GATE - 2022 - ST

EE1030 : Matrix Theory
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1) If the line $y = \alpha x$, $\alpha \geq \sqrt{2}$, divides the area of the region
$$R := \{(x, y) \in \mathbb{R}^2 |\ 0 \leq x \leq \sqrt{y},\ 0 \leq y \leq 2\}$$
into two equal parts, then the value of α is equal to
   a) $\frac{3}{\sqrt{2}}$
   b) $2\sqrt{2}$
   c) $\sqrt{2}$
   d) $\frac{5}{2\sqrt{2}}$

2) Let (X, Y, Z) be a random vector with the joint probability density function
$$f_{X,Y,Z}(x, y, z) = \begin{cases} \frac{1}{3}(2x + 3y + z), & 0 < x < 1, 0 < y < 1, 0 < z < 1, \\ 0, & \text{elsewhere.} \end{cases}$$
Then which one of the following points is on the regression surface of X on (Y, Z) ?
   a) $\left(\frac{4}{7}, \frac{1}{3}, \frac{1}{3}\right)$
   b) $\left(\frac{6}{7}, \frac{2}{3}, \frac{2}{3}\right)$
   c) $\left(\frac{1}{2}, \frac{1}{3}, \frac{2}{3}\right)$
   d) $\left(\frac{1}{2}, \frac{2}{3}, \frac{1}{3}\right)$

3) A random sample X of size one is taken from a distribution with the probability density function
$$f(x; \theta) = \begin{cases} \frac{2x}{\theta^2}, & 0 < x < \theta, \\ 0, & \text{elsewhere.} \end{cases}$$
If $\frac{X}{\theta}$ is used as a pivot for obtaining the confidence interval for θ, then which one of the following is an 80% confidence interval (confidence limits rounded off to three decimal places) for θ based on the observed sample value $x = 10$?
   a) $(10.541, 31.623)$
   b) $(10.987, 31.126)$

c) $(11.345, 30.524)$
d) $(11.267, 30.542)$

4) Let X_1, X_2, $\cdots$, X_7 be a random sample from a normal population with mean 0 and variance $\theta > 0$. Let

$$K = \frac{X_1^2 + X_2^2}{X_1^2 + X_2^2 + \cdots + X_7^2}.$$

Consider the following statements:

(I) The statistics K and $X_1^2 + X_2^2 + \cdots + X_7^2$ are independent.
(II) $\frac{7K}{2}$ has an F-distribution with 2 and 7 degrees of freedom.
(III) $E\left(K^2\right) = \frac{8}{63}$.

Then which of the above statements is/are true ?
a) (I) and (II) only
b) (I) and (III) only
c) (II) and (III) only
d) (I) only

5) Consider the following statements:

(I) Let a random variable X have the probability density function

$$f_X(x) = \frac{1}{2}e^{-|x|}, \quad -\infty < x < \infty.$$

Then there exist *i.i.d.* random variables X_1 and X_2 such that X and $X_1 - X_2$ have the same distribution.
(II) Let a random variable Y have the probability density function

$$f_Y(y) = \begin{cases} \frac{1}{4}, & -2 < y < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

Then there exist *i.i.d.* random variables Y_1 and Y_2 such that Y and $Y_1 - Y_2$ have the same distribution.

Then which of the above statements is/are true ?
a) (I) only
b) (II) only
c) Both (I) and (II)
d) Neither (I) nor (II)

6) Suppose X_1, X_2, $\cdots$, X_n, $\cdots$ are independent exponential random variables with the mean $\frac{1}{2}$. Let the notation *i.o.* denote 'infinitely often'. Then which of the following is/are true ?

a) $P\left(\left\{X_n > \frac{\epsilon}{2}\log_e n\right\} i.o.\right) = 1$ for $0 < \epsilon \leq 1$
b) $P\left(\left\{X_n < \frac{\epsilon}{2}\log_e n\right\} i.o.\right) = 1$ for $0 < \epsilon \leq 1$
c) $P\left(\left\{X_n > \frac{\epsilon}{2}\log_e n\right\} i.o.\right) = 1$ for $\epsilon > 1$
d) $P\left(\left\{X_n < \frac{\epsilon}{2}\log_e n\right\} i.o.\right) = 1$ for $\epsilon > 1$

7) Let $\{X_n\}, n \geq 1$, be a sequence of random variables with the probability mass functions

$$p_{X_n}(x) = \begin{cases} \frac{n}{n+1}, & x = 0, \\ \frac{1}{n+1}, & x = n, \\ 0, & \text{elsewhere.} \end{cases}$$

Let X be a random variable with $P(X = 0) = 1$. Then which of the following statements is/are true ?

a) X_n converges to X in distribution
b) X_n converges to X in probability
c) $E(X_n) \to E(X)$
d) There exists a subsequence $\{X_{n_k}\}$ of $\{X_n\}$ such that X_{n_k} converges to X almost surely

8) Let $\mathbf{M}$ be any 3×3 symmetric matrix with eigenvalues 1, 2 and 3. Let $\mathbf{N}$ be any 3×3 matrix with real eigenvalues such that $\mathbf{MN} + \mathbf{N}^T\mathbf{M} = 3\mathbf{I}$, where $\mathbf{I}$ is the 3×3 identity matrix. Then which of the following cannot be eigenvalue(s) of the matrix $\mathbf{N}$?

a) $\frac{1}{4}$
b) $\frac{3}{4}$
c) $\frac{1}{2}$
d) $\frac{7}{4}$

9) Let $\mathbf{M}$ be a 3×2 real matrix having a singular value decomposition as $\mathbf{M} = \mathbf{USV}^T$, where the matrix $\mathbf{S} = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^T$, $\mathbf{U}$ is a 3×3 orthogonal matrix, and $\mathbf{V}$ is a 2×2 orthogonal matrix. Then which of the following statements is/are true ?

a) The rank of the matrix $\mathbf{M}$ is 1
b) The trace of the matrix $\mathbf{M}^T\mathbf{M}$ is 4
c) The largest singular value of the matrix $\left(\mathbf{M}^T\mathbf{M}^{-1}\right)\mathbf{M}^T$ is 1
d) The nullity of the matrix $\mathbf{M}$ is 1

10) Let X be a random variable such that

$$P\left(\frac{a}{2\pi}X \in \mathbb{Z}\right) = 1, \ a > 0,$$

where $\mathbb{Z}$ denotes the set of all integers. If $\phi_X(t), \ t \in \mathbb{R}$, denotes the characteristic function of X, then which of the following is/are true ?

a) $\phi_X(a) = 1$
b) $\phi_X(\cdot)$ is periodic with period a

c) $|\phi_X(t)| < 1$ for all $t \neq a$

d) $\int_0^{2\pi} e^{-itn}\phi_X(t)\,dt = \pi P\left(X = \frac{2\pi n}{a}\right),\ n \in \mathbb{Z},\ i = \sqrt{-1}$

11) Which of the following real valued functions is/are uniformly continuous on $[0, \infty)$?

a) $\sin^2(x)$

b) $x\sin(x)$

c) $\sin(\sin(x))$

d) $\sin(x\sin(x))$

12) Two independent random samples, each of size 7, from two populations yield the following values :

Population 1	18	20	16	20	17	18	14
Population 2	17	18	14	20	14	13	16

If Mann-Whitney U test is performed at 5% level of significance to test the null hypothesis H_0: Distributions of the populations are same, against the alternative hypothesis H_1: Distributions of the populations are not same, then the value of the test statistic U (**in integer**) for the given data, is ____

13) Consider the multiple regression model

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \epsilon,$$

where ϵ is normally distributed with mean 0 and variance $\sigma^2 > 0$, and β_0, β_1, β_2, β_3 are unknown parameters. Suppose 52 observations of (Y, X_1, X_2, X_3) yield sum of squares due to regression as 18.6 and total sum of squares as 79.23. Then, for testing the null hypothesis $H_0\colon \beta_1 = \beta_2 = \beta_3 = 0$ against the alternative hypothesis $H_1\colon \beta_i \neq 0$ for some $i = 1, 2, 3$, the value of the test statistic (**rounded off to three decimal places**), based on one way analysis of variance, is ____