COL703: Logic for Computer Science (Aug-Nov 2022)

Lectures 22 & 23 (Modal Logic)

Kumar Madhukar

madhukar@cse.iitd.ac.in

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Why?

- In propositional logic, a valuation is a static assignment of truth values to atomic propositions.
- We may use atoms to describe properties of the current state of a program.
- In that case, the truth of an atom varies as the state changes.
- Modal logic is a framework to describe such a situation.

How?

- the idea is to look at a collection of possible valuations simultaneously
- each valuation represents a possible state of the world
- seperately, we specify how the possible worlds are connected to each other
- and enrich our logical language with a way of referring to truths across possible worlds

Syntax

Semantics

- a frame is a structure $F = (W, R)$
- W is a set of possible worlds; $R \subseteq W \times W$ is the accessibility relation
- in familiar terms, a frame is just a directed graph over a set of nodes W

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- a model is a pair (F, V) where $F = (W, R)$ is a frame and $V : W \to 2^{\mathcal{P}}$ be a valuation
- the notion of truth is localised to each world in a model
- $M, w \models \alpha$ denotes that α is true at the world w in the model M

Satisfaction

Exercise

Verify that $M, w \vDash \Diamond\alpha$ iff there exists w' such that wRw', and $M, w' \vDash \alpha$.

Satisfiability and Validity

Examples of valid formulas

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- $\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$ is valid
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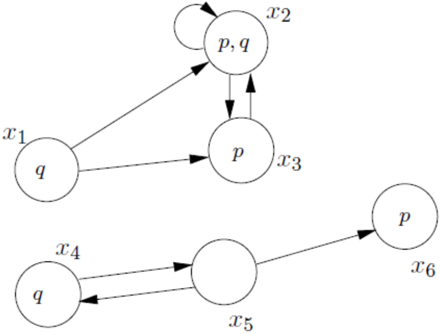
- if α is valid, $\Box\alpha$ is also valid
- what about substituition instances of propositional tautologies (e.g. $\Box\alpha \vee \neg\Box\alpha$)?

Example



Correspondence theory

- the modalities $\Box$ and $\Diamond$ can be used to describe interesting propoerties of the accessibility relation R of a frame (W, R)

- for a modal logic formula α, we identify a class of frames $\mathcal{C}_\alpha$ as follows:
  $F = (W, R) \in \mathcal{C}_\alpha$ iff for every valuation V over W, for every $w \in W$, and for every substituition instance β of α,
  $((W, R), V), w \vDash \beta$

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- the class of reflexive frames is characterised by the formula $\Box\alpha \to \alpha$

Exercise

Prove that the class of transitive frames is characterized by the formula $\Box\alpha \to \Box\Box\alpha$.

Next class

- Correspondence theory
- Sound and Complete Axiomatization

Next week

- Binary Decision Diagrams
- FOL: Soundness and Completeness, Decidable Theories

Thank you!