9 - Intersection of Conics

EE1030:Matrix Theory

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Question:9.5.2
Find the area of the region $\{(x, y) : y^2 \le 4x, 4x^2 + 4y^2 \le 9\}$. (12, 2015)
Solution:

Variables	Description
$\mathbf{V}_1, \mathbf{u}_1, f_1$	Parameters of the parabola $y^2 = 4x$
$\mathbf{V}_2, \mathbf{u}_2, f_2$	Parameters of the circle $4x^2 + 4xy^2 = 9$
$\mathbf{x}^\top (\mathbf{V}_1 + \mu\mathbf{V}_2) \mathbf{x} + 2 (\mathbf{u}_1 + \mu\mathbf{u}_2)^\top \mathbf{x} + (f_1 + \mu f_2)$	Intersection of two conics

Table 9.5.2.1 0: Variables and their description

The parameters of the given parabola are

$$\mathbf{V}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u}_1 = \begin{pmatrix} -2 \\ 0 \end{pmatrix}, f_1 = 0 \tag{0.1}$$

The parameters of the given circle are

$$\mathbf{V}_2 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, f_2 = -9 \tag{0.2}$$

For finding the intersection of two conics

$$\begin{vmatrix} \mathbf{V}_1 + \mu\mathbf{V}_2 & \mathbf{u}_1 + \mu\mathbf{u}_2 \\ (\mathbf{u}_1 + \mu\mathbf{u}_2)^\top & f_1 + \mu f_2 \end{vmatrix} = 0 \tag{0.3}$$

$$\begin{vmatrix} 4\mu & 0 & -2 \\ 0 & 1 + 4\mu & 0 \\ -2 & 0 & -9\mu \end{vmatrix} = 0 \tag{0.4}$$

$$(1 + 4\mu)\left(-36\mu^2 - 4\right) = 0 \tag{0.5}$$

The real value of μ is $\frac{-1}{4}$,
Therefore the equation of conic intersection is

$$\mathbf{x}^\top \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} -2 & 0 \end{pmatrix} \mathbf{x} + \frac{9}{4} = 0 \tag{0.6}$$

$$-x^2 - 4x + \frac{9}{4} = 0 \tag{0.7}$$

$$4x^2 + 16x - 9 = 0 \tag{0.8}$$

$$x_1 = \frac{1}{2}, x_2 = -\frac{9}{2} \tag{0.9}$$

x_2 is rejected because it can't lie on the parabola
Therefore, the points of intersection are

$$\mathbf{x}_1 = \begin{pmatrix} \frac{1}{2} \\ \sqrt{2} \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} \frac{1}{2} \\ -\sqrt{2} \end{pmatrix} \tag{0.10}$$

The required area is

$$2\left(\int_0^{\frac{1}{2}} 2\sqrt{x}dx + \int_{\frac{1}{2}}^{\frac{3}{2}} \sqrt{\frac{9}{4} - x^2 dx}\right) \tag{0.11}$$

$$2\left(\left[\frac{4x^{\frac{3}{2}}}{3}\right]_0^{\frac{1}{2}} + \left[\frac{x}{2}\sqrt{\frac{9}{4} - x^2} + \frac{9}{8}\sin^{-1}\left(\frac{2x}{3}\right)\right]_{\frac{1}{2}}^{\frac{3}{2}}\right) \tag{0.12}$$

$$\frac{2\sqrt{2}}{3} + \frac{9\pi}{8} - \frac{1}{\sqrt{2}} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) \tag{0.13}$$

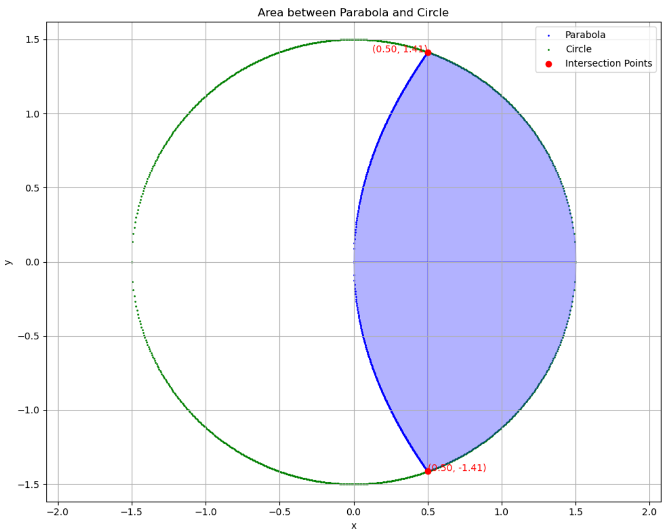

Fig. 0.1: Area between Circle and Parabola