GATE - 2021 - ST

EE1030 : Matrix Theory
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1) Let A be a 3×3 real matrix such that I_3+A is invertible and let $B = (I_3 + A)^{-1} (I_3 - A)$, where I_3 denotes the 3×3 identity matrix. Then which one of the following statements is true?
   a) If B is orthogonal, then A is invertible
   b) If B is orthogonal, then all the eigenvalues of A are real
   c) If B is skew-symmetric, then A is orthogonal
   d) If B is skew-symmetric, then the determinant of A equals -1

2) Let X be a random variable having Poisson distribution such that $E\left(X^2\right) = 110$. Then which one of the following statements is NOT true?
   a) $E(X^n) = 10E\left[(X + 1)^{n-1}\right]$, for all $n = 1, 2, 3, \cdots$
   b) $P(X \text{ is even}) = \frac{1}{4}\left(1 + e^{-20}\right)$
   c) $P(X = k) < P(X = k + 1)$, for $k = 0, 1, \cdots, 8$
   d) $P(X = k) > P(X = k + 1)$, for $k = 10, 11, \cdots$

3) Let X be a random variable having uniform distribution on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Then which one of the following statements is NOT true?
   a) $Y = \cot(X)$ follows standard Cauchy distribution
   b) $Y = \tan(X)$ follows standard Cauchy distribution
   c) $Y = -\log_e\left(\frac{1}{2} + \frac{X}{\pi}\right)$ has moment generating function $M(t) = \frac{1}{1-t},\ t < 1$
   d) $Y = -2\log_e\left(\frac{1}{2} + \frac{X}{\pi}\right)$ follows central chi-square distribution with one degree of freedom

4) Let $\Omega = \{1, 2, 3, \cdots\}$ represent the collection of all possible outcomes of a random experiment with probabilities $P(\{n\}) = \alpha n$ for $n \in \Omega$. Then which one of the following statements is NOT true?
   a) $\lim_{n\to\infty} \alpha_n = 0$
   b) $\sum_{n=1}^{\infty} \sqrt{\alpha_n}$ converges
   c) For any positive integer k, there exist k disjoint events $A_1, A_2, \cdots, A_k$ such that $P\left(\bigcup_{i=1}^{k} A_i\right) < 0.001$

d) There exists a sequence $\{A_i\}_{i\geq 1}$ of strictly increasing events such that $P\left(\bigcup_{i=1}^{k} A_i\right) < 0.001$

5) Let (X, Y) have the joint probability density function

$$f_{X,Y}(x, y) = \begin{cases} \frac{4}{(x+y)^3}, & x > 1, y > 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then which one of the following statements is NOT true?

a) The probability density function of $X + Y$ is

$$f_{X+Y}(z) = \frac{4}{z^3}(z-2), z > 2,$$
$$0, \text{otherwise.}$$

b) $P(X + Y > 4) = \frac{3}{4}$
c) $E(X + Y) = 4\log_e 2$
d) $E(Y|X = 2) = 4$

6) Let X_1, X_2 and X_3 be three uncorrelated random variables with common variance $\sigma^2 < \infty$. Let $Y_1 = 2X_1 + X_2 + X_3$, $Y_2 = X_1 + 2X_2 + X_3$ and $Y_3 = X_1 + X_2 + 2X_3$. Then which of the following statements is/are true?
P : The sum of eigenvalues of the variance covariance matrix of (Y_1, Y_2, Y_3) is $18\sigma^2$.
Q : The correlation coefficient between Y_1 and Y_2 equals that between Y_2 and Y_3.

a) P only
b) Q only
c) Both P and Q
d) Neither P nor Q

7) Let $\{X_n\}_{n\geq 0}$ be a time-homogeneous discrete time Markov chain with either finite or countable state space S. Then which one of the following statements is true?

a) There is at least one recurrent state
b) If there is an absorbing state, then there exists at least one stationary distribution
c) If all the states are positive recurrent, then there exists a unique stationary distribution
d) If $\{X_n\}_{n\geq 0}$ is irreducible, $S = \{1, 2\}$ and $[\pi_1], [\pi_2]$ is a stationary distribution, then $\lim_{n\to\infty} P(X_n = i|X_0 = i) = \pi_i$ for $i = 1, 2$

8) Let customers arrive at a departmental store according to a Poisson process with rate 10. Further, suppose that each arriving customer is either a male or a female with probability $\frac{1}{2}$ each, independent of all other arrivals. Let $N(t)$ denote the total number of customers who have arrived by time t. Then which one of the following statements is NOT true?

a) If S_2 denotes the time of arrival of the second female customer, then $P(S_2 \leq 1) = 25\int_0^1 se^{-5s}ds$

b) If $M(t)$ denotes the number of male customers who have arrived by time t, then $P\left(M\left(\frac{1}{3}\right) = 0 | M(1) = 1\right) = \frac{1}{3}$
c) $E\left[(N(t))^2\right] = 100t^2 + 10t$
d) $E[N(t)N(2t)] = 200t^2 + 10t$

9) Let $X_{(1)} < X_{(2)} < X_{(3)} < X_{(4)} < X_{(5)}$ be the order statistics corresponding to a random sample of size 5 from a uniform distribution on $[0, \theta]$, where $\theta \in (0, \infty)$. Then which of the following statements is/are true?
P : $3X_{(2)}$ is an unbiased estimator of θ.
Q : The variance of $E[2X_{(3)}|X_{(5)}]$ is less than or equal to the variance of $2X_{(3)}$.
a) P only
b) Q only
c) Both P and Q
d) Neither P nor Q

10) Let $X_1, X_2, \cdots, X_n$ be a random sample of size $n (\geq 2)$ from a distribution having the probability density function

$$f(x; \theta) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}}, & x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $\theta \in (0, \infty)$. Let $X_{(1)} = \min\{X_1, X_2, \cdots, X_n\}$ and $T = \sum_{i=1}^{n} X_i$. Then $E(X_{(1)}|T)$ equals
a) $\frac{T}{n^2}$
b) $\frac{T}{n}$
c) $\frac{(n+1)T}{2n}$
d) $\frac{(n+1)^2 T}{4n^2}$

11) Let $X_1, X_2, \cdots, X_n$ be a random sample of size $n (\geq 2)$ from a uniform distribution on $[-\theta, \theta]$, where $\theta \in (0, \infty)$. Let $X_{(1)} = \min\{X_1, X_2, \cdots, X_n\}$ and $X_{(n)} = \max\{X_1, X_2, \cdots, X_n\}$. Then which of the following statements is/are true ? P : $(X_{(1)}, X_{(n)})$ is a complete statistic.
Q : $X_{(n)} - X_{(1)}$ is an ancillary statistic.
a) P only
b) Q only
c) Both P and Q
d) Neither P nor Q

12) Let $\{X_n\}_{n \geq 1}$ be a sequence of independent and identically distributed random variables having common distribution function $F(\cdot)$. Let $a < b$ be two real numbers such that $F(x) = 0$ for all $x \leq a$, $0 < F(x) < 1$ for all $a < x < b$, and $F(x) = 1$ for all $x \geq b$. Let $S_n(x)$ be the empirical distribution function at x based on $X_1, X_2, \cdots, X_n$, $n \geq 1$. Then which one of the following statements is NOT true ?

a) $P\left[\lim_{n\to\infty}\sup_{-\infty<x<\infty}|S_n(x)-F(x)|=0\right]=1$

b) For fixed $x \in (a,b)$ and $t \in (-\infty,\infty)$, $\lim_{n\to\infty} P\left[\frac{\sqrt{n}|S_n(x)-F(x)|}{\sqrt{S_n(x)(1-S_n(x))}} \leq t\right] = P(Z \leq t))$, where Z is the standard normal random variable

c) The covariance between $S_n(x)$ and $S_n(y)$ equals $\frac{1}{n}F(x)(1-F(y))$ for all $n \geq 2$ and for fixed $-\infty < x, y < \infty$

d) If $Y_n = \sup_{-\infty<x<\infty}(S_n(x)-F(x))^2$, then $\{4n\ Y_n\}_{n\geq 1}$ converges in distribution to a central chi-square random variable with 2 degrees of freedom

13) Let the joint distribution of random variables X_1, X_2, X_3 and X_4 be $N_4\left(\underline{\mu}, \Sigma\right)$, where

$$\underline{\mu} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \text{ and } \Sigma = \begin{bmatrix} 1 & 0.2 & 0 & 0 \\ 0.2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0.2 \\ 0 & 0 & 0.2 & 1 \end{bmatrix}.$$

Then which one of the following statements is true ?

a) $\frac{5}{17}\left[(X_1+X_2)^2 + (X_3+X_4-1)^2\right]$ follows a central chi-square distribution with 2 degrees of freedom

b) $\frac{1}{3}\left[(X_1+X_2)^2 + (X_3+X_4-1)^2\right]$ follows a central chi-square distribution with 2 degrees of freedom

c) $E\left[\sqrt{\left|\frac{X_1+X_2-1}{X_3+X_4-1}\right|}\right]$ is NOT finite

d) $E\left[\left|\frac{X_1+X_2+X_3+X_4-2}{X_1+X_2-X_3-X_4}\right|\right]$ is NOT finite