b^2 - 1)a^2}{b^2}.$$

5.19 Points on a Square Grid

A. Bogomolny

n points are chosen randomly at the nodes of a square grid. The points are joined pairwise. What is the probability that at least one of the midpoints of the so-obtained segments lies at a grid node? Consider $n = 2, 3, 4, 5$.

5.20 A Triangle out of Three Broken Sticks

[103, Problem 96]

Three sticks of equal length are each uniformly randomly broken into two pieces.

1. What is the probability that from the three left pieces of the three sticks it is possible to form a triangle?

2. What is the probability that from the three longest pieces of the three sticks it is possible to form a triangle?

3. What is the probability that from the three shortest pieces of the three sticks it is possible to form a triangle?

4. What if the pieces are chosen randomly with the probability of $\frac{1}{2}$ for each stick?

5. What is the expected number of triangles that can be so formed?

5.21 Probability in Dart Throwing

[46, Problem 1999–5]

What is the probability that a dart, hitting a square board at random, lands nearer the center than an edge?

5.22 Probability in Triangle

A. Bogomolny

Point P is uniformly distributed in $\triangle ABC$. AA', BB', CC' are the cevians through P. K, L, M are the midpoints of AA', BB', CC', respectively, Figure 5.1.

Find the probability that P is within $\triangle KLM$.

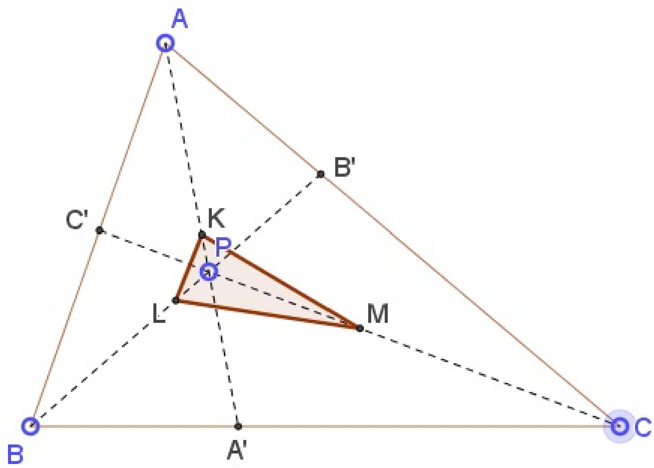


Figure 5.1: Probability in triangle.

Solutions

Three Numbers

Solution 1

A. Bogomolny

The shape described by $0 \le x, y, z \le 1$ and $x+y+z \le 1$ is a pyramid with a triangular base defined by three points, (0,0,0), (0,1,0) and (0,0,1), and the apex at (0,0,1), see Figure 5.2.

The volume of that pyramid is $\frac{1}{3} \cdot 1 \cdot \left(\frac{1}{2} \cdot 1 \cdot 1\right) = \frac{1}{6}$. This is also the volume of the pyramid relative to that of the cube, which is 1.

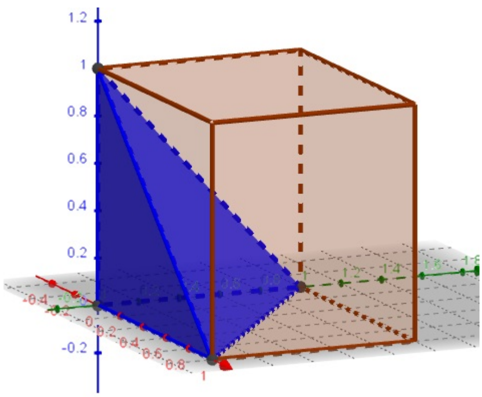


Figure 5.2: Three numbers sum to less than 1.

Solution 2

N.N. Taleb

The probability density function of the uniform distribution on $[0,1]$ is

$$f(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & \text{elsewhere.} \end{cases}$$

Given two random variables, the probability density function of their sum is the convolution of their probability density functions. It is the same for three:

$$\begin{aligned} g(z) &= \int_0^\infty f(z-y) \left(\int_0^\infty f(x) f(y-x) dx \right) dy \\ &= \begin{cases} \frac{z^2}{2}, & 0 \le z \le 1 \\ \frac{1}{2}(-3+6z-2z^2), & 1 \le z \le 2 \\ \frac{1}{2}(9-6z+z^2), & 2 \le z \le 3 \\ 0, & \text{elsewhere.} \end{cases} \end{aligned}$$

Due to the condition $x + y + z \leq 1$, we are only concerned with the distribution over $[0, 1]$ such that the sought probability can be found from $\int_0^1 \frac{z^2}{2} dz = \frac{1}{6}$.

Solution 3

Thamizh Kudimagan

I recast the riddle as follows to help me visualize it better: if a lizard going up a wall takes three consecutive steps, each of with a random value between 0 and 1 unit of length, what is the probability it will end up traveling less than 1 unit of length?

The lizard's first step has a box car probability distribution function between 0 and 1. For each of the values of its first step, you can add the second step. The second step by itself has a similar box car probability distribution function.

The sum of the first two steps has a probability distribution function that can be visualized by sliding the box car probability density function of the second over that of the first and noting the overlapping area, which increases linearly until 1 and then decreases linearly until 2—basically convolution.

The probability density function of the sum of the three steps (again, convolution) can be visualized by sliding the box car probability density function of the third step on the linear probability density function of the sum of the first two steps and noting how the overlap area evolves. Its area is under $\frac{x^2}{2}$ for $0 < x < 1$, so the answer is $\frac{1}{6}$.

To elaborate on the last part, the probability density function for the sum of the three steps (let us call it x for now) is roughly bell-shaped between 0 and 3 but is just $\frac{x^2}{2}$ in the range $0 < x < 1$. Thus, $P(x < 1)$ is the area under $\frac{x^2}{2}$ for $0 < x < 1$, so the answer is $\frac{1}{6}$.

Three Points on a Circle I

Solution 1

Fix point C. The positions of points A and B are then defined by arcs α and β extending from C in two directions. *A priori* we know that $0 < \alpha + \beta < 2\pi$. The values of α and β favorable for our situation (as subtending acute angles satisfy) are $0 < \alpha < \pi$ and $0 < \beta < \pi$. Their sum could not be less than π as this would make angle C obtuse. Therefore, $\alpha + \beta > \pi$. The situation is presented in Figure 5.3 where the square has the side 2π.

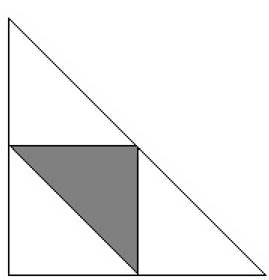


Figure 5.3: A region that yields an acute triangle.

Region D is the intersection of three half-planes: $0 < \alpha$, $0 < \beta$ and $\alpha+\beta < 2\pi$. This is the big triangle in Figure 5.3. The favorable events belong to the shaded triangle which is the intersection of the half-planes $\alpha < \pi$, $\beta < \pi$ and $\alpha + \beta > \pi$. The ratio of the areas of the two is obviously $\dfrac{1}{4}$.

Now observe that, unless the random triangle is acute, it can be thought of as obtuse since the probability of two of the three points A, B, C forming a diameter is 0. (For BC to be a diameter, one should have $\alpha + \beta = \pi$, which is a straight line, with 0 as the only possible assignment of area.) Thus, we can say that the probability of $\triangle$ ABC being obtuse is $\dfrac{3}{4}$. For an obtuse triangle, the circle can be divided into two halves with the triangle lying entirely in one of the halves. It follows that $\dfrac{3}{4}$ is the answer to the following question:

Three points A, B, C are placed at random on a circle of radius 1. What is the probability that all three lie in a semicircle?

Solution 2

[93] credits V.V. Fok and Yu.V. Chekanov with the following argument

We shall select a triangle in two steps. First, we get three points A, B, C and their antipodes A′, B′, C′ across the center of the circle (Figure 5.4). Selecting a triangle consists of selecting one of the points in each of the three pairs A, A', B, B', C, C'. There are eight possible combinations.

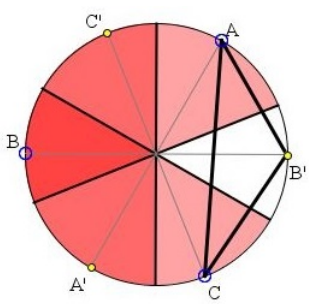


Figure 5.4: How to find an obtuse triangle.

With each of the six points, we associate the semicircle around its antipode. Three points form an obtuse triangle if and only if they lie in a semicircle, and this is only if their associated semicircles have a nonempty intersection. The diameters of the associated semicircles form six sectors. Each sector serves as an intersection of the associated semicircles for exactly one selection of the vertices, meaning that out of eight possible selections there are always six obtuse (and two acute) triangles. The probability that one is obtuse is therefore $\dfrac{6}{8} = \dfrac{3}{4}$.

It follows that the probability of a triangle being acute is $1 - \dfrac{3}{4} = \dfrac{1}{4}$.

Three Points on a Circle II

Solution 1

A. Bogomolny

We shall reformulate the riddle and then refer to another one, Riddle 5.6 on page 74.

For three uniformly random points on a circle these two statements are equivalent:

1. The semicircles of which the three points are the centers cover the circle.
2. The triangle with vertices at the three points has the center of the circle in its interior.

Indeed, consider a borderline case where two of the points are collinear with the center of the circle. Denote the points A, B, C and the center of the circle O (Figure 5.5). Assume that, say AC, is a diameter of the circle.

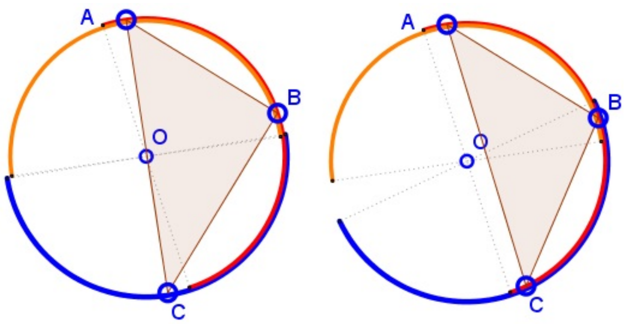


Figure 5.5: A triangle inscribed in a circle may or may not include the center.

Point B is on one of the semicircles defined by AC. Let D be the midpoint of the semicircle defined by AC that does not contain B. Then D is outside the semicircle defined by B. Shifting A and/or C toward B will leave D exposed, not covered by any of the three semicircles. Simultaneously, the center O will drop out of $\triangle ABC$.

Now, with reference to the problem of covering a circle with three semicircles, the probability of having O in the interior of $\triangle ABC$ equals $\dfrac{1}{4}$.

Note

The two statements are also equivalent to Riddle 5.3 on page 73 about the probability of a triangle with vertices at the three points having the center of the circle in its interior.

Solution 2

A. Bogomolny

The center O is inside $\triangle ABC$ if and only if that is an acute triangle, which means that all three central angles AOB, BOC, COA are less than $180°$ (Figure 5.6). It follows that their semicircles intersect pairwise and thus cover the whole of the circle.

Now, with reference to the problem of circle coverage with three semicircles, Riddle 5.6 on page 74, the probability of having O in the interior of $\triangle ABC$ equals $\dfrac{1}{4}$.

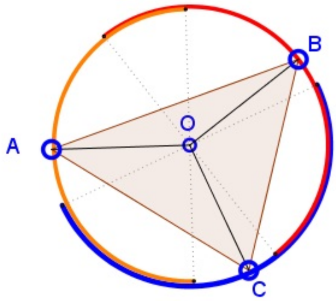


Figure 5.6: Three semicircles that completely cover the circle.

Solution 3

Achraf on Twitter

Let us say that O is the first point we generate. This will be our point of reference. From this point, we will use the polar coordinate system (counterclockwise) to determine the position of A (our second point, with $\widehat{\mathrm{OCA}} = \alpha$) and then the position of B (our third point, with $\widehat{\mathrm{OCB}} = \beta$). Let C be the center of the circle.

Let us call $\breve{A}$ and $\breve{B}$ the two random variables (uniform on $[0, 2\pi]$). Let us distinguish between two cases (there are two letters B in each graph for the two "extreme" letters B, see Figure 5.7):

1. If $0 < \alpha \leq \pi$, then the β that allow C to be in the triangle are those in $[\pi, \pi+\alpha]$.
2. If $\pi < \alpha \leq 2\pi$, then the β that allow C to be in the triangle are those in $[\alpha-\pi, \pi]$.

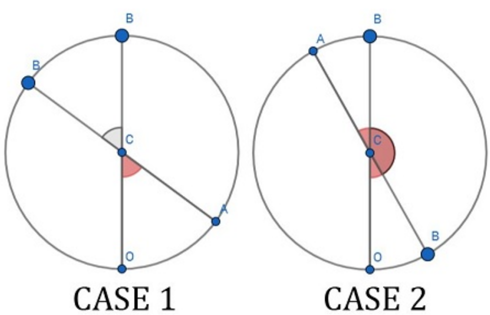


Figure 5.7: Two random points on a circle.

The probability p that C is inside the triangle OAB is (with convenient notations for probability density functions):

$$
\begin{aligned}
p &= \int_0^{\pi} \mathbb{P}(\check{B} \in [\pi, \pi+\alpha] | \check{A} = \alpha)\mathbb{P}(\check{A} = \alpha) d\alpha \\
&\qquad + \int_{\pi}^{2\pi} \mathbb{P}(\check{B} \in [\alpha-\pi, \pi] | \check{A} = \alpha)\mathbb{P}(\check{A} = \alpha) d\alpha \\
&= \int_0^{\pi} \Big(\frac{\pi+\alpha-\pi}{2\pi}\Big)\Big(\frac{1}{2\pi}\Big) d\alpha + \int_{\pi}^{2\pi} \Big(\frac{\pi-(\alpha-\pi)}{2\pi}\Big)\Big(\frac{1}{2\pi}\Big) d\alpha \\
&= \int_0^{\pi} \Big(\frac{\alpha}{2\pi}\Big)\Big(\frac{1}{2\pi}\Big) d\alpha + \int_{\pi}^{2\pi} \Big(\frac{2\pi-\alpha}{2\pi}\Big)\Big(\frac{1}{2\pi}\Big) d\alpha \\
&= \int_0^{\pi} \Big(\frac{\alpha}{2\pi}\Big)\Big(\frac{1}{2\pi}\Big) d\alpha + \int_{\pi}^{0} \Big(\frac{\delta}{2\pi}\Big)\Big(\frac{1}{2\pi}\Big)(-1) d\delta \\
&= \int_0^{\pi} \Big(\frac{\alpha}{2\pi}\Big)\Big(\frac{1}{2\pi}\Big) d\alpha + \int_{0}^{\pi} \Big(\frac{\delta}{2\pi}\Big)\Big(\frac{1}{2\pi}\Big) d\delta \\
&= 2\int_0^{\pi} \Big(\frac{\alpha}{2\pi}\Big)\Big(\frac{1}{2\pi}\Big) d\alpha \\
&= \frac{1}{4}
\end{aligned}
$$

(where we used the change of variable $\delta = 2\pi - \alpha$).

Two Friends Meeting

In a Cartesian system of coordinates (s,t), a square of side 20 (minutes) represents all the possibilities of the morning arrivals of the two friends at the metro station.

The gray area A, Figure 5.8, is bounded by two straight lines, $t = s+5$ and $t = s-5$, so that inside A, $|s-t| \leq 5$. It follows that the two friends will meet only provided their arrivals s and t fall into region A. The probability of this happening is given by the ratio of the area of A to the area of the square:

$$
\frac{400 - \left(15 \times \frac{15}{2} + 15 \times \frac{15}{2}\right)}{400} = \frac{175}{400} = \frac{7}{16}.
$$

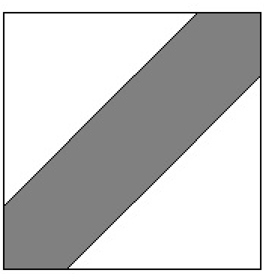


Figure 5.8: Two friends meeting.

Hitting a Dart Board

Mathcount 2018

With a reference to Figure 5.9, if the probability of hitting a region is proportional to its area, then the odds of hitting one in the outer ring to those of hitting one in the middle ring to those of hitting one in the central circle are $5:3:1$.

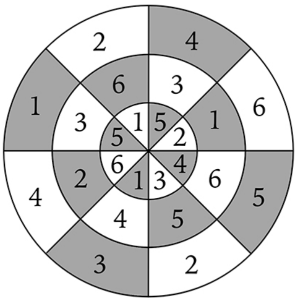


Figure 5.9: Probabilities on a dart board.

Summing up the probabilities for each number, we get $P(1) = \frac{10}{72}$, $P(2) = \frac{14}{72}$, $P(3) = \frac{12}{72}$, $P(4) = \frac{14}{72}$, $P(5) = \frac{10}{72}$ and $P(6) = \frac{10}{12}$.

As the die is fair, the probability of coincidence is proportional to the probability of hitting a number on the dartboard. 2 and 4 have the highest probabilities of being hit so that they are most likely to hit the outcome of the die roll.

For the first question, the distribution on the dartboard is inconsequential. Once a number is hit, there is the same probability of $\frac{1}{6}$ that it will be matched by the outcome of a die roll.

Circle Coverage

Each semicircle is bounded by two diametrically opposite points. The n pairs of points divide the circle into $2n$ regions (which can be proved by induction.) A point on the circle may belong to a given semicircle or to its complement. Hence, the probability that a given point on the circle is in one of the semicircles is 2^{-1}. The probability that it is covered by none of the n semicircles is 2^{-n}. Further, a point belongs to exactly one of the $2n$ regions. If the point is not covered by the semicircles, neither is the region it belongs to. Thus, the probability that at least one of those regions is not covered by the semicircles is $2^{-n}(2n) = 2^{-n+1}n$. The probability that the entire circle is covered is then $1 - 2^{-n+1}n$.

For $n = 1$, the formula gives the probability of 0, which is obviously true because a single semicircle only covers half a circle, not the whole of it. For $n = 2$, the probability is also 0, although two semicircles with the same endpoints cover the entire circle. The probability of this is clearly 0. For $n = 3$, the formula gives the probability $\frac{1}{4}$.

Points in a Semicircle

Solution 1 to Question 1

Associate with every point X on a circle the semicircle arc of which X is the midpoint. We claim that a set of points $\mathcal{X}$ on a circle lie in a semicircle only if the union of the associated semicircles does not completely cover the circle.

Let (C) be the circumference of a circle (to avoid ambiguity); $\mathcal{X} = \{X_k : X_k \in (C),\ k = 1, \ldots, n\}$, $n > 1$ and is an integer. The following two statements are equivalent:

1. There is point $P \in (C)$ such that $\mathcal{X} \subset P$.
2. $(C) \setminus \bigcup\limits_{k=1}^{n} \quad (X_k \neq \emptyset)$.

Thus, the original problem is equivalent to Riddle 5.6 on page 74, where the probability was found to be $1 - 2^{-n+1}n$ for the semicircles to cover (C) completely. This implies that for a set $\mathcal{X}$ of n random points on (C) the probability of them all being covered by a semicircle is $\dfrac{n}{2^{n-1}}$.

Solution 2 to Question 1

Joshua Jordan

Case 1

n is 1 or 2.

By inspection, the answer is 1.

Case 2

$n > 2$.

Let P be any set of $n > 2$ points distributed uniformly at random on the circumference of a circle. With probability 1, the n points are distinct. For any point $p \in P$, let $S(p)$ be the semicircle starting at p (inclusive) and continuing in the counterclockwise direction. If all n points lie on the same semicircle, then there exists a point $p \in P$ such that $P \subseteq S(p)$.

Choose a point $p \in P$ arbitrarily, and let q be any one of the other $n - 1$ points. The probability that $q \in S(p)$ is $\dfrac{1}{2}$. Therefore, the probability that $S(p)$ contains all the other $n-1$ points is $\left(\dfrac{1}{2}\right)^{n-1}$. Since, by definition, $S(p)$ contains p, the probability that $S(p)$ contains all n points, that is that $P \subseteq S(p)$, is also $\left(\dfrac{1}{2}\right)^{n-1}$.

When $n > 2$, there is at most one point $p \in P$ such that $P \subseteq S(p)$. Therefore, we can sum the individual probabilities for each point $p \in P$ to obtain the probability that there exists any $p \in P$ such that $P \subseteq S(p)$. This sum is $n\left(\dfrac{1}{2}\right)^{n-1}$. Therefore, the probability that all $n > 2$ points lie on the same semicircle is $n\left(\dfrac{1}{2}\right)^{n-1}$.

Combining Cases

Note that $n\left(\frac{1}{2}\right)^{n-1} = 1$ when n is either 1 or 2. That is the same answer obtained in case 1. Therefore, the expression from case 2 gives the correct result for any positive integer n, and the probability that all n points lie on the same semicircle is $n\left(\frac{1}{2}\right)^{n-1}$.

Solution to Question 2

Follows Josh Jordan's reasoning

We make an assumption that $\phi < \pi$.

For a point $X \in (C)$, let $\Phi(X)$ be the arc subtending angle ϕ that extends counterclockwise from X, inclusive of X. The probability of a point uniformly distributed on (C) to fall into $\Phi(X)$ equals $\frac{\phi}{2\pi}$.

Given $n > 2$ points $X_1, X_2, \ldots, X_n$, the probability that $X_i \in \Phi(X_k)$, $i \neq k$, is the same $\frac{\phi}{2\pi}$ and, for all $i \neq k$, is $\left(\frac{\phi}{2\pi}\right)^{n-1}$.

For different k, the events E_k: $(\forall i \neq k) X_i \in \Phi(X_k)$ do not intersect, since $X_i \in \Phi(x_k)$ is incompatible with $X_k \in \Phi(X_i)$. It follows that the probability of the event $\sum_{k=1}^{n} E_k$ equals $n\left(\frac{\phi}{2\pi}\right)^{n-1}$.

Now, if for some point Y, $(\forall i) X_i \in \Phi(Y)$, then, for X_k nearest to Y in the counterclockwise direction, $(\forall i \neq k) X_i \in \Phi(X_k)$. As we found, this event has the probability of $n\left(\frac{\phi}{2\pi}\right)^{n-1}$.

Flat Probabilities on a Sphere

Suppose you choose two points with central angle α. By symmetry, we may assume they have coordinates $(0, b, c)$ and $(0, b, -c)$. Figure 5.10 shows a 2D cross section.

You lose your bet if your friend selects a point in one of three caps: above the plane $z = c$, below $z = -c$ or to the right of $y = b$.

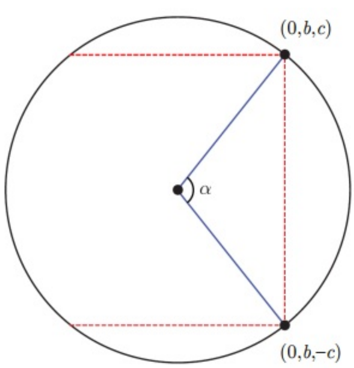


Figure 5.10: A 2D cross section.

Thus, we need to compute the areas of the three spherical caps shown in Figure 5.11. We do that using a theorem discovered by Archimedes:

Note that if two parallel planes h units apart slice a sphere of radius r, the surface area between the planes is $2\pi rh$.

Using these calculations, we conclude that the probability that you win the bet is

$$p(\alpha) = \sin\left(\frac{\alpha}{2}\right) + \frac{1}{2}\left(\cos\left(\frac{\alpha}{2}\right) - 1\right).$$

The function attains its maximum for $\alpha_o = 2\arctan(2) \approx 126.9^\circ$. In addition, $p(\alpha_o) = \dfrac{1}{\varphi} = \dfrac{\sqrt{5}-1}{2}$.

Steve Phelps [76] created a beautiful dynamic illustration for the riddle that is the source for Figure 5.11.

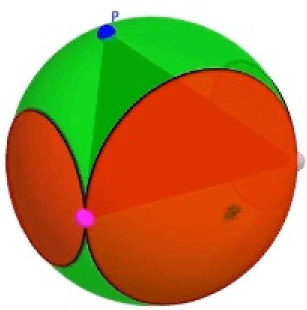


Figure 5.11: Flat probabilities on a sphere: a 3D view.

Four Random Points on a Sphere

Solution 1

[63, Problem A-6]

Four points A, B, C, D define a system of barycentric coordinates in the space such that every point E can be uniquely expressed as a linear combination with real coefficients

$$E = w\mathrm{A} + x\mathrm{B} + y\mathrm{C} + z\mathrm{D}, \qquad w + x + y + z = 1.$$

Point E lies in the interior of the tetrahedron ABCD if and only if all the coefficients are positive. Thus, if O stands for the origin and serves as the center of the sphere, the riddle inquires of the probability that $\mathrm{O} = w\mathrm{A} + x\mathrm{B} + y\mathrm{C} + z\mathrm{D}$, with $w + x + y + z = 1$ and $w, x, y, z > 0$.

Let us fix one of the points and denote it P. The other three points will be denoted $\mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_3$ and will each be selected in two steps. First we select three random diameters $Q_{i1} + Q_{i2}$, $i = 1, 2, 3$. Next, we randomly pick one of the endpoints of each of the three diameters and take them for $\mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_3$, respectively. There are $2^3 = 8$ ways to perform the second step. The three diameters define eight tetrahedra. Observe that since, for $i = 1, 2, 3$, $Q_{i1} + Q_{i2} = 2\mathrm{O}$, the coefficients in $\mathrm{O} = w\mathrm{P} + x\mathrm{P}_1 + y\mathrm{P}_2 + z\mathrm{P}_3$ change signs every time one of the P's is replaced with another endpoint of the corresponding diameter. It follows that, disregarding the cases of zero probability, like that one of the faces of the tetrahedron passes through the origin, only for one out of eight tetrahedra do all the coefficients have the same sign and can be chosen positive. Thus, the probability of including the origin is $\frac{1}{8}$ for any three random diameters. It follows that the sought probability is $\frac{1}{8}$.

Solution 2

A. Bogomolny

Let us associate with every point X on the sphere S the semisphere $\overline{\mathrm{X}}$ that has that point as its center. We claim the following:

A tetrahedron ABCD inscribed into a sphere contains the center of the sphere in its interior if and only if the four hemispheres $\overline{\mathrm{P}_1}$, $\overline{\mathrm{P}_2}$, $\overline{\mathrm{P}_3}$, $\overline{\mathrm{P}_4}$ cover the sphere entirely.

Let O be the center of the sphere and, for a point X on the sphere, let $\underline{\mathrm{X}}$ be the half-space bounded by a plane through O that does not include $\overline{\mathrm{X}}$. If point P on the sphere is not covered by any of the four semispheres $\overline{\mathrm{P}_i}$, $i = 1, 2, 3, 4$, it lies in the interior of the four-sided cone with the apex at O: $\mathrm{P} \in \cap_{i=1}^{4} \underline{\mathrm{P}_i}$. This means that all the angles POP_i, $i = 1, 2, 3, 4$, are obtuse, implying that $\mathrm{P}_i \in \underline{\mathrm{P}}$, $i = 1, 2, 3, 4$. In other words, the tetrahedron $\mathrm{P_1P_2P_3P_4} \subset \underline{\mathrm{P}}$ and, therefore, does not contain O.

This means that if $\mathrm{O} \in \mathrm{P_1P_2P_3P_4}$ then $\mathrm{S} \subset \cup_{i=1}^{4} \underline{\mathrm{P}_i}$. We need to prove the converse.

Assume $\mathrm{O} \notin \mathrm{P_1P_2P_3P_4}$. There exists a plane π that separates P_1, P_2, P_3, P_4 and O. The perpendicular from O to π crosses the sphere at two points; let P be the farthest of the two from π. Then all angles POP_i are obtuse so that $\mathrm{P} \notin \cup_1^4 \underline{\mathrm{P}_i}$.

The answer $\frac{1}{8}$ is now retrieved from the formula $1 - 2^n(n^2 - n + 2)$, with $n = 4$, as the probability of n hemispheres covering the whole of the sphere.

Random Numbers and Obtuse Triangle

For $x, y, 1$ to form a triangle at all, one needs these three inequalities:

$$1 + x > y, \quad 1 + y > x, \quad x + y > 1.$$

The first two inequalities are satisfied automatically. Thus, we should only consider the last one, $x + y > 1$.

Being the side of length 1 is the longest, the angle that lies opposite this side is the largest. It should not be acute or right, although the latter possibility has the probability of zero and can be disregarded.

The angle in question, say α, is obtuse, $\cos\alpha < 0$, so that, from the law of cosines, $x^2 + y^2 < 1$. Graphically, the point (x, y) that satisfies $x + y > 1$ and $x^2 + y^2 < 1$ lies in the blue region shown in Figure 5.12.

The sought probability is the area of that region:

$$\frac{\pi}{4} - \frac{1}{2} = \frac{\pi - 2}{4}.$$

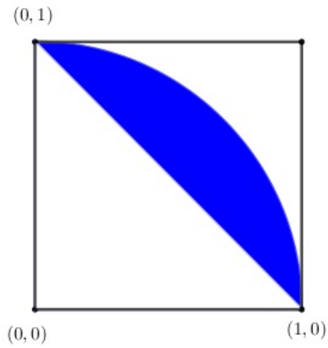


Figure 5.12: Random numbers and obtuse triangle.

Random Intervals with One Dominant

Solution 1

A. Bogomolny

Our sample space consists of $(2n)!$ permutations of the given point indices. An interval J_j is dominant in $2(2n-2)!$ cases, as the $2n-2$ indices can be arbitrarily permuted between the two extremes. One of the intervals is dominant in $n \cdot 2(2n-2)! = 2n(2n-2)!$ cases. Thus, the probability of there being a dominant interval is $\dfrac{2n(2n-2)!}{(2n)!} = \dfrac{1}{2n-1}$.

Solution 2

N.N. Taleb

The riddle is reduced to finding adjacent $X_{\max}$ and $X_{\min}$ (since only the interval $[X_{\max}, X_{\min}]$ covers all other intervals) in permutations of $X_1, \dots, X_{2n}$.

Note that, given that $2j$ is even, to be compatible with the previous formulation of the riddle, we consider intervals $[X_1, X_2]$ and$[X_3, X_4]$ but not $[X_2, X_3]$.

- $X_{\max}$ followed by $X_{\min}$

  There are $(2n-2)!$ sequences of each

$$\text{lines} \begin{pmatrix} (X_{\max}, X_{\min}, X_3, X_4, \dots, X_{2n}) \\ (X_1, X_2, X_{\max}, X_{\min}, \dots, X_{2n}) \\ \dots \\ (X_1, X_2, \dots, \dots, X_{\max}, X_{\min}) \end{pmatrix}$$

  for a total of $n(2n-2)!$

- $X_{\min}$ followed by $X_{\max}$

  By mirroring, same $n(2n-2)!$

- Thus,

  Numerator $= 2n(2n-2)!$
  Denominator(total permutations) $= (2n)!$

  $$p = \frac{1}{2n-1}.$$

Thanks to Michael Wiener for correcting the count of the number of lines.

Solution 3
Michael Weiner

The easier path is to consider that $X_{\min}$ is equally likely to be paired with any of the other $2n-1$ values. Only 1 is $X_{\max}$.

Solution 4
Zhuo Xi

Locate the point on the left (minimum). It has $\frac{1}{2}$ chance to be odd, with $\frac{1}{2n-1}$ probability that the next index point is maximum. Similarly, it has $\frac{1}{2}$ chance to be even with $\frac{1}{2n-1}$ chance the previous index point is maximum.

Distributing Balls of Two Colors in Two Bags

[7]

Let W_1, W_2 stand for the number of white balls placed into two bags; similarly, let B_1, B_2 stand for the number of black balls. If $P(W)$ is the probability of drawing a white ball, we have several examples:

W_1	B_1	W_2	B_2	$P(W)$
5	5	5	5	$\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{2}$
10	0	0	10	$\frac{1}{2}\cdot 1+\frac{1}{2}\cdot 0=\frac{1}{2}$
3	7	7	3	$\frac{1}{2}\cdot\frac{3}{10}+\frac{1}{2}\cdot\frac{7}{10}=\frac{1}{2}$
3	3	7	7	$\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{2}$

The best way to find the solution is to set $W_1 = 1$, $B_1 = 0$, $W_2 = 9$ and $B_2 = 10$. Then

$$P(W) = \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot\frac{9}{19} = \frac{14}{19} \approx 0.7368.$$

Figure 5.13 shows a graph of the function $f(x, y) = \frac{1}{2}\left(\frac{x}{x+y} + \frac{10-x}{20-x-y}\right)$ in the region $(0, 10) \times (0, 10)$.

Here x corresponds to W_1, y to B_1, etc.

For integer values, the maximum of f is attained for $f(1,0)$ and $f(9,10)$ which are practically the same distribution.

The maximum probability could not exceed 75% because there is always a bag where at most 50% of the balls are white and another where at most 100% of the balls are white.

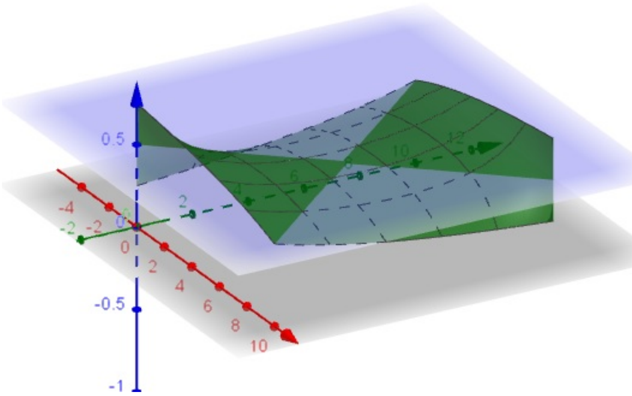


Figure 5.13: Distributing balls of two colors in two bags.

Notes

In Figure 5.13, the function $f(x,y) = \dfrac{1}{2}\left(\dfrac{x}{x+y} + \dfrac{10-x}{20-x-y}\right)$ appears to max out at the corners $(0,0)$ and $(10,10)$. This is probably due to software interpretation of one of the values, $f(0,0)$, $f(10,10)$, as $\dfrac{1}{2}$, the other as 1. From the viewpoint of probabilities, neither makes sense. Drawing a ball from an empty bag could not be assigned either $\dfrac{1}{2}$ or 1 as a probability.

Hemisphere Coverage

For a start, let us observe that some configurations, like when, for example, three or more of the hemisphere-bounding great circles concur, may not fall in line with general reasoning, but such configurations occur with probability 0 and may be ignored.

Each hemisphere is completely bounded by a great circle. The n great circles divide the surface of the sphere into $n^2 - n + 2$ regions (which can be proved by induction). Each great circle bounds two hemispheres. Hence, the probability that a given point, not on the great circle, is in one of the hemispheres is 2^{-1}. The probability that it is covered by none of the n hemispheres is 2^{-n}. The probability that the point lies on one of the great circles is 0. Other than that, a point belongs to exactly one of the $n^2 - n + 2$ regions. If the point is not covered by the hemispheres, neither is the region it belongs to. Thus, the probability that at least one of those regions is not covered by the hemispheres is $2^{-n}(n^2 - n + 2)$. The probability that the entire sphere is covered is then $1 - 2^{-n}(n^2 - n + 2)$.

For $n = 1$, the formula gives the probability of 0, which is obviously true because a single hemisphere only covers half a sphere, not the whole of it. For $n = 2$, the probability is also 0, although two hemispheres, if bounded by the same great circle, cover the entire sphere. The probability of this is the same as that of having a random point on the sphere fall on a given great circle, which is 0. For $n = 3$, the formula again gives zero probability.

Random Points on a Segment

Solution to Question 1

The expected value of each of the pieces, due to the symmetry principle, is $\frac{1}{2}$. It is not so for the expected value of the shorter of the two. (This is a curiosity because one of the two is surely shorter!)

If the point has landed on number t, then, provided $t < \frac{1}{2}$, the left piece is the shortest; otherwise, the shortest piece is on the right. Assuming the shortest piece is the left one, its maximum length is $\frac{1}{2}$ and the average is half that, i.e., $\frac{1}{4}$. The same number is found if $t > \frac{1}{2}$. Then the average length of the longer piece is $\frac{3}{4}$, and the ratio of the two is $1 : 3$. This is the ratio of the expected values. The expected value of the ratio is different.

Assume $t > \frac{1}{2}$. Then t is the length of the longer piece; $1 - t$ is the length of the shorter one. Since t is distributed uniformly on $\left[\frac{1}{2}, 1\right]$, the expected value of the ratio is

$$2\int_{1/2}^{1} \frac{1-t}{t}dt = 2(\ln t - t)\Big|_{1/2}^{1} = 2\ln 2 - 1 \approx 0.386.$$

The expected value of the ratio of the longest segment to the shortest one is infinite for it is equal to the divergent integral $2\displaystyle\int_{1/2}^{1} \frac{t}{1-t}dt$.

Solution 1 to Question 2

Let L be the expected value of the length of the first piece. Since, according to the symmetry principle (Appendix B), the lengths of all three segments have the same distribution, their lengths have the same expected value, namely, L. However, in any experiment, the lengths of the three pieces add up to 1. It follows that $3L = 1$ and $L = \frac{1}{3}$.

Solution 2 to Question 2

Joshua Bowman

Let the two points have coordinates x and y. Then the expected value of the middle segment is

$$\iint\limits_{[0,1]\times[0,1]} |x - y|dxdy = \frac{1}{3}.$$

By symmetry (Appendix B), the expected values of the other two segments are also $\frac{1}{3}$.

Solution to Question 3

As in question 1, the expected value of each of the pieces, due to the symmetry principle, is $\frac{1}{3}$. However, if we talk of the expected values of the shortest, middle and the longest segments, then all three numbers are different.

I will use geometric insight. Assume that the left point is a distance X from the left end and the right point, a distance Y, such that $X < Y$. We can think of the pair (X, Y) as being uniformly distributed in the unit square, with the condition $X < Y$ satisfied by the points above the main diagonal.

The task is to compare the three lengths: X, $Y - X$ and $1 - Y$. If X is the least one, then $X < Y - X$ and $X < 1 - Y$, i.e., $2X < Y$ and $X + Y < 1$. The area where the two conditions are satisfied is the dark triangle shown in Figure 5.14.

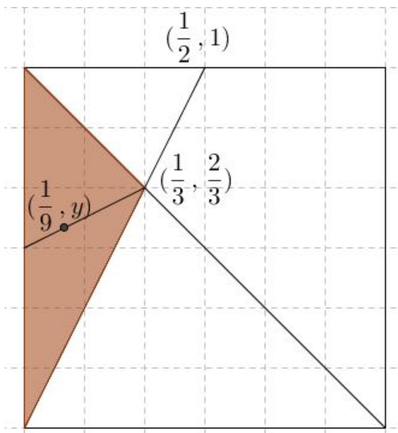


Figure 5.14: Random points on a segment 1.

The average value of X is attained at the center of gravity of the triangle, that is $\frac{1}{3}$ way from the base such that it is equal to $\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$.

Points (X, Y) that correspond to the case where X is the largest length lie in the quadrilateral marked in Figure 5.15.

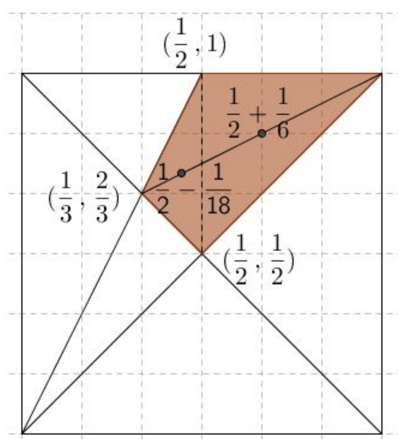


Figure 5.15: Random points on a segment 2.

To find the average value of X in the quadrilateral, it is split into two triangles. For the right triangle, the average value is $\frac{1}{2}+\frac{1}{6}$; for the left one, it is $\frac{1}{2}-\frac{1}{18}$. The areas of the triangles are in proportion $3:1$, so that the average value of X in the quadrilateral is the weighted sum:

$$\frac{3\cdot\left(\frac{1}{2}+\frac{1}{6}\right)+1\cdot\left(\frac{1}{2}-\frac{1}{18}\right)}{3+1}=\frac{3\cdot\frac{4}{6}+1\cdot\frac{8}{18}}{4}=\frac{11}{18}.$$

For the case where X is the middle length, we can shorten the computation by subtracting from the total of 1 the two fractions, $\frac{1}{9}=\frac{2}{18}$ and $\frac{11}{18}$, to obtain $\frac{5}{18}$. It follows that the expected lengths of the shortest to the middle to the longest segments are in proportion $2:5:11$.

Probability of First Digit in Product

Solution 1

We shall represent integers with d digits as $N=m\cdot 10^{d-1}$, with $1\le m<10$. Then for two integers, $N_1=m_1\cdot 10^{d-1}$ and $N_2=m_2\cdot 10^{d-1}$, $N_1N_2=m_1m_2\cdot 10^{2d-2}$, with $1\le m_1m_2<100$. The first digit of the product is n, provided $m_1m_2\in[n,n+1)\bigcup[10n,10(n+1))$.

The totality of the products m_1m_2 fills proportionally the bands defined by the hyperbolas $xy=n$ and $xy=10n$ in the interior of the square $(1,1)-(10,10)$. Thus, the sought limit probabilities (that I will denote P^n) are the areas of those bands relative to the area of the square, which is 81, Figure 5.16.

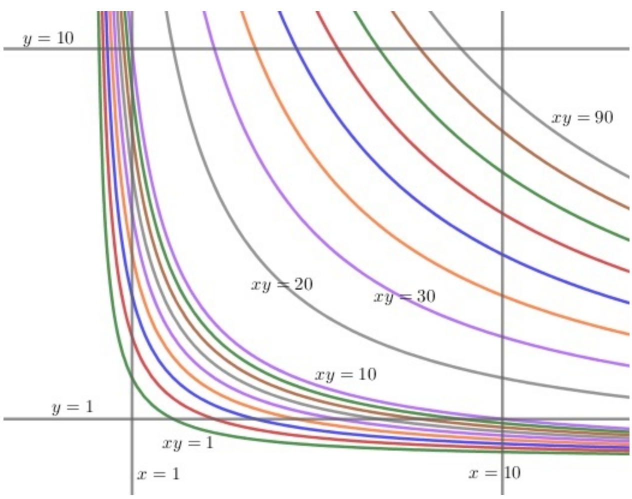


Figure 5.16: Probability of first digit in product.

For $n = 9$, the calculations differ somewhat from the rest of the digits:

$$\begin{aligned} 81P^9 &= \left[\int_1^9 \left(\frac{10}{x} - \frac{9}{x}\right) dx + \int_9^{10} \left(\frac{10}{x} - 1\right) dx\right] + \int_9^{10} \left(10 - \frac{90}{x}\right) dx \\ &= \int_1^9 \frac{1}{x} dx + \int_9^{10} \left[\left(\frac{10}{x} - 1\right) + \left(10 - \frac{90}{x}\right)\right] dx \\ &= \ln x\Big|_1^9 + (9x - 80\ln x)\Big|_9^{10} \\ &= 9 + 81\ln 9 - 80\ln 10. \end{aligned}$$

For $n \neq 9$, we will run the calculations piecemeal:

$$\begin{aligned} [\text{Area below } xy = 10n] = U(n) &= 9n + \int_n^{10} \frac{10n}{x} dx \\ &= 9n + 10n\ln x\Big|_n^{10} = 9n + 10n\ln\frac{10}{n}. \end{aligned}$$

$$\begin{aligned} [\text{Area below } xy = n] = L(n) &= \int_1^n \frac{n}{x} dx \\ &= n\ln x\Big|_1^n = n\ln n. \end{aligned}$$

Then

$$\begin{aligned} P^n &= [U(n+1) - U(n)] + [L(n+1) - L(n)] \\ &= 9n\ln n - (n+1)\ln(n+1) + 9 + 10\ln 10. \end{aligned}$$

Solution 2

Amit Itagi

If the d-digit number is written in scientific notation, the significand is uniformly distributed between 1 and 10 (as $d \to \infty$). For illustrative purposes, 2936 is 2.936×10^3 in scientific notation; 2.936 is the significand.

The mantissa of the $\log_{10}$ of the number or the $\log_{10}$ of the significand has a probability density function of $f(x) = C \cdot 10^x$ over $x \in [0, 1)$, where C is a normalization constant $\dfrac{1}{\int_0^1 10^x dx}$ that turns out to be $\dfrac{\ln(10)}{9}$. If m_1 and m_2 are the mantissas for the two numbers being multiplied, the mantissa of the product is $M = (m_1 + m_2) - \lfloor m_1 + m_2 \rfloor$. If $M \in [\log_{10} D, \log_{10}(D+1))$, then the first digit of the product is D.

Figure 5.17 shows the joint space of m_1 and m_2. The partitions of the shaded regions are at $m_1 + m_2$ values of $\log_{10} 2 - \log_{10} 10$ and $(1 + \log_{10} 2) - (1 + \log_{10} 19)$. The partitioned regions come in pairs where regions with the same color have the same value of M and thus the same first digit D. The red region has first digit 1, the blue region has the first digit 2 and so on until the last green region has first digit 9. Thus, the probability of having digit D as the first digit is

$$\begin{aligned}
P(D) = \Bigg\{ & \int_{m_2=0}^{\log_{10}(D+1)} \int_{m_1=0}^{\log_{10}(D+1)-m_2} - \int_{m_2=0}^{\log_{10} D} \int_{m_1=0}^{\log_{10}(D)-m_2} \\
& + \int_{m_2=\log_{10} D}^{1} \int_{m_1=1+\log_{10} D-m_2}^{1} \\
& - \int_{m_2=\log_{10}(D+1)}^{1} \int_{m_1=1+\log_{10}(D+1)-m_2}^{1} \Bigg\} f(m_1)f(m_2)dm_1dm_2 \\
= & \frac{1}{81}\left[(D+1)\ln(D+1) - D\ln(D) - 1\right] \\
& + \frac{10}{81}\left[D\ln(D) - (D+1)\ln(D+1) + 1 + \ln(10)\right] \\
= & \frac{1}{9}\left[D\ln(D) - (D+1)\ln(D+1) + 1\right] + \frac{10\ln(10)}{81}.
\end{aligned}$$

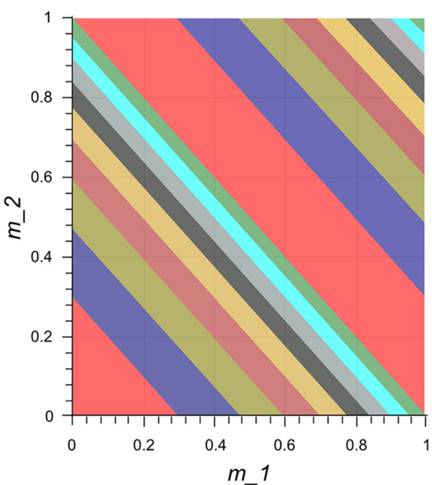


Figure 5.17: Probability of first digit in product, logarithmic scale.

The computed values are shown in the table:

Leading Digit	Probability
1	0.241348
2	0.183209
3	0.145454
4	0.117380
5	0.095007
6	0.076402
7	0.060474
8	0.046549
9	0.034178

Solution 3
N.N. Taleb

$x, y \approx$ UniformDistribution$[\{L, H\}]$ and it is independent. The product has for its probability density function $\varphi(.)$:

$$\varphi(z) = \begin{cases} \dfrac{\log\left(\dfrac{H^2}{z}\right)}{(H-L)^2} & H^2 > z \wedge HL < z \\ \dfrac{\log\left(\dfrac{z}{L^2}\right)}{(H-L)^2} & HL \geq z. \end{cases}$$

We have

$$\int_{L^2}^{H^2} \varphi(z) = 1 \text{ (verification for paranoid people)}$$

$$P_1 = \int_{9L^2}^{10L^2} \varphi(z)\, dz = -\frac{L^2(1 + 9\log(9) - 10\log(10))}{(H-L)^2}$$

$$P_2 = \int_{9/10(H+1)^2}^{H^2} \varphi(z)\, dz$$

$$= \frac{H^2 - 18H - 9(H+1)^2 \log(10) + 18(H+1)^2 \log\left(\dfrac{3}{H} + 3\right) - 9}{10(H-L)^2}.$$

We replace $L = 10^{d-1}$, $H = 10^d - 1$

$$P_d = (P_1 + P_2)|_d$$

$$= \frac{-2^{d+3}5^{d+2} + 100^d \left(180 \log\left(\dfrac{3}{10^d - 1} + 3\right) + 9 - 9\log(9) - 80\log(10)\right) + 100}{\left(10 - 9\ 10^d\right)^2}$$

$$\lim_{d\to\infty} P_d = \frac{1}{9} + \log(9) - \frac{80\log(10)}{81} \approx 0.0341776.$$

Illustrations
Marcos Carreira

I looked at z being the fractional part of $\log_{10}(x) + \log_{10}(y)$, then the frequency distribution of $\lfloor 10^z \rfloor$. The results were very close to the exact count for $d = 2$ (Figures 5.18–5.21).

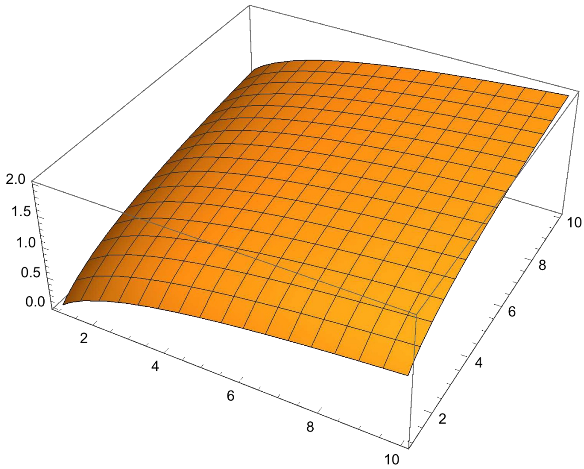


Figure 5.18: Sum of two logarithms 1.

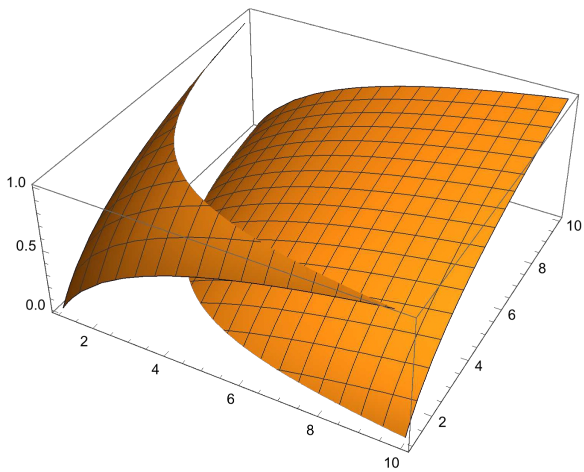


Figure 5.19: Sum of two logarithms 2.

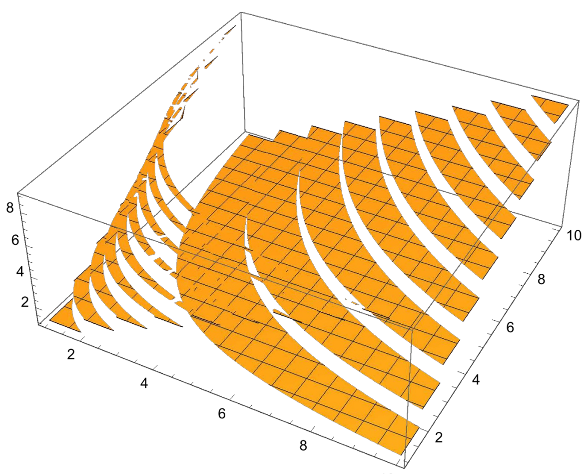


Figure 5.20: Sum of two logarithms 3.

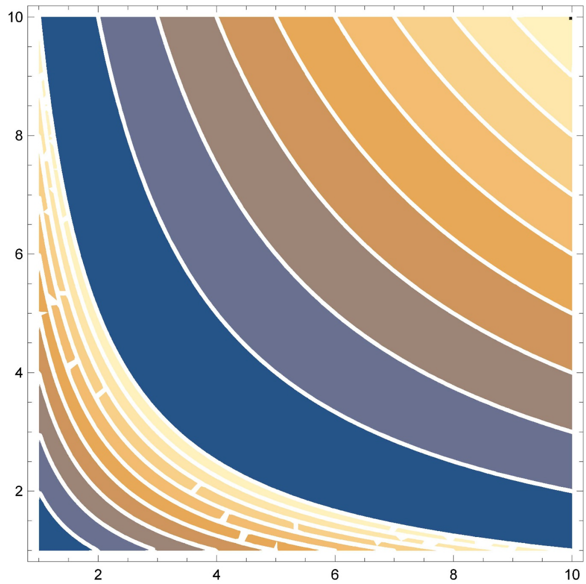


Figure 5.21: Sum of two logarithms: contour plot.

Birds on a Wire

Solution 1

Nathan Bowler

The following argument, although not rigorous, gives the right answer and shows why it is true (and I guess with a bit of effort you could make it rigorous). Consider a wire with lots of birds on it and scale it up so that there is one bird per unit length. Consider a bird at b. I shall assume that b is far enough away from the ends that they make no difference. For any interval, the number of birds in that interval has about a Poisson distribution with parameter x. In particular, the interval is empty with probability $1 - e^{-x}$, and the distance to the nearest bird to the right has an exponential distribution with parameter 1. The same may be said for the nearest bird to the left. So the distance to the nearest bird overall has an exponential distribution with parameter $1 + 1 = 2$. Without loss of generality [**WLOG**], the nearest bird at b' satisfies $b' - b = x > 0$. Then b is the nearest bird to b' with probability e^{-x}.

Give bird b a paintbrush, and let it paint half of the interval $(b, b + x)$ if it is the nearest bird to b' and the whole interval otherwise. A simple combinatorial argument shows that if each bird on the wire does this, then the wire will be painted as in the riddle. So we are interested in the expected length painted by bird b since this is (after scaling) the required value. However, the comments in the first paragraph allow us to evaluate it as a simple integral:

$$\begin{aligned} p &= \int_0^\infty x \cdot e^{-2x} \cdot \left(\frac{1}{2}(e^{-x} + (1 - e^{-x})\right) dx \\ &= \int_0^\infty x \cdot (2 * e^{-2x} - e^{-3x}) dx \\ &= \frac{2}{4} - \frac{1}{9} \\ &= \frac{7}{18} \end{aligned}$$

using

$$\int_0^\infty x e^{-kx} dx = \frac{1}{k^2}.$$

This value is 0.38888888..., which seems to fit with the results produced by the applet. I must admit, at first I thought the answer was going to be something nice like e^{-1}, but $\frac{7}{18}$ is what the algebra gives.

Solution 2

Mark Huber

Here is another way of thinking about it: suppose that for a bird at position a on the wire we assign a random variable L_a that is the length of the yellow line to the left of a. So if a is closest to its bird on the right and if the bird to the left of a is closer to its left bird, $L_a = 0$; otherwise it is some positive number.

What is the probability that L_a is in some tiny little interval around h? Or, in probability notation, that $P(L_a \in dh) =?$

There are two ways L_a can be close to h. One case is when there is a bird at $a - h$, no bird in the interval $(a - h, a)$ and no bird in $(a, a + h)$. This occurs with probability $(1 - 2h)^{n-2} n dh$.

Another case is when there is a bird at $a - h$, no bird from $(a - 2h, a)$ and at least one bird in $(a, a + h)$. This occurs with probability $((1 - 2h)^{n-2} - (1 - 3h)^{n-2}) n dh$.

To find the expected value of L_a, this probability has to be multiplied by h, then integrated for h from 0 to $\frac{1}{2}$. This gives an expected value of about $\frac{7}{18} \cdot \frac{1}{n}$. Since there are n birds, the total yellow line is expected to be $\frac{7}{18}$.

A variant of the strong law of large numbers then completes the proof, showing that as the number of birds goes to infinity, the amount of line colored yellow is $\frac{7}{18}$.

Solution 3

Moshe Eliner

Let us pass through all birds, starting, say from the left, stopping at every fourth bird. I will call this segment $4b$. Notice a delicate point:

Denote the shortest connecting segments sc (i.e., yellow lines). Each four birds has three inner connecting segments that average 2.

1. The states of $sc = 1$ and $sc = 3$ force inner point positions on the neighboring (fourth) bird.
2. On a uniform wire, birds tend to have uniform distribution, hence inner point positions cannot be predefined.
3. There cannot be $sc = 1$ or three more than average. This implies that each four birds have on average $sc = 2$.

Now it is enough to calculate the density when $sc = 2$:

There are three possible $sc = 2$, but, since one can rearrange segments inside the four-bird groups, all have the same density. Let us choose one of them, e.g., the one with the two shortest connecting segments.

Integrate $ydxdy$:

1. y from $y = \frac{1}{2}$ to $\frac{2}{3}$, x from $2y - 1$ to $1 - y$
2. y from 0 to $\frac{1}{2}$, x from 0 to y.

This yields $\frac{7}{108}$, but, since the total integration area is $\frac{1}{6}$, the resulting density is $\frac{7}{18}$.

Solution 4

Stuart Anderson

Looking at these solutions, I was particularly struck by solution 3, which did not involve the notion of an exponential distribution at all. I found it somewhat hard to follow, so I started looking at the assumptions behind the other solutions to see how you could solve the riddle without the exponential distribution.

Here is the surprising result: part of the solution REALLY IS independent of the distribution, although the final answer does depend on the distribution.

Let L be the length of an interval between adjacent birds. Suppose that the $P(L < x) = F(x)$ for an arbitrary cumulative probability function F. Then the density function $f(x) = F'(x)$ gives the probability density for the interval to have length x.

A given interval will be painted twice if it is "small," i.e., shorter than both its neighbors; once if "medium," i.e., shorter than one and longer than the other neighbor and not painted if "big," i.e., longer than both neighbors. What is the probability of each case and its expected length?

For the interval that is longer than its neighbors, the probability is

$$\int_0^{\infty} (F(x) \cdot F(x) \cdot f(x))dx = \int_0^1 (F^2)dF = \frac{1}{3}.$$

Similar calculations for the other two cases, using $1 - F(x)$ in place of $F(x)$ when we want the neighboring interval to be longer instead of shorter, give the same result. This is rather surprising in itself: regardless of the distribution, the probabilities $P(\text{big})$, $P(\text{medium})$ and $P(\text{small})$ are all exactly $\frac{1}{3}$.

The expected length of an interval that is longer than both neighbors is

$$\frac{\int_0^\infty (x \cdot F(x) \cdot F(x) \cdot f(x))dx}{\left(\frac{1}{3}\right)},$$

which we integrate by parts (very carefully!): let $u = x$, and $dv = (F(x))^2 \cdot f(x)dx = \frac{d((F(x))^3 - 1)}{3}$. (We can choose any antiderivative we like when integrating by parts, and this one allows the $u \cdot v$ term to vanish at ∞.) Then we get

$$(x \cdot (F^3 - 1))\big|_0^\infty - \int_0^\infty (F^3 - 1)dx = \int_0^\infty (1 - F^3)dx$$

if $F(x)$ approaches 1 sufficiently quickly for the leftmost term to vanish and the integral to converge. Let us call this number B (for "big" interval).

A similar but simpler calculation shows that the expected length of an arbitrary interval (no conditions as to its size relative to its neighbors) gives $\int_0^\infty (1 - F)dx$. Let us call this number A (for "average" interval).

Since all but the big intervals are painted, the expected fraction painted is $\frac{1 - B \cdot P(\text{big})}{A} = 1 - \frac{B}{3A}$. When $F(x) = 1 - e^{-kx}$ is the exponential distribution,

$$B = \int_0^\infty (3e^{-kx} - 3e^{-2kx} + e^{-3kx})dx = \frac{3 - \frac{3}{2} + \frac{1}{3}}{k} = \frac{11}{6k},$$

while $A = \int_0^\infty (e^{-kx})dx = \frac{1}{k}$. Therefore, the expected fraction painted is

$$1 - \frac{\frac{11}{6k}}{\frac{3}{k}} = 1 - \frac{11}{18} = \frac{7}{18}.$$

A more detailed calculation shows that $\frac{2}{18}$ of the length is painted twice (small intervals), $\frac{5}{18}$ is painted once (medium intervals) and $\frac{11}{18}$ is unpainted (big intervals). Since these types of intervals occur with equal probability, this also shows that the ratio of mean sizes of small, medium and big intervals is $2 : 5 : 11$. Of course, if we took $\frac{1}{3}$ of the intervals completely at random, we would expect them to cover $\frac{1}{3} = \frac{6}{18}$ of the wire, so we can extend the ratio to include the overall average length: $S : M : B : A = 2 : 5 : 11 : 6$. Therefore, medium intervals are slightly shorter than the average length.

Solution 5

Bogdan Lataianu

Let $U_{(i)}$ be the order statistics of uniform distribution, $i = 1 \dots n-1$ and $v_r = u_i - u_{i-1}$ with $u_0 = 0$. Then the probability density function of $v_1, \dots, v_r$ for $\sum_{i=1}^r v_i \leq 1$, as shown in [30], is

$$f(v_1, \dots, v_r) = \frac{(n-1)!}{(n-r-1)!}(1 - v_1 - \dots v_r)^{n-r-1}, r = 1 \dots n-1.$$

When $r = 3$, $f(v_1, v_2, v_3) = (n-1)(n-2)(n-3)(1-v_1-v_2-v_3)^{n-4}$. Let Z_i be the painted length of the i^{th} interval after a realization of the order statistics and denote $X = V_{i-1}$, $Y = V_i$, $Z = V_{i+1}$, A the event $V_i < V_{i-1}$ or $V_i < V_{i+1}$. Then $Z_i = 1_A Y$ and

$$E(Z_i) = \int 1_A y f(x,y,z) dxdydz = \int_A y f(x,y,z) dxdydz.$$

We have

$$P(A) = P(Y < X) + P(Y < Z) - P(Y < X \text{ and } Y < Z).$$

Therefore,

$$\begin{aligned} E(Z_i) = (n-1)(n-2)(n-3)\Bigg(&\int_0^{1/2} \int_y^{1-y} \int_0^{1-x-y} y(1-x-y-z)^{n-4} dzdxdy \\ &+ \int_0^{1/2} \int_0^{1-2y} \int_y^{1-x-y} y(1-x-y-z)^{n-4} dzdxdy \\ &- \int_0^{1/3} \int_y^{1-2y} \int_y^{1-x-y} y(1-x-y-z)^{n-4} dzdxdy \Bigg) \\ = \frac{1}{4n} + \frac{1}{4n} - \frac{1}{9n} = \frac{7}{18n}. \end{aligned}$$

In our case, suppose we have n birds. Then $E(\text{painted length}) = E(\text{two intervals never painted}) + E(\text{two intervals always painted}) + (n-3)E(Z_i) = \dfrac{2}{n+1} + (n-3)\dfrac{7}{18n}$ with the limit $\dfrac{7}{18}$.

Lucky Times at a Moscow Math Olympiad

Solution to Question 1

For a moment defined as $h : m : s$, consider the moment $11 - h : 59 - m : 60 - s$. The two constitute reflections of each other across the vertical diameter (the line 12–6), see Figure 5.22.

For example, $7 : 42 : 12$ reflects into $4 : 17 : 48$. The latter is lucky; the former is not.

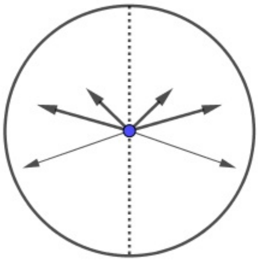


Figure 5.22: Lucky times at a Moscow math olympiad 1.

In general, a legitimate position of the three hands reflects into another legitimate position. However, one is lucky; the other is not. It follows that there is a 1-1 correspondence between lucky and unlucky moments. More than that, two discrete successive lucky moments, like say $h : m : s$ and $h : m : s+1$, correspond to two successive unlucky moments $11-h : 59-m : 60-s-1$ and $11-h : 59-m : 60-s$. Thus, the two define one-second intervals corresponding to each other, though one is entirely lucky and the other is entirely unlucky. The argument could be expanded to intervals longer than one second. In general, the reflection of a lucky interval is an unlucky interval of the same length, making the two events equiprobable: the probability of each is $\frac{1}{2}$.

Solution 1 to Question 2

A. Bogomolny

Consider any two hands, e.g., the hour and the minute hands, and the clock face sector formed by their extensions across the center of the face as in Figure 5.23.

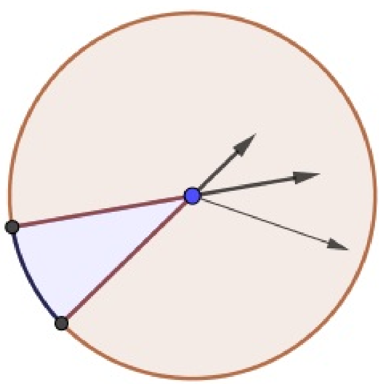


Figure 5.23: Lucky times at a Moscow math olympiad 2.

Whenever the second hand falls into that sector, the moment is unlucky; a moment is lucky if the second hand falls outside the sector.

We can simulate the situation with two random points on the circle. Two points split the circle into two arcs and it appears that we are interested in their relative lengths. The unlucky sector is never bigger than 180°.

Cut the circle at one of the points. The result is a segment with a random point inside. The point divides the segment into two pieces. In the solution of Riddle 5.2 on page 79, we found that the average length of the longer piece is $\frac{3}{4}$ of the whole segment while that of the shorter piece is $\frac{1}{3}$.

Back to the original riddle, the lucky moments of a day come up with the probability of $\frac{3}{4}$, three times more often than the unlucky moments.

Note that an objection may be raised as to the reasoning behind dropping of two points. If, as above we associate the two points with the positions of the hour and minute hands, we should admit that not all positions correspond to legitimate time moments. However, for any such position, the clock face can be rotated to make the hands show a real time. This leads to the idea of reducing the riddle to that on a segment.

Solution 2 to Question 2

Thamizh Kudimagan

For angle θ between the hour and minute hands, the moment is unlucky only if the second hand is in the diametrically opposite sector of angle θ, so it is lucky over $2\pi - \theta$ out of 2π, and

$$P(\text{lucky}) = \int_0^{\pi} \frac{2\pi - \theta}{2\pi} \frac{d(\theta)}{\pi} = \frac{3}{4}.$$

Solution 3 to Question 2

Thamizh Kudimagan

In fact, the integration does not even seem necessary, based on the argument of Solution 2.

- When hour and minute hands are closest, $P(\text{lucky}) = 1$.
- When hour and minute hands are farthest, $P(\text{lucky}) = \frac{1}{2}$.

So, $P(\text{lucky})$ for all hour and minute hands positions is $\frac{1}{2}\left(1 + \frac{1}{2}\right) = \frac{3}{4}$.

Probability of a Random Inequality

Solution 1

Substituting $y = 2a - x$ into the inequality gives

$$b^2x(2a - x) > (b^2 - 1)a^2, \text{ or,}$$

$$b^2x^2 - 2ab^2x + (b^2 - 1)a^2 < 0, \text{ or,}$$

$$(bx - (b - 1)a)(bx - (b + 1)a) < 0, \text{ or,}$$

$$\frac{(b - 1)a}{b} < x < \frac{(b + 1)a}{b}.$$

The length of the interval where the given inequality holds is $\dfrac{(b+1)a}{b} - \dfrac{(b-1)a}{b} = \dfrac{2a}{b}$. The sought probability is the ratio of that length to the length of the interval of the distribution of x, i.e., $2a$. Thus, we have $\dfrac{2a}{b} \div 2a = \dfrac{1}{b}$.

Solution 2

N.N. Taleb

We have the transformation $x \sim U[0, 2a]$. The equation $z = xy = (2a - x)x$ has probability density function:

$$\varphi(z) = \frac{1}{2a\sqrt{a^2 - z}}, \qquad 0 < z \le a^2.$$

It also has the survival function:

$$\mathbb{P}(Z > k) = \int_k^{a^2} \frac{1}{2a\sqrt{a^2 - z}} \qquad dz = \frac{\sqrt{a^2 - k}}{a}.$$

Replacing k with $\dfrac{a^2\left(b^2 - 1\right)}{b^2}$, we get $\dfrac{1}{b}$.

Points on a Square Grid

Solution 1

Given two grid points, their x coordinates have the same parity with the probability of $\dfrac{1}{2}$. It is the same for the y coordinates. Both coordinates have (pairwise) the same parity with the probability of $\dfrac{1}{2} \cdot \dfrac{1}{2} = \dfrac{1}{4}$.

For $n = 3$, for any combination of the coordinates of one point, there are six combinations of parities for the other two that do not lead to a midpoint at a grid node. For example, if one point has both coordinates even, denoted e/e, then for the other two there are the following possibilities:

$$\begin{cases} e/o, & o/e, o/o \\ o/e, & o/o, o/e \\ o/o, & o/e, e/o. \end{cases}$$

Since there are four combinations of parities for the chosen point, there are $4 \cdot 6 = 24$ possible combinations that do not place a midpoint at a grid node. For three points, there are 4^3 possible combinations so that the probability of getting a midpoint on a grid node equals $1 - \dfrac{24}{64} = \dfrac{5}{8}$.

For $n = 4$, there is no midpoint on a grid node if and only if the four points have all four distinct parity combinations: o/o, o/e, e/o, e/e. There are $4!$ ways to distribute these combinations between the four points, out of the total of $4^4 = 256$ combinations. This happens with the probability $\dfrac{24}{256} = \dfrac{3}{32}$. Thus, at least one of the midpoints falls on a grid node with the probability of $1 - \dfrac{3}{32} = \dfrac{29}{32}$.

For $n = 5$, it is shown elsewhere with the pigeonhole principle that, in this case, at least one of the midpoints falls onto a grid node. On the other hand, since there are only four possible parity combinations for a 2D grid point, if there are five points, the probability is 1 that two of them have the same combination of parities. This may be considered a *probabilistic* solution to an otherwise deterministic problem.

Solution 2

dash_amitabh on Twitter

Each point has four equally likely parity combinations (odd/even on both coordinates). No midpoint will be a grid node when no two points have the same parity, for which the probability is

$$\begin{cases} n = 2, & \left(\frac{1}{4}\right)^2 \cdot 4 \cdot 3 = \frac{3}{4} \\ n = 3, & \left(\frac{1}{4}\right)^3 \cdot 4 \cdot 3 \cdot 2 = \frac{3}{8} \\ n = 4, & \left(\frac{1}{4}\right)^4 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = \frac{3}{32} \\ n = 5, & \left(\frac{1}{4}\right)^5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 0 = 0 \ \ \text{(unauthorized addition).} \end{cases}$$

The probability that a midpoint will fall at the node of the grid is $1 - \frac{3}{4} = \frac{1}{4}$, $1 - \frac{3}{8} = \frac{5}{8}$, $1 - \frac{3}{32} = \frac{5}{32}$, $1 = 1 - 0$, for $n = 2, 3, 4, 5$, respectively.

Solution 3

Amit Itagi

The nodes can be considered to have integral x and y coordinates. For midpoints to have integral coordinates (with probability denoted by P), the coordinates of both points should have same parity—even or odd (a requirement for both x and y coordinates). For each point, there are four possibilities (four bins) with equal probability: (e, e), (e, o), (o, e) and (o, o).

For $n > 4$, the pigeonhole principle guarantees that there will be at least two points in the same bin and $P(n : n > 4) = 1$.

For the remaining cases, P is one minus the probability of assigning distinct bins (assign the first point to a random bin, the second point to one of the remaining three bins, the third point to one of the remaining two bins and so on). Thus,

$$P(n) = \begin{cases} 1 - \prod_{k=1}^{n-1} \frac{(4-k)}{4} & (n \leq 4), \\ 1 & (n > 4). \end{cases}$$

Hence,

$$
\begin{aligned}
P(2) &= 1 - \frac{3}{4} = \frac{1}{4},\\
P(3) &= 1 - \frac{3}{4}\cdot\frac{2}{4} = \frac{5}{8},\\
P(4) &= 1 - \frac{3}{4}\cdot\frac{2}{4}\cdot\frac{1}{4} = \frac{29}{32},\\
P(5) &= 1.
\end{aligned}
$$

A Triangle out of Three Broken Sticks

Solution 1 to Question 1

A. Bogomolny

Assume the sticks are of length 1 and consider the cube with vertices A$(0,0,0)$, B$(1,0,0)$, D$(0,1,0)$, E$(0,0,1)$, C$(1,1,0)$, F$(1,0,1)$, H$(0,1,1)$ and G$(1,1,1)$ as shown in Figure 5.24.

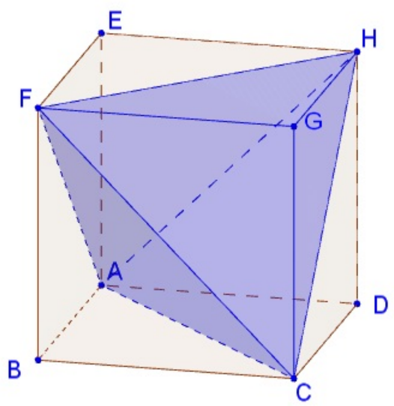


Figure 5.24: A triangle out of three broken sticks.

The left pieces of the sticks are defined by their right points and I shall describe their lengths with the real $x, y, z \in (0,1)$. Thus, the cube constitutes the sample space for all possible combinations of the breaking points. For x, y, z to form a triangle, we need three inequalities:

$$
\begin{aligned}
&z < x + y\\
&y < x + z\\
&x < y + z.
\end{aligned}
$$

Note that, say, line AF has the equation $z = x$ and line AH has the equation $z = y$ so that the plane through $\mathrm{A, F, H}$ is described by $z = x + y$. The plane divides the space into two half-spaces. In the one that contains G, $x + y > z$.

Similarly, the plane AFC is described by $x = y + z$ and the plane ACH by $y = x + z$. The set of points that satisfies the three inequalities belongs to the intersection of the three half-spaces and the cube: that is, the figure V = ACFHG. It consists of two pyramids with the base CFH and apices at A and G.

V is obtained from the cube by cutting corner pyramids ACFB, ACHD and AFHE. Each of these has volume $\frac{1}{3} \cdot \frac{1}{2} \cdot 1 = \frac{1}{6}$. It follows that the volume of V is $1 - 3 \cdot \frac{1}{6} = \frac{1}{2}$.

Solution 2 to Question 1

Amit Itagi

Without loss of generality, let the length of the sticks be unity. Let the length of the three pieces (sorted) be $x \geq y \geq z$. Let the regions in the X, Y, Z space (and the projections of the space) where the sorting holds and where forming a triangle is possible be termed "feasible region" and "positive region," respectively.

Figure 5.25 shows the feasible region in the YZ plane in white with the conditions on x. The conditions in black and blue are for the positive and feasible regions, respectively.

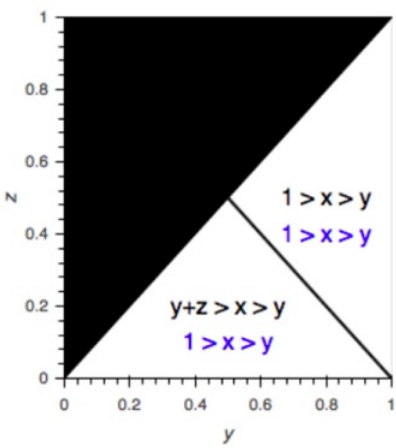


Figure 5.25: Left pieces.

The probability of triangulation is given by

$$\begin{aligned} P &= \frac{\displaystyle\int_{z=0}^{1/2}\int_{y=z}^{1-z}\int_{x=y}^{y+z} dxdydz + \int_{y=1/2}^{1}\int_{z=1-y}^{y}\int_{x=y}^{1} dxdzdy}{\displaystyle\int_{z=0}^{1/2}\int_{y=z}^{1-z}\int_{x=y}^{1} dxdydz + \int_{y=1/2}^{1}\int_{z=1-y}^{y}\int_{x=y}^{1} dxdzdy} \\ &= \frac{\frac{1}{24} + \frac{1}{24}}{\frac{1}{8} + \frac{1}{24}} \\ &= \frac{1}{2}. \end{aligned}$$

Solution 1 to Question 2
A. Bogomolny

If now x, y, z stand for the lengths of the longest pieces, then $\frac{1}{2} < x, y, z < 1$. Then, for example, $x + y > \frac{1}{2} + \frac{1}{2} = 1 > z$, and the other two inequalities are similar. We conclude that the probability in this case is 1.

Solution 2 to Question 2
Amit Itagi

Figure 5.26 corresponds to this case. Clearly, the probability is 1.

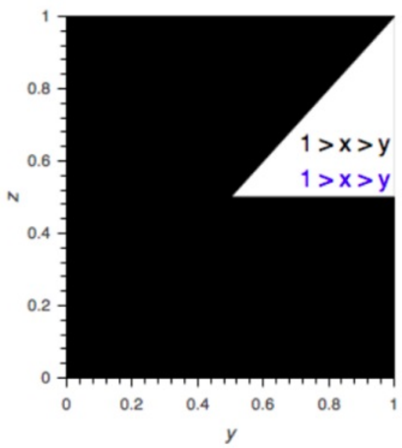


Figure 5.26: Longer pieces.

Solution 1 to Question 3
A. Bogomolny

If now x, y, z stand for the lengths of the shortest pieces, then $0 < x, y, z < \frac{1}{2}$ and each of the three is drawn uniformly randomly from the interval $\left(0, \frac{1}{2}\right)$. The situation is similar to question 1 but in a smaller cube. However, since all that matters is the relative lengths of the sides, question 1 applies directly. The probability in this case is $\frac{1}{2}$.

Solution 2 to Question 3
Amit Itagi

This case is the same as the case of "left pieces" except that x, y and z are drawn from $(0, 0.5]$ instead of $(0, 1.0)$. The probability does not change by this scaling and is, therefore, $\frac{1}{2}$.

Solution 1 to Question 4

A. Bogomolny

First, I should mention quite a few people observed that it does not matter whether we choose left or right pieces in question 1 because both side lengths of a broken stick have the same probability distribution. This implies that when a random choice is made between the left and right pieces for each stick, the probability of getting a triangle is no different than in question 1, i.e., $\frac{1}{2}$.

It occurred to me that there might be a difference with the case where the choice is made between the long and the short pieces, rather than between the left and the right ones. Many people insisted there should not be any difference. I made the mistake of thinking that there should be no difference since $\min(x, 1-x)$ is distributed on $\left[0, \frac{1}{2}\right]$ while $\max(x, 1-x)$ is distributed on $\left[\frac{1}{2}, 1\right]$. Tangentially, support also came from the difference in probabilities in questions 2 and 3. That was explained and corrected by Timon Klugge.

Where the three numbers x, y, z satisfy $0 \le x, y, z \le \frac{1}{2}$, the broken sticks form a triangle. Let $\triangle(x, y, z) = 1$ if x, y, z form a triangle and 0 otherwise. We could denote the probability we found as

$$P\left(\triangle(x, y, z) = 1 \,|\, 0 \le x, y, z \le \frac{1}{2}\right) = \frac{1}{2}.$$

Now, if $\triangle(x, y, z) = 1$ then also $\triangle(2x, 2y, 2z) = 1$, such that we were able to conclude that in the domain of definition, $[0, 1]^3$, $\triangle(x, y, z) = 1$ half the time.

Now, there is a harder road to arrive at the same conclusion. Assuming $0 \le x, y, z \le \frac{1}{2}$, what is the probability of $P(\triangle(x, y, z) = 1)$? In other words, what is $P\left(\triangle(x, y, z) = 1 \text{ and } 0 \le x, y, z \le \frac{1}{2}\right)$? The formula for conditional probability gives

$$\begin{aligned}
&P\left(\triangle(x, y, z) = 1 \text{ and } 0 \le x, y, z \le \frac{1}{2}\right) \\
&\quad = P\left(\triangle(x, y, z) = 1) \,|\, 0 \le x, y, z \le \frac{1}{2}\right) \cdot P\left(0 \le x, y, z \le \frac{1}{2}\right) \\
&\quad = \frac{1}{2} \cdot \frac{1}{8} = \frac{1}{16}.
\end{aligned}$$

Similarly, with the result of question 2, we conclude that

$$\begin{aligned}
&P\left(\triangle(x, y, z) = 1 \text{ and } \frac{1}{2} \le x, y, z \le 1\right) \\
&\quad = P\left(\triangle(x, y, z) = 1) \,|\, \frac{1}{2} \le x, y, z \le 1\right) \cdot P\left(\frac{1}{2} \le x, y, z \le 1\right) \\
&\quad = 1 \cdot \frac{1}{8} = \frac{1}{8}.
\end{aligned}$$

Now we have to consider six additional cases that come in two groups of three. If S and L stand for short and long, respectively, and we define $S_x = \min(x, 1-x)$ and $L_x = \max(x, 1-x)$, and similarly for y and z, then there are eight combinations SSS, $SSL, \ldots, LLL$. The first, SSS, and the last, LLL, have been just treated.

We need to find $P(\triangle(x,y,z) = 1 \,|\, SSL)$ and then also $P(\triangle(x,y,z) = 1 \,|\, SLL)$.

SSL

We have three variables x, y, z that satisfy

$$x < \frac{1}{2}, \quad y < \frac{1}{2}, \quad z > \frac{1}{2}.$$

In addition, the variables need to satisfy $x + y > z$. Points (x, y, z) that satisfy the four conditions are located inside the corner pyramid in a small cube as shown in Figure 5.27.

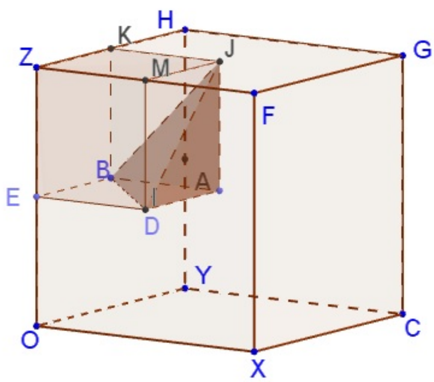


Figure 5.27: Splitting probabilities for two short and one long pieces.

The volume of this pyramid, relative to the larger cube, is $\frac{1}{8} \cdot \frac{1}{6} = \frac{1}{48}$. Cycling through the three variables gives $3 \cdot \frac{1}{48} = \frac{1}{16}$.

SLL

In this case we need to satisfy the following inequalities:

$$x < \frac{1}{2}, \quad y > \frac{1}{2}, \quad z > \frac{1}{2}, \quad x + y > z, \quad x + z > y.$$

The points (x, y, z) that satisfy all five inequalities lie outside two pyramids in the small cube shown in Figure 5.28.

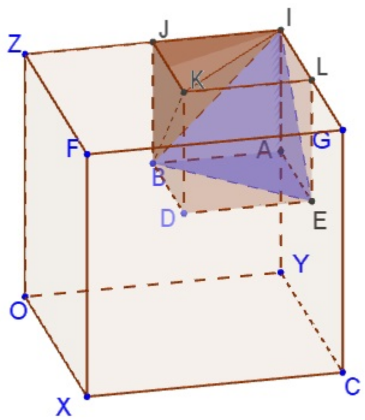


Figure 5.28: Splitting probabilities for one short and two long pieces.

The relative volume of that space is $\frac{1}{8}\left(1 - 2 \cdot \frac{1}{6}\right) = \frac{1}{12}$. Cycling through the three variables gives $3 \cdot \frac{1}{12} = \frac{1}{4}$. Putting everything together, denote $\mathcal{S} = \{SSS, SSL, \ldots, SSS\}$, then, the formula of total probability is

$$\begin{aligned}
P(\triangle(x,y,z) = 1) &= \sum_{\omega \in \mathcal{S}} P(\Delta)x, y, z) = 1 \,|\, \omega) \\
&= \frac{1}{16} + \frac{1}{16} + \frac{1}{4} + \frac{1}{8} \\
&= \frac{1}{2}.
\end{aligned}$$

As Long Huynh Huu has observed, that was clear to start with.

Solution 2 to Question 4
Amit Itagi

For the case of "left pieces," the length of the stick has a uniform distribution of $(0, 1)$. Choosing one side or the other does not change the distribution for the chosen stick as both sides are identically distributed. Thus, the probability for this case remains $\frac{1}{2}$.

Solution 1 to Question 5
A. Bogomolny

The expectation that $x, y, z \in (0, 1)$ form side lengths of a triangle is $E_{x,y,z} = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2}$.

The same is true if we replace x with $1 - x$, y with $1 - y$ or z with $1 - z$ so that, for example, $E_{1-x,y,z} = \frac{1}{2}$. There are eight such combinations. The eight events are not independent; however, the individual expectations can be summed up, which gives the total expectation of $8 \cdot \frac{1}{2} = 4$.

Solution 2 to Question 5

Amit Itagi

There are eight combinations depending on which side is chosen for each stick. For each combination, the chosen side of a stick can be arbitrarily labeled "left." This approach maps every combination to the "left pieces" case, so the probability of forming a triangle is $\frac{1}{2}$. Thus, the expected number of triangles is four.

Probability in Dart Throwing

Solution 1

The region where points are closer to the center of the square than to an edge is the intersection of the interiors of four parabolas, as shown in Figure 5.29.

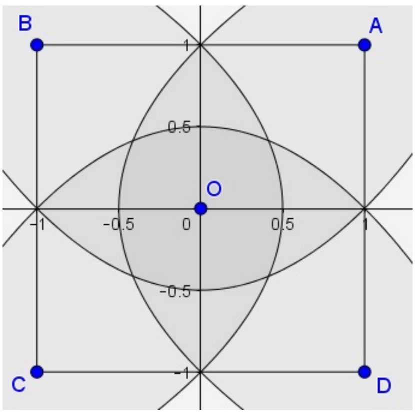


Figure 5.29: Dart throwing: the whole picture.

The probability sought is the ratio of the area of that region to the area of the square whose side was chosen to be 2.

The region can be seen to consist of eight congruent pieces, as in Figure 5.30.

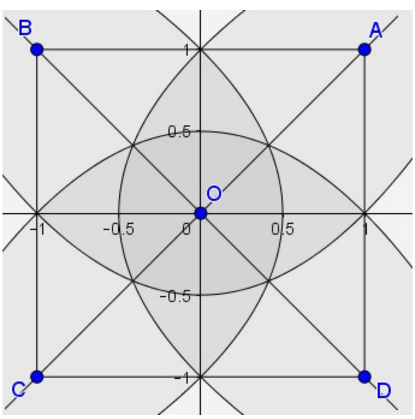


Figure 5.30: Dart throwing: the whole picture. Showing symmetry.

Thus, suffice it to calculate the area of one octant (Figure 5.31). Note that the square also consists of eight parts, each housing one part of the region, so we need only compare the two areas of the paraboloa and the triangle.

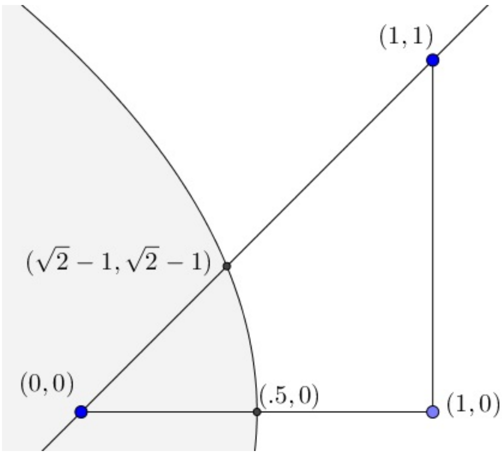


Figure 5.31: Dart throwing: one octant.

The parabola is defined by the equation $1 - x = \sqrt{x^2 + y^2}$, or $x = \dfrac{1-y^2}{2}$. Its intersection with the diagonal $y = x$ is found from $x^2+2x-1 = 0$ to be $(\sqrt{2}-1, \sqrt{2}-1)$.

It is easier to integrate by y than by x. The area of the shaded region within the triangle is

$$\begin{aligned}
\int_0^{\sqrt{2}-1} \left(\frac{1-y^2}{2} - y\right) dy &= \left[\frac{y}{2} - \frac{y^2}{2} - \frac{y^3}{6}\right]_0^{\sqrt{2}-1} \\
&= \frac{\sqrt{2}-1}{2} - \frac{(\sqrt{2}-1)^2}{2} - \frac{(\sqrt{2}-1)^3}{6} \\
&= \frac{4\sqrt{2}-5}{6}.
\end{aligned}$$

The area of the triangle is $\dfrac{1}{2} \cdot 1 \cdot 1 = \dfrac{1}{2}$ and the ratio of the two is $\dfrac{4\sqrt{2}-5}{3} \approx 0.21895$.

Solution 2

Amit Itagi

Using symmetry, the riddle can be analyzed in the first octant. Let the side of the square be 2. The curve that divides the region into points closer to either vertex of the vertical edge is the parabola with focus at the vertex and directrix as the vertical edge (Figure 5.32).

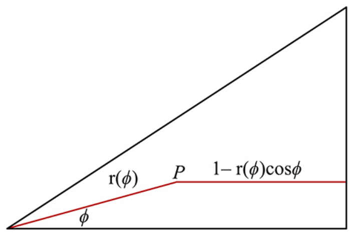


Figure 5.32: Dart throwing in polar coordinates.

Let a point P in the octant have polar coordinates (r, ϕ). Its distance from the vertex is r and that from the edge is $r(1-\cos\phi)$. Thus, the equation of the parabola is

$$r(\phi) = \frac{1}{1+\cos\phi}.$$

Hence, the required probability is

$$P = \frac{\text{area on the vertex side of the parabola}}{\text{area of the triangle}} = 2\int_0^{\pi/4} \frac{1}{2} r^2(\phi)d\phi$$

$$= \int_0^{\pi/4} \frac{d\phi}{(1+\cos\phi)^2} = \left.\frac{\sin\phi(2+\cos\phi)}{3(1+\cos\phi)^2}\right|_0^{\pi/4} = \frac{(4\sqrt{2}-5)}{3} \sim 0.219.$$

Notes

From the last formula,

$$\lim_{n\to\infty} \frac{1}{3}\left(1 - \frac{1}{\left(1+\cos\left(\frac{\pi}{n}\right)\right)^2}\right) = \frac{1}{3}\cdot\frac{1}{4} = \frac{1}{4}.$$

Interestingly, this limit could be found with additional calculations. As n grows, n-gon approaches a circle and, due to symmetry, the same happens with the intersection of the parabolas. The radius of the former is twice the radius of the latter and their areas are in the ratio $4:1$ (Figure 5.33).

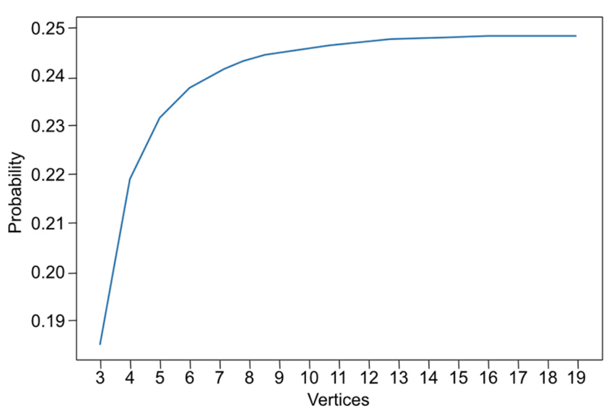


Figure 5.33: Dart throwing in polar coordinates, n-gon board.

Solution 3

Derek Boeckner

We generalize the question to n-gons. The region of points that are closer to the center than to the boundary of the n-gon is the intersection of the interiors of n parabolas with a $2n$-fold symmetry. Thus, we need only to focus on one of $2n$ wedges.

Let the center of a regular n-gon be at the origin. Take an edge as the line $x = 1$. Mixing in polar coordinates, cut the line off at $\theta = 0$ and at $\theta = \frac{\pi}{n}$. The boundary of the bullseye/target where the center is nearer than the edge is given by a parabolic arc with focus at 0 and directrix at $x = 1$. The equation for this is $r = \dfrac{1}{1+\cos(\theta)}$. We only look at the right triangular wedge between $\theta = 0$ and $\theta = \frac{\pi}{n}$ and then exploit symmetry.

We want the probability of being closer to the origin than this boundary, so we use polar integration:

$$\begin{aligned}
\frac{1}{2}\int_0^{\pi/n}\left(\frac{1}{1+\cos\theta}\right)^2 d\theta &= \frac{1}{8}\int_0^{\pi/n}\sec^4\frac{\theta}{2}d\theta \\
&= \frac{1}{16}\left(\frac{4}{3}\tan^3\left(\frac{\pi}{2n}\right)+4\tan\left(\frac{\pi}{2n}\right)\right) \\
&= \frac{1}{12}\cdot\tan\left(\frac{\pi}{2n}\right)\left(\sec^2\left(\frac{\pi}{2n}\right)+2\right) \\
&= \frac{1}{12}\left(\frac{\sin\left(\frac{\pi}{n}\right)}{1+\cos\left(\frac{\pi}{n}\right)}\right)\left(\frac{2}{1+\cos\left(\frac{\pi}{n}\right)}+2\right) \\
&= \frac{1}{6}\sin\left(\frac{\pi}{n}\right)\left(\frac{2+\cos\left(\frac{\pi}{n}\right)}{\left(1+\cos\left(\frac{\pi}{n}\right)\right)^2}\right).
\end{aligned}$$

Since the area of the triangle is $\dfrac{\tan\left(\frac{\pi}{n}\right)}{2}$ we have a probability of

$$\begin{aligned}
\frac{1}{3}\cos\left(\frac{\pi}{n}\right)\left(\frac{2+\cos\left(\frac{\pi}{n}\right)}{\left(1+\cos\left(\frac{\pi}{n}\right)\right)^2}\right) &= \frac{1}{3}\left(\frac{(1+2\cos\left(\frac{\pi}{n}\right)+\cos^2\left(\frac{\pi}{n}\right)-1)}{\left(1+\cos\left(\frac{\pi}{n}\right)\right)^2}\right) \\
&= \frac{1}{3}\left(1-\frac{1}{\left(1+\cos\left(\frac{\pi}{n}\right)\right)^2}\right).
\end{aligned}$$

Notes

These notes follow the tweets of N.N. Taleb (more authentically: “This is the curse of dimensionality behind $P \neq NP$ we used with GMO idiots”); the calculations and the graph in Figure 5.34 are by Marcos Carreira.

An analogous problem formulated in higher dimensions leads to a surprising result: the probability of getting nearer to center than to the edge tends to zero.

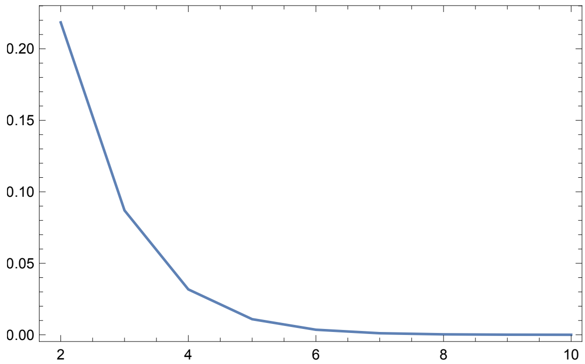


Figure 5.34: Dart throwing in polar coordinates. The bigger the dimension, the less the chance of hitting the center.

Probability in Triangle

Solution 1

Leo Giugiuc

Let x, y and z be the areas of triangles PBC, PCA and PAB. We have

$$\begin{aligned} K &= \frac{(y+z)A + yB + zC}{2(y+z)}, \\ L &= \frac{xA + (z+x)B + zC}{2(z+x)}, \\ M &= \frac{xA + yB + (x+y)C}{2(x+y)}. \end{aligned}$$

P is inside $\triangle$KLM if and only if $P = \alpha K + \beta L + \gamma M$, with $\alpha, \beta, \gamma > 0$ and $\alpha + \beta + \gamma = 1$, i.e.,

$$\begin{aligned} \frac{\alpha}{2} + \beta \cdot \frac{x}{2(z+x)} + \gamma \cdot \frac{x}{2(x+y)} &= \frac{x}{x+y+z} \\ \alpha \cdot \frac{y}{2(y+z)} + \frac{\beta}{2} + \gamma \cdot \frac{y}{2(x+y)} &= \frac{y}{x+y+z} \\ \alpha \cdot \frac{z}{2(y+z)} + \beta \cdot \frac{z}{2(z+x)} + \frac{\gamma}{2} &= \frac{z}{x+y+z}. \end{aligned}$$

Solving that system shows that the signs of α, β and γ are determined by the signs of $(x+y-z)(x-y+z)$, $(-x+y+z)(x+y-z)$ and $(x-y+z)(-x+y+z)$, respectively. Introduce $X = -x+y+z$, $Y = x-y+z$ and $Z = x+y-Z$.

Next, we show that $X, Y, Z > 0$. Indeed, assuming to the contrary that, e.g., $Z \le 0$ we get $Y > 0$ and $X > 0$, from which not all α, β and γ are positive—a contradiction. We have

$$\begin{aligned} P &= \frac{1}{x+y+z} \cdot (xA + yB + zC) \\ &= \frac{1}{x+y+z} \cdot \left(X\frac{B+C}{2} + Y\frac{C+A}{2} + Z\frac{A+B}{2} \right). \end{aligned}$$

This exactly means that P is interior to $\triangle$ KLM if and only if $Z > 0$, $Y > 0$ and $X > 0$.

In other words, P is interior to $\triangle$ KLM if and only if P is interior to the medial triangle of $\triangle$ ABC, so the probability of P being in $\triangle$ KLM is the same as the probability of its being in the medial triangle, i.e., $\frac{1}{4}$.

Solution 2

Francisco Javier García Capitán

If $P = (x : y : z)$ in homogeneous barycentric coordinates, then $A' = (0 : y : z)$, and $K = (y + z : y : z)$. From the identity

$$\{y+z, y, z\} = (x+y+z)\frac{\{x, y, z\}}{x+y+z} + (-x+y+z)\{1, 0, 0\},$$

we have the ratio

$$\frac{PK}{AK} = \frac{-x+y+z}{x+y+z}.$$

The equation $-x + y + z = 0$ corresponds to the line parallel to BC through the midpoints of AB and AC. The ratio $PK : AK$ is positive if and only if P lies in the half plane determined by this parallel not containing A.

Therefore, the point P is interior to $\triangle$ KLM when P is interior to the medial triangle of $\triangle$ ABC and we get that the sought probability is $\frac{1}{4}$.

Chapter 6

Combinatorics

> Probability is expectation founded upon partial knowledge. A perfect acquaintance with *all* the circumstances affecting the occurrence of an event would change expectation into certainty, and leave neither room nor demand for a theory of probabilities.
>
> George Boole, 1854, *An Investigation of the Law of Thought*

Riddles

6.1 Shuffling Probability

[91, Problem 91]

Integers 1 through 53 are written on cards, one per card. The stack is thoroughly shuffled. Five cards are drawn. What is the probability that the cards are drawn in their natural order, the smallest first and the rest in increasing order of magnitude?

6.2 Probability with Factorials

[4, Problem 10]

Let R be the set of rational fractions from $(0,1)$ in lowest terms so that $\frac{n}{d}$ only if $0 < n < d$ and $\gcd(n,d) = 1$. For $K \in \mathbb{N}$, define

$$A(K) = \{x : x \in R, nd = K!\}$$

and set $A = A(20) \cup A(21) \cup A(22)$.

What is the probability that a random number drawn from A belongs to $A(20)$? Answer the same question for

$$A = \bigcup_{K=20}^{25} A(K).$$

6.3 Two Varsity Divisions

[103]

20 varsity teams are assigned randomly to two divisions. What is the probability that the two strongest teams will happen to be

- in different divisions?
- in the same division?

6.4 Probability of Having 5 in the Numerator

[4, Problem 10]

Let set S consist of all rational fractions in lowest terms from $(0,1)$ such that if $\dfrac{a}{b} \in S$ then $ab = 10!$

What is the probability that the numerator of a random member of S is divisible by 5?

6.5 Random Arithmetic Progressions

[54, pp. 37–40]

Five numbers are drawn at random without replacement from the set $\{1, \ldots, n\}$. Show that the probability that the first three numbers drawn, as well as all five numbers, can be arranged to form arithmetic progressions is not less than

$$\frac{1}{\binom{n-1}{3}}.$$

6.6 Red Faces of a Cube

[56, p. 78]

The outside of an $n \times n \times n$ cube is painted red. The cube then is chopped into n^3 unit cubes. The latter are thoroughly mixed up and put into a bag. One small cube is withdrawn at random from the bag and tossed across a table. What is the probability that the cube stops with the red face on top?

6.7 Red and Green Balls in Red and Green Boxes

[33, C.W. Trigg, Problem E1400, p. 698], [57]

There are six red balls and eight green balls in a bag. Five balls are drawn randomly and placed in a red box; the remaining nine balls are placed in a green box. What is the probability that the number of red balls in the green box plus the number of green balls in the red box is not prime?

6.8 Probability of Equal Areas on a Chessboard

A random rectangle is drawn on an 8×8 chessboard with the sides of the former parallel to the sides of the latter. What is the probability that the rectangle contains the same amount of both colors?

Consider two cases (where measurements are relative to a 1×1 square side length):

- The side lengths of the rectangle are real numbers.
- The side lengths of the rectangle are integers.

This problem arose in the discussion following Stan Wagon's article on the problem of integer rectangles [95]. The follow-up article, by Nathan Bowler, is available online [22].

6.9 Random Sum

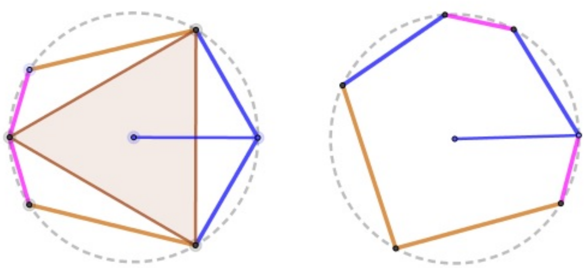
[77]

Ten coins are labeled with numbers 1 through 10 on one side. All 10 coins are tossed and the sum of the numbers landing face up is calculated. What is the probability that this sum is at least 45?

6.10 Probability of Equilateral Triangle

[60]

In a circle of radius R there are six chords, equal in pairs: $\{R, R, a, a, b, b\}$. The chords may form inscribed hexagons in a variety of ways.



What is the probability that some three endpoints form an equilateral triangle?

6.11 Shelving an Encyclopedia

[74]

An encyclopedia with 26 volumes $(A, B, C, \ldots, Z)$ is put randomly on a shelf.

1. What is the probability that volumes A and B appear next to one another in the correct order AB?
2. What is the probability that volumes A, B and C appear next to one another in the correct order ABC?
3. What is the probability that any two successive volumes appear next to one another in the correct order?

6.12 Loaded Dice I

[21, Problem 10]

Prove that no matter which two loaded dice we have, it cannot happen that each of the sums $2, 3, \ldots, 12$ comes up with the same probability.

6.13 Loaded Dice II

It is impossible to load two dice so as to have all 11 outcomes equiprobable. Is it possible for 10 selected outcomes?

This is a continuation of Riddle 6.12. It appeared in the chapter “Mathematical Creativity in Problem Solving and Problem Proposing II” by Murray S. Klamkin in [5].

6.14 Dropping Numbers into a 3×3 Square

[27, Problem CC246]

Place the numbers $1, 2, \ldots, 9$ at random so that they fill a 3×3 grid.

1. What is the probability that each of the row sums and each of the column sums is odd?
2. What if both diagonal sums are also required to be odd?
3. What if exactly one of the diagonal sums is also required to be odd?

6.15 Probability of Matching Socks

In a drawer, there are r red, b blue and g green socks. Two are drawn at random. What is the probability of getting a matching pair?

6.16 Numbered Balls Out of a Box

[65, Problem 320]

A box contains 20 balls numbered $1, 2, \ldots, 20$. If three balls are randomly taken from the box without replacement, what is the probability that one of them is the average of the other two? Answer the same question for 21 balls. Which probability is larger?

6.17 Planting Trees in a Row

[56, Section 10]

A gardener plants three maple trees, four oak trees and five birch trees in a row. He plants them in random order with each arrangement being equally likely. What is the probability that no two birch trees are next to each other?

Consider two cases:

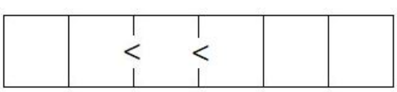
1. If trees of the same type are indistinguishable?
2. If all trees are different?

6.18 Six Numbers, Two Inequalities

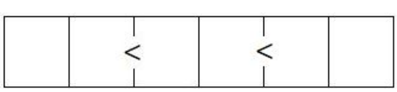
[75, Donald Knuth's puzzles]

Six distinct digits, say 1, 2, 3, 4, 5, 6, are placed randomly into the boxes below.

1. What is the probability that the indicated inequalities hold between the digits adjacent to the inequality symbols?



2. Answer the same question concerning another placement of the inequality signs:

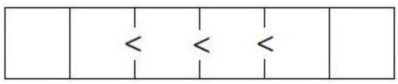


6.19 Six Numbers, Three Inequalities

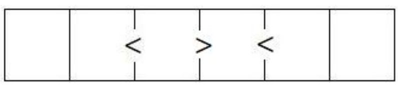
[75, Donald Knuth's puzzles]

Six distinct digits, say 1, 2, 3, 4, 5, 6, are placed randomly into the boxes below.

1. What is the probability that the indicated inequalities hold between the digits adjacent to the inequality symbols?



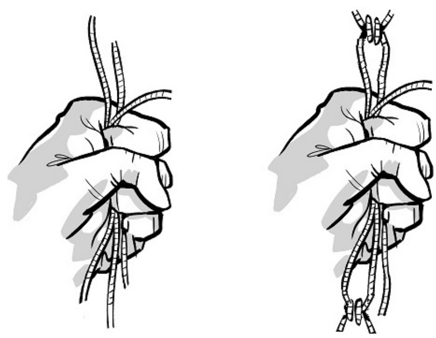
2. Answer the same question concerning another placement of the inequality signs:



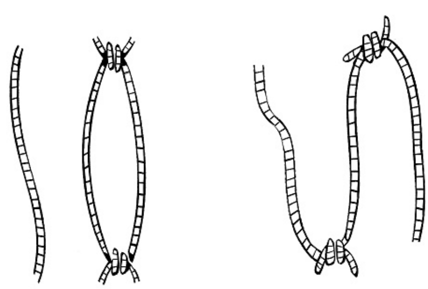
6.20 Tying Knots in Brazil

[10]

$\mathcal{A}$ is clutching three identical pieces of string in her fist, as illustrated below on the left.



She asks $\mathcal{B}$ to tie two ends of the string, chosen at random, at either side of her fist, as illustrated above on the right, so that there is one free end on either side. The result could be as shown below:



1. What is the probability that all the pieces are joined in one long piece?
2. With four strings?
3. With five strings?

6.21 Tying Knots in Russia

[103, Problem 76]

In some rural Russian communities, there existed the following form of divination: a girl would clutch in her fist six grass blades so that the ends of every one of them stuck out at both sides. A friend would then tie pairwise the top ends and then also the low ends. If, as a result, the grass blades formed a ring, it was a sign that the girl would be married within a year.

1. What is the probability that the grass blades would be tied into a single ring?
2. Answer the same question with $2n$ blades.

6.22 Guessing Hat Numbers

[64, Problem 112]

1. Five students each blindly choose a hat among six numbered $1, \dots, 6$. Each announces the largest and the smallest number he/she sees.

   What is the probability that each will be able to guess his/her own number?
2. Answer the same question with n hats and $k < n$ students.

6.23 Probability of an Odd Number of Sixes

[8, Problem 470]

1. What is the probability of an odd number of sixes turning up in a random toss of n fair dice?
2. What is the probability of an even number of sixes turning up in a random toss of n fair dice?
3. Are the two numbers equal? Why or why not?

6.24 Probability of Average

[65, Problem 320]

A box contains 20 balls numbered $1, 2, \dots, 20$. If three balls are randomly taken out of the box without replacement, what is the probability that the number on one of the balls will be the average of the other two?

6.25 Marking and Breaking Sticks

[71, pp. 36–41]

A person makes two marks—randomly and independently—on a stick, after which the stick is broken into n pieces. What is the probability that the two marks are found on the same piece?

Compare two cases: when the pieces are equal and when the division is random.

As usual, when no distribution is specified, the word "random" refers to the uniform distribution. "Independently" means independent of any previous action. This is especially important in the second part of the riddle. To avoid ambiguity, assume that, prior to breaking the stick, the $n - 1$ marks are made randomly and independently of all the marks already made.

6.26 Bubbling of Sorts

[54, pp. 21–22]

A given sequence $r_1, r_2, \ldots, r_n$ of distinct real numbers can be put in ascending order by means of one or more "bubble passes." A bubble pass through a given sequence consists of comparing the second term with the first term and exchanging them if and only if the second term is smaller, then comparing the third term with the second term and exchanging them if and only if the third term is smaller and so on in order, through comparing the last term r_n with its current predecessor and exchanging them if and only if the last term is smaller.

The following example shows how the sequence 1, 9, 8, 7 is transformed into the sequence 1, 8, 7, 9 by one bubble pass. The numbers compared at each step are underlined.

$$\underline{1 \quad 9} \quad 8 \quad 7 \qquad 1 \quad \underline{9 \quad 8} \quad 7 \qquad 1 \quad 8 \quad \underline{9 \quad 7} \qquad 1 \quad 8 \quad 7 \quad 9$$

Suppose that $n = 40$ and that the bubble sort algorithm applies to the random sequence of distinct numbers $r_1, r_2, \ldots, r_{40}$ in order to place them in ascending order.

1. Find the probability that the number that begins as r_{20} will end up, after one bubble pass, in poisition #30.
2. Find the probability that the number that begins as r_{20} will end up in poisition #30 after two bubble passes.
3. Find the probability that the number that begins as r_{20} will end up in poisition #30 after $n \leq 10$ bubble passes.

6.27 Probability of Successive Integers

[11, Problem 2]

Five distinct numbers are chosen from 10, 11, 12,..., 99. What is the probability that at least two of them are consecutive?

Solutions

Shuffling Probability

You have five cards which may have been drawn in any of $5! = 120$ possible ways. Only one of these is the natural increasing order. The probability of this happening is $\dfrac{1}{120}$.

The specific number of cards is a red herring [19]. No matter which number you start with, as long as you draw five cards, there are $5! = 120$ variants. Each comes with probability of $\dfrac{1}{120}$.

Probability with Factorials

Let us list the prime factors of, say, $20!$. All are less than 20; there are eight of them: 2, 3, 5, 7, 11, 13, 17, 19. Since we are only concerned with the fractions in reduced form, these factors are somehow distributed between their numerators n and denominators d (in whatever power it takes), but no factor appears in both n and d. Therefore, there are $2^8 = 256$ possible values for pairs n, d. For only half of them $n < d$, making the number of eligible fractions in $A(20!) = 128$.

Since there are no primes between 19 and 23, $A(21!)$ and $A(22!)$ also have 128 elements each. Since the pairwise intersections of the three sets are empty, a number drawn from their union has equal probability to be drawn from either of them, i.e., $\dfrac{1}{3}$.

$23!$, $24!$ and $25!$ include an additional prime—23, implying that each contains $\dfrac{1}{2}(2^9) = 256$ elements. Thus, in this case, $|A| = 3 \cdot 128 + 3 \cdot 256 = 9 \cdot 128$. Thus, the probability of the drawn number to be from $A(20)$ is $\dfrac{1}{9}$.

Two Varsity Divisions

Solution 1

20 teams may be split into two divisions of 10 teams each in $\dfrac{1}{2}\dbinom{20}{10}$ ways. With two teams removed, 18 are left to be split between the two divisions. For the first question, we need nine teams to complete the division with one of the strongest teams (the other division will be filled automatically). Thus, the probability in question 1 is

$$P = \frac{\dbinom{18}{9}}{\dfrac{1}{2}\dbinom{20}{10}} = \frac{10}{19}.$$

For the second question, we need eight teams to complete the division of the two strongest teams. Thus, the probability in question 2 is

$$P = \frac{\dbinom{18}{8}}{\dfrac{1}{2}\dbinom{20}{10}} = \frac{9}{19}.$$

Solution 2

Denote, for convenience, the two strongest teams A and B. When A is chosen into one of the divisions, there are 19 slots that B can fit in. For the first question, 10 of these slots are in the "other" division, giving the probability as $\frac{10}{19}$.

For the second question, there are nine slots in the same division as A, making the probability of B falling there equal to $\frac{9}{19}$.

Naturally, if A and B are not in the same division, then they are in different ones so that $\frac{10}{19} + \frac{9}{19} = 1$.

Probability of Having 5 in the Numerator

Solution 1

There are four prime numbers below 10: 2, 3, 5, 7. Their factors may appear in either numerator or denominator of $\frac{a}{b}$ from S but not both. All in all, there are 2^4 fractions $\frac{a}{b}$ that satisfy $ab = 10!$. Half of them are less than one; half of them are more than one. It follows that $|S| = 2^3$.

Now $10! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7 = 256 \cdot 81 \cdot 25 \cdot 7$. There are only two fractions below one with 5 in the numerator:

$$\frac{25}{256 \cdot 81 \cdot 7} \text{ and } \frac{25 \cdot 7}{256 \cdot 81},$$

for, already $25 \cdot 81 > 256 \cdot 7$. Thus, the probability in question is $\frac{2}{8} = \frac{1}{4}$.

Solution 2

Amit Itagi

$ab = 10! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7 = 256 \cdot 81 \cdot 25 \cdot 7$. For $\frac{a}{b}$ to be in the lowest form, there should be no common prime factor of a and b. Moreover, $a < b$. Thus,

$$a \in \{1, 256, 81, 25, 7, 256 \cdot 7, 81 \cdot 7, 25 \cdot 7\},$$

with $b = \frac{10!}{a}$. The cardinality of the set is eight and the number of elements divisible by 5 is two. Hence, the required probability is $\frac{1}{4}$.

Random Arithmetic Progressions

Solution 1

There are $\binom{n}{5}$ ways to choose four numbers out of n. This is the size of our sample space. Not all five-term sequences admit an arrangement into an arithmetic progression. Those that do can be identified by two numbers, say x and y, playing the role of the smallest and the largest terms of the progression. x and y need to satisfy $x \equiv y \pmod 4$.

Let $N(n)$ be the number of such pairs. Thus, the probability of getting a five-term arithmetic progression equals $P_5 = \dfrac{N(n)}{\binom{n}{5}}$.

If five numbers $\{a_0, a_1, a_2, a_3, a_4\}$ form an arithmetic progression, then so do $\{a_0, a_1, a_2\}$, $\{a_1, a_2, a_3\}$, $\{a_2, a_3, a_4\}$ and $\{a_1, a_3, a_5\}$. In other words, out of $10 = \binom{5}{3}$ possible selections of three elements, only four amount to an arithmetic progression. Combining the two estimates in one, the probability in question equals

$$P_{5,3} = \frac{N(n) \cdot \dfrac{4}{10}}{\binom{n}{5}}.$$

If $n = 4m$, $N(4m) = 4 \cdot \dfrac{m(m-1)}{2} = \dfrac{n(n-4)}{8}$ because there are four remainders modulo 4 and, for each, m terms that can be paired with other $m-1$ terms.

Now if $n = 4m - k$, where $k = 1, 2, 3$, define $n' = n + k$. As before, there are $\dfrac{n'(n'-4)}{8}$ pairs of numbers, but several have been just added. So to correct the mishap,

$$\begin{aligned}
N(n) &= \frac{n'(n'-4)}{8} - k\left(\frac{n'}{4} - 1\right) \\
&= \frac{(n+k)(n+k-4) - 2k(n+k-4)}{8} \\
&= \frac{(n+k-4)(n-4)}{8} = \frac{n^2-4n}{8} + \frac{4k-k^2}{8} \\
&\ge \frac{n(n-4)}{8},
\end{aligned}$$

because $k < 4$. Thus, in any event $N(n) \ge \dfrac{n(n-4)}{8}$, implying

$$\begin{aligned}
P_{5,3} &= \frac{N(n) \cdot \dfrac{4}{10}}{\binom{n}{5}} \\
&\ge \frac{4 \cdot 12 \cdot n(n-4)}{8 \cdot n(n-1)(n-2)(n-3)(n-4)} \\
&= \frac{6}{(n-1)(n-2)(n-3)} = \frac{1}{\binom{n-1}{3}}.
\end{aligned}$$

Solution 2

N.N. Taleb

Denominator: There are $\frac{n!}{(n-5)!}$ permutations of subsample $S = \{1, \ldots, n\}$.

Numerator: We look for $\xi = \left\lfloor \frac{n}{4} \right\rfloor$.

General solution for 5: We have arithmetic sequences (five members) separated by $\zeta = \{1, 2, \ldots, \xi\}$, such that the sequence will be expressed as a_1, $a_1 + \zeta$, $a_1 + 2\zeta \ldots a_1 + 4\zeta$, and constrained by $a_1 + 4\zeta \leq n$.

We can show that there are $n - 4\zeta$ possible increasing sequences of progression of difference ζ.

There are 5! permutations for any five selected numbers. The number of sequences is

$$\sum_{\zeta=1}^{\xi} (n - 4\zeta) = -2\xi^2 + n\xi - 2\xi.$$

The probability of having a progression becomes

$$p = \frac{5! \left(-2\xi^2 + n\xi - 2\xi\right)}{\frac{n!}{(n-5)!}}.$$

For large n or values of n divisible by 4,

$$p = \frac{15}{(n-3)(n-2)(n-1)},$$

which is greater than

$$\frac{1}{\binom{n-1}{3}} = \frac{6}{(n-3)(n-2)(n-1)}.$$

General solution for 3 then 5: Conditional on having the draws with arithmetic sequence $\{X_{(1)}, X_{(2)}, \ldots, X_{(5)}\}$ in various permutations of $\frac{5!}{(5-3)!} = 60$ counts of the first three draws, how many include a sequence?

We have six counts for $\{X_{(1)}, X_{(2)}, X_{(3)}\}$ (that is 3!) plus six counts of $\{X_{(2)}, X_{(3)}, X_{(4)}\}$, plus six counts of $\{X_{(3)}, X_{(4)}, X_{(5)}\}$, plus six counts of $\{X_{(1)}, X_{(3)}, X_{(5)}\}$ for a total of 24 counts, so the correction is

$$p \times \frac{24}{60}.$$

Red Faces of a Cube

Solution 1

All faces are equiprobable, making the sought probability equal to $\frac{6n^2}{6n^3} = \frac{1}{n}$.

Solution 2

Amit Itagi

For $n = 1$, the probability is 1 as all faces of the cube are painted.

For $n > 1$, let $f(k, n)$ be the number of unit cubes that have k faces painted:

$$f(k,n) = \begin{cases} (n-2)^3, \text{ for } k = 0 \\ 6(n-2)^2, \text{ for } k = 1 \\ 12(n-2), \text{ for } k = 2 \\ 8, \text{ for } k = 3 \\ 0, \text{ for } 4 \le k \le 6. \end{cases}$$

Thus, the required probability is

$$\begin{aligned} P(n) &= \sum_{k=0}^{6} P(\text{red face up}|\text{cube has } k \text{ faces painted}) P(\text{cube has } k \text{ faces painted}) \\ &= \sum_{k=0}^{6} \frac{k}{6} \cdot \frac{f(k,n)}{n^3} \\ &= \frac{1}{6n^3} \left[6(n-2)^2 + 24(n-2) + 24 \right] \\ &= \frac{1}{n}. \end{aligned}$$

I was wondering how we got such a nice-looking answer. There are a total of $6n^3$ faces, out of which $6n^2$ are on the outer surface. Therefore, choosing a random cube and a random face is equivalent to choosing a face randomly from the $6n^3$ possibilities.

Solution 3

Joshua Zucker

Think of the cube as a conglomeration of n^2 columns of $1 \times 1 \times 1$ cubes stacked on top of each other. Of the n small cubes in each such column, only one has its top face painted red.

Red and Green Balls in Red and Green Boxes

Let g stand for the number of green balls in the red box. Then, since the green box ought to have at least one green ball, we have the following distribution of colors:

Red Box (5 Balls)	Green Box (9 Balls)
g green	$8 - g$ green
$5 - g$ red	$g + 1$ red

Therefore, the number of green balls in the red box plus the number of red balls in the green box equals $g + (g + 1) = 2g + 1$.

Note that, since $0 \le g \le 5$, $1 \le 2g + 1 \le 11$. The only non-prime odd numbers in this range are 1 and 9; for these, $g = 0$ or $g = 4$. The probability of drawing all five red balls or just one red ball is given by

$$\frac{\binom{6}{5}+\binom{8}{4}\binom{6}{1}}{\binom{14}{5}}=\frac{6+420}{2002}=\frac{213}{1001}.$$

Probability of Equal Areas on a Chessboard

First of all, we shall prove Nathan Bowler's claim.

A rectangle with sides parallel to the sides of a chessboard covers equal areas of both colors if and only if one of the two conditions holds:

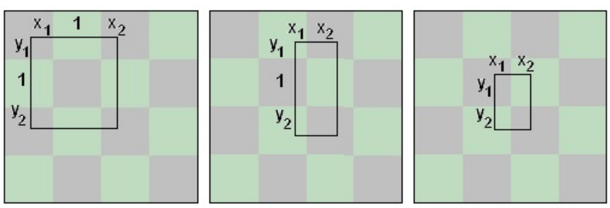
- One of the sides of the rectangle is an even multiple of the side of a chessboard square. The dimension of the other side is irrelevant.
- The center of the rectangle lies on a grid line.

For simplicity, let us call any segment double a square's side in length even.

The sufficiency (the "if" part) is pretty obvious. If, say, the vertical dimension of the rectangle is double a square's side, then in every vertical cross-section the rectangle segments of different colors add up to the same lengths. If the center of the rectangle lies on a grid line, then, by symmetry, the statement is also true.

Observe that every rectangle can be split into at most three parts: one with an even horizontal dimension, one with an even vertical dimension and one with both dimensions less than two.

The problem of showing the necessity, i.e., the "only if" part, is thus reduced to showing that, in a rectangle with sides less than two, there is more of one color than the other, provided the center of the rectangle does not lie on a grid line. Here are the three prototypical possibilities:



In all cases,

$$0 < x_1, x_2 < 1, \text{ and}$$

$$0 < y_1, y_2 < 1.$$

Also,

$$0 < x_1 + x_2 < 1$$

in the first case, and

$$0 < y_1 + y_2 < 1$$

in cases 1 and 2.

If A_1 and A_2 are total areas of the regions of the light and dark colors inside the rectangle, then, in the first case, we get

$$A_1 = x_1 + x_2 + y_1 + y_2,$$

$$A_2 = (x_1 + x_2)(y_1 + y_2) + 1.$$

The difference of the two is

$$A_1 - A_2 = -(x_1 + x_2 - 1)(y_1 + y_2 - 1),$$

which is never zero.

In the second case, we get

$$A_1 = x_1 y_1 + x_2 + x_1 y_2 = x_1(y_1 + y_2) + x_2,$$

$$A_2 = x_2 y_1 + x_1 + x_2 y_2 = x_2(y_1 + y_2) + x_1.$$

The difference of the two is

$$A_1 - A_2 = (x_1 - x_2)(y_1 + y_2 - 1),$$

which may be zero if and only if $x_1 = x_2$, i.e., in the excluded case where the center of the rectangle lies on a grid line. The last case is treated similarly.

Due to the fact that the probability of a random rectangle having its center on a grid line is zero, the riddle has an unexpected solution: for a rectangle with an even side, the probability of covering equal white and black areas is one; for the less fortunate rectangles, the probability is zero.

As having an integer, let alone an even, side has the probability of zero, zero is the answer for the first case.

For the second case, the sides of the rectangle may be independently odd or even. There are several interpretations.

One answer is that in three out of four possibilities at least one side is even, making the probability of having equal areas $\frac{3}{4}$.

However, this is a simplified treatment of the riddle. The availability of even and odd side lengths may depend on the distance of a selected corner to the sides of the chessboard.

On an 8×8 board, integer rectangles may be of size $M \times N$, where $1 \le M, N \le 7$. Thus, for $M = 2, 4, 6$, there are seven rectangles with at least one side even, 21 in all. For each of $M = 1, 3, 5, 7$, there are three rectangles with at least one side even, 12 in all. Therefore, the total number of integer rectangular sizes that cover equal white and black areas is $21 + 12 = 33$. This is out of the total of $7 \times 7 = 49$ possible sizes, giving the probability of $\frac{33}{49}$.

If we count the frequencies of various rectangle sizes, the answer changes again. The sizes $M \times N$ are tabulated separately depending on odd/even combination of side lengths (the tables are by Thamizh Kudimagan):

M/N	1	3	5	7
1	64	48	32	16
3	48	36	24	12
5	32	24	16	8
7	16	12	8	4
	Total :		400	

M/N	1	3	5	7
2	56	42	28	14
4	40	30	20	10
6	24	18	12	6
8	8	6	4	2
	Total :	$320 \times 2 =$	640	

M/N	2	4	6	8
2	49	35	21	7
4	35	25	15	5
6	21	15	9	3
8	7	5	3	1
	Total :		256	

Thus, an integer rectangle covers equal areas of black and white in $896 = 640+256$ out of $400 + 896 = 1296$ cases, i.e., with the probability of $\dfrac{896}{1296} = \dfrac{56}{81}$.

Random Sum

Solution 1

The maximum possible sum is $1+2+\cdots+10 = 55$. $45 = 55-10$. It is easier to calculate the probability that the sum is at most 10. This would be the sum of numbers written on the coins that fell face down. Just by inspection, there are 43 combinations of numbers from 1 to 10, each taken at most once, that sum to at most 10.

The sample space consists of 2^n possible outcomes, making the desired probability equal to $\dfrac{43}{2^n}$.

Solution 2

Mike Lawler

Sequence A000009 from the On-Line Encyclopedia of Integer Sequences [85] lists, for a given index N, the number of partitions of N into distinct parts:

$$\begin{array}{cccccccccccc} N & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ & 1 & 1 & 1 & 2 & 2 & 3 & 4 & 5 & 6 & 8 & 10. \end{array}$$

For example,

$$\begin{aligned}7 &= 7+0 = 6+1 = 5+2 = 4+3\\ &= 1+2+4\\ 8 &= 8+0 = 7+1 = 6+2 = 5+3\\ &= 1+2+5 = 1+3+4.\end{aligned}$$

Summing up the 11 numbers in the second row of the above table gives 43.

Solution 3
Robert Frey and, independently, N.N. Taleb

The expansion of the generating function $\prod_{n\in S}(1+q^n)$ where S is the set under consideration (here $1, 2, \dots, 10$) into polynomials, such as $a_1q+a_2q^2\cdots+a_mq^m$, will produce coefficients that equal the number of distinct partitions of m.

We are considering values $45 \le j \le \sum_{i=1}^{10} i = 55$:

$$\prod_{i=1}^{10}\left(1+q^i\right) = q^{55}+q^{54}+q^{53}+2q^{52}+2q^{51}+3q^{50}+4q^{49}+5q^{48}+6q^{47}+8q^{46}+10q^{45}+\cdots,$$

with coefficients $\{1,1,1,2,2,3,4,5,6,8,10\}$ that sum to 43. The total number of tuples under consideration is 2^{10}. Hence the probability is $\frac{43}{2^{10}} \approx .041992$.

Probability of Equilateral Triangle

Amit Itagi

Let $\Phi(x)$ be the angle subtended by a chord of length x at the center. $\Phi(R) = 60°$. Let us work under the assumption that $a \neq R$. This also implies that $a \neq b$ for, if $a = b$, then the only way $\Phi(R)+\Phi(a)+\Phi(b) = 180°$ is if $a = b = R$. Without loss of generality, let $a < R < b$ (one of a and b has to be smaller than R and the other greater than R because $\Phi(a)+\Phi(b) = 120°$). Let $\Phi(a) = \alpha$ and $\Phi(b) = 120° - \alpha$. The only additive combinations of the Φs that result in $120°$ are $2\Phi(R)$ and $\Phi(a)+\Phi(b)$. Thus, the allowed permutations of $\{R, R, a, a, b, b\}$ to get us an equilateral triangle are the permutations of (R, R, a, b, a, b) where permutations are allowed within each of the two (a, b) units and the entire set can be permuted in a cyclic manner.

The total number of permutations of the set is

$$P_{\text{total}} = \frac{6!}{2\times 2\times 2} = 90,$$

with each two in the denominator discounting the permutations within the same kind of chords. The number of valid permutations of the kind $(R, R, \dots)$ is $2\times 2 = 4$ (two permutations within each (a, b) unit). Six cyclic permutations can be generated from each of the four permutations. Thus, $P_{\text{valid}} = 4\times 6 = 24$. Hence, the required probability is

$$\text{Prob} = \frac{P_{\text{valid}}}{P_{\text{total}}} = \frac{24}{90} = \frac{4}{15} \sim 0.267.$$

Shelving an Encyclopedia

Solution 1 to Question 1

Vincent Pantaloni

All said, there are 26! ways to randomly shelve 26 volumes. If A and B are considered as an immutable pair, like being glued together, the whole of 25 volumes could be placed randomly in 25! ways. Thus, the probability of having volumes A and B next to each other in the right order is $\frac{25!}{26!} = \frac{1}{26}$.

Solution 2 to Question 1

Pair AB could be in any of 25 positions while the remaining 24 volumes can be permuted arbitrarily, giving the probability of $\frac{25 \cdot 24!}{26!} = \frac{1}{26}$.

Solution 1 to Question 2

Employing the reasoning from the first solution to the first question, glue volumes A, B and C together to obtain a collection of 24 books. The answer then is $\frac{24!}{26!} = \frac{1}{26 \cdot 25}$.

Solution 2 to Question 2

Employing the reasoning from the second solution to the first question, the sequence ABC can be located in any of 24 places. The other 23 volumes can be permuted arbitrarily, giving the probability of $\frac{24 \cdot 23!}{26!} = \frac{1}{26 \cdot 25}$.

Solution to Question 3

Let $A(n)$ denote the set of permutations of $\mathbb{N}_{n+1} = \{1, 2 \ldots, n, n+1\}$ that do not contain two successive numbers next to each other in the right order, i.e., the permutations with no substrings $k(k+1)$, whatever k, and let $a(n) = |A(n)|$ be the number of elements of $A(n)$. (This is sequence A000255 from [85].) Note that we are looking for the number $(n+1)! - a(n)$, $(n = 25)$ of permutations that do contain such pairs of successive numbers.

A permutation $\sigma_{n+1} \in A(n)$ can be associated with a number of permutations from $A(n-1)$ when $n+1$ is excluded. Permutations from $A(n-1)$ with n last can be looked at as members of $A(n-2)$ and $n=1$ can be properly inserted in any of $n-1$ positions, $0, 1, \ldots, n-2$, which produces $(n-1) \cdot a(n-2)$ permutations from $A(n)$. On the other hand, given a permutation from $A(n-1)$ with n in, say, position $k \neq n$, the term $n+1$ can be inserted in n positions: $0, 1, \ldots, k-1, k+1, \ldots, n$, leading to the total of $n \cdot a(n-1)$ permutations. Thus, we are led to a recurrence relation:

$$a(n) = n \cdot a(n-1) + (n-1) \cdot a(n-2),$$

$$a(0) = a(1) = 1.$$

Thus, one answer to question 3 is $\frac{n! - a(n-1)}{n!}$.

Given the formula for the number of *derangements* d_n of n elements, it is straightforward to verify that

$$a(n-1) = \frac{d_{n+1}}{n},$$

which gives another answer to question 3, $1 - \dfrac{d_{n+1}}{n \cdot n!}$.

Loaded Dice I

Solution 1

[21]

Let p_i be the probability of i, $i = 1, \ldots, 6$, coming up for the first die and q_i be the probability of i coming up for the second. Assuming all the sums come up with the same probability, the latter equals $\frac{1}{11}$. Then p_1q_1 is the probability of the sum being 2 such that $p_1q_1 = \frac{1}{11}$. Similarly, $p_6q_6 = \frac{1}{11}$ is the probability of the sum being 12.

The probability of the sum being 7 is

$$\begin{aligned}\frac{1}{11} &= \sum_{k+j=7} p_kq_j > p_1q_6 + p_6q_1 \\ &= \frac{p_1}{11p_6} + \frac{p_6}{11p_1} = \frac{1}{11}\left(\frac{p_1}{p_6} + \frac{p_6}{p_1}\right) \geq \frac{1}{11} \cdot 2,\end{aligned}$$

by the *AM-GM inequality*. That is a contradiction.

Solution 2

[21]

Let $P(x) = \displaystyle\sum_{k=1}^{6} p_ix^{i-1}$ be a probability generating function for the first die, $Q(x) = \displaystyle\sum_{k=1}^{6} q_ix^{i-1}$ that for the second. Since $p_p, q_p \neq 0$, each of $P(x), Q(x)$ is a polynomial of degree five and, as such, has a real root.

On the other hand,

$$P(x)Q(x) = \sum_{k=2}^{12} \frac{1}{11}x^{k-2} = \frac{1}{11} \cdot \frac{x^{11}-1}{x-1}.$$

The contradiction comes from the fact that the *cyclotomic polynomial* $\displaystyle\sum_{k=0}^{10} x^k$ does not have real roots.

Notes

The two solutions are automatically extended to dice of different shapes, e.g., all Platonic solids. However, the second solution would not work for a polyhedron with an odd number of faces, e.g., for a cube with a corner sawed off.

Loaded Dice II

Let $P(x) = \sum_{k=1}^{6} p_i x^i$ be a probability generating function for the first die, $Q(x) = \sum_{k=1}^{6} q_i x^i$ that for the second. The polynomial coefficients are the "weights" for each of the indices. Let us set $p_1 = 0$, meaning that the first die never comes up with 1 on top. Factor out x^2 from P and x from Q:

$$P(x) = x^2 P_0(x) = x^2(p_2 + p_3 x + p_4 x^2 + p_5 x^3 + p_6 x^4),$$
$$Q(x) = x Q_0(x) = x(q_1 + q_2 x + q_3 x^2 + q_4 x^3 + q_5 x^4 + q_6 x^5).$$

Assuming that the outcomes $3, 4, \ldots, 12$ are equiprobable,

$$\begin{aligned} P_0(x)Q_0(x) &= \frac{1}{10}(1 + x + x^2 + \cdots + x^9) \\ &= \frac{1}{10} \cdot \frac{x^{10} - 1}{x - 1} = \frac{1}{10} \frac{(x^5 - 1)(x^5 + 1)}{x - 1} \\ &= \frac{1}{10} \cdot \frac{1 + x + x^2 + x^3 + x^4}{5} \cdot \frac{1 + x^5}{2}. \end{aligned}$$

This means that there is a choice:

$$p_1 = 0, \qquad p_2,\, p_3,\, p_4,\, p_5,\, p_6 = \frac{1}{5}$$
$$q_1 = q_6 = \frac{1}{2}, \qquad q_2 = q_3 = q_4 = q_5 = 0.$$

The second die is easily created by taking a regular cube and placing numbers 1 and 6 on three faces each. Both of them can be simulated by a pair of regular icosahedra (each with 20 faces): one with 10 faces bearing 1 and 10 faces bearing 6, the other with numbers (or number of dots) 2, 3, 4, 5, 6 on four faces each.

Dropping Numbers into a 3×3 Square

Solution 1 to Question 1

Missouri State Problem Solving Group

In order for the sum of three numbers to be odd, they must either all be odd or exactly one number must be odd. There are five odd numbers available, so one row must have three odd numbers and the other two rows must have exactly one odd number each, with the analogous result for columns. Once we choose a row with three odd numbers and a column with three odd numbers, we have used up all five available odd numbers. There are $3 \times 3 = 9$ ways of choosing the row and column, $5!$ ways of determining where the odd numbers are, and $4!$ ways of determining where the even numbers are. This gives a total of $9 \times 5! \times 4!$ matrices with the desired property.

There are a total of $9!$ matrices with entries from $1, 2, \ldots, 9$, so the probability we seek is

$$\frac{9 \times 5! \times 4!}{9!} = \frac{1}{14}.$$

Solution 2 to Question 1

Amit Itagi

We have four even and five odd numbers. The parity combinations of three numbers that result in an odd sum are $\{O, E, E\}$ and $\{O, O, O\}$.

The only possibility is that two rows have two even numbers each and the third row has no even number. The same logic applies to the columns. Thus, we have to choose the two rows and the two columns. This can be done in $3 \times 3 = 9$ ways. The two rows and two columns set the positions for the evens. The remaining positions are for the odds. Thus, there are 4! ways of permuting the evens and 5! ways of permuting the odds. The total number of arbitrary permutations is 9!. Thus, the probability is

$$P_1 = \frac{9 \cdot 4! \cdot 5!}{9!} = \frac{1}{14}.$$

Solution 1 to Question 2

Missouri State Problem Solving Group

If there is a column and a row with three odd numbers each, there is no way to place three odd numbers on a diagonal. In any event, there is a unique odd number that is shared by the chosen row and column (of each of the three odd numbers). For a diagonal sum to be odd, this number needs to be on the diagonal. Only in one case out of nine are both the diagonal sums odd; this is when the common odd element is at the center of the matrix. The probability of this is

$$\frac{5! \times 4!}{9!} = \frac{1}{126}.$$

Solution 2 to Question 2

Amit Itagi

For one diagonal to have an odd sum, two of the four even positions in the preceding question need to have the same row and column number. If those positions are (i, i) and (j, j), the remaining evens have to occupy (i, j) and (j, i) to satisfy the conditions in question 1. Now the only way the other diagonal sums to an odd number is if (i, j) and (j, i) fall on the other diagonal and (i, i), (j, j) do not fall on that diagonal. Thus, $\{i, j\} = \{1, 3\}$. The nine possible ways of choosing the even positions drops to one and the probability shrinks by a factor of nine to

$$P_2 = \frac{1}{9 \cdot 14} = \frac{1}{126}.$$

Solution 1 to Question 3

Missouri State Problem Solving Group

If the selected point is in one of the corner matrices then the corresponding diagonal's sum is odd while that of the other is even. There are four corners and the probability of such an event is

$$\frac{4 \times 5! \times 4!}{9!} = \frac{4}{126} = \frac{2}{63}.$$

Solution 2 to Question 3

Amit Itagi

For the main diagonal to have an odd sum, we place the two evens at (i,i) and (j,j) as in question 2. However, to keep the sum of the other diagonal even, (i,j) and (j,i) need to be off that diagonal. Thus, $\{i,j\} = \{1,2\}$ or $\{2,3\}$. Now instead of nine ways of choosing the positions of evens in question 1, we drop to two ways of choosing the positions of evens and the probability drops by a factor of $\frac{9}{2}$. However, we could as well have chosen the second diagonal to give an odd sum instead of the first. This doubles the probability. Thus,

$$P_3 = \frac{2}{\frac{9}{2} \cdot 14} = \frac{2}{63}.$$

Probability of Matching Socks

Solution 1

To start with, instead of looking for a matching pair, let us find the probability that both socks are red.

The probability of getting one red sock is

$$\frac{r}{r+b+g}.$$

Assuming that the first sock is red, the probability of getting a second red sock is

$$\frac{r-1}{r+b+g-1}.$$

When it comes to calculating probabilities, colors do not make much difference: an analogous argument applies to blue and green socks, implying that the probability of getting two blue socks is

$$\frac{b(b-1)}{(r+b+g)(r+b+g-1)}$$

and that of getting two green socks is

$$\frac{g(g-1)}{(r+b+g)(r+b+g-1)}.$$

The answer to the question is then

$$\frac{r(r-1)+b(b-1)+g(g-1)}{(r+b+g)(r+b+g-1)}.$$

The formula is eminently reasonable for, if any of the three numbers is 1, the effect of having an unmatched sock in a drawer is to increase the denominator without adding anything to the numerator.

Solution 2

There are $\binom{r+b+g}{2}$ ways to select two socks out of the total of $r+b+g$. There are $\binom{r}{2}$ ways to select two out of r red socks. Therefore, the probability of selecting two red socks is

$$\begin{aligned}P(2 \text{ red socks}) &= \frac{\binom{r}{2}}{\binom{r+b+g}{2}}\\ &= \frac{r!}{2!(r-2)!}\cdot\frac{2!(r+b+g-2)!}{(r+b+g)!}\\ &= \frac{r(r-1)}{(r+b+g)(r+b+g-1)}.\end{aligned}$$

This is the same result as before, and it is similar for the other colors.

Numbered Balls Out of a Box

Solution 1

The average of two integers is an integer if the two summands are of the same parity: either both are odd or both are even. These two are the largest and the smallest of the three while the middle one is uniquely determined by these two.

The number of ways to choose two even numbers out of 20 is $\binom{10}{2} = 45$. Similarly, there are 45 odd pairs.

The total number of possible selections of three balls out of 20 is $\binom{20}{3} = 1140$, giving the required probability as

$$\frac{45+45}{1140} = \frac{9}{114} = \frac{3}{38}.$$

If there are 21 balls, then the corresponding probability is

$$\frac{\binom{10}{2}+\binom{11}{2}}{\binom{21}{3}} = \frac{45+55}{1330} = \frac{10}{133} < \frac{3}{38}.$$

Solution 2

Let the total number of balls be N. Out of the three numbers chosen, let i be the middle number. If the smallest number is $i-k$, then the largest number has to be $i+k$. Thus, for a fixed i, the number of ways of choosing the other pair of numbers is equal to the number of ways of choosing a valid k and is given by $\min\{i-1, N-i\}$.

Thus, the number of ways of choosing the balls satisfying the constraints is

$$\begin{aligned}
N_c &= \sum_{i=1}^{N} \min\{i-1, N-i\} \\
&= \sum_{i=1}^{\lceil \frac{N}{2} \rceil} (i-1) + \sum_{i=1+\lceil \frac{N}{2} \rceil}^{N} (N-i).
\end{aligned}$$

Thus, the required probabilities are

$$\begin{aligned}
P(20) &= \frac{\sum_{i=1}^{10}(i-1) + \sum_{i=11}^{20}(20-i)}{C(20,3)} = \frac{45+45}{1140} = \frac{3}{38} \text{ and} \\
P(21) &= \frac{\sum_{i=1}^{11}(i-1) + \sum_{i=12}^{21}(21-i)}{C(21,3)} = \frac{55+45}{1330} = \frac{10}{133}.
\end{aligned}$$

$P(20) > P(21)$.

Planting Trees in a Row

Solution 1 to Question 1

Let us solve the riddle in general with m maple trees, o oak trees and b birch trees.

There is a total T of arrangements of the three kinds of trees:

$$T = \frac{(m+o+b)!}{m!\; o!\; b!}.$$

There are

$$M = \binom{m+o}{o} = \frac{(m+o)!}{m!\, o!}$$

ways to arrange maple and oak trees and

$$N = \binom{m+o+1}{b} = \frac{(m+o+1)!}{(m+o+1-b)!\, b!}$$

ways to place b birch trees so that no two are adjacent. The probability we are interested in is

$$\begin{aligned}
P = \frac{M \cdot N}{T} &= \frac{(m+o)!}{m!\, o!} \cdot \frac{(m+o+1)!}{(m+o+1-b)!\, b!} \cdot \frac{m!\, o!\, b!}{(m+o+b)!} \\
&= \frac{(m+o)!\,(m+o+1)!}{(m+o+1-b)!\,(m+o+b)!} = \frac{(m+o+1)!}{(m+o+1-b)!\, b!} \cdot \frac{(m+o)!\, b!}{(m+o+b)!} \\
&= \frac{\binom{m+o+1}{b}}{\binom{m+o+b}{b}}.
\end{aligned}$$

Solution 2 to Question 1

Amit Itagi

First, order the maple and the oak trees (7! ways). Plant the birch trees in five of the possible eight locations (on either side of the planted trees or in the middle of two planted trees) for each birch tree ($P(8,5)$ ways). Without restrictions, there are 12! ways to plant. Thus, the desired probablility is

$$P_d = \frac{7! \cdot P(8,5)}{12!} = \frac{7}{99}.$$

Solution 1 to Question 2

There is a total $T = (m+o+b)!$ of ways to plant the trees. Maples and oaks can be planted in $M = (m+o)!$ distinct ways, creating $m+o+1$ places for single birch trees. These can be planted there in $N = \dfrac{(m+o+1)!}{(m+o+1-b)!}$ unique ways. The probability we are after is

$$\begin{aligned} P = \frac{M \cdot N}{T} &= \frac{(m+o)! \cdot (m+o+1)!}{(m+o+1-b)!} \cdot \frac{1}{(m+o+b)!} \\ &= \frac{(m+o)! \cdot (m+o+1)! \cdot b!}{(m+o+1-b)!} \cdot \frac{1}{(m+o+b)! \cdot b!} \\ &= \frac{(m+o+1)!}{(m+o+1-b)!\, b!} \cdot \frac{(m+o)!\, b!}{(m+o+b)!} \\ &= \frac{\binom{m+o+1}{b}}{\binom{m+o+b}{b}}. \end{aligned}$$

Solution 2 to Question 2

Amit Itagi

This is the same as the distinguishable case other than the fact that the number of ways with and without the constraint both get scaled down by 3!4!5!—the number of permutations among the trees of the same kind. As a result, the ratio and the probability do not change.

Notes

Perhaps surprisingly, the two cases produce exactly the same result. No less surprisingly, the answer is a fraction of two binomial coefficients which are the staples of problems dealing with indistinguishable objects. As a matter of fact, the distinction between maples and oaks is a red herring [19]. The only quantity that matters is their total amount. However, the distinction between maples and oaks on one hand and birch trees on the other has to be accounted for in the solution.

There are a total of $\binom{m+o+b}{b}$ ways select "birches" among the whole bunch of the trees. The remaining trees create $m+o+1$ spaces to be filled individually by those selected.

The answer in both cases is $\frac{7}{99}$. Finding it with the specific number of each type of tree makes that fact more surprising than it deserves to be. The general (algebraic) solution sheds some light on the commonality of the two cases.

Six Numbers, Two Inequalities

Solution 1 to Question 1

A. Bogomolny

Unlike the earlier Riddle 1.9 on page 3, the two questions have different answers.

For both questions, the sample space counts 6! elements. The task is to count how many of these satisfy both inequalities. What is important is how many combinations of the numbers adjacent to the inequality symbols satisfy the inequalities. For the first question, we multiply that number by 3! to account for the remaining three digits. For the second question, the factor is 2!.

For the first question, if the first number is 1, there are 10 choices for the other two: $(2,3)$, $(2,4)$, $(2,5)$, $(2,6)$, $(3,4)$, $(3,5)$, $(3,6)$, $(4,5)$, $(4,6)$, $(5,5)$. Similarly, if the first number is 2, there are six variants for the other two numbers. Then there are three variants if the first number is 3 and only one if the first number is 4. In all, there are $(10+6+3+1)\cdot 3!$ combinations that satisfy both inequalities. The probability of this happening is $\frac{20\cdot 3!}{6!}=\frac{1}{6}$.

Solution 2 to Question 1

Leon Stein

For any triplet, there are six ways to put the numbers in three spots, but only one holds ordering. Thus, only $\frac{1}{6}$ of total arrangements are valid.

Solution 1 to Question 2

A. Bogomolny

For the second question, there are $\binom{6}{4}$ ways to fill the slots around the inequality symbols. For every choice of the four numbers, there are six ways to satisfy the two inequalities, e.g., $(1,2,3,4)$, $(1,3,2,4)$, $(1,4,2,3)$, $(2,3,1,4)$, $(2,4,1,3)$, $(3,4,1,2)$. Thus, the total number of "successful" combinations is $\binom{6}{4}\cdot 6\cdot 2$, giving the sought probability as $\frac{6\cdot 5\cdot 6\cdot 2}{2\cdot 6!}=\frac{1}{4}$.

Solution 2 to Question 2

Long Huynh Huu

Let us observe that the two inequalities are independent (Appendix A) because swapping the digits in one pair does not affect the other. Thus, borrowing from the logic of the first question, the answer is $\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$.

Six Numbers, Three Inequalities

The answers to the questions are $\frac{1}{24}$ and $\frac{5}{24}$, respectively.

Solution to Question 1

A. Bogomolny

For the first question, we choose four digits to place next to the inequality signs. These can be permuted in $4! = 24$ ways, of which only one satisfies all three inequalities. We conclude that the probability of this happening is $\frac{1}{24}$.

We can do a more complete counting: there are $\binom{6}{4} = \frac{6!}{2!4!}$ ways to choose four numbers out of six. The other two can be permuted in $2! = 2$ ways. Thus, the sought probability is $\frac{6!2!}{2!4!6!} = \frac{1}{4!}$, as before.

Solution to Question 2

A. Bogomolny

For the second question, observe that if four numbers u, v, w, x fit the inequalities in the first question, i.e., $u < v < w < x$, then

$$\begin{aligned} u < w > v < x \\ u < x > v < w \\ v < w > u < x \\ v < x > u < w \\ w < x > u < v. \end{aligned}$$

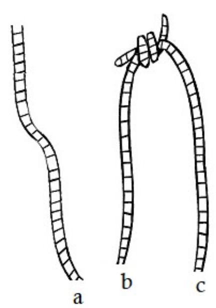
Thus, to every solution of the first question there are five corresponding solutions to the second. It follows that the probability of a random sequence satisfying the inequalities in the second question is $\frac{5}{24}$.

Tying Knots in Brazil

A. Bogomolny

Solution to Question 1

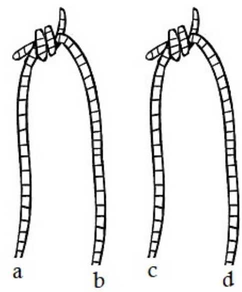
There are three ways to tie the loose ends below the fist: $a \leftrightarrow b$, $a \leftrightarrow c$, $b \leftrightarrow c$.



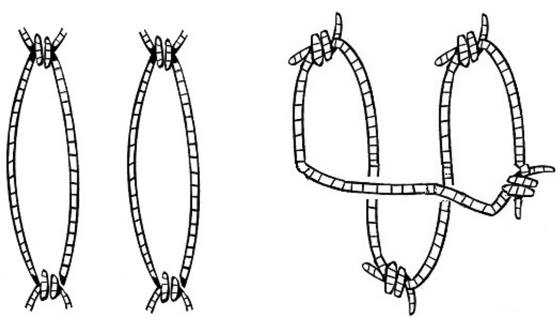
Two of them ($a \leftrightarrow b$, $a \leftrightarrow c$) lead to one long string so that the probability of that event is $\frac{2}{3}$.

Solution to Question 2

Given four string ends, when one knot is tied, the remaining two ends are left to be automatically tied. There is no choice.



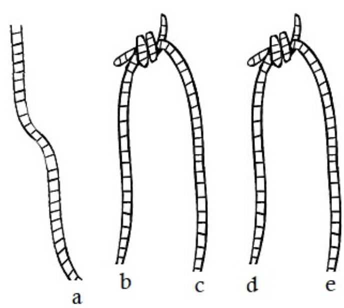
There are $\binom{4}{2} = 6$ ways to choose two strings out of two, but there is doubling of the count as any two pairs of strings are counted twice. All in all, there are just three distinct ways to tie the lower ends: $a \leftrightarrow b$ + $c \leftrightarrow d$, $a \leftrightarrow c$ + $b \leftrightarrow d$, $a \leftrightarrow d$ + $b \leftrightarrow c$.



One of these creates two small rings with two knots each whereas two yield a single ring with four knots. Thus, the probability of getting a long ring is $\frac{2}{3}$.

Solution to Question 3

Tying two knots above the fist leaves us with this configuration:



There are $\frac{1}{2}\binom{5}{2}\binom{3}{2} = 15$ ways to form two distinct pairs of knots. To obtain one long string, the end a needs to be tied. There are four ways to accomplish this. Joining it with either b, c, d or e will leave one end loose. Say, if a is tied to b, c needs to be tied next. There are two ways to accomplish this. Thus, in all, there are $4 \cdot 2 = 8$ ways to tie two knots so as to form one long string. Thus, the probability of this happening equals $\frac{8}{15}$.

Tying Knots in Russia

Solution 1 to Question 1

Six grass blade ends could be tied pairwise in $5 \cdot 3 \cdot 1$ different ways (the first end could be tied with any of the remaining five; the second, with any of the remaining three). This is true both for top and low ends, giving a total of 15×15 possible outcomes.

The question is how many of these have a favorable outcome, i.e., result in a single ring? Let us enumerate the blades 1, 2, 3, 4, 5, 6, and assume that, at the top, 1 is tied to 2, 3 to 4 and 5 to 6. At the bottom, 1 may be tied to 3, 4, 5 or 6; assume it is tied to 3. Then 2 may be tied to 5 or 6. In all, there are 2×4 different ways to tie the bottom ends for each of the 15 different knot combinations at the top, giving 15×8 combinations that produce a single ring. Thus, the probability of this happening is $\frac{120}{225} = \frac{8}{15}$.

Solution 2 to Question 1

From Riddle 6.20 we know that the ends of five strings are tied into one long string with the probability of $\frac{8}{15}$. Add an additional string to tie a knot at the top and another at the bottom to obtain a complete ring. This shows that there is a 1-1 correspondence between arrangements of five strings that result in one long string and the arrangement of six strings that results in a single ring.

Solution to Question 2

For $2n$ grass blades, the top strings can be tied in $(2n-1)!!$ ways, where

$$k! = k \cdot (k-2) \cdot (k-4) \dots X,$$

where $X = 1$ if k is odd and 2 if k is even. Thus, there is a total of $((2n-1)!!)^2$ possible knot arrangements. As in solution 1 to the first question, for each of $(2n-1)!!$ arrangements at the top, there are $(2n-2)!!$ knot arrangements at the bottom, giving the probability of getting a single ring as

$$P = \frac{(2n-1)!! \ \times \ (2n-2)!!}{((2n-1)!!)^2} = \frac{(2n-2)!!}{(2n-1)!!}.$$

Notes

The bride-to-be may be lucky to see a single loop formed by the grass blades, but, in general, there will be more loops than one. The expected number of such loops is discussed elsewhere, Riddle 8.22 on page 196.

Guessing Hat Numbers

Solution to Question 1

First, let us solve a more straightforward problem:

> Five students blindly choose a hat among six numbered $1, \ldots, 6$. Each announces the largest and the smallest number he/she sees. What is the probability that each will be able to deduce his/her own number?

As there are six hats and five students, exactly one of the hats is bound to be left over. The sample space for the problem consists of six elements, which could be the omitted elements or the sequences of numbers on the hats that the students are wearing.

As a general remark, if the selected hats bear numbers $a_1 < a_2 < a_3 < a_4 < a_5$, then the numbers a_1, a_2, a_4, a_5 will always be called out. (For example, a_4 will be called by the wearer of the hat with a_5.) The question is whether the number a_3 can be deduced with certainty. This will be the case when $a_4 - a_2 = 2$, i.e., when a_3 is tight between a_2 and a_4. This will happen if the omitted number is one of 1, 2, 5, 6; the ambiguity will occur when either 3 or 4 is left over.

Thus, every student will be able to deduce his/her number in four out of six cases, giving the probability of the correct deduction as $\frac{4}{6} = \frac{2}{3}$.

For the question where students need to guess their numbers, at least four of them will always be able to deduce theirs, i.e., to guess them with the probability of 1. In four out of six cases, the remaining person will also be able to deduce his/her number. In two out of six cases, the wearer of a_3 will have $1:1$ chance to make a correct guess. The total probability then comes to:

$$\frac{4}{6} \cdot 1 + \frac{2}{6} \cdot \frac{1}{2} = \frac{5}{6}.$$

Solution to Question 2

As in the solution to the first question, we first look at the case where all the students may or may not **deduce** the numbers on their hats. We consider $n \geq 6$ and, for k, several cases. Let P be the sought probability.

If $k = 1$, $P = 0$ since a single fellow has no way to deduce the number on his/her hat.

If $k = 2$, $P = 1$ because two students call out the other's number.

If $k = 3$ or $k = 4$, $P = 1$ because all numbers will be eventually called out, and each student may pick the number he/she does not see.

Therefore, we focus on $k > 4$. Let the numbers on the hats, in the increasing sequence, be

$$a_1 < a_2 < \cdots < a_{k-1} < a_k.$$

Observe that the numbers a_1 and a_k will be called out $k - 1$ times each. Numbers a_2 and a_{k-1} will be called out just once. If all k numbers are consecutive, i.e., if the above sequence has no gaps, then, as in the case $k = 3$ or $k = 4$ above, each of the students will be able to deduce his/her number. It will be the one he/she does not see.

If there are gaps, some of the students will not be able to deduce their numbers.

The gaps are absent if $a_{k-1} - a_2 = k - 3$ so that between a_2 and a_{k-1} there is enough room for exactly $k - 4$ numbers. For a_2 we must have $2 \leq a_2 \leq n - k + 2$; if $a_2 = i \geq 2$, then for a_1 there are $i - 1$ choices and for a_k, $n - k - i + 3$ choices. The total number of sequences that allow the students to surmise their hat number is

$$\begin{aligned}
&\sum_{i=2}^{n-k+2}(i-1)(n-k-i+3)\\
&\qquad=\sum_{i=2}^{n-k+2}(-i^2+(n-k+4)i-(n-k+3))\\
&\qquad=-\frac{(n-k+2)(n-k+3)(2n-2k+5)}{6}\\
&\qquad\quad+\frac{(n-k+4)(n-k+2)(n-k+3)}{2}-(n-k+2)(n-k+3)\\
&\qquad=(n-k+2)(n-k+3)\left[-\frac{1}{3}n+\frac{1}{3}k-\frac{5}{6}+\frac{1}{2}n-\frac{1}{2}k+2-1\right]\\
&\qquad=\frac{(n-k+3)(n-k+2)(n-k+1)}{6}=\binom{n-k+3}{3}.
\end{aligned}$$

It follows that the probability of all the students being able to deduce their hat number is

$$P=\frac{\binom{n-k+3}{3}}{\binom{n}{k}}.$$

Now let us turn to the "guessing" problem. The latest formula is the key. Let j be the total length of the gaps in the sequence $a_2<\cdots<a_{k-1}$. $1\le j\le n-k$. This is the same as the number of students who have no choice but to make a guess. For every j, there are

$$\binom{n-(k+j)+3}{3}\binom{k-4+j}{j}$$

such sequences, making the total equal to

$$\sum_{j=1}^{n-k}\binom{n-(k+j)+3}{3}\binom{k-4+j}{j}.$$

In itself, this sum does not make much sense because, for different j, the probabilities of guessing right differ; in every case they are $(j+1)^{-j}$. However, the total can be verified manually. For example, as we saw, for $n=6$ and $k=5$, the only possible value for j is 1, so we have

$$\binom{n-(k+j)+3}{3}\binom{k-4+j}{j}=\binom{3}{3}\binom{2}{1}=2,$$

as expected. For $n=7$ and $k=5$, there are two possibilities: $j=1$ and $j=2$. The formula gives

$$\binom{4}{3}\binom{2}{1}+\binom{3}{3}\binom{3}{2}=4\cdot 2+1\cdot 3=11.$$

Here are all 11 cases (an asterisk denotes the omitted numbers that ought to be guessed):

$$\begin{array}{ccccccccc}
1 & 2 & 3^* & * & * & 6 & 7 & & (j=2) \\
1 & 2 & * & 4^* & * & 6 & 7 & & (j=2) \\
1 & 2 & * & * & 5^* & 6 & 7 & & (j=2) \\
1 & - & 3 & 4^* & * & 6 & 7 & & (j=1) \\
1 & - & 3 & * & 5^* & 6 & 7 & & (j=1) \\
- & 2 & 3 & 4^* & * & 6 & 7 & & (j=1) \\
- & 2 & 3 & * & 5^* & 6 & 7 & & (j=1) \\
1 & 2 & 3^* & * & 5 & - & 7 & & (j=1) \\
1 & 2 & * & 4^* & 5 & - & 7 & & (j=1) \\
1 & 2 & 3^* & * & 5 & 6 & - & & (j=1) \\
1 & 2 & * & 4^* & 5 & 6 & - & & (j=1).
\end{array}$$

The general formula for the probability of all students being able to guess correctly appears to be

$$P = \frac{\binom{n-k+3}{3} + \sum_{j=1}^{n-k} (j+1)^{-j} \binom{n-(k+j)+3}{3} \binom{k-4+j}{j}}{\binom{n}{k}}.$$

Probability of an Odd Number of Sixes

If $0 \le k \le n$, the probability of k sixes turning up in a random toss of n fair dice is

$$\binom{n}{k} \left(\frac{5}{6}\right)^{n-k} \left(\frac{1}{6}\right)^{k}.$$

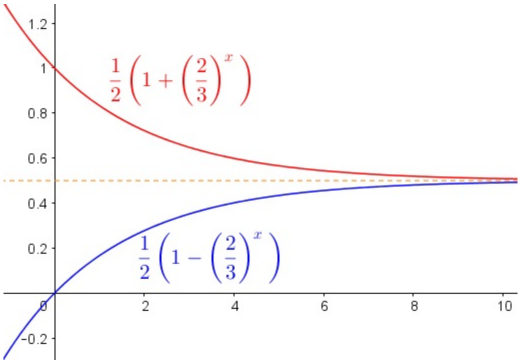
Hence, with $a = \frac{5}{6}$ and $b = \frac{1}{6}$, the required probability is

$$\begin{aligned}
P &= \sum_{k \text{ is odd}} \binom{n}{k} \left(\frac{5}{6}\right)^{n-k} \left(\frac{1}{6}\right)^{k} \\
&= \text{sum of every second term in the expansion of } (a+b)^n \\
&= \frac{1}{2}\left[(a+b)^n - (a-b)^n\right] \\
&= \frac{1}{2}\left[1 - \left(\frac{2}{3}\right)^n\right].
\end{aligned}$$

In a similar manner, the probability of an even number of sixes equals

$$\begin{aligned}
Q &= \frac{1}{2}\left[(a+b)^n + (a-b)^n\right] \\
&= \frac{1}{2}\left[1 + \left(\frac{2}{3}\right)^n\right].
\end{aligned}$$

Naturally, $P+Q=1$. Why is $Q>P$? The graphics below may be suggestive.



Zero is an even number always lurking in the background.

Probability of Average

Solution 1

Vrund Shah

To have a whole average, the largest and the smallest numbers of the three drawn need to have the same parity. If they do, the third number is determined uniquely.

There are $\binom{10}{2}=45$ pairs of even numbers in the set $1,2,\ldots,20$ and as many pairs of odd numbers. This means there are exactly 90 ways to choose three numbers out of 20 so that one of them is the average of the other two. Altogether, there are $\binom{20}{3}$ ways to choose three numbers, all equiprobable, giving the sought probability as

$$P=\frac{\binom{10}{2}+\binom{10}{2}}{\binom{20}{3}}=\frac{90}{1140}=\frac{3}{38}.$$

Solution 2

Hélvio Vairinhos

For each ball $i=2,\ldots,10$, the number of pairs which have i as their average is $i-1$, which totals $\dfrac{9\cdot 10}{2}=45$. The same is true for $i=11,\ldots,19$, due to symmetry: $i\mapsto 20-i$. Hence, the total number of choices is $45\cdot 2=90$, out of $\binom{20}{3}$.

Solution 3

Amit Itagi

Let the smallest number chosen be k. To satisfy the average condition, the other two balls need to be numbered $k+d$ and $k+2d$ for some positive integer d. Moreover,

$k + 2d \le 20$. Thus, if k is odd, $d \in \{1, 2, \ldots, 10 - \frac{k}{2}\}$; if k is even, $d \in \{1, 2, \ldots, 10 - \frac{k+1}{2}\}$. Thus, the required probability is

$$\begin{aligned}
P &= \frac{1}{C(20,3)} \left[\sum_{k\in\{1,3,5,\ldots,19\}} \left(10 - \frac{k+1}{2}\right) + \sum_{k\in\{2,4,6,\ldots,20\}} \left(10 - \frac{k}{2}\right) \right] \\
&= \frac{2}{C(20,3)} \sum_{m=1}^{10} (10 - m) \\
&= \frac{2 \cdot 6 \cdot 45}{20 \cdot 19 \cdot 18} = \frac{3}{38} \approx 0.079.
\end{aligned}$$

Solution 4

Joshua B. Miller

The number of overlapping triples $(i - 2k, i - k, i)$

1. for $k = 1, \ldots, 9$
2. for $k = 1$, $i \in [3, 20]$, so the number is 18
3. for $k = 2$, $i \in [5, 20]$, so the number is 16
4. … and so on
5. for $k = 9$, $i \in [19, 20]$, so the number is 2.

The number of overlapping triples is $2 \cdot \frac{9 \cdot 10}{2} = 90$ while the total number possible is $\binom{20}{3}$.

Marking and Breaking Sticks

For the case of equal pieces, one of the marks ought to be located on one of the pieces. The second mark is located on the same piece with the probability of $\frac{1}{n}$.

For the random lengths, imagine that first $n - 1$ marks are placed at the break points, making the total number of marks $n + 1$. There are $\binom{n+1}{2}$ ways to pick two marks out of $n+1$. Of interest are those in which two original marks follow each other, with no "break" marks in between. In other words, if the marks are numbered from 1 through $n + 1$, we are interested in the distributions where the original marks bear successive numbers: 1 and 2, or 2 and 3, …, or n and $n + 1$. There are n such cases (out of $\binom{n+1}{2}$), implying that the sought probability is

$$\frac{n}{\binom{n+1}{2}} = \frac{n \cdot 2!(n-1)!}{(n+1)!} = \frac{2}{n+1}.$$

This shows a rather remarkable increase compared to the case of equal pieces.

Bubbling of Sorts

Solution 1 to Question 1

In order not to be pulled down, a number needs to be larger than the ones preceding it; to be drawn up, it also needs to be larger than some following it. To get into position #30, a number needs to be larger than the previous 29 terms while smaller than term #31. If we denote the two numbers r_{30} and r_{31}, then this means that r_{31} is the largest of the 31 numbers and r_{30} is the next largest. There are 31! permutations of 31 numbers and 29! permutations that fix two numbers out of 31. The probability is thus

$$P = \frac{29!}{31!} = \frac{1}{30 \cdot 31} = \frac{1}{930}.$$

Solution 2 to Question 1

To get into position #30, a number needs to be larger than the previous 29 terms while smaller than term #31, but nothing makes r_{20} especially remarkable. The largest of the first 30 numbers could have been in any position, not necessarily position #20. Thus, the probability that r_{20} is the largest among the first 30 numbers is $\frac{1}{30}$. Similarly, The number in position #31 that stopped the advancement of r_{20} is the largest among the first 31, while the probability of the largest number among 31 to be in the position #31 is $\frac{1}{31}$. The two events being independent, the probability of both happening is $\frac{1}{30 \cdot 31} = \frac{1}{930}$.

Solution to Question 2

On the first pass, as the advancement of r_{20} was stopped by r_{31}, the latter could have been up on the next step so as to bring into position #31 a number smaller than r_{20} (the one that is in position #30 after the first pass). For this not to happen, the number r_{32} needs to be larger than r_{20}. Thus, the sample space consists of 32! sequences in which one position is fixed and two are interchangeable, making the probability for the second question

$$P = \frac{29! \cdot 2}{32!} = \frac{2}{30 \cdot 31 \cdot 32} = \frac{1}{14880}.$$

Solution to Question 3

Christopher D. Long

For n bubble passes, r_{20}, in order to stay in position #30, needs protection of n numbers, each larger than r_{20}. These are located in positions $31, 32, \ldots, 30 + n$. Their order is inconsequential. This gives the probability $P = \frac{29! \cdot n!}{(30+n)!}$.

After 10 passes (or less) the terms in positions 31–40 will be correctly ordered. If the same is true (which is not assured) for positions 1–29, the algorithm will stop. If it does not, it will continue to reorder the terms in those lower positions without affecting r_{20} in position #30.

Notes

Antonio Catalano and later Mathew Crawford have interpreted the second question as independent from the first:

> Assume the first question was never asked. Then what is the probability that the number that begins as r_{20} will end up in position #30 after two bubble passes?
>
> We might as well ignore positions 33–40. Relabel 1–32, for convenience, with 30 in spot 20 and either 31 or 32 in spot 32. Bubbling small numbers backward from the end helps perspective. The answer is $\dfrac{2}{32 \cdot 31}$.

Probability of Successive Integers

Solution 1

Josh Jordan

The answer is $1 - \dfrac{\binom{86}{5}}{\binom{90}{5}} \approx 0.2076.$

The tricky part is counting the number of ways to choose five non-adjacent numbers from 10 through 99. Below I show that this is $\binom{86}{5}$ using a 1-1 correspondence involving adding/removing padding to bit-strings.

For simplicity, it is helpful to first subtract 9 from each of the numbers, so they range from 1 through 90.

Choose five non-adjacent numbers from 1 through 86. Write them as a bit string in which there is a 1 in position N just when N was chosen.

For example, the choice $\{2, 3, 6, 8, 86\}$ would look like this:

$$\overbrace{0110010100\ldots001}^{\text{(86 bits)}}.$$

Now add a zero between each pair of ones: that will be four additional zeros. The string becomes the following:

$$\overbrace{01010001001000\ldots001}^{\text{(90 bits)}}.$$

The second string represents five numbers chosen from 1 through 90 with no adjacent numbers. To convert any such string to a selection of numbers from 1 through 86, remove a zero between each pair of ones.

Solution 2

Amit Itagi

First, consider the event that no two chosen numbers are adjacent. Let the five chosen numbers sorted in increasing order be a_j $(j = 1, 2, \ldots, 5)$. Define five groups of consecutive numbers: $G_1 = (a_1, \ldots, a_2 - 1)$, $G_2 = (a_2, \ldots, a_3 - 1)$, …, $G_4 = (a_4, \ldots, a_5 - 1)$ and $G_5 = (a_5, \ldots, 99)$. Note, numbers less that a_1 do not fall in any of the five groups.

Lump those numbers in a group G_0. For no two of the five numbers to be adjacent, the groups G_1–G_4 need to have at least two elements. G_5 will have at least one element, as $a_5 \le 99$, and G_0 can have no element. If N_i is the number of elements in group G_i, the number of ways of choosing the numbers under the constraint is the number of solutions of $N_0 + N_1 + N_2 + \cdots + N_5 = 90$, such that $N_i \ge 2$ for $0 < i < 5$, $N_5 \ge 1$ and $N_0 \ge 0$. This is equivalent to the number of solutions of $N_0+N_1+N_2+\cdots+N_5 = 90-4+1 = 87$ under the modified constraint $N_i \ge 1$. The number of solutions is $\binom{87-1}{6-1} = \binom{86}{5}$.

Thus, the required probability is $1 - \dfrac{\binom{86}{5}}{\binom{90}{5}} \approx 0.207$.

Solution 3

Stephen Morris

Count the choices that fail. These all start with even numbers. Starting with the highest and working down, randomly decide whether to increment it and all higher numbers. If the highest is 96 or 98 we can choose to increment max 3 or 1 times:

$$1 - \frac{\binom{43}{5} \times 2^5 + \binom{43}{4} \times 26 + \binom{44}{4} \times 6}{\binom{90}{5}}.$$

The exact result is $\dfrac{106081}{511038}$.

Solution 4

Christopher D. Long

We can use an urn model. Let $x_2, \ldots, x_5 \ge 1$, $x_1 \ge 9$, $x_6 \ge 0$ and $x_1 + \cdots + x_5 + x_6 = 99 - 5 = 94$. Our numbers y_i for $i = 1, 2, 3, 4, 5$ equal $y_i = x_1 + \cdots + x_i + i$. The number of ways to assign these values is $\binom{94-4-9+5}{5} = \binom{86}{5}$; hence $p = 1 - \dfrac{\binom{86}{5}}{\binom{90}{5}}$.

Chapter 7

Conditional Probability

How dare we speak of the laws of chance? Is not chance the antithesis of all law?

Joseph Bertrand, 1889, *Calcul des Probabilités*

Riddles

7.1 Two Liars

[36, Problem 2.4.9]

A and B tell the truth with the probability of $\frac{1}{3}$ and lie with the probability of $\frac{2}{3}$.

A makes a statement, and B confirms that the statement made by A is true.

What is the probability that A is actually telling the truth?

7.2 Three Liars

[36, Problem 2.4.10]

A, B and C tell the truth with the probability of $\frac{1}{3}$ and lie with the probability of $\frac{2}{3}$.

A makes a statement and B makes an observation about whether A's statement is true or not. C tells us that B confirms that A is telling the truth.

What is the probability that A is actually telling the truth?

7.3 Taking Turns to Toss a Die

[71, pp. 59–61]

Three players—A, B, C—take turns tossing a fair die, following the sequence $ABCABCABC\ldots$. This continues until one of them comes up with a 6 on top. That player drops out of the game while the other two continue to take turns until one of them tosses a 6, at which point the game ends.

- What are the probabilities for each of A, B, C to be the first to exit the game?
- What are the probabilities of the exit sequences AB, BA and CB?

7.4 Two Coins: One Fair, One Biased

[46, Problem 1989–4]

Two coins are given. One is fair: $P(\text{heads}) = \frac{1}{2}$. The other coin is biased: $P(\text{heads}) = \frac{2}{3}$. One of the coins is tossed once, resulting in heads. The other coin is tossed three times, resulting in two heads.

Which coin is more likely to be the biased coin, the first or the second?

7.5 Probability of the Second Marble

[65, Problem 318]

A bag contains marbles of two colors, black and white; all combinations of colors (including all marbles being of the same color) are equiprobable. You randomly (blindly) pick a marble from the bag. It is black.

What is the probability that if you picked a second marble at random it would also be black?

7.6 Quotient Estimates I

[71]

Two numbers X and Y are randomly chosen from the interval $[0,1]$. Let r denote the quotient of the larger of the two by the smaller: $r = \dfrac{\max\{X,Y\}}{\min\{X,Y\}}$. Depending on the assumptions, there are three questions. For a given $k > 0$, what is the probability that $r \geq k$

1. when X is uniformly distributed on $[0,1]$ and Y is uniformly distributed on $[X,1]$?
2. when X, Y are jointly uniformly distributed on $\{(x,y) : 0 \leq x \leq y \leq 1\}$?
3. when X and Y are uniformly and independently distributed on $[0,1]$?

7.7 Quotient Estimates II

[71]

$n \geq 2$ random variables X_i, $i = 1, 2, \ldots, n$, are each uniformly distributed on the interval $[0,1]$. If $r = \dfrac{\max\limits_{1\leq i\leq n} \{X_i\}}{\min\limits_{1\leq i\leq n} \{X_i\}}$, what is the probability that $r \geq k$ for a given $k \geq 1$?

7.8 The Lost Boarding Pass

On a sold out flight, 100 people line up to board the plane. The first passenger in the line has lost his boarding pass but is allowed in, regardless. He takes a random seat. Each subsequent passenger takes his or her assigned seat if available or a random, unoccupied seat otherwise. What is the probability that the last passenger to board the plane finds his seat unoccupied?

7.9 Lucky Contest Winners

[89]

Students at your school have just finished competing in the qualifying round of a nationally sponsored contest on mathematical reasoning and sense making. When the work is scored, it turns out that four students at your school all have perfect preliminary papers—two girls and two boys. The school decides to hold a random drawing among these four students to select two of them to send to the national finals. The drawing takes place in the school auditorium. You show up late to the drawing, just as one of the winners—a girl—is leaving the stage amid cheers.

1. Suppose that the girl you saw leaving the stage is the first winner. What is the probability that the second winner will also be a girl?

2. Suppose that the girl you saw leaving the stage is the second winner. What is the probability that the first winner was also a girl?

3. You pass a door to the auditorium, peek in, see a girl descending the stage but, being pressed for time, close the door and continue on your errand. You do not know whether the girl you saw leaving the stage is the first or the second winner. What is the probability that the other winner is also a girl?

4. Having missed the whole ceremony, you eventually call the principal and inquire whether there was a girl among the winners. The principal responds in the affirmative. What is the probability that both girls were selected?

7.10 Diminishing Hopes

[36, Problem 2, 4, 13a]

A fellow's desk sports eight drawers where he randomly (but with equal probabilities) stores his documents. This would be the whole condition of the riddle except that in two out of 10 cases the fellow simply forgets to store a document, and eventually the latter gets lost.

When the fellow needs a document, he starts a search from the first drawer and proceeds sequentially until the document is found or it becomes clear (after checking all the drawers) that the document has not been stored in the desk in the first place.

There are several questions:

1. The fellow checks and finds no documents in the first drawer. What is the probability that the document will be found in the remaining seven drawers?

2. The fellow checks and finds no documents in the first four drawers. What is the probability that the document will be found in the remaining four drawers?

3. The fellow checks and finds no documents in the first seven drawers. What is the probability that the document will be found in the last remaining drawer?

7.11 Incidence of Breast Cancer

[81]

In a study, physicians were asked what the odds of breast cancer would be in a woman who was initially thought to have a 1% risk of cancer but who ended up with a positive mammogram result (a mammogram accurately classifies about 80% of cancerous tumors and 90% of benign tumors). Ninety-five out of a hundred physicians estimated the probability of cancer to be about 75%. Do you agree?

7.12 A Search for Heads and Its Consequences

[67]

Jack flips a coin three times and writes down the outcomes. Next, Jack circles one of those outcomes that is immediately preceded by heads. What is the probability that the marked flip is a heads, conditional on

1. its being the second flip?

2. its being the third flip?

3. there being no additional information on the selection?

7.13 A Three Group Split

[23]

If the integers 1 to 9 are randomly distributed into three sets of three integers, what is the probability that at least one of the sets contains only odd integers?

7.14 Lewis Carroll's Pillow Problem

[43, pp. 188–189], [41, pp. 129–132]

A bag contains a counter, known to be either white or black. A white counter is put in, the bag is shaken and a counter is drawn out, which proves to be white. What is the chance of drawing a white counter now?

7.15 A Follow Up on Lewis Carroll's Pillow Problem

This problem was inspired by Chris Conradi's letter.

Suppose a counter is drawn at random from a bag containing b black and w white counters. Without revealing its color, that counter is placed in a second, empty bag. Then, a white counter is added to this bag. Now, you draw a counter from this second bag and it turns out to be white. What is the probability that the remaining counter is white?

7.16 Sick Child and Doctor

[36, p. 48]

A doctor is called to see a sick child. The doctor has prior information that 90% of sick children in that neighborhood have the flu, while the other 10% are sick with measles. Let F stand for an event of a child being sick with the flu and M stand for an event of a child being sick with measles. Assume for simplicity that $F \cup M = \Omega$, i.e., that there are no other maladies in that neighborhood.

A well-known symptom of measles is a rash (the event of having which we denote R). $P(R|M) = .95$. However, occasionally children with flu also develop a rash so that $P(R|F) = 0.08$.

Upon examining the child, the doctor finds a rash. What is the probability that the child has measles?

7.17 Right Strategy for a Weaker Player

[65, Problem 316]

A and B are to play a two game chess match. If they are tied after two games, they will continue until the first win. A is the weaker of the two players. With a daring game, A's probability of winning is 45% and of losing is 55%; with a conservative game, A's probability of a draw is 90% and of losing is 10%. Which of A's strategies could lead to his winning the match? With what probability?

7.18 Chickens in Boxes

[71]

Four boxes are arranged in an L shape, Figure 7.1.

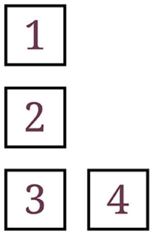


Figure 7.1: Four boxes.

Each box has only enough room for a single chicken. Each of the boxes has the probability of $\frac{1}{2}$ to receive a chicken, subject to the requirement that exactly one of the vertical boxes (1, 2 or 3) contains a chicken and that exactly one of the horizontal boxes (3 or 4) contains a chicken.

1. What are the chances that the corner box contains a chicken?

2. Which of the boxes is most likely to contain a chicken?

7.19 Two Chickens in Boxes

[71]

Two chickens are randomly placed into four boxes that are arranged in an L shape, Figure 7.1.

Each box has only enough room for a single chicken. Each of the boxes has the probability of $\frac{1}{2}$ to receive a chicken, subject to the requirement that at least one of the vertical boxes (1, 2 or 3) contains a chicken and that at least one of the horizontal boxes (3 or 4) contains a chicken.

1. What are the chances that the corner box contains a chicken?

2. Which of the boxes is most likely to contain a chicken?

7.20 Two Chickens in Bigger Boxes

[71]

Two chickens are randomly placed into four boxes that are arranged in an L shape, Figure 7.1.

Each box has enough room for two chickens. Chicken placement is subject to the requirement that there be at least one nonempty vertical box (1, 2 or 3) and that there be at least one nonempty horizontal box (3 or 4).

1. What are the chances that the corner box contains a chicken?

2. Which of the boxes is most likely to contain a chicken?

Solutions

Two Liars

Solution 1

A. Bogomolny

Both A and B tell the truth in one of three cases and lie in two of three cases. The three cases for A and B can be visualized as a 3×3 square.

B could claim that A's statement is true if both were telling the truth or if both lied. There are five red squares that correspond to these conditions. In only one of these (the upper left corner) was A actually telling the truth (Figure 7.2). This happens in one of five cases, i.e., with the probability of $\frac{1}{5}$.

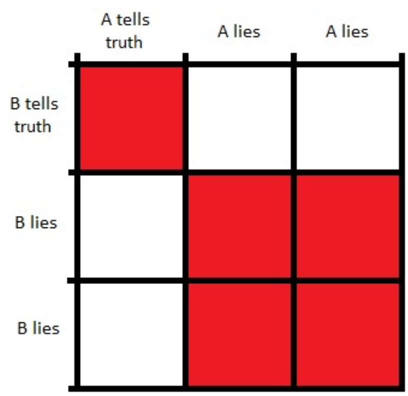


Figure 7.2: Two liars.

Solution 2

Christian Adib

Let E be the event that A makes a statement and B confirms. This can happen in two ways:

- T, both A and B are telling the truth;
- L, both A and B are lying.

$$P(E) = P(T) + P(L) = \frac{1}{3} \cdot \frac{1}{3} + \frac{2}{3} \cdot \frac{2}{3} = \frac{5}{9}.$$

Given that we observe E, we are interested in finding the probability that the underlying process is in fact T, that is, $P(T|E)$. From the formula for conditional probabilities, $P(T|E) = \dfrac{P(T \cap E)}{P(E)}$ and $P(T \cap E) = P(T)$ because $T \subset E$. In conclusion,

$$P(T|E) = \frac{P(T)}{P(E)} = \frac{\frac{1}{9}}{\frac{5}{9}} = \frac{1}{5}.$$

Three Liars

Solution 1
A. Bogomolny

The tree shown in Figure 7.3 represents all possible cases wherein C tells us that B confirmed A's statement.

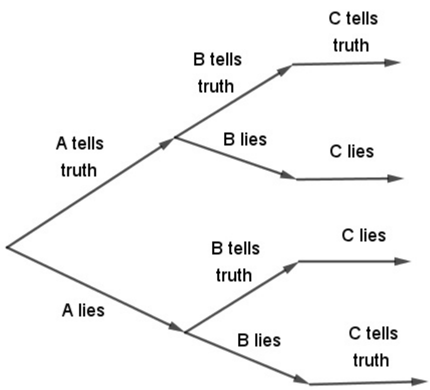


Figure 7.3: Three liars.

Counting the probabilities gives the total,

$$P = \frac{1}{3} \cdot \left(\frac{1}{3} \cdot \frac{1}{3} + \frac{2}{3} \cdot \frac{2}{3}\right) + \frac{2}{3} \cdot \left(\frac{1}{3} \cdot \frac{2}{3} + \frac{2}{3} \cdot \frac{1}{3}\right)$$
$$= \frac{13}{27}.$$

This corresponds to 13 cases out of 27 wherein C tells us that B confirmed A's statement. The first addend in the equation above corresponds to $1 \cdot 1 + 2 \cdot 2 = 5$ cases wherein A's statement was indeed true. Therefore, with the information at hand, we may conclude that the probability that A was telling the truth is $\dfrac{\frac{5}{27}}{\frac{13}{27}} = \dfrac{5}{13}$.

Solution 2
Amit Itagi

Let $t = \dfrac{1}{3}$ and $f = \dfrac{2}{3}$.

A Speaks Truth	B Speaks Truth	C Speaks Truth	C Can Make the Statement	Probability
0	0	0	0	f^3
0	0	1	1	tf^2
0	1	0	1	tf^2
0	1	1	0	t^2f
1	0	0	1	tf^2
1	0	1	0	t^2f
1	1	0	0	t^2f
1	1	1	1	t^3

$$\begin{aligned}
&P(A \text{ speaks truth}|C \text{ can make the statement})\\
&\qquad = \frac{P(A \text{ speaks truth} \cap C \text{ can make the statement})}{P(C \text{ can make the statement})}\\
&\qquad = \frac{tf^2+t^3}{3tf^2+t^3} = \frac{t^2+f^2}{t^2+3f^2} = \frac{1^2+2^2}{1^2+3\cdot 2^2} = \frac{5}{13}.
\end{aligned}$$

If the properties in the first four columns of the above truth table are labeled by P_k, where k is the column number, then the above truth table gives P_4 as

$$\overline{P_1}.\overline{P_2}.P_3 + \overline{P_2}.\overline{P_3}.P_1 + \overline{P_3}.\overline{P_1}.P_2 + P_1.P_2.P_3$$

using logic gates or as

$$\begin{aligned}
&((\neg P_1) \wedge (\neg P_2) \wedge P_3) \vee ((\neg P_2) \wedge (\neg P_3) \wedge P_1) \vee\\
&\qquad ((\neg P_3) \wedge (\neg P_1) \wedge P_2) \vee (P_1 \wedge P_2 \wedge P_3)
\end{aligned}$$

using propositional logic.

Taking Turns to Toss a Die

Solution 1

Let P_A, P_B, P_C be the probabilities of A, B, C, respectively, being the first to exit the game. Concerning P_A, when it is A's turn, he either tosses 6 and the game stops or all three toss anything but 6 and the game continues:

$$P_A = \frac{1}{6} + \left(\frac{5}{6}\right)^3 P_A,$$

from which $P_A = \dfrac{36}{91}$. For B to be the first to exit the game, A needs to miss 6 followed by B tossing 6. If this does not happen and they all (in the BAC sequence) miss 6, the game continues:

$$P_B = \frac{5}{6} \cdot \frac{1}{6} + \left(\frac{5}{6}\right)^3 P_B,$$

from which $P_B = \dfrac{30}{91}$. For P_C, we have

$$P_C = \left(\frac{5}{6}\right)^2 \cdot \frac{1}{6} + \left(\frac{5}{6}\right)^3 P_C,$$

which resolves to $P_C = \dfrac{25}{91}$. As a confirmation, $P_A + P_B + P_C = 1$.

With A out of the game, the sequence of throws becomes $BCBC\ldots$. If p_1 is the probability of B being the first to toss a 6 then, as before,

$$P_1 = \frac{1}{6} + \left(\frac{5}{6}\right)^2 P_1,$$

yielding $p_1 = \dfrac{6}{11}$. Obviously for C to get ahead of B the probability $p_2 = \dfrac{5}{11}$. These probabilities remain valid for a two person sequence, regardless of their identities. Observe that when, e.g., B exits first, the game proceeds in the sequence $CACA\ldots$. Thus,

$$\begin{aligned}
P_{AB} &= P_A \cdot p_1 = \frac{36}{91} \cdot \frac{6}{11} = \frac{216}{1001} \\
P_{BA} &= P_B \cdot p_2 = \frac{30}{91} \cdot \frac{5}{11} = \frac{150}{1001} \\
P_{CA} &= P_C \cdot p_1 = \frac{25}{91} \cdot \frac{6}{11} = \frac{150}{1001} \\
P_{CB} &= P_A \cdot p_2 = \frac{25}{91} \cdot \frac{5}{11} = \frac{125}{1001} \\
P_{AC} &= P_A \cdot p_2 = \frac{36}{91} \cdot \frac{5}{11} = \frac{180}{1001} \\
P_{BC} &= P_A \cdot p_1 = \frac{30}{91} \cdot \frac{6}{11} = \frac{180}{1001}.
\end{aligned}$$

It can easily be checked that all six add up to 1, as they should. Perhaps surprisingly $P_{BA} = P_{CA}$ and $P_{AC} = P_{BC}$.

Solution 2

Amit Itagi

Let A_k be the event that A is the k^{th} person to exit (with a similar notation for B and C):

$$P(A_1) = \frac{1}{6}\left[1 + \left(\frac{5}{6}\right)^3 + \left(\frac{5}{6}\right)^6 + \cdots\right] = \frac{\frac{1}{6}}{1 - \left(\frac{5}{6}\right)^3} = \frac{36}{91},$$

$$P(B_1) = \left(\frac{5}{6}\right) P(A_1) = \frac{30}{91},$$

$$P(C_1) = \left(\frac{5}{6}\right) P(B_1) = \frac{25}{91}.$$

$$\begin{aligned}
P(B_2|A_1) = P(C_2|B_1) = P(A_2|C_1) \\
&= \frac{1}{6}\left[1 + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^4 + \cdots\right] \\
&= \frac{\frac{1}{6}}{1 - \left(\frac{5}{6}\right)^2} = \frac{6}{11},
\end{aligned}$$

$$P(C_2|A_1) = P(A_2|B_1) = P(B_2|C_1) = 1 - P(B_2|A_1) = \frac{5}{11}.$$

Thus,

$$P(A_1 \cap B_2) = P(B_2|A_1)P(A_1) = \left(\frac{6}{11}\right)\left(\frac{36}{91}\right) = \frac{216}{1001},$$

$$P(A_1 \cap C_2) = P(C_2|A_1)P(A_1) = \left(\frac{5}{11}\right)\left(\frac{36}{91}\right) = \frac{180}{1001},$$

$$P(B_1 \cap C_2) = P(C_2|B_1)P(B_1) = \left(\frac{6}{11}\right)\left(\frac{30}{91}\right) = \frac{180}{1001},$$

$$P(B_1 \cap A_2) = P(A_2|B_1)P(B_1) = \left(\frac{5}{11}\right)\left(\frac{30}{91}\right) = \frac{150}{1001},$$

$$P(C_1 \cap A_2) = P(A_2|C_1)P(C_1) = \left(\frac{6}{11}\right)\left(\frac{25}{91}\right) = \frac{150}{1001},$$

$$P(C_1 \cap B_2) = P(B_2|C_1)P(C_1) = \left(\frac{5}{11}\right)\left(\frac{25}{91}\right) = \frac{125}{1001}.$$

Solution 3

N.N. Taleb

The probability of the event occurring at period i is $p(1-p)^{i-1}$. Let $P(.)$ be the probability that player A, B or C exits first.

Skipping three for each, we get

$$P(A) = \sum_{i=0}^{\infty} p(1-p)^{3i} = \frac{1}{(p-3)p+3} = \frac{36}{91},$$

$$P(B) = \sum_{i=0}^{\infty} p(1-p)^{3i+1} = \frac{1-p}{(p-3)p+3} = \frac{30}{91},$$

$$P(C) = \sum_{i=0}^{\infty} p(1-p)^{3i+2} = \frac{(p-1)^2}{(p-3)p+3} = \frac{25}{91}.$$

We note that events A, B and C are separable and exhaustive; hence, the probabilities sum to unity. This will help in what follows because every combination starting with A, B or C will entail $P(A)$, $P(B)$ or $P(C)$, respectively.

Now we condition AB. Since we have the sequence $BCBCBCBC\ldots$ (after A) and need to skip two for each, starting with p

$$P(AB) = P(A)\sum_{i=0}^{\infty} p(1-p)^{2i} = P(A)\frac{1}{2-p} = \frac{216}{1001}.$$

We also condition BA. Since we have the subsequent sequence $ACACACAC\ldots$ (after B), we need to skip two for each, but starting with $p(1-p)$

$$P(BA) = P(B)\sum_{i=0}^{\infty} p(1-p)^{2i+1} = P(B)\frac{1-p}{2-p} = \frac{150}{1001}.$$

As for CB, we have the subsequent sequence $ABABABABABAB\ldots$ (after C):

$$P(CB) = P(C)\sum_{i=0}^{\infty} p(1-p)^{2i+1} = P(B)\frac{1-p}{2-p} = \frac{125}{1001}.$$

Two Coins: One Fair, One Biased

Solution 1

Assuming the fair coin is the one that was tossed, the probability of it showing heads and of the other coin showing two heads is given by

$$\frac{1}{2}\cdot\binom{3}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^1 = \frac{2}{9}.$$

On the other hand, if the biased coin was tossed once then the probability of the observed result is

$$\frac{2}{3}\cdot\binom{3}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^1 = \frac{2}{8},$$

telling us that, more likely, it was the biased coin that was tossed once.

Solution 2

Amit Itagi

Without any *a priori* knowledge, let us assume that the first coin is equally likely to be either of the two coins. Let D denote the observation and B the event that the first coin is biased. Thus, $P(B) = P(\overline{B}) = \frac{1}{2}$,

$$P(D|B) = \left[\frac{2}{3}\right]\cdot\left[C(3,2)\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)\right] = \frac{1}{4},$$

$$P(D|\overline{B}) = \left[\frac{1}{2}\right]\cdot\left[C(3,2)\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)\right] = \frac{2}{9}.$$

Using the Bayes theorem,

$$\begin{aligned}P(B|D) &= \frac{P(D|B)P(B)}{P(D|B)P(B)+P(D|\overline{B})P(\overline{B})}\\ &= \frac{\frac{1}{4}}{\frac{1}{4}+\frac{2}{9}} = \frac{9}{17} > \frac{1}{2}.\end{aligned}$$

Thus, the observation makes the first coin more likely to be biased.

Solution 3

Joshua B. Miller

Prior (relative) odds are even, while the likelihood ratio is

$$\begin{aligned}LR &= \frac{\Pr(E|B,F)}{\Pr(E|F,B)} = \frac{\frac{2}{3}\cdot\left(\frac{1}{2}\right)^3}{\frac{1}{2}\cdot\left(\frac{2}{3}\right)^2\cdot\frac{1}{3}} = \frac{\frac{2}{3}\cdot\frac{1}{8}}{\frac{1}{2}\cdot\frac{4}{27}}\\ &= \frac{9}{8}.\end{aligned}$$

Odds in favor of the first coin being biased and the second coin being fair are $9:8$, i.e., a probability of $\frac{9}{17}$.

Probability of the Second Marble

Solution 1

A. Bogomolny

Assume the bag at hand contains $n>0$ marbles. There are $n+1$ possibilities for the number of black marbles, ranging from 0 to n; all $n+1$ are said to be equiprobable. Let N_k denote the event that there were k black marbles to start with. Let B_1 denote the event that the first drawn marble was black; B_2 is the event that the second marble was black.

Thus, the probability that the bag at hand carries i black marbles is $P(N_i)=\frac{1}{n+1}$ for any $i=0,\ldots,n$, and the probability that a randomly drawn marble from that bag is black is $P(B_1|N_i)=\frac{i}{n}$. Then, by the Bayes theorem, the probability that the bag from which a black marble was drawn has k black marbles is

$$\begin{aligned}P(N_k|B_1) &= \frac{P(B_1|N_k)P(N_k)}{\sum_{i=0}^{n}P(B_1|N_i)P(N_i)} = \frac{\frac{k}{n(n+1)}}{\sum_{i=0}^{n}\frac{i}{n(n+1)}} = \frac{k}{\sum_{i=0}^{n}i}\\ &= \frac{2k}{n(n+1)}.\end{aligned}$$

Now, if the event N_k took place the probability of the second marble being black is $\dfrac{k-1}{n-1}$ and the total probability of the second marble being black is

$$\begin{aligned}
P(B_2) &= \sum_{k=1}^{n} \frac{2k}{n(n+1)} \cdot \frac{k-1}{n-1} = \frac{2}{n(n+1)(n-1)} \sum_{k=1}^{n} k(k-1) \\
&= \frac{2}{n(n+1)(n-1)} \sum_{k=1}^{n} (k^2 - k) \\
&= \frac{2}{n(n+1)(n-1)} \left(\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} \right) \\
&= \frac{1}{n-1} \left(\frac{2n+1}{3} - 1 \right) = \frac{1}{n-1} \cdot \frac{2(n-1)}{3} \\
&= \frac{2}{3}.
\end{aligned}$$

Note that the probability is independent of n.

Solution 2

[65]

There is a shortcut for evaluating $P(N_k|B_1)$. Since we do not know how many black marbles there are in the bag at hand, we should consider all $n+1$ possible configurations. The total amount of black marbles for all eventualities is $\sum_{i=0}^{n} i = \dfrac{n(n+1)}{2}$, making the probability of getting a black marble at all equal to $\dfrac{2}{n(n+1)}$ and the probability of getting one from a bag with k marbles $P(N_k|B_1) = \dfrac{2k}{n(n+1)}$. From here we proceed as in the first solution.

Solution 3

Amit Itagi

Let D_i denote the event that the i^{th} draw is black and B_k denote the event that there are k black marbles out of a total of n:

$$\begin{aligned}
P(D_1) &= \sum_{k=0}^{n} P(D_1|B_k)P(B_k) = \frac{1}{n+1} \cdot \sum_{k=0}^{n} \frac{k}{n} \\
P(D_2 \cap D_1) &= \sum_{k=0}^{n} P(D_2 \cap D_1|B_k)P(B_k) = \frac{1}{n+1} \cdot \sum_{k=0}^{n} \frac{k}{n} \cdot \frac{k-1}{n-1}. \\
P(D_2|D_1) &= \frac{P(D_2 \cap D_1)}{P(D_1)} = \frac{1}{n-1} \cdot \frac{\sum_{k=0}^{n} k(k-1)}{\sum_{k=0}^{n} k} \\
&= \frac{2}{3}.
\end{aligned}$$

The answer brings to mind the Monty Hall problem [100]. First grab two marbles blindfolded and then apply $n = 2$ on those two. After the first marble is black, we know there are one or two black marbles, but not zero. The first one could have been black in BW or either one could have been black in BB. Therefore, $P(B_1B_2) = \frac{2}{3}$.

Quotient Estimates I

Solution 1 to Question 1

Faryad D. Sahneh

Suppose X is uniformly randomly chosen in $[0, 1]$ and Y is uniformly randomly chosen in $[X, 1]$. The ratio $r = \frac{Y}{X}$ is a random variable with the cumulative distribution function

$$P(r \geq k) = P(Y \geq kX) = \int P(Y \geq kX|X = x)dP(X = x).$$

Since Y is drawn from $[X, 1]$, $P(Y \geq kX|X = x) = 1 - \frac{x}{1-x}(k-1)$ for $1 \leq k \leq \frac{1}{x}$ and 0 for $k > \frac{1}{x}$.

Therefore, for a fixed $k \geq 1$, the interval of the integration above is $0 \leq x \leq \frac{1}{k}$, and $dP(X = x) = dx$:

$$\begin{aligned} P(r \geq k) = P(Y \geq kX) &= \int_0^{\frac{1}{k}} 1 - \frac{x}{1-x}(k-1)dx \\ &= 1 - (k-1)\ln\frac{k-1}{k}. \end{aligned}$$

Solution 2 to Question 1

Bogdan Lataianu

Given x,

$$\begin{aligned} P(Y > kX) = P(y \in (kx, 1)) &= \int_{kx}^{1} \frac{1}{1-x}dy \\ &= \frac{1-kx}{1-x}, 0 < x < \frac{1}{k}. \end{aligned}$$

Hence, in the general case,

$$\begin{aligned} P(Y > kX) &= \int_0^{1/k} \frac{1-kx}{1-x}dx \\ &= 1 + (1-k)\ln\left(1 - \frac{1}{k}\right). \end{aligned}$$

Solution 3 to Question 1

N.N. Taleb

We can reduce this case to the distribution of

$$\omega := \max\left\{\frac{X}{(1-X)Y+X}, \frac{(1-X)Y+X}{X}\right\} = \frac{(1-X)Y+X}{X}$$

where both variables are uniform in $[0,1]$, with

$$f_\omega(\omega) = \log\left(\frac{\omega}{\omega-1}\right) - \frac{1}{\omega}.$$

Then

$$\mathbb{P}(r > k) = \int_k^\infty \log\left(\frac{\omega}{\omega-1}\right) - \frac{1}{\omega}\, d\omega = (1-k)(\log(k-1) - \log(k)).$$

Solution to Question 2

Bogdan Lataianu

With reference to Figure 7.4, $\{(x,y) : 0 \le x \le y \le 1\} = \triangle\,\mathrm{ABC}$.

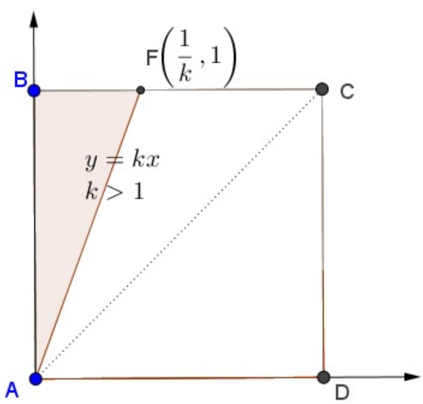


Figure 7.4: Quotient estimate 1.

In $\triangle\,\mathrm{ABC}$, $y \ge x$; hence, always $r \ge 1$. It follows that, for $k \le 1$, the probability $P(r \ge k) = 1$. For the case $k \ge 1$, having a joint uniform distribution means that the probability of (X,Y) falling into a given subset of $\triangle\,\mathrm{ABC}$ is proportional to the area of the subset. Now, for a fixed $k \ge 1$, $r \ge k$ in $\triangle\,\mathrm{ABF}$, where $F = \left(\frac{1}{k}, 1\right)$. It follows that $P(r \ge k) = \frac{[\triangle\,\mathrm{ABF}]}{[\triangle\,\mathrm{ABC}]} = \frac{1}{k}$. (Here $[F]$ denotes the area of shape F.)

Solution 1 to Question 3

A. Bogomolny

X and Y being uniformly distributed on $[0,1]$ means that their probability densities are defined by $f_X(x) = 1$, $x \in [0,1]$, and 0 elsewhere. Similarly, $f_Y(y) = 1$, $y \in [0,1]$, and 0 elsewhere. Being the two are independent, their joint density is the product of the individual two: $f_{X,Y}(x,y) = f_X(x)f_Y(y)$. It follows that $f_{X,Y}(x,y) = 1$ for $(x,y) \in [0,1] \times [0,1]$ and 0 elsewhere. In other words, the joint distribution is uniform, and the reasoning used to answer question 1 applies here as well.

With reference to Figure 7.5, for a given $k \geq 1$, $\dfrac{Y}{X} \geq k$ in $\triangle$ ABF and $\dfrac{X}{Y} \geq k$ in $\triangle$ ADE.

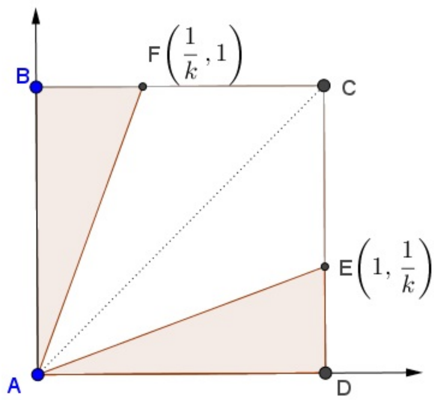


Figure 7.5: Quotient estimate 2.

It follows that $\left\{(x,y) : \dfrac{\max\{X,Y\}}{\min\{X,Y\}}\right\} = \triangle\,\text{ABF} \cup \triangle\,\text{ADE}$. As a consequence, for $k \geq 1$,

$$P(r \geq k) = \frac{[\triangle\,\text{ABF}] + [\triangle\,\text{ADE}]}{[\triangle\,\text{ABCD}]} = \frac{\frac{1}{2k} + \frac{1}{2k}}{1} = \frac{1}{k}.$$

For $k < 1$, the corresponding probability is always 1.

Solution 2 to Question 3

N.N. Taleb

$r = \dfrac{\max\{X,Y\}}{\min\{X,Y\}}$ can be rewritten as

$$r = \max\left\{\frac{X}{Y}, \frac{Y}{X}\right\} = \mathbf{1}_{\{X/Y<1\}}\frac{Y}{X} + \mathbf{1}_{\{X/Y\geq 1\}}\frac{X}{Y}.$$

Yet the distribution of ω, a ratio of independent uniform variables, in $[0,1]$ is

$$f_\omega(\omega) = \begin{cases} \dfrac{1}{2} & \text{if } w \leq 1 \\ \dfrac{1}{2\omega^2} & \text{if } w > 1. \end{cases}$$

Normalizing for values > 1 since r is necessarily > 1, we end up with $f_r(r) = \dfrac{1}{r^2}$ and

$$\mathbb{P}(r > k) = \int_k^\infty \frac{1}{r^2}\,dr = \frac{1}{k}.$$

Quotient Estimates II

Solution 1
[71]

Let $U = \min_{1 \le i \le n} \{X_i\}$ and $V = \max_{1 \le i \le n} \{X_i\}$. The question requests that we find $P(V \ge kU)$, $k \ge 1$.

To solve the problem we have to integrate the joint distribution $f_{U,V}(u,v)$ over the shaded triangle (Figure 7.6). The triangle is a part of the region $\{(u,v) : u < v\}$. Thus, we only need to find $f_{U,V}(u,v)$ for $u < v$.

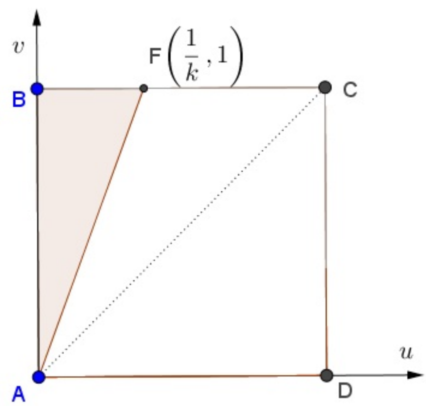


Figure 7.6: Quotient estimate 3.

Let $f(x)$ be the uniform density function on $[0,1]$. For all $i = 1, \ldots, n$, $f_{X_i}(x) = f(x)$. Let $F(x)$ be the uniform distribution function $F(x) = \int_{-\infty}^{x} f(\xi)d\xi$:

$$F(x) = \begin{cases} 0, & x \le 0, \\ x, & 0 \le x \le 1, \\ 1, & x \ge 1. \end{cases}$$

For all $i = 1, \ldots, n$, $F_{X_i}(x) = F(x)$. Now, since $X_1, \ldots, X_n$ are independent,

$$\begin{aligned} F_V(v) = P(V < v) &= P(X_1 < v)P(X_2 < v) \cdots P(X_n < v) \\ &= F_{X_1}(v)F_{X_2}(v) \cdots F_{X_n}(v) = F^n(v) \end{aligned}$$

so that

$$F_V(v) = \begin{cases} 0, & v \le 0, \\ v^n, & 0 \le v \le 1, \\ 1, & v \ge 1. \end{cases}$$

For the cumulative distribution function,

$$\begin{aligned} F_{U,V}(u,v) &= P(U \le u, V \le v) \\ &= P(U \le u, X_1 \le v, X_2 \le v, \ldots, X_n \le v). \end{aligned}$$

As a reminder, $U > u$, if and only if $X_i > u$ for all $i = 1, \ldots, n$. $U \le u$ means the opposite, i.e., that $X_i \le u$ for some X_i. Therefore, we may continue (with $u > v$):

$$\begin{aligned}
F_{U,V}(u,v) &= P(U \le u, V \le v)\\
&= P(U \le u, X_1 \le v, X_2 \le v, \ldots, X_n \le v)\\
&= P(X_1 \le v, X_2 \le v, \ldots, X_n \le v) - \prod_{i=1}^{n} P(u < X_i \le v)\\
&= F^n(v) - P(u < X_1 \le v)P(u < X_2 \le v)\cdots P(u < X_n \le v)\\
&= F^n(v) - (F(v) - F(u))^n\\
&= v^n - (v-u)^n.
\end{aligned}$$

Next, $f_{U,V}(u,v) = \dfrac{\partial^2 F_{U,V}(u,v)}{\partial u \partial v} = n(n-1)(v-u)^{n-2}$. Finally,

$$\begin{aligned}
P(V > kU) &= \int_0^1 \int_0^{v/k} n(n-1)(v-u)^{n-2} du dv\\
&= \int_0^1 n(v-u)^{n-1}\Big|_{v/k}^{0} du dv\\
&= \int_0^1 n\left(v^{n-1} - v^{n-1}\left(1 - \frac{1}{k}\right)^{n-1}\right) dv\\
&= \int_0^1 n\left(v^{n-1} - \left(v - \frac{v}{k}\right)^{n-1}\right) dv\\
&= \int_0^1 n\left(v^{n-1} - v^{n-1}\left(1 - \frac{1}{k}\right)^{n-1}\right) dv\\
&= \int_0^1 nv^{n-1}\left(1 - \left(1 - \frac{1}{k}\right)^{n-1}\right) dv\\
&= 1 - \left(1 - \frac{1}{k}\right)^{n-1}.
\end{aligned}$$

Solution 2

Amit Itagi

Let x and y be the smallest and largest of the chosen numbers. These numbers are shown on the x and y axes in Figure 7.7. The red region is not feasible as $y > x$. The region where the ratio is less than k is the green region.

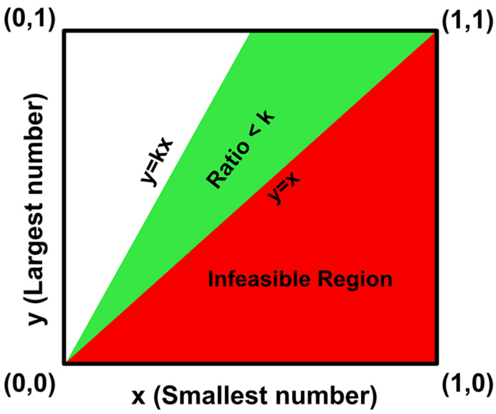


Figure 7.7: Quotient estimate 4.

The probability density of the chosen set of points being in a window $dxdy$ around (x, y) is

$$f(x, y)dxdy = n(n-1)(y-x)^{n-2}dxdy.$$

The $(y-x)^{n-2}$ is the probability of choosing the $(n-2)$ points between the two fixed endpoints. The $n(n-1)$ comes from choosing the endpoints.

The required probability is one minus the integral of this density over the green region and is given by

$$\begin{aligned}P &= 1 - \int_{y=0}^{1}\int_{x=\frac{y}{k}}^{y} f(x,y)dxdy = 1 - \int_{y=0}^{1}\int_{x=\frac{y}{k}}^{y} n(n-1)(y-x)^{n-2}dxdy \\ &= 1 - \int_{y=0}^{1} ny^{n-1}\left[\left(1-\frac{1}{k}\right)^{n-1}\right]dy \\ &= 1 - \left(1-\frac{1}{k}\right)^{n-1}.\end{aligned}$$

Solution 3

N.N. Taleb

We consider

$$r = \frac{\max\limits_{1\le i\le n}\{X_i\}}{\min\limits_{1\le i\le n}\{X_i\}} = \max_{0\le i\ne j\le n}\frac{X_i}{X_j}.$$

We know the distribution of $\omega = \dfrac{X_i}{X_j}$ and can arrange m permutations, with $m = \dfrac{n!}{(n-2)!}$. If we assume that $n-1$ ratios are independent, with probability distribution function $f_\omega(\omega) = \dfrac{1}{2\omega^2}$ and cumulative distribution function $F_\omega(\omega) = \dfrac{1}{2\omega}$, then

$$\mathbb{P}(\max\{\omega_1, \omega_2, \ldots, \omega_m\} > k) = 1 - \left(1-\frac{1}{k}\right)^{n-1}, \qquad k \ge 1.$$

The Lost Boarding Pass

Solution 1

[102, pp. 35–37]

When the last passenger boards the plane, there are just two possibilities: the one remaining seat may be his assigned seat or that of the first passenger. Under the assumption that no preference has been exhibited by the boarding passengers towards either of the two seats, they each have the same probability to become the last unoccupied seat: 50%.

Solution 2

[21, p. 176], B. Bollobás credits Oliver Riordan with the proof.

Suppose there are $n \geq 2$ passengers and n seats. If, for any $k < n$, the first k passengers occupy the seats assigned to them (as a group, not individually), then from then on everybody, including the last passenger, sits in his own seat. Also, if one of the first $k < n$ passengers occupies the seat of the last passenger then the last passenger will not sit in his own seat. Now, if, for $k-1$, neither of these events holds, then the k^{th} passenger's choice has exactly as much chance of leading to the first event as to the second. If his own seat is unoccupied, neither of the events will happen. If his own seat seat is occupied, then there is exactly one (unoccupied) seat, the first, that leads to the first event and exactly one (unoccupied) seat, the last, that leads to the second.

Solution 3

Sergej Saletic

There is $\dfrac{1}{100}$ chance the first passenger sits in his own seat, ensuring the last passenger's seat will be available. The first passenger has a $\dfrac{1}{100}$ chance to sit in the last passenger's seat, ensuring it will be occupied. Finally, the first passenger has a $\dfrac{98}{100}$ chance to sit in someone else's seat and, whenever that happens, the first "middle" passenger who takes either the first or last passenger's seat will do so with a probability of $\dfrac{1}{2}$ for each. Therefore,

$$\frac{1}{100} \cdot 1 + \frac{98}{100} \cdot \frac{1}{2} + \frac{1}{100} \cdot 0 = \frac{1}{2}.$$

Solution 4

Giorgos Papadopulos

Define:

A_k as the event that the k^{th} passenger sits in his own seat.
I_m as the event that the first passenger takes the m^{th} seat.

Then,

$$
\begin{aligned}
P(A_k) &= \frac{n-k+1}{n} \cdot 1 + \frac{1}{n} \cdot 0 + \frac{k-2}{n} \cdot \frac{n-k+1}{n-k+2} \\
&= \frac{n-k+1}{n(n-k+2)}[(n-k+2)+(k-2)] \\
&= \frac{n-k+1}{n-k+2}.
\end{aligned}
$$

For example, for $n = 8$, $k = 6$, $m = 1 \ldots, 8$,

1. I_1, I_6, I_7, I_8 events are obvious.
2. in event I_5, $P(A_2) = P(A_3) = P(A_4) = 1$, so the fifth person has only four seats available: $P(A_6|I_5) = \dfrac{3}{4}$.
3. in event I_4, $P(A_2) = P(A_3) = 1$, so we have two cases.
   In the first I_4 case, the fourth person goes to seat #5 and the fifth person has only four seats available: $P(A_6|I_4) = \dfrac{3}{4}$.
   In the second I_4 case, the fourth person goes to a seat different than #5. Then $P(A_5) = 1$ and four seats are available: $P(A_6|I_4) = \dfrac{3}{4}$.
4. in event I_3, $P(A_2) = 1$, so we have three cases.
   In the first I_3 case, the third person goes to seat #4 and then we proceed similarly to event I_4: $P(A_6|I_3) = \dfrac{3}{4}$.
   In the second I_3 case, the third person goes to seat #5. Then $P(A_4) = 1$ and we proceed similarly to event I_5: $P(A_6|I_3) = \dfrac{3}{4}$.
   In the third I_3 case, the third person goes to a seat different than #4 or #5. Then $P(A_4) = P(A_5) = 1$ and there are four seats available: $P(A_6|I_3) = \dfrac{3}{4}$.
5. in event I_2, with similar reasoning as above, we have four cases and $P(A_6|I_2) = \dfrac{3}{4}$.

Adding probabilities, we get

$$
P(A_6) = \sum_{m=1}^{8} P(A_6|I_m) = \frac{1}{8} \cdot \left(1 + \frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4} + 0 + 1 + 1\right) = \frac{3}{4}
$$

and

$$
P(A_6^c) = 1 - \frac{3}{4} = \frac{1}{4}.
$$

Similarly, for $k = 8$,

$$
P(A_8) = \frac{1}{8} \cdot 1 + \frac{1}{8} \cdot 0 + \frac{6}{8} \cdot \frac{1}{2} = \frac{1}{2}.
$$

Solution 5

Jim Totten, [65, Problem 324], considers the situation when the passengers board a plane—a bus actually—in numerical order of their seats.

Let P_n be the probability that passenger p_n will sit in seat s_n. We will use mathematical induction to show that $P_n = \frac{1}{2}$ for $n > 1$. First, with $n = 2$, there are two equiprobable events: p_1, p_2 sit in s_1, s_2 or in s_2, s_1, making $P_2 = \frac{1}{2}(1+0) = \frac{1}{2}$.

Now, suppose $n > 2$ and that $P_m = \frac{1}{2}$ for $1 < m < n$. If passenger p_1 chooses s_1, then passenger p_n sits in s_n with probability 1. If p_1 chooses seat s_n, then p_n sits in s_n with probability 0. If p_1 chooses s_k with $1 < k < n$, passenger p_i sits in s_i for $1 < i < k$, and passenger p_k will have to choose from seats s_1 or s_j with $k+1 \le j \le n$.

Since this is essentially the same situation as if passenger p_k were p_1 with $n-k+1$ passengers and seats, we conclude from the induction hypothesis that the probability that passenger p_n sits in seat s_n is $\frac{1}{2}$ for each of the $n-2$ choices of k, $1 < k < n$. Therefore,

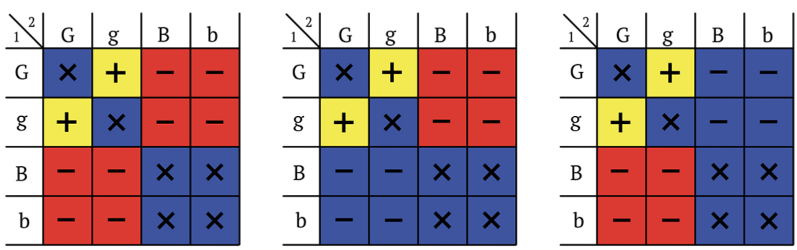
$$P_n = \frac{1 + 0 + (n-2)\cdot\frac{1}{2}}{n} = \frac{1}{2}.$$

Lucky Contest Winners

Solution 1

The simplest way to approach the riddle is to consider the associated sample space. There are two girls and two boys from which a pair of winners is going to be drawn. Let us denote the girls G and g, and the boys B and b. Formally speaking, there are $16 = 4 \times 4$ possible ordered pairs formed by the four letters. However, from the context of the problem, the repetitions, like GG, make no sense and should be excluded outright.

The first three questions (with the last coming first) are collected in the following tables:



One of the winners is a girl

First winner is a girl

Second winner is a girl

The legend is this: The impossible pairs (two boys or the same girl twice) are marked with a "×". The two girl combinations are marked with "+" and all the others with "−". In the first question (the middle table), the first member of the pair is known to be a girl, making the four combinations in the lower left corner impossible. These are also colored blue. A similar treatment applies to the upper right corner in the last table (second question). Of the remaining squares, the ones with favorable outcomes—Gg or gG—are colored yellow and the unfavorable ones are colored red.

For the first two questions there are six possible outcomes: two favorable and four unfavorable. In both cases then the probability that two girls have been selected to advance to the national finals is $\frac{2}{6} = \frac{1}{3}$.

For the third question, you were as likely to cast your eyes on the first girl as on the second. By the symmetry principle, the probabilities of the two events are equal to $\frac{1}{2}$, whereas each of the girls has an individual probability of $\frac{1}{3}$ to be a winner. Thus, the total probability of both girls is $\frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{3}$.

For the fourth question, there are 10 possible outcomes with only two favorable. The probability of having two lucky girls is then $\frac{2}{10} = \frac{1}{5}$.

Solution 2

Mike Shaughnessy

If the girl that you saw on the stage was the first winner, then, out of the three students left for the random drawing for a second winner, only one is a girl and two are boys. The chances that the second winner is also a girl will be 1 in 3, or $\frac{1}{3}$.

However, if the girl you saw on stage was actually the second winner, then the chance that the first winner was also a girl is $\frac{1}{2}$, because before the first girl was picked there were two girls and two boys, so the chance of choosing a girl was then 2 in 4, or $\frac{1}{2}$.

Similiarly, for question 2, if the girl you saw on the stage was the first winner, then, out of the three remaining students, one is a girl and two are boys, so the chances that the second winner will also be a girl is $\frac{1}{3}$.

Unlike question 1, it does not make any difference whether you saw the first girl or the second girl; the fact that you saw a girl winning at all means that there is only a 1 in 3 chance that the other winner is a girl. Therefore, the probability that the first winner was also a girl, if the girl that you saw on the stage is the second winner, is also $\frac{1}{3}$.

Notes

The solutions are followed by advice to run a simulation. Before you do, give a thought to the fact that the second part of the first solution does not exploit the fact that the student you saw as the second winner was a girl. The argument would be the same if you saw a boy. This should make one suspicious of this line of reasoning.

Solution 3

To strengthen the second solution, we may apply the Bayes theorem. In our notations, we would like to evaluate $P(G|g)$—the probability that the first winner was a girl provided the second winner was also a girl. We accept as known (obvious or from the argument above) that

1. $P(G) = P(B) = \frac{1}{2}$, i.e., the probabilities of the first winner being a girl or a boy are both equal to $\frac{1}{2}$.

2. $P(g|G) = \frac{1}{3}$, i.e., the probability of the second winner being a girl provided the first one was a girl is $\frac{1}{3}$ (solutions 1 and 2).

3. $P(g|B) = \frac{2}{3}$, i.e., the probability of the second winner being a girl provided the first one was a boy is $\frac{2}{3}$. This is because, when the first boy was out after the first drawing, there were two girls and one boy remaining for the second drawing.

Now, let us use Bayes's formula:

$$\begin{aligned} P(G|g) &= \frac{P(Gg)}{P(g)} = \frac{P(g|G)P(G)}{P(g|G)P(G) + P(g|B)P(B)} \\ &= \frac{\frac{1}{3} \cdot \frac{1}{2}}{\frac{1}{3} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{1}{2}} = \frac{\frac{1}{3}}{\frac{1}{3} + \frac{2}{3}} \\ &= \frac{1}{3}. \end{aligned}$$

Solution 4

Looking back, it might have been easier to evaluate $P(Gg)$ outright. This is the probability that the two winners were both girls. There are $\binom{4}{2} = 6$ cases to choose two items out of four. We are only interested in one selection. Therefore, $P(Gg) = \frac{1}{6}$. Divide by $P(g)$ which is $\frac{1}{2}$ and we obtain $P(G|g) = \frac{1}{6} \div \frac{1}{2} = \frac{1}{3}$.

Diminishing Hopes

The answers to the three questions are $\frac{7}{9}$, $\frac{2}{3}$ and $\frac{1}{3}$, respectively. The result may be surprising. Chances of locating the document decrease with every checked drawer. The general formula for the probability of finding the document after unsuccessfully checking n drawers is $P_n = \frac{8-n}{10-n}$. This is a decreasing function of n for $n < 10$. You can check the solutions below.

Solution 1, Clever

[36] and, independently, Michael Wiener

Let us add two imaginary drawers to the desk and assume that this is where the documents get lost. We may even think of them as being locked during the search.

With k drawers to go, of which two are unavailable, the probability of a successful outcome is $\frac{k-2}{k}$. With $k = 10 - n$, this reduces to $P_n = \frac{8-n}{10-n}$. This is a decreasing function of n for $n < 10$.

It is also curious that, if we only ask about the probability of finding the document in the next available drawer (after checking the first n drawers), the probability will be $Q_n = \frac{1}{10-n}$, an increasing function of n for $n < 10$.

Solution 2, Counting Odds
Joshua B. Miller

Not finding the document after k checks is $\frac{8}{8-k}$ times more likely if it is not in the desk (not found with probability 1) versus in desk (not found with probability $\frac{8-k}{8}$). Prior odds it is not in the desk are $1:4$, so posterior odds it is not in the desk are $\frac{8}{8-k}:4$, i.e., $2:(8-k)$. Therefore, odds in favor are $(8-k):2$, i.e., (1) $7:2$, (2) $4:2$, (3) $1:2$, which translates into the probabilities (1) $\frac{7}{9}$, (2) $\frac{4}{6}$, (3) $\frac{1}{3}$.

Solution 3, Conditional Probabilities
Amit Itagi

Let us denote the events "document is lost" by L and "document is in drawer i" by D_i. Let the unconditional probability of D_i be q; $q = \frac{1}{8} \cdot \frac{4}{5} = \frac{1}{10}$. The overbar designates a complementary event. Then

1. $P(\overline{L}|\overline{D_1}) = \dfrac{P(\overline{D_1}|\overline{L})}{P(\overline{D_1})} = \dfrac{(7/8)(8q)}{1-q} = \dfrac{7}{9}$.

2. $P(\overline{L}|\cap_{i=1}^4 \overline{D_i}) = \dfrac{P(\cup_{i=1}^4 \overline{D_i}|\overline{L})}{P(\cup_{i=1}^4 D_i)} = \dfrac{(1/2)(8q)}{1-4q} = \dfrac{2}{3}$.

3. $P(\overline{L}|\cap_{i=1}^7 \overline{D_i}) = \dfrac{P(\cup_{i=1}^7 \overline{D_i}|\overline{L})}{P(\cup_{i=1}^7 D_i)} = \dfrac{(1/8)(8q)}{1-7q} = \dfrac{1}{3}$.

Solution 4, Bayesian Odds
A. Bogomolny

We assume that the search for a document is thorough enough to ensure that a document that was stored in the desk will be necessarily found.

Let D denote the event that the sought document is in one of the drawers; $\overline{D}$ that it is not: $P(D) = \frac{4}{5}$, $P(\overline{D}) = \frac{1}{5}$. Let K be the event of not finding the document in the first k drawers. Then, computing the Bayesian odds,

$$\frac{P(\overline{D}|K)}{P(D|K)} = \frac{P(K|\overline{D})}{P(K|D)} \cdot \frac{P(\overline{D})}{P(D)},$$

where $\dfrac{P(\overline{D})}{P(D)} = \dfrac{1}{4}$, $P(K|\overline{D}) = 1$ and $P(K|D)$ is the probability of the document not being found in the first k drawers, even though it is somewhere in the desk. This exactly means that the document is in the $8-k$ remaining drawers, making $P(K|D) = \dfrac{8-k}{8}$. Thus,

$$\frac{P(\overline{D}|K)}{P(D|K)} = \frac{8}{8-k} \cdot \frac{1}{4} = \frac{2}{8-k}.$$

Since $P(\overline{D}|K) + P(D|K) = 1$, the above resolves into $P(D|K) = \dfrac{8-k}{10-k}$.

This gives the three answers above.

Solution 5, Bayes's Theorem

A. Bogomolny

In the notations of solution 3,

$$P(D|K)a = \frac{P(K|D) \cdot P(D)}{P(K|D) \cdot P(D) + P(K|\overline{D}) \cdot P(\overline{D})} = \frac{\dfrac{8-k}{8} \cdot \dfrac{4}{5}}{\dfrac{8-k}{8} \cdot \dfrac{4}{5} + 1 \cdot \dfrac{1}{5}}$$

$$a = \frac{\dfrac{8-k}{2}}{\dfrac{8-k+2}{2}} = \frac{8-k}{10-k}.$$

Incidence of Breast Cancer

Introduce the events:

- P—mammogram result is positive,
- B—tumor is benign,
- M—tumor is malignant.

Bayes's formula in this case is

$$\begin{aligned} P(M|P) &= \frac{P(P|M)P(M)}{P(P|M)P(M) + P(P|B)P(B)} \\ &= \frac{.80 \times .01}{.80 \times .01 + .10 \times .99} \\ &\approx 0.075 \\ &= 7.5\%. \end{aligned}$$

The result is a far cry from a common estimate of 75%.

A Search for Heads and Its Consequences

With assistance from Joshua B. Miller, Alejandro Rodríguez, Thamizh Kudimagan and Zhuo Xi.

Solution to Question 1

There are four possible outcomes for the first two flips: TT, TH, HT, HH. Since Jack selects a flip immediately preceded by a heads, the first two sequences could not have led to his selection of the second flip. Two possibilities remain: HT and HH, which are equiprobable. However, in the second case, Jack had a choice between selecting the second or the third flips. Thus, the probability of selecting heads is split. Hence, the answer is $\frac{1}{3}$.

Solution to Question 2

Since the third flip was selected, the second one is bound to be heads. Thus, we have four possibilities: THT, THH, HHT, HHH. They are all equiprobable, so that the answer in this case is also $\frac{1}{2}$.

Solution 1 to Question 3

We do not know what Jack's selection was, but we do know that heads came up in the first flip or the second (or both). Thus, we have six possible sequences (i.e., excluding TTT and TTH): HTT, HTH, THT, THH, HHT, HHH. Each of these has the probability of $\frac{1}{6}$ of occurring. In the first four Jack has no choice and is bound to selecting a unique one. His election is a heads in just one case, THH, with the probability of $\frac{1}{6}$. Of the remaining two cases, where Jack has a choice to make, heads would be selected in three out of four possibilities: $H\boldsymbol{H}T$, $HH\boldsymbol{T}$, $H\boldsymbol{H}H$, $HH\boldsymbol{H}$. Thus, the probability of a heads flip being selected is $\frac{1}{2} \cdot \frac{3}{6} = \frac{1}{4}$.

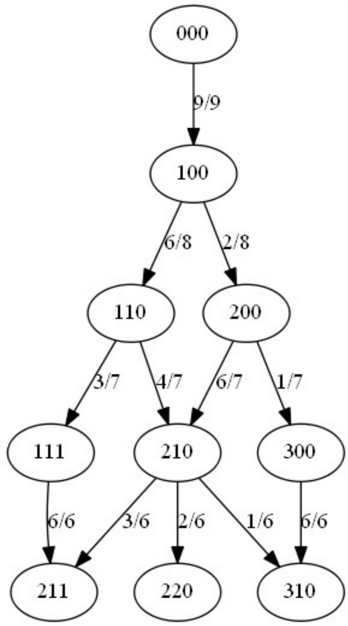
Combining the two cases, heads would be selected with the probability of $\frac{1}{6} + \frac{1}{4} = \frac{5}{12}$.

Solution 2 to Question 3

A shorter way is to combine the answers to the first two questions, since it is equally likely to be the first or second flip: $\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3} = \frac{5}{12}$.

A Three Group Split

Let us represent the three group split by a sequence of three digits, each of which standing for the amount of even numbers in a group. The order of the groups is of no consequence. The starting configuration is 000. There are nine open slots to place the first even number in. However it is done, the next state is 100.



With one even number in place, there are eight slots—two in the same group as the first even number. Placing the second even number there moves to the state 200; placing it elsewhere, with probability $\frac{6}{8}$, will move to the state 110. We continue in this manner, distributing even numbers, of which there are four.

When all is said and done, there are only three possible states with four even numbers: 211 (meaning no group has three odd numbers), 220 and 310. In each of the latter two cases, there is one group (corresponding to 0) in which all numbers are odd.

The probability of getting into the state 220 is

$$\begin{aligned}P(220) &= \left(\frac{6}{8}\cdot\frac{4}{7}+\frac{2}{8}\cdot\frac{6}{7}\right)\cdot\frac{2}{6}\\ &= \frac{24+12}{56}\cdot\frac{1}{3}=\frac{3}{14}.\end{aligned}$$

The probability of getting into the state 310 is

$$\begin{aligned}P(310) &= \frac{2}{8}\cdot\left(\frac{1}{7}\cdot\frac{6}{6}+\frac{6}{7}\cdot\frac{1}{6}\right)+\frac{6}{8}\cdot\frac{4}{7}\cdot\frac{1}{6}\\ &= \frac{1}{4}\cdot\frac{6+6}{42}+\frac{1}{14}=\frac{2}{14}.\end{aligned}$$

Thus, the probability of having a group of odd numbers is $\frac{2}{14}+\frac{3}{14}=\frac{5}{14}$.

Lewis Carroll's Pillow Problem

According to Martin Gardner:

> In 1893 Carroll published a little book of seventy-two original mathematical puzzles.... The most interesting puzzles in *Pillow-Problems* concern probability. The first one, problem 5, is simple to state but extremely confusing to analyze correctly.

To help the reader see the difficulty and perhaps make the presence of the confusion clear, Carroll gave two solutions.

Solution 1

Lewis Carroll

As the state of the bag after the operation is necessarily identical with its state before it, the chance is just what it was, viz. $\frac{1}{2}$.

Solution 2

Lewis Carroll, Martin Gardner

Let B and W_1 stand for the black or white counter that may be in the bag at the start and W_2 for the added white counter. After removing a white counter there are three equally likely states:

Inside Bag	Outside Bag
W_1	W_2
W_2	W_1
B	W_2

In two of these states, a white counter remains in the bag, so the chance of drawing a white counter the second time is $\frac{2}{3}$.

Notes

Some time after putting the problem on the web I (AB) received a letter from Chris Conradi:

Carroll's Pillow Problem says that we know a bag contains one counter, and it is either black or white. The solution presumes that the bag is equally likely to contain a black counter or a white counter, although neither Carroll nor Bogomolny makes that clear. It would be helpful to make that point in the statement of the problem. Otherwise, the answer cannot be determined from the statement of the problem.

This is just a particular instance of a more general problem. Suppose that the first counter is drawn at random from another bag containing d counters, n of which are white and $d - n$ of which are black. Without revealing its color, this first counter is placed in a second, empty bag. Then a white counter is added to this bag. Now you draw a counter from this second bag and it turns out to be white. What is the probability that the remaining counter is white? The general answer is $\frac{2n}{n+d}$, i.e.,

$n = 1$ and $d = 2$. The answer is $\dfrac{2}{3}$, as given, but we can also determine the probability if $n = 1$ and $d = 4$ $\left(\dfrac{2}{5}\right)$, if $n = 3$ and $d = 4$ $\left(\dfrac{6}{7}\right)$ or if $n = 5$ and $d = 18$ $\left(\dfrac{10}{23}\right)$.

A Follow Up on Lewis Carroll's Pillow Problem

Solution 1, Correct and Short

Joshua B. Miller

Prior odds in favor of two white counters in the second bag are $w : b$, but drawing w is twice as likely under ww as under bw, so posterior odds in favor are $2w : b$, i.e., $\text{Prob}(ww|\text{drawn } w) = \dfrac{2w}{2w+b}$.

Solution 2, Correct and Long

Introduce $P(W) = \dfrac{w}{w+b}$, probability of the counter removed from the first bag being white. For the black counter, it is $P(W) = \dfrac{b}{w+b}$. $P(W|W)$ and $P(B|W)$ are the probabilities that the remaining counter in the second bag is white and black, respectively; $\overline{P}(W|W)$ and $P(W|B)$ are the probabilities of removing of a white counter from the second bag, provided that it contains two white counters or a black and a white counter.

The Bayesian odds are expressed as $\dfrac{P(W|W)}{P(B|W)} = \dfrac{\overline{P}(W|W)}{P(W|B)} \cdot \dfrac{P(W)}{P(B)}$. It is obvious that $\dfrac{\overline{P}(W|W)}{P(W|B)} = 2$, implying

$$\frac{P(W|W)}{P(B|W)} = 2 \cdot \frac{w}{b} = \frac{2w}{b}.$$

Since $P(W|W) + P(B|W) = 1$, we derive $P(W|W) = \dfrac{2w}{2w+b}$.

Solution 3, Correct and Long

We will use Bayes's theorem. The notations are from the previous solution:

$$\begin{aligned}
P(W|W) &= \frac{\overline{P}(W|W) \cdot P(W)}{\overline{P}(W|W) \cdot P(W) + P(W|B) \cdot P(B)} \\
&= \frac{1 \cdot \dfrac{w}{w+b}}{1 \cdot \dfrac{w}{w+b} + \dfrac{1}{2} \cdot \dfrac{b}{w+b}} \\
&= \frac{\dfrac{w}{w+b}}{\dfrac{1}{2} \cdot \dfrac{2w+b}{w+b}} = \frac{2w}{2w+b}.
\end{aligned}$$

Solution 4, Incorrect

There is a 50% chance that the remaining counter is from the original bag, so $P = \frac{w}{w+b}$. There is another 50% chance that the remaining counter is the added white one. Together, these have a proability of 1. Therefore, the sought probability is $0.5\frac{w}{w+b} + 0.5$, which simplifies to $P = \frac{2w+b}{2(w+b)}$.

How to See That Solution 4 Is Incorrect

Assume that there were no white counters in the first bag: $w = 0$. Then the probability of the leftover counter being white is zero. However, for $w = 0$, the formula produces $\frac{1}{2}$.

Sick Child and Doctor

This is a direct application of Bayes's theorem:

$$\begin{aligned} P(M|R) &= \frac{P(R|M)P(M)}{(P(R|M)P(M) + P(R|F)P(F))} \\ &= \frac{.95 \times .10}{.95 \times .10 + .08 \times .90} \\ &\approx 0.57. \end{aligned}$$

The result is nowhere close to 95% of $P(R|M)$.

Right Strategy for a Weaker Player

Solution 1

The strategy is to play a daring game when tied or losing and a conservative game when ahead in the match.

After two games, there are four possible outcomes (from A's perspective, with D, W, L standing for Draw, Win, Lose):

Result	Conclusion	Probability
LL	match lost	$0.55 \cdot 0.55 = 0.3025$
LW	tiebreaker required	$0.55 \cdot 0.45 = 0.2475$
WL	tiebreaker required	$0.45 \cdot 0.10 = 0.0450$
WD	match won	$0.45 \cdot 0.90 = 0.4050$

The tiebreaker happens with the probability of 0.2925, so A is going to play a daring game, winning with the probability of 0.45. This makes the total probability of his winning the match equal to

$$0.4050 + 0.45 \cdot 0.2925 = 0.536625 \approx 54\%.$$

Solution 2

N.N. Taleb

The approach is to use a simplified form of stochastic dynamic programming, given that we are initially facing a 2 period decision tree.

We have two regimes indexed by 1 (aggressive), 2 (conservative) and their corresponding probabilities:

$$P_1 = \begin{pmatrix} p_{w_1} \\ p_{d_1} \\ p_{l_1} \end{pmatrix} \qquad P_2 = \begin{pmatrix} p_{w_2} \\ p_{d_2} \\ p_{l_2} \end{pmatrix}.$$

We also have a payoff vector per period for win, draw or lose of $\lambda = (1, 0, -1)$.

We look at the expectation at every step. Note that for the dot product $P_1.(1+\lambda)$ can have for notation $P_1.(1+\lambda^T)$.

$$\max \begin{pmatrix} p_{w_1}.P_1.(1+\lambda)(\text{aggressive, win, then aggressive}) \\ p_{w_1}.P_2.(1+\lambda)(\text{aggressive, win, then conservative}) \\ \\ p_{d_1}.P_1.(\lambda)(\text{aggressive, draw, then aggressive}) \\ p_{d_1}.P_2.(\lambda)(\text{aggressive, draw, then conservative}) \\ \\ p_{l_1}.P_1.(-1+\lambda)(\text{aggressive, loss, then aggressive}) \\ p_{l_1}.P_2.(-1+\lambda)(\text{aggressive, loss, then conservative}) \\ \\ p_{w_2}.P_1.(1+\lambda)(\text{conservative, win, then aggressive}) \\ p_{w_3}.P_2.(1+\lambda)(\text{conservative, win, then conservative}) \\ \\ p_{d_2}.P_1.(\lambda) \\ p_{d_2}.P_2.(\lambda) \\ \\ p_{l_2}.P_1.(-1+\lambda) \\ p_{l_2}.P_2.(-1+\lambda) \end{pmatrix}$$

Chickens in Boxes

[71]

Solution 1

All in all, there are 16 equiprobable ways to place chickens into four boxes. Only three of them satisfy the conditions of the problem:

Box	1	2	3	4
	chicken			chicken
		chicken		chicken
			chicken	

Thinking of the three possibilities as equiprobable, the corner box 3 contains a chicken only once; hence, the chance of this happening is 1 : 2. The same is true of boxes 1 and 2. Box 4, on the other hand, contains a chicken twice; hence, it contains a chikcen with the chances of 2 : 1.

Solution 2

Probability that the corner box contains a chicken, provided there is only one chicken, $P(B_3|1\text{ chicken}) = 1$; in the case of two chickens, $P(B_3|2\text{ chickens}) = 0$. The first event occurs in one case (see the table above); the second event occurs in two cases. Therefore, the chances that the corner box contains a chicken are 1 : 2.

Similarly, for box 4, $P(B_4|1\text{ chicken}) = 0$; $P(B_4|2\text{ chickens}) = 1$. The first event occurs in one case, the second in two. Hence, for box 4, the chances are 2 : 1.

By the same reasoning, $P(B_1|1\text{ chicken}) = 0$; $P(B_1|2\text{ chickens}) = \frac{1}{2}$. Therefore, the chances for B_1 (and, similarly, B_2) are $\frac{1}{2} : 1 = 1 : 2$.

Two Chickens in Boxes

Solution 1

As in the previous riddle, there are 16 equiprobable chicken placements, of which five satisfy the conditions of the problem:

Box	1	2	3	4
	chicken			chicken
		chicken		chicken
			chicken	chicken
	chicken		chicken	
		chicken	chicken	

Thus, boxes 1 and 2 contain a chicken with the probability $\frac{2}{5}$; for boxes 3 and 4, the probability is $\frac{3}{5}$.

Solution 2

Joshua B. Miller

Chicken placements $CCxx$, $CxCx$, $CxxC$, $xCCx$, $xCxC$, $xxCC$ are equally likely. It follows that prior odds in box 3 are 1 : 1,

$$\Pr(\text{“yes”}|\text{in box 3}) = 1,$$

$$\Pr(\text{“yes”}|\text{not in box 3}) = \frac{2}{3}$$

and posterior odds are $\frac{3}{2} : 1$, or 3 : 2, i.e.,

$$\Pr(\text{in box 3}) = \frac{3}{5}.$$

Two Chickens in Bigger Boxes

Solution 1

As in the earlier riddles, there are 16 equiprobable chicken placements, of which five satisfy the conditions of the problem:

Box	1	2	3	4
	chicken			chicken
		chicken		chicken
			chicken	chicken
	chicken		chicken	
		chicken	chicken	
			2 chickens	

Thus, boxes 1 and 2 contain a chicken with the probability $\frac{1}{3}$; for box 3, the probability is $\frac{2}{3}$; for box 4, it is $\frac{3}{6} = \frac{1}{2}$.

Solution 2

Let us follow where the chickens go. Each has equal probabilities of getting into any of the four boxes. Some configurations are OK, while some violate the constraints of the riddle:

chick 1 in box / chick 2 in box	1	2	3	4
1	X	X	OK!	OK
2	X	X	OK!	OK
3	OK!	OK!	OK!	OK!
4	OK	OK	OK!	X

There are 11 legitimate, equiprobable situations: in seven of them, the corner box is not empty; in six, box 4 is not empty; in four, boxes 1 and 2 are each not empty. Thus, the probabilities for the four boxes not to be empty come out as $\frac{4}{11}, \frac{4}{11}, \frac{7}{11}, \frac{6}{11}$.

Notes

Why are there two solutions and which is correct, as the two cannot be correct at the same time? The difference between the two is in that the first one treats the chickens as indistinguishable while the second presumes that they are different: chicken 1 and chicken 2.

I do not believe that the problem is suggestive of the idea of two different chickens, although solution 2 has been posted on Twitter several times. I included it here to give a reason to remark on that topic.

Chapter 8

Expectation

> If you hear a "prominent" economist using the word 'equilibrium,' or 'normal distribution,' do not argue with him; just ignore him, or try to put a rat down his shirt.
>
> Nassim Nicholas Taleb, *The Black Swan: The Impact of the Highly Improbable*

What Is Expectation?

In a sample space of equiprobable outcomes, the probability of an event is the ratio of the number of favorable outcomes to the total size of the space. This means that the probabilities of events are defined in relation to each other. If there is a finite number of exhaustive (i.e., at least one always happens) and mutually exclusive (i.e., no two happen simultaneously) events A_k, $k = 1, 2, \ldots, K$, with n_k being the number of favorable outcomes in A_k, then

$$P(A_k) = \frac{n_k}{N},$$

where $N = n_1 + n_2 + \cdots + n_K$.

Borrowing an example from the classic text by W. Feller [38], in a certain population, n_k is the number of families with k children. N is then the total number of families and, assuming the most benign of circumstances, there are $2N$ adults. How many children are there? To state the obvious, each of the n_k families with k kids has k kids, so the number of kids in such families is kn_k. The total number of kids is the sum of such products over all the various family sizes:

$$T = 1 \cdot n_1 + 2 \cdot n_2 + \cdots + k \cdot n_k + \cdots + K \cdot n_K,$$

where K is the number of children in the largest family. On average, every family has $E = \dfrac{T}{N}$ kids:

$$\begin{aligned} E &= \frac{T}{N} \\ &= 1 \cdot \frac{n_1}{N} + 2 \cdot \frac{n_2}{N} + \cdots + K \cdot \frac{n_K}{N} \\ &= 1 \cdot p_1 + 2 \cdot p_2 + \cdots + K \cdot p_K \\ &= \sum_{k=0}^{K} k \cdot p_k \end{aligned}$$

where $p_k = \dfrac{n_k}{N}$ is the probability for a family to have k children. Expectation is one of the most important notions in the theory of probabilities. We shall give a more general definition.

Let $\mathbf{X}$ be a random variable that takes values x_k, $k = 1, 2, \ldots, K$, with the probability p_k: $P(\mathbf{X} = x_k) = p_k$. The sum

$$E(\mathbf{X}) = \sum_{k=1}^{K} x_k p_k$$

is known as the *mathematical expectation* of $\mathbf{X}$ (and often the *expected value*, the *mean* or simply the *average*).

The main feature of the expectation is its linearity: for two random variables $\mathbf{X}$ and $\mathbf{Y}$, and two real numbers a and b,

$$E(a\mathbf{X} + b\mathbf{Y}) = aE(\mathbf{X}) + bE(\mathbf{X}).$$

This is actually a direct consequence of the additive definition.

Riddles

8.1 Randomly Placed Letters in Envelopes

[72, Problem 104]

N letters are randomly placed into N envelopes, one per envelope. What is the expected number of letters which get into the correct envelopes?

8.2 Expected Number of Fixed Points

The previous riddle of misplaced letters/envelopes admits a more formal formulation [51, Chapter 5]:

> Let $\mathbb{N}_n = \{1, 2 \ldots, n\}$. Consider a permutation p of $\mathbb{N}_n$. A point k is *fixed* by p if $p(k) = k$. Permutations with no fixed points are known as *derangements*.

Considering all possible permutations, what is the expected number of fixed points?

8.3 An Average Number of Runs in Coin Tosses

A coin is tossed several times and the outcomes are recorded in a string of H (heads) and T (tails). For example

is recorded as $HHTTTHT$. Now, a *run* is any part of such a record that consists of a maximal number of contiguous equal outcomes. In the example, we have four runs: HH, TTT, H, T. Their lengths are 2, 3, 1, 1, respectively. Now, here is the question:

> A coin is tossed N times. What is the expected value (average) of the number of runs?

8.4 How Long Will It Last?

[72, Problem 98]

In a coin tossing game, you win if our tosses match; I win if our tosses differ. The winner collects both coins. You start with m coins; I start with n coins.

How long, on average, will the game last until one of us is wiped out?

8.5 Expectation of Interval Length on Circle

[38, Volume 2, Chapter 1]

Two points, X and Y are uniformly distributed on a circle of circumference 1.

- What is the expected arc length of interval |XY|?
- Point P is fixed on the circle. What is the expected arc length of interval |XY| that contains point P?
- What if there are three random points?
- What if there are n random points?

8.6 Waiting for a Train

[12, pp. 22–23]

Every morning at a railroad station there is just one train arriving between 8:00 and 9:00 and another arriving between 9:00 and 10:00. The arrival times and their probabilities are shown in the following table:

Train A	8:10	8:30	8:50
Train B	9:10	9:30	9:50
Probability	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$

A traveler arrives at 8:20. What is the expectation of his waiting time?

8.7 Average Number of Runs in a Sequence of Random Numbers

[54, pp. 29–33]

M numbers are randomly selected from the set $\{1, \ldots, N\}$. A run is either a decreasing or an increasing subsequence of maximum stretch, i.e., “subruns” do not count as runs.

Assuming no number was selected more than once in succession, what is the average number of runs?

8.8 Training Bicyclists on a Mountain Road

[26, Problem 65352]

In the movie *GoldenEye*, racing James Bond startles a group of bicyclists so that they fall over in a chain reaction. Relaxing the story, assume each of the n bicyclists in a line tumbles with the probability of p. Once one of them tumbles, all following him tumble too.

What is the expected number of bicyclists who fall on the road?

8.9 Number of Trials to First Success

Let V be the event that occurs in a trial with probability p. Is the mathematical expectation of the number of trials to the first occurrence of V in a sequence of trials $\frac{1}{p}$?

8.10 Waiting for an Ace

[69, Problem 40]

After a thorough shuffle of a deck of 52 cards, you deal cards from the top of the deck. How long—on average—will you have to wait for an ace?

8.11 Two in a Row

[46, Problem P1983–2]

Suppose you repeatedly toss a fair coin until you get two heads in a row. What is the probability that you stop on toss number 10?

8.12 Waiting for Multiple Heads

Flipping a coin, heads comes up with the probability of p, $0 < p < 1$.

- What is the expected number of flips before getting three heads for the first time?
- What is the expected number of flips before getting four heads for the first time?

8.13 Probability of Consecutive Heads

Josh Jordan

A biased coin takes 100 flips on average to get 10 heads in a row. What is the bias?

8.14 Two Dice Repetition

[66, Problem 409]

There are two fair dice, one marked as usual with 1, 2, 3, 4, 5, 6, the other with $B, I, N, G, O, *$. Both dice are tossed on a move. What is the expected number of the move at which the first repetition occurs in each of these cases:

1. All 36 combinations ($1B$ through $6*$) are considered to be different (and equally likely)?
2. As before, but all combinations $1*, 2*, 3*, 4*, 5*, 6*$ are considered the same?

8.15 Expected Number of Happy Passengers

This is a follow-up to Riddle 7.8 on page 158; the current riddle was suggested to me by Konstantin Knop and to him by Alexander Pipersky.

On a sold out flight, 100 people line up to board the plane. The first passenger in the line has lost his boarding pass, but is allowed in, regardless. He takes a random seat. Each subsequent passenger takes his or her assigned seat if available, or a random, unoccupied seat, otherwise.

A passenger is happy if he/she occupies his/her own seat, unhappy otherwise. What is the expected number of happy passengers?

8.16 Expectation of the Largest Number

n balls are drawn, without replacement, from an urn containing $N > n$ balls, numbered 1 to N.

What is the expected value of the largest number drawn?

8.17 Waiting for a Larger Number

[70, pp. 31–32]

In what follows all numbers are uniformly distributed on $[0, 1]$. Start with a number S_0 and then keep generating the sequence $S_1, S_2, \ldots$ until, for some N, $S_N > S_0$ for the first time.

What is the expected value $E(N)$ of that N?

8.18 Waiting to Exceed 1

[70, pp. 31–32]

In what follows all numbers are uniformly distributed on $[0, 1]$. Start with a number S_0 and then keep generating the sequence $S_1, S_2, \ldots$ until, for some N, $\sum_{k=0}^{N} S_k > 1$ for the first time.

What is the expected value $E(N)$ of that N?

8.19 Waiting for All Six Outcomes

[102, pp. 35–37]

On average, how many times do you need to roll a die before all six different numbers show up?

8.20 Walking Randomly—How Far?

[64, Problem 115]

A fair coin is flipped repeatedly. Starting from $x = 0$, each time the coin comes up heads, 1 is added to x, and each time the coin comes up tails, 1 is subtracted from x. Let a_n be the expected value of $|x|$ after n flips of the coin.

Does $a_n \to \infty$ as $n \to \infty$?

8.21 Expectation of Pairings

[4, Problem 22]

Suppose that eight boys and 12 girls line up in a row. Let S be the number of places in the row where a boy and a girl are standing next to each other. For example, for the row

$$GBBGGGBGBGGGBGBGGBGB,$$

we have $S = 13$.

Find the average value of S (if all possible orders of these 20 people are considered).

8.22 Making Spaghetti Loops in the Kitchen

This riddle relates to Riddles 6.20 and 6.21. In [101, pp. 3, 7] P. Winkler also refers to Martin Gardner's [42].

The 100 ends of 50 strands of cooked spaghetti are paired at random and tied together. How many pasta loops should you expect to result from this process, on average?

8.23 Repeating Suit When Dealing Cards

[68, Exercise 9.10.13]

Consider a standard deck of 52 cards, thoroughly shuffled, making all card orderings equally likely. We pick cards one at a time until we get two cards of the same suit. What is the expected number of cards drawn before two cards of the same suit are drawn? Consider two cases:

1. Drawing cards with replacement.
2. Drawing cards without replacement.

Before running the calculations, guess which is smaller.

8.24 Family Size

[70]

A young couple plans to start a family. They consider five different strategies. What is the expected family size

1. if they stop having children after having one boy?
2. if they stop having children after having one girl?
3. if they stop after having children of both sexes?
4. if they stop after they have a child of the same sex as the first one?
5. if they stop after they have a child of the sex different from that of their first child?

8.25 Averages of Terms in Increasing Sequence

[53, B–425]

Let k and n be positive integers with $k < n$ and let S consist of all k-tuples $X = (x_1, x_2, \ldots, x_k)$, with each x_j an integer and $1 \le x_1 < x_2 < \ldots < x_k \le n$.

For $j = 1, 2, \ldots, k$, find the average value $\overline{x_j}$ of x_j over all X in S.

Solutions

Randomly Placed Letters in Envelopes

Each letter has the probability of $\frac{1}{N}$ to get into the right envelope. If E_k is the event of the k^{th} letter getting into the right envelope,

$$E_k = \begin{cases} 1, & \text{if the } k^{\text{th}} \text{ letter gets into the right envelope} \\ 0, & \text{otherwise.} \end{cases}$$

The expectation of E_k is $E(E_k) = 1 \cdot \frac{1}{N} + 0 \cdot \frac{N-1}{N} = \frac{1}{N}$, for all k.

Although the events are not independent, the expectation being linear, we still have

$$E\left(\sum_{k=1}^{N} E_k\right) = \sum_{k=1}^{N} E(E_k) = N \cdot \frac{1}{N} = 1.$$

Thus, for all N, the expected number of correct fits is one. For $N = 2$, either both letters get into the right envelopes, or none does. Both events have the probability of $\frac{1}{2}$, with the expectation of

$$E(E_1 \cup E_2) = 0 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} = 1.$$

Expected Number of Fixed Points

Solution 1

[51, Chapter 5]

The expected number in question, which we shall denote $E(N)$, equals

$$E(n) = \sum_{k=0} kq_k,$$

where q_k is the probability of there being exactly k fixed points. The number of permutations with exactly k fixed points is the number of ways of choosing k points out of n times the number of derangements of the remaining $n-k$ points. It follows that

$$q_k = \frac{\binom{n}{k} \cdot !(n-k)}{n!},$$

where $!m$ is the *subfactorial* of m, the number of derangements of a set of m elements. Thus, with a substitution $m = n - k$ and using the symmetry of the binomial coefficients, $\binom{n}{n-m} = \binom{n}{n}$, we get

$$\begin{aligned}
E(n) &= \sum_{k=0} k \cdot \frac{\binom{n}{k} \cdot !(n-k)}{n!} = \sum_{m=0}^{n} (n-m) \frac{\binom{n}{m} \cdot !m}{n!} \\
&= \sum_{m=1}^{n} (n-m) \frac{\binom{n}{m} \cdot !m}{n!} = \sum_{m=1}^{n} (n-m) \frac{(n-m) \cdot !m}{m!(n-m)!} \\
&= \frac{!m}{m!(n-m-1)!}.
\end{aligned}$$

Multiplying both sides by $(n-1)!$,

$$\begin{aligned}
(n-1)! \cdot E(n) &= \sum_{m=1}^{n} (n-m) \frac{(n-1)!}{m!(n-m-1)!} \cdot !m \\
&= \sum_{m=1}^{n} \binom{n-1}{m} \cdot !m.
\end{aligned}$$

Still, it is easily seen that $\sum_{m=1}^{n} \binom{n-1}{m} \cdot !m = (n-1)!$ because every permutation has a certain number of fixed points with the remaining points deranged. (As above, but with $n-1$ points, there are $\binom{n-1}{n-m-1} \cdot !m = \binom{n-1}{m} \cdot !m$ permutations with m fixed points.) Thus, from $(n-1)! \cdot E(n) = (n-1)!$, we get $E(n) = 1$, independent of n.

Solution 2

Roland van Gaalen

Use the indicator function $\mathbb{I}(k) = 1$ if there is a fixed point at k and 0 otherwise.

$E[\mathbb{I}(k)] = \dfrac{1}{n}$ for each k. By the additivity of the expected value,

$$E(n) = \sum_{k=1}^{n} E[\mathbb{I}(k)] = n \cdot \frac{1}{n} = 1.$$

Solution 3

Amit Itagi

Of $n!$ permutations, $(n-1)!$ fix a particular point. Thus, the total number of fixed points over all permutations is $n \times (n-1)!$. It follows that the expected number is $\dfrac{n \times (n-1)!}{n!} = 1$.

An Average Number of Runs in Coin Tosses

Eli Bogomolny

Instead of counting the number of runs, we will count the number of breaks between the runs, i.e., the number of pairs HT or TH. Obviously, the number of breaks is one less than the number of runs.

As has been noted, a break occurs when two successive tosses have different outcomes. All in all, two coin tosses may have four outcomes: the two tosses are different in two cases and equal also in two. Thus, the probability of a break occurring over a particular pair of tosses is $\frac{1}{2}$.

For N tosses of the coin, there are $N-1$ successive pairs of tosses, each having an expectation of $0 \times \frac{1}{2} + 1 \times \frac{1}{2} = \frac{1}{2}$ of constituting a break. The *expectation* being a linear function, the expected number of breaks among $N-1$ pairs is $\frac{N-1}{2}$.

It follows that the average number of runs is $\frac{N-1}{2} + 1 = \frac{N+1}{2}$.

How Long Will It Last?

Solution 1

Let us denote $K = m + n$. There are two ways the game may end: the winner is either you or me. Conditioned on me being the winner, let $L(x)$, $0 \le x \le K$, be the expected length of the game if I have x coins. There is an obvious recurrence relation

$$L(x) = \frac{1}{2}L(x-1) + \frac{1}{2}L(x+1) + 1,$$

as I can win or lose a coin with the same probability of $\frac{1}{2}$. One is added to the account for the present toss.

Thus, we have an inhomogeneous difference equation of order two:

$$L(x+1) - 2L(x) + L(x-1) + 2 = 0.$$

The characteristic polynomial of the homogeneous equation has a double root 1 such that the general solution of the inhomogeneous equation is a quadratic polynomial, say $L(x) = u + vx + wx^2$. Substituting that into the recurrence yields $2w = -2$ so that $w = -1$.

For the boundary conditions, we have two: $L(0) = 0$ and $L(K) = 0$. From the first one, $u = 0$, and from the second, $v = K$. Thus, the solution is $L(x) = x(K - x)$. If I start with n coins, $L(n) = n(K - n) = mn$.

Naturally, the expectation is symmetric in n and m.

Solution 2

[72, p. 106]

Let $E(n, m)$ be the expected length of the game in the case I start with n and you start with m coins. On a toss, we will move to either $E(n-1, m+1)$ or $E(n+1, m-1)$, each with the probability of $\frac{1}{2}$, which leads to a recurrence relation:

$$E(n, m) = 1 + \frac{1}{2}E(n-1, m+1) + \frac{1}{2}E(n+1, m-1),$$

1 indicating the first toss. Of course this assumes $n, m > 0$. The boundary conditions are $E(0, m) = 0$ and $E(n, 0) = 0$.

If we view $E(n, m)$ as the function of one variable, say, n along the line $m + n =$ constant, then the formula says that the second difference is a constant (-2), and so $E(n, m)$ is a quadratic function. Vanishing it at the endpoints forces this to be cmn, and direct evaluation shows that $c = 1$, implying $E(n, m) = mn$.

Solution 3

Christopher D. Long

Let $E(k)$ be the expected value of the length of the game, if, say, I am in possession of k coins. Then

$$E(k) = \frac{1}{2}(E(k+1) + E(k-1)).$$

It is convenient to reduce that to a difference equation:

$$(E(k+1) - E(k)) - (E(k) - E(k-1)) = -2.$$

In terms of the difference operator Δ, $\Delta^2 E = -2$, implying that E is a quadratic function:

$$E(k) = -k(k-1) + ak + b$$

with the boundary conditions $E(0) = 0$ and $E(m + n) = 0$. From the former, $b = 0$ and, from the latter, $a = m + n - 1$ so that $E(k) = -k(k-1) + k(m + n - 1)$. At the beginning of the game,

$$E(m) = -m(m-1) + m^2 + mn - m = mn.$$

Expectation of Interval Length on Circle

Solution 1

Fix a direction on the circle. If |XY| denotes the arc length from point X to point Y, then |XY| + |YX| = 1 and, by symmetry, the two intervals have the same distribution. Thus, the average length of either is $\frac{1}{2}$.

If there are three random points then, by the same token, the expected length of any of the intervals is $\frac{1}{3}$. The same holds for two random points dropped onto the segment $[0, 1]$. Dropping n points on $[0, 1]$ is equivalent to dropping $(n+1)$ points onto the circle. This means that if one point is fixed on the circle, it falls into an interval which is the union of the two intervals with that point as one of the ends. It follows that P belongs to the interval whose average length is twice that of other intervals.

Thus, if there are n random points, the average arc length of an interval is $\frac{1}{n}$. With n random points and one fixed, the expected length of the interval that contains the fixed point is $\frac{2}{n+1}$.

Solution 2

N.N. Taleb

We are looking for $f(.)$, the probability distribution function of the distribution of $z = \max\left(|x-y|, |-x-y+1|\right)$:

$$f(z) = \begin{cases} 4-4z & \frac{1}{2} < z < 1 \\ 4z & 0 < z \le \frac{1}{2} \\ 0 & \text{elsewhere.} \end{cases}$$

Finally,

$$\int_0^1 z f(z)\, dz = \frac{1}{2}.$$

Waiting for a Train

There is a probability of $\frac{1}{6}$ of the traveler missing train A by 10 minutes. The distribution of the waiting times is shown in the following table:

Time in min	10	30	50	70	90
Probability	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}\cdot\frac{1}{6}$	$\frac{1}{6}\cdot\frac{1}{2}$	$\frac{1}{6}\cdot\frac{1}{3}$

The expectation is:

$$10\cdot\frac{1}{2} + 30\cdot\frac{1}{3} + 50\cdot\frac{1}{36} + 70\cdot\frac{1}{12} + 90\cdot\frac{1}{18} = \frac{980}{36} \approx 27 \text{ min.}$$

Notes

A curious thing about this riddle is that, assuming that the traveler comes a little earlier as not to miss train A, say at 8:05, then the expected waiting time will grow:

$$5\cdot\frac{1}{6} + 25\cdot\frac{1}{2} + 45\cdot\frac{1}{3} \approx 28.$$

Average Number of Runs in a Sequence of Random Numbers

Solution 1

A run can start at any one term of the sequence but the last one and there is always a run that starts at the very first term. If p_k is the probability of a run starting in position k, then clearly $p_1 = 1$ while $p_M = 0$. The average number of runs is simply $R = \sum_{k=1}^{M} p_k$, or $R = 1 + \sum_{k=2}^{M-1} p_k$.

Let n_k be the number in the k^{th} position. The probability p_k of a run starting at position k is complementary to the event that, in magnitude, n_k is between its two neighbors: either $n_{k-1} < n_k < n_{k+1}$ or $n_{k-1} > n_k > n_{k+1}$. The latter two events are equiprobable and each comes with the probability of $\dfrac{\binom{N}{3}}{N(N-1)^2}$ because there are $\binom{N}{3}$ ways to select three numbers out of N and only one way to order them in a given order, either increasing or decreasing. There are N ways to choose the first number and $N-1$ ways to choose each of the other two. Thus, the probability of a run starting at the k^{th} position is

$$\begin{aligned}
p_k &= 1 - 2\frac{\binom{N}{3}}{N(N-1)^2} = 1 - 2\frac{N(N-1)(N-2)}{3! \cdot N(N-1)^2} \\
&= 1 - \frac{N-2}{3(N-1)} \\
&= \frac{2N-1}{3(N-1)}.
\end{aligned}$$

The average number of runs, then, is simply

$$R = 1 + \sum_{k=2}^{M-1} p_k = 1 + (M-2)\frac{2N-1}{3(N-1)}.$$

Solution 2

Joshua B. Miller

An alternation is a transition from an up run to a down run. The number of runs is one plus the number of alternations:

$$R(x_1, \ldots, x_M) := 1 + \sum_{i=2}^{M-1} [x_i < \min\{x_{i-1}, x_{i+1}\}] + \sum_{i=2}^{M-1} [x_i > \max\{x_{i-1}, x_{i+1}\}]$$

$$\begin{aligned}
E[R(x_1, \ldots, x_M)] = 1 &+ \sum_{i=2}^{M-1} \Pr\left(x_i < \min\{x_{i-1}, x_{i+1}\}\right) \\
&+ \sum_{i=2}^{M-1} \Pr\left(x_i > \max\{x_{i-1}, x_{i+1}\}\right).
\end{aligned}$$

Note 1

$$\begin{aligned}
\Pr\left(x_i < \min\{x_{i-1}, x_{i+1}\}\right) &= \sum_{k=1}^{N-1} \Pr(\min\{x_{i-1}, x_{i+1}\} > k \mid x_i = k)\Pr(x_i = k) \\
&= \sum_{k=1}^{N-1} \left(\frac{N-k}{N-1}\right)^2 \frac{1}{N} \\
&= \frac{2N-1}{6N-6},
\end{aligned}$$

where the second line follows from conditional independence.

Note 2

$$\begin{aligned}\Pr\left(x_i > \max\{x_{i-1}, x_{i+1}\}\right) &= \sum_{k=2}^{N} \Pr(\max\{x_{i-1}, x_{i+1}\} < k \mid x_i = k)\Pr(x_i = k)\\ &= \sum_{k=2}^{N} \left(\frac{k-1}{N-1}\right)^2 \frac{1}{N}\\ &= \frac{2N-1}{6N-6}.\end{aligned}$$

Combining these together,

$$\begin{aligned}E[R(x_1, \ldots, x_M)] &= 1 + 2(M-2)\frac{2N-1}{6N-6}\\ &= 1 + (M-2)\frac{2N-1}{3(N-1)}.\end{aligned}$$

Note 3

For $N = 2$, the formula reduces to $1 + (M-2)\dfrac{2 \cdot 2 - 1}{3(2-1)} = M - 1$, implying that the experiment is only possible if $M \le N$. There are just two possible outcomes: $1, 2, \ldots, N$ and $N, N-1, \ldots, 1$. This is due to the stipulation that no two successive selections may be equal. If the stipulation is removed, then the riddle becomes equivalent to finding the average length of the runs in coin tossing, which, as we know (Riddle 8.7 on page 193), equals $\dfrac{M+1}{2}$.

Training Bicyclists on a Mountain Road

Solution 1

A. Bogomolny

Let $q = 1 - p$ be the probability that a bicyclist will not tumble on his own. Then the probability that bicyclist number k (counting from the head of the line) does not fall equals q^k.

Let I_k be the indicator function for the event that bicyclist number k tumbles down:

$$I_k = \begin{cases} 1, & \text{if the } k^{\text{th}} \text{ bicyclist tumbles} \\ 0, & \text{otherwise.} \end{cases}$$

Then $P(I_k = 0) = q^k$ so that $P(I_k = 1) = 1 - q^k$ and $\mathbb{E}(I_k) = 1 - q^k$.

We are interested in the expected number of bicyclists who eventually tumble, $\mathbb{E}(n)$:

$$\begin{aligned}\mathbb{E}(n) &= \sum_{k=1}^{n} \mathbb{E}(I_k) = \sum_{k=1}^{n} (1 - q^k)\\ &= n - q\frac{1-q^n}{1-q} = n - \frac{1-p}{p}(1 - (1-p)^n).\end{aligned}$$

Solution 2

N.N. Taleb

n tumble with probability p (after the first does). $n-1$ tumble with probability $p(1-p)$ (in the case where the second one tumbles). $n-2$ tumble with probability $p(1-p)^2$ and so on. 1 tumbles with probability $p(1-p)^{n-1}$. Summing up,

$$\sum_{i=0}^{n-1} p(n-i)(1-p)^i = \frac{-1+(1-p)^{n+1}+p+np}{p}.$$

Solution 3

Alejandro Rodríguez

The probability that cyclist i is the first one to fall is equal to the probability that the previous $i-1$ cyclist did not fall, $(1-p)^{i-1}$, multiplied by the individual probability of falling p:

$$P_i = p(1-p)^{i-1}.$$

If cyclist i falls then the remaining $n-i$ clyclists fall too for a total of $x_i = n+1-i$. The expected value of x is then equal to

$$E(x) = \sum_{i=1}^{n} P_i x_i.$$

Substitute and split the sum:

$$\begin{aligned} E(x) &= \sum_{i=1}^{n} p(1-p)^{i-1}(n+1-i) \\ &= \frac{p}{1-p}[(n+1)\sum_{i=1}^{n}(1-p)^i - \sum_{i=1}^{n} i(1-p)^i]. \end{aligned}$$

Now, we use the following results:

$$\sum_{i=1}^{n} a^i = \frac{a(1-a^n)}{1-a}$$

$$\sum_{i=1}^{n} ia^i = \frac{a(1-a^n)}{(1-a)^2} - \frac{n(1-a^{n+1})}{1-a}$$

$$\begin{aligned} E(x) = \frac{p}{1-p}\Bigg[&(n+1)\frac{(1-p)(1-(1-p)^n)}{1-(1-p)} - \frac{(1-p)(1-(1-p)^n)}{(1-(1-p))^2} \\ &+ \frac{n(1-(1-p)^{n+1})}{1-(1-p)}\Bigg] \end{aligned}$$

which yield

$$E(x) = n - \frac{1-p}{p}(1-(1-p)^n).$$

Number of Trials to First Success

Solution 1

We shall call an occurrence of V in a trial a *success*; a trial is a *failure* otherwise.

The event will occur on the first trial with probability p. If that fails, it will occur on the second trial with probability $(1-p)p$. If that also fails, the probability of the event coming up on the third trial is $(1-p)^2p$.

More generally, the probability of the first success on the n^{th} trial is $(1-p)^{n-1}p$. We are interested in the expected value:

$$E = p + 2(1-p)p + 3(1-p)^2p + \cdots + n(1-p)^{n-1}p + \ldots.$$

One way to determine E is to use a probability generating function:

$$f(x) = p + 2(1-p)px + 3(1-p)^2px^2 + \ldots + n(1-p)^{n-1}px^{n-1} + \ldots.$$

By definition, $E = f(1)$. (Note that, since, the series converges for $x = 1$, $f(1)$ does exist.) To find f in closed form, we first integrate term by term and then differentiate the resulting function:

$$\begin{aligned}
F(x) = \int f(x)dx &= \int \sum_{n=0}(n+1)(1-p)^n px^n dx \\
&= \sum_{n=0}(1-p)^n p \int (n+1)x^n dx \\
&= \sum_{n=0}(1-p)^n px^{n+1} + C \\
&= \frac{p}{1-p}\sum_{n=0}(1-p)^{n+1}x^{n+1} + C \\
&= \frac{p}{1-p}\frac{(1-p)x}{1-(1-p)x} + C \\
&= \frac{px}{1-(1-p)x} + C.
\end{aligned}$$

Now differentiate

$$f(x) = F'(x) = \frac{p}{(1-(1-p)x)^2},$$

from which $f(1) = \dfrac{p}{p^2} = \dfrac{1}{p}$, as required.

Solution 2, Shortcut

There is a shortcut that makes the finding of E more transparent. Obviously, the expectation of the first success counting from the second trial is still E. Taking into account the first trial, we can say that, with probability $1-p$, the expected number of trials to the first success is $E+1$, while it is just 1 with probability p. This leads to a simple equation:

$$E = p + (1-p)(E+1) = 1 + E(1-p).$$

Solving this equation for E gives $E = \dfrac{1}{p}$:

$$\begin{aligned}
E &= p\sum_{n=0}(n+1)(1-p)^n \\
&= p\sum_{n=0} n(1-p)^n + p\sum_{n=0}(1-p)^n \\
&= p\sum_{n=1} n(1-p)^n + p\cdot\frac{1}{1-(1-p)} \\
&= (1-p)p\sum_{n=0}(n+1)(1-p)^n + 1 \\
&= (1-p)E + 1.
\end{aligned}$$

Waiting for an Ace

Solution 1

[69, Problem 40]

Four aces divide the deck into five parts, each of which may contain 0 to 48 cards. According to the symmetry principle (Appendix B), all five have, on average, the same length: $\dfrac{48}{5} = 9.6$. The next card after an aceless run should be an ace—a card number 10.6.

For this riddle, the symmetry principle may be formulated as follows:

> When n (uniformly) random points are picked on a segment, the $n+1$ parts so created have the same length distribution.

Solution 2

N.N. Taleb

Note the successive waiting times, excluding events. We have the following probabilities:

1. $\dfrac{4}{52}$, waiting time 0,
2. $\dfrac{4}{51}\left(1-\dfrac{4}{52}\right)$, waiting time 1,
3. $\dfrac{4}{50}\left(1-\dfrac{4}{52}\right)\left(1-\dfrac{4}{51}\right)$, waiting time 2

and so on. The expectation becomes

$$\sum_{i=1}^{48} i\frac{4}{52-i}\prod_{j=0}^{i-1}\left(1-\frac{4}{52-j}\right) = \frac{48}{5}.$$

The expression could also be written as

$$\sum_{i=0}^{47} \frac{4(i+1)\prod_{j=0}^{i}\left(1-\dfrac{4}{52-j}\right)}{52-1-i} = \frac{48}{5}.$$

The former seems self-explanatory.

Notes

N.N. Taleb, M. Wiener

For a total sample size of n with the initial probability of p, the expected time to the first success is

$$E(\tau) = \frac{n+1}{pn+1}.$$

Note that, as $n \to \infty$, this converges to the solution $\frac{1}{p}$ of the aforementioned problem with replacement.

Two in a Row

Solution 1

Let X_n stand for the result of the n^{th} toss. We will look into the more general problem of having to stop at the n^{th} toss. This would mean $X_n = X_{n-1} = H$ and $X_{n-2} = T$. In addition, the sequence of the previous tosses should not contain two consecutive heads.

Such a sequence may start with either T or HT, and the remainder ought to be a sequence with the same property—no HH. Thus, we get a recurrence: $S_k = S_{k-1} + S_{k-2}$, where S_k is the number of sequences with no HH of length k. Obviously, $S_1 = 2$, $S_2 = 3$, meaning $S_k = F_{k+2}$, where F_n is the Fibonacci sequence

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots \text{ and } 0 = F_0.$$

There are 2^n possible outcomes for n tosses of a coin. Of these, S_{n-3} end with THH and have no repeated heads beforehand. Therefore, the sought probability is $P_n = \frac{F_{n-1}}{2^n}$, which for $n = 10$ gives $P_{10} = \frac{34}{1024}$.

Solution 2

Amit Itagi

Let Q_k be the event that two consecutive heads do not occur in k consecutive tosses.

A Q_k event ends in one of the following patterns: TH, TT or HT. Suppose the coin is tossed one more time. The probability of Q_{k+1} would be the same as Q_k if not for a possible ending pattern of THH arising from the TH ending pattern in Q_k. Thus, we need to subtract the probability of the THH ending. Hence,

$$P(Q_{k+1}) = P(Q_k) - \frac{1}{8}P(Q_{k-2}).$$

Solving for this recurrence relation,

$$P(Q_k) = c_1\left(\frac{1}{2}\right)^k + c_2\left[\frac{(1-\sqrt{5})}{4}\right]^k + c_3\left[\frac{(1+\sqrt{5})}{4}\right]^k.$$

c_1, c_2 and c_3 are undetermined coefficients that can be solved for using the initial conditions: $P(Q_1) = 1$, $P(Q_2) = \frac{3}{4}$ and $P(Q_3) = \frac{5}{8}$ (obtained by enumerating the cases). This results in

$$P(Q_k) = \frac{(5-3\sqrt{5})}{10}\left[\frac{(1-\sqrt{5})}{4}\right]^k + \frac{(5+3\sqrt{5})}{10}\left[\frac{(1+\sqrt{5})}{4}\right]^k.$$

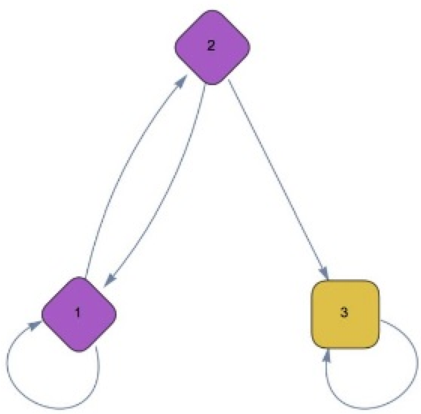
In order to end with two consecutive heads for the first time on toss number 10, Q_7 has to happen followed by THH. Thus, the required probability is

$$\frac{1}{8}P(Q_7) = \frac{(5-3\sqrt{5})}{80}\left[\frac{(1-\sqrt{5})}{4}\right]^7 + \frac{(5+3\sqrt{5})}{80}\left[\frac{(1+\sqrt{5})}{4}\right]^7 = \frac{17}{512}.$$

Solution 3

N.N. Taleb

We have the following Markov chain:



It has the following transition matrix M:

$$\begin{array}{cccc} & \text{N} & \text{H} & \text{HH} \\ \text{N} & \frac{1}{2} & \frac{1}{2} & 0 \\ \text{H} & \frac{1}{2} & 0 & \frac{1}{2} \\ \text{HH} & 0 & 0 & 1 \end{array}$$

The matrix M to the i^{th} power is

$$\left(\begin{array}{ccc} \frac{2^{-2i-1}\left(\sqrt{5}-1\right)\left(1-\sqrt{5}\right)^i}{\sqrt{5}}+\frac{2^{-2i-1}\left(\sqrt{5}+1\right)^{i+1}}{\sqrt{5}} & \frac{2^{-2i-2}\left(\sqrt{5}-1\right)\left(\sqrt{5}+1\right)^{i+1}}{\sqrt{5}}-\frac{2^{-2i-2}\left(1-\sqrt{5}\right)^i\left(\sqrt{5}-1\right)\left(\sqrt{5}+1\right)}{\sqrt{5}} & \frac{2^{-2i-2}\left(\sqrt{5}-1\right)^2\left(1-\sqrt{5}\right)^i}{\sqrt{5}}-\frac{2^{-2i-2}\left(\sqrt{5}+1\right)^{i+2}}{\sqrt{5}}+1 \\ \frac{\left(\frac{1}{4}\left(\sqrt{5}+1\right)\right)^i}{\sqrt{5}}-\frac{\left(\frac{1}{4}\left(1-\sqrt{5}\right)\right)^i}{\sqrt{5}} & \frac{2^{-2i-1}\left(\sqrt{5}+1\right)\left(1-\sqrt{5}\right)^i}{\sqrt{5}}+\frac{2^{-2i-1}\left(\sqrt{5}-1\right)\left(\sqrt{5}+1\right)^i}{\sqrt{5}} & -\frac{2^{-2i-1}\left(\sqrt{5}-1\right)\left(1-\sqrt{5}\right)^i}{\sqrt{5}}-\frac{2^{-2i-1}\left(\sqrt{5}+1\right)^{i+1}}{\sqrt{5}}+1 \\ 0 & 0 & 1 \end{array}\right).$$

The matrix M after i steps, starting from position 1, is

$$\begin{pmatrix}1\\0\\0\end{pmatrix}^T \cdot \mathrm{M}^i =$$

$$\begin{pmatrix} \dfrac{2^{-2i-1}\left(\left(1+\sqrt{5}\right)^{i+1}-\left(1-\sqrt{5}\right)^{i+1}\right)}{\sqrt{5}} \\ \dfrac{4^{-i}\left(\left(1+\sqrt{5}\right)^{i}-\left(1-\sqrt{5}\right)^{i}\right)}{\sqrt{5}} \\ \frac{1}{5}2^{-2i-1}\left(5\cdot 2^{2i+1}+(3\sqrt{5}-5)\left(1-\sqrt{5}\right)^i-(5+3\sqrt{5})\left(1+\sqrt{5}\right)^i\right) \end{pmatrix}.$$

Hence, the cumulative probability P of having HH in i steps or fewer is

$$P(i)=\frac{1}{5}2^{-2i-1}\left(\left(3\sqrt{5}-5\right)\left(1-\sqrt{5}\right)^i+5\;2^{2i+1}-\left(3\sqrt{5}+5\right)\left(\sqrt{5}+1\right)^i\right),$$

which yields the mass generating function,

$$p(i)=P(i)-P(i-1)=\frac{1}{5}2^{-2i-1}\left(\left(\sqrt{5}+5\right)\left(1-\sqrt{5}\right)^i-\left(\sqrt{5}-5\right)\left(\sqrt{5}+1\right)^i\right).$$

By the Binet identity, where F is the Fibonacci number,

$$\begin{aligned} F_{i-1} =& \frac{\left(\sqrt{5}+5\right)^{i-1} - \left(\sqrt{5}-5\right)^{i-1}}{2^{i-1}\sqrt{5}} \\ p(i) =& \frac{F_{i-1}}{2^i}. \end{aligned}$$

Here

$$p(10) = \frac{34}{1024}.$$

The unconditional probability of being in the second game (conditional on both winning previous games but regardless of having been matched) is

$$p_2 = \left(\frac{1}{2^2}\right) \frac{\frac{n}{2^2}}{\binom{\frac{n}{2}}{2}}.$$

The unconditional probability of being in the i^{th} game is

$$p_i = \frac{1}{2^i} \cdot \frac{\frac{n}{2^i}}{\binom{\frac{n}{2^{i-1}}}{2}}.$$

The conditional probability of being in the i^{th} game is

$$\begin{aligned} p_i' =& (1-p_{i-1})\frac{1}{2^i} \cdot \frac{\frac{n}{2^i}}{\binom{\frac{n}{2^i}}{2}} p_1 + (1-p_1)p_2 + (1-p_1)(1-p_2)p_3 \\ P(n) =& \sum_{k=1}^{m} p_k \prod_{j=1}^{k-1} (1-p_j), \\ m =& \frac{\log n}{\log 2}. \end{aligned}$$

Solution 4

Giorgos Papadopulos

For 10 tosses, the ninth and the tenth should be heads (H). Therefore, we concentrate on the first eight tosses:

1. All tails: $\binom{8}{0} = 1$.
2. One head, seven tails. There are seven possible positions of H before and after T, excluding the very last, after all tails: $\binom{7}{1} = 7$.

3. $2H$, $6T$. With the same reasoning as above, there are six distinct positions for H between T's, including the front: $\binom{6}{2} = 15$.

4. And so on.

The total probability is the sum of the cases above divided by 2^{10}:

$$P(10) = \frac{\binom{8}{0} + \binom{7}{1} + \binom{6}{2} + \binom{5}{3} + \binom{4}{4}}{2^{10}} = \frac{34}{1024}.$$

In general, the formula is

$$P(n) = \frac{1}{2^n} \sum_{i=0}^{(n-2)/2} \binom{n-2-i}{i}.$$

Waiting for Multiple Heads

Solution 1, Case $n = 3$

For the first appearance of three heads in n tosses,

1. the last one needs to be a heads,
2. the total number of heads needs to be three (in the limit, the rare event of no heads will play no role),
3. the first two heads could be in any of the first $n - 1$ tosses.

Thus, we get the expectation of the first occurrence of three heads:

$$E_3(p) = \sum_{n=3}^{\infty} n \frac{(n-1)(n-2)}{2!} (1-p)^{n-3} p^3 = \frac{3}{p}.$$

To see that, introduce $f(x) = \sum_{n=3}^{\infty} n \frac{(n-1)(n-2)}{2!} x^{n-3}$. Integrate three times from 0 to x to get $\sum_{n=3} x^n = \frac{x^3}{1-x}$. Differentiate three times to find $f(x) = \frac{3}{(1-x)^4}$. Then substitute $x = 1 - p$ and multiply by $\frac{p^3}{2!}$.

Solution 2, Case $n = 4$

For the first appearance of four heads in n tosses,

1. the last one needs to be a heads,
2. the total number of heads needs to be four (in the limit, the rare event of no heads will play no role),
3. the first three heads could be in any of the first $n - 1$ tosses.

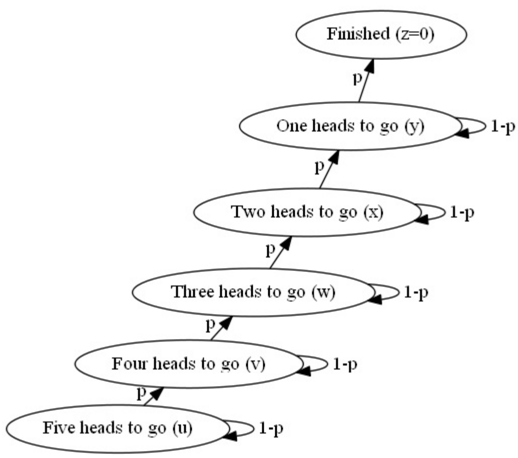
Thus, we get the expectation of the first occurrence of four heads:

$$E_4(p) = \sum_{n=4}^{\infty} n \frac{(n-1)(n-2)(n-3)}{3!}(1-p)^{n-4}p^4 = \frac{4}{p}.$$

Follow the steps described above, but with four integrations followed by four differentiations.

Solution 3, Markov Chain
Josh Jordan

It seems reasonable to conjecture that on average one will need to wait $\frac{N}{p}$ flips before getting N heads the first time. I will describe a process of obtaining this result for $N = 5$ that will make the assertion obvious. The process is based on constructing the Markov chain that describes flipping a coin and counting the number of heads.



There are six nodes, corresponding to the number of heads from zero to five. The one at the bottom corresponds to a zero count of heads. This is, say, state 1. We move from one state to the next with the probability of p and stay in the same state with the probability of $1 - p$.

There are six variables u, v, w, x, y, z corresponding to the average number of coin flips that are yet required to get to the last state of having five heads. These six variables are connected as in the following system:

$$z = 0$$

$$y = 1 + (1-p)y + pz$$

$$x = 1 + (1-p)x + py$$

$$w = 1 + (1-p)w + px$$

$$v = 1 + (1-p)v + pw$$

$$u = 1 + (1-p)u + pv.$$

The free coefficient 1 tells us that the count of heads changes with every transition from state to state. The system can be rewritten:

$$py = 1$$

$$px = 1 + py$$

$$pw = 1 + px$$

$$pv = 1 + pw$$

$$pu = 1 + pv.$$

A step-by-step substitution from the top down shows that $py = 1$, $px = 2$, $pw = 3$, $pv = 4$, $pu = 5$. The first four confirm our previous derivations; the last one confirms the conjecture, $E_5(p) = u = \frac{5}{p}$. The way it was obtained shows how the general conjecture can be proved by induction.

Solution 4, Markov Chain by Mathematica
Marcos Carreira

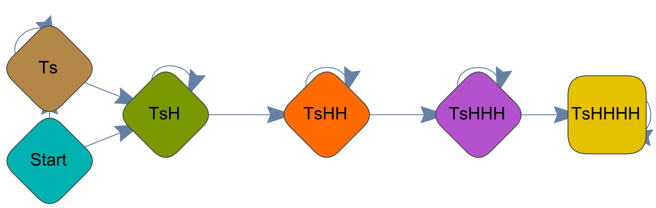
stts = {"Start", "Ts", "TsH", "TsHH", "TsHHH", "TsHHHH"};

$$\mathbf{tmat} = \begin{pmatrix} 0 & 1-p & p & 0 & 0 & 0 \\ 0 & 1-p & p & 0 & 0 & 0 \\ 0 & 0 & 1-p & p & 0 & 0 \\ 0 & 0 & 0 & 1-p & p & 0 \\ 0 & 0 & 0 & 0 & 1-p & p \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

proc = DiscreteMarkovProcess[1, tmat];

MarkovProcessProperties[proc];

Graph[stts, proc]



Grid[Table[{j − 2, stts[[j]], Mean[FirstPassageTimeDistribution[proc, j]]},

{j, 3, Length[stts]}], Frame → All]

1	TsH	$\frac{1}{p}$
2	TsHH	$\frac{2}{p}$
3	TsHHH	$\frac{3}{p}$
4	TsHHHH	$\frac{4}{p}$

Solution 5

Amit Itagi

Let $q = 1 - p$. Suppose we get k heads for the first time after n flips. Then the first $n-1$ flips should have exactly $k-1$ heads and the last flip has to result in heads. The probability of this happening is

$$Z(n,k) = C(n-1,k-1)p^k q^{n-k}.$$

Thus, the expected number of flips before getting k heads (assuming $k > 0$) is

$$\begin{aligned}
f(k) &= \sum_{n=0}^{\infty} nZ(n,k) = \sum_{n=0}^{\infty} nC(n-1,k-1)p^k q^{n-k} \\
&= \sum_{n=0}^{\infty} n\left[\frac{k}{n}C(n,k)\right]p^k q^{n-k} \\
&= kp^k \sum_{n=0}^{\infty} C(n,k)q^{n-k} \\
&= \frac{kp^k}{k!}\sum_{n=0}^{\infty} n(n-1)\dots(n-k+1)q^{n-k} \\
&= \frac{p^k}{(k-1)!}\frac{\partial^k}{\partial q^k}\sum_{n=0}^{\infty} q^n \\
&= \frac{p^k}{(k-1)!}\frac{\partial^k}{\partial q^k}\frac{1}{1-q}. \\
f(3) &= \frac{p^3}{2!}\frac{\partial^3}{\partial q^3}\frac{1}{1-q} \\
&= \frac{p^3}{2!}\frac{3!}{(1-q)^4} = \frac{3}{p}. \\
f(4) &= \frac{p^4}{3!}\frac{\partial^4}{\partial q^4}\frac{1}{1-q} \\
&= \frac{p^4}{3!}\frac{4!}{(1-q)^5} = \frac{4}{p}.
\end{aligned}$$

Solution 6, Mathematica Again

N.N. Taleb

Generalization of a problem: flipping a coin, with probability of heads p, $0 < p < 1$, what is the expected number of flips before getting $3, 4, m$ heads?

Nonconsecutive Heads

For three nonconsecutive heads, we denote by x an event we do not count and H the heads on the coin:

	x	xH	$xHxH$	$xHxHxH$
x	$1-p$	p	0	0
xH	0	$1-p$	p	0
$xHxH$	0	0	$1-p$	p
$xHxHxH$	0	0	0	1

with transition matrix

$$\pi = \begin{pmatrix} 1-p & p & 0 & 0 \\ 0 & 1-p & p & 0 \\ 0 & 0 & 1-p & p \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We denote the dot product of the matrix $\pi^n = \underbrace{\pi.\pi\ldots\pi}_{n}$:

$$\pi^n = \left(\begin{matrix} (1-p)^n & n(1-p)^{n-1}p & \frac{1}{2}(n-1)n(1-p)^{n-2}p^2 \\ 0 & (1-p)^n & n(1-p)^{n-1}p \\ 0 & 0 & (1-p)^n \\ 0 & 0 & 0 \end{matrix} \right.$$

$$\left. \begin{matrix} 1-\frac{1}{2}(1-p)^{n-2}((n-2)p((n-1)p+2)+2) \\ (-np+p-1)(1-p)^{n-1}+1 \\ 1-(1-p)^n \\ 1 \end{matrix} \right).$$

We have the transition probability from the initial state $S_0 = x$ to $S_n = xHxHxH$ in n throws:

$$p(S_n|S_0) = 1 - \frac{1}{2}(1-p)^{n-2}((n-2)p((n-1)p+2)+2).$$

Hence, we can derive the probability mass function (pmf) of the stopping (first passage) time distribution (indexed for three heads):

$$\phi_{3,n} = p(S_{n-1}|S_0) - p(S_n|S_0) = \frac{1}{2}(n-2)(n-1)p^3(1-p)^{n-3}, \quad n \geq 3$$

and the expected stopping time:

$$\sum_{n=3}^{\infty} n\phi_n = \frac{3}{p}.$$

For four and more nonconsecutive heads,

$$\pi_4 = \begin{pmatrix} 1-p & p & 0 & 0 & 0 \\ 0 & 1-p & p & 0 & 0 \\ 0 & 0 & 1-p & p & 0 \\ 0 & 0 & 0 & 1-p & p \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and $\phi_{4,n} = \frac{1}{6}(n-3)(n-2)(n-1)p^4(1-p)^{n-4}$. The expected stopping time is

$$\sum_{n=4}^{\infty} n\phi_{4,n} = \frac{4}{p}.$$

Generalizing, for m heads,

$$\phi_{m,n} = \frac{1}{m!}(n-m)(n-m-1)\ldots(n-3)(n-2)(n-1)p^m(1-p)^{n-m},$$

or, more compactly, using the Pochhammer symbol for the rising factorial,

$$\phi_{m,n} = \frac{1}{m!}(n-m)_m p^m(1-p)^{n-m}.$$

Finally,

$$\sum_{n=m}^{\infty} n\phi_{m,n} = \frac{m}{p}.$$

Consecutive Heads

For three consecutive heads,

$$\begin{array}{ccccc} & x & H & HH & HHH \\ x & 1-p & p & 0 & 0 \\ H & 1-p & 0 & p & 0 \\ HH & 1-p & 0 & 0 & p \\ HHH & 0 & 0 & 0 & 1. \end{array}$$

The matrix and the pmf of the first passage time distribution are unwieldy, but we get for the expected stopping time

$$\frac{1}{p^3} + \frac{1}{p^2} + \frac{1}{p}.$$

Generally, for m consecutive heads,

$$\sum_{i=1}^{m} \frac{1}{p^m}.$$

Probability of Consecutive Heads
Signor Ernesto

As was observed in the previous riddle (see also [92]), the expectation of the number of tosses until 10 successive heads is

$$\sum_{k=1}^{10} \frac{1}{p^{10}} = \frac{p^{10}-1}{(p-1)p^{10}}.$$

We are looking for P that satisfies $E = \dfrac{p^{10}-1}{(p-1)p^{10}} = 100$. This can be found approximately: $E \approx 0.712212$.

Two Dice Repetition
[66, Problem 409]

Solution to Question 1

This is very much like in the Birthday Problem [97], but with 36 distinct equiprobable selections.

A repetition occurs on move n, $2 \leq n \leq 37$, for the first time if, until then, there were no repetitions, but then on the n^{th} move there was one. The probability of this event is

$$\begin{aligned} p(n) &= \frac{35}{36} \cdot \frac{34}{36} \cdots \frac{36-(n-2)}{36} \cdot \frac{n-1}{36} \\ &= \frac{(n-1)36 \cdot 35 \cdots (38-n)}{36^n}. \end{aligned}$$

The expectation for the number of moves is then

$$E = \sum_{k=2}^{37} kp(k) \approx 8.203.$$

Solution to Question 2

Now, for n, $2 \leq n \leq 32$, the first repetition can occur at move n in one of two ways:

1. there was no $*$ in the first $n-1$ moves or
2. there was exactly one $*$ in the first $n-1$ moves, whereas $*$ may or may not appear on move n.

Let $q(n)$ be the probability of the first repetition happening on move n. Then, to start with,

$$q(2) = \frac{30}{36} \cdot \frac{1}{36} + \frac{6}{36} \cdot \frac{6}{36} = \frac{11}{216}.$$

For $3 \le n \le 32$,

$$\begin{aligned} q(n) &= \frac{30 \cdot 29 \cdots (32-n)}{36^{n-1}} \cdot \frac{n-1}{36} \\ &\qquad + (n-1) \cdot \frac{6}{36} \cdot \frac{30 \cdot 29 \cdots (33-n)}{36^{n-2}} \cdot \frac{6+(n-2)}{36} \\ &= \frac{30 \cdot 29 \cdots (33-n)}{36^n} \cdot (n-1)(5n+56). \end{aligned}$$

The expected number of moves at which the first repetition occurs is then

$$\sum_{k=2}^{32} kq(k) \approx 6.704.$$

Expected Number of Happy Passengers

Solution 1

Konstantin Knop

Assuming the plane seats n passengers, let $F(n)$ be the expectation of the number of unhappy passengers and $F^*(n)$ be a similar expectation, conditioned on the first passenger getting a wrong seat, so that the two are related as follows:

$$F(n) = \frac{1}{n} \cdot 0 + \frac{n-1}{n} \cdot F^*(n).$$

$F^*(n)$ is thus the number of unhappy passengers in an n-seat plane with an *a priori* misplaced seat.

Let us now start from the beginning, but counting passengers from the end of the line. When the n^{th} passenger (the one without the boarding pass) enters the plane, he chooses any of the n available seats—in particular, his own—with probability $\frac{1}{n}$. With the same probability, he lands in the seat of the k^{th} passenger (from the end of the line). If this happens, all the passengers entering before the k^{th} passenger will be happy. The latter will have a choice of k seats, of which one is that of the first passenger. With the probability of $\frac{1}{k}$, he will choose the latter, making the total number of unhappy passengers two. With the probability of $\frac{k-1}{k}$, he will choose a seat of one of the passengers yet in line, adding one to the count of unhappy passengers. This would reduce the problem to that of a k-seat plane (with one seat *a priori* misplaced), which may be described as adding to the total expectation of unhappy passengers the quantity

$$\begin{aligned} \frac{2}{k} + \frac{k-1}{k}(1 + F^*(k) + 1) &= \frac{k+1}{k} + \frac{k-1}{k}F^*(k) \\ &= \frac{k+1}{k} + F(k). \end{aligned}$$

Summing up for all possible choices of k,

$$F(n) = \frac{1}{n} \cdot 0 + \frac{1}{n} \sum_{k=1}^{n-1} \left(\frac{k+1}{k} + F(k) \right),$$

or

$$n \cdot F(n) = \sum_{k=1}^{n-1} \left(\frac{k+1}{k} + F(k) \right).$$

For $F(n-1)$ we have a similar expression:

$$(n-1) \cdot F(n-1) = \sum_{k=1}^{n-2} \left(\frac{k+1}{k} + F(k) \right).$$

Substitution of the latter into the former produces

$$nF(n) = (n-1)F(n-1) + \frac{n+1}{n} + F(n-1),$$

or

$$F(n) = F(n-1) + \frac{1}{n-1}.$$

With $F(1) = 0$, we are able to solve the recurrence:

$$F(n) = \sum_{k=1}^{n-1} \frac{1}{k},$$

so that $F(100) \approx 5.177$.

Solution 2

A. Bogomolny

Passenger number n, counting from the end, gets his/her own seat with the probability $\frac{n}{n+1}$. (This is because the previous passenger was facing $n+1$ choices, and it does not matter whether the n^{th} passenger's seat was among them—another previous passenger could have taken it. See also solution 4 in the predecessor riddle on page 158.) Define the random variable $X(n)$ that takes value 1 with the probability $\frac{n}{n+1}$ and value 0 with the probability $\frac{1}{n+1}$. Then the expected value is $E(X(n)) = \frac{n}{n+1}$.

The first (to board the plane) has the probability of $\frac{1}{100}$ to be happy. Then (using the linearity of the expectation) the expected value $E(100)$ of happy passengers is

$$E(100) = \frac{1}{100} + \sum_{n=1}^{99} E(X(n)) = \sum_{n=1}^{99} \frac{n}{n+1} + \frac{1}{100} \approx 94.823.$$

The number of unhappy passengers is then about $100 - 94.823 = 5.177$.

Expectation of the Largest Number

Solution 1

The following is an anonymous letter submitted as a solution to Riddle 8.17.

Dear Alexander,

I am writing about your symmetry principle (Appendix B) as you applied it to the riddle of the expected number of cards to be dealt in order to get the first ace (Riddle 8.10 on page 194).

I came up with the probability puzzle (this riddle) and had been working on it. After tediously solving it by normal means, I realized that my problem was the same as your ace riddle. I used the same symmetry-principle logic as you did to solve your ace riddle: the n drawn balls divide, by rank, the remaining $N-n$ balls in the urn into $n+1$ groups of average size $\dfrac{N-n}{n+1}$. The last group is from ball number $\left\lfloor \dfrac{n}{n+1}\cdot(N+1)\right\rfloor+1$ to ball number N. The ball just before the first ball of the last group is the expected highest ball drawn:

$$E(\max) = \left\lfloor \frac{n}{n+1}\cdot(N+1)\right\rfloor.$$

It occurred to me that a little more can be squeezed out of these riddles with more use of symmetry. By symmetry, the expected minimum is given by

$$\begin{aligned} E(\min) &= \left\lfloor 1-\frac{n}{n+1}\cdot(N+1)\right\rfloor \\ &= \left\lfloor \frac{1}{n+1}\cdot(N+1)\right\rfloor, \end{aligned}$$

which is the same as the answer to your ace riddle. The formulas for expected minimum and maximum can be generalized to any rank:

$$E(k) = \left\lfloor \frac{k}{n+1}\cdot(N+1)\right\rfloor,$$

for $k=1$ to n, where k is the rank from smallest to largest.

Solution 2

Amit Itagi

For k to be the largest ball, choose k and $n-1$ balls from the first $k-1$ balls. Thus, the required expectation value is

$$\begin{aligned} E &= \frac{\sum_{k=n}^{N} k\binom{k-1}{n-1}}{\binom{N}{n}} = \frac{\sum_{k=n}^{N} n\binom{k}{n}}{\binom{N}{n}} = \frac{n}{\binom{N}{n}}\left[\binom{N}{n}+\binom{n+1}{n}+\cdots+\binom{N}{n}\right] \\ &= \frac{n}{\binom{N}{n}}\binom{N+1}{n+1} = \frac{n(N+1)}{(n+1)}. \end{aligned}$$

Solution 3

N.N. Taleb

Drawing without replacement, we can make use of the hypergeometric distribution for the distribution of Z, the max of an n-size subsample of $1, 2, \ldots, N$ with mass function $\mathbb{P}(Z = z) = \phi(.)$:

$$\phi(n, N, z) = \frac{\binom{N-z-1}{n-1}}{\binom{N-z}{n}} = \frac{n}{N-z}.$$

The expectation can be written as follows:

$$\begin{aligned}
\mathbb{E}(Z) &= \sum_{i=0}^{N-n} (N-i)\phi(n, N, i) \prod_{j=0}^{i-1} (1 - \phi(n, N, j)) \\
&= \frac{n\left(nN - \frac{\Gamma(-N)}{\Gamma(-n-1)\Gamma(n-N)} + n + N + 1\right)}{(n+1)^2} \\
&= \frac{nN}{n+1} + \frac{n}{n+1} + \underbrace{\frac{n\sin(\pi n)\Gamma(n+1)\csc(\pi N)\sin(\pi(n-N))\Gamma(-n+N+1)}{\pi(n+1)\Gamma(N+1)}}_{=0} \\
&= \frac{n(N+1)}{n+1}.
\end{aligned}$$

Drawing with replacement, we have a discrete uniform distribution with cumulative $F(n) = \frac{x}{N}$. The cumulative probability of Z is, by standard argument (all x below z), $F(Z \geq z) = \left(\frac{z}{N}\right)^n$.

The expectation becomes

$$\mathbb{E}(Z) = \sum_{i=1}^{N} i\left(\left(\frac{i}{N}\right)^n - \left(\frac{i-1}{N}\right)^n\right).$$

Waiting for a Larger Number

Solution 1

[70]

Talking of probabilities, we disregard the possibility of getting two equal numbers as having zero probability. Further,

$$P(N > n) = P(S_0 > S_1, S_0 > S_2, \ldots, S_0 > S_n)$$

because, from the definition of N, none of the previous numbers exceeded S_0. Put another way,

$$P(N > n) = P(\max\{S_0, S_1, S_2, \ldots, S_n\} = S_0).$$

Now, here is a crucial observation: all $n+1$ numbers are uniformly distributed on $[0, 1]$, so there is nothing special about S_0, and, in principle, any of them could be the largest. Thus, $P(N > n) = \dfrac{1}{n+1}$. (This is an instance of invocation of the principle of symmetry. See Appendix B.) Similarly, $P(N > n-1) = \dfrac{1}{n}$. Now,

$$P(N = n) = P(N > n-1) - P(N > n) = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}.$$

To continue,

$$E(N) = \sum_{n=1}^{\infty} n \cdot \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty,$$

strange as it may appear.

On one hand, $P(N > n) = \dfrac{1}{n+1}$ indicates that not finding a maximum after, say, 99 attempts has a fairly low probability of $\dfrac{1}{100}$. On the other hand, the expected waiting interval for that to happen is infinite.

Solution 2

Josh Jordan

By symmetry, $E[N]$ is unchanged if we stop instead when $S_N < S_0$. For any given S_0, this takes (Riddle 8.9 on page 194) $\dfrac{1}{S_0}$ tries on average, so $E[N] = \dfrac{1}{S_0}$. Finally, since S_0 is uniformly distributed over $[0, 1]$, we have

$$E[N] = (\text{area under the curve } \frac{1}{S_0} \text{ for } 0 \le S_0 \le 1) = \infty.$$

Solution 3

Amit Itagi, Marcos Carreira

The integral of $(1-x)(1+2x+3x^2+\ldots)$ from 0 to 1 is the solution. The second term is just the derivative of $1+x+x^2+\ldots$ with respect to $x = \dfrac{1}{(1-x)^2}$. The integral of $\dfrac{1}{1-x}$ from 0 to 1 diverges.

Solution 4

N.N. Taleb

The probability of first exit on the first shot is $P(S_1 > S_0) = 1 - S_0$. The probability of first exit on the second shot is $P(S_1 < S_0, S_2 > S_0) = S_0(1 - S_0)$. The probability of first exit on the t^{th} shot is $P(S_1 < S_0, \ldots, S_{t-1} < S_0, S_t > S_0) = S_0^{t-1}(1 - S_0)$.

The conditional stopping time (knowing S_0) is

$$E(N|S_0) = \sum_{t=1}^{\infty} t S_0^{t-1}(1 - S_0) = \frac{1}{1 - S_0}.$$

The unconditional stopping time (not knowing S_0) is

$$\lim_{H\to 1}\lim_{L\to 0}\int_L^H \frac{dx}{1-x} = \lim_{H\to 1}\lim_{L\to 0}\ln\left(\frac{1-L}{1-H}\right) = \infty.$$

Waiting to Exceed 1

Solution 1

Marcos Carreira with Mathematica

Refine[Table[CDF[UniformSumDistribution[n], x], {n, 1, 6}], $0 \le x < 1$]

$$\left\{x, \frac{x^2}{2}, \frac{x^3}{6}, \frac{x^4}{24}, \frac{x^5}{120}, \frac{x^6}{720}\right\}$$

Table[CDF[UniformSumDistribution[n], 1], {n, 1, 6}]

$$\left\{1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \frac{1}{720}\right\}$$

1 − Table[CDF[UniformSumDistribution[n], 1], {n, 1, 6}]

$$\left\{0, \frac{1}{2}, \frac{5}{6}, \frac{23}{24}, \frac{119}{120}, \frac{719}{720}\right\}$$

Differences[1 − Table[CDF[UniformSumDistribution[n], 1], {n, 1, 6}]]

$$\left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{8}, \frac{1}{30}, \frac{1}{144}\right\}$$

Differences[Table[CDF[UniformSumDistribution[n], 1], {n, 1, 6}]]

$$\left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{8}, \frac{1}{30}, \frac{1}{144}\right\}$$

$\boldsymbol{\sum_{n=1}^{\infty}\left((n-1) * \left(\frac{1}{(n-1)!} - \frac{1}{n!}\right)\right)}$

$-1 + e$

Solution 2

Amit Itagi

Let $T_n = \sum_{k=0}^{n} S_n$:

$$P(T_n < 1) = \int_{S_0=0}^{1}\int_{S_1=0}^{1-T_0}\int_{S_2=0}^{1-T_1}\cdots\int_{S_n=0}^{1-T_{n-1}} dS_n dS_{n-1}\ldots dS_0. \tag{8.1}$$

Let us make an observation here. For some natural numbers k and j,

$$\begin{aligned}\int_{S_k=0}^{1-T_{k-1}} \frac{(1-T_k)^j}{j!} dS_k &= \int_{S_k=0}^{1-T_{k-1}} \frac{(1-T_{k-1}-S_k)^j}{j!} dS_k \\ &= -\frac{(1-T_{k-1}-S_k)^{j+1}}{(j+1)!}\Bigg|_0^{1-T_{k-1}} \\ &= \frac{(1-T_{k-1})^{j+1}}{(j+1)!}.\end{aligned}$$

Note, this result is also valid if $j = 0$.

The integrand for the rightmost integral in equation 8.1 can we written as $\dfrac{(1-T_n)^j}{j!}$ with $j = 0$. Thus, the integrand keeps propagating through the integrals from right to left with k decrementing by 1 and j incrementing by 1 with every integral. Thus, we are left with

$$P(T_n < 1) = \int_{S_0=0}^{1} \frac{(1-T_0)^n}{n!} dS_0 = \int_{S_0=0}^{1} \frac{(1-S_0)^n}{n!} dS_0 = \frac{1}{(n+1)!}.$$

The desired expected value is

$$\begin{aligned}
E(n) &= \sum_{n=1}^{\infty} n \cdot P(T_{n-1} < 1 \cap T_n > 1) \\
&= \sum_{n=1}^{\infty} n \cdot [P(T_n > 1) - P(T_{n-1} > 1 \cap T_n > 1)] \\
&= \sum_{n=1}^{\infty} n \cdot [P(T_n > 1) - P(T_{n-1} > 1)] \\
&= \sum_{n=1}^{\infty} n \cdot \left[\left(1 - \frac{1}{(n+1)!}\right) - \left(1 - \frac{1}{(n)!}\right)\right] \\
&= \sum_{n=1}^{\infty} n \cdot \left[\frac{1}{n!} - \frac{1}{(n+1)!}\right] \\
&= 1 \cdot \left(\frac{1}{1!} - \frac{1}{2!}\right) + 2 \cdot \left(\frac{1}{2!} - \frac{1}{3!}\right) + \cdots \\
&= \frac{1}{1!} + (2-1) \cdot \frac{1}{2!} + (3-2) \cdot \frac{1}{3!} + \cdots \\
&= \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots = e - 1 \sim 1.72.
\end{aligned}$$

Solution 3

[70, pp. 94–98]

Let $T_n = \sum_{k=0}^{n} S_n$ and N be the first value of n for which $T_n > 1$:

$$P(N > n) = P(T_0 < 1, T_1 < 1, \ldots, T_n < 1) = P(T_n < 1).$$

It follows that

$$\begin{aligned}
P(N = n) &= P(N > n-1) - P(N > n) \\
&= P(T_{n-1} < 1) - P(T_n < 1).
\end{aligned}$$

We also know that $P(N=0)=0$. Thus,

$$\begin{aligned}E(N) &= \sum_{n=1}^{\infty} nP(N=n) = \sum_{n=1}^{\infty} n(P(T_{n-1}<1) - P(T_n<1)) \\ &= \sum_{n=1}^{\infty} nP(T_{n-1}<1) - \sum_{n=1}^{\infty} nP(T_n<1)) \\ &= \sum_{n=0}^{\infty}(n+1)P(T_n<1) - \sum_{n=0}^{\infty} nP(T_n<1)) \\ &= \sum_{n=0}^{\infty} P(T_n<1).\end{aligned}$$

Now, $P(T_0<1) = P(S_0<1) = 1$. By induction, $P(T_n<1) = \dfrac{1}{(n+1)!}$ so that

$$E(N) = \sum_{n=0}^{\infty} P(T_n<1) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} = e-1.$$

Notes

Previously T_n was defined as the sum of $n+1$ random variables. In a more common formulation, we would consider T_n as the sum of n random variables. In that case, the result would appear slightly different because then $P(T_n<1) = \dfrac{1}{n!}$, so that we would have

$$E(N) = \sum_{n=0}^{\infty} P(T_n<1) = \sum_{n=0}^{\infty} \frac{1}{n!} = e.$$

Solution 4

[72, Problem 100]

Call $E(x)$ the expected number of number selections for the sum to exceed x. Since the first number is uniformly distributed on $[0,1]$,

$$E(x) = 1 + \int_0^1 E(x-t)dt,$$

but $E=0$ for $x<0$, so if $0 \le x \le 1$ this relation can be converted into

$$E(x) = 1 + \int_0^x E(x-t)dt = 1 + \int_0^u E(u)du.$$

It follows that $E'(x) = E(x)$, $E(0)=1$ and so $E(x)=e^x$. In particular, $E(1)=e$.

Waiting for All Six Outcomes

Solution 1

Waiting time for an event that occurs with probability p is known to be $\dfrac{1}{p}$ (Riddle 8.9 on page 194).

Assume we already saw $k < 6$ numbers. The probability of seeing one of the remaining $6-k$ numbers after that is $\dfrac{6-k}{6}$, making the waiting time equal to $\dfrac{6}{6-k}$.

Naturally, we start counting from $k = 0$ and the first roll of a die is always a success: one of the numbers shows up. Knowing that the expectation is additive, the expected number of rolls until all six numbers show up is

$$\sum_{k=0}^{5} \frac{6}{6-k} = 6\left(\frac{1}{6} + \frac{1}{5} + \frac{1}{4} + \frac{1}{3} + \frac{1}{2} + \frac{1}{1}\right) \approx 14.7.$$

Solution 2

David Koops

Let $t(n)$ be the number of throws until all outcomes appear, starting from n distinct outcomes. On the next throw, with the probability $\frac{n}{6}$, we do not get an outcome that we have not seen before, and, with probability $\dfrac{6-n}{6}$, we get a new outcome. Solve

$$t(n) = 1 + \frac{n}{6} \cdot t(n) + \frac{6-n}{6} \cdot t(n+1)$$

for $t(0)$ with $t(6) = 0$. Explicitly, $t(n) - t(n+1) = \dfrac{6}{6-n}$. Thus,

$$\begin{aligned}
t(5) - t(6) &= \frac{6}{1}\\
t(4) - t(5) &= \frac{6}{2}\\
t(3) - t(4) &= \frac{6}{3}\\
t(2) - t(3) &= \frac{6}{4}\\
t(1) - t(2) &= \frac{6}{5}\\
t(0) - t(1) &= \frac{6}{6}.
\end{aligned}$$

Summing up, $t(0) = \displaystyle\sum_{n=0}^{5} \frac{6}{6-n} \approx 14.7.$

Walking Randomly—How Far?

Solution 1

If $x \neq 0$ for some n, then $a_{n+1} = a_n$ because, with equal probabilities, x changes to either $x+1$ or $x-1$. If $x = 0$, then necessarily $|x|$ grows by 1.

If n is odd, the probability of $x = 0$ is 0. If $n = 2k$, there are $\dbinom{2k}{k}$ equiprobable ways to have the same number of tails as of heads. Thus, the probability of $x = 0$ if n

is even is

$$\frac{\binom{n}{\frac{n}{2}}}{2^n}.$$

This leads to the following recurrence:

$$a_{n+1} = \begin{cases} a_n, & \text{if } n \text{ is odd} \\ a_n + \dfrac{\binom{n}{\frac{n}{2}}}{2^n}, & \text{if } n \text{ is even.} \end{cases}$$

Solving that, we obtain

$$a_{2n+2} = a_{2n+1} = \sum_{k=0}^{n} \frac{\binom{2k}{k}}{2^{2k}}.$$

It follows that

$$\lim_{n\to\infty} a_n = \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{2^{2k}}.$$

It is not hard to roughly estimate the central binomial coefficient $\binom{2k}{k}$. It is the largest among $2k+1$ coefficients in the sum $\sum_{s=0}^{2k} \binom{2k}{s} = 2^{2k}$ so that $\binom{2k}{k}(2k+1) > 2^{2k}$.

In other words, $\dfrac{\binom{2k}{k}}{2^{2k}} > \dfrac{1}{2k+1}$. We conclude that $a_n \to \infty$, for

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{2^{2k}} > \sum_{k=0}^{\infty} \frac{1}{2k+1}$$

and the latter series is known to converge to infinity.

Solution 2

Amit Itagi, Hélvio Vairinhos, Keith Dawid

Let y be that random variable that represents the incremental change in x because of a coin toss. Thus, y takes the following two values: -1 and 1. It has a mean of $\mu = 0$ and a variance of $\sigma^2 = 1$. The central limit theorem implies that after a large number n, of coin tosses, x has a normal distribution with mean $\tilde{\mu} = n\mu = 0$ and variance $\tilde{\sigma}^2 = n\sigma^2$. Thus, $z = |x|$ has a half normal distribution with expected value

$$a_n = \frac{\tilde{\sigma}\sqrt{2}}{\sqrt{\pi}} = \frac{\sigma\sqrt{2n}}{\sqrt{\pi}} = \sqrt{\frac{2n}{\pi}}.$$

As $n \to \infty$, $a_n \to \infty$.

Solution 3

N.N. Taleb

Let x be S_n, the summed variables; let X follow a binomial with probability density function $\phi_n(x) = 2^{-n}\binom{n}{n-x}$.

We have

$$\mathbb{E}(|X|) = \sum_{x=0}^{n} \left| x - \frac{n}{2} \right| \phi_n(x).$$

Assume n is sufficiently large that

$$\begin{aligned}\left\lceil \frac{n}{2} \right\rceil \approx \frac{n}{2} &= 2 \sum_{x=n/2}^{n} \left(x - \frac{n}{2} \right) \phi_n(x) \\ &= \frac{\Gamma\left(\frac{n}{2} + \frac{1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)}.\end{aligned}$$

We have

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \frac{\Gamma\left(\frac{n}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} = \frac{1}{\sqrt{2}}.$$

In other words

$$\lim_{n\to\infty} \mathbb{E}(|X|) = \sqrt{\frac{n}{2\pi}}.$$

Solution 4

N.N. Taleb

The binomial probability density function is

$$\phi_n(x) = 2^{-n}\binom{n}{x}.$$

Adjusting for $2\left| x - \frac{n}{2} \right|$ to adapt it to an absolute sum, we have the characteristic function Ψ:

$$\begin{aligned}\Psi_X(t) &= \sum_{x=n/2}^{n} 2^{-n}\binom{n}{x} \left(\exp\left(i2t \left(x - \frac{n}{2} \right) \right) \right) \\ &= \frac{\Gamma\left(\frac{n}{2} + \frac{1}{2}\right) {}_2F_1\left(1, -\frac{n}{2}; \frac{n+2}{2}; -e^{2it}\right)}{\sqrt{\pi}\Gamma\left(\frac{n}{2} + 1\right)}\end{aligned}$$

and the mean

$$-i\Psi'_X(t)|_{t=0} = \frac{n\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{2\sqrt{\pi}\Gamma\left(\frac{n}{2}+1\right)},$$

as before.

Solution 5

N.N. Taleb

The characteristic function of a Bernoulli distribution that delivers $\{-1, 1\}$ is $\Psi_X(t) = \frac{1}{2}(e^{it} + e^{-it}) = \cos(t)$. Hence, the moment of order p of the sum is

$$m_p = i^p \left.\frac{\partial^p \cos^n(t)}{\partial t^p}\right|_{t=0},$$

with $m_1 = 0$, $m_2 = n$, $m_3 = 0$, $m_4 = n(3n-2)$.

Using kurtosis as a benchmark of the speed of convergence to Gaussian, $\kappa = \frac{m_4}{m_2^2}$, we have $\kappa = 3 - \frac{2}{n}$.

For n large enough, the distribution is effectively Gaussian. Let $\varphi(x)$ be the normal distribution with mean 0 and standard deviation $\sqrt{n}$. We have

$$\int_{-\infty}^{\infty} |x|\varphi(x)\, dx = \sqrt{\frac{2}{\pi}}\sqrt{n}$$

as the converging expectation.

Expectation of Pairings

Solution 1

In general, let us consider a group of m boys and n girls. Let P_k be the probability of having a couple boy/girl or girl/boy in positions $(k, k+1)$, $k = 1, 2, \ldots, m+n-1$. From the point of view of chance, all P_k are the same; $P_k = P$, $k = 1, 2, \ldots, m+n-1$. Thus, similar to a problem of coin tossing (Riddle 8.3 on page 192), the expectation E in question is $E = (m+n-1)P$.

To find P, note that once a boy and a girl find themselves next to each other, the remaining boys and girls can be arranged in $\binom{n+m-2}{n-1}$ ways. This should be multiplied by 2 because we do not distinguish the order in which the selected couple comes in the sequence. On the other hand, there are $\binom{n+m}{n}$ possible arrangements, implying

$$P = \frac{2\binom{n+m-2}{n-1}}{\binom{n+m}{n}} = \frac{2nm}{(n+m)(n+m-1)}.$$

We conclude that

$$E = (m+n-1)P = \frac{2nm}{n+m}.$$

For $m = 8$ and $n = 12$, this gives $\frac{48}{5}$.

Solution 2

There are $\frac{m}{n+m}$ ways to choose a boy and $\frac{m}{n-m-1}$ ways to choose a girl from the remaining group. The probability of their being next to each other is

$$2 \cdot \frac{m}{n+m} \cdot \frac{m}{n-m-1} = \frac{2nm}{(n+m)(n+m-1)}.$$

This could happen in $n-1$ positions, making the expectation equal to

$$E = (n+m-1) \cdot \frac{2nm}{(n+m)(n+m-1)} = \frac{2nm}{n+m},$$

the harmonic mean of m and n.

Making Spaghetti Loops in the Kitchen

Solution 1

P. Winkler

We compute the probability of making a loop at every step and then, using linearity of expectation, obtain the expectation by summing up the probabilities.

Having tied $k-1$ knots, you pick up an end, and, of the remaining $100-2(k-1)-1 = 101-2k$ ends, you can choose another to tie to this one. Only one choice (the other end of the chain) results in a loop. It follows that the probability that your k^{th} knot makes a loop is $\frac{1}{101-2k}$, and the expected number of loops is

$$\frac{1}{99} + \frac{1}{97} + \frac{1}{95} + \cdots + \frac{1}{1} \approx 2.93777485,$$

which is less than three loops! As n grows, the expected number of loops approaches the n^{th} harmonic number H_n.

Solution 2

A. Bogomolny

Let $N(m)$ be the expected number of loops after tying m knots. Assume we have tied $k-1$ of them so that $102-2k$ loose ends remain. We pick one arbitrarily and then another one from the remaining $101-2k$ to tie to the first selection. With the probability of $\frac{1}{101-2k}$, we will get an additional loop, and, with the probability of $\frac{100-2k}{101-2k}$, the number of loops will remain the same. Thus, we get a recurrence relation:

$$N(k) = \frac{100-2k}{101-2k}N(k) + \frac{1}{101-2k}(N(k-1)+1) = N(k-1) + \frac{1}{101-2k}.$$

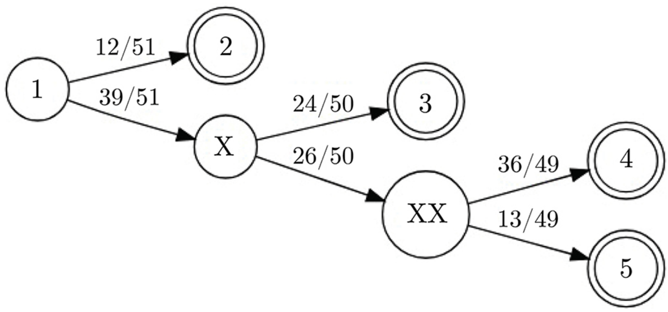
Unfolding the recurrence, we get

$$N(k) = \frac{1}{99} + \frac{1}{97} + \frac{1}{95} + \cdots + \frac{1}{1} \approx 2.93777485.$$

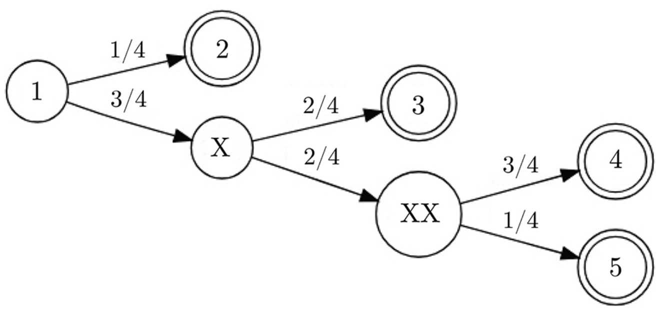
Repeating Suit When Dealing Cards

The two cases are best represented with tree diagrams.

Drawing cards without replacement:



Drawing cards with replacement:



In the first case, the expected number of cards is

$$2 \cdot \frac{12}{51} + 3 \cdot \frac{39}{51} \cdot \frac{24}{50} + 4 \cdot \frac{39}{51} \cdot \frac{26}{50} \cdot \frac{36}{49} + 5 \cdot \frac{39}{51} \cdot \frac{26}{50} \cdot \frac{13}{49} = \frac{68053}{20825} \approx 3.27.$$

In the case of drawing with replacement, the expectation is

$$2 \cdot \frac{1}{4} + 3 \cdot \frac{3}{4} \cdot \frac{2}{4} + 4 \cdot \frac{3}{4} \cdot \frac{2}{4} \cdot \frac{36}{49} + 5 \cdot \frac{3}{4} \cdot \frac{2}{4} \cdot \frac{1}{4} = \frac{103}{32} \approx 3.22 < 3.27.$$

Family Size

Solution to Questioins 1 and 2

The first two questions are two specific cases of the general riddle of "Number of Trials to First Success" (Riddle 8.9 on page 194). With equal probabilities of birth for boys and girls, i.e., with $\frac{1}{2}$, the expected waiting time is $\frac{1}{p} = 2$.

Solution to Question 3

Assume p is the probability of having a boy, $1-p$ that of having a girl. The sample space for our riddle consists of just two kinds of sequences:

$BB\dots BG$ and $GG\dots GB$.

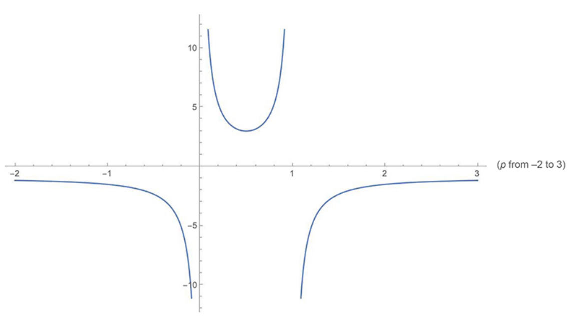
For a sequence of length n, the probabilities are $1-p^{n-1}p$ and $p^{n-1}(1-p)$, respectively, for $n\geq 2$. The mathematical expectation is then

$$\begin{aligned}\sum_{n=2}^{\infty}&n((1-p^{n-1}p+p^{n-1}(1-p))\\ &=p(1-p)\sum_{n=2}^{\infty}n((1-p)^{n-2}+p^{n-2})\\ &=p(1-p)\left(\frac{2-p}{(1-p)^2}+\frac{2-(1-p)}{(1-(1-p))^2}\right)\\ &=\frac{1-p+p^2}{p(1-p)}.\end{aligned}$$

This result sits well with the formula $E=\dfrac{1}{p}$ for the expected length of a sequence of trials until the first success. Since the series in the latter starts with p which is omitted as meaningless in the present problem, we would get

$$\left(\frac{1}{p}-p\right)+\left(\frac{1}{1-p}-(1-p)\right)=\frac{1-p+p^2}{p(1-p)},$$

confirming our calculations. The graph of $f(p)=\dfrac{1-p+p^2}{p(1-p)}$ shows that the expectation is at its minimum when $p=\dfrac{1}{2}$ and goes to infinity as p approaches either 0 or 1.



For $p=\dfrac{1}{2}$, the expectation equals 3.

Solution to Question 4

The sample space for this riddle consists of sequences

$GBB\dots BG$ and $BGG\dots GB$

that, for length $n \geq 2$, come with probabilities $(1-p)^2 p^{n-2}$ and $p^2(1-p)^{n-2}$, respectively. The expectation is then given by

$$\begin{aligned} E &= \sum_{n=2}^{\infty} n((1-p)^2 p^{n-2} + p^2(1-p)^{n-2}) \\ &= \frac{(1-p)^2}{p^2} \sum_{n=0}^{\infty} np^n + \frac{p^2}{(1-p)^2} \sum_{n-2}^{\infty} n(1-p)^n \\ &= \frac{(1-p)^2}{p^2} S(p) + \frac{p^2}{(1-p)^2} S(1-p), \end{aligned}$$

where $S(q) = \sum_{n=2}^{\infty} nq^n$.

This leaves computing $S(q) = \dfrac{q^2(2-q)}{(1-q)^2}$ so that the expectation in this case is given by

$$\begin{aligned} E &= \frac{(1-p)^2}{p^2} \frac{p^2(2-p)}{(1-p)^2} + \frac{p^2}{(1-p)^2} \frac{(1-p)^2(1+p)}{p^2} \\ &= (2-p) + (1+p) = 3. \end{aligned}$$

This expectation is independent of p, which is a great surprise in its own right, but there is more. Note that the formula has been derived under an implicit assumption that p is neither 0 nor 1. What if, say, $p = 0$? What if, by a fluke of fate, women stopped producing boys. Then obviously the only possible child component in a family would be two girls. In another extreme ($p = 1$), all families would consist of two boys. Although neither case is plausible, I think that the circumstance merits being mentioned as a naturally occurring discontinuity.

Solution to Question 5

This is just a reformulation of question 3.

Averages of Terms in Increasing Sequence

Solution 1

Graham Lord, Faryad Sahneh

The number of k-tuples X in which $x_j = m$ is $\binom{m-1}{j-1}\binom{n-m}{k-j}$. Indeed, we are free to choose $j-1$ numbers from the first $m-1$ and $k-j$ numbers from among $m+1, \ldots, n$. Thus, the total number of k-tuples equals

$$\sum_{m=j}^{n-k+j} \binom{m-1}{j-1}\binom{n-m}{k-j} = \binom{n}{k}.$$

For the average then,

$$\begin{aligned}
\overline{x_j} &= \sum_{m=j}^{n-k+j} m\binom{m-1}{j-1}\binom{n-m}{k-j}\Big/\binom{n}{k}\\
&= \sum_{m=j}^{n-k+j} j\binom{m}{j}\binom{n-m}{k-j}\Big/\binom{n}{k}\\
&= j\sum_{m=j}^{n-k+j} \binom{(m+1)-1}{(j+1)-1}\binom{(n+1)-(m+1)}{(k+1)-(j+1)}\Big/\binom{n}{k}\\
&= j\sum_{m'=j\,'}^{n-k+j\,'} \binom{m'-1}{j\,'-1}\binom{(n+1)-m'}{(k+1)-j\,'}\Big/\binom{n}{k}\\
&= j\cdot\binom{n+1}{k+1}\Big/\binom{n}{k}\\
&= j\cdot\frac{n+1}{k+1},
\end{aligned}$$

where $m' = m+1$ and $j\,' = j+1$.

Solution 2

Long Huynh Huu

First, we use the bijection

$$\begin{aligned}
\phi : S &\to S\\
(x_1,\dots,x_k) &\mapsto (x_2-x_1,\dots,x_n-x_1,n+1-x_1).
\end{aligned}$$

$$\begin{aligned}
\bar{x}_1 =& \frac{1}{S}\sum_{x\in S} x_1\\
=& \frac{1}{S}\sum_{x\in S} \phi(x)_1\\
=& \frac{1}{S}\sum_{x\in S} x_2 - x_1\\
=& \bar{x}_2 - \bar{x}_1.
\end{aligned}$$

Likewise, we obtain $\bar{x}_{j+1} - \bar{x}_j = \bar{x}_1$ by using ϕ^j instead of ϕ where $j = 1,\dots,k-1$. This shows that

$$\bar{x}_j = j\bar{x}_1 \quad \text{for } j = 1,\dots,k. \tag{8.2}$$

Now consider the bijection

$$\begin{aligned}
\psi : S &\to S\\
(x_1,\dots,x_k) &\mapsto (n+1-x_k,\dots,n+1-x_1).
\end{aligned}$$

Summation is commutative, so we get, for each $j = 1, \ldots, k$,

$$\begin{aligned}\bar{x}_1 =& \frac{1}{S} \sum_{x \in S} x_1 \\ =& \frac{1}{S} \sum_{x \in S} \psi(x)_1 \\ =& \frac{1}{S} \sum_{x \in S} n + 1 - x_k \\ =& n + 1 - \bar{x}_k \\ =& n + 1 - k\bar{x}_1\end{aligned}$$

by equation (8.2).

We conclude $\bar{x}_1 = \dfrac{n+1}{k+1}$, so that $a = \dfrac{n+1}{k+1}$.

Solution 3

Christopher D. Long

For the number of different such sequences, consider k red 1s and $n - k$ blue 1s, each of which may be placed at one of the $k + 1$ locations between the red 1s. This gives $\dbinom{n}{k}$ such sequences, but it shows us more.

There is only one location for blue 1s that increases the value of x_1, but two for x_2, etc. Thus, $E(x_i) = i \cdot E(x_1)$. What is $E(x_1)$?

Think of the value of each x_i as equal to the number of red + blue 1s up to and including the i^{th} red 1. Each blue 1 has $k+1$ locations (before the first red 1, between the first and second red 1s, etc.). The first location increments all x_i, the last none.

Thus, the expected value each adds to $x_1 + \cdots + x_k$ is the average of $0, 1, \ldots, k$, or $\dfrac{k}{2}$.

Thus, each blue 1 adds an expected value of $\dfrac{k}{2}$ to $\sum E(x_i)$, so $E(x_1) \cdot k \cdot \dfrac{k+1}{2} = k \cdot \dfrac{k+1}{2} + (n-k) \cdot \dfrac{k}{2}$. Thus, $E(x_1) = 1 + \dfrac{n-k}{k+1} = \dfrac{n+1}{k+1}$, and $E(x_j) = j \cdot \dfrac{n+1}{k+1}$.

Solution 4

Christopher D. Long

Casting the riddle as $k+1$ sequential urns makes it visually clearer. The value of x_i is $i+$the sum of the balls in urns $1, \ldots, i$, and we are placing $n-k$ balls in the $k+1$ urns.

The probability that each ball increases the value of x_i is $\dfrac{i}{k+1}$ (placed in urn $x_1, \ldots, x_i$); hence, the expected value of $E(x_i)$ is immediately

$$i + (n-k) \cdot \frac{i}{k+1} = i \cdot \frac{n+1}{k+1}.$$

Chapter 9

Recurrences and Markov Chains

> …the probable, I have said, is that which is it rational for us to believe that the probable is true.
>
> John Maynard Keynes, 2014, *A Treatise on Probability*

Riddles

9.1 A Fair Game of Chance

[8, Problem 494]

Alice and Bob play a fair game repeatedly for one nickel each game. If originally Alice has a nickels and Bob has b nickels, what are Alice's chances of winning all of Bob's money, assuming the play goes on until one person has lost all his or her money?

9.2 Amoeba's Survival

[13]

A population starts with a single amoeba. For this one and for the generations thereafter, there is a probability of $\frac{3}{4}$ that an individual amoeba will split to create two amoebas and a $\frac{1}{4}$ probability that it will die without producing offspring. What is the probability that the family tree of the original amoeba will go on forever?

9.3 Average Visibility of Moviegoers

[46]

At a movie theater, moviegoers line up to buy tickets. The ticket seller calls a patron *viewable* if he/she is taller than all the people in front of him/her in line; otherwise, the patron is *hidden*. Given that no two patrons are precisely the same height, what is the average number of viewable patrons among all possible permutations in a line of n moviegoers?

9.4 Book Index Range

[25]

The index of a book lists every page on which certain words appear. To save space, these are listed in ranges; for example, if a word occurs on pages 1, 2, 3, 5, 8 and 9, then its index contains ranges: 1–3, 5, 8–9.

A certain word appears on each page of an n-page book, $n > 0$, independently with probability p. Find the expected number of entries in its index entry.

9.5 Probability of No Two-Tail Runs

[46]

A fair coin is tossed n times. What is the probability that two tails do not appear in succession if $n = 10$?

9.6 Artificially Unintelligent

[71, Chapter 11]

An electronic device is being devised for transmitting 1 bit of information at a time. So far, the device is not 100% reliable. With the probability p, the bit is transmitted correctly; otherwise, it is flipped.

n such devices are chained output to input and the first receives a bit. What is the probability that the bit will be transmitted correctly through the chain of n devices?

9.7 Two Sixes in a Row

[71, pp. 54–55]

What is the expected number of tosses of a fair die until two successive sixes show up for the first time?

9.8 Matching Socks in Dark Room

[64, Problem 184]

The colors of 10 pairs of socks range from medium gray through black. Washed and dried in a dark room, they have to be matched. Two socks form an unacceptable pair if their colors differ by more than one shade.

What is the probability of having 10 acceptable pairs when they are being matched at random, due to the light conditions?

9.9 Number of Wire Loops

[62, Problem 7]

Start with 2018 pieces of wire. Next, attach each end of a piece of wire to another end, choosing randomly so that all pairings are equally likely (including attaching the two ends of a piece to one another). When this process is completed, there will be loops (no wire ends unattached), and the number of loops will range from 1 to 2018.

What is the average (expected) number of loops?

9.10 Determinants in $\mathbb{Z}_2$

The problem was offered at the 1976 Student Olympiad at the Department of Mathematics of Moscow State University.

Find the probability p_n that a randomly selected determinant of order n with elements from $\mathbb{Z}_2$ is not 0. Prove there is a limit $\lim\limits_{n\to\infty} p_n$.

9.11 Gambling in a Company

[21, pp. 149–150]

In a game of n gamblers, the i^{th} gambler starts the game with a_i dollars. In each round, two gamblers selected at random make a fair bet, and the winner gets a dollar from the loser. A gambler losing all his money leaves the game. The game continues as long as possible, i.e., until one of the gamblers has all the money.

1. What is the expected number of rounds to the end of the game?
2. What the probability that the i^{th} gambler ends up with all the money?
3. For $n = 3$, what is the expected number of rounds until the first loser quits the game?

9.12 Probability in the World Series

[46, Problem 1981–4]

The winning team in the World Series must win four out of seven games. The two teams are equally matched.

What is the expectation of the number of games played?

9.13 Probability of a Meet in an Elimination Tournament

[8, Problem 297]

A tennis club invites 32 players of equal ability to compete in an elimination tournament (the players compete in pairs, with only the winner staying for the next round). What is the probability that a particular pair of players will meet during the tournament?

9.14 Rolling a Die

[57, Problem 20]

A normal die with numbers 1 through 6 on its faces is thrown repeatedly until the running total first exceeds 12.

1. What is the most likely total that will be obtained?
2. What is the expected total?

9.15 Fair Duel

[52, Problem 31]

Smith and Brown have challenged each other to a duel. They will take turns shooting at one another until one has been hit. Smith, who can hit Brown only 40% of the time, is the weaker shot, so he will be allowed to go first. They have determined that the duel favors neither shooter. What is Brown's probability of hitting Smith?

Solutions

A Fair Game of Chance

Let $p(n)$ be Alice's chances of winning the total amount of $a+b$, provided she has n nickels in her possession. Obviously $p(0)=0$. If she is left with a nonzero capital, Alice may, at every trial, win or lose one nickel, each with the probability of $\frac{1}{2}$:

$$p(n)=\frac{p(n+1)}{2}+\frac{p(n-1)}{2}, \qquad n>0.$$

In other words, $2p(n)=p(n+1)+p(n-1)$, or $p(n+1)-p(n)=p(n)-p(n-1)$. From here, recursively,

$$\begin{aligned}
p(n+1)-p(n) &= p(n)-p(n-1)\\
&= p(n-1)-p(n-2)\\
&= p(n-2)-p(n-3)\\
&\dots\\
&= p(2)-p(1)\\
&= p(1)-p(0)\\
&= p(1).
\end{aligned}$$

It follows that $p(n)=np(1)$, and, since $p(a+b)=1$, $p(1)=\dfrac{1}{a+b}$. It follows that $p(a)=\dfrac{a}{a+b}$.

Amoeba's Survival

Solution 1

Good and general notations help solve the problem. Let p be the probability of a successful split for a single amoeba and P be the probability in question, the probability that an amoeba's family tree is infinite.

With the probability p, we have a second generation of two amoebas. The probability that at least one of them will have an infinite family tree is $1-(1-P)^2$ because $(1-P)^2$ is the probability that both of them will perish undivided. Therefore,

$$P=p(1-(1-P)^2)$$

because both sides of the equation represent the probability of the long term survival of the original amoeba.

Simplification yields $pP^2+(1-2p)P=0$, or $P\cdot(pP+(1-2p))=0$. Since $P\neq 0$, $pP+(1-2p)=0$ or $P=\dfrac{2p-1}{p}$.

By experimental verification, if $p>\frac{1}{2}$, there is a true population explosion that suggests that, in this case, $P\neq 0$. The formula we obtained also confirms that, if a generic amoeba divides with the probability not exceeding $\frac{1}{2}$, it stands no chance to survive forever. However, for the specific case where $p = \frac{3}{4}$, the probability of survival is $P = \frac{2}{3}$.

Notes

There is a nagging question whether relying on experimental results is a valid argument against $P = 0$. Let us modify the above a little. Let P_n be the probability that the n^{th} generation of amoebas is not empty. Then we have a recurrent relation:

$$P_{n+1} = p\left(1 - (1 - P_n)^2\right).$$

We shall prove by induction that $P_n > \frac{2p-1}{p}$. First off, $P_0 = 1$, as we are given that an amoeba is present at the outset. Let $P_n > \frac{2p-1}{p}$. We proceed in several steps:

$$\begin{aligned}
&P_n > \frac{2p-1}{p} \\
&1 - P_n < 1 - \frac{2p-1}{p} = \frac{1-p}{p} \\
&(1-P_n)^2 < \left(\frac{1-p}{p}\right)^2 \\
&1 - (1-P_n)^2 > 1 - \left(\frac{1-p}{p}\right)^2 = \frac{2p-1}{p^2} \\
&P_{n+1} = p\left(1 - (1-P_n)^2\right) > \frac{2p-1}{p},
\end{aligned}$$

as promised.

Solution 2

Christopher D. Long

This is a disguised biased random walk with an absorbing state at 0 which starts at 1. If this random walk is at N, considering N moves corresponds to considering the fate of N amoebas simultaneously. This yields $1 - \frac{1-p}{p} = 1 - \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{2}{3}$ by the usual arguments.

Average Visibility of Moviegoers

Solution 1

Amit Itagi

For n patrons, let the number of viewable patrons summed over all permutations be $g(n)$. Thus, the average number of patrons over the permutations is $\frac{g(n)}{n!}$ (denote this by $f(n)$).

There are $(n-1)!$ permutations where the tallest person is last in line. For this case, the sum of viewable patrons over all permutations is $g(n-1)+(n-1)!$ (note $g(n-1)$ for the first $n-1$ in line and $(n-1)!$ because the tallest patron is a viewable patron for all the permutations).

If a patron other than the tallest patron is last in line, the sum of viewable patrons over all permutations where the last patron is held fixed is $g(n-1)$ (the last patron is not a viewable patron in this case). However, the last patron can be chosen in $(n-1)$ ways for this case. Thus,

$$\begin{aligned} g(n) &= g(n-1)+(n-1)!+(n-1)g(n-1) \\ &= (n-1)!+ng(n-1) \\ \Rightarrow\ f(n) &= \frac{1}{n}+f(n-1) \text{ (dividing both sides by } n!). \end{aligned}$$

Noting that $f(1)=1$, the solution of this recurrence relation is

$$f(n)=\sum_{k=1}^{n}\frac{1}{k}=H_n \ (n^{\text{th}} \text{ harmonic number}).$$

Solution 2
[46]
Observe that the i^{th} tallest person is viewable if and only if he is in front of all the people $1,2,\ldots,i-1$. This happens with probability $\frac{1}{i}$. Therefore, the answer is

$$\sum_{i=1}^{n}\frac{1}{i}\approx \ln n+\gamma,$$

where $\gamma=0.5772157\ldots$, Euler's constant.

Book Index Range

Let $r_n(p)$ be the sought expectation. We shall show that

$$r_n(p)=p+(n-1)p(1-p) \tag{9.1}$$

by induction on n.

When $n=1$, we have $r_1(p)=p$, which is clearly true.

Suppose $n>1$ and assume equation 9.1 holds for $r_{n-1}(p)$, which is the expected number of ranges for an $(n-1)$-page book. The addition of one more page may or may not change the total number of ranges. When page n is added, the number of ranges increases by one if the term occurs on page n and does not occur on page $n-1$; this happens with probability $p(1-p)$. Otherwise, the number of ranges does not change. Therefore,

$$\begin{aligned} r_n(p) &= p(1-p)\cdot[r_{n-1}(p)+1]+[1-p(1-p)]\cdot r_{n-1}(p) \\ &= r_{n-1}(p)+p(1-p) \\ &= p+(n-1)p(1-p). \end{aligned}$$

Probability of No Two-Tail Runs

Solution 1

Let X_n be the number of H/T strings with no two successive T. $X_n = T_n + H_n$ to distinguish the strings that end in T from those that end with H.

Obviously, $H_1 = T_1 = 1$, $H_n = H_{n-1} + T_{n-1}$ and $T_n = H_{n-1}$. Thus, we may compute

n	1	2	3	4	5	6	7	8	9	10
T_n	1	2	3	5	8	13	21	34	55	89
H_n	1	1	2	3	5	8	13	21	34	55

It follows that there are $X_{10} = T_{10} + H_{10} = 89 + 55 = 144$ 10-letter-long strings with no TT. There are $2^{10} = 1024$ 10-letter strings in all. Thus, the probability of not seeing TT is $\frac{144}{1024} = \frac{9}{64}$.

Solution 2

N.N. Taleb

Let $\mathbf{M}$ be the Markov chain,

$$\mathbf{M} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix},$$

representing transitions between the following states:

	N	T	TT
N	$\frac{1}{2}$	$\frac{1}{2}$	0
T	$\frac{1}{2}$	0	$\frac{1}{2}$
TT	0	0	1

The matrix at the 10^{th} step is $\mathbf{M}^{10}$ and the probabilities are

$$(1,0,0).\mathbf{M}^{10} = \begin{pmatrix} p(N)_{10} \\ p(T)_{10} \\ p(TT)_{10} \end{pmatrix} = \begin{pmatrix} \frac{89}{1024} \\ \frac{55}{1024} \\ \frac{55}{64} \end{pmatrix}.$$

Thus, the probability of never getting two tails in 10 throws is

$$1 - p(TT)_{10} = 1 - \frac{55}{64} = \frac{9}{64}.$$

Artificially Unintelligent

Solution 1

Amit Itagi

The number of bit flips has to be even. Let $q = 1 - p$:

$$P = \sum_{\{k|k\equiv 0 \pmod 2),\ k\le n\}} \binom{n}{k} q^k p^{n-k}.$$

Using the binomial expansion,

$$P = \begin{cases} \dfrac{(q+p)^n + (q-p)^n}{2} = \dfrac{1+(1-2p)^n}{2}, & n \text{ even} \\ \dfrac{(q+p)^n - (q-p)^n}{2} = \dfrac{1-(1-2p)^n}{2}, & n \text{ odd}. \end{cases}$$

Since, for odd n, $(-1)^n = -1$, $\dfrac{1+(2p-1)^n}{2}$ so that we can combine the two expressions into one:

$$\frac{1+(2p-1)^n}{2}.$$

Solution 2

Hélvio Vairinhos

Let $\begin{pmatrix}1\\0\end{pmatrix}$ represent the correct bit and $\begin{pmatrix}0\\1\end{pmatrix}$ the flipped bit. The transition probabilities between two states are given by the (diagonalizable) transfer matrix:

$$T = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix} = X^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 2p-1 \end{pmatrix} X,$$

where

$$X = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

and

$$X = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

For n transmissions, the matrix is

$$T^n = X^{-1} \begin{pmatrix} 1 & 0 \\ 0 & (2p-1)^n \end{pmatrix}$$

$$X = \frac{1}{2} \begin{pmatrix} 1+(2p-1)^n & 1-(2p-1)^n \\ 1-(2p-1)^n & 1+(2p-1)^n \end{pmatrix}.$$

The probability of measuring the correct bit after n transmissions is

$$(T^n)_{11} = \frac{1}{2}(1 + (2p-1)^n).$$

Solution 3
Timon Knigge

Using the binomial theorem $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$, we want only the terms with an even number of failures:

$$\begin{aligned}
P &= \sum_{k=0}^{n} \binom{n}{k} p^{n-k}(1-p)^k \cdot \frac{1+(-1)^k}{2} \\
&= \frac{1}{2}\left(\sum_{k=0}^{n} \binom{n}{k} p^{n-k}(1-p)^k\right) + \frac{1}{2}\left(\sum_{k=0}^{n} \binom{n}{k} p^{n-k}(p-1)^k\right) \\
&= \frac{1}{2}\left((p+(1-p))^n + (2p-1)^n\right) \\
&= \frac{1}{2} + \frac{1}{2}(2p-1)^n.
\end{aligned}$$

Solution 4
N.N. Taleb

We need either 0 or an even number of switches for transmission to be correct. Let $\phi = p^x \binom{n}{x}(1-p)n-x$ be the probability distribution function of a binomial distribution $N(n,p)$ with n trials and the probability of success p, for x successes. We need x to be even: $x = 0, 2, \ldots, n$.

The probability under consideration is

$$\begin{aligned}
P(n) &= \sum_{x=0}^{n/2} p^{2x} \binom{n}{2x} (1-p)^{n-2x} = \frac{1}{2}\left(1 + (2p-1)^n\right) \\
&\approx \frac{1}{2},
\end{aligned}$$

for large values of n. Since $0 \le p < 1$,

$$\lim_{n\to\infty} \frac{1}{2}\left(1 + (2p-1)^n\right) = \frac{1}{2}.$$

Solution 5
Alejandro Rodríguez

Let $P(n,1)$ be the probability that the message gets transmitted correctly given that it is correct and it has to be transmitted n more times. Let $P(n,0)$ be the probability that the message gets transmitted correctly given that it is incorrect and it has to be transmitted n more times. Recursively, we can compute these probabilities as

$$P(n,1) = pP(n-1,1) + (1-p)P(n-1,0)$$
$$P(n,0) = pP(n-1,0) + (1-p)P(n-1,1)$$

with boundary conditions

$$P(0,1) = 1$$
$$P(0,0) = 0.$$

From the previous equations, we can show that

$$P(n,1) + P(n,0) = P(n-1,1) + P(n-1,0),$$

which implies that

$$P(n,0) = 1 - P(n,1).$$

Substitute into the first equation and solve to get

$$P(n,1) = (2p-1)P(n-1,1) + 1 - p$$
$$P(n,1) = (2p-1)^n + (1-p)\sum_{i=0}^{n-1}(2p-1)^i$$
$$P(n,1) = (2p-1)^n + (1-p)\frac{1-(2p-1)^n}{1-(2p-1)}$$
$$P(n,1) = \frac{(2p-1)^n}{2} + \frac{1}{2}.$$

Notes

As $n \to \infty$, the probability tends to $\frac{1}{2}$, regardless of p.

Two Sixes in a Row

Solution 1

Let $P(n)$ denote the probability that, for the first time, two successive sixes showed up on the tosses $n-1$ and n. Obviously, $P(1) = 0$ and $P(2) = p^2$, with $p = \frac{1}{6}$.

Let $\mathbb{E}(n)$ be the expected number of tosses. There is no doubt that $\mathbb{E}(n) > 2$. We denote the toss that came up with 6 on top as S; all other tosses are denoted collectively as F. Any sequence of S and F starts with one of the four combinations: SS, SF, FS, FF. With the probability of p^2 of the first two successes, we achieve the goal in just two throws. In case of either SF or FF, we have to add two to the number of tosses and start counting again. SF has the probability of $p(1-p)$; FF has the probability $(1-p)^2$. Concerning the first two tosses being FS, which has the probability of $(1-p)p$, the third toss may have two outcomes. If it is S, we achieve our goal in three tosses—this with the probability of $(1-p)p^2$. If it is F, we count three steps and continue as from the beginning. This happens with the probability of

$(1-p)^2p$. Thus, we are led to the recurrence relation

$$\begin{aligned}\mathbb{E}(n) &= 2\cdot p^2 \\ &\quad + (\mathbb{E}(n)+2)(p(1-p)+(1-p)^2) \\ &\quad + 3\cdot(1-p)p^2 \\ &\quad + (\mathbb{E}(n)+3)(1-p)^2p.\end{aligned}$$

A few algebra steps yield

$$\mathbb{E}(n) = \frac{2+p-p^2}{p^2(2-p)} = \frac{(2-p)(p+1)}{p^2(2-p)} = \frac{1}{p^2}+\frac{1}{p}.$$

For $p=\frac{1}{6}$, $\mathbb{E}(n)=36+6=42$.

In passing, for a fair coin, $p=\frac{1}{2}$ so that $\mathbb{E}(n)=4+2=6$.

Solution 2

Christopher D. Long

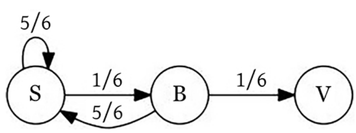
The expected number of throws $\mathbb{E}(t)$ until a 6 is on top is six, so

$$\mathbb{E} = \mathbb{E}(t\mathbb{E}(t)+t) = \mathbb{E}(t)^2+\mathbb{E}(t) = 42.$$

Solution 3

Alejandro Rodríguez

Using the Markov chain, we need three nodes: S stands for any sequence of throws that does not end with a 6; B is an intermediate node that denotes sequences that end with a single 6; V is the final node for the sequences that end in two 6s.



We derive two equations (borrowing the notations from the associated nodes). The equation

$$V = 1+\frac{1}{6}B+\frac{5}{6}V$$

tells us that moving from a node via an attached arrow takes one throw of a die (+1) in the diagram. Then with the probability of $\frac{1}{6}$, i.e., of getting one 6, we move to a new state B, while with the probability of $\frac{5}{6}$ we return to where we started (but still wasting one toss).

The other equation is

$$B = 1+\frac{5}{6}V$$

which tells us that with one toss we either get to the goal V or back to the start S.

We can find V from the two equations: $V = 42$, which conventionally counts the last throw.

Solution 4

Issam Eddine

Let us call $T(n)$, for $n \in \mathbb{N}$, a time at which n successive 6s show up for the first time. For $n \in \mathbb{N}$, $T(n)$ is a random variable. We want to find $\mathbb{E}(T(2))$. We know (Riddle 8.9 on page 194) that $\mathbb{E}(T(1)) = 6$ and that the game is a Markov process. For $n > 1$, we have the following relation:

$$\mathbb{E}(T(n+1)) = \frac{1}{6}\left(\mathbb{E}(T(n)) + 1\right) + \frac{5}{6}\left(\mathbb{E}(T(n)) + 1 + \mathbb{E}(T(n+1))\right).$$

We then deduce that $\mathbb{E}(T(n+1)) = 6(\mathbb{E}(T(n)) + 1)$. Thus, $\mathbb{E}(T(2)) = 42$.

Solution 5

Michael Wiener

Using Nielsen's result [73], we have an L-letter alphabet, with $L = 6$, and a target string S of n letters. Let S' be the longest common prefix and suffix of S (excluding the case of $S' = S$). Then $\mathbb{E}(S) = L^n + \mathbb{E}(S')$. Thus, $\mathbb{E}(\text{first } 66) = 6^2 + \mathbb{E}(\text{first } 6) = 36 + 6 = 42$.

Solution 6

N.N. Taleb

$$\mathbf{M} = \begin{pmatrix} \frac{5}{6} & \frac{1}{6} & 0 \\ \frac{5}{6} & 0 & \frac{1}{6} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{M}'_n = (1,0,0).\mathbf{M}^n = \begin{pmatrix} \dfrac{2^{-2n-1}3^{-n-1}\left(\left(5+3\sqrt{5}\right)^{n+1} - \left(5-3\sqrt{5}\right)^{n+1}\right)}{\sqrt{5}} \\ \dfrac{3^{-2n-1}4^{-n}\left(\left(15+9\sqrt{5}\right)^{n} - \left(15-9\sqrt{5}\right)^{n}\right)}{\sqrt{5}} \\ \frac{1}{5}2^{-2n-1}3^{-n-1}(A+B-C) \end{pmatrix}$$

where

$$A = 5\ 2^{2n+1}3^{n+1},$$

$$B = \left(7\sqrt{5} - 15\right)\left(5 - 3\sqrt{5}\right)^n,$$

$$C = \left(15 + 7\sqrt{5}\right)\left(5 + 3\sqrt{5}\right)^n.$$

We take the third row and get $\varphi(t)$, the probability distribution function of the stopping time with the difference of the third rows of $\mathbf{M}'_n$ and $-\mathbf{M}'_{n-1}$:

$$\begin{aligned}\varphi(t) =& \frac{1}{\sqrt{5}} 2^{-2t-1} 3^{-t-2} \left(\left(3\sqrt{5}+5\right)^t - \left(5-3\sqrt{5}\right)^t \right)\\ \mathbb{E}(n) =& \sum_{t=2}^{n} t\varphi(t)\\ =& \frac{1}{5} 2^{-2n-1} 3^{-n-1} \Big(- \left(7\sqrt{5}+15\right) n \left(3\sqrt{5}+5\right)^n\\ & - 6\left(47\sqrt{5}+105\right)\left(3\sqrt{5}+5\right)^n + \left(7\sqrt{5}-15\right) n \left(5-3\sqrt{5}\right)^n\\ & + 6\left(47\sqrt{5}-105\right)\left(5-3\sqrt{5}\right)^n + 35\ 3^{n+2} 4^{n+1} \Big)\end{aligned}$$

$$\lim_{n\to\infty} \mathbb{E}(n) = 42.$$

Notes

N.N. Taleb

The number of steps to k sixes in a row is

$$\mathbb{E}(n,k) = \frac{1}{p^k} + \frac{1}{p^{k-1}} + \frac{1}{p^{k-2}} + \cdots + \frac{1}{p},$$

with $p = \dfrac{1}{6}$.

Matching Socks in Dark Room

Solution 1

Let $f(n)$ be the number of pairings of n pairs of socks into acceptable pairs. For $n \le 2$, any pairing is acceptable. Furthermore, $f(1) = 1$; $f(2) = 3$.

In general, let us start with a sock of the darkest shade. If we happen to pick a sock of the same shade, we are left with computing $f(n-1)$. Otherwise, there is a pair of socks of the next shade; we can choose either of the two and match the other one with the remaining sock of the darkest shade. This gives a recurrence:

$$f(n) = f(n-1) + 2f(n-2).$$

The characteristic equation of the latter is $x^2 - x - 2 = 0$, with roots $x = 2$ and $x = -1$. The general solution of the recurrence depends on two constants: $f(n) = c_1 2^n + c_2(-1)^n$, which are found from the initial conditions $f(1) = 1$ and $f(2) = 3$. Thus, we arrive at

$$f(n) = \frac{2^{n+1} + (-1)^n}{3}.$$

For 10 pairs of socks, there are $f(10) = \dfrac{2048+1}{3} = 683$ possible matchings.

To find the sought probability, we have to find the total number of pairings, absent any restrictions.

One sock can be matched with any of 19, but when that pair is picked, the next sock may match with any of the remaining 17, and so on. Thus, the total number of pairings is

$$19!! = 19 \cdot 17 \cdot 15 \cdot \dots \cdot 5 \cdot 3 \cdot 1.$$

The probability we seek equals $\dfrac{683}{19!!} \approx 10^{-6}$.

Solution 2

$f(n)$ could have been found via generating functions. Let us set

$$\begin{aligned} F(x) &= \sum_{n=1}^{\infty} f(n)x^n \\ &= 1 + 3x + \sum_{n=3}^{\infty} f(n-1) + 2\sum_{n-3} f(n-2)x^n \\ &= 1 + 3x + x\left(F(x) - x\right) + 2x^2F(x), \end{aligned}$$

from which

$$\begin{aligned} F(x) &= \frac{x+2x^2}{1-x-2x^2} = \frac{x}{3}\left(\frac{4}{1-2x} - \frac{1}{1+x}\right) \\ &= \frac{x}{3}\left(4\sum_{n=0}^{\infty}(2x)^n - \sum_{n=0}^{\infty}(-x)^n\right) \\ &= \frac{x}{3}\sum_{n=0}^{\infty}\left(2^{n+2} - (-1)^n\right)x^n \\ &= \sum_{n=0}^{\infty}\frac{2^{n+2} - (-1)^{n+2}}{3}x^{n+1} \\ &= \sum_{n=1}^{\infty}\frac{2^{n+1} + (-1)^n}{3}x^n. \end{aligned}$$

By definition, $f(n) = [x^n]F(x)$, the coefficient of the series by the term x^n. Thus, as before,

$$f(n) = \frac{2^{n+1} + (-1)^n}{3}.$$

Number of Wire Loops

Amit Itagi

Let the expected number of loops for n wires be $f(n)$. There are $\dbinom{2n}{2} = 2n^2 - n$ ways of choosing two ends from n wires. Out of these, n cases will result in a wire forming a self loop with $n-1$ wire segments remaining. The other $2n^2 - 2n$ cases will result in no self loop with just $n-1$ wire segments remaining. Thus,

$$\left[2n^2 - n\right] f(n) = n\left[1 + f(n-1)\right] + \left[2n^2 - 2n\right] f(n-1)$$
$$\Rightarrow\ f(n) = f(n-1) + \frac{1}{2n-1}.$$

$f(1) = 1$. Thus,

$$f(n) = \sum_{k=1}^{n} \frac{1}{2k-1} = \sum_{k=1}^{2n} \frac{1}{k} - \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k} = H_{2n} - \frac{H_n}{2}.$$

Thus,

$$f(2018) = H_{4036} - \frac{H_{2018}}{2} \sim 4.787,$$

where H_n is the n^{th} harmonic number:

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Determinants in $\mathbb{Z}_2$

Solution 1

Observe that the total number of $n \times n$ matrices with elements in $\mathbb{Z}_2$ equals 2^{n^2}. The number of nonzero $n \times n$ determinants over $\mathbb{Z}_2$ equals the number of n linearly independent (and ordered) rows of n elements from $\mathbb{Z}_2$, i.e., vectors in $\mathbb{Z}_2^n$. Note also that if $a_1, \ldots, a_k$, $k \le n$, are linearly independent in $\mathbb{Z}_2^n$, then there are exactly $\mathbb{Z}_2^k$ linear combinations $\sum_{i=1}^{k} \lambda_i a_i \in \mathbb{Z}_2^n$. Thus, there are exactly $2^n - 2^k$ ways to expand the system of k linearly independent vectors $a_1, \ldots, a_k$ to a system of $k+1$ linearly independent vectors $a_1, \ldots, a_k, a_{k+1}$. It follows that the number of nonzero determinants over $\mathbb{Z}_2$ of order n is

$$(2^n - 1)(2^n - 2)(2^n - 2^2)\cdots(2^n - 2^{n-1}) = 2^{n^2} \prod_{k=1}^{n} \left(1 - \frac{1}{2^k}\right).$$

From here, $p_n = \prod_{k=1}^{n} \left(1 - \frac{1}{2^k}\right)$. Thus, $p_n > p_{n+1}$. In addition,

$$\prod_{k=1}^{n} \left(1 - \frac{1}{2^k}\right) \ge \frac{1}{2}\left(1 - \sum_{k=2}^{n} \frac{1}{2^k}\right) = \frac{1}{2}\left(1 - \frac{2^{k+1}-1}{2^{k+2}}\right),$$

implying

$$\begin{aligned}\lim_{n\to\infty}\prod_{k=1}^{n}\left(1-\frac{1}{2^k}\right) &\ge \frac{1}{2}\lim_{n\to\infty}\left(1-\frac{2^{k+1}-1}{2^{k+2}}\right)\\ &= \frac{1}{2}\left(1-\lim_{n\to\infty}\frac{2^{k+1}-1}{2^{k+2}}\right)\\ &= \frac{1}{2}\left(1-\frac{1}{2}\right)=\frac{1}{4}.\end{aligned}$$

Thus, $p=\lim_{n\to\infty}p_n$ exists and $p\ge\dfrac{1}{4}$.

Solution 2

Long Huynh Huu

The problem reduces to computing the number of ordered bases of $V=F_2^n$. This can be done by choosing

- a vector $b_1\in V-\text{span}(0)$
- a vector $b_2\in V-\text{span}(b_1)$
- a vector $b_3\in V-\text{span}(b_1,b_2)$
- and so on.

Considering that the number of $n\times n$ matrices over F_2 is $2^{n^2}=(2^n)^n$, we obtain

$$\frac{\prod_{i=0}^{n-1}(2^n-2^i)}{(2^n)^n}=\prod_{i=0}^{n-1}\left(1-2^{i-n}\right)=\prod_{j=1}^{n}\left(1-\frac{1}{2^j}\right)$$

as the probability of the random determinant to be nonzero.

The probability of a random determinant being 0 is, therefore,

$$1-\prod_{j=1}^{n}\left(1-\frac{1}{2^j}\right).$$

Gambling in a Company

The expected number of rounds is $\sum_{i<j}a_ia_j$. The probability that the i^{th} gambler ends up with all the money is $\dfrac{a_i}{a_1+a_2+\cdots+a_n}$. For $n=3$, the expected number of rounds until the first loser quits the game is $\dfrac{3a_1a_2a_3}{a_1+a_2+a_3}$.

$n=2$ Analog

The riddle is an obvious extension of the game of two players (Riddle 9.1 on page 237). The latter is usually modeled by the well-known one-dimensional walk where a point on an axis moves one step at a time—left or right—with probabilities $p = q = \frac{1}{2}$. The two players start with the capitals of, say, u and v dollars and lose or win one dollar from the other player with the probability of $\frac{1}{2}$. On average, the game lasts uv rounds in which the probabilities of winning are $\frac{u}{u+v}$ and $\frac{v}{u+v}$, respectively.

Definitions

Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ denote the state of the game where the i^{th} player owns $u_i \geq 0$ dollars. We will write $|\mathbf{u}| = u_1 + u_2 + \cdots + u_n$. The quantity $a = |\mathbf{u}|$ remains unchanged during the game.

Let $N(\mathbf{u})$ be the set of all states that can be attained from $\mathbf{u}$ in one round. If $\mathbf{u}$ has k nonzero components, then $|N(\mathbf{u})| = k(k-1)$. Define $V(\mathbf{u})$ as the set of all states that could be attained starting with $\mathbf{u}$:

$$V(\mathbf{u}) = \{\mathbf{v} : \forall i \, v_i \geq 0, |\mathbf{u}| = |\mathbf{v}|, (u_i = 0) \Rightarrow (v_i = 0)\}.$$

Solution 1 to Question 1

Let $R(\mathbf{u})$ be the expected number of rounds, starting with $\mathbf{u}$, until one of the players has all the money. (We are to prove that $\mathbf{a} = \sum_{j<j} a_i a_j$.) There is a recurrent relation: $R(\mathbf{u})$ is the average of all $\mathbf{x} \in N(\mathbf{u})$ plus 1. We also have boundary conditions: $R(\mathbf{u}) = 0$ only if all components of $\mathbf{u}$ but one are zero.

The function $R(\mathbf{u}) = \sum_{i<j} a_i a_j$ satisfies the recurrence and the boundary condition. Indeed, for any two components u and v, the product uv appears in every term of the sums over the neighborhood $N(\mathbf{u})$. The same holds for -1; the linear terms u and v cancel out. We only need to prove that no other function satisfies the recurrence, along with the boundary conditions.

If there are two such solutions, then the difference at any point is the average of its values in the point's neighborhood and, therefore, could not be either maximum or minimum (in the neighborhood) unless the function is constant. Then the function is constant over the whole play field $V = V(\mathbf{x})$, and, being 0 on the boundary is 0 on V, the two solutions coincide.

Solution 2 to Question 1

Hélvio Vairinhos

Let $a = \sum_{i=1}^{n} a_i$ be the total capital at play. Let R_i be the number of rounds played by the i^{th} gambler. The remaining $(n-1)$ gamblers are undistinguishable from a single gambler with $(a - a_j)$ dollars. Hence, we only need to consider two gamblers, for which $E(R_i) = a_i(a - a_i)$. The total number of rounds is $R = \frac{1}{2}\sum_{i=1}^{n} R_i$; hence,

$$E(R) = \frac{1}{2}\sum_{i=1}^{n} a_i(a - a_i) = \sum_{i<j} a_i a_j.$$

Solution 1 to Question 2

Let $p_i(\mathbf{u})$ be the probability that the i^{th} player ends up with all the money, starting with the state $\mathbf{u}$. (We have to prove that $p_i(\mathbf{u}) = \dfrac{u_i}{|\mathbf{u}|}$.) $p_i(\mathbf{u})$ is the average of all $p_i(\mathbf{x})$, for $x \in N(\mathbf{u})$, with the following boundary conditions: $p_i(\mathbf{u}) = 0$ if $u_i = 0$ and $p_i(\mathbf{u}) = 1$ if $u_i = a$ so that all other components vanish automatically.

As before, we simply must prove uniqueness.

Solution 2 to Question 2

Hélvio Vairinhos

Consider a players with 1 dollar each. Before the first round, the probability of winning the game is uniform among the players: $\dfrac{1}{a}$. Partition the players into n teams so that the i^{th} team has a_i members/dollars. Then, the probability of the i^{th} team (i.e., gambler) winning the game is $p_i = a_i \cdot \dfrac{1}{a} = \dfrac{a_i}{a}$.

Solution to Question 3

Here the recurrence is exactly the same as for the first question, but the boundary condition is different: the game stops when one of the three components vanishes.

Probability in the World Series

Solution 1

The probability of winning the World Series in exactly n games is

$$P(n) = \frac{1}{2} \cdot \frac{1}{2^{n-1}} \cdot \binom{n-1}{3} \cdot 2 = \frac{1}{2^{n-1}} \cdot \binom{n-1}{3}.$$

Because $\dfrac{1}{2}$ is the probability of winning the n^{th} game, $\dfrac{\binom{n-1}{3}}{2^{n-1}}$ is the probability of winning three games out of the previous $n-1$, and there are two teams to choose from.

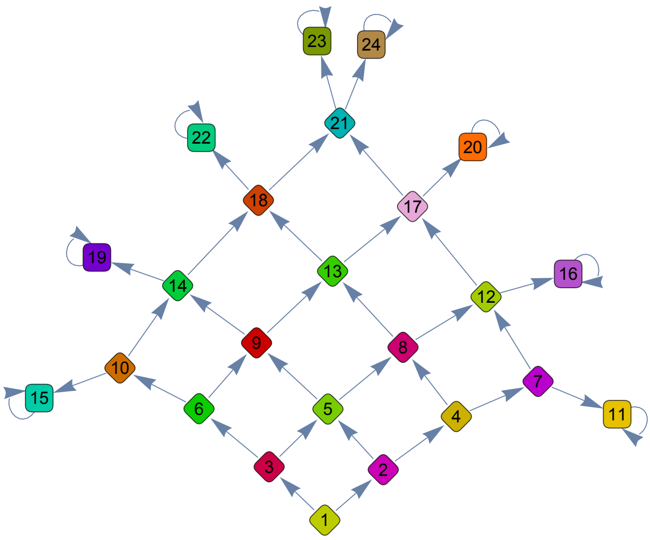
We have $P(4) = \dfrac{1}{8}$, $P(5) = \dfrac{1}{4}$, $P(6) = \dfrac{5}{16}$, $P(7) = \dfrac{5}{16}$. The expectation E is then

$$E = 4 \cdot \frac{1}{8} + 5 \cdot \frac{1}{4} + 6 \cdot \frac{5}{16} + 7 \cdot \frac{5}{16} = \frac{93}{16} = 5.8125.$$

Solution 2

Marcos Carreira

Form a Markov chain with absorbing states $\{11, 15, 16, 19, 20, 22, 23, 24\}$ as shown here:



Those states are attained with probabilities

$$\left\{\frac{1}{16}, \frac{1}{16}, \frac{1}{8}, \frac{1}{8}, \frac{5}{32}, \frac{5}{32}, \frac{5}{32}, \frac{5}{32}\right\}$$

in $\{4, 4, 5, 5, 6, 6, 7, 7\}$ games, respectively. Thus, we obtain the expectation of

$$\frac{1}{4} + \frac{1}{4} + \frac{5}{8} + \frac{5}{8} + \frac{30}{32} + \frac{30}{32} + \frac{35}{32} + \frac{35}{32} = 5.8125.$$

Solution 3

Amit Itagi

For the series to end in n games ($n \geq 4$), the winning team has to win three of the first $n-1$ games and the n^{th} game. Moreover, either one of the two teams could be the winning team. Thus, the expected number of games is given by

$$\begin{aligned} E &= \sum_{n=4}^{7} n \cdot 2 \cdot \binom{n-1}{3} \left(\frac{1}{2}\right)^n = \frac{93}{16} \\ &= 5.8125. \end{aligned}$$

Probability of a Meet in an Elimination Tournament

Solution 1

Let us consider a more general problem of 2^k players. We shall denote the two players A and B.

If $k = 1$, the two players necessarily play each other: $P_1 = 1$.

If $k = 2$, there are three possible pairings, implying that A and B meet in the first round with the probability of $\frac{1}{3}$. If not, each wins his first round match with probability

$\frac{1}{2}$ such that the probability of their meeting in the second round is $\frac{2}{3} \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{6}$. Thus, $P_2 = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$.

In general, with 2^k players, A and B meet in the first round with the probability of $\frac{1}{2^k - 1}$. With the probability of $\frac{2^k - 2}{2^k - 1}$, they do not meet in the first round but continue to the second with the probability of $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$. For the total probability P_k, we get a recurrence:

$$P_k = \frac{1}{2^k - 1} + \frac{1}{4} \cdot \frac{2^k - 2}{2^k - 1} \cdot P_{k-1},$$

i.e.,

$$2(2^k - 1)P_k = 2 + (2^{k-1} - 1) \cdot P_{k-1}.$$

After multiplying both sides by 2^{k-1}, we get

$$2^k(2^k - 1)P_k = 2^k + 2^{k-1}(2^{k-1} - 1) \cdot P_{k-1}.$$

This recurrence is easily solved:

$$\begin{aligned} 2^k(2^k - 1)P_k &= 2^k + 2^{k-1}(2^{k-1} - 1) \cdot P_{k-1} \\ &= 2^k + 2^{k-1} + 2^{k-2}(2^{k-2} - 1)P_{k-2} \\ &= 2^k + 2^{k-1} + 2^{k-2} + 2^{k-3}(2^{k-3} - 1)P_{k-3} \\ &\qquad \cdots \\ &= \sum_{i=k}^{2} 2^i + 2(2-1)P_1 = \sum_{i=k}^{1} 2^i = 2\sum_{i=0}^{k-1} 2^i = 2(2^k - 1). \end{aligned}$$

Thus, $2^k(2^k - 1)P_k = 2(2^k - 1)$ so that $P_k = \frac{1}{2^{k-1}}$. For $k = 5$, the probability is $\frac{1}{16}$.

Solution 2

Xavier Faure

With 2^k particpants $2^k - 1$ games will be played out. On the other hand, the total amount of possible games equals $\frac{2^k(2^k - 1)}{2} = 2^{k-1}(2^k - 1)$. Thus, for each game to be played in the tournament, the probability equals $P_k = \frac{2^k - 1}{2^{k-1}(2^k - 1)} = \frac{1}{2^{k-1}}$.

Solution 3

Amit Itagi

Suppose there are N players. When two players meet in round k, they are effectively the chosen two from a pool of 2^k players who competed in the sub-bracket leading to that particular match. Let us call such a sub-bracket a k-sub-bracket.

There are $M_k = \dfrac{2^N}{2^k}$ k-sub-brackets. The probability that both players end up in a particular k-sub-bracket is

$$P_1(k) = \frac{2^k}{2^N} \cdot \frac{2^k - 1}{2^N - 1}.$$

The probability that two players from a k-sub-bracket meet in round k is

$$P_2(k) = 2 \cdot \frac{1}{2^k} \cdot \frac{1}{2^k - 1} = \frac{1}{2^{k-1}} \cdot \frac{1}{2^k - 1}.$$

Thus, the probability that the two players play each other is

$$\begin{aligned} P_N &= \sum_{k=1}^{N} M_k \cdot P_1(k) \cdot P_2(k) \\ &= \sum_{k=1}^{N} \left(\frac{2^N}{2^k}\right) \left(\frac{2^k}{2^N}\right) \left(\frac{2^k - 1}{2^N - 1}\right) \left(\frac{1}{2^{k-1}}\right) \left(\frac{1}{2^k - 1}\right) \\ &= \frac{1}{2^N - 1} \sum_{k=1}^{N} \frac{1}{2^{k-1}} = \frac{1}{2^{N-1}}. \end{aligned}$$

For $N = 5$,

$$P_5 = \frac{1}{16}.$$

Rolling a Die

Solution to Question 1

[57, Problem 20], N.J. Fine

There are six possible outcomes before the running total exceeds 12: 12, 11, 10, 9, 8, 7. Here is the list of possible outcomes for any of these:

12 :	13	14	15	16	17	18
11 :	12	13	14	15	16	17
10 :	11	12	13	14	15	16
9 :	10	11	12	13	14	15
8 :	9	10	11	12	13	14
7 :	8	9	10	11	12	13

The most likely total appears to be 13 as it is the only one with a chance to be reached in all six cases.

Solution to Question 2

Josh Jordan

The exact answer for question 2 can be found by setting up a Markov chain and solving the resulting set of linear equations. Let $x, a, b, c, \ldots, l$ be the expected sum if the running total is $0, 1, 2, 3, \ldots, 12$:

$$6x = a + b + c + d + e + f$$

$$6a = b + c + d + e + f + g$$

$$6b = c + d + e + f + g + h$$

$$6c = d + e + f + g + h + i$$

$$6d = e + f + g + h + i + j$$

$$6e = f + g + h + i + j + k$$

$$6f = g + h + i + j + k + l$$

$$6g = h + i + j + k + l + 13$$

$$6h = i + j + k + l + 27$$

$$6i = j + k + l + 42$$

$$6j = k + l + 58$$

$$6k = l + 75$$

$$6l = 93.$$

Then

$$x = \frac{63954819943}{4353564672} \approx 14.6902.$$

Solution to Question 2, The Long Route (Unrefined but Rigorous Solution)

N.N. Taleb

The most likely total is trivial (we leave the solution for later, in the implementation section). The expected total is more involved.

Let $\theta = \{1, 2, \ldots, 6\}$. Let $\chi = \{\chi_1, \chi_2, \ldots\}$ be the set of D-tuples T_6^D, ordered multisets generated from θ. If we pick the exhaustive solution of $D = 13$, the multiset total number of members is $d = 6^{13}$. We have $\chi_m = (\chi_m(1), \chi_m(2), \ldots, \chi_m(13))$, say $\chi_1 = (1, 1, \ldots, 1)$, $\chi_2 = (1, 1, \ldots, 2)$, etc. Each tuple χ_i has a probability of occurrence of $\frac{1}{d}$. For each tuple, the corresponding stopping time condition is $\mathbf{1}_{t<\tau}$, with τ the first time the preceding sum exceeds 13:

$$\tau(m) = \left\{ \inf t : \sum_{i=1}^{t} \chi_m(i) \geq 13 \right\}.$$

$$\chi'_m = \left(\sum_{i=1}^{j} \chi_m(i) \right)_{j=1}^{3} \sqcup \left(\mathbf{1}_{t-1<\tau(m)} \sum_{i=3}^{j} \chi_m(i) \right)_{j=4}^{13}.$$

Let Σ_τ be the cumulative tally on the die at stopping time τ. Therefore, we have

$$\mathbb{E}(\Sigma_\tau) = \frac{1}{d} \sum_{m=1}^{D} \max(\chi'_m)$$

and the average time to stopping is

$$\mathbb{E}(\tau) = \frac{1}{d} \sum_{m=1}^{d} \tau(m).$$

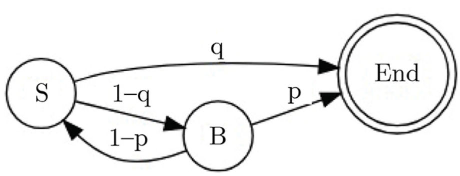
Implementation and Refinement

The previous method, while simple to implement, is computationally heavy. We know that the gain in stopping time from $D = 12$ to $D = 13$ is 6^{-13} (a single tuple has a stopping time of 13), and the gain from $D = 11$ to $D = 12$ is 6^{-12} (only six additional tuples have a stopping time of 12). We can safely reduce D by opting for a length of integers $\approx \frac{D}{2}$ instead of D for the tuples and test for convergence. Thus, we can compute the bias:

$$\left(\begin{array}{ccc}
D=5 & \begin{pmatrix} \text{Expected Total} & 14.6447 \\ \text{Expected } \tau & 3.97605 \\ \text{Quantile} & 14 \end{pmatrix} & \begin{pmatrix} \text{Total Tuples} & 7776 \\ \text{Stopped Tuples} & 7014 \\ \text{Precision:} & 0.902006 \end{pmatrix} \\
D=6 & \begin{pmatrix} \text{Expected Total} & 14.68 \\ \text{Expected } \tau & 4.13774 \\ \text{Quantile} & 14 \end{pmatrix} & \begin{pmatrix} \text{Total Tuples} & 46656 \\ \text{Stopped Tuples} & 45738 \\ \text{Precision:} & 0.980324 \end{pmatrix} \\
D=7 & \begin{pmatrix} \text{Expected Total} & 14.6886 \\ \text{Expected } \tau & 4.1861 \\ \text{Quantile} & 14 \end{pmatrix} & \begin{pmatrix} \text{Total Tuples} & 279936 \\ \text{Stopped Tuples} & 279144 \\ \text{Precision:} & 0.997171 \end{pmatrix} \\
D=8 & \begin{pmatrix} \text{Expected Total} & 14.69 \\ \text{Expected } \tau & 4.19577 \\ \text{Quantile} & 14 \end{pmatrix} & \begin{pmatrix} \text{Total Tuples} & 1679616 \\ \text{Stopped Tuples} & 1679121 \\ \text{Precision:} & 0.999705 \end{pmatrix}
\end{array}\right).$$

Fair Duel

Generalizing a little, assume Smith hits Brown with probability q and Brown hits Smith with probability p. There is a three state Markov chain: S(mith) shoots, B(rown) shoots, the duel End(s).



If P is the probability of Smith winning the duel (and we know that, the duel being fair, $P = \frac{1}{2}$), then

$$P = q + (1-q)(1-p)P,$$

a recurrence which says that Smith wins on the first shot or gets an equal chance after he and Brown both miss.

This leads to $P(q+p-qp)=q$, or $(q+p-qp)=2q$, i.e., $p=\dfrac{q}{1-q}$. In the specific case where $q=\dfrac{4}{10}$, $p=\dfrac{4}{6}=\dfrac{2}{3}$.

We may represent the chain with a transition matrix, e.g., with three states S, B, End (in that order):

$$M=\begin{pmatrix} 0 & 1-q & q \\ 1-p & 0 & p \\ 0 & 0 & 1 \end{pmatrix}.$$

The last row simply means that, once we get to state End, this is where we stay with the probability of 1:

$$M^2=\begin{pmatrix} (1-q)(1-p) & 0 & q \\ 0 & (1-q)(1-p) & p \\ 0 & 0 & 1 \end{pmatrix}.$$

This tells us that, after two attempts, B is back in B with the probability $(1-q)(1-p)$. The same is true of the state S.

The state of affairs after two steps is read from the first row and the recurrence is obtained by multiplying vector (1 0 0) by M^2:

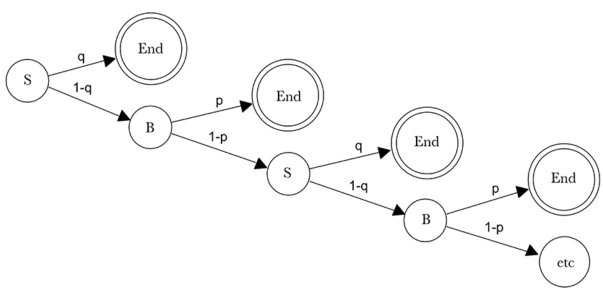
$$(1\ 0\ 0)\begin{pmatrix} (1-q)(1-p) & 0 & q \\ 0 & (1-q)(1-p) & p \\ 0 & 0 & 1 \end{pmatrix}=((1-q)(1-p)\ 0\ q),$$

and, as a consequence, $P=q+(1-p)(1-q)P$.

Solution 2

Amit Itagi

The problem can be described with a decision tree:



Equivalently,

$$P(S\text{ win})=0.4\cdot[1+0.6\cdot(1-p)+0.6^2\cdot(1-p)^2+\ldots]$$

$$P(B\text{ win})=(1-0.4)\cdot p\cdot[1+0.6(1-p)+0.6^2\cdot(1-p)^2+\ldots]$$

$$P(S\text{ win})=P(B\text{ win})\ \Rightarrow p=\frac{0.4}{1-0.4}=\frac{2}{3}.$$

Chapter 10

Sampling of American Mathematics Competition Problems

> Probability is a liberal art; it is a child of skepticism, not a tool for people with calculators on their belts to satisfy their desire to produce fancy calculations and certainties.
>
> Nassim Nicholas Taleb, 2007, *The Black Swan*

Riddles

10.1 Five-Digit Numbers Divisible by Eleven

AHSME 1970, [80, Problem 31]

If a number is selected at random from the set of all five-digit numbers in which the sum of the digits is equal to 43, what is the probability that the number will be divisible by 11?

10.2 The Odds for Two Teams

AHSME 1971, [80, Problem 23]

Teams A and B are playing a series of games. If the odds for either team to win any game are even and Team A must win two or Team B three games to win the series, then what are the odds for the teams to win the series?

10.3 Cutting a String into Unequal Pieces

AHSME 1972, [80, Problem 17]

A piece of string is cut in two at a point selected at random. What is the probability that the longer piece is at least x times as large as the shorter piece ($x \leq 1$)?

10.4 Getting an Arithmetic Progression with Three Dice

AHSME 1977, [6, Problem 17]

Three fair dice are tossed (all faces have the same probability of coming up). What is the probability that the three numbers turned up can be rearranged to form an arithmetic progression with common difference one?

10.5 Probability of No Distinct Positive Roots

AHSME 1979, [6, Problem 27]

An ordered pair (b, c) of integers, each of which has absolute value less than or equal to five, is chosen at random, with each such ordered pair having an equal likelihood of being chosen.

What is the probability that the equation $x^2 + bx + c = 0$ will not have distinct positive roots?

10.6 Rolling a Defective Die

2005 AMC 12A, [34, Problem 14]

On a standard die, one of the dots is removed at random with each dot equally likely to be chosen. The die is then rolled.

What is the probability that the top face has an odd number of dots?

10.7 Counting Coins

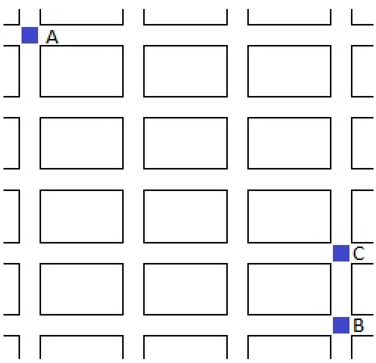
AHSME 1980, [6, Problem 20]

A box contains 2 pennies, 4 nickels and 6 dimes. Six coins are drawn without replacement, with each coin having an equal probability of being chosen. What is the probability that the value of the coins drawn is at least 50 cents?

10.8 The Roads We Take

AHSME 1982, [6, Problem 25]

The figure below is a map of part of a city: the small rectangles are blocks and the spaces in between are streets. Each morning a student walks from intersection A to intersection B, always walking along the streets shown, always going east or south. For variety, at each intersection where he has a choice, he chooses with probability $\frac{1}{2}$ (independent of all other choices) whether to go east or south.



Find the probability that, on any given morning, he walks through intersection C.

10.9 Marbles of Four Colors

AHSME 1996, [79, Problem 26]

An urn contains marbles of four colors: red, white, blue and green. When four marbles are drawn without replacement, the following events are equally likely:

1. the selection of four red marbles;
2. the selection of one white and three red marbles;
3. the selection of one white, one blue and two red marbles;
4. the selection of one marble of each color.

What is the smallest number of marbles satisfying the given conditions?

10.10 Two Modified Dice

AHSME 1997, [79, Problem 10]

Two six-sided dice are fair in the sense that each face is equally likely to turn up. However, one of the dice has the 4 replaced by 3 and the other die has the 3 replaced by 4. When these dice are rolled, what is the probability that the sum is an odd number?

10.11 Exchanging Balls of Random Colors

AMC 2006, [35, Problem 17B]

Bob and Alice each have a bag that contains one ball of each of the colors blue, green, orange, red and violet. Alice randomly selects one ball from her bag and puts it into Bob's bag. Bob then randomly selects one ball from his bag and puts it into Alice's bag. What is the probability that after this process the contents of the two bags are the same?

Solutions

Five-Digit Numbers Divisible by Eleven

The maximum sum of five decimal digits is 45. To obtain 43, we should allow either one 7 or two 8s. There are five integers of the first kind, 79999, 97999, 99799, 99979, 99997, and 10 integers of the second kind, 88999, 89899, 89989, 89998, 98899, 98989, 98998, 99889, 99898, 99988.

Numbers are divisible by 11 if their alternating sums of digits are divisible by 11. There are two numbers of the first kind that satisfy this criterion: 97999 and 99979. Indeed,

$$9 - 7 + 9 - 9 + 9 = 11,$$

$$9 - 9 + 9 - 7 + 9 = 11.$$

Of the second kind, only one number, 98989, is divisible by 11. Indeed,

$$9 - 8 + 9 - 8 + 9 = 11.$$

Out of the total of 15 numbers, three are divisible by 11. The probability of this event is $\frac{3}{15} = \frac{1}{5}$.

The Odds for Two Teams

The series would last at most four games before Team A either wins two games or loses (to Team B) three games. Team A wins the series in two games with probability $\frac{1}{4}$. There are two ways for it to win in three games: LWW or WLW. Each occurs with probability $\frac{1}{8}$ so that the probability that Team A wins in exactly three games is also $\frac{1}{4}$. Team A wins in exactly four games if one of the following events takes place: WLLW, LWLW, LLWW. Each of these happens with probability $\frac{1}{16}$ for a total of $\frac{3}{16}$. The total probability to win the series for Team A is, therefore, $\frac{1}{4} + \frac{1}{4} + \frac{3}{16} = \frac{11}{16}$. Team B wins with probability $\frac{5}{16}$. The odds are 11 to 5.

Cutting a String into Unequal Pieces

Think of the string as a segment AB of length $1 + x$. Let P be a point inside AB such that AP : PB $= x$. To satisfy the condition of the riddle, the random point should fall to the left of P. This happens with probability $\frac{1}{1+x}$. This is the probability that the longer piece is at least x times as long as the shorter piece, provided the longer piece is to the right of the shorter one. A similar reasoning applies to the case where the longer piece is to the left of the shorter one. This happens with the same probability, $\frac{1}{x+1}$. The answer is, therefore, $\frac{2}{x+1}$.

Getting an Arithmetic Progression with Three Dice

There are four possible arithmetic progressions with common difference one that could be formed by the numbers 1 to 6: $\{1,2,3\}$, $\{2,3,4\}$, $\{3,4,5\}$, $\{4,5,6\}$. Of 6^3 possible outcomes, each of these occurs in $3! = 6$ cases, making the total probability

$$4 \cdot \frac{6}{6^3} = \frac{1}{9}.$$

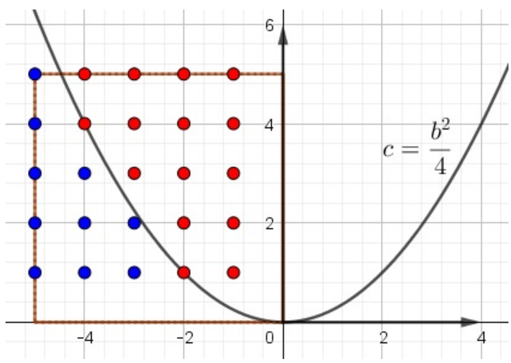
Probability of No Distinct Positive Roots

The basis for the solution is naturally the quadratic formula: $x_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4c}}{2}$.

Now, it may be easier to determine when the equation does have distinct positive roots. This happens when

1. $c > 0$,
2. $b < 0$ and
3. $b^2 - 4c > 0$.

With b on the horizontal axis and c on the vertical one, only the blue points satisfy the three conditions:



There are 10 such points and $121 = 11^2$ ordered pairs that satisfy $|b|, c \leq 5$, making the probability of distinct positive roots equal to $\dfrac{10}{121}$. The probability of not having distinct positive roots is then $1 - \dfrac{10}{121} = \dfrac{111}{121}$.

Rolling a Defective Die

Solution 1

Thamizh Kudimagan

Sum over all faces:

$$\begin{aligned}
&P(\text{face up}) \times [P(\text{being odd/no change}) \times P(\text{no change})\\
&\qquad\qquad + P(\text{being odd/change}) \times P(\text{change})]\\
&= \frac{1}{6}\left(1\cdot\frac{20}{21} + 0\cdot\frac{1}{21} + 0\cdot\frac{19}{21} + 1\cdot\frac{2}{21} + 1\cdot\frac{18}{21} + 0\cdot\frac{3}{21}\right)\\
&\qquad\qquad + \frac{1}{6}\left(0\cdot\frac{17}{21} + 1\cdot\frac{4}{21} + 1\cdot\frac{16}{21} + 0\cdot\frac{5}{21} + 0\cdot\frac{15}{21} + 1\cdot\frac{6}{21}\right)\\
&= \frac{11}{21}.
\end{aligned}$$

Solution 2

A. Bogomolny, Amit Itagi, Christopher D. Long, Aaron Haspel

This is a shortcut from solution 1.

Originally, there are $1+2+3+4+5+6=21$ dots of which $1+3+5=9$ are on the odd numbered faces and $2+4+6$ are on the even numbered faces. After a dot is removed, there are either two even faces and four odd faces or four even faces and two odd faces. The probability of the first event is $\frac{12}{21}$; the probability of the second one is $\frac{9}{21}$. In the first case, the probability of having an odd face on top is $\frac{4}{6}$; it is $\frac{2}{6}$ for the second case. Since all the variants are mutually exclusive, the total probability is

$$\frac{4}{6}\cdot\frac{12}{21} + \frac{2}{6}\cdot\frac{9}{21} = \frac{2\cdot 12 + 9}{3\cdot 21} = \frac{11}{21}.$$

Counting Coins

Solution 1

[34, p. 143]

There are 12 coins and $\binom{12}{6} = 924$ equiprobable ways to draw six of them. There are only two cases where the value of the drawn coins exceeds 50 cents:

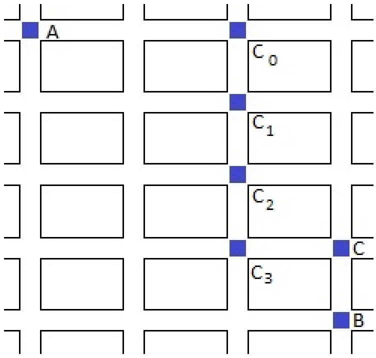
1. 6 dimes
2. 5 dimes plus any other coin but a dime
3. 4 dimes plus 2 nickels.

The number of ways the cases may occur are 1, $\binom{6}{5}\binom{6}{1} = 36$ and $\binom{6}{4}\binom{4}{2} = 90$, making the sought probability equal to $\frac{1+36+90}{924} = \frac{127}{924}$.

The Roads We Take

Solution 1

[6, p. 174]



The probability that the student passes through C is the sum of the probabilities that he passes through intersections C_i, $i = 0, 1, 2, 3$, and, at each, continues eastward. The number of ways of getting from A to C_i equals $\binom{2+i}{2}$ because each such path has two horizontal streets which may come in any order. The probability of taking each of the passes is $\left(\frac{1}{2}\right)^{3+i}$ due to the available number of choices on the way there. Thus, the total probability comes to

$$\sum_{i=0}^{3} \binom{2+i}{2} \left(\frac{1}{2}\right)^{3+i} = \frac{1}{8} + \frac{3}{16} + \frac{6}{32} + \frac{10}{64} = \frac{21}{32}.$$

Notes

Just counting the number of ways from A to B and from A to C would not work. The numbers are $\binom{6}{3} = 20$ and $\binom{7}{3} = 35$, allegedly giving the sought probability as $\frac{20}{35} = \frac{4}{7}$. The reason that this does not work is that different pathways come with different probabilities such that simple comparison of their numbers is misleading.

Solution 2

[6, p. 174]

It is easy to construct a tree diagram to count probabilities of getting at every possible intersection:

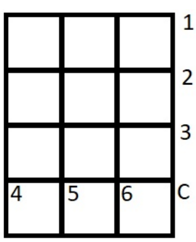
$$\begin{array}{ccccccc}
1 & \rightarrow & \frac{1}{2} & \rightarrow & \frac{1}{4} & \rightarrow & \frac{1}{8} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\frac{1}{2} & \rightarrow & \frac{1}{2} & \rightarrow & \frac{3}{8} & \rightarrow & \frac{5}{16} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\frac{1}{4} & \rightarrow & \frac{3}{8} & \rightarrow & \frac{3}{8} & \rightarrow & \frac{1}{2} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\frac{1}{8} & \rightarrow & \frac{1}{4} & \rightarrow & \frac{5}{16} & \rightarrow & \frac{21}{32} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\frac{1}{16} & \rightarrow & \frac{3}{16} & \rightarrow & \frac{11}{32} & \rightarrow & 1.
\end{array}$$

Summing up the probabilities, we observe that for the nodes on the right edge going southward is the only chance. Similarly, bottom nodes also offer only one option. At all other nodes the chances are evenly split between the neighbors on the right and below.

Solution 3

Christopher D. Long

The first location for the walker to reach the outside of the 3×3 subsquare must be one of the six numbered spots:



By symmetry, the probabilities on each side must sum to $\frac{1}{2}$. Spots 1, 2 and 3 must pass through C. From the other direction, the walker must pass through spot 6 to reach C.

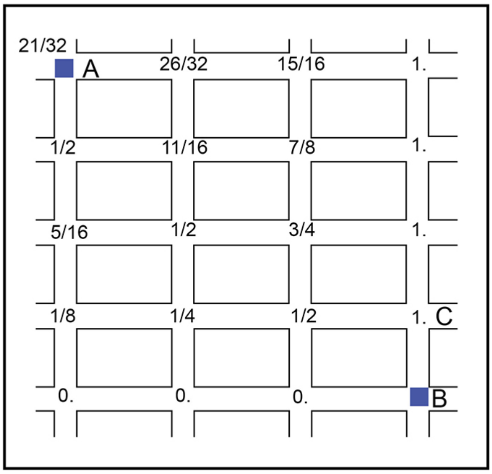
Thus, the total probability is

$$\frac{1}{2} + \binom{5}{2} \cdot \left(\frac{1}{2}\right)^5 \cdot \frac{1}{2} = \frac{21}{32}.$$

Solution 4

Josh Jordan

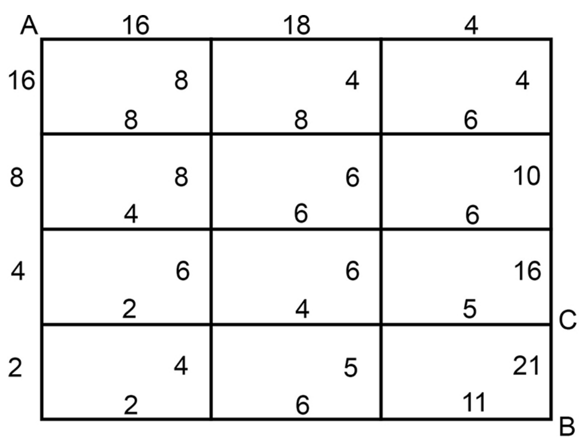
Solve using Markov chains. Label each intersection with the probability that a path through it will pass through C before reaching B. Each intersection's probability is the average of its south and east neighbors. Working backwards from B, we find the probability for A is $\frac{21}{32}$.



Solution 5

Amit Itagi

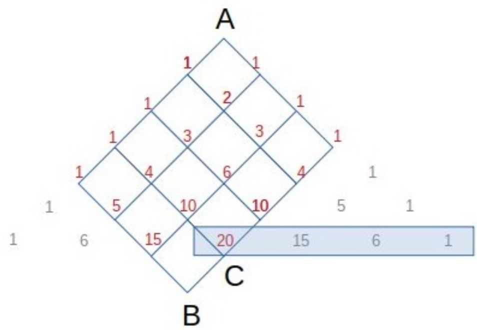
Start with three paths from A, dividing paths into equals at each step and conserving the number of paths at each vertex. 21 of the 32 paths go through C. The answer is: $\frac{21}{32}$.



Solution 6

Daren Chapin

A key observation is that Pascal's triangle—the number of ways to reach the intersection—continues beyond the eastern edge; everything that crosses that edge also gets to C. Therefore, $\dfrac{20+15+6+1}{2^6} = \dfrac{21}{32} = 0.65625$.



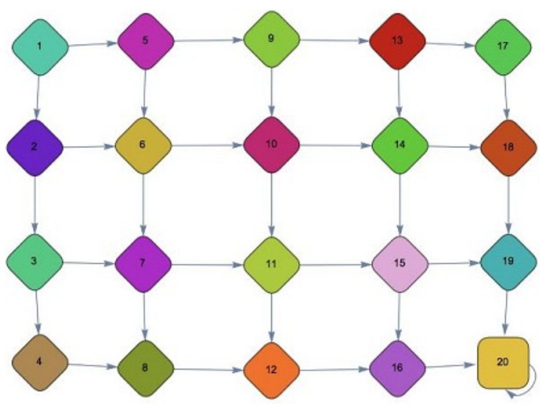
Solution 7

N.N. Taleb

We introduce the following node (intersection) enumeration:

1	5	9	13	17
2	6	10	14	18
3	7	11	15	19
4	8	12	16	20

with the connectivity graph,



so that the role of C is taken by node number 16. Here are the probabilities of passing through each of the 20 nodes:

Table[{z, Table[PDF[proc[i], z], {i, 0, 7}]//Total}, {z, 1, 20}]

$$\left\{\{1,1\},\left\{2,\frac{1}{2}\right\},\left\{3,\frac{1}{4}\right\},\left\{4,\frac{1}{8}\right\},\left\{5,\frac{1}{2}\right\},\left\{6,\frac{1}{2}\right\},\left\{7,\frac{3}{8}\right\},\right.$$
$$\left\{8,\frac{5}{16}\right\},\left\{9,\frac{1}{4}\right\},\left\{10,\frac{3}{8}\right\},\left\{11,\frac{3}{8}\right\},\left\{12,\frac{1}{2}\right\},\left\{13,\frac{1}{8}\right\},\left\{14,\frac{1}{4}\right\},$$
$$\left.\left\{15,\frac{5}{16}\right\},\left\{16,\frac{21}{32}\right\},\left\{17,\frac{1}{16}\right\},\left\{18,\frac{3}{16}\right\},\left\{19,\frac{11}{32}\right\},\{20,1\}\right\}$$

Marbles of Four Colors

Let r, w, b, g be the numbers of red, white, blue and green marbles, respectively. Denote the total number of marbles as $n = r + w + b + g$. The sample space of all possible four marble selections consists of $\binom{n}{4}$ elementary events. We just count the number of elementary events that combine into the four events described in the problem:

1. $\binom{r}{4}$,
2. $\binom{r}{3}\binom{w}{1}$,
3. $\binom{r}{2}\binom{w}{1}\binom{b}{1}$,
4. $\binom{r}{1}\binom{w}{1}\binom{b}{1}\binom{g}{1}$.

The four quantities are equal:

$$\binom{r}{4} = \binom{r}{3}\binom{w}{1} = \binom{r}{2}\binom{w}{1}\binom{b}{1} = \binom{r}{1}\binom{w}{1}\binom{b}{1}\binom{g}{1}.$$

The equalities give successively

$$\frac{r-3}{4} = w, \qquad \frac{r-2}{3} = b \qquad \text{and} \qquad \frac{r-1}{2} = g.$$

As a reminder, we are looking for integer solutions to the three equations, implying $r = 3 \pmod 4$, $r = 2 \pmod 3$, $r = 1 \pmod 2$. The Chinese Remainder Theorem [14] tells us how to find r that satisfy all three conditions. Finding the smallest such r could also be accomplished by direct inspection: $r = 11$. From here, $w = 2$, $b = 3$ and $g = 5$ so that $n = 11 + 2 + 3 + 5 = 21$, for the smallest possible r, implying that n is the smallest possible total.

All other possible values of r are given by $r = 11 + 12k$, where $k \geq 0$ is an integer.

Two Modified Dice

An odd sum may result in one of the two dice showing an even number, the other an odd number and vice versa. There are four even faces and two odd faces on one die and two even and four odd on the other.

Thus, the probabilities come out to be

$$\frac{4}{6} \cdot \frac{4}{6} + \frac{2}{6} \cdot \frac{2}{6} = \frac{20}{36} = \frac{5}{9}.$$

Exchanging Balls of Random Colors

Regardless of which color ball Alice placed into Bob's bag, there are now six balls of which two are of the same color in Bob's bag. Only if Bob picks one of these and puts it into Alice's bag will the two bags have the same contents. This happens with the probability $\frac{2}{6} = \frac{1}{3}$.

Chapter 11

Elementary Statistics

> Statistical thinking will one day be as necessary for efficient citizenship as the ability to read and write!
>
> Samuel S. Wilks, 1951, Presidential Address to the American Statistical Association

Riddles

11.1 A Question about Median

[26, Problem 65312]

One number from a given set of 100 numbers was removed, and it was found that the median of the remaining 99 was 74. The number was put back and another was removed. The median became 68.

What is the median of the whole set of 100 numbers?

11.2 Family Statistics

[36, p. 45]

Do men have more sisters than women?

11.3 How to Ask an Embarrassing Question

[24, Section 7.6], [70, pp. 15–16]

How could one conduct a poll with an absolute guarantee of anonymity?

11.4 An Integer Sequence with Given Statistical Parameters

[36, Problem 1.1.5]

Find an integer sequence of seven integers that satisfies the following conditions simultaneously:

$$\begin{cases} \text{Mean:} & 10 \\ \text{Median:} & 9 \\ \text{Mode:} & 7 \\ \text{Range:} & 15. \end{cases}$$

In addition, assume that both 9 and 10 are terms of the sequence.

11.5 Mean, Mode, Range and Median

[79, Problem 46]

A list of five positive integers has mean 12 and range 18. The mode and the median are both 8.

How many different values are possible for the second largest element of the list?

Solutions

A Question about Median

Solution 1

Place the numbers in increasing order, say, $a_1 \le a_2 \le \cdots \le a_{100}$. The median of the set is $\dfrac{a_{50}+a_{51}}{2}$.

If a number, say a_j with $j \le 50$, was removed, then a_{51} becomes the median; with $j > 50$, a_{50} becomes the median. It follows that $a_{50} = 68$ and $a_{51} = 74$.

The median of all 100 numbers is then $\dfrac{68+74}{2} = 71$.

Solution 2

N.N. Taleb

Let $S = \{a_1, a_2, \ldots, a_{50}, a_{51}, \ldots, a_{100}\}$, arranged by rank. Let $M(T)$ be the median of set T. Note that the removal of **any** term above the median has the same effect; the removal of any term below the median also has the same effect. It follows that in the two experiments the items have been removed from different halves of the set S:

$$M(S \setminus \{a_{>50}\}) = a_{50} = 68,$$

$$M(S \setminus \{a_{\le 50}\}) = a_{51} = 74.$$

The median $M(S)$ is therefore 71.

Family Statistics

An answer may be surmised from a few examples. Take a small family with two siblings: a boy and a girl. The boy has a sister, while the girl does not. In a family with a son and two daughters, the boy has two sisters, while the girls have only one each. A rule seems to emerge: a girl is excluded from sister counting, while boys count all the female siblings there are. From this, the conclusion that men should have more sisters than women seems to follow naturally. However, this conclusion is wrong. Men have as many sisters as do women. The arguments below seem sufficiently convincing to seal the result.

Solution 1

A. Bogomolny

In a family with s girls and $n-s$ boys, the boys have a total of $s(n-s)$ sisters, while the girls have $s(s-1)$ sisters. Assume that boys and girls come into the world with equal probabilities of $\dfrac{1}{2}$ and that the birth events are independent. Then there are 2^n ways a family with n children might have come about. Of this, $\dbinom{n}{s}$—the binomial coefficient "n choose s"—is the number of n-children families with s daughters. Therefore, the average number of sisters boys from n-children families have is given by

$$B_n = 2^{-n} \sum_{s=0}^{n} \binom{n}{s} \cdot s(n-s).$$

Similarly, the average number of girls' sisters in such families is

$$G_n = 2^{-n} \sum_{s=0}^{n} \binom{n}{s} \cdot s(s-1).$$

Both sums are easily computed with generating functions [16]. Let $B_n(x,y) = (x+y)^n 2^{-n}$ and $G_n(x) = (1+x)^n 2^{-n}$. Then

$$B_n = \frac{\partial^2}{\partial x\, \partial y} B_n(1,1),$$

while

$$G_n = \frac{d^2}{dx^2} G_n(1),$$

which immediately implies $B_n = G_n = \dfrac{n(n-1)}{4}$.

Solution 2

[36]

The following questions are equivalent:

1. Do men have more sisters than women?
2. Do men have more sisters than brothers?
3. Does a random child have as many sisters as brothers?
4. Does an average family have as many daughters as sons?

Yes is an obvious answer to the last question.

Solution 3

In a letter from many years ago, a correspondent described a reaction of a friend of his in these words:

> For any one person (gender unspecified), the gender distribution of their siblings is unaffected by their own gender. There are the same number of men and women to be that one person. Therefore men and women have the same number of sisters.

How to Ask an Embarrassing Question

[96]

Let us start with an abstract of the groundbreaking paper by Stanley L. Warner [96]:

> A survey technique for improving the reliability of responses to sensitive interview questions is described. The technique permits the respondent to answer "yes" or "no" to a question without the interviewer knowing what information is being conveyed by the respondent. The privacy of the interviewee is protected by randomizing his response. For example if all members of a population belong either to group A or to group B and the investigator wants to determine the proportion of group A individuals in the population, this information can be elicited by using the following procedures. Before each interview, the respondent is provided with a spinner marked

with a point A and a point B. The spinner is marked off in such a way that the spinner's marker will stop at point A with a probability of p and at point B with a probability of $1-p$. When the interviewer asks the sensitive question concerning group membership, the respondent spins the spinner out of the sight of the interviewer. The marker will either stop at point A or point B. The respondent then indicates whether or not he belongs to the group to which the marker is pointing. The respondent does not tell the interviewer where the marker is pointing. Assuming that these responses are truthful, it is then possible to determine maximum likelihood estimates of the true proportion of A's in the population. The formula for calculating these estimates is provided.

Let N be the population of respondents, P the number of those that belong to group A and M the number of yes answers received. The task is to estimate P.

Solution 1

[96]

M is comprised of truthful answers from a proportion of P and truthful answers from a proportion of $(N-P)$:

$$M = pP + (1-p)(N-P) = P(2p-1) + N(1-p)$$

from which $P = \dfrac{M - N(1-p)}{2p-1}$. Obviously, $p = \dfrac{1}{2}$ should be avoided.

Solution 2

[24, Section 7.6]

Instruct each respondent to toss two fair coins secretly. If the coins both land heads up, then that person reports the opposite of the truthful answer. Anyone who flips at least one tail answers truthfully.

As there are four equiprobable outcomes for a toss of two coins, one fourth of the population will lie, while three fourths will answer truthfully. Assuming that members of group A are in the same proportion as the general population, we obtain the following equation:

$$M = \frac{1}{4}(N-P) + \frac{3}{4}P$$

from which $P = 2\left(M - \dfrac{1}{4}N\right)$.

Solution 3

[70, pp. 15–16]

Instruct each respondent to toss a fair coin (out of sight of the interviewer). If the coin shows tails, ask the person to truthfully answer whether he/she belongs to group A. If the coin shows heads, instruct the person to record the result of the second toss, with "yes" for, say, heads and "no" for tails. Ask the person to report the final result:

$$M = \frac{1}{2}P + \frac{1}{4}N$$

from which we get the same formula as in solution 2, $P = 2\left(M - \dfrac{1}{4}N\right)$.

Notes

Formulas in solutions 2 and 3 can both be obtained formally from that in solution 1, setting $p = \frac{3}{4}$.

An Integer Sequence with Given Statistical Parameters

The riddle, as stated, has three solutions that were found—one at a time—by just playing with pencil and paper, until Amit Itagi undertook a rigorous investigation.

Solution 1

By inspection, with a little effort we find

$$4, 7, 7, 9, 10, 14, 19.$$

Solution 2

Gökhan Karahan

One solution is

$$5, 7, 7, 9, 10, 12, 20.$$

Solution 3

Bryan Johnson

One solution is

$$3, 7, 7, 9, 12, 14, 18.$$

Solution 4

Amit Itagi

Let the sequence sorted from smallest to largest values be labeled $a_i,\ i \in \{1, 2, \ldots, 7\}$. From the median, $a_4 = 9$, and one of a_5, a_6, a_7 is 10. From the mode, either two or three of a_1, a_2, a_3 are 7.

Suppose all three are 7. Then from the range, $a_7 = 7 + 15 = 22$. From the mean, the last undetermined number is $70 - 3 \times 7 - 9 - 10 - 22 = 8$. This is not possible as the undetermined number has to be greater than $a_4 = 9$.

Suppose there are two 7s in the sequence. First, suppose that 7 is the smallest number. Thus, $a_3 = 8$ or $a_3 = 9$. From the range, $a_7 = 7 + 15 = 22$. Then, from the mean, the remaining two undetermined numbers (a_3 and another one) have to add up to $70 - 2 \times 7 - 9 - 10 - 22 = 15$. Thus, the last number is either $15 - 8 = 7$ or $15 - 9 = 6$. Neither is greater than $a_4 = 9$. Thus, this case is not possible.

The remaining case is that there are two 7s and $a_1 < 7$. From the range, $a_7 = a_1 + 15$. Thus, from the mean, a_1, a_7 and the other undetermined number (denoted by b) add up to $70 - 2 \times 7 - 9 - 10 = 37$. Thus, $b + 2a_1 + 15 = 37$ or $b + 2a_1 = 22$. Note $b \le a_7 = a_1 + 15$. Thus, $22 = b + 2a_1 \le 3a_1 + 15$ or $a_1 > 2$. Noting that $a_1 < 7$, the possible combinations of (a_1, b) are $(3, 16)$, $(4, 14)$, $(5, 12)$ and $(6, 10)$. The last combination is disallowed as 7 is the mode. Thus, the possible sequences are: $(3, 7, 7, 9, 10, 16, 18)$, $(4, 7, 7, 9, 10, 14, 19)$ and $(5, 7, 7, 9, 10, 12, 20)$.

Solution 5

N.N. Taleb

Let $X = \{x_{(1)}, x_{(2)}, x_{(3)}, x_{(4)}, x_{(5)}, x_{(6)}, x_{(7)}\}$ ordered such that

$$x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(7)}.$$

We have $x_{(7)} = x_{(1)} + 15$, $x_{(4)} = 9$, and we need to solve for:

$$\frac{1}{7}\left(2x_{(1)} + x_{(2)} + x_{(3)} + x_{(5)} + x_{(6)} + 24\right) = 10,$$

with

$$(A \vee B \vee C) \wedge D \wedge E \wedge F,$$

where

$$\begin{aligned}
A &=: \ x_1 = x_2 = 7 \wedge x_5 < x_6 < x_1 + 15,\\
B &=: \ x_2 = x_3 = 7 \wedge x_5 < x_6 < x_1 + 15,\\
C &=: \ x_1 = x_2 = x_3 = 7 \wedge x_5 \leq x_6 \leq x_1 + 15,\\
D &=: \ 9 \leq x_5 \leq 10,\\
E &=: \ x_1 \leq x_2 \leq x_3 \leq 9 \leq x_5 \leq 10 \leq x_6,\\
F &=: \ ((x_5 = 9 \wedge x_6 = 10) \vee x_5 = 10)\,.
\end{aligned}$$

The acceptable sequences are therefore

$$\begin{pmatrix} 3 & 7 & 7 & 9 & 10 & 16 & 18\\ 4 & 7 & 7 & 9 & 10 & 14 & 19\\ 5 & 7 & 7 & 9 & 10 & 12 & 20 \end{pmatrix}.$$

Mean, Mode, Range and Median

Let the numbers be a, b, c, d, e, in increasing order. First, as the mean is 12, $a+b+c+d+e = 60$. Second, since the median is 8, $c = 8$. Third, since the range is 18, $e-a = 18$. Thus, we are looking at the sequence $a, b, 8, d, a+18$, satisfying $2a+b+d+26 = 60$, or $2a+b+d = 34$. In addition, 8 is a unique mode so that either $b = 8$ or $d = 8$ (or both). We come to two possibilities:

1. $2a + d = 26$, with $d \geq 8$, or
2. $2a + b = 26$, with $b \leq 8$.

In the first case, $d \geq 8$, implying $2a \leq 18$, i.e., $a \leq 9$. However, for a, $a \leq b = 8$ is a stronger inequality. At this point, we have the following information concerning the five numbers: $a \leq 8$, $b = c = 8$, $d \geq 8$, $e = a + 18$. The least value a can be is 3, for, with $a = 2$, $e = 19$, implying $D \geq 60 - 2 - 8 - 8 - 20 = 22 > 20$. Therefore, we have the following table:

a	b	c	d	e
3	8	8	20	21
4	8	8	18	22
5	8	8	16	23
6	8	8	14	24
7	8	8	12	25
8	8	8	10	26

The second largest element may be one of 10, 12, 14, 16, 18, 20.

In the second case, where $2a + b = 26$ with $a \leq b \leq 8$, we have an obvious impossibility.

Appendix A

Dependent and Independent Events

The word *independent* is ubiquitous in the study of probabilities in at least two circumstances.

- **Independent experiments**: The same or different experiments may be run in a sequence, with the sequence of outcomes being the object of interest. For example, we may be interested to study patterns of heads and tails in successive throws of a coin. We then talk of a singular *compound* experiment that combines a sequence of constituent trials. The trials—the individual experiments—may or may not affect the outcomes of later trials. If they do, the experiments are called *dependent*; otherwise, they are *independent*. The sample space of the compound experiment is formed as a product of the sample spaces of the constituent trials.

- **Independent events**: An event is a subset of a sample space. Events may or may not be independent; according to the definition, two events, A and B, are independent only if $P(A \cap B) = P(A)P(B)$.

It is a common practice to blur the distinction between these circumstances. When running *independent experiments*, the usage of the product formula $P(A \cap B) = P(A)P(B)$ is justified on combinatorial grounds. For a pair of independent events, the formula serves as a definition. An association between the two, as discussed here, provides a justification for the latter.

Consider tossing a coin three times in a row. Since each of the throws is independent of the other two, we consider all $8 = 2^3$ possible outcomes as equiprobable and assign each the probability of $\frac{1}{8}$. The following is the sample space of a sequence of three coin tosses:

$$\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

There are 2^8 possible events, but we are presently interested in, say, two:

$$A = \{HHH, HTH, THH, TTH\} \text{ and}$$

$$B = \{HHH, HHT, THH, THT\}.$$

A is the sequence of tosses in which the third one comes up heads. B is the event in which heads comes up on the second toss. Since each contains four outcomes out of the equiprobable eight,

$$P(A) = P(B) = \frac{4}{8} = \frac{1}{2}.$$

The result may have been expected: after all, $\frac{1}{2}$ is the probability of heads on a single toss. Are events A and B independent according to the definition? Indeed they are. To see that, observe that

$$A \cap B = \{HHH, THH\},$$

the event of having heads on the second and third tosses. $P(A \cap B) = \frac{2}{8} = \frac{1}{4}$. Further, let us find the conditional probability $P(A|B)$:

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{1}{4} \div \frac{1}{2} = \frac{1}{2} \\ &= P(A). \end{aligned}$$

$P(A|B) = P(A)$, and, according to the definition, events A and B are independent, as expected.

This is in fact always the case. Assume we run a sequence of (independent) experiments with, among others, two possible outcomes x and y with probabilities $P(x) = p$ and $P(y) = q$. The event that the first outcome in a sequence of experiments happens to be x has the probability of p because something happens on every trial with the probability of 1 and the combinatorial product rule applies. Similarly, the event of having the outcome of y on the second trial (in any sequence of experiments) has the probability of q. Using *random variables* V_1 and V_2 for the outcomes of the first and second experiments, we may express this in the following manner:

$$P(V_1 = x) = p \text{ and}$$

$$P(V_2 = y) = q.$$

If A and B are the corresponding events, $P(A) = p$, $P(B) = q$. The event $A \cap B$ of having x on the first experiment and y on the second has the probability of $\frac{1}{pq}$. It follows that

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} = \frac{pq}{q} = p \\ &= P(A), \end{aligned}$$

making the events A and B independent.

The following remark may be surprising but is worth being noted. Two events A and B may belong (both of them) to different sample spaces. Moreover, their dependency may be a function of the sample space at hand.

For example, assume the sample space is depicted as a rectangular part of a square grid as shown in Figure A.1. Event A is described by a 6×10 rectangle, event B by a 5×5 rectangle. The two intersect at a 3×1 rectangle.

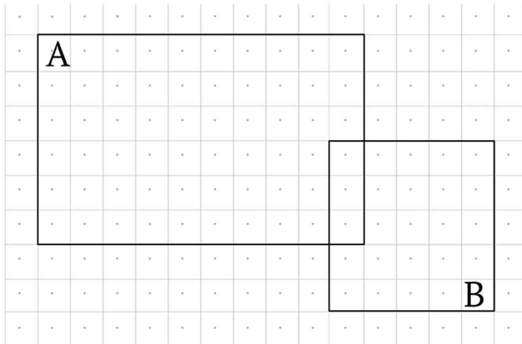


Figure A.1: Dependence depends on sample space.

First, consider the ambient sample space of size 25×20. Then

$$\frac{60}{500} = P(A) = P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{3}{25}.$$

We see that A and B are independent. However, if the sample space measures 24×20, the situation is different:

$$\frac{60}{480} = P(A) \neq P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{3}{25}.$$

Appendix B

Principle of Symmetry

According to the *Principle of Symmetry*, also known as the *Principle of Indifference*, equal or uniform probabilities should be assigned to equipossible events so that equipossible becomes equiprobable [47]. We will leave it at that with a quote from Bas C. van Fraassen's book, [40]:

> Since its inception in the seventeenth century, probability theory has often been guided by the conviction that symmetry can dictate probability. The conviction is expressed in such slogan formulations as that equipossibility implies equal probability, and honoured by such terms as indifference and sufficient reason. As in science generally we can find here symmetry arguments proper that are truly a priori, as well as arguments that simply assume contingent symmetries, and "arguments" that reflect the thirst for a hidden, determining reality. The great failure of symmetry thinking was found here, when indifference disintegrated into paradox; and great success as well, sometimes real, sometimes apparent. The story is especially important for philosophy, since it shows the impossibility of the ideal of logical probability.

Example 1

When n points are dropped on a unit segment, the distributions of lengths of the so obtained $n+1$ pieces are all the same.

To see why this indeed may be so, drop $n+1$ points on a circle. Then, because of the symmetry, all the arcs so obtained have the same distribution. Now, tear the circle at one of the points. It must be true for the resulting segment with n marked points that all its pieces have the same length distribution.

See Riddles 3.8, 5.14, 8.10, 8.16 and 8.17.

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