

9 - Intersection of Conics

EE1030:Matrix Theory

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Question:9.5.2

Find the area of the region $\{(x, y) : y^2 \leq 4x, 4x^2 + 4y^2 \leq 9\}$.

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Solution:

Variables	Description
$\mathbf{V}_1, \mathbf{u}_1, f_1$	Parameters of the parabola $y^2 = 4x$
$\mathbf{V}_2, \mathbf{u}_2, f_2$	Parameters of the circle $4x^2 + 4xy^2 = 9$
$\mathbf{x}^\top (\mathbf{V}_1 + \mu \mathbf{V}_2) \mathbf{x} + 2(\mathbf{u}_1 + \mu \mathbf{u}_2)^\top \mathbf{x} + (f_1 + \mu f_2)$	Intersection of two conics

Table 9.5.2.1 0: Variables and their description

The parameters of the given parabola are

$$\mathbf{V}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u}_1 = \begin{pmatrix} -2 \\ 0 \end{pmatrix}, f_1 = 0 \quad (0.1)$$

The parameters of the given circle are

$$\mathbf{V}_2 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, f_2 = -9 \quad (0.2)$$

For finding the intersection of two conics

$$\begin{vmatrix} \mathbf{V}_1 + \mu \mathbf{V}_2 & \mathbf{u}_1 + \mu \mathbf{u}_2 \\ (\mathbf{u}_1 + \mu \mathbf{u}_2)^\top & f_1 + \mu f_2 \end{vmatrix} = 0 \quad (0.3)$$

$$\begin{vmatrix} 4\mu & 0 & -2 \\ 0 & 1 + 4\mu & 0 \\ -2 & 0 & -9\mu \end{vmatrix} = 0 \quad (0.4)$$

$$(1 + 4\mu)(-36\mu^2 - 4) = 0 \quad (0.5)$$

The real value of μ is $-\frac{1}{4}$,

Therefore the equation of conic intersection is

$$\mathbf{x}^\top \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -2 & 0 \end{pmatrix} \mathbf{x} + \frac{9}{4} = 0 \quad (0.6)$$

$$-x^2 - 4x + \frac{9}{4} = 0 \quad (0.7)$$

$$4x^2 + 16x - 9 = 0 \quad (0.8)$$

$$x_1 = \frac{1}{2}, x_2 = -\frac{9}{2} \quad (0.9)$$

x_2 is rejected because it can't lie on the parabola

Therefore, the points of intersection are

$$\mathbf{x}_1 = \left(\frac{1}{2}, \frac{1}{\sqrt{2}} \right), \mathbf{x}_2 = \left(-\frac{1}{2}, -\sqrt{2} \right) \quad (0.10)$$

The required area is

$$2 \left(\int_0^{\frac{1}{2}} 2\sqrt{x} dx + \int_{\frac{1}{2}}^1 \sqrt{\frac{9}{4} - x^2} dx \right) \quad (0.11)$$

$$2 \left(\left[\frac{4x^{\frac{3}{2}}}{3} \right]_0^{\frac{1}{2}} + \left[\frac{x}{2} \sqrt{\frac{9}{4} - x^2} + \frac{9}{8} \sin^{-1} \left(\frac{2x}{3} \right) \right]_{\frac{1}{2}}^1 \right) \quad (0.12)$$

$$\frac{2\sqrt{2}}{3} + \frac{\sqrt{5}}{2} + \frac{9}{4} \sin^{-1} \left(\frac{2}{3} \right) - \frac{1}{\sqrt{2}} - \frac{9}{4} \sin^{-1} \left(\frac{1}{3} \right) \quad (0.13)$$

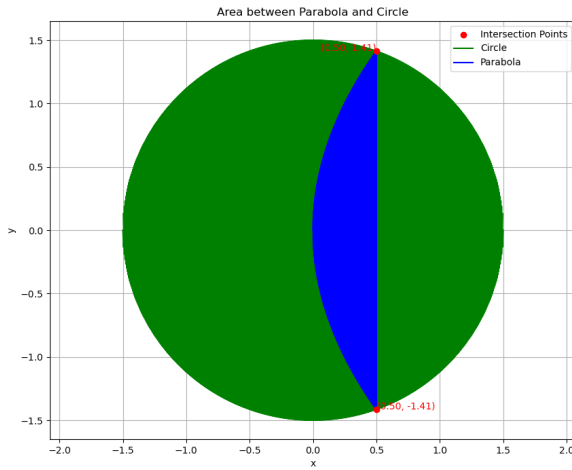


Fig. 0.1: Area between Circle and Parabola